PROBABILITY - Basics

Compilation of Important Topics

CONTENTS

Experiment, Sample Space and Event

DEFINITIONS

Experiment

An experiment is a set of processes terms, which are carried out under stipulated conditions to study the phenomena associated with it. Broadly, there can be two types of experiments:

(i)Experiments with definite outcome: These types of experiments are certain in nature. Outcomes of experiments are known in advance.

(ii)Experiments with indefinite outcome: These type of experiment are uncertain in nature, we cannot predict the outcome with certainty.

Random Experiment

An experiment whose all possible outcomes (results) are known in advance, but the result of any specific performance cannot be predicted before completion of the experiment.

Illustration:

Consider the following experiments.

(i) Tossing a coin

(ii) Rolling a die

(iii) Drawing a card from a pack of well-shuffled pack of 52 cards.

Solution:

All the experiments have more than one possible outcome. All outcomes are known in advance. Hence all these experiments are random experiments.

Sample Space

The set of all possible outcomes of an experiment (or trial) is called the sample space. It is usually denoted by s.

A die is thrown. The upper face can show any of the six numbers (1, 2, 3, 4, 5, 6). Due to this, we get a sample space consisting of 6 elements.

A card is drawn from a well-shuffled pack of cards. We get a sample space consisting of 52 elements.

For example, for the experiment of tossing a coin, the sample space is a set S = {head, tail}. For the experiment of tossing two coins, the sample space is S = {head - head, head - tail, tail - head, tail - tail}.

Consider the experiment of shooting a tiger until there is a hit. The sample space is the countably infinite set S = {h, mh, mmh, ...} where h denotes a hit, m denotes a miss and mh denotes a miss followed by a hit and so on.

Event

The appearance of a particular outcome, which we may find in the sample space, is an event. Any subset of the sample space is called an event.

Choosing an ace from a pack of well-shuffled cards is an event. This event occurs 4 times in a pack of card i.e. in sample space. Observing an even number when a dice is thrown is an event. This event occurs three times in the sample space of 6 elements. So, event is a subset of sample space.

Complement of an Event:

Let S be the sample space and A be some events, then the set of all out comes which are in S but not in A is called the complete of event A. It is denoted by A^- or A^c.

Illustration:

In rolling a die event A is described as numbers

A = {2, 4, 6}

A^c = {1, 3, 5}

Mutually Exclusive Events:

Let the two events be A and B. If occurrence of event A excludes the possibility of occurrence of B and vice-versa, then we say that A and N are mutually exclusive event i.e. they cannot occur simultaneously. It (i.e. mutual exclusion) can happen only when the sets of two events have no common point.

Illustration:

In rolling a die, event A is described as appearance of an even number and event B is described as appearance of an odd number.

A = {2, 4, 6}

B = {1, 3, 5}

Clearly if one says that A has occurred we will instantly conclude that B cannot occur (in that trial of course). So,

$A \cap B = \Phi$

Equally likely Events:

Two or more events are said to be equally likely when there is no reason to prefer one event over other i.e. they have equal (not same) number of points in their sets.

Illustration:

In rolling a die event A is described as number showing less than 4 and event B is described as number appearing greater than 3.

A = {1, 2, 3}

B = {4, 5, 6}

Clearly A and B have equal number of points in the sample space and hence are equally likely.

Exhaustive Events:

If n events $A_1, A_2, \ldots, A_n$ related to any particular sample space are such that if we take union of the sets of all the n events, sample space is formed. i.e.

$A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$

Illustration:

In rolling a die three events are described as follows

A_1 : even number appears

A_2 : less than 4 appears

A_3 : greater than 4 appears

Hence A_1 : {2, 4, 6}

A_2 : {1, 2, 3}

A_3 : {5, 6}

$A_1 \cup A \cup A_3$ = {1, 2, 3, 4, 5, 6} which is the sample space for the experiment of rolling a die.

Mutually Exclusive and Exhaustive Events.

Events $A_1, A_2, \ldots A_n$ are said to be mutually exclusive and exhaustive if they satisfy the condition for mutual exclusion and exhaustiveness both.

i.e. $A_1 \cup A_2 \cup A_3 \ldots \cup A_n = S$

and $A_i \cap A_j = \Phi$, where $i = 1, 2, \ldots, n$

$j = 1, 2, \ldots n$ and $i \neq j$.

i.e. mutually exclusive and exhaustive events are those events of which union is equal to sample space and occurrence of any one of them excludes the possibility of occurrence of all others.

Illustration:

In rolling a die, events A_1, A_2, A_3 and A_4 are described as follows

A_1 : Number less then 2 occurs.

A_2 : 2 occurs

A_3 : odd number greater that 1 occur.

A4 : even number greater than 2 occur.

A1 : {1}

A2 : {2}

A3 : {3, 5}

A4 : {4, 6}

Clearly, $A1 \cup A2 \cup A3 \cup A4 = \{1, 2, 3, 4, 5, 6\} = S$

and $A1 \cap A2 = A1 \cap A3 = A1 \cap A4 = A2 \cap A3 = A2 \cap A4 = A3 \cap A4 = \Phi$

i.e. $Ai \cap Aj = \Phi$, $i \neq j$, $i, j = 1, 2, 3, \ldots\ldots\ldots\ldots n$.

Independent Events

If the occurrence or non occurrence of any event does not affect the sample space of occurrence of the other, then two events are said to independent.

Equally likely events/outcomes

In an experiment, two or more event/outcomes are said to be equally likely, if they have the same chances associated with them. i.e. no one of them has more chance of occurrence than others.

Introduction To Probability

PROBABILITY

If n represents the total number of equally mutually exclusive and exhaustive possible outcomes of an experiment and m of them are favourable to the event A, then the probability of the event A is defined as

$P(A) = n(E)/n(S) = m/n$

This is known as classical definition of probability

Note:

(i) for any event A $\subseteq$ S

(ii) $0 \leq P(A) \leq 1$, $P(A)$ ε R

(iii) $P(A) + P(\overline{A}) = 1$

(iv) If a cases are favourable to an event A and b cases are favourable to an event $\overline{A}$ (i.e. unfavourable to A) then P(A) = a/(a+b) and $P(\overline{A})$ = b/(b+a).

We say that odds in favour of A are a : b

and odds against of A are b : a

Rule 1: In the case of mutually exclusive events the chance of happening of one or other of them is the sum of the chances of the separate events. e.g. if A and B are mutually exclusive events, P(A∪B) = P(A) + P(B). If E_1, E_2, ..., E_n are mutually exclusive events, then $P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$.

Therefore, if there are n possible outcomes of an experiment which are mutually exclusive, exhaustive and equally likely then, since the probability associated with each outcome is the same (say x) and since they are mutually exclusive, the probability of occurrence of one of them is the sum nx which must be equal to 1(because they are also exhaustive).

That is, nx = 1 => x = 1/n.

Corollary: Since A and A' are mutually exclusive events, P(AUA') = P(A')=1.

If out of m + n equally likely, mutually exclusive and exhaustive cases, m cases are favourable to an event A and n are not favourable to an event A. m : n is called odds in favour of A, n : m is called odds against A, probability of occurrence of event A is m/(m+n) and that of non-occurrence of it is m/(m+n).

Notation: Let A and B by two events, then

(i) A' or $\overline{A}$ or A^c stands for the non-occurrence of negation of A.

(ii) AUB stands for the occurrence of at least one of A or B.

(iii) $A \cap B$ stands for the simultaneous occurrence of A and B.

(iv) $A' \cap B'$ stands for the non-occurrence of both A and B.

(v) $A \subseteq B$ stands for "the occurrence of A implies occurrence of B".

Illustration:

A die is rolled. What is the probability that outcome will be an even number?

Solution:

When the die is rolled the possible outcomes are

S = {1, 2, 3, 4, 5, 6,} and A = outcome will be even number i.e. {2, 4, 6}

So, P(A) = n(E)/n(S) = 3/6 = ½

Illustration:

Two dice are thrown simultaneously. What is the probability of obtaining a total score of seven?

Solution:

There are six possible ways as to the number of points on the first die; and to each of these ways, there corresponding 6 possible number of points on second die. Hence total number of ways

S = 6 × 6 = 36

We now find out how many ways are favourable to the total of 7 points. This may happen only in following ways : (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3).

Hence, required Probability = 6/36 =1/6. **Illustration:**

Suppose a die is rolled, can we find the probability that either odd number or a number divisible by 4 comes?

