PERMUTATIONS & COMBINATIONS

-BASICS-

***CONTENTS*:**

FUNDAMENTAL PRINCIPAL OF COUNTING

Rule of Product

If one experiment has n possible outcomes and another experiment has m possible outcomes, then there are m × n possible outcomes when both of these experiments are performed.

In other words if a job has n parts and the job will be completed only when each part is completed and the first part can be completed in a_1 ways, the second part can be completed in a_2 ways and so on ...... then nth part can be completed in an ways, then the total number of ways of doing the job is $a_1a_2a_3 \dots a_n$. This is known as the rule of product.

Illustration:

A college offers 7 courses in the morning and 5 in the evening. Find the possible number of choices with the student who wants to study one course in the morning and one in the evening.

Solution:

The student has seven choices from the morning courses out of which he can select one course in 7 ways.

For the evening course, he has 5 choices out of which he can select one in 5 ways.

Hence the total number of ways in which he can make the choice of one course in the morning and one in the evening = 7 × 5 = 35.

Illustration:

A man has five friends. In how many ways can be invite one or more of them to a tea party.

Solution:

Methods I

Case I: The man can invite one friend in ${}^5C_1$ ways

Case II: The man can invite two friend in ${}^5C_2$ ways

Case III: The man can invite three friend in ${}^5C_3$ ways

Case IV: The man can invite four friend in ${}^5C_4$ ways

Case V: The man can invite five friend in ${}^5C_5$ ways

All these five cases are mutually exclusive i.e. the man can either invite one friend or two friends or ...., but cannot simultaneously invite one friend and two friends.

Remember: Whenever there is 'OR', we add.

Therefore Total number of ways in which the man can invite his friends

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 5 + 10 + 10 + 5 + 1 = 31$$

Method II.

This problem can also be solved as follows:

Each of his friends may be dealt in two ways - either invited or not invited.

Therefore, The total number of ways of inviting all friends is

$= 2 \times 2 \times 2 \times 2 \times 2 = 32$ ways.

But this includes the case, in which all the friends are not invited. Hence, the total number of ways in which the man invites one or more friends is 25 - 1 = 31 ways.

Note: In the above illustration, we have found the total number of combinations of n dissimilar things taking any number of them at a time, which is $2^n - 1$.

Rule of Sum

If one experiment has n possible outcomes and another has m possible outcomes, then there are (m + n) possible outcomes when exactly one of these experiments is performed.

In other words if a job can be done by n methods and by using the first method can be done in a1 ways or by second method in a2 ways and so on ... by the nth method in an ways, then the number of ways to get the job done is $(a_1 + a_2 + ... + a_n)$.

Illustration:

How many straight lines can be formed from six points, no three of which are collinear?

Solution:

To form a straight line, we need to select two points out of six points. This can be done in ${}^6C_2$ ways = 6.5/2.1 = 15 ways.

Illustration:

In how many ways can a committee of five be formed from amongst four boys and six girls so as to include exactly two girls?

Solution:

We have to select two girls from six girls and three boys from four boys.

Number of ways of selecting girls = ${}^6C_2$ = 6.5/2.1 = 15

Number of ways of selecting boys = ${}^4C_3$ = 4

Number of ways of forming the committee = $15 \times 4 = 60$.

Note: Here we have used multiplication rule.

Illustration:

A college offers 7 courses in the morning and 5 in the evening. Find the number of ways a student can select exactly one course, either in the morning or in the evening.

Solution:

The student has seven choices from the morning courses out of which he can select one course in 7 ways.

For the evening course, he has 5 choices out of which he can select one in 5 ways.

Hence he has total number of 7 + 5 = 12 choices.

Illustration:

How many (i) 5 - digit, (ii) 3 - digit numbers can be formed by using 1, 2, 3, 4, 5 without repetition of digits.

Solution:

(i) Making a 5-digit number amounts to filling 5 places.

Places:					
Number of Choices:	**5**	**4**	**3**	**2**	**1**

The first place can be filled in 5 ways using any of the given digits.

The second place can be filled in 4 ways using any of the remaining 4 digits.

Similarly, we can fill the 3^{rd}, 4^{th} and 5^{th} place.

No. of ways to fill all the five places.

$= 5 \times 4 \times 3 \times 2 \times 1 = 120$

=> 120 5-digit numbers can be formed.

(ii) Making a 3 - digit number amounts to filling 3 places.

Places:			
Number of choices:	**5**	**4**	**3**

Number of ways to fill all the three places = $5 \times 4 \times 3 = 60$.

Hence the total possible 3 - digit numbers = 60.

Illustration:

How many 4-letter words can be formed using a, b, c, d, e

(i) Without repetition (ii) with repetition

Solution:

(i) The number of words that can be formed is equal to the number of ways to fill the three places.

Places:				
Number of Choices:	**5**	**4**	**3**	**2**

=> 5 × 4 × 3 × = 120 words can be formed when repetition is not allowed.

(ii) The number of words that can be formed is equal to the number of was to fill the three places.

Places:				
Number of Choices:	**5**	**5**	**5**	**5**

First place can be filled in 5 ways. If repetition is allowed, all the remaining places can be filled in 5 ways each.

=> 5 × 5 × 5 × 5 = 625 words can be formed when repetition is allowed.

Exercise

1. 1 Plane, 2 trains and 3 buses ply between Delhi and Agra.

   (a) In how many ways can you to Agra from Delhi.

   (b) In how many ways can you go and come back if you go by train,

2. Total number of combinations of n dissimilar things, taken at least one at a time ________.

Ans.1 (a) 6,

(b) 2 × = 12

2. $2^n - 1$

PERMUTATIONS (ARRANGEMENT OF OBJECTS)

The number of permutations of n objects, taken r at a time, is the total number of arrangements of n objects, in groups of r where the order of the arrangement is important.

(i) Without repetition

(a) Arranging n objects, taking r at a time in every arrangement, is equivalent to filling r places from n things.

r-Places	1	2	3	4	———	r
Number of Choices:	n	n-1	n-2	n-3		n - (r - 1)

Number of ways of arranging = Number of ways of filling r places

$= n(n - 1)(n - 2) \ldots (n - r + 1)$

$= (n(n - 1)(n - 2) \ldots (n - r + 1)((n-r)!))/((n-r)!) = n!/((n-r)!) = {}^nP_r$.

(b) Number of arrangements of n different objects taken all at a time $= {}^np_n = n!$

(ii) With repetition

(a) Number of permutations (arrangements) of n different objects, taken r at a time, when each object may occur once, twice, thrice ...... up to r times in any arrangements

= Number of ways of filling r places, each out of n objects.

r-Places:	1	2	3	4	———	r
Number of Choices :	n	n	n	n		n

Number of ways to arrange = Number of ways to fill r places $= (n)^r$.

(iii) Number of arrangements that can be formed using n objects out of which p are identical (and of one kind), q are identical (and of one kind) and rest are different $= n!/p!q!r!$.

Illustration:

How many 7 - letter words can be formed using the letters of the words:

(a) BELFAST, (b) ALABAMA

Solution:

(a) BELFAST has all different letters. Hence, the number of words

${}^7P_7 = 7! = 5040$.

(b) ALABAMA has 4 A's but the rest are all different. Hence the number of words formed is $7!/4! = 7 \times 6 \times 5 = 210$.

Illustration:

(a) How many anagrams can be made by using the letters of the word HINDUSTAN?

(b) How many of these anagrams begin and end with a vowel.

(c) In how many of these anagrams all the vowels come together.

(d) In how many of these anagrams none of the vowels come together.

(e) In how many of these anagrams do the vowels and the consonants occupy the same relative positions as in HINDUSTAN?

