Five Algorithmic Puzzles

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Abstract

Many fascinating mathematical puzzles revolve around algorithms. Typically, you (the victim) are presented with a 'situation' together with a collection of possible operations, and a target state. You may or may not be able to exercise choice in applying the operations. You are asked: Can you reach the target state? Or perhaps: Can you *avoid* reaching the target state? And sometimes: In how many operations?

Listed here are five such problems, all of which can be solved using a principle elucidated below. Solutions are provided for all but one, together with limited information about the problems' sources. The collection is dedicated to Martin Gardner in connection with the *Gathering for Gardner V*, Atlanta GA, April 5–7 2002.

1 Operations and Targets

A classic paradigm for some of the best mathematical puzzles goes something like this: You are presented with a "current situation", a "target state" and a set of "operations" which you can use to modify a situation. You are asked to prove one of these statements (but not necessarily told which):

1. There is a (finite) sequence of operations which reaches the target state;

2. Any sequence of operations will eventually reach the target;

3. Every sequence of operations reaches the target in the same number of steps;

4. No sequence of operations can reach the target.

Typically the operation changes some aspect of the situation for the better, while possibly losing ground elsewhere. How can you determine whether the target is reachable?

Here is a practice problem from an old Russian Olympiad.

> Suppose that you are given an $m \times n$ array of numbers and permitted at any time to reverse the signs of all the numbers in any row or column. Prove that you can arrange matters so that all the row sums and column sums are non-negative.

Of course, flipping a row that has a negative sum will fix that sum but possibly ruin some column sums. How can you be sure to make progress?

Your goal in this and many other problems should be to find a parameter P—some kind of numerical rating of states—which somehow encapsulates progress toward the target.

To prove (1), you want to show that until the target is reached there is always an operation (or sequence of operations) available which improves P. To make sure that you don't get caught in Zeno's paradox (making smaller and smaller steps, and never reaching the target value) you may have to show that P can always be improved by at least a certain amount, or that there are only finitely many possible situations.

To prove (2), you do the same except that now you show that *every* choice of operation improves P.

$$
\begin{array}{c}
\begin{array}{|rrrrr|}
\hline
2 & -3 & -1 & 0 & 3 \\
-1 & 2 & -1 & -1 & 2 \\
1 & 3 & 2 & -4 & 2 \\
2 & -1 & 0 & 3 & 1 \\
-3 & -2 & -2 & 4 & 4 \\
\hline
\end{array}
\begin{array}{r}
1 \\ 1 \\ 4 \\ 5 \\ 1
\end{array}
\\[4pt]
\begin{array}{rrrrr}
1 & -1 & -2 & 2 & 12
\end{array}
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
\begin{array}{|rrrrr|}
\hline
2 & 3 & -1 & 0 & 3 \\
-1 & -2 & -1 & -1 & 2 \\
1 & -3 & 2 & -4 & 2 \\
2 & 1 & 0 & 3 & 1 \\
-3 & 2 & -2 & 4 & 4 \\
\hline
\end{array}
\begin{array}{r}
7 \\ -3 \\ -2 \\ 7 \\ 5
\end{array}
\\[4pt]
\begin{array}{rrrrr}
1 & 1 & -2 & 2 & 12
\end{array}
\end{array}
$$

Figure 1: Flipping a column

To prove (3), you show that every operation improves P by the same amount.

To prove (4), you show that *no* operation improves P, yet attaining the target requires improvement.

Now let us return to the array problem. We see that letting P be the number of lines (rows and columns) with non-negative sum is the wrong parameter; it could go down even when a line with negative sum is flipped. Instead, let's try setting P equal to the sum of all the entries in the array. Flipping a row with sum $-s$ increases P by $2s$, since P can be written as the sum of all the row sums (and similarly for columns). Since there are only finitely many reachable situations (actually, no more than 2^{m+n}), and P goes up every time you flip a negative-sum line, you must reach a time when all the line sums are non-negative.

This was a Type (1) problem but as you see it could also have been phrased as a Type (2) problem, by specifying that only negative-sum lines may be flipped, then asking you to show that you *will* reach a point when all the line-sums are non-negative.

For the problems below, considerably more imagination may be required to find a parameter P that works.

2 Five Problems

2.1 The Infected Checkerboard

An infection spreads among the squares of an $n \times n$ checkerboard in the following manner: if a square has two or more infected neighbors, then it becomes infected itself. (Neighbors are orthogonal only, so each square has at most 4 neighbors.)

For example, suppose that we begin with the n squares on the main diagonal infected. Then the infection will spread to neighboring diagonals and eventually to the whole board. Prove that you *cannot* infect the whole board if you begin with fewer than n infected squares.

