<u>COURSE SYNOPSIS</u>
<u>Linear Algebra and Numerical Analysis (MA2030)</u>

Vector Spaces

- <u>Definition</u> :

  A set V with two operations + (addition) and · (scalar multiplication) is said to be a vector space if it satisfies the following axioms:

  - (a) $x + y = y + x, \forall x, y \in V$.
  - (b) $(x + y) + z = x + (y + z), \forall x, y, z \in V$.
  - (c) $\exists\, 0 \in V$ such that $x + 0 = x \ \forall x \in V$.
  - (d) for each $x \in V$, $\exists\, \tilde{x}$ such that $x + \tilde{x} = 0$.
  - (e) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y \ \forall \alpha \in F$ and $\forall x, y \in V$.
  - (f) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot y$, $\forall \alpha, \beta \in F$ and $\forall x \in V$.
  - (g) $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x) \ \forall \alpha, \beta \in F, \forall x \in V$.
  - (h) $1 \cdot x = x \ \forall x \in V$.

- <u>Basic Properties</u> :

  - The zero element is unique, i.e., if there exists θ_1 , θ_2 such that $x + \theta_1 = x$ and $x + \theta_2 = x, \forall x \in V$, then $\theta_1 = \theta_2$.
  - Additive inverse for each vector is unique, i.e., for $x \in V$, if there exist x_1 & x_2 such that $x + x_1 = 0 = x + x_2$, then $x_1 = x_2$.
  - In a vectorspace V over F, for any $x \in V$ and $\alpha \in F$:

    (a) $0 \cdot x = 0$

    (b) $(-1) \cdot x = -x$

    (c) $\alpha \cdot 0 = 0$.

Vector Subspaces

- <u>Definition</u> :

  Let W be a subset of a vector space V . Then W is called a subspace of V if W is a vector space with respect to the operations of addition and scalar multiplication as in V .

- *<u>Theorem (i)</u>* : Let W be a subset of a vector space V . Then W is a subspace of V if and only if W is non-empty and $x + \alpha y \in W$ for all $x, y \in W$ and $\alpha, \in F$.
- *<u>Theorem (ii)</u>* : Let V_1 and V_2 are subspace of a vector space V . Then $V_1 \cap V_2$ is a subspace of V .
- *<u>Theorem (iii)</u>* : Let V_1 and V_2 be subspaces of a vector space. Then $V_1 \cup V_2$ is a subspace if and only if either $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$.

Linear Combination of a set of vectors

- <u>Definition</u> :

  Let V be a vector space and $v_1, \ldots, v_n \in V$. Then, by a linear combination of $v_1, \ldots, v_n$, we mean an element in V of the form $\alpha_1 v_1 + \cdots + \alpha_n v_n$ with $\alpha_j \in F, j = 1, \ldots, n$.

Span of a set of vectors

- Definition :

  Let S be a subset of V . Then the set of all linear combinations of elements of S is called the span of S, and is denoted by span S.

- *Theorem (i)* : Let V be a vector space and $S \subseteq V$. Then span S is a subspace of V and it is the smallest one containing S.
- *Theorem (ii)* : Let V_1 and V_2 be subspaces of a vector space V . Define $V_1 + V_2 = \{u + v : u \in V_1 \,\&\, v \in V_2\}$. Then, $V_1 + V_2 = \mathrm{span}(V_1 \cup V_2)$.

Linearly Dependent and Linearly Independent sets

- Definition :
  - A set of vectors $\{v_1 , . . . , v_n\}$ is said to be *linearly dependent* if one of the vectors can be written as a linear combination of others.
  - A set of vectors $\{v_1 , . . . , v_n\}$ is said to be linearly independent if the set is not linearly dependent

- Properties :
  - If $\{u_1 , . . . , u_n\}$ is linearly dependent, then for any vector $v \in V$, the set $\{u_1 , . . . , u_n , v\}$ is linearly dependent.
  - If $E = \{u_1 , . . . , u_n\}$ is linearly independent, then any subset of E is linearly independent.

Basis of Vector Spaces and Dimensions

- Definition :
  - Let V be an F-vector space. A set which is linearly independent and spans a vector space V is called an *F-basis* of V .
  - Consider set $\{M_{ij} : i = 1, . . . , m; j = 1, . . . , n\}$, where M_{ij} is the matrix with (i, j)-th entry 1 and all other entries 0. Then this is a basis of $M_{mxn}(F)$, called the standard basis.

