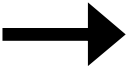


CSE 548: Analysis of Algorithms

Lecture 7
(Generating Functions)

Rezaul A. Chowdhury
Department of Computer Science
SUNY Stony Brook
Fall 2017



An Impossible Counting Problem

Suppose you went to a grocery store to buy some fruits. There are some constraints though:

- A. The store has only two **apples** left: one red and one green.
So you cannot take more than 2 apples.
- B. All but 3 **bananas** are rotten. You do not like rotten bananas.
- F. **Figs** are sold 6 per pack. You can take as many packs as you want.
- M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.
- P. They sell 4 **peaches** per pack. Take as many packs as you want.

Now the question is: in how many ways can you buy n fruits from the store?



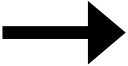
Generating Functions

Generating functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series.

For example, one can represent a sequence s_0, s_1, s_2, \dots as:

$$S(z) = s_0 + s_1z + s_2z^2 + s_3z^3 + \dots + s_nz^n + \dots$$

So s_n is the coefficient of z^n in $S(z)$.



An Impossible Counting Problem

- A. The store has only two **apples** left: one red and one green.
So you cannot take more than 2 apples.
- B. All but 3 **bananas** are rotten. You do not like rotten bananas.
- F. **Figs** are sold 6 per pack. You can take as many packs as you want.
- M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.
- P. They sell 4 **peaches** per pack. Take as many packs as you want.

$$A(z) = 1 + 2z + z^2 = (1 + z)^2$$

$$B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}$$

$$F(z) = 1 + z^6 + z^{12} + z^{18} + \dots = \frac{1}{1 - z^6}$$

$$M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}$$

$$P(z) = 1 + z^4 + z^8 + z^{12} + \dots = \frac{1}{1 - z^4}$$

(Lecture 7) Generating Functions

An Impossible Counting Problem

Suppose you can choose n fruits in s_n different ways.

Then the generating function for s_n is:

$$\begin{aligned} S(z) = A(z)B(z)F(z)M(z)P(z) &= (1+z)^2 \times \frac{1-z^4}{1-z} \times \frac{1}{1-z^6} \times \frac{1-z^6}{1-z^2} \times \frac{1}{1-z^4} \\ &= \frac{1+z}{(1-z)^2} \\ &= (1+z) \sum_{n=0}^{\infty} (n+1)z^n \\ &= \sum_{n=0}^{\infty} (2n+1)z^n \end{aligned}$$

Equating the coefficients of z^n from both sides:

$$s_n = 2n + 1$$

Fibonacci Numbers

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f_{n-1} + f_{n-2} & \text{otherwise.} \end{cases}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2} + [n = 1]$$

Generating function: $F(z) = f_0 + f_1z + f_2z^2 + f_3z^3 + \dots$

$$\begin{aligned} F(z) = \sum_n f_n z^n &= \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n = 1] z^n \\ &= \sum_n f_n z^{n+1} + \sum_n f_n z^{n+2} + z \\ &= zF(z) + z^2F(z) + z \end{aligned}$$

Fibonacci Numbers

$$\begin{aligned} F(z) &= zF(z) + z^2F(z) + z \\ \Rightarrow F(z) &= \frac{z}{1-z-z^2} \\ &= \frac{z}{(1-\varphi z)(1-\hat{\varphi} z)}, \text{ where } \varphi = \frac{1+\sqrt{5}}{2} \text{ \& } \hat{\varphi} = \frac{1-\sqrt{5}}{2} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1}{1-\varphi z} - \frac{1}{1-\hat{\varphi} z} \right) \\ &= \frac{1}{\sqrt{5}} \sum_n (\varphi^n - \hat{\varphi}^n) z^n \end{aligned}$$

Equating the coefficients of z^n from both sides:

$$f_n = \frac{1}{\sqrt{5}} (\varphi^n - \hat{\varphi}^n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Average Case Analysis of Quicksort

(Lecture 7) Generating Functions

Quicksort

- Input:** An array $A[1:n]$ of n distinct numbers.
- Output:** Numbers of $A[1:n]$ rearranged in increasing order of value.
- Steps:**
1. **Pivot Selection:** Select pivot $x = A[1]$.
 2. **Partition:** Use a stable partitioning algorithm to rearrange the numbers of $A[1:n]$ such that $A[k] = x$ for some $k \in [1, n]$, each number in $A[1:k-1]$ is smaller than x , and each in $A[k+1:n]$ is larger than x .
 3. **Recursion:** Recursively sort $A[1:k-1]$ and $A[k+1:n]$.
 4. **Output:** Output $A[1:n]$.