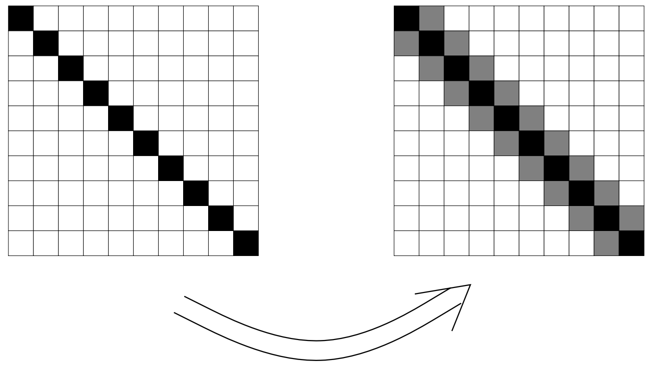

Figure 2: Infecting the checkerboard from the main diagonal

2.2 Emptying a Bucket

You are presented with 3 large buckets each containing an integral number of ounces of some non-evaporating fluid. At any time you may double the contents of one bucket by pouring into it from a fuller one; in other words, you may pour from a bucket containing x ounces into one containing $y \leq x$ ounces until the latter contains $2y$ (and thus the former $x - y$).

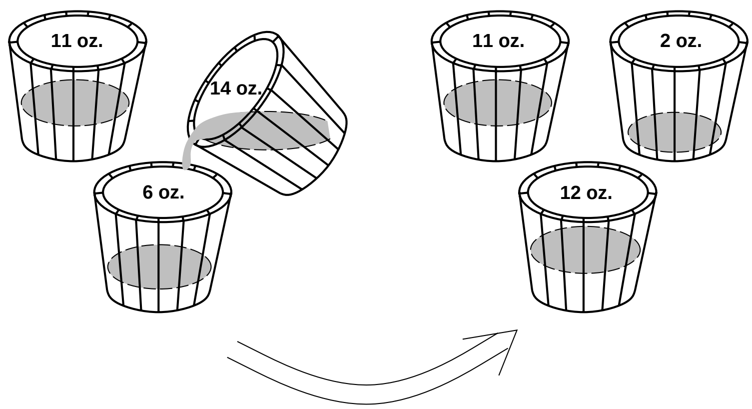

Figure 3: One step toward emptying a bucket

Prove that no matter what the initial contents, you can, eventually, empty one of the buckets.

2.3 Pegs on the Half-Plane

Each grid point on the XY plane on or below the X-axis is occupied by a peg. At any time a peg can be made to jump over a neighbor peg (horizontal, vertical or diagonal) and onto the next grid point in line, provided that point was unoccupied, after which the neighbor is removed.

Can you get a peg arbitrarily far above the X-axis?

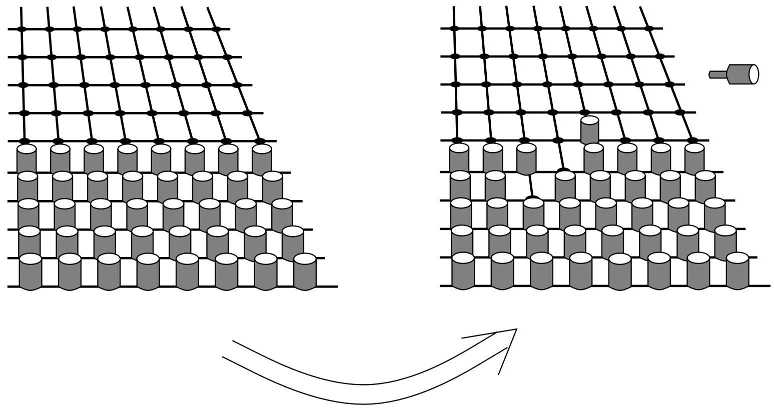

Figure 4: A diagonal jump puts a peg on the line $y = 1$

2.4 Flipping the Polygon

The vertices of a polygon are labeled with numbers, the sum of which is positive. At any time you may change the sign of a negative label, but then the new value is subtracted from both neighbor's values so as to maintain the same sum.

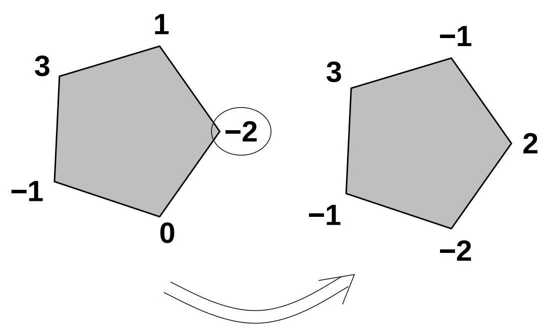

Figure 5: One flip on a pentagon

Prove that inevitably, no matter which labels are flipped, the process will terminate after finitely many flips, with all values non-negative.