- *Theorem (i)* : Let V be a vector space and $B \subset V$. Then TFAE:

  (i) B is a basis of V

  (ii) B is a maximal linearly independent set in V ,i.e., B is linearly independent and $B \cup \{u\}$ is linearly dependent for any $u \in V$.

  (iii) B is a minimal spanning set of V , i.e., span(B) = V and no proper subset of B can span V .

- ***Dimensions***
- *Definition :* A vector space V is said to be finite dimensional if there exists a finite basis for V, otherwise it is called infinite dimensional vector space.

  Suppose V is a finite dimensional vector space. Then the cardinality of a basis is said to be the dimension of V, *denoted by* ***dim V*** .

- *Theorem (ii)* : If a vector space has a finite spanning set, then it has a finite basis.
- *Theorem (iii)* : Let V be a vector space with basis consisting of n elements.

Then any subset of V having n + 1 vectors is linearly dependent.

- ***Corollary** (a):* Any two bases of a finite dimensional vector space has same cardinality.
- ***Corollary** (b):* If a vector space contains an infinite linearly independent subset, then it is an infinite dimensional space.

➢ *Theorem (iv) :* Let V be a vector space of dimension n and A be a subset of V containing m vectors.
(a) If A is linearly independent, then $m \leq n$.
(b) If $m > n$, then A is linearly dependent.
(c) If A is linearly independent and $m = n$, then A is a basis of V .

➢ *Theorem (v) :* If V is a finite dimensional vector space and W is a proper subspace of V , then $\dim W < \dim V$.

➢ *Theorem (vi)* : Let W be a subspace of a finite dimensional vector space V and $B_W = \{u_1 , . . . , u_m \}$ be a basis of V . Then there exists a basis B_V of V such that $B_W \subseteq B_V$.

➢ *Theorem (vii)* : Let V_1 and V_2 be finite dimensional subspaces of a vector space. If $V_1 \cap V_2 = \{0\}$, then $\dim(V_1 + V_2) = \dim V_1 + \dim V_2$.

➢ *Theorem (viii) :* Let V_1 and V_2 be finite dimensional subspaces of a vector space V . Then
$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2).$$

✯ Linear Transformations

➢ Definition :
Let V_1 and V_2 be vector spaces over F. A function $T : V_1 \to V_2$ is said to be a linear transformation (or a linear map) if
$T (x + y) = T (x) + T (y)$ and $T (\alpha x) = \alpha T (x)$ for every $x, y \in V_1$ and for every $\alpha \in F$.

➢ *Theorem (i)* : Let $T : V \to W$ be a linear transformation, then for all vectors $u, v , v_1 , . . . , v_n \in V$ and scalars $\alpha_1 , . . . , \alpha_n \in F$:
(i) $T (0) = 0$
(ii) $T (u - v) = T (u) - T (v)$
(iii) $T (\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha_1 T (v_1) + \cdots + \alpha_n T (v_n)$.

➢ *Theorem (ii)* : Let V be a finite dimensional vector space with basis $B = \{v_1 , . . . , v_n \}$. If T_1 and T_2 are two linear maps from V to another vector space W such that $T_1 (v_i) = T_2 (v_i)$ for all $i = 1, . . . , n$, then $T_1 (v) = T_2 (v)$ for all $v \in V$.

- *Theorem (iii)* : Let V be a finite dimensional vector space with basis $B = \{v_1, \ldots, v_n\}$. Let W be a vector space containing the n vectors $w_1, \ldots, w_n$. Then, there exists a unique linear map $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, \ldots, n$.

Matrices of Linear Maps (Properties)

- *Theorem (i)* : Let V, W be finite dimensional vector spaces. Let $A = \{v_1, \ldots, v_n\}$ be an ordered basis of V and $B = \{w_1, \ldots, w_m\}$ be an ordered basis of W. Let $T : V \to W$ be a linear transformation. Then for every $x \in V$
$$[T(x)]_B = [T]_A^B\, [x]_A$$
i.e., every linear transformation can be realized as a matrix multiplication.