Solution:

(a) The total number of anagrams

= Arrangements of nine letters taken all at a time = $9!/2! = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $7!/2!$ ways. Hence the total number anagrams $= 3 \times 2 \times 7!/2! = 15210$.

(c) Assume the vowels (IUA) as a single letter. The letters (IUA) H, D, S, T, N, N can arranged in $7!/2!$ ways. Also IUA can be arranged, among themselves, in $3! = 6$ ways.

Hence the total number of anagram $= 7!/2! \times 6 = 15120$.

(d) Let us divide the task in two parts. In the first, we arrange the 6 consonants as shown in $6!/2!$ ways.

× C × C × C × C × C × C × (C stands for consonants and × stand for blank spaces between them)

3 vowels can be arranged in 7 places (between the consonants) in $7p_3 = 7!/2! = 210$.

(e) In this case the vowels are arranged among themselves in $3! = 6$ ways. Also the consonants are arranged among themselves in $6!/2!$ ways.

Hence the total number of anagrams $= 6!/2! \times 6 = 2160$.

Illustration:

How many 3 digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 where

(a) digits may not be repeated,

(b) digits may be repeated

Solution:

(a) Let the 3 digit number by XYZ.

Position (X) can be filled by 1, 2, 3, 4, 5 but not 0, so it can be filled in 5 ways.

Position (Y) can be filled in 5 ways again. Since 0 can be placed in this position.

Position (Z) can be filled in 4 ways.

Hence, by the fundamental principal of counting, total number of ways is $5 \times 5 \times 4 = 100$ ways.

(b) Let the 3 digit number be XYZ.

Position (X) can be filled in 5 ways.

Position (Y) can be filled in 6 ways

Position (Z) can be filled in 6 ways.

Hence, by the fundamental principle of counting, total number of ways is $5 \times 6 \times 6 = 180$.

Illustration:

Find the number of ways in which 6 letters can be posted in 10 letterboxes.

Solution:

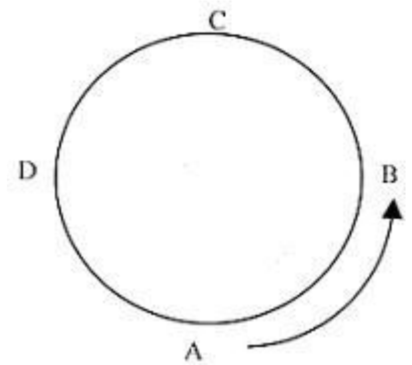
For every letter, we have 10 choices (i.e. 10 letterboxes).

Hence the total number of ways = 10^6 = 1,000,000.

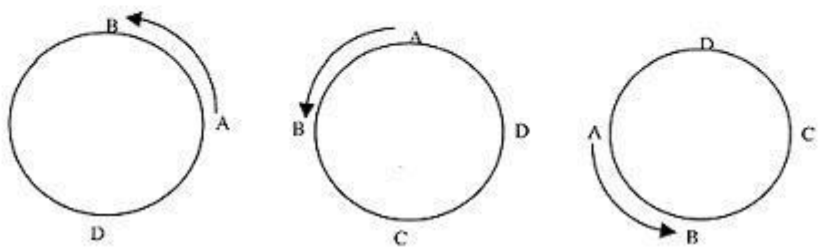
CIRCULAR PERMUTATIONS

The arrangements we have considered so far are linear. There are also arrangements in closed loops, called circular arrangements.

Consider four persons A, B, C and D, who are to be arranged along a circle. It's one circular arrangement is as shown in adjoining figure.



Shifting A, B, C, D one position in anticlockwise direction we will get arrangements as follows.



Arrangements as shown in figure (I) (II) (III) and (IV) are not different as relative position of none of the four persons A, B, C, D is changed. But in case of linear arrangements the four arrangements are.

Thus, it is clear that corresponding to a single circular arrangement of four different things there will be 4 different linear arrangements. Let the number of different things be n and the number of their circular permutations be x.

Now for one circular permutation, number of linear arrangements is n

For x circular arrangements number of linear arrangements

$= nx.$ .... (1)

But number of linear arrangements of n different things

$= n!$ .... (2)

From (1) and (2) we get

$Nx = n! \Rightarrow x = n!/n = (n - 1)!$.

Suppose n persons ($a_1, a_2, a_3, ..., a_n$) are to be seated around a circular table. There are n! ways in which they can be seated in a row. On the other hand, all the linear arrangements

$a_1, a_2, a_3, ..., a_n$

$a_n, a_1, a_2, ..., a_{n-1}$

$a_{n-1}, a_n, a_1, a_2, ..., a_{n-2}$

....

....

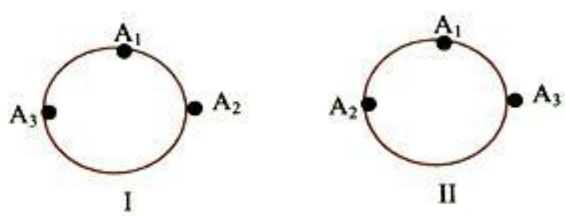
$a_2, a_3, a_4, ..., a_1$

will lead to the same arrangements for a circular table. Hence one circular arrangement corresponds to n unique row (linear) arrangements. Hence the total number of circular arrangements of n persons is = (n - 1)!.

In other words the permutation in a row has a beginning and an end, but there is nothing like beginning or end in circular permutation. Hus, in circular permutation, we consider one object is fixed and the remaining objects are arranged in (n - 1)! ways (as in the case of arrangement in a row).

Distinction between clockwise and Anti-clockwise Arrangements

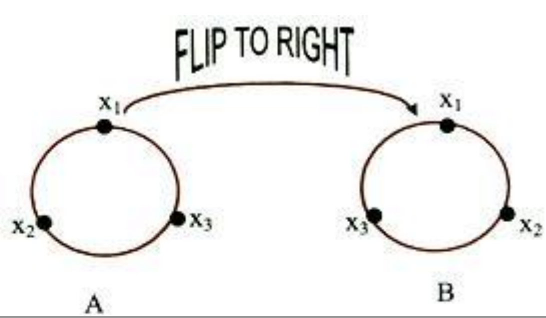
Consider the following circular arrangements:



In figure I the order is clockwise whereas in figure II, the other is anti-clock wise. These are two different arrangements. When distinction is made between the clockwise and the anti-clockwise arrangements of n different objects around a circle, then the number of arrangements = (n - 1)!.

But if no distinction is made between the clockwise and anti-clockwise arrangements of n different objects around a circle, then the number of arrangements is (n - 1)!.

As an example consider the arrangements of beads (all different) on a necklace as shown in figure A and B.



Look at (A) having 3 beads x_1, x_2, x_3 as shown. Flip (A) over on its right. We get (B) at once. However, (A) and (B) are really the outcomes of one arrangement but are counted as 2 different arrangements in our calculation. To nullify this redundancy, the actual number of different arrangements is
(n - 1)!/2.

Note: **(i)** When the positions are numbered, circular arrangements is treated as a linear arrangement.

(ii) In linear arrangements it does not make difference whether the positions are numbered or not.

Illustration:

20 persons we invited to a party. In how many ways can they be seated in a round table such that two particular persons sit on either side of the host?

Solution:

After fixing the places of three persons (1 host + 2 persons) and treating them as 1 unit we can arrange the total (20 - 2 + 1) = 19 units in 18! ways. Again these particular persons can sit on either side of the host in 2 ways.

Hence the total number of ways is 18! × 2.

Illustration:

In how many ways 10 boys and 5 girls can sit around a circular table, so that no two girls sit together.