2.5 Breaking the Chocolate Bar

You have a rectangular chocolate bar marked into mxn squares, and you wish to break up the bar into its constituent squares. At each step you may break one piece along any of its marked vertical or horizontal lines.

Prove that every method finishes in the same number of steps.

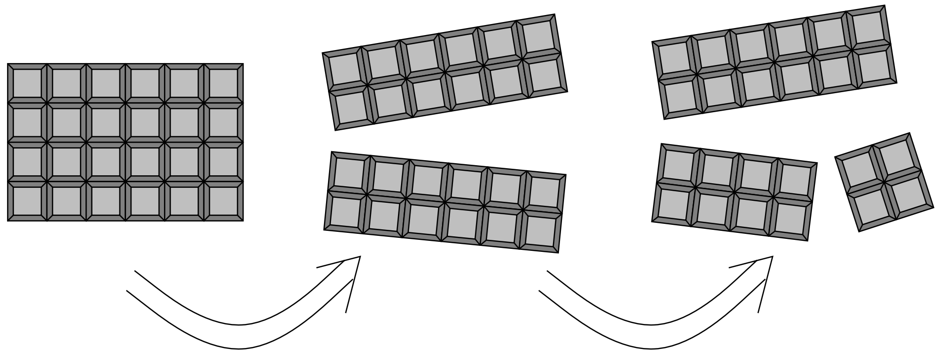

Figure 6: Two steps toward breaking up a chocolate bar

3 Solutions

3.1 The Infected Checkerboard

This lovely problem appeared in the Soviet magazine KVANT around 1986, then migrated to Hungary; a probabilistic version is under study by mathematicians Gabor Pete and Jozsef Balogh. The problem reached me through Joel Spencer of the Courant Institute, who claimed there was a "one-word proof"! As you will see, this is only a mild exaggeration.

Would-be solvers, misled by the diagonal example, often try to show that there must be an initially infected square in each row or column; but that is far from true. Note for example that the configuration of sick squares shown in Fig. fig:sick spreads to the whole board.

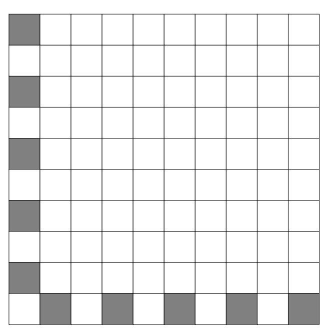

Figure 7: Another way to infect the whole board

Indeed there are myriad ways to infect the whole board with n sick squares, but apparently no way to do it with fewer. Some magic parameter P is needed here, but what?

The parameter is the perimeter! When a square is infected at least two of its boundary edges are absorbed into the interior of the infected area, and at most two added to the boundary of the infected area. Hence the perimeter of the infected area cannot increase. Since the perimeter of the whole board is $4n$ (assuming unit-length edges), the initial infected area must have contained at least n squares.

An additional exercise for those interested: prove that n initial sick squares are necessary even when the top and bottom of the board are joined to form a cylinder. If the sides are joined as well, forming a torus, then $n-1$ initial sick squares are sufficient (and necessary).

The perimeter no longer works but another approach, found by Bruce Richter (U. of Waterloo) and me, does the trick.

3.2 Emptying a Bucket

Yet another beauty from the former Soviet Union, this problem appeared in the 5th U.S.S.R. Mathematics Competition, Riga, 1971. The problem reached me via Christian Borgs, of Microsoft Research. I will give two solutions: a combinatorial one of my own, and an elegant number-theoretic one found by Svante Janson of Uppsala University, Sweden. I do not know which, if either, solution was the intended one.

In Svante's solution, P is the content of a particular bucket and we show how P can always be reduced until it is zero. In my solution, however, we show that P can always be *increased* until one of the *other* buckets is empty!

To do this, we first note that we can assume there is exactly one bucket containing an odd number of ounces of fluid; for, if there are none we can scale down by a power of 2, otherwise one step with two odd buckets will reduce their number to one or none.

Second, note that with an odd and an even bucket we can always do a reverse step, i.e. get half the contents of the even bucket into the odd one. This is because each state can be reached from at most one state, thus if you take enough steps you must cycle back to your original state; the state *just before* you return is the result of your "reverse step".