- *Theorem (ii)* : Let V, W, Z be finite dimensional vector spaces with A, B, C their respective ordered bases. Let $T_1 : V \to W$ and $T_2 : W \to Z$ be linear transformations. Then
$$[T2 \circ T1]^C{}_A = [T2]^C{}_B [T1]^B{}_A$$

Properties of Linear Transformations

- Kernel of $T = N(T) = \{v \in V : T(v) = 0\}$.

- Range of $T = R(T) = \{w \in W : w = T(v) \text{ for some } v \in V\}$.

- *Theorem (iii)* : Let $T : V \to W$ is a linear transformation. Then
    1. ker T is a subspace of V.
    2. Im T is a subspace of W.

- Nullity of $T = \dim N(T)$

- Rank of $T = \dim R(T)$

- Bijective Transformation : A $f : X \to Y$ is called one-one if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. f is called onto if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. If f is one-one and onto, then it is called **bijective**. A bijective linear transformation is called an **isomorphism**.

- *Theorem (iv)* : A linear transformation $T : V \to W$ is one-one if and only if $N(T) = \{0\}$.

- *Theorem (v)* : Let $T : V \to W$ be a bijective linear transformation. Then $T^{-1} : W \to V$ is linear.

- **Rank Nullity Theorem**
  Let V be an n-dimensional vector space and $T : V \to W$ be a linear transformation. Then
  $$\text{Rank}(T) + \text{Nullity}(T) = n.$$

- Two vector spaces V and W are isomorphic if there exists an isomorphism $T : V \to W$.

- *Theorem (vi)* : V and W are isomorphic if and only if $\dim V = \dim W$.

System of Linear Equations

- Definition :
  Let $A \in F_{m\times n}$, $b \in F_m$. Then
  $$\begin{matrix} a_{11} x_1 + \cdots + a_{1n} x_n = b_1 \\ \vdots \qquad \vdots \qquad \vdots \\ a_{m1} x_1 + \cdots + a_{mn} x_n = b_n \end{matrix}$$
  is called a system of linear equations for the unknowns $x_1 , \ldots , x_n$ with coefficients in F.
  - The solution set of the system $Ax = b$ is
    $$\text{Sol}(A, b) = \{x \in F^n : Ax = b\}.$$
    The system $Ax = b$ is solvable if $\text{Sol}(A, b) = \emptyset$.

- *Proposition 1* : $Ax = b$ is solvable iff rank (A) = rank (A|b).

- *Proposition 2* : Let $x_0 \in F^n$ be a solution of $Ax = b$. Then
  $$\text{Sol}(A, b) = x_0 + N(A) = \{x_0 + x : x \in N(A)\}.$$

- *Corollaries* :
  1. If x_0 is a solution of $Ax = b$ and $\{v_1 , \ldots , v_r \}$ is a basis for N(A), then $\text{Sol}(A, b) = \{x_0 + \lambda_1 v_1 + \cdots + \lambda_r v_r : \lambda_i \in F\}$. Here, r = nullity (A) = n − rank (A).
  2. A solvable system $Ax = b$ is uniquely solvable iff $N(A) = 0$ iff rank (A) = n.
  3. If A ia a square matrix, then $Ax = b$ is uniquely solvable iff $\det(A) = 0$.

- ***Cramer's Rule*** : Let $\det(A) = 0$. Denote by C_i the ith column of A. Then the solutions of $Ax = b$ is given by $x_i = \det(A[C_i \leftarrow b])/\det(A)$.

- *Proposition 3* : Elementary row operations preserve the row rank of a matrix. Elementary column operations preserve the column rank of a matrix.

- *Proposition 4* : For a matrix A,
  $$\text{rowrank}(A) = \text{colrank}(A) = \text{rank}(A).$$

- *Proposition 5* : If we change the augmented matrix (A|b) by elementary row transformations into a matrix (A |b), then Sol(A, b) = Sol(A , b).

- ***Gaussian Elimination when det(A)<>0***

  1. Start with the augmented matrix (A|b). If the (1,1) entry is 0, then interchange first row with another to bring the (1,1) entry of the new matrix nonzero. Replace all other rows by that row minus a suitable multiple of the first row to kill all (j, 1) entries for j > 1.

  2. After the k th step, Do not touch the first k rows. Continue as in Step 1 with the rest of the matrix.

  3. After the (n − 1)th step, the matrix is of the form (A |b), where A is upper triangular. Use back-substitution to solve it.

