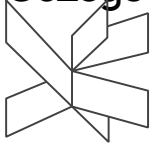


Gør tanke til handling

**VIA University  
College**



# Set theory

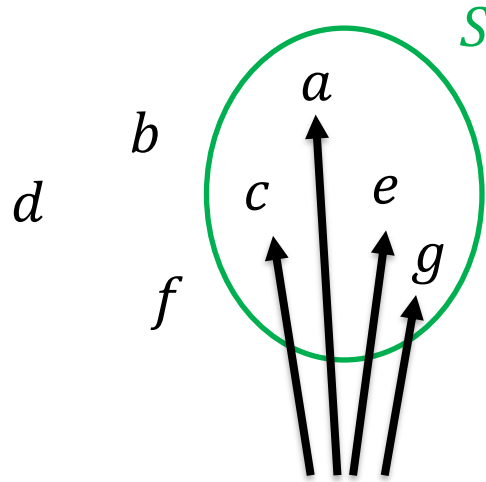


# Sets



# Sets

Collections of definite and separate objects



The **elements** of the **set**  $S$ .



# Specifying a set 1: Listing notation

One way of specifying a set is by simply listing its elements.

## Examples:

$$A = \{a, b, c, d, e\}$$

$$B = \{\text{elephant, rhino, hippo, giraffe, buffalo}\}$$

$$C = \{1, 2, 3\}$$

$$D = \{0.1, 1, \pi, 8.2939, \dots\}$$

Note: order doesn't matter!  $\{a, b, c\} = \{c, a, b\}$

Note: each element only appears once:  $\{a, a, a, b, b, c, a, b\} \rightarrow \{a, b, c\}$ .



# Specifying a set 2: Set builder notation

Another way to specify a set is by *describing* it.

This is done using the symbol  $|$ , which means “such that”.

Examples:

$$A = \{l | l \text{ is a vowel in the English alphabet}\}$$

$$B = \{a | a \text{ is one of the five largest animals living in Africa}\}$$

$$C = \{n | n \text{ is an integer greater than 0 and less than 4}\}$$

$$D = \{x | x \in \mathbb{R}, 0 < x < 9\}$$

An equivalent way to write  $C$  and  $D$ :

$$C = \{n \text{ is an integer} | n \text{ is greater than 0 and less than 4}\}$$

$$D = \{x \in \mathbb{R} | 0 < x < 9\}$$



# A few important sets



# The sets $\mathbb{N}$ , $\mathbb{Z}$ , $\mathbb{Q}$ and $\mathbb{R}$

In the previous slide, we referred to the set  $\mathbb{R}$  - the set of real numbers.

This is one of the fundamental number sets, which we will be referring to a lot during the course. Below these sets are listed:

- The set of natural numbers (positive integers):  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$  sometimes, 0 is included as well

- The set of integers:  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

All the numbers that can be written as fractions

- The rational numbers:  $\mathbb{Q} = \{-\frac{2}{5}, \frac{3}{3}, \frac{23}{393}, \frac{239,129}{2093}, \dots\}$