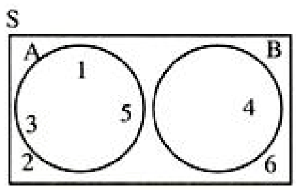
Solution:

When a die is rolled 1, 2, 3, 4, 5 and 6 are possible outcomes. So, S = {1, 2, 3, 4, 5, 6}.

Let event A : odd number i.e. A = {1, 3, 5}

event B : number divisible by 4 i.e. B = {4}

Here, we have represented the sample space and event in Venn Diagram.



We have to find probability of either A or B, i.e. P(A U B)

Clearly, A U B = {1, 3, 4, 5} i.e. A + B

Hence, P(A U B) = P(A) + P(B)

= 3/6 +1/6 = 2/3.

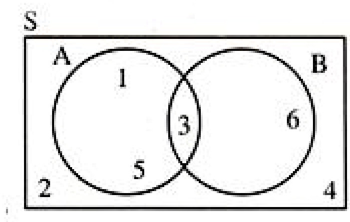
Now, think about a different situation: A die is rolled, can we calculate the probability that the number, which comes is either odd or divisible by 3?

Sample space is S = {1, 2, 3, 4, 5, 6}.

Define, event A : odd number i.e. A = {1, 3, 5}

event B : divisible by 3 i.e. B = {3, 6}

We can represent sample space and even in Venn diagram.



Now, we have to find probability that either A or B occurs i.e. P(A U B)

Clearly, A U B = P{1, 3, 5, 6}

Can we write A U B = A + B?

Let's do it, A U B = 1, 3, 5, 3, 6; this is incorrect for obvious reasons since we counted 3 twice.

From above illustration, it can be concluded that when we write A U B = A + B we count the common element twice, once while counting the elements of A and once while counting the elements of B. Therefore in order to get correct result we should subtract it (common) element once i.e.

A U B = A + B - A ∩ B {A ∩ B element common in both A and B}

= 1, 3, 5, 4, 6, - 3

= 1, 3, 5, 6

Hence the probability

P(A U B) = P(A) + P(B) - P(A ∩ B) = (3/6) + (2/6) - (1/6) = 2/3

Note:

1. In previous case, $P(AUB) = P(A) + P(B)$ was correct because $A \cap B = f$. So, we get a very important result.

$P(AUB) = P(A) + P(B)$ when A and B are mutually exclusive events.

$P(AUB) = P(A) + P(B) - P(A\cap B)$, when A and B are not mutually exclusive events.

2. This particular case of two events can easily be generalized for the three events A, B and C

$P(AUBUC) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$.

Illustration:

The probabilities that a student passes in Mathematics, Physics and Chemistry are m, p and c respectively. Out of these subjects, the student has a 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Prove that $m + p + c = 27/20$.

Solution:

$1 - (1 - m)(1 - p)(1 - c) = 0.75$ .... (i)

$(1 - m) pc + m (1 - p) c + mp (1 - c) + mpc = 0.5$ .... (ii)

$(1 - m) pc + m (1 - p) c + mp (1 - c) = 0.4$ .... (iii)

(ii) - (iii) gives : $mpc = 0.1 = 1/10$ From equation (i), we get

$p + c + m - (mp + mc + pc) + mpc = 0.75$ .... (iv)

From (ii),

$pc + mc + mp - 3mpc + mpc = 0.5$

$\Rightarrow pc + mc + mp = 0.7$

Substituting in (iv), we get

$m + p + c = 0.75 - 0.1 + 0.7 = 1.35 = 135/100 = 27/20$

INTERSECTION AND UNION OF SETS OF EVENTS

Let $E_1, E_2, ..., E_n$ be n events, then $P(E_1UE_2U ...... UE_n)$, represents the probability of occurrence of at least one of the events from $E_1, E_2, ..., E_n$ and $P(E_1\cap E_2\cap ...... \cap E_n)$ represents the occurrence of all the events together.

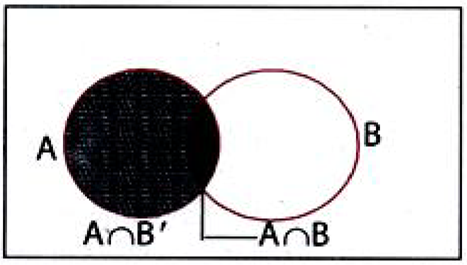
In general, we have

$P(E_1 \cup E_2 \cup \ldots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{1 \le i \le j \le n}$ $P(E_i \cap E_j \cap E_k) - \ldots + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots \cap E_n)$.

Corollary: If A and B are any two events,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Note that $P(A \cap B') = P(A) - P(A \cap B)$.

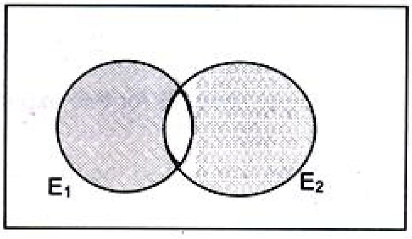


Also for any two events A and B,

P(exactly one of them occurs)

$= P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= P(E_1 \cup E_2) - P(E_1 \cap E_2)$.



If E_1, E_2 and E_3 be three events, then

(i) P(at least two of E_1, E_2, E_3 occur)

$= P(E_2 \cap E_3) + P(E_3 \cap E_1) + P(E_1 \cap E_2) - 2P(E_1 \cap E_2 \cap E_3)$,

(ii) P(exactly two of E_1, E_2, E_3 occur) $= P(E_2 \cap E_3) + P(E_3 \cap E_1) + P(E_1 \cap E_2) - 3P(E_1 \cap E_2 \cap E_3)$,

(ii) P(exactly one of E_1, E_2, E_3 occur)

$= P(E_1) + P(E_2) + P(E_3) - 2P(E_3 \cap E_1) + 2P(E_1 \cap E_2) + 3P(E_1 \cap E_3)$.

If events are mutually exclusive, $P(E_1 \cup E_2 \cup \ldots\ldots \cup E_n) = \sum_{i=1}^{n} P(E_i)$

Illustration:

In a single cast with fair dice, what is the chance of throwing

(i) two 4's, (ii) a doublet,

(iii) five - six, (iv) a sum of 7.

Solution:

(i) There are 6 × 6 equally likely cases

(as any face of any die may turn up)

=> 36 possible outcomes. For this event, only one outcome (4, 4) is favourable

=> probability = 1/36

(ii) A doublet can occur in six ways {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}

Therefore probability of a doublet = 6/36 = 1/6.

(iii) Two favourable outcomes {(5, 6), (6, 5)}:

This probability = 2/36 = 1/18.

(iv) A sum of 7 can occur in the following cases: {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}which are 6 in number.Therefore probability = 6/36 = 1/6.

Illustration:

Seven accidents occur in a week. What is the probability that they happen on the same day.

Solution:

Total number of cases = Total no. of ways in which 7 accidents can happen in a week = 7^7.

Favourable number of cases out of these = number of those in which all 7 happen on one day (any day of the week) = 7. Hence the required probability = 7/7^7 = 1/7^6 .

Illustration:

From a bag containing 5 white, 7 red and 4 black balls a man draws 3 balls at random. Find the probability that all are white.

Solution:

Total number of balls in the bag = 5 + 7 + 4 = 16

=> The total number of ways in which 3 balls can be drawn

= 16C3 = (16*15*14)/(3*2*1) = 560.

Thus sample space S for this experiment has 560 outcomes i.e. n(S) = 560.

Let E be the event of all the three balls being white. Total number of white balls in 5.

So the number of ways in which 3 white balls can be drawn = 5C3 = (5*4*3)/(3*2*1) = 10.

Thus E has 10 elements of S, i.e. n(E) = 10

=> Probability of E, P(E) = n(E)/n(S) = 10/560 = 1/56.

Illustration:

From a pack of 52 cards two cards are drawn at random. Find the probability of the following events:

(i) Both the cards are spades.

(ii) One card is of spade and the other is a diamond.

Solution:

The total number of ways in which 2 cards can be drawn

= $^{52}C_2$ = (52*51)/(1*2) = 26 × 51 = 1326.

=> Number of elements in the sample space S are n(S) = 1326.

(i) Let the event that both cards are of spade, be denoted by E_1. Then $n(E_1)$ = Number of elements in E_1 = Number of ways in which 2 cards can be selected out of 13 cards of spade = $^{13}C_2$ = (13*12)/(1*2) = 78.

Hence the probability of E_1 = $P(E_1) = n(E_1)/n(S)$ = 78/1326 = 1/17.