- ***Gaussian Elimination when $A \in F^{m\times n}$***

  (A) Start as in Gaussian Elimination as long as possible. After, say, t steps, you find that the first t diagonal elements are nonzero, but no interchange among the last m − t rows can fill the (t + 1, t + 1) entry with a nonzero element.

  (B) Use interchange of columns from among the last n − t columns and keep track of the corresponding variables. Continue to proceed as in Step (A). Reach finally at a matrix where the first r diagonal elements are nonzero, and all the rest n − r rows are zero rows in the matrix A. Call the b-entries as b'_i

  (C) If one of b_r +1 , . . . , bn is nonzero, then Ax = b is not solvable.

  (D) Otherwise, omit the last n − r rows. You now end up with the augmented matrix as (T |S|b), where T is an upper triangular r × r invertible matrix, S is an r × k matrix, b is an r vector.

  (E) Write the unknowns as y_1 , . . . , y_r , z_1 , . . . , z_k . It is a permutation of the original x_1 , . . . , x_n . The system is Ty + Sz = b .

  (F) Solve Ty = b and adjoin k zeros to get w_0 . For jth vector w_j solve Ty = −Sj , the jth column of S. Adjoin one 1 at jth position rest 0 for getting w_j . (We want to get k linearly independent solutions, since nullity of T + S is k .)

  (G) Finally reorder the vectors w_0 , . . . , w_k for v_0 , . . . v_k taking care of the reordering of the unknowns done in Step (F).

  (H) Sol(A, b) = $\{v_0 + \lambda_1 v_1 + \cdots + \lambda_k v_k : \lambda_i \in F\}$.

Eigenvalues and Eigenvectors

- *Definition* :

  Let V be a vector space over F, and $T : V \to V$ be a linear operator. By an eigenvector associated with the eigenvalue $\lambda \in F$ we mean a nonzero vector $v \in V$ with the property that $Tv = \lambda v$.

- *Proposition 1* : A vector v is an eigenvector of $T : V \to V$ for the eigenvalue $\lambda \in F$ iff $v \in N(T - \lambda I)$. Thus λ is an eigenvalue of T iff the operator $T - \lambda I$ is not one-one.

- *Proposition 2* : Let V over F be finite dimensional. Let A be the matrix for an operator $T : V \to V$ with respect to some basis. Then $\lambda \in F$ is an eigenvalue of T iff $\det(A - \lambda I) = 0$.

- **Characteristic Polynomial** : The polynomial $\det(A - \lambda I)$ in the variable λ is called the characteristic polynomial of the matrix A.

- *Proposition 3* : A and A^t have the same eigenvalues.

- *Proposition 4* : Similar matrices have the same eigenvalues.

- *Proposition 5 :* If A is diagonal or upper triangular or lower triangular, then its diagonal elements are precisely its eigenvalues.

- *Proposition 6* : det(A) equals the product of all eigenvalues.

  tr (A) equals the sum of all eigenvalues.

- **Cayley-Hamilton Theorem**

  Any square matrix satisfies its characteristic polynomial.

- **Hermitian Matrices**

  Proposition 7 : Eigenvalues of a real symmetric matrix or of a Hermitian matrix are real.

  Eigenvalues of a skew-hermitian matrix are purely imaginary or zero.

- *Proposition 8* : Eigenvectors associated with distinct eigenvalues of an $n \times n$ matrix are linearly independent.

- *Proposition 9* : If an $n \times n$ matrix has n distinct eigenvalues, then A is similar to a diagonal matrix.

Inner Product of Vector Spaces

- *Definition* : An inner product on a vector space V is a map x , y which associates a pair of vectors in V to a scalar (x , y) x , y satisfying

  (a) $x , x \geq 0$ for each $x \in V$.

  (b) $x , x = 0$ i# $x = 0$.

  (c) $x + y , z = x , z + y , z$ for all $x , y , z \in V$.

  (d) $\alpha x , y = \alpha\, x , y$ for each $\alpha \in F$ and for all $x , y \in V$.

  (e) $y , x = x , y$ for all $x , y \in V$.

- A vector space with an inner product on it is called an **inner product space** (ips).

- *Proposition 1* : Let V be an ips. For all x , y , z ∈ V and for all α ∈ F,
  $<x, y+z> = <x, y> + <x, z>$, $<x,\alpha y> = \alpha^c<x,y>$ (α^c is the conjugate of α).