Solution:

(● stands for boys)

10 boys can be seated in a circle in 9! ways. There are 10 spaces between the boys, which can be occupied by 5 girls in ${}^{10}p_5$ ways. Hence total number of ways = 9! ${}^{10}p_5$ = (9!10!)/5! .

Number of circular permutations of n different things taken r at a time

= $\frac{{}^{n}_{r}P}{r}$ (if clockwise and anticlockwise orders are taken as different)

= $\frac{{}^{n}_{r}P}{r}$ (if clockwise and anticlockwise orders are not taken to be different)

Illustration:

In how many ways can 20 persons be seated round a table if there are 9 chairs?

Solution:

In case of circular table the clockwise and anticlockwise arrangements are different.

Hence the total number of ways = $\frac{{}^{20}_{0}P}{9}$

Illustration:

How many necklaces of 10 beads each can be made from 20 beads of different colours?

Solution:

In case of necklace there is no distinction between the clockwise and anticlockwise arrangements. Then the required number of circular permutations

$\frac{{}^{20}_{10}P}{2 \times 10}$ = $19!/(10!)^2$.

COMBINATIONS

Meaning of combination is selection of objects.

Selection of objects without repetition:

The number of selections (combinations or groups) that can be formed form n different objects taken r($0 \le r \le n$) at a time is $^nC_r = n!/(r!(n-r))!$.

Explanation: Let the total number of selection (or groups) = x. Each group contains r objects, which can be arranged in r! ways. Hence number of arrangements of r objects = x × (r!).

But the number of arrangements = ^np_r

=> $x \times (r!) = {}^np_r$ => $\frac{{}^n_rP}{r!}$ x = => $x = n!/(r!(n-r)!) = {}^nC_r$.

Selection of objects with repetition

The number of combination of n distinct objects taken r at a time when each may occur once, twice, thrice, ... upto r times, in any combination = $^{n+r-1}C_r$.

Explanation: Let eh n objects a1, a2, a3, ... an. In a particular group of r objects, let

a_1 occur x_1 times,

a_2 occur x_2 times,

a_3 occur x_3 times,

....

....

a_n occur x_n times,

such that $x_1 + x_2 + x_3 + \ldots\ldots\ldots + x_n = r$ .... (1)

$0 < x_i < r$ $\forall$ $i \in \{1, 2, 3, \ldots\ldots, n\}$.

Now the total number of selections of r objects, out of n

= number of non-negative integral solution of equation (1)

$= {}^{n+r-1}C_{n-1} = {}^{n+r-1}C_r$.

Note: Details of finding the number of integral solutions of equation (1) are given on page 12 (Multinomial theorem).

Illustration:

Let 15 toys be distributed among 3 children subject to the condition that any child can take any number of toys. Find the required number of ways to do this if

(i) toys are distinct, (ii) if toys are identical

Solution:

(i) Toys are distinct

Here we have 3 children and we want the 15 toys to go to the 3 children with repletion. In other words it is same as selecting and arranging children 15 times out of 3 children with the condition that any children can be selected any no. of time which can be done in 315 ways (n = 3, r = 15).

(ii) Toys are identical

Here we only have to select children 15 times out of 3 children with the condition that any children can be selected any number of times which can be done in

${}^{3+5-1}C_{15} = {}^{17}C_{2}$ way (n = 3, r = 5).

PERMUTATIONS VS COMBINATIONS

DIFFERENCE BETWEEN PERMUTATION AND COMBINATION:

In combination we are concerned only with the number of things n each selection whereas in permutation we also consider the order of things which makes each arrangement different The symbol ${}^nP_r$, denotes the number of permutations of n different things taken r at a time, whereas ${}^nC_r$ denotes the number of combinations of n different things taken r at a time.

Enquiry: How can we find the number of ways of selection n things taken r at a time, when their order also matters?

This can be done by finding the number of permutations of n different things taken r at a time (n > r) or to find the value of ${}^nP_r$.

This is same as finding the number of ways in which we can fill up r places when we have n different things at our disposal. The first place may be filled up in n-ways, because we may take any one of the n things. When it has been filled up in any one of these ways, the second place can then be filled up in (n - 1) ways, and since each way of filling up the first place can be associated with each way of filling up the second, the number of ways in which the first two places can be filled up is given by the product n(n - 1) ways, And when the first two places have been filled up in any way, the third place can be filled up in (n - 2) ways. And reasoning as before, the number of ways in which three places can be filled up is n(n - 1)(n - 2).

Proceeding thus, and noticing that a new factor is introduced with each new place filled up, and that at any stage the number of factors is the same as the number of places filled up, we shall have the number of ways in which r places can be filled up, we shall have the number of ways in which r places can be filled up equal to n(n - 1)(n - 2) .... to r factor, and the rth factor is n - (r - 1), or n - r + 1. Therefore the number of permutations of n things taken r at a time is

$${}^nP_r = n(n - 1)(n - 2)......(n - r + 1).$$

RESTRICTED SELECTION AND ARRANGEMENT

(a) The number of ways in which r objects can be selected form n different objects if k particular objects are

(i) always included = ${}^{n-k}C_{r-k}$.

(ii) never included = ${}^{n-k}C_r$.

(b) The number of arrangement of n distinct objects taken r at a time so that k particular objects are

(i) always included = ${}^{n-k}C_{r-k}.r!$,

(ii) never included = ${}^{n-k}C_r.r!$.

Illustration:

A delegation of four students is to be selected form a total of 12 students. In how many ways can the delegation be selected if

(a) all the students are equally willing.

(b) two particular students have to be included in the delegation.

(c) two particular students do not wish to be together in the delegation.

(d) two particular students wish to be included together only,

(e) two particular students refuse to be together and two other particular student wish to be together only in the delegation.

Solution:

(a) Formation of delegation means selection of 4 out of 12. Hence the number of ways = ${}^{12}C_4 = 495$.

(b) Two particular students are already selected. Hence we need to select 2 out of the remaining 10. Hence the number of ways = ${}^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 - 45 = 450.

(d) There are two possible cases

(i) Either both are selected. In this case the number of ways in which this selection can be made = 45.

Or (ii) both are selected. In this case all the four students are selected from the rest of ten students.

This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255.

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i) (A, B, C) selected (D) not selected

(ii) (A, B, D) selected (C) not selected

(iii) (A, B) selected (C, D) not selected

(iv) (C) selected (A, B, D) not selected

(v) (D) selected (A, B, C) not selected

(vi) A, B, C, D not selected

For (i) the number of ways selection = ${}^{8}C_1 = 8$

For (ii) the number of ways selection = ${}^{8}C_1 = 8$

For (iii) the number of ways selection = ${}^{8}C_2 = 28$

For (iv) the number of ways selection = ${}^{8}C_3 = 56$

For (v) the number of ways selection = ${}^{8}C_3 = 56$

For (vi) the number of ways selection = ${}^{8}C_4 = 70$

Hence, total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226.

Some results related to ${}^{n}C_r$

(i) ${}^{n}C_r = {}^{n}C_{n-r}$

(ii) If ${}^{n}C_r = {}^{n}C_k$, then r = k or n-r = k

(iii) ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$

(iv) ${}^{n}C_r = n/r\ {}^{n-1}C_{r-1}$

(v) $$\frac{{}^{n}_{r}C}{{}^{n}_{r-1}C} = \frac{n-r+1}{r}$$

(vi) (a) If n is even ${}^{n}C_r$ is greatest for r = n/2

(b) If n is odd, is greatest for r = (n-1)/2,(n+1)/2

Illustration:

(a) How many diagonals are there in as n-sided polygon (n > 3?

(b) How many triangles can be formed by joining the vertices of an n-sided polygon?

How many these triangles have

(i) exactly one side common with that of the polygon?

