1.1 Recap Of Generating Functions

Generating Functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series. You can represent a sequence $s_0, s_1, s_2, s_3, ...$ as:

$$S(z) = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + ... + s_n z^n + ...$$

In the above equation., s_n is the coefficient of z^n.

1.2 Generating Sequence for Fibonacci Numbers

The general form of generating functions is as follows:

$$F(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + ... + f_n z^n$$

The general sequence for Fibonacci numbers is given by 0,1,1,2,3,5,8,13.....

The recurrence for Fibonacci sequence is given by:

$$T(x) = \begin{cases} (0), & if x == 0 \\ (1), & if n == 1 \\ f(n-1) + f(n-2) & otherwise \end{cases}$$

The recurrence can be rewritten as :

$$fn = f(n-1) + f(n-2) + [n == 1] \tag{1.2.1}$$

The algorithm got Fibonacci sequence is as follows:

Algorithm 1 Fib(N)

- *IF N==0 return 0*
- $f_n = 1, f_{n-1} = 0, f_{n-2} = 0$
- $f_{n-2} = f_{n-1}$
- $f_n = f_{n-1}$
- $f_n = f_{n-1} + f_{n-2}$
- *return* f_n

The complexity of the above algorithm is $\theta(n)$ and also it is space efficient.

The generating function of Fibonacci series can be written from recurrence relation as follows:

$$F(z) = \sum_{n=0}^{\infty}(f_n)z^n = \sum_{n=0}^{\infty}(f_{n-1} + f_{n-2} + [n = 1])z^n$$

$$F(z) = \sum_{n=0}^{\infty}(f_{n-1})z^n + \sum_{n=0}^{\infty}(f_{n-2})z^n + \sum_{n=0}^{\infty}([n = 1])z^n \tag{1.2.2}$$

The $\sum_{n=0}^{\infty}([n = 1])z^n$ is valid only for n=1

Therefore for n=1;

$$\sum_{n=0}^{\infty}([n = 1])z^n = z \tag{1.2.3}$$

Now for $\sum_{n=0}^{\infty}(f_{n-1})z^n$:

$$\begin{aligned}\sum_{n=0}^{\infty}(f_{n-1})z^n &= z\sum_{n=0}^{\infty}(f_{n-1})z^{n-1} \\ &= z\sum_{n=0}^{\infty}(f_n)z^n \end{aligned} \tag{1.2.4}$$

Now for $\sum_{n=0}^{\infty}(f_{n-2})z^n$:

$$\begin{aligned}\sum_{n=0}^{\infty}(f_{n-2})z^n &= z^2\sum_{n=0}^{\infty}(f_{n-2})z^{n-2}\\ &= z^2\sum_{n=0}^{\infty}(f_n)z^n \qquad (1.2.5)\end{aligned}$$

Thus the equation (1.2.2) using equation(1.2.3),equation(1.2.4),equation(1.2.5) :

$$F(z) = zF(z) + z^2F(z) + z \qquad (1.2.6)$$

Therefore,

$$F(z) - zF(z) - z^2F(z) = z$$

Therefore,

$$F(z) = \frac{z}{(1 - z - z^2)}$$

$$F(z) = \frac{z}{(1 - \phi z)(1 - \hat{\phi} z)}$$

where,
$\phi = \frac{1+\sqrt{5}}{2}$ ......(also called golden ratio)
$\hat{\phi} = \frac{1-\sqrt{5}}{2}$

On comparing we get,

$$(1 - \phi z)(1 - \hat{\phi} z) = (1 - z - z^2)$$

$$1 - (\phi + \hat{\phi})z + \phi\hat{\phi}z^2 = (1 - z - z^2)$$

On comparing L.H.S with R.H.S we get,

$\phi + \hat{\phi} = 1$
$\phi\hat{\phi} = -1$

Solving above 2 equations we get,
$\phi = \frac{1+\sqrt{5}}{2}$ $\hat{\phi} = \frac{1-\sqrt{5}}{2}$

$$F(z) = \frac{1}{\sqrt{5}}\left(\frac{1}{(1-\phi z)} - \frac{1}{(1-\hat{\phi} z)}\right)$$
$$= \frac{1}{\sqrt{5}}\left((1-\phi z)^{-1} - (1-\hat{\phi} z)^{-1}\right)$$

On simplifying,

$$F(z) = \frac{1}{\sqrt{5}}\left(\left(\phi + \phi z + \phi z^2 + \phi z^3 + ... + \phi z^n\right) - \left(\hat{\phi} + \hat{\phi} z + \hat{\phi} z^2 + \hat{z}^3 + ... + \hat{\phi} z^n\right)\right)$$

$$F(z) = \frac{1}{\sqrt{5}}\left(((\phi - \hat{\phi}) + (\phi - \hat{\phi})z + ...(\phi - \hat{\phi})z^n\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{(1+\sqrt{5})}{2}\right)^n - \left(\frac{(1-\sqrt{5})}{2}\right)^n\right)$$

The above calculation requires $\theta(\log(n))$ time which is exponentially smaller than the $\theta(n)$ i.e time required for recursion for Fibonacci series.