- Let V be an ips. For any x∈V, the length of x, also called the **norm** of x is
  $||x||$ = square root of $<x,x>$.

- *Proposition 2* : Let x,y∈V, an ips. The parallelogram law holds:
  $$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2$$

- *Proposition 3* : Let x,y∈V, an ips. $|<x,y>| \le ||x||\,||y||$. Further, the equality holds iff {x,y} is linearly dependent.

- *Proposition 4* : (Triangle Inequality) For all x,y ∈ V, an ips,
  $$||x+y|| \le ||x|| + ||y||.$$

Orthogonality

- *Definition* : Let x , y ∈ V , an ips. The vector x is orthogonal to y, i.e., $x \perp y$ iff $<x,y> = 0$.

- *Proposition 1 (Pythagoras)* : Let V be an ips. Let x , y ∈ V.
  (a) If $x \perp y$, then $||x + y||^2 = ||x||^2 + ||y||^2$.
  (b) Suppose V is a real vector space.
  If $||x + y||^2 = ||x||^2 + ||y||^2$, then $x \perp y$.

- **Orthogonal Set**
  Let V be an ips, $S \subseteq V$, and x ∈ V .
  (a) $x \perp S$ iff for each y ∈ S , $x \perp y$.
  (b) $S^{\perp} = \{x \in V : x \perp S\}$.
  (c) S is called an orthogonal set when
  x , y ∈ S , x ≠ y implies $x \perp y$.
  An orthogonal set of nonzero vectors is lineraly independent.

- **Orthonormal Sets**
  Let V be an ips. A set $S \subseteq V$ is called an orthonormal set if S is orthogonal and $||x|| = 1$ for each x ∈ S. In addition, if an orthonormal set S is also a basis for V , then S is called an orthonormal basis.

- *Proposition 2 (Fourier expansion and Parseval's Identity)* : Let $S = \{u_1 , u_2 , . . . , u_n \}$ be an orthonormal set in an ips V . Let x ∈ span(S). Then
  $$x = \sum_{j=1 \text{ to } n} <x,u_j>u_j \quad \text{and} \quad ||x||^2 = \sum_{j=1 \text{ to } n} |<x,u_j>|^2$$

- *Bessel's Inequality* : Let $\{u_1, u_2, \dots, u_n\}$ be an orthonormal set in an ips V . Let $x \in V$. Then

$$\sum_{j=1 \text{ to } n} |\langle x,u_j\rangle|^2 \le \|x\|^2$$

- **Gram-Schmidt Orthogonalization**

  Let $V = \text{span}\{u_1, \dots, u_n\}$. Define

  $v_1 = u_1$

  $v_2 = u_2 - (\langle u_2,v_1\rangle/\langle v_1,v_1\rangle)*v_1$

  .

  .

  .

  $v_{n+1} = u_{n+1} - (\langle u_{n+1},v_1\rangle/\langle v_1,v_1\rangle)*v_1 - \dots - (\langle u_{n+1},v_n\rangle/\langle v_n,v_n\rangle)*v_n$

- **Best Approximation**

  Let U be a subspace of an ips V . Let $v \in V$. A vector $u \in U$ is a best approximation of v if $\|v - u\| \le \|v - x\|$ for each $x \in U$.

- *Proposition 3* : Let U be a subspace of an ips V . A vector $u \in U$ is a best approximation of $v \in V$ iff $v - u \perp U$. Moreover, a best approximation is unique.

- **Best Approximation Solution**

  Let U be a vector space, V an ips, $A : U \to V$ a linear transformation. A vector $u \in U$ is a Best Approximate Solution (also called a Least Square Solution) of the equation $Ax = y$ if

$$\|Au - y\| \le \|Az - y\| \text{ for all } z \in U.$$

- *Proposition 4* : Let U be a vector space, V an ips, $A : U \to V$ a linear transformation.

  (a) A vector $u \in U$ is a best approximate solution iff $Au - y \perp R(A)$.

  (b) If $\dim(R(A)) < \infty$, then $Ax = y$ has a best approximate solution.

  (c) A best approximate solution is unique iff A is one-one.

- <u>Proposition 5</u> : Let $A \in R^{m\times n}$. A vector $u \in R^n$ is a best approximate solution of $Ax = y$ iff $A^tAu = A^ty$.