Stable Partitioning: If two numbers p and q end up in the same partition and p appears before q in the input, then p must also appear before q in the resulting partition.

Average Number of Comparisons by Quicksort

We will average the number of comparisons performed by *Quicksort* on all possible arrangements of the numbers in the input array.

Let t_n = average #comparisons performed by *Quicksort* on n numbers.

Then

$$t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{1}{n} \sum_{k=1}^n (t_{k-1} + t_{n-k}) & \text{otherwise.} \end{cases}$$

The recurrence can be rewritten as follows.

$$t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise.} \end{cases}$$

Average Number of Comparisons by Quicksort

The recurrence: $t_n = \begin{cases} 0 & \text{if } n < 1, \\ n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k & \text{otherwise.} \end{cases}$

Let $T(z)$ be an ordinary generating function for t_n 's:

$$\begin{aligned} T(z) &= t_0 + t_1 z + t_2 z^2 + \dots + t_n z^n + \dots \\ &= t_0 + \sum_{n=1}^{\infty} t_n z^n \\ &= t_0 + \sum_{n=1}^{\infty} \left(n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n \end{aligned}$$

Average Number of Comparisons by Quicksort

We have: $T(z) = t_0 + \sum_{n=1}^{\infty} \left(n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} t_k \right) z^n$

Differentiating:

$$\begin{aligned} T'(z) &= \sum_{n=1}^{\infty} \left(n(n-1) + 2 \sum_{k=0}^{n-1} t_k \right) z^{n-1} \\ &= z \sum_{n=2}^{\infty} n(n-1) z^{n-2} + 2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^n t_k \right) z^n \\ &= z \frac{d^2}{dz^2} \left(\left(\sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left(t_n z^n \left(\sum_{k=0}^{\infty} z^k \right) \right) \end{aligned}$$

(Lecture 7) Generating Functions

Average Number of Comparisons by Quicksort

$$T'(z) = z \frac{d^2}{dz^2} \left(\left(\sum_{n=0}^{\infty} z^n \right) - 1 - z \right) + 2 \sum_{n=0}^{\infty} \left(t_n z^n \left(\sum_{k=0}^{\infty} z^k \right) \right)$$

$$= z \frac{d^2}{dz^2} ((1 - z)^{-1} - 1 - z) + 2(1 - z)^{-1} \sum_{n=0}^{\infty} t_n z^n$$

$$= \frac{2z}{(1 - z)^3} + \frac{2}{1 - z} T(z)$$

Rearranging: $(1 - z)^2 T'(z) - 2(1 - z) T(z) = \frac{2z}{1 - z}$

$$\Rightarrow \frac{d}{dz} ((1 - z)^2 T(z)) = \frac{d}{dz} (-2 \ln(1 - z) - 2z)$$

Integrating: $(1 - z)^2 T(z) = -2 \ln(1 - z) - 2z + c$ (c is a constant)



Average Number of Comparisons by Quicksort

We have, $(1 - z)^2 T(z) = -2 \ln(1 - z) - 2z + c$ (c is a constant)

Putting $z = 0$, $T(0) = c \Rightarrow t_0 = c \Rightarrow c = 0$

Hence, $(1 - z)^2 T(z) = -2 \ln(1 - z) - 2z$

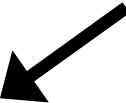
$$\Rightarrow T(z) = 2(-\ln(1 - z) - z)(1 - z)^{-2}$$

$$= 2 \left(\sum_{j=2}^{\infty} \frac{z^j}{j} \right) \left(\sum_{k=0}^{\infty} (k + 1) z^k \right)$$

Equating coefficients of z^n from both sides,

$$t_n = 2 \left(\sum_{k=2}^n \frac{n + 1 - k}{k} \right) = 2(n + 1) \sum_{k=1}^n \frac{1}{k} - 4n = 2(n + 1) H_n - 4n,$$

where $H_n = \sum_{k=1}^n \left(\frac{1}{k} \right)$ is the n^{th} harmonic number.



Average Number of Comparisons by Quicksort

We have, $t_n = 2(n + 1) H_n - 4n$,

where $H_n = \sum_{k=1}^n \left(\frac{1}{k} \right)$ is the n^{th} harmonic number.

But we know, $H_n = \ln n + O(1)$ (prove it)

Hence, $t_n = 2(n + 1)(\ln n + O(1)) - 4n = \Theta(n \log n).$