(ii) Let E_2 be the event that one card is of spade and one is of diamond. Then $n(E_2)$ = number of elements in E_2 = number of ways in which one card of spade can be selected out of 13 spades and one card of diamond can be selected out of 13 diamond cards

= $^{13}C_1 \times {}^{13}C_1$ = 13×13 = 169

=> $P(E_2) = n(E_2)/n(S)$ = 169/1326 = 13/102.

Illustration:

A has 3 shares in a lottery containing 3 prizes and 6 blanks: B has one share in a lottery containing one prize and 2 blanks. Compare their chances of success.

Solution:

Total number of tickets in the first lottery = 3 + 6 = 9

A may select any three tickets out of these 9 tickets. Therefore, the number of element in the sample space S is given by

= n(S) = $^{9}C_3$ = (9*8*7)/(3*2*1) = 84.

Let E_1 be the event of winning the prize in the first lottery by A. So,. $\overline{E_1}$ is the event of not winning the prize by A. Then number of elements

in $\overline{E_1}$ is given n($\overline{E_1}$) = number of ways of selecting 3 tickets out of six blank tickets = $^6C_3$ = (6*5*4)/(1*2*3) = 20.

=> The probability of not winning the prize by A = P($\overline{E_1}$) = n($\overline{E_1}$)/n(S) = 20/84 = 5/21.

Since P(E_1) + P($\overline{E_1}$) = 1, P(E_1) = 1 - P($\overline{E_1}$) or P(E_1) = 1 - 5/21 = 16/21.

For second lottery

n(S) = number of ways of selecting 1 ticket out of 3 tickets.

Or n(S) = $^3C_1$ = 3.

Let E_2 be the event of winning the prize by B.

n(E_2) = number of ways of selecting one ticket out of 1 prize ticket = $^1C_1$ = 1

=> The probability of winning the prize by B

= P(E_2) = n(E_2)/n(S) = 1/3 so that the ratio of the probabilities of winning the prizes by A and B = P(E_2)/P(E_2) = (16/21)/(1/3) = 16/7.

Illustration:

From a pack of cards, four are drawn at random. What is the chance that there is one card of each suit?

Solution:

A pack of cards has 52 cards. Let S be the sample space. Then

n(S) = $^{52}C_4$ = (52*51*50*49)/(4*3*2*1) = 13 × 17 × 25 × 49.

Let E be the event of drawing one card from each suit.

There are four different suits and each suit has 13 cards.

n(E) = number of total ways of drawing one card from each suit.

= $^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1$ = 13 × 13 × 13 × 13

=> Required probability = P(E) = n(E)/n(S) = (13*13*13*13)/(13*17*25*49) = 2197/20825.

Illustration:

In a given race the odds in favour of four horses A, B, C, D are 1:3, 1:4, 1:5, 1:6 respectively. Assuming that, a dead heat is impossible, find the chance that one of them wins the race.

Solution:

Let P(A), P(B), P(C) and P(D) be the responsibilities of winning of the horses A, B, C and D respectively. Then

P(A) = 1/4, P(B) = 1/5, P(C) = 1/6, P(D) = 1/7.

Since the above events are mutually exclusive, the chance that one of them wins

= P(AUBUCUD) = P(A) + P(B) + P(C) + P(D)

= (1/4) + (1/5) + (1/6) + (1/7).

Illustration:

Ten passengers get into an elevator on the ground floor of a 20 floor building. What is the probability that they will all get out at different floors?

Solution:

Total no. of ways in which the 10 passengers can get out = 20^{10}.

Favourable no. of ways = number of those cases in which not more than one passenger gets out at each floor = ${}^{20}P_{10}$ = 20!/10!.

Required probability = 20! / (10!$(20)^{10}$).

Illustration:

There are three events A, B and C out of which one and only one can happen. The odds are 8 to 3 against A, 5 to 2 against B. Find odds against C.

Solution:

Let the total no. of cases = m + n + p:

m are in favour of A, n in favour of B & p in favour of C.

We have (n+p)/m = 8/3 , (m+p)/n = 5/2, so that

P(A') = (n+p)/(m+n+p) = 8/11, P(B') = (m+p)/(m+n+p) = 5/7

=> P(A) = 3/11, P(B) = 2/7

Also P(A) + P(B) + P(C) = 1 => P(C) = 34/77

Conditional Probability

$P(A/B)$ denotes the probability of event A happening, given that event B has happened. It is the conditional probability of A, given B. While calculating $P(A/B)$, we assume that event B has occurred. It implies that the outcomes favourable to B become the total outcomes and hence outcomes favourable to $P(A/B)$ are outcomes common to A and B.

$P(A/B) = P(A \text{ given } B)$

= (Total no. of favourable cases)/(Total no. of cases)= $n(A \cap B)/n(B) = P(A\cap B)/P(B)$.

Thus $P(A/B) = P(A\cap B)/P(B)$.

=> $(P(A\cap B) = P(B).(P(A/B) = P(A).(P(B/A))$.

Note that if A and B are independent events then

$P(A\cap B) = P(A)\ P(B)$

=> $P(A/B) = P(A\cap B)/P(B)\ \ P(A)P(B)/P(B) = P(A)$ which should be the case as occurrence of A does not depend upon B. Similarly $P(B/A) = P(B)$.

More often than not the probability of one event is affected by occurrence of other events. Let a bag contains 3 red, 2 green and 4 yellow balls. A ball is drawn and found to be red. Consider the following cases:

(i) If the red ball is replaced then find the probability that the next ball to be drawn is yellow.

(ii) What happens when red ball will not be replaced?

Let, R : red ball is first drawn

Y : yellow ball in next drawn.

In first case when the red ball is replaced.

$P(Y)$ = (Total number of yellow balls)/(total number of balls)=4/9

In second case the red is ball is not replaced, so,

$P(Y)$ = (Total number of yellow balls)/(Total number of balls left)=4/8

The probability in these two cases is different. In the first case when the first ball was red and it replaced it does not affect the probability of yellow ball in second draw but in second case when the ball was not replaced the probability of yellow ball is changed. It happened because in second case the sample space is changed. So from here we can draw a conclusion that when occurrence of any event changes the sample space, the probability of other event change. Hence the probability also changes.

It means that, occurrence of any event change the sample space. Then Probability of other events change, in other words probability of some event depend or the occurrence of other events. This is known as "Conditional Probability".

Let A and B be the two events. If A has already happened (i.e. given) then probability of B can be found by using the formula

$P(B/A) = P(A \cap B)/P(A)$

If A and B are independent events, then

$P(B/A) = P(B)$ i.e. $P(A \cap B) = P(A)\,P(B)$

or, $P(A/B) = P(A)$

(i) n events $A_1, A_2, A_3, \ldots\ldots\ldots\ldots A_n$ are said to be pair-wise independent tiff P

$(A_i \cap A_j) = P(A_i)\,P(A_j)$, where 1, I = 1, 2, 3, ..., n

(ii) These n events are said to be mutually independent iff

$P(A_1 \cap A_2 \cap \ldots\ldots\ldots \cap A_n) = P(A_1)\,P(A_2)\ldots\ldots\ldots P(A_n)$.

Note : If the set of n events related to a sample space are pair-wise independents, they must be mutually independent, but vice versa is not always true.

To apply total probability formula we must check whether A_1 and A_2 fulfil the following three conditions or not.

$A_1 \cap A_2 = \Phi$

and $A_1 \cup A_2 = S$.

$A_1 \subset S$ and $A_2 \subset S$.

Probability of an event which depends on more than one event

Suppose we have n events $A_1, A_2, \ldots\ldots\ldots, A_n$ related to a sample space such that:

(i) They are mutually exclusive i.e.

$A_i \cap A_j = \Phi$, I, j = 1, 2 .... n, I $\neq$ j.

(ii) They are exhaustive i.e.

$A_1 \cup A_2 \cup A_3 \ldots\ldots\ldots \cup A_n = S$.

(iii) They are proper subsets of sample space S i.e.

$A_i \subset S, i = 1, 2, \ldots\ldots\ldots n$.

Then probability of an event A which depends on the events $A_1, A_2, \ldots\ldots\ldots$, An can be calculated as

$P(A) = \sum^{n}_{(i=1)} P(A/A_i)\, P(A_i)$

$= P(A/A_1)\, P(A_1) + P(A/A_2)\, P(A_2) + \ldots\ldots + P(A/A_n)\, P(A_n)$

This is a formula for total probability.