(ii) exactly two sides common with that of the polygon?

(iii) no side common with that of the polygon?

Solution:

(a) Number of lines formed by joining the vertices of a polygon

= number of selection of 2 points each selected from the given n points

$= {}^nC_2 = (n(n-1)!)/2.1$.

Out of ${}^nC_2$ lines, n are the sides of the polygon.

Hence the number of diagonals = ${}^nC_{2-n} = ((n(n-1))/2) - n = (n(n-3))/2$.

(b) Number of triangles formed by joining the vertices of the polygon = number of selection of 3 points from n points.

${}^nC_3 = (n(n-1)(n-2))/3.2.1$.

Let the vertices of the polygon be marked as $A_1, A_2, A_3, \ldots\ldots, A_n$.

(i) Select two consecutive vertices A_1, A_2 of the polygon. For the required triangles we can select the third vertex from the $A_4, A_5, \ldots, A_{n-1}$. This can be done in ${}^{n-4}C_1$ ways. Also two consecutive points (end points of a side of polygon) can be selected in n ways.

(ii) For the required triangle, we have to select three consecutive vertices of the polygon. i.e. $(A_1\ A_2\ A_3)$, $(A_2\ A_3\ A_4)$ $(A_3\ A_4A_5)$ ... $(A_n\ A_1\ A_2)$. This can be done in n ways.

(iii) Triangle having no side common + triangle having exactly one side common + triangle having exactly two sides common

(with those of the polygon) = Total number triangles formed

=> Triangle having no side common with those of the polygon.

$= {}^nC_3 - n(n - 4) - n = ((n(n-1)(n-2))/6) - n(n - 4) - n$.

$= n/6\ [n^2 + 3n + 2 - 6n + 24 - 6] = [n^2 + 9 + 20] = (n(n-4)(n-5))/6$.

All possible selections

(i) Selection from distinct objects :

The number of selections from n different objects, taking at least one = ${}^nC_1 + {}^nC_2 + {}^nC_3 + ... + {}^nC_n = 2^n - 1$.

In other words, for every object we have two choices, either select or reject in a particular group. Total number of choice (all possible selections) = 2.2.2. ...n times = 2^n. But this also includes the case when none is selected.

And the number of such cases is $2^n - 1$.

(ii) Selection from identical objects:

(a) The number of selections of r objects out of n identical objects is 1.

(b) Total number of selections of zero or more objects from n identical objects is n+1.

(c) The total number of selections of at least one out of $a_1 + a_2 + a_3 + ... +$ an objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on ...... a_n are alike (of nth kind), is

$[(a_1 + 1)(a_2 + 1)(a_3 + 1).........(a_n + 1)] - 1$.

(iii) Selection when both identical and distinct objects are present:

The number of selections, taking at least one out of $a_1 + a_2 + a_3 + ... a_n + k$ objects, where a_1 are alike (of one kind), a_2 are alike (of second kind) and so on ... a_n are alike (of nth kind), and k are distinct =
$\{[(a_1 + 1)(a_2 + 1)(a_3 + 1) ... (a_n + 1)]2^k\} - 1$.

Illustration:

In how many ways can a person having 3 coins of 25 paise, 4 coins of 50 paise and 2 coins of 1 rupee give none or some coins to a beggar?

Solution:

The person has 3 coins of 25 paise, 4 coins of 50 paise and 2 coins of 1 rupee.

The number of ways in which he can give none or some coins to a beggar is (3 + 1)(4 + 1)(2 + 1) = 60 ways.

(iv) (a) Total number of divisor's of a given natural number

To find the number of factors of a given natural number greater than 1 we can write n as n = $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots\dots p_n^{\alpha_n}$, where $p_1, p_2, ..., p_n$ are distinct prime numbers and $\alpha_1, \alpha_2 ...... \alpha_n$ are non-negative integers. Now any divisor of n will be of the form d = $p_1^{\beta_1} p_2^{\beta_2} \dots\dots p_k^{\beta_k}$; here number of factors will be equal to numbers of ways in which we can choose β_1s' which can be done in $(\alpha_1 + 1)(\alpha_2 + 1)......(\alpha_n + 1)$ ways.

Sum of all the divisors of n is given by

$$\left(\frac{p_1^{\alpha_1+1}-1}{p_1-1}\right).\left(\frac{p_2^{\alpha_2+1}-1}{p_2-1}\right).\left(\frac{p_3^{\alpha_3+1}-1}{p_3-1}\right)\dots\left(\frac{p_n^{\alpha_3+1}-1}{p_n-1}\right).$$

(b) The exponent of a prime number p_1 in n = $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots \ldots p_n^{\alpha_n}$ **is given by**

$$\alpha_1 = \left[\frac{n}{p_1}\right] + \left[\frac{n}{p_1^2}\right] + \left[\frac{n}{p_1^3}\right] + \ldots$$

Illustration:

How many positive factors are there of the number 360 and find the sum of all these factors.

Solution:

Let n = 360 = 23.32.5

=> No. of factors of 360 = (3 + 1)(2 + 1)(1 + 1) = 24.

Sum of all the factors =$((2^4-1)/1) . ((3^2-1)/2) . ((5^2-1)/4)$ = 15. 13. 6 = 1170.

DIVISION AND DISTRIBUTION OF OBJECTS

(With fixed number of objects in each group)

(i) Into groups of unequal size (different number of objects in each group)

(a) Number of ways, in which n distinct objects can be divided into r unequal groups containing a_1, a_2, a_3, ..., a_r

things ($a_1 \neq a_j$)

$= {}^{n}C_{a1} . {}^{n-a1}C_{a2} . {}^{n-a1-a2}C_{ar} . = n!/a_1!a_2!a_3!...a_r!$

Here $a_1 + a_2 + a_3 + ... + a_r = n$.

(b) Number of ways in which n distinct objects can be distributed among r persons such that some person get a_1 objects,

another person get a_2 objects .... and similarly someone gets a_r objects $= n!r! /a_1!a_2!a_3!...a_r!$.

Explanation: Let us divide the task into two parts. In the first part, we divide the objects into groups. In the second part, these r groups can be assigned to r persons in r! ways.

(ii) Into groups of equal size (each group containing same number of objects)

(a) Number of ways in which m × n distinct objects can be divided equally into n groups (unmarked) = $(mn)!/(m!)^n n!$.

(b) Number of ways in which m × n different object can be distributed equally among n persons (or numbered groups)

= (number of ways of dividing) × (number of groups)! $= (mn)!n!/(m!)^n n! = (mn)!/(m!)^n$.

DERANGEMENTS AND MULTINOMIAL THEOREM

DERANGEMENTS

Any change in the existing order of things is called a derangement.

If 'n' things are arranged in a row, the number of ways in which they can, be deranged so that none of them occupies its original place is

$$n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots\dots+(-1)^n\frac{1}{n!}\right)=n!\sum_{r=0}^{n}(-1)^r\frac{1}{r!}$$

and it is denoted by D(n).

A question on derangement can be of the following kind:

Illustration:

Supposing 4 letters are placed in 4 different envelopes. In how many ways can be they be taken out from their original envelopes and distributed among the 4 different envelopes so that no letter remains in its original envelope?

Solution:

Using the formula for the number of derangements that are possible out of 4 letters in 4 envelopes, we get the number of ways as :

$4!(1 - 1 + 1/2! - 1/3! + 1/4!) = 24(1 - 1 + 1/2 - 1/6 + 1/24) = 9$.

