Basic Graph Theory

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Contents

1 Defining a graph

- A **graph** is a triple (V, E, Ψ_G).
- Adjacent vertices : vertices having an edge between them.
- A **simple graph** is graph without multiple edges.

2 Representing a graph

- Adjacency matrix A(G) = $[a_{ij}]$
  - $a_{ij} = \begin{cases} \text{number of edges joining } v_i \text{ and } v_j & \text{if } i \neq j \\ \text{twice the number of loops incident with } v_i & \text{if } i = j \end{cases}$
- Incedence matrix B(G) = $[b_{ij}]$
  - $b_{ij} = \begin{cases} 0 & \text{if } v_i \text{ is not incedent with } e_j. \\ 1 & \text{if } v_i \text{ is incedent with } e_j \text{ and } e_j \text{ is not a loop.} \\ 2 & \text{if } v_i \text{ is incedent with } e_j \text{ and } e_j \text{ is a loop.} \end{cases}$

3 Properties of a graph

- Degree of a vertex :
  - $n = |V|$
    $m = |E|$
  - $deg_G(v)$ = number of edges incident with v (loops counted twice)
  - $\sum_{i=1}^{n} a_{ij} = deg_G(v_j)$
  - $\sum_{j=1}^{m} b_{ij} = deg_G(v_i)$
  - $\sum_{i=1}^{n} b_{ij} = 2$
  - $\sum_{i=1}^{n} v_i = 2m$ (Euler's theorem)
  - Corollory to Euler's theorem : In any graph G, the number of vertices of odd degree is even.
- Isomorphism :
  Two graphs $G(V, E, \Psi_G)$ and $H(W, F, \Psi_H)$ are said to be isomorphic if the following conditions hold.

- There exist bijections f and g such that
  $f : V \to W$
  $g : E \to F$
- For all $e \in E$, $\Psi_G(e) = (u, v) \Leftrightarrow \Psi_H(g(e)) = (f(u), f(v))$

4 Supply demand theorem

- A sequence $\{d_i\}$ is said to be **grphic** if there exists a simple graph whose degree sequence is identical with $\{d_i\}$.
- **Theorem :** A sequence $\{d_i\}$ is graphic if and only if
  for all k $\leq$ n : $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min(k, d_i)$
- LHS can be seen as demand for edges.
- RHS can be seen as maximum supply of edges possible.

5 Havel-Hakimi theorem

- Sequence S = $\{d_i\}$ is in **non increasing** order.
- **Theorem :** S is graphic if and only if
  S' = $(d_2 - 1, d_3 - 1, ..., d_{d_1+1} - 1, d_{d_1+2}, d_{d_1+3}, ..., d_n)$ is graphic.
- This theorem is useful for constructing graphs from a given degree sequence.

6 Subgraphs

- Spanning subgraph : Vertex set same as original graph.
- Induced subgraph : For all the vertices u,v of subgraph, if (u,v) is edge in original graph, then (u,v) in edge in subgraph as well.
- Note : Induced spanning subgraph is the graph itself.

7 Connected graphs and shortest paths

- Definitions of walk, trail, path, closed walk, closed trail and closed path.
- u and v are said to be **connected** iff there exists a (u,v) path.
- A graph is connected iff every pair of vertices is connected.
- A **component** is a maximal connected subgaph.

- Number of components = 1 iff the graph is connected.
- Dijkstra's shortest path algorithm.

8 Bipartite graphs

- Graph is called bipartite iff V can be partitioned in A and B such that no two vertices in A are adjacent and no two vertices in B are adjacent.
- Every even cycle is bipartite.
- Every odd cycle in non-bipartite.
- Subgraph of a bipartite graph is bipartite.
- A graph is bipartite **iff** it contains no odd cycles.

9 Trees

- Connected and acyclic graph is called a tree.
- Any two of the following three imply the third.
  - G is connected.
  - G in acyclic.
  - number of edges of G + 1 = number of vertices of G
- A simple graph is a tree **iff** any two vertices are connected by a unique path.
- **Cut edge** is an edge, removal of which will increase the number of components.
- G is tree iff every edge of G is its cut edge.
- **Cut vertex** is an vertex, removal of which will increase the number of components.
- Every connected graph contains at least two non cut vertices.
- Every tree has at least two leaves.

10 Spanning trees

- A graph is connected iff it contains a spanning tree.
- Edge contraction : G.e is G after contracting e.
- $n(G.e) = n(G) - 1$
  $m(G.e) = m(G) - 1$
  n and m denote numbers of vertices and edges respectively.
- Number of spanning trees in G : $\tau(G) = \tau(G - e) + \tau(G.e)$
  - Kirchoff's recursion for spanning trees.
- Kruskal's algorithm for finding out minimum spanning tree (it is a greedy algorithm).

11 Connectivity

- **Vertex cut** S is a set of vertices such that G - S is disconnected.
- S is called k-vertex-cut if it contains k vertices.
- $k_0(G)$: vertex-connectivity = number of vertices in smallest vertex cut.
- G is **t-vertex-connected** if $k_0(G) \geq t$
  (deletion of any $t - 1$ vertices will not disconnect the graph).
- **Edge cut** F is a set of edges such that G - F is disconnected.
- F is called l-edge-cut if it contains l edges.
- $k_1(G)$: edge-connectivity = number of edges in smallest edge cut.
- G is **l-edge-connected** if $k_1(G) \geq l$
  (deletion of any $l - 1$ edges will not disconnect the graph).
- $\delta(G)$ is smallest degree among the degrees of vertices of G.
- Whitney's inequality : $k_0(G) \leq k_1(G) \leq \delta(G)$ is smallest degree among the degrees of vertices of G.
- Menger's theorem : A graph is t-vertex-connected connected iff any two vertices are connected by t internally disjoint paths.

12 Eulerian graphs

- If there exists a closed trail containing all the edges of a graph, that graph is called as Eulerian graph.
- A connected graph is Eulerian iff all vertices are of even degree.
- A cennected graph has an open Eulerian trail iff it has exactly two vertices of odd degree.
- G is Eulerian if can be expressed as an union of edge-disjoint cycles.
- Fleury's algorithm to generate a closed Eulerian trail.

13 Hamiltonian graphs

- If there exists a spanning cycle in a graph, the graph is called a Hamiltonian graph.
- **Necessary condition :** G is Hamiltonian $\Rightarrow C(G-S) \leq |S|\ \forall S \subseteq V(G)$. C(G) is number of components of G.
- **Sufficient conditions :**
  - $deg_G(v) \geq n/2\ \forall v \in V(G) \Rightarrow G$ is Hamiltonian.
  - $deg_G(v) + deg_G(u) \geq n\ \forall$ non-adjacent pair of vertices u and $v \Rightarrow G$ is Hamiltonian.
  - Let d_i be the degree sequence of G in non decreasing order.
    $(d_k \leq k < n/2 \Rightarrow d_{n-k} \geq n-k)\forall k$
    $\Rightarrow G$ is Hamiltonian.
- **Necessary and sufficient condition :**
  - Join all the non-adjaent pairs of vertices u v with an edge if they satify $deg(u) + deg(v) \geq n$ in a stepwise manner, checking for the condition in each step. The resulted graph is called the **Closure** of graph G.
  - G is Hamiltonian $\Leftrightarrow Closure(G)$ is Hamiltonian.