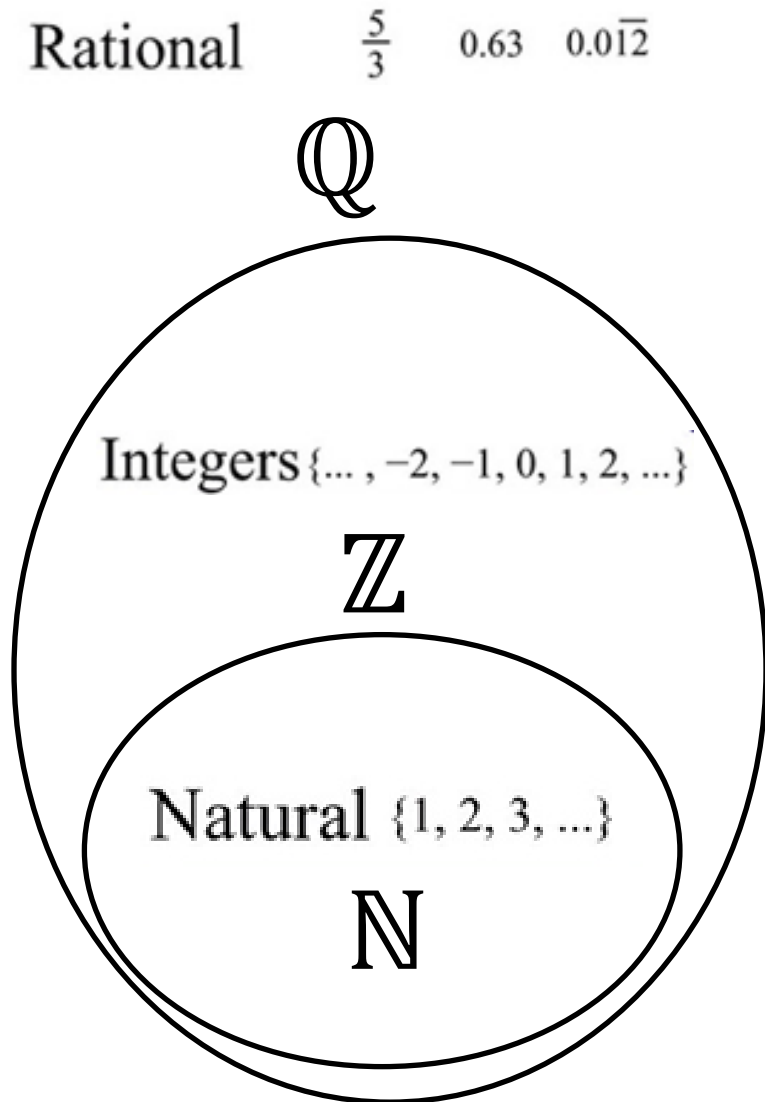
All the numbers that cannot be written as fractions

- The set of irrational numbers:  $\mathbb{I} = \{\dots, \sqrt{2}, e, \pi, \dots\}$

- The set of real numbers :  $\mathbb{R} = \{-17.8923, -10, \dots\}$

The continuum of all numbers from  $-\infty$  to  $+\infty$  - this consist of  $\mathbb{Q}$  together with  $\mathbb{I}$ .

# The real numbers: $\mathbb{R}$



Irrational

$\sqrt{3}$   $\pi$  0.10010001...

$\mathbb{I}$

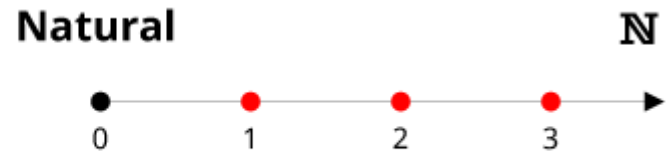




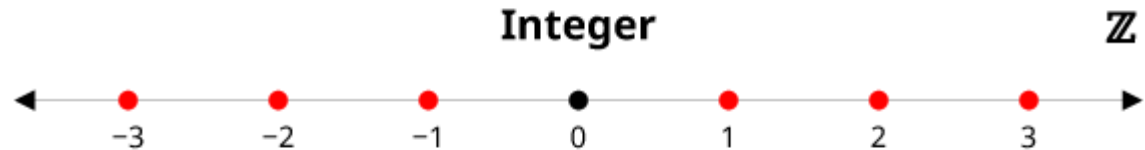
# The number line

We can imagine that all numbers lie on a line (or axis) from  $-\infty$  to  $\infty$ :

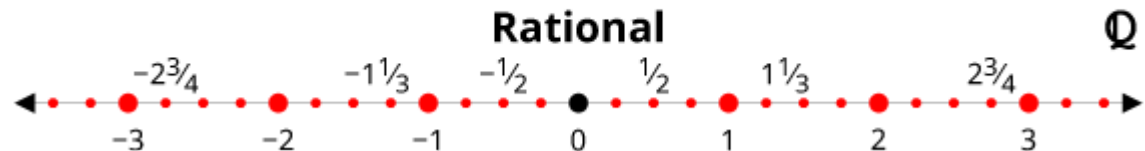
Start with the counting numbers  
(zero may be included):



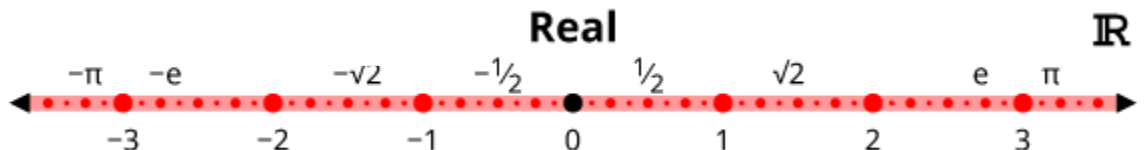
Extend the line backward to include  
the negatives:



Insert all the fractions:

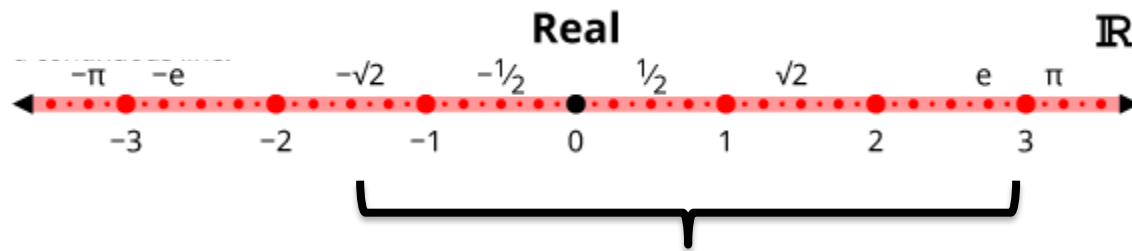


Fill in all the irrational numbers to  
make a continuous line:



# Interval notation

Consider an interval of the numbers on the real line – say from  $-\sqrt{2}$  to 3:



We use the following notation to denote this interval:

$$[-\sqrt{2}; 3]$$

if the endpoints are included

$$(-\sqrt{2}; 3) \quad \text{or} \quad ]-\sqrt{2}; 3[$$

if the endpoints are not included

We may also consider intervals that extends all the way to  $-\infty$  for example. For this, we use the same notation – however, note that  $\infty$  or  $-\infty$  are never included:

$$(-\infty; 3]$$

$\leftarrow$  interval from minus infinity to 3.

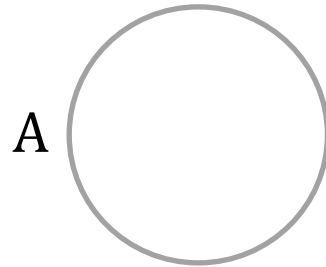


# Venn diagrams and subsets

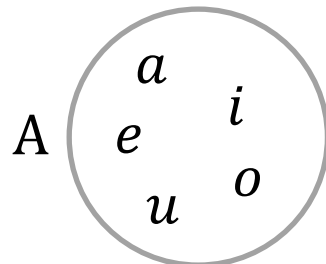


# Venn diagrams

Graphical representations of sets as geometrical shapes (circles, ellipses, rectangles, ...):



Sometimes the set's elements are also shown:



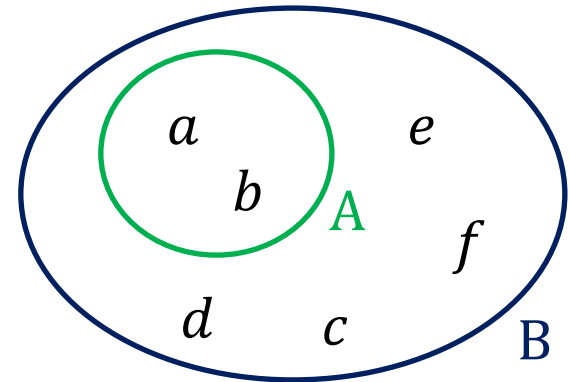
# Subsets

A set  $A$  is a subset of another set  $B$  if all its elements are within it.  
It is written  $A \subseteq B$ .

Note that this is fulfilled if  $A$  and  $B$  are equal!

Example 1:

$$A = \{a, b\}; B = \{a, b, c, d, e, f\} \Rightarrow A \subseteq B$$



Example 2:

$$A = \{1, 2, 3, 4\}; B = \{4, 3, 2, 1\} \Rightarrow A \subseteq B$$



# Proper subsets

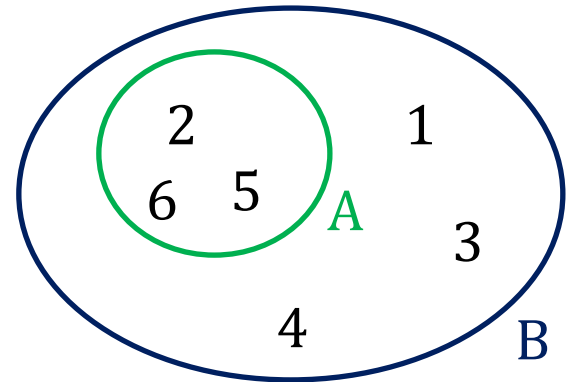
A set  $A$  is a **proper** subset of  $B$  if  $A$  is a subset of  $B$  but  $A \neq B$ .