Illustration:

Take the case of previous illustration in which a bag contains 3 Red, 2 Green and 4 Yellow balls. A ball is drawn and if it is Red or Yellow it is replaced otherwise not. Find the probability that second ball drawn is a yellow ball.

Solution:

First of all, we should be very clear about the event of which probability is to be found and the events on which it depends.

Let's A_1 : First ball is either red or yellow

A_2 : First ball is green

A : Second ball drawn is yellow

We have to calculate P(A) which definitely depends on A_1 and A_2.

Therefore, $P(A) = P(A/A_1)\, P(A_1) + P(A/A_2)\, P(A_2)$

$P(A/A_2)$ = Probability of A when A_1 is already happen.

= 4/9 (i.e. probability of A given A_1)

$P(A_1) = 7/9$

$P(A/A_2)$ = Probability of A when A_2 is already happened (Probability of A given A_2). = 4/8

$P(A_2) = 2/9$

Hence, $P(A) = 4/9 \times 7/9 + 4/8 \times 2/9 = 37/81$

Illustration:

If a dice is thrown, what is the probability of occurrence of a number greater than 1, if it is known that only odd numbers can come up.

Solution:

S=the sample space = {1, 2, 3, 4, 5, 6}

A=the event of occurrence of an odd number = {1, 3, 5}

B=the event of occurrence of a number greater than 1={2, 3, 4, 5, 6}

Here $A \cap B = \{3, 5\}$

so that $P(B/A) = P(A \cap B)/(P(A)) = (n(A \cap B))/(n(A)) = 2/3$.

Illustration:

In a college, 25% students failed in Mathematics, 155 students failed in Physics, and 10% failed in Mathematics and Physics. A student is selected at random:

(i) If he failed in Physics, then find the chance of his failure in Mathematics,

(ii) If he failed in Mathematics, then find the chance of his failure in Physics,

(iii) Find the chance of his failure in Mathematics or Physics.

Solution:

Let E1 and E2 be the events of failure in mathematics and physics respectively. Let the total number of students appearing in the examination be 100.

Since 25% students failed in Mathematics, n(E1) = 25.

=> $P(E1) = n(E1)/(n(S)) = 25/100 = 1/4$.

Since 15% students failed in Physics, n(E2) = 15

=> $P(E2) = n(E2)/(n(S)) = 15/100 = 3/20$.

Again 10% students failed in Physics and Mathematics both, so n(E1ÇE2) = 10

=> $P(E1 \cap E2) = n(E1 \cap E2)/(n(S)) = 10/100 = 1/10$..

(i) The chance of failure in Mathematics while he has failed in Physics is given by

$P(E1/E2) = = (P(E1 \cap E2))/P(E2) = (1/10)/(3/20) = 2/3$.

(ii) The chance of failure in Physics while he has failed in Mathematics is given

by $P(E2/E1) = (P(E1 \cap E2))/P(E2) = (1/10)/(1/4) = 2/5$..

(iii) The chance of failure in Mathematics or Physics is:

$P(E1 \cup E2) = P(E1) + P(E2) - P(E1 \cap E2)$

[since both the events are not mutually exclusive]

Independent Events

Independent Events

Events are said to be independent when the happening of any one of them does not affect the happening of any of the others/

(i) A and B are independent if

$P(B/A) = P(B)$ and $P(A/B) = P(A)$.

(ii) The probability of the occurrence of several independent events is the product of their individual probabilities.

i.e. If E_1, E_2, ... E_n are n independent events then $P(E_1 \cap E_2 \cap \dots \cap E_n)$

$= P(E_1)P(E_2)P(E_3)\dots P(E_n)$.

Remark: The converse of this is also true i.e. if n given events satisfy the above condition then they will be independent.

Pairwise Independent Events

Three events E_1, E_2 and E_3 are said to pairwise independent if

$P(E_1 \cap E_2) = P(E_1)P(E_2)$, $P(E_2 \cap E_3) = P(E_2) P(E_3)$ and $P(E_3 \cap E_1) = P(E_3)P(E_1)$

i.e. Events E_1, E_2, E_3, ... E_n will be pairwise independent if

$P(A_i \cap A_j) = P(A_i) P(A_j)$ $i \neq j$.

Three events are said to be mutually independent if

$P(E_1 \cap E_2) = P(E_1) P(E_2)$, $P(E_2 \cap E_3) = P(E_2) P(E_3)$ and $P(E_3 \cap E_1) = P(E_3) P(E_1)$

and $P(E_1 \cap E_2 \cap E_3) = P(E_1) P(E_2) P(E_3)$.

If two events A and B are mutually exclusive,

$P(A \cap B) = 0$ but $P(A) P(B)$ ¹ 0 (In general).

=> $P(A \cap B) \neq P(A) P(B)$

=> mutually exclusive events will not be independent.

So if two events are independent, they have to have some common element between them i.e. they cannot be mutually exclusive. Mutually exclusiveness is used when the events are taken from the same experiment and independence is used when the evens are taken from different experiments.

For example

(i) Two fair dies are thrown. Let two events be 'first die shows an even number' and second die shows an odd number'. These two events are independent events since the result of the first die does not depend upon the result of 2^{nd} die. But these events are not mutually exclusive since both the events may simultaneously occur.

(ii) From a pack of cards, (a) Two cards are drawn in succession without replacement. Let an event be 'card drawn is Ace'. Then the two events (i.e. two cards being aces) are dependent since drawing of 2^{nd} card very much depends upon which is the first drawn card. (b) If the cards are drawn with replacement, the drawing of second card will not depend upon first card. So, two events will be independent.

Illustration:

An event A_1 can happen with probability p_1 and event A_2 can happen with probability p_2. What is the probability that

(i) exactly one of them happens.

(ii) at least one of them happens (Given A_1 and A_2 are independent events).

Solution:

(i) The probability that A_1 happens is p_1

=> The probability that event A_1 fails $1-p_1$. Also the probability that A_2 happens is p_2.

Now, the chance that A_1 happens and A_2 fails is $p_1(1 - p_2)$ and the chance that A_1fails and A_2 happens is $p_2(1 - p_1)$.

=> The probability that one and only one of them happens is

$$p_1(1 - p_2) + p_2(1 - p_1) = p_1 + p_2 + 2p_1p_2.$$

(ii) The probability that both of them fail = $(1 - p_1)(1 - p_2)$

=> Probability that at least one happens = $1-(1 - p_1)(1 - p_2) = p_1 + p_2 + p_1 p_2$.

Illustration:

A person draws a card from a pack of 52, replaces it and shuffles it. He continues doing it until he draws a spade. What is the chance that he has to make?
(i) At least 3 trials
(ii) Exactly 3 trials

Solution:
(i) For at least 3 trials, he has to fail at the first 2 attempts and then after that it doesn't make a difference if he fails or wins at the 3rd or the subsequent attempts.
Chance of success at any attempt = 1/4
=> chance of failure = 3/4
=> chance of failing in first 2 attempt = $(3/4)^2$=Chance of success at any attempt = 1/4
=> chance of failure = 3/4
=> chance of failing in first 2 attempt = $(3/4)^2 = 9/16$.

(ii) For exactly 3 attempts, he has to fail in the first two attempts and succeed in the 3rd attempt. The required probability = $(3/4)^2.1/4 = 9/64$.

Illustration:

A, B and C in order, toss a coin. The one who gets a head first wins. Find their respective probabilities of winning.

Solution:

P(Head) = 1/2 = P(Tail) in any toss of the coin. A wins if he gets a head in the first trial or if he fails, then B & C fail and then he gets a head and so on.

Therefore, $P(A) = 1/2 + (1/2)^3.1/2 + (1/2)^6.1/2 + ...;$

$= 1/2\ [1 + 1/8 + 1/64 + ...] = 1/2.1/(1-1/8) = 4/7.$

Similarly $P(B) = 1/2.1/2 + (1/2)^4.1/2 + (1/2)^7.1/2+..... = 2/7$

and $P(C) = (1/2)^2.1/2 + (1/2)^5.1/2 +..... = 1/7.$

Total Probability Theorem

Let A_1, A_2, ... A_n be a the set of mutually exclusive and exhaustive events and E be some event which is associated with A_1, A_2, ..., An. Then probability that E occurs is given by

$P(E) = \sum^n_{(i=1)} P(A_i)P(E/A_i).$

Illustration:

A bag contains 3 white balls and 2 black balls another contains 5 white and 3 black balls. If a bag is chosen at random and a ball is drawn from it, what is the probability that is white?