MULTINOMIAL THEOREM

Let $x_1, x_2, \dots\dots, x_m$ be integers. Then number of solutions to the equation

$x_1 + x_2 + \dots + x_m = n$ ... (1)

subject to the conditions $a_1 < x_1 < b_1, a_2 < x_2 < b_2, \dots, a_m < x_m < b_m$ ... (2)

is equal to the coefficient of x^n in

$$(x^{a_1}+x^{a_1+1}+\dots\dots+x^{b_1})(x^{a_2}+x^{a_2+1}+\dots\dots+x^{b_1})\dots\dots(x^{a_m}+x^{a_m+1}+\dots+x^{b_m}) \quad \dots(3)$$

This is because the number of ways in which sum of m integers in (1) subject to given conditions (2) equals n is the same as the number of times x^n comes in (3). Using this we get the number of non negative integral solutions of (1) is given by ${}^{n+m-1}C_{m-1}$ and number of positive integral solutions of (1) is given by ${}^{n-1}C_{m-1}$.

Illustration:

In how many different ways three persons A, B, C having 6, 7 and 8 one rupee coins respectively can donate Rs.10 collectively.

Solution:

The number of ways in which they can denote Rs.10 is the same as the number of solutions of the equation

$x_1 + x_2 + x_3 = 10$

subject to conditions $0 \leq x_1 \leq 6$, $0 \leq x_2 \leq 7$, $0 \leq x_3 < 8$

Hence the required number of ways

= coefficient of x^{10} in $(1+x+x^2+...+x^6)(1+x+x^2+...+x^7)(1+x+x^2+...+x^8)$

= coefficient of x^{10} in $(1-x^7)(1-x^8)(1-x^9)(1-x)^{-3}$

= coefficient of x^{10} in $(1-x^7-x^8-x^9)(1+{}^3C_1x + {}^4C_2x^2+ {}^5C_3x^3+...+{}^{12}C_{10}x^{10})$

(Ignoring powers higher than 10)

$= {}^{12}C_2 - {}^5C_3 - {}^4C_2 - {}^3C_1$

= 66 - 10 - 6 - 3 = 47.

USEFUL TIPS

(i) Number of combinations of dissimilar things taken r at a time when p particular things always occur=$^{n-p}C_{r-p}$.

Explanation: Here, actually we are making a selection of (r - p) things out of (n - p) things which can be done in $^{n-p}C_{r-p}$ ways.

(ii) Number of permutations of n dissimilar things taken r at a time when p particular things always occur = $^{n-p}C_{r-p}r!$

Explanation: Here, number of combinations is the same as above but every combinations of r things can be permutated in $\lfloor r$ ways and that's why total number of permutations = $^{n-p}C_{r-p}r!$

(iii) Number of combinations of n dissimilar things taken r at a time when p particular things never occur = $^{n-p}C_{r-p}$.

Explanation: Here, actual selection of r things is being made out of (n - p) things and that's why total number of selections = $^{n-p}C_{r-p}$.

(iv) Number of permutations of n dissimilar things taken r at a time when p particular things never occur = $^{n-p}C_{r-p}\lfloor r$.

Explanation: Here, number of combinations is the same as above but every selection made of r things can be permutated in $\lfloor r$ ays and therefore then total no. of permutations = $^{n-p}C_{r-p}\lfloor r$.

(v) Gap Method: If there are m men and n women (m > n) and they have to sit in a row in such a way that no two women sit together then total no. of such arrangements = $^{m+1}C_n$. m!

Explanation: If we denote men by m and women by w then there are exactly (m + 1) places in which women can be placed such that no two women will be together. This can be done in $^{m+1}C_n$ ways. Moreover, m men can be arranged among themselves in m! ways. Therefore, total number of arrangements : $^{m+1}C_n$.m!

w m w m w .... m w m w

(vi) String method: Many a times, one may encounter a problem of arranging n number of persons in a row such that m of them is always together. For this we tie all these m persons with a string i.e. we treat them as one, then we have (n - m + 1). Therefore total number of arrangements = (n - m + 1)!.m!.

Never together: If question is to find the number of arrangements such that m out of n are never all together then such total number of arrangements.

Total possible arrangements of a n persons without any restrictions - Total arrangements when m out of them are always together.

SOLVED EXAMPLES

Example 1

A letter lock consists of three rings each marked with 10 different letters. In how many ways is it possible to make an unsuccessful attempt to open the block?

Solution:

Two rings may have same letter at a time but same ring cannot have two letters at time, therefore, we must proceed ring wise.

Each of the three rings can have any one of the 10 different letters in 10 ways.

Therefore Total number of attempts = $10 \times 10 \times 10 = 1000$.

But out of these 1000 attempts only one attempt is successful.

Therefore Required number of unsuccessful attempts

= 1000 - 1 = 999.

Example 2

Find the total number of signals that can be made by five flags of different colour when any number of them may be used in any signal.

Solution:

Case I : When only one flag is used.

No. of signals made = ${}^5P_1 = 5$.

Case II : When only two flag is used.

Number of signals made = ${}^5P_2 = 5.4 = 20$.

Case III : When only three flags are used.

Number of signals is made = ${}^5P_3 = 5.4.3 = 60$.

Case IV : When only four flags are used.

Number of signals made = ${}^5P_4 = 5.4.3.2 = 120$.

Case V : When five flags are used.

Number of signals made = ${}^5P_5 = 5! = 120$.

Hence, required number = 5 + 20 + 60 + 120 + 120 = 325.

Example 3

Prove that if each of the m points in one straight line be joined to each of the n points on the other straight line, the excluding the points on the given two lines. Number of points of intersection of these lines is 1/4 mn (m-1(n-1).

Solution:

To get one point of intersection we need two points on the first line and two points on the second line. These can be selected out of n-points in $^nC_2$ ways and for m points in $^mC_2$ ways.

Therefore Required number = $^mC_2 \times {}^nC_2$

= (m(m-1))/2! . (n(n-1))/2!

= 1/4 m n (m - 1)(n - 1)

Example 4

There are ten points in a plane. Of these ten points four points are in a straight line and with the exception of these four points, no other three points are in the same straight line. Find

(i) the number of straight lines formed.

(ii) the number of triangles formed.

(iii) the number of quadrilaterals formed by joining these ten points.

Solution:

(i) For straight line, we need 2 points

No. of point selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of straight line formed
0	2	$^4C_0 \times {}^6C_2 = 15$
1	1	$^4C_1 \times {}^6C_1 = 24$
2	0	1

(In last case only one straight line is formed)

Therefore Required number = 15 + 24 + 1 = 40

(ii) For triangle, we need 3 points

No. of point selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of triangles formed
0	3	${}^4C_0 \times {}^6C_3 = 20$
1	2	${}^4C_1 \times {}^6C_2 = 60$
2	1	${}^4C_2 \times {}^6C_1 = 36$
3	0	0

(In last case number of triangles formed is 0)

Therefore Required number = 20 + 60 + 36 + 0 = 116

(iii) For a quadrilateral, we need 4 points

No. of point selected out of 4 collinear points	No. of points selected out of remaining 6 points	No. of quadrilateral formed
0	4	${}^4C_0 \times {}^6C_4 = 15$
1	3	${}^4C_1 \times {}^6C_3 = 80$
2	2	${}^4C_2 \times {}^6C_2 = 90$
3	1	0
4	0	0

(In these cases number quadrilateral is formed

Therefore Required number = 15 + 80 + 90 = 185.

Example 5

A student is allowed to select at most n-blocks from a collection of (2n + 1) books. If the total number of ways in which he can select a book is 63, find the value of n.

Solution:

Since the student is allowed to select at the most n-books out of (2n + 1) books, therefore, he can choose, one book, two books or at the most n books. The number of ways of selecting at least one books are

${}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots\ldots\ldots {}^{2n+1}C_n = 63 = S$ (Say)

Again, we know that

${}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots\ldots\ldots {}^{2n+1}C_n + {}^{2n+1}C_{2n+1} = 2^{2n+1}$

Now $^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$

$^{2n+1}C_1 = {}^{2n+1}C_{2n}$ etc.........