This is written  $A \subset B$ .

## Example

$$A = \{2,5,6\}, B = \{1,2,3,4,5,6\} \Rightarrow A \subset B$$

Note: in this case, it is also true that  $A \subseteq B$ .



## Counterexample

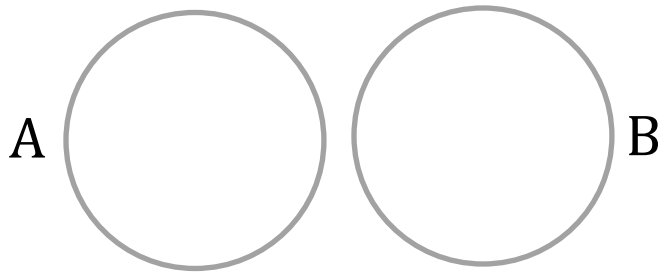
$$S = \{1,2\}, T = \{1,2\} \Rightarrow S \subseteq T, \text{ but } S \not\subset T.$$



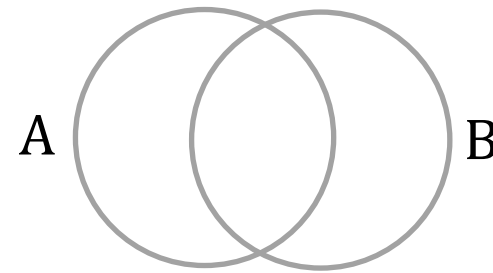
# Disjoint sets

Two sets are ***disjoint*** if they do not have anything in common.

Disjoint:



Not disjoint:



Examples:

$\{1,2,3\}$  and  $\{3,4,5\}$  are not disjoint.

$\{1,3,5\}$  and  $\{2,4,6\}$  are disjoint.

$\{1\}$  and  $\{a, b, \{1\}\}$  are disjoint.

$\emptyset$  and any set (even  $\emptyset$  itself!) are disjoint.



# The empty set

$$\emptyset = \{ \}$$

The empty set is a subset of every set!

To illustrate this, imagine a set  $A$  and assume that  $\emptyset$  is not a subset of  $A$ . That means that  $\emptyset$  has at least one element which is not in  $A$ . But that's a contradiction, because  $\emptyset$  has no elements!





# The size – or order - of a set

The size (or order) of a set is the number of elements in the set.  
It is written as  $|A|$ .

## Example

$$A = \{1, 2, 3, 4\} \Rightarrow |A| = 4$$

$$B = \{6, 2, g, 9, \pi\} \Rightarrow |B| = 5$$

$$C = \{x \in \mathbb{N} \mid 0 < x < 3\} \Rightarrow |C| = 2$$

$$D = \{x \in \text{The Alphabet} \mid x \text{ is a vowel}\} \Rightarrow |D| = 5$$

$$|\emptyset| = 0$$



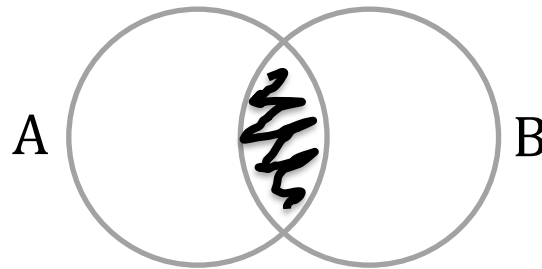
# Set operations



# Intersection

The intersection between two sets is the set of elements they have in common.

It is written  $A \cap B$ .



## Example:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

$\Rightarrow A \cap B = \{\text{whale, sea lion}\}$

Note: This is analogous to the Boolean AND-operator.