Solution:

The probability that the first bag is chosen is 1/2 and the chance of drawing a white ball from it is 3.5.

=> Chance of choosing the first bag and drawing a white ball is 1/2.3/5.

Similarly, the chance of choosing the second bag and drawing a white ball is 1/2.5.8.

Hence the chance of randomly choosing a bag and drawing a white ball
is = 1/2.3/5+1/2.5/8 (Mutually exclusive cases)

= 49/80.

Illustration:

Find the probability that a year chosen at random has 53 Sundays.

Solution:

Let P(L) be the probability that a year chosen at random is leap

$P(L) = 1/4$
$\Rightarrow P(L^c) = 3/4$.

Let P(S) be the probability that an year chosen at random has 53 Sundays

Now P(S/L) = Probability that a leap year has 53 Sundays.

A leap year has 366 days, 52 weeks + 2 days; the remaining 2 days may be Sunday-Monday, M–T, T–W.

W–Th, Th–F, F-Sat or Sat–S.

Out of the 7 possibilities, 2 are favourable.

$\Rightarrow P(S/L) = 2/7$. Similarly $P(S/L^c) = 1/7$.

$\Rightarrow P(S) = 1/4.2/7 + 3/4.1/7 = 5/28$

Baye's Theorem

Baye's theorem revises (reassigns) the probabilities of the events $A_1, A_2, ..., A_n$, related to a sample space, when there is an information about the outcome beforehand. The earlier probabilities of the events $P(A_i)$, I = 1, 2, ..., n are called a priori probabilities and the probabilities of events calculated after the information A is received i.e. $P(A_i/A)$ is called posterior probabilities.

Conditions for the application of Baye's formula is that priori events i.e. $A_1, A_2, ..., A_n$ of the sample space are exhaustive and mutually exclusive i.e.

$A_1 \cup A_2 \cup .... \cup A_n = S$

and $A_i \cap A_j = \Phi$ j, i = 1, 2, ... n and $i \neq j$

It's derivation is as follows

Let $B_1, B_2, ..., B_n$ be n mutually exclusive and exhaustive events and A be any event in sample space, then

$A = (A \cap B_1) \cup (A \cap B_2) \cup .... (A \cap B_n)$

$\Rightarrow P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) ....+ P(A \cap B_n)$

$= P(B_1).P(A/B_1) + P(B_2).P(A/B_2) +.........+ P(B_n).P(A/B_n)$

Hence, $P(B_j/A) = = (P(B_j).P(A/B_j))/(\sum^n_{(i=1)} P(B_i).P(A/B_i))$

If in a problem some event has already happened and then the probability of another event is to be found, it is an application of Baye's Theorem

Note: To recognize the question in which Baye's theorem is to be used, the key word is "is found to be".

Illustration:

A die is rolled and it is found that number turned up is an even number. Find the probability that it is 2.

Solution:

All possible events when we roll a die are

A_1 : 1 appears $P(A_1) = 1/6$

A_2 : 2 appears $P(A_2) = 1/6$

A_3 : 3 appears $P(A_3) = 1/6$

A_4 : 4 appears $P(A_4) = 1/6$

A_5 : 5 appears $P(A_5) = 1/6$

A_6 : 6 appears $P(A_6) = 1/6$

An even number is turned up this is an information to us. You should revise the probability of priori events A_i in the light of information received. (It will be totally foolish if we don't revise the probabilities. Since the information of even numbers is available, it makes the probabilities of 1, 3, and 5 equal to zero).

Let, A : Even number has turned up.

We have to calculate the probability of A_2 when A is given. Since A_1, A_2, ..., A_6 are exhaustive and mutually exclusive we can apply Baye's formula

$P(A_2/A) == (P(A)P(A/A_2))/(P(A_1)P(A/A_1)+P(A_2)P(A/A_2)+.......+P(A_6)P(A/A_6))$

$P(A/A_1)$ means probability of A when A_1 is given i.e. probability of coming of an event number when 1 has appeared. Obviously it is zero.

Hence $P(A/A_1) = 0$

Similarly, $P(A/A_2) = 1$, $P(A/A_3) = 0$, $P(A/A_4) = 1$, $P(A/A_5) = 0$, $P(A/A_6) = 1$

Therefore, $P(A_2/A) = (1/6\times1)/(1/6\times0+1/6\times1+1/6\times0+1/6\times1+1/6\times0+1/6\times1)=1/3$

Note: This problem is very simple and illustrated only to make the application of Baye's formula clear.

Illustration:

In a factor, machines A, B and C manufacture 15%, 25% and 60% of the total production of bolts respectively. Of the bolts manufactured by the machine A, B and C 4%, 2% and 3% are defective A bolt is drawn at random and is found to be defective. What is the probability that it was produced by B?

Solution:

In the formula $P(A_k/A) = (P(A_k)P(A/A_k))/(\sum^{n}_{(i=1)} P(A_i).P(A/A_i))$

A_i means all the possibilities, which can happen w.r.t. to the given event A, while A_k means the particular event whose probability w.r.t. B we are required to find

Let us take A = the event of bolt being defective.

A_1 = the bolt is produced by B.

A_2 = the bolt is produced by A.

A_3 = the bolt is produced C.

Required probability = $P(A_1/A)=(P(A_1)P(A/A_1))/(\sum^{3}_{(i=1)} P(A_i).P(A/A_i))$

$=>P(A_1/A)$

$=(25/100\times2/100)/((15/100\times4/100)+(25/100\times2/100)+(60/100\times3/100))$

$= 50/ (60+50+180) = 5/29.$

Illustration:

A bag contains 5 balls of unknown colours two balls are drawn at random and are found to be red. Find the probability that the bag contains exactly 4 red balls.

Solution:

First of all let's try to find out all the possibilities, which can happen w.r.t. to given events. Since two balls drawn are found to be red, so there are four possibilities. The bag contains only two red balls (say event A_2), bag contains 3 red balls (A_3), bag contains 4 red balls (A_4), or that all the balls are red balls (A_5). Let B : two balls are drawn and found to be red. So, it is obvious that we have to find the probability of A_4 given B i.e. required probability = $P(A_4/B)$

$P(A_4/B)=(P(A_4).P(B/A_4))/(P(A_2).P(B/A_2)+P(A_3).P(B/A_3)+P(A_4).P(B/A_4)+P(A_5).P(B/A_5))$

Now as all these four possible cases are equally likely i.e. we cannot say that bag is more likely to contains 5 red balls, than 3 red balls, so priori probabilities of all these events will be equal i.e.

$P(A_2) = P(A_3) = P(A_4) = P(A_5)$

Since A_2, A_3, A_4 and A_5 are exhaustive

$\therefore \quad P(A_2) + P(A_3) + P(A_4) + P(A_5) = 1$

$\therefore \quad P(A_4) = 1/4$(since A_2, A_3, A_4 and A_5 are equally likely also)

Now $P(B/A_4)$ means that bag contains 4 red balls and we have to find the probability, that two balls drawn at random are red which is obviously ${}^4C_2/{}^5C_2$.

So $P(A_4/B)=(1/4\times(4C_2)/(5C_2))/(1/4\times(2C_2)/(5C_2)+1/4\times(3C_2)/(5C_2)+1/4\times(4C_2)/(5C_2)+1/4\times(5C_2)/(5C_2))$

$= 12/39$

Proof: This is so because

P(Ai/B)= P(Ai ∩ B)/P(B) = P(Ai ∩ B)/P((B ∩ Ai) U P(B ∩ A2) U...(B ∩ An)) = P(Ai ∩ B)/P((B ∩ A1) U P(B ∩ A2) U...(B ∩ An))

= P(Ai).P(B/Ai)/∑ni=1 P(Ai)P(B/A1)

Note : In conditional probability the sample space is reduced to the set of samples/outcomes in the event which is given to have happened.

Illustration:

Each of three bags A, B, C contains white balls and black balls. A has a_1 white & $b_1$black, B has a_2 white & b_2 black and C has a_3 white & b_3 black. A ball is drawn at random and is found to be white. Find the respective probability that it is from A, B & C.

Solution: Here A_1, A_2, A_3 are the events that the bag picked are A, B, C respectively

E is the event that a white ball is drawn.

We are supposed to find $P(A_1/E)$, $P(A_2/E)$, $P(A_3/E)$.