Finally we argue that as long as there is no empty bucket, the odd bucket's contents can always be increased. For, if there is a bucket whose contents are divisible by 4 we can empty half of it into the odd bucket; if not, one forward operation between the even buckets will create such a bucket.

Here is Svante's solution, in his own words:

"Label the buckets A, B, C with, initially, a, b and c ounces of fluid, where $0 < a \leq b \leq c$. I will describe a sequence of moves leading to a state where the minimum of the three amounts is smaller than a. If this minimum is zero we are home, otherwise we relabel and repeat.

Let $b = qa + r$, where $0 \leq r < a$ and $q \geq 1$ is an integer. Write q in binary form: $q = q_0 + 2q_1 + \cdots + 2^n q_n$ where each q_i is 0 or 1 and $q_n = 1$.

Do $n+1$ moves, numbered $0, \ldots, n$, as follows: in move i we pour from B into A if $q_i = 1$ and from C into A if $q_i = 0$. Since we always pour into A, its content is doubled each time so A contains 2^i a before the ith move. Hence the total amount poured from B equals qa, so at the end there remains $b - qa = r < a$ in B. Finally observe that the total amount poured from C is at most

$$\sum_{i=0}^{n-1} 2^i a < 2^n a <= qa <= b <= c$$

so there will always be enough fluid in C (and in B) to do these moves."

As far as I know, no one knows even approximately how many steps are required for this problem (in whatever is the worst starting state involving a total of n ounces of fluid). My solution shows that order n^2 steps suffices, but Svante's does better, bounding the number by a constant times $n \log n$. The real answer might be still smaller.

3.3 Pegs on the Half-Plane

This is a variation of a problem is described in *Winning Ways for Your Mathematical Plays*, Vol. 2, by Berlekamp, Conway and Guy (Academic Press, 1982). We believe the problem was invented originally by the second author, John H. Conway of Princeton University. In

Conway's problem diagonal jumps were not permitted; one can nonetheless get a peg to the line $y = 4$ without much difficulty, but an argument like the one below shows that no higher position can be reached.

With or without diagonal jumps, the difficulty is that as pegs rise higher, grid points beneath them are denuded. What is needed is a parameter P which is rewarded by highly-placed pegs but compensatingly punished for holes left behind. A natural choice would be a sum over all pegs of some function of the peg's position. Since there are infinitely many pegs, we must be careful to ensure that the sum converges.

We could, for example, assign value r^y to a peg on $(0, y)$, where r is some real number greater than 1, so that the values of the pegs on the lower Y-axis sum to the finite number $\sum_{y=-\infty}^{0} r^y = r/(r-1)$. Values on adjacent columns will have to be cut, though, to keep the sum over the whole plane finite; if we cut by a factor of r for each step away from the Y-axis, we get a weight of $r^{y-|x|}$ for the peg at (x, y) and a total weight of

$$\frac{r}{r-1} + \frac{1}{r-1} + \frac{1}{r-1} + \frac{1}{r(r-1)} + \frac{1}{r(r-1)} + \cdots$$

$$= \frac{r^2 + r}{(r-1)^2} < \infty$$

for the initial position.

If a jump is executed, then at best (when the jump is diagonally upward and toward the Y-axis) the gain to P is vr^4 and the loss $v + vr^2$, where v is the previous value of the jumping peg. As long as r is at most the square root of the "golden ratio" $\theta = (1 + \sqrt{5})/2 \approx 1.618$, which satisfies $\theta^2 = \theta + 1$, this gain can never be positive.

If we go ahead and assign $r = \sqrt{\theta}$ then the initial value of P works out to about 39.0576; but the value of a peg at the point $(0, 16)$ is $\theta^8 \approx 46.9788$ *by itself*. Since we cannot increase P, it follows that we cannot get a peg to the point $(0, 16)$. But if we could get a peg to *any* point on or above the line $y = 16$ then we could get one to $(0, 16)$ by stopping when some peg reaches a point $(x, 16)$, then redoing the whole algorithm shifted left or right by $|x|$.

We do not know the highest value of y for which the point $(0, y)$ is reachable, allowing diagonal jumps. Perhaps an industrious reader can close this gap.

3.4 Flipping the Polygon

This problem generalizes one which appeared in the International Mathematics Olympiad (submitted by a composer from East Germany, I am told) and subsequently termed "the pentagon problem".

The problem has many solutions, and can even be generalized further from n-gons to arbitrary connected graphs. However, the solution below, due to Bernard Chazelle (Computer Science Department, Princeton), stands out for its combination of elegance and strong conclusion.