$A \cap B = \{x | x \in A \wedge x \in B\}$

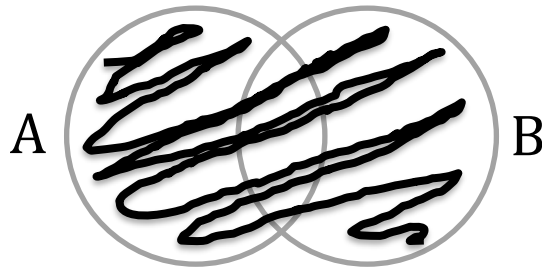
= “The set of elements which are in both  $A$  AND  $B$ ”



# Union

The union between two sets,  $A$  and  $B$ , is the set of all elements in  $A$  and all elements in  $B$ .

It is written  $A \cup B$ .



Note: This is analogous to the Boolean OR-operator.

$$A \cup B = \{x | x \in A \vee x \in B\}$$

= “The set of elements which are in either  $A$  OR  $B$ ”

Example:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

$$\Rightarrow A \cup B = \{\text{shark, whale, penguin, sea lion, dog, cat, bear}\}$$

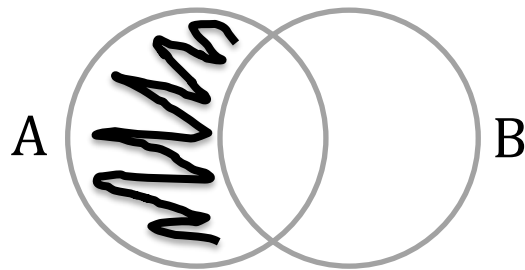


# Set difference

The set difference between two sets  $A$  and  $B$  is the set of all the elements in  $A$  which are not elements in  $B$ .

This is also called the relative complement between  $A$  and  $B$ .

It is written  $A - B$ .



$$A - B = \{x | x \in A \wedge x \notin B\}$$

Example:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

$$\Rightarrow A - B = \{\text{shark, penguin}\}$$

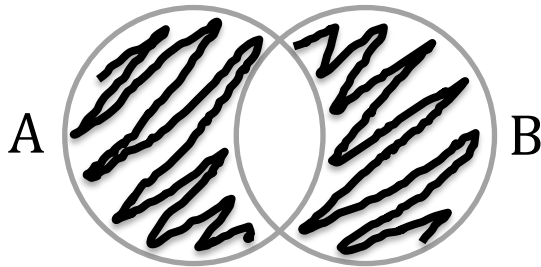


# The symmetric difference

The symmetric difference between two sets  $A$  and  $B$  is the set of all the elements in  $A$  and all the elements in  $B$  except the part they have in common.

It is written  $A \oplus B$ .

Note: This is analogous to the Boolean XOR-operator.



$$A \oplus B = \{x | x \in (A \cup B) \wedge x \notin (A \cap B)\}$$

= “The set of elements which are in either  $A$  or  $B$ , but not both”

Example:

$A = \{\text{shark, whale, penguin, sea lion}\};$

$B = \{\text{dog, cat, whale, bear, sea lion}\}$

$$\Rightarrow A \oplus B = \{\text{shark, penguin, dog, cat, bear}\}$$

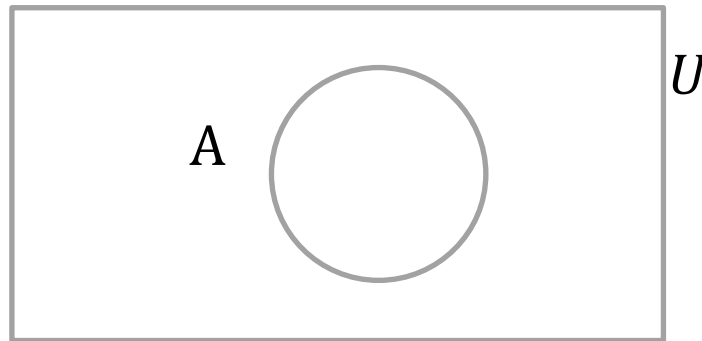


# Universal set

The **universal set** – also called “the universe” – is the *set of all elements* from which any given set is drawn.

This depends on the context!

The universal set is usually denoted  $U$  and is drawn as a rectangle.



## Examples:

For the set  $\{-5, -3, 2, 3, 4\}$ ,  $U = \mathbb{Z}$ .