$P(A_1/E) = P(A_1 U B)/P(E)$ = ((Prob. that bag A is chosen and white is drawn)/(Prob. that a bag is chosen at random and white is drawn))

$= (P(A_1).P(E/A_1))/(P(A_1).P(E/A_1)+P(A_2).P(E/A_2)+P(A_3).P(E/A_3))$

$= (1/3.a_1/(a_1+b_1))/(1/3.[a_2/(a_2+b_2)+a_2/(a_2+b_2)+a_3/(a_3+b_3)]) = p_1/(p_1+p_2+p_3)$

Similarly, $P(A_2/E) = P_2/P_1+P_2+P_3$, $P(A_3/E) = P_3/P_1+P_2+P_3$

where $p_1 = a_1/a_1+b_1$, $p_2 = a_2/a_2+b_3$, $p_3 = a_3/a_3+b_3$

Binomial Distribution

Binomial Distribution for Successive Events

As experiment may be repeated n times and each of these n trials are independent of one another. If each trial only gives one of the two possible outcomes, say 'success' (when required event take place) and failure (when required event does not take place) then it is called a Binomial Experiment. Let probability of success in any trial be p and that of failure be q, then p + q = 1

Then $(p + q)^n = C_0\ P^n + C_1P^{n-1}q + \ldots\ldots C_rp^{n-r}q^r + \ldots + C_nq^n$

Then the probability of exactly k successes in n trials is given by

$P_k = {}^nC_kq^{n-k}p^k$

Some important facts related with binomial distribution

(i) The probability of getting at least k successes is

$P(x \geq k) = \Sigma^n_{x=k}\ {}^nC_x\ p^x\ q^{n-x}$

(ii) $\Sigma^n_{x=k}\ {}^nC_x\ q^{n-x}\ p^x = (q + p)^n = 1$

Illustration:

In a rainy season, there is 60% chance that it will rain on a particular day. What is the probability that there will exactly 4 rainy days in a week?

Solution:

Let probability of raining on a particular day P(success) = 3/5

And probability that no rain on a particular day q(failure) = 2/5

Probability or exactly 4 rainy day i.e. 4 success out of 7 trials

$= {}^7C_4\ (p)^4(q)^3 = (\ 7!/4!.3!\ ) * (\ 3/5\ )^4 * (\ 2/5\ )^3$

Binomial Distribution for Successive Events

As experiment may be repeated n times and each of these n trials are independent of one another. If each trial only gives one of the two possible outcomes, say 'success' (when required event take place) and failure (when required event does not take place) then it is called a Binomial Experiment. Let probability of success in any trial be p and that of failure be q, then p + q = 1

Then (p + q)n = C0 Pn + C1Pn-1q +...... Crpn-rqr +...+ Cnqn

Then the probability of exactly k successes in n trials is given by

Pk = nCkqn-kpk

Some important facts related with binomial distribution

(i) The probability of getting at least k successes is

$P(x > k) = \Sigma nx=k \ nCx \ px \ qn-x$

(ii) $\Sigma nx=k \ nCx \ qn-x \ px = (q + p)n = 1$

Illustration:

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Probability or exactly 4 rainy day i.e. 4 success out of 7 trials

$= 7C4 \ (p)4(q)3 = (7!/4!.3!) * (3/5)4 * (2/5)3$

Solved Examples

Example 1:

A pair of dice is thrown. Find the probability of obtaining a sum of 8 or getting an even number on both the dice.

Solution:

Let the events be defined as:

A : obtaining a sum of 8

B : getting an even number on both dice

We are required to find out the total Probability A and B, i.e.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Now cases favourable to A are

(3, 5) (5, 3) (2, 6) (6, 2) (4, 4)

So, $P(A) = 5/36$

Cases favourable to B : (2, 2), (2, 4), (2, 6),

(4, 2), (4, 4), (4, 6),

(6, 2), (6, 4), (6, 6).

$P(B) = 9/36$

Now, (2, 6) (6, 2) and (4, 4) are common to both events A and B

So $P(A \cap B) = 3/36$

$\Rightarrow P(A \cup B) = = 5/36+9/36-3/36=11/36$ (Ans.)

Example 2:

A number x is selected from first 100 natural numbers. Find the probability that x satisfy the condition $x + 100/x > 50$.

Solution:

Total number of ways of selecting x is 100.

Now the given condition is $x + 100/x > 50$, on analyzing this equation, carefully, we see that this equation is satisfied for all the numbers x such that $x \geq 48$ and also for x = 1 and 2

So, favourable number cases is 55

Therefore, probability = 55/100 = 11/20 (Ans.)

Example 3:

'A' has three share in a lottery in which there are 3 prizes and 6 blanks' 'B' has one share in a lottery in which there is 1 prize and 2 blanks. Show that A's chance of winning a prize of B's chance of winning a prizes in ratio 16:7

Solution:

Method 1:

'A' may win one, two or all the three prizes, the total probability is

=(3C16C2)/(9C3)+(3C26C1)/(9C3)+(3C36C0)/(9C3)=(45+18+1)/84=64/84=16/21

P(B) = (1C1)/(3C1)=1/3

(P(A))/(P(B))=(16×3)/21=16/7=> P(A) : P(B) = 16 : 7 (Proved)

Method 2:

A can win one, two or all the prizes in the following manner

1 prize, 2 blanks	2 prizes, 1 blank	3 prizes
I	II	III

All the three cases where 'A' can win is independent and mutually exclusive. Hence probability that A can win prize is given as

P(A) = P(I) + P(II) + P(III)

Let's calculate P(I): it is something like this:

A bag contains 3 white balls (prizes and 6 black balls (blanks) then what is the probability that out of 3 draws random, there is 1 white ball (prize) and 2 black balls (blanks).

Which one can calculate very easily.

Favourable ways for this event = 3C1 × 6C2

Total number of ways = 9C3

P(I) = (3C1× 6C2)/(9C3),, similarly we can calculate P(II) and P(III)

Hence, P(A) = (3C1× 6 C2)/(9C3)+(3C2× 6C1)/(9C3)+(3C3× 6C0)/(9C3) = 16/21

B has only one share in a lottery, which consists 1 prize and 2 blanks' therefore B can at most win only one prize. Hence

P(B) = 1C1/3C1 = 1/3

(P(A))/(P(B))=(16×3)/21=16/7

P(A) : P(B) = 16:7 (Ans.)

Example 4:

There are three events A, B and C one of which must, and only one can happen, the odd are 8 to 3 against and 2 to 5 for B. Find the odd, against C.

Solution:

P(A) = 3/11, P(B) = 2/7, P(C) = x (say)

Since one most and only one can happen therefore A, B, C are mutually exclusive and exhaustive events.

So, P(A) + P(B) + P(C) = 1

=> 3/11+2/7+x=1

x = (77-21-22)/77=34/77

Odd against C = (77 - 34) : 34 or 43 : 34 (Ans.)

Example 5:

An unbiased coin is tossed. If the result is head, a pair of unbiased dice are rolled and the number obtained by adding the number on the two faces are noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4……, 12 is picked & the number on the card is noted. What is the probability that the number noted is 7 or 8.

Solution:

Let us define the events

A : head appears.

B : Tail appears

C : 7 or 8 is noted.

We have to find the probability of C i.e. P(C)

P(C) = P(A) P (C/A) + P(B) P(C/B)

Now we calculate each of the constituents one by one

P(A) = probability of appearing head = 1/2

P(C/A) = Probability that event C takes place i.e. 7 or 8 being noted when head is already appeared. (If something is already happen then it becomes certain, i.e. now it is certain that head is appeared we have to certainly roll a pair of unbiased dice).

= 11/36 (since (6, 1) (1, 6) (5, 2) (2, 5) (3, 4) (4, 3) (6, 2) (2, 6) (3, 5) (5, 3) (4, 4) i.e. 11 favourable cases and of course 6 × 6 = 36 total number of cases)

Similarly, P(B) = 1/2

P(B/C) = 2/11 (Two favourable cases (7 and 8) and 11 total number of cases).

Hence, P(C)=1/2×11/36+1/2×2/11=193/792 (Ans.)

Example 6:

If the probability for A to win a game against B is 0.4. If A has an option of playing either a "best of 3 games" or a "best of 5 games" match against B, which option should A choose so that the probability of his winning the match is higher?

Solution:

Lets define the events

A : A wins a game against B => P(A) = 0.4

$\bar{A}$: A losses a game against B => P ($\bar{A}$) = 0.6

B : A wins match against B

Case I: When A plays a "Best of three games" match.

In this case A will have to win at least two of three games if he has to win the match.