Let $x(0), \ldots, x(n-1)$ be the labels, summing to $s > 0$, with indices taken modulo n, and define the doubly infinite sequence $b(\cdot)$ by $b(0) = 0$ and $b(i) = b(i-1) + x(i \mod n)$. The sequence $b(\cdot)$ is not periodic, but periodically ascending; $b(i+n) = b(i) + s$.

If $x(i)$ is negative, $b(i) < b(i-1)$ and flipping $x(i)$ has the effect of switching $b(i)$ with $b(i-1)$ so that they are now in ascending order. It does the same for all pairs $b(j)$, $b(j-1)$ shifted from these by multiples of n. Thus, flipping labels amounts to sorting $b(\cdot)$ by adjacent transpositions!

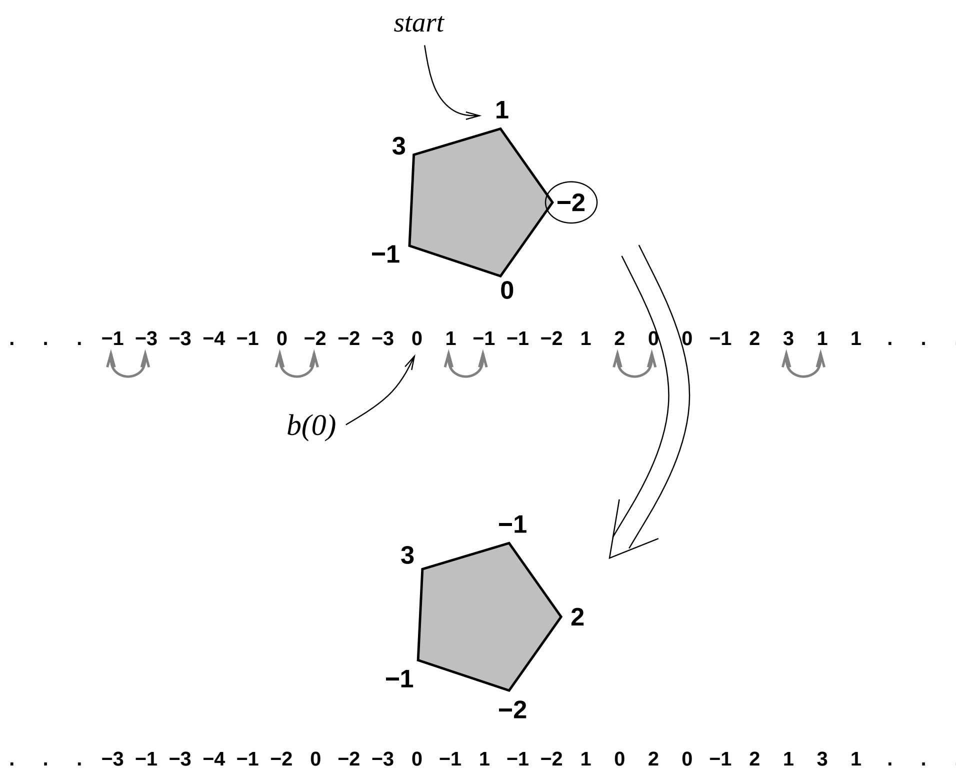

Figure 8: Flipping on a pentagon, while sorting a sequence

To track the progress of this sorting process we need a parameter P which measures the degree to which $b(\cdot)$ is out of order, but is still finite. To do this let i^+ be the number of indices $j > i$ for which $b(j) < b(i)$, and i^- the number of indices $j < i$ for which $b(j) > b(i)$. Note that i^+ and i^- are finite and depend only on $i \mod n$. Observe that $\sum_{i=0}^{n-1} i^+ = \sum_{i=0}^{n-1} i^-$; we let this sum be our magic parameter P.

When $x(i+1))$ is flipped, i^+ decreases by one and every other j^+ is unchanged, so P goes down by *exactly one*. When P hits 0 the sequence is fully sorted so all labels are non-negative and the process terminates.

We have shown more than asked: the process terminates in exactly the same number (P) of steps regardless of choices, and moreover, the final configuration is independent of choices as well! The reason is that there is only one sorted version of $b(\cdot)$; entry $b(i)$ from the original sequence must wind up in position $i + i^+ - i^-$ when the sorting is complete.

3.5 Breaking the Chocolate Bar

This one has been known to stump some *very* high-powered mathematicians for as much as a full day, until the light finally dawns amid groans and beatings of the head against the wall. We wouldn't want to deprive you, dear reader, of the opportunity to try it for yourself.

Acknowledgment

Inspiration for collecting these puzzles and much more comes from Martin Gardner's famous *Mathematical Games* column in *Scientific American*. Thanks also to my (former) friends who served as test victims for countless puzzles.