For the set  $\{0.1, \pi, 5, 8.358\}$ ,  $U = \mathbb{R}$ .

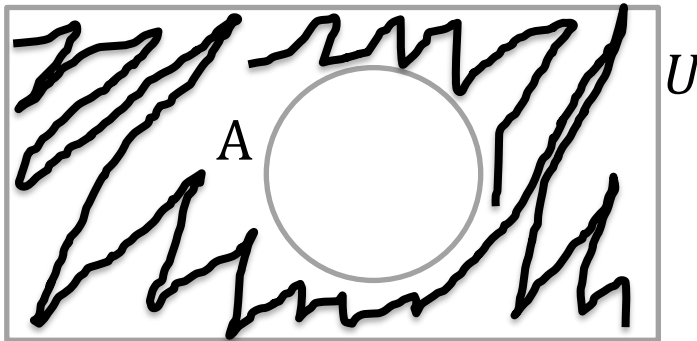
For the set  $\{\text{Congo, Togo, Egypt}\}$ ,  $U = \text{countries}$   
or maybe  $U = \text{African countries}$ .



# The complement

The complement to a set is everything in “the universe”, or the universal set, that is not in the set.

It is written  $A^C$  (or sometimes  $\bar{A}$  or  $A'$ ).



Note: This is analogous to the Boolean NOT-operator.

$A^C = \{x \in U \mid x \notin A\}$

= “The set of elements which are NOT in  $A$ ”

## Examples:

$A = \{3, 5\}$  and  $U = \text{primes below } 10 \Rightarrow A^C = \{2, 7\}$

$B = \{\text{red, blue}\}$  and  $U = \text{primary colors} \Rightarrow A^C = \{\text{yellow}\}$

$C = \{x \mid 0 \leq x \leq 2\}$  and  $U = \mathbb{R} \Rightarrow C^C = \{x \mid x < 0 \vee 2 < x\}$





# Membership tables



# Membership tables

Consider the following question

“An element is in the sets  $B$  and in the set  $C$ , but not in the set  $A$ . Is it in the set  $(A \cup C) \cap B$ ?”

One way to answer any question of this type is to use a so-called **membership table**. A membership table for a combination of the sets  $A$ ,  $B$  and  $C$  contains one row for all possible combinations of membership in each of these three sets, as you can see in the next slide.



# Example of a membership table

This column represent the set we are interested in.

To answer whether an element is in  $(A \cup C) \cap B$ , it is helpful first to determine if it is in  $A \cup C$ .

This row represent elements which are neither in  $A, B$  nor  $C$ .

This row represent elements which are both in  $B$  and  $C$  but not in  $A$ .

This row represent elements which are both in  $A, B$  and  $C$ .

$A$	$B$	$C$	$A \cup C$	$(A \cup C) \cap B$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

If an element is neither in  $A$  nor  $C$ , it is not in  $A \cup C$ .

If an element is in  $A$  but not in  $C$ , it is in  $A \cup C$ .

If an element is in  $B$  but not in  $A \cup C$ , it is not in  $(A \cup C) \cap B$ .

If an element is in both  $A \cup C$  and in  $B$ , it is in  $(A \cup C) \cap B$ .

# Example of a membership table

This column represent the set we are interested in.



$A$	$B$	$C$	$A \cup C$	$(A \cup C) \cap B$
0	0	0	0	0
0	0	1	1	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

This row represent elements which are both in  $B$  and  $C$  but not in  $A$ .



**“An element is in the sets  $B$  and in the set  $C$ , but not in the set  $A$ . Is it in the set  $(A \cup C) \cap B$ ?”**

**From the table, it is easy to see that the answer is yes!**



# Showing equalities with membership tables

We can use a membership-table to show that two sets are the same. This will be the case if the columns corresponding to the two sets are identical.

Example: Showing that  $A \cap B = B - (B - A)$

$A$	$B$	$A \cap B$	$B - A$	$B - (B - A)$
1	1	1	0	1
1	0	0	0	0
0	1	0	1	0
0	0	0	0	0

For all combinations of membership in  $A$  and  $B$ , an element is in  $B - (B - A)$  if and only if it is in  $A \cap B$ . This proves that  $A \cap B = B - (B - A)$ .