This he can do by winning either 2 games or all of the 3 games in the manner.

$$\underset{\text{I}}{A\,\bar{A}\,A} + \underset{\text{II}}{\bar{A}\,A\,A} + \underset{\text{III}}{A\,A\,\bar{A}} + \underset{\text{IV}}{A\,A\,A}$$

P(B) = 3C2 A2 ($\bar{A}$) + 3C3 (A)3

= 3 × (0.4)2 × (0.6) + (0.4)3 = 0.352

Case II: When A play a "Best of 5 games against" B. Then

P(B) = 5C3 A3 ($\bar{A}$)2 + 5C4 (A)4 $\bar{A}$ + 5C5 (A)5

= 10 × (0.4)3 (0.6)2 + 5 × (0.4)4 (0.6) + (0.4)5 = 0.23424

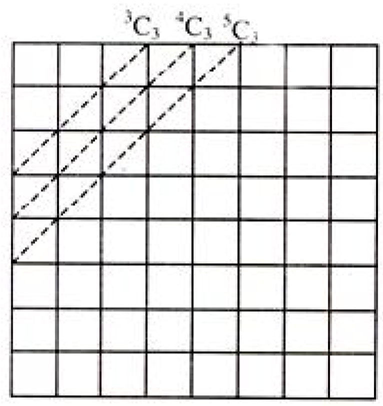
Therefore, A should choose Best of three games (Ans.)

Example 7:

Three squares are chosen at random on a chessboard. Find the probability that they lie in a diagonal line.

Solution:

Total number of cases is 64C3. It is clear from the figure that favourable numbers of case are 2(3C3 + 4C3 + 5C3 + 6C3 + 7C3) + 8C3. Now, diagonal can also be in the direction of NE to SW i.e. favourable cases are 2|2 (3C3 + 4C3 + 5C3 + 6C3 + 7C3) +8C3_|.



So required probability = (favourable cases)/(Total cases)

=(2|2(3C3 + 4C3 + 5C3 + 6C3 + 7C3) + 8C3_|)/64C3 = 7/744 (Ans.)

Example 8:

A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of the subset of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements.

Solution:

In set P we can have no element i.e. Φ, 1 element, 2 element, ... upto n elements. If we have no element in P, we will leave by all the elements and number of set Q formed by those elements will have no common element in common with P. Similarly, it there are r elements in P we are left with rest of (n - r) element to form Q, satisfying the condition that P and Q should be disjoint.

Hence, the total number of ways in which P and Q are disjoint can be given by

$$= \sum_{r=0}^{n} {}^{n}C_r (2)^{(n-r)} = (3)^n$$

[Suppose you have a 5 element set {1, 2, 3, 4, 5} and you formed a subset of 2 elements, which you can do in $^5C_2$ ways. Let's select 1 and 2, then we are left with 3, 4 and 5. Now we can form $(2)^3 = 8$ i.e. {Φ}, {3}, {4}, {5}, {3, 4}, {3, 5}, {4, 5}, {3, 4, 5} subsets which will have no common element with the previously chosen subset. So the total number of ways in which you can form two disjoint set is $^5C_0(2)^5 + {}^5C_1(2)^4 + {}^5C_2(2)^3 + {}^5C_3(2)^2 + {}^5C_4(2)^1 + {}^5C_5(2)^0 = (3)^5$]

Total number of ways in which we can form P and Q

$$= {}^nC_0 (2)^n + {}^nC_1(2)^n + {}^nC_2(2)^n + \dots + {}^nC_n(2)^n = (4)^n$$

or simply total number of ways = Total number of ways of forming P × total number of ways of forming Q.

$$= 2^n \times 2^n = (4)^n.$$

So, required probability = $(3/4)^n$. (Ans.)

Example 9:

In a test an examinee either guesses or copies or knows that answer to a multiple choice question which has 4 choices. The probability that he makes a guess is 1/3 and the probability that he copies is 1/6. The probability that his answer is correct, given the copied it, is 1/8. Find the probability that he knew the answer to the question, given that he answered it correctly.

Solution:

P(g) = probability of guessing = 1/3

P(c) = probability of copying = 1/6

P(k) = probability of knowing = 1 - 1/3 - 1/6 = 1/2

(Since the three-event g, c and k are mutually exclusive and exhaustive)

P(w) = probability that answer is correct

$$P(k/w)=\frac{P(w/k).P(k)}{P(w/c)P(c)+P(w/k)P(k)+P(w/g)P(g)}$$ (using Baye's theorem)

$$= \frac{1\times 1/2}{(1/8,1/6)+(1\times 1/2)+(1/4\times 1/3)} = 24/29$$ (Ans.)

Example 10:

A speak truth 3 out of 4 times. He reported that Mohan Bagan has won the match. Find the probability that his report was correct.

Solution:

Method 1: Let T : A speaks the truth

B : Mohan Bagan won the match

Given, P(T) =3/4 ∴ P($\overline{T}$) = 1 - 1/3 = 1/4

A match can be won, drawn or loosen

∴ P(B/T) = 1/3 P(B/$\overline{T}$) = 2/3 .

Using Baye's theorem we get

P(T/B) = (P(T).P(B/T))/(P(T).P(B/T)+P($\overline{T}$)P(B/$\overline{T}$))

= 3/4×1/3)/(3/4×1/3+1/4×2/3)=(1/4)/(5/12)=3/5

Method 2: Let, T : The man speak truth

A : Mohan Bagan won the match

B : He reported that Mohan Bagan has won.

P(A) = 1/3(the match may also end in a draw)

P(T) = 3/4

P(B) = P(A) P(T) + P($\overline{A}$) P($\overline{T}$)

= 1/3×3/4+2/4×1/4=1/4+1/6=(3+2)/12=5/12

P(T/B)=(P(B/T).P(T))/(P(B))=(1/3×3/4)/(5/12)=3/5 (Ans.)

Example 11:

A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random one at a time with replacement. The event A, B, C are defined as:

A : the first bulb is defective

B : the second bulb is non defective

C : both the bulbs are defective or both are non-defective,

Determine whether

(i) A, B, C are pair wise independent.

(ii) A, B, c are mutually independent.

Solution:

Let probability of defective bulb is denoted by Pd and non-defective bulb is denoted by Pn.

So, Pd = Pn = 1/2

Let, D1 : First bulb is defective

D2 : Second bulb is defective

N1 : First bulb is non-defective

N2 : Second bulb is non-defective

Now event A is First bulb is defective and second bulb is defective or First bulb is defective and second bulb is non-defective.

i.e. A is D1D2, D1N2

B is D1 N2, N1 N2

and C is D1D2, N1N2

=> $A \cap B$ = D1N2

$B \cap C$ = N1N2

$C \cap A$ = D1D2

=> $P(A \cap B) = 1/4\ P(B \cap C) = P(C \cap A)$

also $P(A) = 1/2*1/2 + 1/2*1/2 = 1/2 = P(B) = P(C)$

=> $P(A \cap B) = P(A)\ P(B)$

$P(B \cap C) = P(B)\ P(C)$

$P(C \cap A) = P(C)\ P(A)$

This condition is insufficient for A, B, C to be pair wise independent

Now, $A \cap B \cap C = \Phi \Rightarrow P(A \cap B \cap C) \neq P(A)\ P(B)\ P(C)$

Since, there is no element, which is common to all A, B, C.

So, A, B, C are not mutually independent events. (Ans.)

Example 12:

A sample consists of integer 1, 2……… 2n. The probability of choosing the integer k is proportional to log k. Find the conditional probability of choosing the integer 2 given that an even integer is chosen.

Solution:

P(integer k is chosen)k = C log k (∵ it is proportional to log k), where C is some constant.

Now let even

A : 2 is chosen

B : An even integer is chosen, we have to find conditional probability of A given event B has happened i.e. P(A/B)

$P(A/B) = (P(A \cap B))/(P(B)) = (P(A).P(B/A))/(\Sigma P(A)P(B/A))$

$= (C.\log 2)/(C \log 2 + C \log 4 + C \log 6 + \ldots\ldots\ldots + C \log 2n)$

$= (\log 2)/\log(2n.(|n)) = (\log 2)/(n \log 2 + \log(|n))$ (Ans)

Example 13:

Two players A and B toss a coin alternatively, with A beginning the game. The players who first throw a head is deemed to be the winter. B's coin is fair and A's is biased and has a probability p showing a head. Find the value of p so that the game is equi-probable to both the players.

Solution:

Player 'A' wins if he gets head in the first trial or in third [If B does not get head in his first trial] and so on.