Hence, we have

$1 + 1 + 2S = 2^{2n+1}$

or $2 + 2 . 63 = 2^{2n+1}$

or $128 = 2^{2n+1}$ or $27 = 2^{2n+1}$

=> $2^{n+1} = 7$

or $2n = 6$

$n = 3$

Example 6

A family consists of a grandfather, 6 sons and daughters and 5 grand children. They are to be seated in a row for dinner. The grand children wish to occupy the two seats at each end and the grandfather refuses to have a grandchild on either side of him. In how many ways can the seating arrangements be made for the dinner?

Solution:

There are 6 adults, 4 grand children and 1 grand father.

Let us mark the seat for 11 persons from 1 to 11.

S S S S S S

1 2 3 ....9 10 11

Seats number 1, 2 and 10, 11 at the ends are to be occupied by 4 grand children and it can be done in $^4P_4 = 4! = 24$ ways.

Now, we will seat the grandfather who cannot occupy seat number 3 or seat number 9 because he does not want to have a child by his side. Hence, he has to choose any of five seats 4, 5, 6, 7, 8 i.e. can seat himself in 5 ways.

Example 7

In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the fourth paper is 100. Find the number of ways in which the candidate can score 60% marks in the aggregate.

Solution:

Aggregate of marks $50 \times 3 + 100 = 250$

Therefore 60% of the aggregate = $3/5 \times 250 = 150$

Now the number of ways of getting 150 marks in the aggregate

= coefficient of x^{150} in $(x^0 + x^1 + ...+ x^{50})^3 (x^0 + x^1 + x^2 + ...+ x^{100})$

= coeff. of x^{150} in $((1 - x^{51})/(1 - x))^3 ((1 - x^{101})/(1 - x))$

= coeff. of x^{150} in $(1-x^{51})^3(1-x^{101})(1-x)^{-4}$

= coeff. of x^{150} in $(1-3x^{51} + 3.x^{102} - x^{153})(1-x^{101})(1-x)^{-4}$

coeff. of x^{150} in

$$\lfloor 1 - 3.x^{51} - x^{101} + 3.x^{102} + 3x^{152} + ...\rfloor \times \lfloor 1 + 4x + x^{(r+3)} C_r x^r \rfloor$$

= 1.(151.152.153)/6 - 3(100.101.102)/6 - 1(50.51.52)/6 + 3(49.50.51)/6

= 151.76.51 - 100.101.51 - 50. 17.26 + 49.25.51

= 57 (151.76 - 100 - 101) + 17 (49.25 - 50.26)

= 51 (11476 - 10100) + 17 (3675 - 1300)

= 51.1376 + 17.2375 = 70176 + 40375

= 110551.

Example 8

Find the number of 4 lettered words that can be formed form the letters of the word PROPORTION?

Solution:

Step 1

Letter	Freq.
P	2
R	2
O	3
T	1
I	1
N	1
	10 (always check total)

Step 2

(i) here we cannot have all the 4 letters alike, so

(ii) 3 alike 1 diff. Selection Remark

Say X Y ${}^1C_1 \times {}^5C_1 = 5$ ${}^1C_1 \rightarrow$ 3O's

${}^5C_1 \rightarrow$ one form {P, R, T, I, N}

Arrangement of these 3X's and 1 Y = $\frac{{}^4_4P}{3!1!}$

= 4!/3! = 4

Hence total such words = 5 × 4 = 20.

(iii) 2 same 2 diff. Selection

2(X) (Y, Z) ${}^3C_1 \times {}^5C_2 = 12$

Arrangement = (p+q+r)/p!q!r! = (2+1+1)2!1!1! = 12

Hence total words = 12 × 30 = 360

(iv) 2 same 2 another same Selection

(2X) (2Y) ${}^3C_2 = 3$

Arrangements = (2+2)!/2!2! = 6

Total words = 3 × 6 = 18

(v) 1 same, 3 diff º 4diff. Selection ${}^6C_4 = {}^6C_2 = 15$

Arrangement = (1+1+1+1)!/1!1!1!1! = 4! = 24

Total words = 15 × 24 = 360

(vi) 1 same, 3 another same º case ii (already considered)

Hence grand total of desired words

= 20 + 360 + 18 + 360 - 758.

Example 9

You are given the responsibility of organizing a fresher's welcome party at IIT-Delhi. Total 496 students will join IIT-D. Six restaurants A, B, C, D, E and F are booked. Capacity of each is 120, 146, 46, 72, 72, 80 persons at a time respectively. You have to group them in such a way that each group has least possible number of students. How would you do so?

Solution:

Remember: Equal division minimizes the number of students in each group.

But: $496/6 = 82.6 > 46$

Therefore C has least capacity of 46 and is not capable to accommodate 82 students. Hence let us group for it first i.e. 46 students get accommodated.

Students left	Restaurant left
450	A B D E and F

But $450/5 = 90 > 72$

Next D and E each require smallest group 72, 72

D → 72 E → 72

Students left	Restaurant left
306	A B and F

Now F is the least left for grouping, so assigning 80 to F.

Students left	Restaurant left
226	A and B

Now Therefore $226/2 = 113 < 120$ and each of A and B is capable of accommodating to this. Hence,

A → 113

B → 113

But not

A → 120

B → 106

As in the case A does not have minimum possible number of students.

Hence, we grouped 496 students into 46, 72, 72, 80, 113 and 113 each.

No. of ways of doing so = $496! / (46!(72!)^2 .2!80!(113!)^2.2!)$

Example 10

How many different numbers can be formed with the digits 1, 3, 5, 7, 9 when taken all at a time and what is their sum?

Solution:

The total number of numbers = $\underline{|5}$ = 120. Suppose we have 9 in the unit's place. We will have $\underline{|4}$ = 24 such numbers. The number of numbers in which we have 1, 3, 5 or 7 in the unit's place is $\underline{|4}$ = 24.

Hence, the sum of the digits in the unit's place in all the 120 numbers

= 24 (1 + 3 + 5 + 7 + 9)

= 600.

The number of numbers when we have any one of the given digits in ten's place is also $\underline{|4}$ = 24 in each case. Hence, the sum of the digits in the ten's place = 24 (1 + 3 + 5 + 7 + 9) tens

= 60 tens = 600 × 10.

Proceeding similarly, the required sum

= 600 units + 600 tens + 600 hundreds + 600 thousands + 60 ten thousands.

= 60 (1 + 10 + 100 + 1000 + 10000) = 600 × 11111

= 6666600.

Example 11

Find the values of r for which the number fo combinations of n things taken r at a time is greatest.

Solution:

Since $\quad {}^nC_r = (n(n-1)(n-2)....(n-r+2)(n-r+1))/(1.2.3...(r-1)r)$

and $\quad {}^nC_{r-1} = (n(n-1)(n-2)....(n-r+2))/(1.2.3...(r-1))$

Therefore $\quad {}^nC_r = {}^nC^{r-1} \times (2-r+1)/r = {}^nC_{r-1}(((n+1)/r)-1).$

The multiplying factor (n-r+1)/r may be written as (((n+1)/r) - 1), which shows that it decreases as r increases. Hence as r resumes the values 1, 2, 3,...... in succession, ${}^nC_r$ is continually increased until (((n+1)/r) - 1) becomes equal to 1 or less than 1.

Now (((n+1)/r) - 1) > 1

so long as (n+1)/r) - 2

that is (n+1)/r) - r .