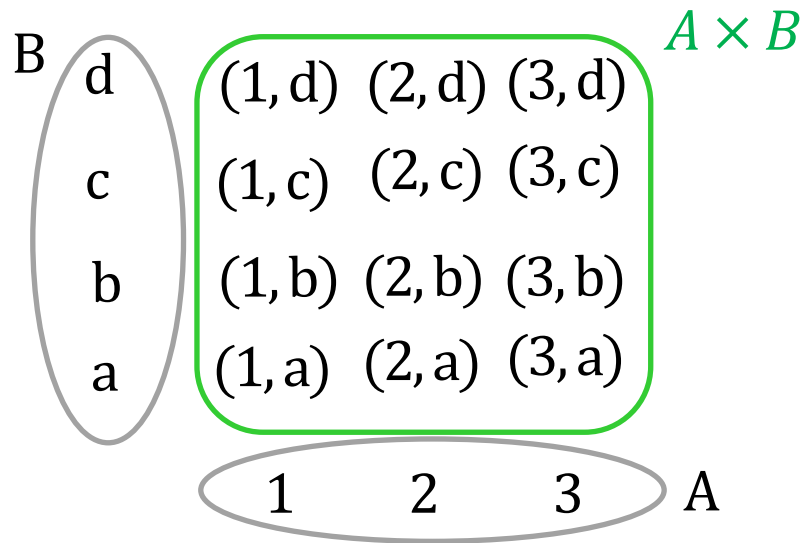


# Cartesian products and tuples



# Cartesian product 1

The Cartesian product between two sets  $A$  and  $B$  is the combination of every element in  $A$  with every element in  $B$ .



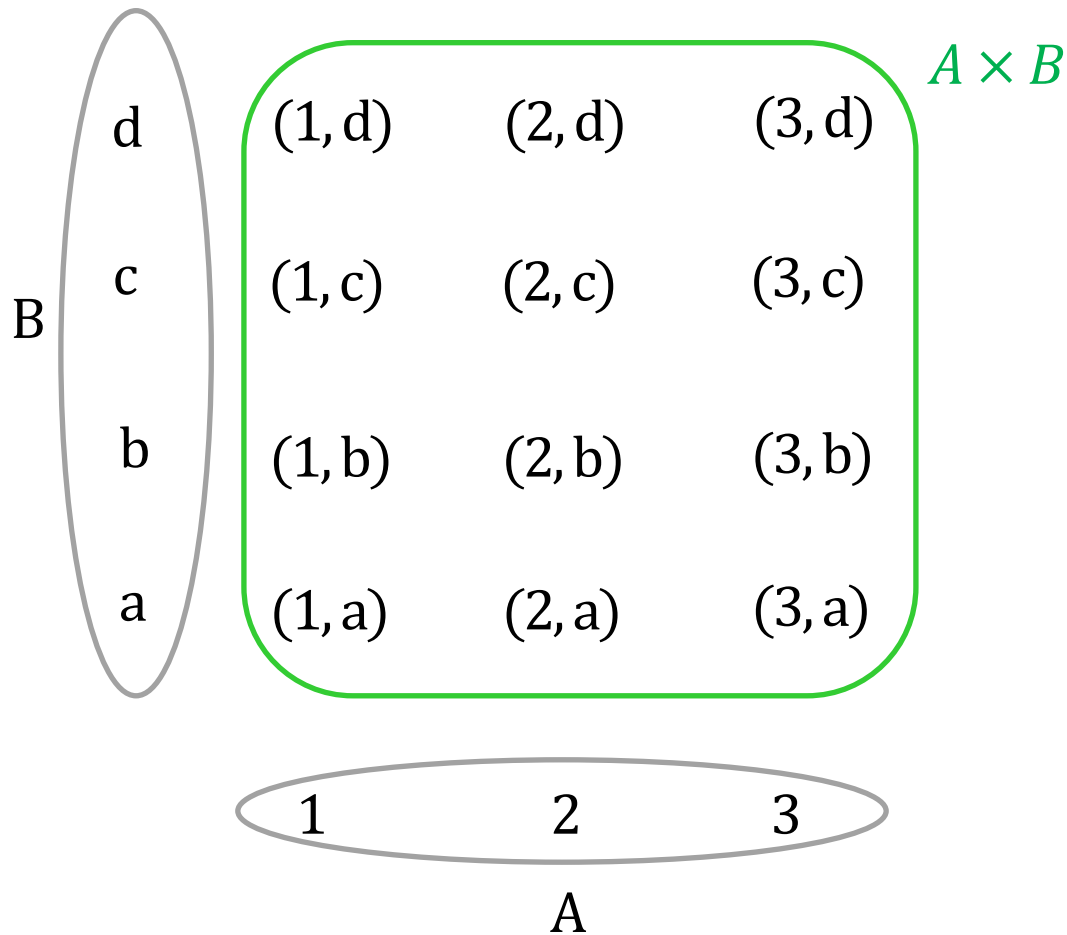
$$A \times B = \{(x, y) | x \in A \wedge y \in B\}$$