$P(A) = p + (1 - p) \times 1/2 \times p + (1 - p) \times 1/2 \times (1 - p) \times 1/2 \times p + \ldots\infty$

This is an infinite G.P. of first term p and common ratio (1-p)/2

$= p/(1-((1-p)/2)) = (2p)/(p+1)$

According to given condition

$P(A) = P(B)$

$\Rightarrow 2p/(1+p) = 1-2p/(p+1)$

$\Rightarrow (4p)/(1+p) = 1 \Rightarrow p=1/3$ (Ans.)

Example 14:

In a multiple choice question, there are four alternative answers, of which one or more are correct.

A candidate will get marks in the question only when if he ticks all the correct answers. The candidate decides to tick answers at random. If he is allowed up to three chances to answer the question, find the probability that he will get marks in the question.

Solution:

Let the multiple choice question has four alternatives in the form

(a) ______________ (b) ______________

(c) ______________ (d) ______________

The possible answers are

a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd, abcd

Hence the total number of possible answers of such questions = 15.

If a student is given 3 chances to make guess, he will definitely choose 3 different answers. Hence number of favourable way = 3.

Therefore, Probability = 3/15 = 1/5 (Ans.)

Alternative Method:

If you are bit practiced you can nearly visualize that there are 4 elements (options) to form number of subsets (possible answers), which we can do in 24 - 1 = 15 ways. (1 is subtracted, as null set in this case means you are not choosing any answer at all).

Total number of favourable way = 3

Hence probability = 3/15 = 1/5 (Ans.)

Example 15:

Four cards are drawn at random from a well-shuffled pack of 52 cards. Find the probability that there is exactly one pair.

Solution:

Let H, C, D, S denotes heat, club, diamond and spade respectively.

We have to draw 4 cards at random, so that it consists of exactly one pair. A pair means two cards of same denomination. i.e. (5H, 5C) or (6C, 6D), so any one of such (i.e. having exactly one pair) out of four-selected card should look like.

2H, 2S, 3C, 5D

Now we will find in how many ways, we can do this.

We have a pack of card like this.

1 H,	1 C,	1 D,	1 S	...... (1)
2 H,	2 C,	2 D,	2 S	...... (2)
3 H,	3 C,	3 D,	3 S	...... (3)
J H,	J C,	J D,	J S	...... (11)
Q H,	Q C,	Q D,	Q S	...... (12)
K H,	K C,	K D,	K S	...... (13)

Firstly, in how many ways can we select a pair? There are 13 rows in above depiction of a pack of card. Each row consists of 4 cards of same denomination. If we select any of the row, which we can do in 13C1 way (say we have selected 3 row) and in that selected row if we select any of the 2 cards (say 3 H and 3 C) out of 4 which we can do in 4C2 ways, we will get total number of selecting a pair.

i.e. $13C1 \times 4C2$

Having selected pair we have to select two cards of two different denominations.

Now we are left with 50 cards but out of those 50 cards 2 cards (in our case 3 D and 3 S) are of no use as they will destroy our pair (they will make either triplet or quartet). Hence we are left with only 48 cards.

Since we have to select 2 more cards and if one says it can be done in 48C2 ways. It will be wrong. In our case, as those 48C2 combinations will have many pairs included in itself, and we need only one pair. So what we need is to select two cards carefully. We will do it one by one. We will select one card, which we can do in 48C1 ways (say we have selected 5S).

Number of ways in which we have selected 3 cards

$= 13C1 \times 4C2 \times 48C1.$

Now we are left with 47 cards but out of this 47 cards there are 3 cards (5H, 5C, 5D) are of no use as selection of any one of them will form a pair. Therefore, we are left with only 44 cards. We can select any of one card without violating any of our conditions in 44C1 ways.

Hence favourable numbers of ways = $13C1 \times 4C2 \times 48C1 \times 44C1$

Total numbers of ways = 52C4

Probability = $(13C1 \times 4C2 \times 48C1 \times 44C1) / (52C4)$ (Ans.)

Example 16:

Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that these three numbers are in A.P.

Solution:

We have 21 tickets. The total number of ways of selecting 3 out of 21 tickets will be = 21C3.

We will calculate the number of favourable ways in the following manner, suppose first we have selected 1. Now other two selection can be done like this

$$\left.\begin{array}{ccc} 1, & 2, & 3 \\ 1, & 3, & 5 \\ 1, & 4, & 7 \\ \vdots & \vdots & \vdots \\ 1, & 11, & 21 \end{array}\right\} \rightarrow 10 \text{ ways.}$$

Similarly, if we select number 2 then,

$$\left.\begin{array}{ccc} 2, & 3, & 4 \\ 2, & 4, & 6 \\ \vdots & \vdots & \vdots \\ 2, & 11, & 20 \end{array}\right\} \rightarrow 9 \text{ ways.}$$

For starting with 3 we have 9 ways.

For starting with 4 we have 8 ways

So, the favourable ways

= 10 + 9 + 9 + 8 + 8 + 7 + 7 + 6 + 6 + 5 + 5 + 4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 = 100

Probability = 100/21C3. (Ans.)

OVERVIEW

Consider an experiment that can produce a number of results. The collection of all results is called the sample space of the experiment. The power set of the sample space is formed by considering all different collections of possible results. For example, rolling a die can produce six possible results. One collection of possible results give an odd number on the die. Thus, the subset {1,3,5} is an element of the power set of the sample space of die rolls. These collections are called "events." In this case, {1,3,5} is the event that the die falls on some odd number. If the results that actually occur fall in a given event, the event is said to have occurred.

A probability is a way of assigning every event a value between zero and one, with the requirement that the event made up of all possible results (in our example, the event {1,2,3,4,5,6}) is assigned a value of one. To qualify as a probability, the assignment of values must satisfy the requirement that if you look at a collection of mutually exclusive events (events with no common results, e.g., the events {1,6}, {3}, and {2,4} are all mutually exclusive), the probability that at least one of the events will occur is given by the sum of the probabilities of all the individual events.

The probability of an event *A* is written as P(*A*), p(*A*) or Pr(*A*). This mathematical definition of probability can extend to infinite sample spaces, and even uncountable sample spaces, using the concept of a measure.

The *opposite* or *complement* of an event *A* is the event [not *A*] (that is, the event of *A* not occurring); its probability is given by P(not *A*) = 1 - P(*A*). As an example, the chance of not rolling a six on a six-sided die is 1 – (chance of rolling a six) $= 1 - \frac{1}{6} = \frac{5}{6}$. See Complementary event for a more complete treatment.

If both events *A* and *B* occur on a single performance of an experiment, this is called the intersection or joint probability of *A* and *B*, denoted as $P(A \cap B)$.

Independent probability

If two events, *A* and *B* are independent then the joint probability is

$$P(A \text{ and } B) = P(A \cap B) = P(A)P(B),$$

for example, if two coins are flipped the chance of both being heads is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Mutually exclusive

If either event *A* or event *B* or both events occur on a single performance of an experiment this is called the union of the events *A* and *B* denoted as $P(A \cup B)$. If two events are mutually exclusive then the probability of either occurring is

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

For example, the chance of rolling a 1 or 2 on a six-sided die is $P(1 \text{ or } 2) = P(1) + P(2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Not mutually exclusive

If the events are not mutually exclusive then

$$\mathrm{P}\left(A \text{ or } B\right) = \mathrm{P}\left(A\right) + \mathrm{P}\left(B\right) - \mathrm{P}\left(A \text{ and } B\right).$$

For example, when drawing a single card at random from a regular deck of cards, the chance of getting a heart or a face card (J,Q,K) (or one that is both) is $\frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{11}{26}$, because of the 52 cards of a deck 13 are hearts, 12 are face cards, and 3 are both: here the possibilities included in the "3 that are both" are included in each of the "13 hearts" and the "12 face cards" but should only be counted once.

Conditional probability

Conditional probability is the probability of some event *A*, given the occurrence of some other event *B*. Conditional probability is written $\mathrm{P}(A \mid B)$, and is read "the probability of *A*, given *B*". It is defined by

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

If $\mathrm{P}(B) = 0$ then $\mathrm{P}(A \mid B)$ is undefined. Note that in this case *A* and *B* are independent.

Summary of probabilities

Summary of probabilities	
Event	**Probability**
A	$P(A) \in [0,1]$
not A	$P(A^c) = 1 - P(A)$
A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
A and B	$P(A \cap B) = P(A\|B)P(B) = P(B\|A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$