

Problems 5

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Exercise 1

Recap ch. 4.

Let $X \sim N(3, 9)$.

- Find $P(X > 0) = \Phi(1) \approx 0.8413$.
- Find $P(-3 < X < 8) = \Phi\left(\frac{5}{3}\right) - \Phi(-2) \approx 0.9295$.
- Find $P(X > 5 | X > 3) = 2 \times (1 - \Phi\left(\frac{2}{3}\right)) \approx 0.5050$.

Exercise 2

Recap ch. 4.

Let $X \sim N(3, 9)$ and $Y = 5 - X$.

- Find $P(X > 2) = \Phi\left(\frac{1}{3}\right) \approx 0.6306$.
- Find $P(-1 < Y < 3) = \Phi\left(\frac{1}{3}\right) - \Phi(-1) \approx 0.4719$.
- Find $P(X > 4 | Y < 2) = 2(1 - \Phi\left(\frac{1}{3}\right)) \approx 0.7389$.

Exercise 3

Recap ch. 5.

Consider the set of points in the set C :

$$C = \{(x, y) | x, y \in \mathbb{Z}, x^2 + |y| \leq 2\}.$$

Suppose that we pick a point (X, Y) from this set completely at random.

- What probability does each point have of being chosen? $\frac{1}{11}$

- Find the joint and marginal PMFs of X and Y .

$$P_{XY}(x, y) = \begin{cases} \frac{1}{11} & (x, y) \in C \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(-2) = P_{XY}(0, -2) = \frac{1}{11}$$

$$P_Y(-1) = P_{XY}(0, -1) + P_{XY}(-1, -1) + P_{XY}(1, -1) = \frac{3}{11}$$

$$P_Y(0) = P_{XY}(0, 0) + P_{XY}(1, 0) + P_{XY}(-1, 0) = \frac{3}{11}$$

$$P_Y(1) = P_{XY}(0, 1) + P_{XY}(-1, 1) + P_{XY}(1, 1) = \frac{3}{11}$$

$$P_Y(2) = P_{XY}(0, 2) = \frac{1}{11}$$

$$P_X(i) = \begin{cases} \frac{3}{11} & \text{for } i = -1, 1 \\ \frac{5}{11} & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Find the conditional PMF of X given $Y = 1$.

$$P_{X|Y}(i | 1) = \frac{P_{XY}(i, 1)}{P_Y(1)}$$

$$= \frac{\frac{1}{11}}{\frac{3}{11}} = \frac{1}{3}, \quad \text{for } i = -1, 0, 1.$$

- Are X and Y independent? No

- e. Find $E[XY^2] = 0$.
- f. Find $E[X|Y = 1] = 0$.
- g. Find $\text{Var}(X|Y = 1) = 2/3$.
- h. Find $E[X| |Y| \leq 1] = 0$.
- i. Find $E[X^2|Y \leq 1] = 2/3$.

Exercise 4

The number of accidents in a certain city is modeled by a Poisson random variable with average rate of 10 accidents per day. Suppose that the number of accidents in different days are independent. Use the central limit theorem to find the probability that there will be more than 3800 accidents in a certain year. Assume that there are 365 days in a year. $P(Y \geq 3800) \approx 0.0065$

$$Y = X_1 + X_2 + \cdots + X_n, \quad n = 365$$

$$X_i \sim \text{Poisson}(\lambda = 10). \quad \text{Thus: } EX_i = 10$$

$$\text{Var}(X_i) = \lambda = 10$$

$$EY = 365 \times 10 = 3650$$

$$\text{Var}(Y) = 365 \times 10 = 3650$$

$$\frac{Y - 3650}{\sqrt{3650}} \text{ is approximately } N(0, 1) \quad (\text{by the CLT})$$

$$\begin{aligned} P(Y \geq 3800) &= P\left(\frac{Y - 3650}{\sqrt{3650}} \geq \frac{3800 - 3650}{\sqrt{3650}}\right) \\ &= 1 - \Phi\left(\frac{3800 - 3650}{\sqrt{3650}}\right) \\ &\approx 1 - \Phi(2.48) \\ &\approx 0.0065 \end{aligned}$$