## Examples:

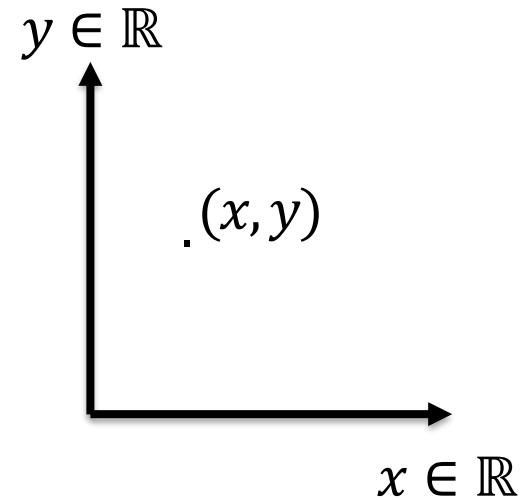
$$C = \{0, 1\}, D = \{x, y\} \Rightarrow C \times D = \{(0, x), (0, y), (1, x), (1, y)\}$$

$$S = \{u, v, z\}, T = \{u, v\} \Rightarrow S \times T = \{(u, u), (u, v), (v, u), (v, v), (z, u), (z, v)\}$$

# Cartesian product : $A \times B$



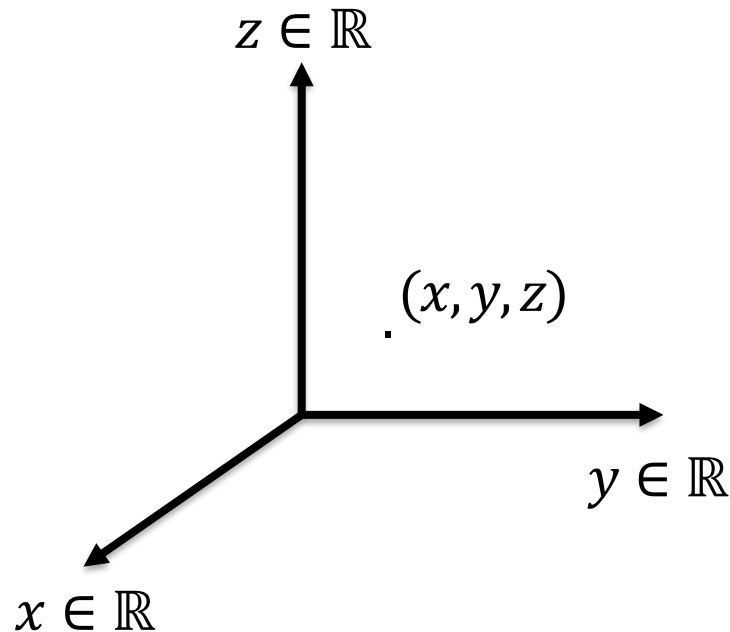
Cartesian coordinate system





# Cartesian product between more than two sets

$$A \times B \times C = \{(x, y, z) | x \in A \wedge y \in B \wedge z \in C\}$$



$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) | a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}$$

# Tuples

The elements of a Cartesian product are called **tuples**.

An n-tuple contains n elements.

Examples:

2-tuple:  $(3,1)$ , 3-tuple:  $(\text{Denmark}, \text{Norway}, \text{Sweden})$ , 5-tuple:  $(a, b, c, d, e)$

A 2-tuple is also called an **ordered pair**.

Unlike sets, tuples are *ordered*, i.e. it matters which element comes first:

For sets:  $\{1,2,3\} = \{2,3,1\}$

For tuples:  $(1,2,3) \neq (2,3,1)$

