Recap - CRV Part 1

Probability Density Function

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b for any a and b (4.1)

If X is a *continuous random variable*, for any x_1 and x_2,

$$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \qquad (4.2)$$

Cumulative Distribution Function

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du \qquad (4.3)$$

for $-\infty < x < \infty$.

Probability Density Function from the Cumulative Distribution Function

Given $F(x)$,

$$f(x) = \frac{dF(x)}{dx}$$

as long as the derivative exists.

Mean and Variance

Suppose that X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx \qquad (4.4)$$

$E[X^2] - (E[X])^2$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \left(\int_{-\infty}^{\infty} x^2 f(x)\,dx\right) - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x) f(x)\,dx \qquad (4.5)$$

LOTUS

Assignment 1 (20%)

Let X denote a continuous stochastic variable with the following probability density function

$$f(x) = \begin{cases} cx^4 & for -1 \le x \le 1 \\ 0 & otherwise \end{cases} \quad \text{, where } c \text{ is a constant}$$

a) Show that the cumulative probability function of X is

$$F(x) = \begin{cases} 0 & for\ x < -1 \\ \frac{1}{5}c(x^5+1) & for -1 \le x \le 1 \\ 1 & for\ x > 1 \end{cases}$$

$$\int_{-1}^{x} c \cdot u^4\, du = \frac{1}{5} c u^5 \Big|_{-1}^{x} = \frac{1}{5} c\left(x^5 - (-1)^5\right) = \underline{\underline{\frac{1}{5}c(x^5+1)}}$$

b) Determine the constant c and restate both the probability density function and the cumulative probability function using the actual value of c

$$\int_{-1}^{1} f(x)\, du = 1 = \int_{-1}^{1} c x^4\, dx = \frac{1}{5} c\left((1)^5 - (-1)^5\right) = \frac{2}{5} c = 1$$

$$\Rightarrow c = 5/2$$

$$\boxed{\begin{aligned} f(x) &= \begin{cases} 5/2\, x^4 & x \in [-1;1] \\ 0 & \text{else} \end{cases} \\ F(x) &= \begin{cases} 0 & x < -1 \\ 1/2\,(x^5+1) & -1 \le x \le 1 \\ 1 & x > 1 \end{cases} \end{aligned}}$$

c) Compute $P\left(-\frac{1}{2} < X < \frac{1}{2}\right)$ and $P(X > 0)$ $\quad F(\tfrac{1}{2}) - F(-\tfrac{1}{2}) = \underline{\underline{\tfrac{1}{32}}}$, $1 - F(0) = \underline{\underline{1/2}}$

d) Find the expected value and variance of X

$$\hookrightarrow E(X) = \int_{-1}^{1} x \cdot \frac{5}{2} x^4\, dx = \frac{5}{12} x^6 \Big|_{-1}^{1} = \underline{\underline{0}}$$

$$Var(X) = \int_{-1}^{1} x^2 \cdot \frac{5}{2} x^4\, dx = \frac{5}{14} x^7 \Big|_{-1}^{1} = \underline{\underline{\frac{5}{7}}}$$

Assignment 3 (25%)

A central database server receives, on the average, 25 requests per second from its clients. Assuming that requests received by a database follow a Poisson distribution

a) What is the probability that the server will receive no requests in a 10-millisecond interval?
b) What is the probability that the server will receive more than 2 requests in a 10-millisecond interval?
c) What is the probability that the server will receive between 2 and 4 (both included) requests in a 20-millisecond interval?

Let T be the time in seconds between requests.

d) What is the probability that less than or equal to 10-milliseconds seconds will elapse between job requests?
e) What is the probability that more than 100-milliseconds will elapse between requests?

Assignment 1 (15%)

Let X denote a continuous stochastic variable with the following density function

$$f(x) = \begin{cases} c(1-x^2) & \text{for } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Determine the value of c and state the cumulative distribution function of X.
b. Determine $P(X \leq \frac{1}{2})$ and $P(X \leq -\frac{1}{4})$
c. Determine the expected value and the variance of X.

a) $\int_{-1}^{1} c(1-x^2)\,dx = 1 \Rightarrow \frac{3}{4} = c$

$\int_{-1}^{x} \frac{3}{4}(1-u^2)\,du \Rightarrow F(x) = -\frac{x^3}{4} + \frac{3x}{4} + \frac{1}{2}$

b) $F(\frac{1}{2}) = \frac{(\frac{1}{2})^3}{4} + \frac{3\cdot\frac{1}{2}}{4} + \frac{1}{2} = \underline{\underline{\frac{27}{32}}}$, $F(-\frac{1}{4}) = \underline{\underline{\frac{81}{256}}}$

Assignment 1 (15%)

Compute the expected value, $E(X)$, if X has a density function as follows:

a. $f(x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}} & x > 0 \\ 0 & otherwise \end{cases}$ $\int_0^{\infty} \frac{1}{4} x^2 e^{-\frac{x}{2}} \, dx = \underline{\underline{4}}$

b. $f(x) = \begin{cases} 5x^{-2} & x > 5 \\ 0 & otherwise \end{cases}$ $\int_5^{\infty} 5x^{-1} \, dx = 5 \cdot \log x \Big|_5^{\infty} = \underline{\underline{\infty}}$

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

c. If $E(X) = \frac{3}{5}$, find a and b.

Assignment 1 (20%)

The length of time X (in hours), needed by students in the SMP course to complete the three hour exam is a continuous random variable with the following density function;

$$f(x) = \begin{cases} q\left(x^2 + x\right) & \text{if } 0 \leq x \leq 3 \\ 0 & \text{else} \end{cases}$$

a. Find the value q that makes f a probability density function. $\rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

b. Find the cumulative distribution function. $\rightarrow \int_{-\infty}^{x} f(x)\,dx =$

c. Find the probability that a student will complete the exam in

  (i) less than an hour $\rightarrow F(1)$

  (ii) between one and two hours $\rightarrow F(2) - F(1)$

  (iii) more than two hours $\rightarrow 1 - F(2)$

  (iv) during the final ten minutes of the exam. $1 - F(\frac{17}{6})$

d. Find the mean time needed to complete the three hour SMP exam. $\rightarrow \int_{-\infty}^{\infty} x \cdot f(x)\,dx$

e. Find the variance and standard deviation of X.

$\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = V$ $\quad \sqrt{V}$

Assignment 1 (15%)

Compute the expected value, $E(X)$, if X has a density function as follows:

a. $f(x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}} & x > 0 \\ 0 & otherwise \end{cases}$

b. $f(x) = \begin{cases} 5x^{-2} & x > 5 \\ 0 & otherwise \end{cases}$

The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

c. If $E(X) = \frac{3}{5}$, find a and b.

1 $\int_0^1 x(a+bx^2)\,dx = \frac{3}{5}$

$\frac{1}{2}a + \frac{1}{4}b = \frac{3}{5}$ 1

2 $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 (a+bx^2)\,dx = 1$

$a + \frac{1}{3}b = 1$ 2

$\begin{cases} \frac{1}{2}a + \frac{1}{4}b = 3/5 \\ a + \frac{1}{3}b = 1 \end{cases} \Rightarrow a = \frac{3}{5}, \; b = \frac{6}{5}$

solve