We have to choose the greatest value of r consistent with this inequality.

(1) Let n be even, and equal to 2m, then

(n+1)/2 = (2m+1)/2 = m+(1/2) ;

and for all values r up to m (inclusive of m) this is greater than r.

Hence by putting r = m = n/2 , we find that the greatest number of combination is ${}^{n}C_{n/2}$.

(2) Let n be odd, and equal to 2m + 1; then

(n+1)/2 = (2m+1)/2 = m+1 ;

and for all values of r up to m (inclusive of m) this is greater than r, but when r = m + 1 the multiplying factor becomes equal to 1, and ${}^{n}C_{m+1} = {}^{n}C_{m}$; that is

${}_{n+1}{}^{n}C/ 2 = {}_{n+1}{}^{n}C/ 2$ and

therefore the number of combinations is greatest when the things are taken (n+1)/2 or (n-1)/2 at a time, the result being the same in the two cases

Example 12

If all of the letters in the word **DIPLOMA** are used, then the number of different 7- letter arrangements that can be made beginning with 3 vowels is

A. 24
* **B.** 144
C. 720
D. 5 040

Solution:

$$\underset{v}{\underline{3}} \cdot \underset{v}{\underline{2}} \cdot \underset{v}{\underline{1}} \cdot \underset{c}{\underline{4}} \cdot \underset{c}{\underline{3}} \cdot \underset{c}{\underline{2}} \cdot \underset{c}{\underline{1}}$$

$$3!4! = 144$$

Example 13

A car manager wants to line up 10 cars that are identical except for colour. There are
3 red cars, 2 blue cars, and 5 green cars. Determine the number of possible arrangements of the 10 cars if they are lined up in a row along one side of a parking lot, and a blue car is parked on each end of the row.

Solution:

$$\frac{2 \times 8! \times 1}{2!3!5!} = 56$$

Example 14

A 6-player volleyball team stands in a straight line for a picture. If 2 particular players, Joan and Emily, must be together, then how many different arrangements can be made for the picture?

Solution:

5!2! = 240

Example 15

A teacher tells his students that on a multiple-choice test with 12 questions, 2 answers are **A**, 3 are **B**, 3 are **C**, and 4 are **D**. How many different answer keys are possible?

Solution:

$$\frac{12!}{2!3!3!4!} = 277\ 200$$

Example 16

Solve $_nC_2 = \frac{_nP_3}{3!}$, and verify your solution.

Solution:

$$\frac{n(n-1)(n-2)!}{(n-2)!\ 2!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!\ 3!}$$

$$\begin{aligned} n(n-1)3! &= n(n-1)(n-2)2! \\ 3! &= (n-2)2!,\ n \neq 0, 1 \\ 6 &= 2n - 4 \\ 10 &= 2n \\ n &= 5 \end{aligned}$$

LHS	RHS
$_5C_2 = 10$	$\frac{_5P_3}{3!} = \frac{60}{6} = 10$

Example 17

A school committee consists of 1 vice-principal, 2 teachers, and 3 students. The number of different committees that can be selected from 2 vice-principals, 5 teachers, and 9 students is

Solution:

$$\underset{\text{(VP)}}{_2C_1} \times \underset{\text{(Teachers)}}{_5C_2} \times \underset{\text{(Students)}}{_9C_3} = 2 \times 10 \times 84 = 1\ 680$$

Example 18

The vertices of an octagon are marked on a circle. Determine the number of triangles that can be formed using any 3 of the vertices.

Solution:

$_8C_3 = 56$

Example 19

In a group of 9 people, there are 4 females and 5 males. Determine the number of 4- member committees consisting of at least 1 female that can be formed.

Solution: Method 1

Total number of committees possible – number of committees with no females.

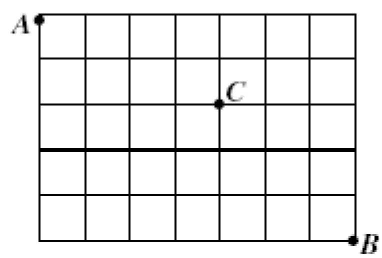
$_9C_4 - {_5C_4} = 126 - 5 = 121$

Method 2
1 female and 3 males or
2 females and 2 males or
3 females and 1 male or
4 females and 0 males

$= ({}_4C_1 \times {}_5C_3) + ({}_4C_2 \times {}_5C_2) + ({}_4C_3 \times {}_5C_1) + ({}_4C_4 \times {}_5C_0)$
$= 40 + 60 + 20 + 1$
$= 121$

Example 20

Given the diagram below, determine the number of pathways starting from *A* and moving to *B* along the gridlines if a pathway must pass through *C* and must always move closer to *B*.

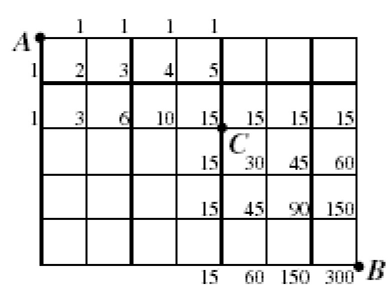


Solution:

Method 1
Using combinations

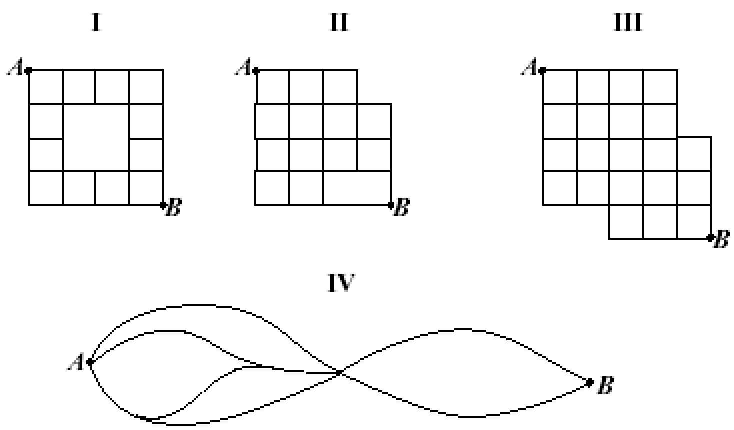
$$\frac{6!}{4!2!}\times\frac{6!}{3!3!} = 15\times 20 = 300$$
$(A \text{ to } C)(C \text{ to } B)$

Method 2
Counting using properties of Pascal's triangle.

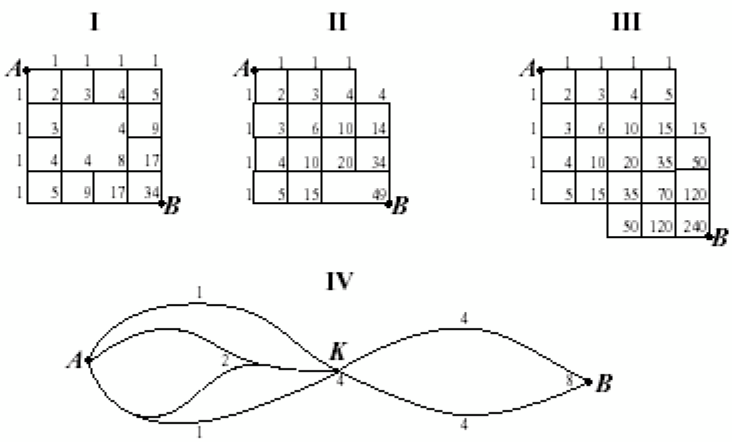


Example 21

For each of the diagrams below, determine the number of pathways starting from *A* and moving to along the gridlines given that a pathway must always move closer to *B*.



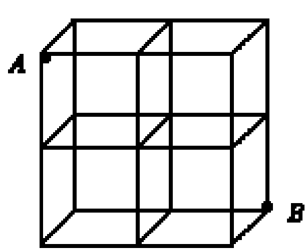
Solution:



Since three paths lead into point *K*, three values must be added.
Since two paths lead into point *B*, two values must be added.

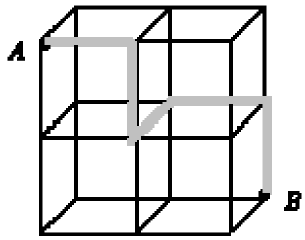
Example 22

Determine the number of possible paths possible from point *A* to point *B* in the following diagram if travel may occur only along the edges of the cubes and if the path must always move closer to *B*.



Solution:

a.



One possible path is shown. Any path from *A* to *B* must travel along 5 edges and must consist of 2 edges to the right (R), 2 edges downward (D), and 1 edge to the back (B).

The number of arrangements of these letters is

$$\frac{5!}{2!2!} = \frac{120}{4} = 30$$

Therefore, 30 paths are possible.

Example 23

Determine the coefficient of the term containing Ax^2y^5 in the expansion of $(2x + y)^7$.

Solution:

$${}_7C_k(2x)^{7-k}(y)^k = Ax^2y^5$$

since $x^{7-k} = x^2$ **or** $y^k = y^5$

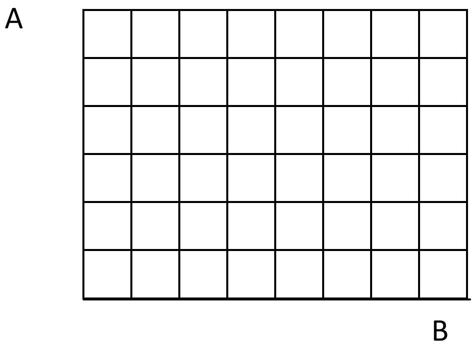
Therefore, $k = 5$

$$\begin{aligned} {}_7C_5(2x)^2(y)^5 &= Ax^2y^5 \\ (21)(4x^2)(y^5) &= Ax^2y^5 \\ 84x^2y^5 &= Ax^2y^5 \\ A &= 84 \end{aligned}$$

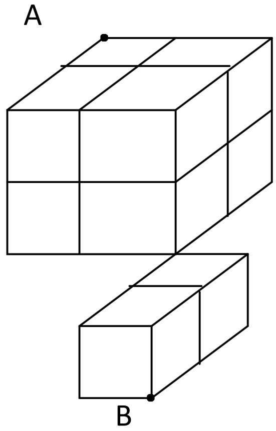
TRY THESE

1. If a vending machine has a button for each of the 26 letters of the alphabet and for each of the digits 0 – 9, how many different types of candy can be dispensed with a letter followed by a digit to be entered?
2. How many 3 digit numbers can be formed if no two digits are alike?
3. Hamburgers can be made with cheese, lettuce, relish, ketchup and mustard. How many different types of burgers can be made if any number of toppings can be used?
4. How many different ways can the letters of the word MONTREAL be arranged if
   a) there are no restrictions?
   b) the first letter must be T?
   c) the 2^{nd}, 5^{th} and 7^{th} letters must be vowels?
   d) The letters E and A must be adjacent?
   e) The letters E and A must not be adjacent?
5. How many arrangements can be made using all the letters of the word CURRICULUM?
6. Expressed as a single factorial $(m - 3)(m - 4)(m - 5)! =$
7. If eight people get on a bus with three empty seats, how many ways can these people arrange themselves in the seats?
8. Seven distinct points are on a circle. How many triangles can be formed using these seven points as vertices?
9. If a five card hand is dealt, how many a) have at least one Ace?
   b) have at least two Aces?
   c) have at most three Aces?
10. A student council consists of five girls and six boys. How many committees of five students can be selected from this council if a) there are no restrictions?
    b) exactly five boys are on the committee?
    c) exactly five girls are on the committee?
    d) at least two girls are on the committee?

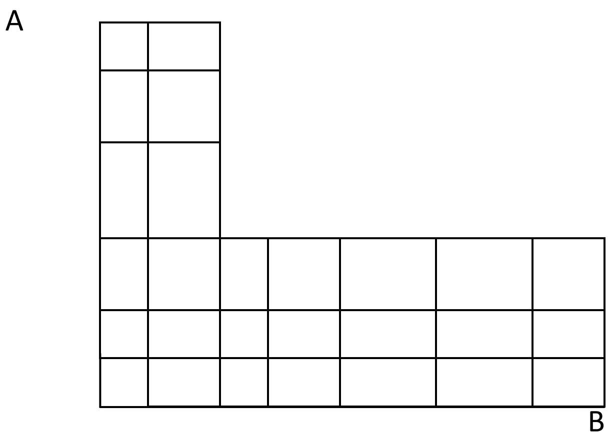
11. A homeowner wants to purchase two different pictures, one to hang above a hall table, and one to hang above a sofa. An interior decorator arrives at the home with several different pictures and shows the owner all 42 different arrangements. How many different pictures did the decorator bring to the house?
12. Determine the number of pathways from A to B. a)



b)



c)



Answers:
1. 260   2. 648   3. 32   4.a) 40 320 b) 5040 c) 720 d) 10 080   e) 30 240
5. 151 200   6. (m – 3)!   7. 336   8. 35   9.a) 886 656 b) 108 336 c) 2 598 912 10. a) 462 b) 6 c) 200 d) 381   11. 7   12.a) 252   b) 1080 c) 707

KEYWORDS AND PHRASES

Pure Math 30 Permutations and Combinations Summary

Fundamental Counting Principle $3 \times 2 \times 1 = 6$

Factorials $3! = 6$

Permutation – arrangement where order is important

$${}_nP_r = \frac{n!}{(n-r)!} \qquad {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

Combination – arrangements where order is not important

$${}_nC_r = \frac{n!}{(n-r)!r!} \qquad {}_{10}C_3 = \frac{10!}{(10-3)!3!} = \frac{10!}{7!3!} = 120$$

Key Words and Phrases

"two red marbles **or** three green marbles" means to <u>add</u> the permutations or combinations

"three actors **and** four actresses" means to <u>multiply</u> the permutations or combinations

"vowels **must be together**" means to place vowels into <u>one unit or space</u>

"vowels **must not be together**" means to have no restrictions subtract vowels kept together

"**at least one** diamond" means to use complement (no restrictions subtract no diamonds)

"**at least** three females" in a five member group means to add up three, four and five females

"**at most** three kings" in a five card hand means to add up 0, 1 ,2 and 3 kings

Pascal's Triangle

" The sum of the numbers in k^{th} row of Pascal's Triangle is 2^{k-1} "

" The sum of the coefficients in the expansion of $(x+y)^n$ is 2^n "

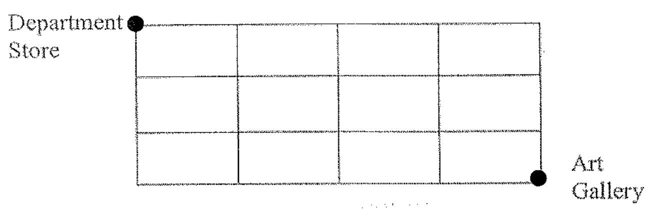
$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Binomial Theorem

$t_{k+1} = {}_nC_k x^{n-k} y^k$

- the general term of the expansion $(x+y)^n$ is used to determine term values, constant terms etc

Pathways



Number of possible routes from Department Store to Art Gallery is

$\frac{7!}{4!3!}$ or ${}_7C_4$ or ${}_7C_3$

Probability $= \frac{favourable outcomes}{total possible outcomes}$