1.3 Average Case Analysis of Quick Sort

For analysis of Quick sort we assume that there are n integers between 1 to n. For arranging n numbers there are n! permutations.That is Quick sort will take one permutation and return one permutation of n numbers.Depending on the permutation of n the running time of Quick Sort will vary.For example, in sorted sequence we get a complexity of $\theta(n^2)$ from Quick Sort.Here we assume the pivot as first number and then keep on sorting other numbers according to pivot such that L.H.S we get numbers lesser than pivot and on R.H.S we get numbers greater than pivot.In sorted sequence all the numbers will go to the right subtree of pivot as we assumed pivot to be first number.

Consider another scenario that the first number will be $\frac{n}{2}$ then Quick sort will be to divide numbers in $\frac{n}{2}$ and $\frac{n}{2}$ groups.Recursively we can solve the problem in $\theta \log n$ steps.

Our goal is to find the average running time of Quick sort.

The average running time of Quick sort for all permutations is given by:

$$t(n) = \frac{Number of permutation}{all permutations}$$

In this analysis of Quick sort we must satisfy the condition of stable partition which states that if a comes before b in the input then in output also a must come before b even after applying partitioning.If this property does not hold it means that the permutation itself changes during the analysis.

For eg:
The numbers to be sorted are:
4,3,1,2,5,6,9,7,10,8

Let the pivot be k=4.Then the there will be at least (k-1) numbers in left partition and (n-k) numbers in right partition.When we select a pivot we compare it with (n-1) numbers.Therefore after we select pivot we have total (n-1)! permutations. Therefore ,

$$t(n) = \frac{n!(n-1) + (n-1)! \sum_{k=1}^{\infty} t_{k-1} + t_{n-k}}{n!}$$

where,

n!=number of comparisons for partitioning the list
(n-1)!= number of permutations after pivot is decided i.e k can be pivot in (n-1)!

$$t(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{\infty} t_{k-1} + t_{n-k}$$

For further simplifying above equation consider,

1	t_0	t_{n-1}
2	t_1	t_{n-2}
3	t_2	t_{n-3}
.	.	.
.	.	.
.	.	.
n	t_{n-1}	t_0

After going through the table we get,

$$t(n) = (n-1) + \frac{1}{n}\sum_{k=0}^{\infty} t_k$$

Writing in generating equations formula:

$$T(z) = t_0 + t_1 z + t_2 z^2 + t_3 z^3 + ... + t_n z^n \tag{1.3.1}$$

In above equation t_0 is 0 because if n=0 then count of numbers to be sorted is zero.

Therefore,

$$t(n) = (n-1) + \frac{2}{n}\sum_{n=0}^{n-1} t_k \tag{1.3.2}$$

Therefore,

$$T(z) = t_0 + \sum_{n=1}^{\infty} t_n z^n$$

$$T(z) = t_0 + \sum_{n=1}^{\infty} (n-1) + \frac{2}{n}\sum_{k=0}^{n-1} t_k z^n$$

To eliminate n we differentiate with respect to n

$$T'(z) = \sum_{n=1}^{\infty}\left((n)(n-1) + 2\sum_{k=0}^{n-1} t_k\right) z^{n-1} \tag{1.3.3}$$

which equals,

$$T'(z) = \sum_{n=1}^{\infty}\left((n)(n-1)z^{n-1} + 2\sum_{k=0}^{n-1} t_k z^{n-1}\right)$$

$$T'(z) = z\sum_{n=1}^{\infty} (n)(n-1)z^{n-2} + 2\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} t_k\right) z^{n-1} \tag{1.3.4}$$

The first term looks like the it has been differentiated twice with z twice and hence,

$$T'(z) = z\sum_{n=1}^{\infty} \frac{d^2}{dz^2} z^n + 2\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} t_k\right) z^{n-1}$$

For the first term

The value of the first term at n=1 is

$$z\sum_{n=1}^{\infty} \frac{d^2}{dx^2} z^n = 0$$

Therefore,

$$z\sum_{n=2}^{\infty} \frac{d^2}{dx^2} z^n = z_2 + z_4 + ...$$

In the above term if we have (1+z) then it is a series of form $\frac{1}{(z-1)}$

So we modify the term as

$$z\frac{d^2}{dz^2}\sum_{n=2}^{\infty}(z^n - 1 - z)$$

$$= z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right)$$

For the second term

$$2\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1} t_k\right) z^{n-1}$$

This can be written as,

$$2\sum_{n=1}^{\infty} t_n z^n (1 + z + z^2 + ...)$$

Substituting each term in original equation (1.3.4) we get,

$$T'(z) = z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right) + 2\sum_{n=1}^{\infty} t_n z^n (1 + z + z^2 + ...)$$

$$= z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right) + 2\sum_{n=1}^{\infty} t_n z^n \sum_{k=0}^{\infty} z^k$$

$$= z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right) + 2\sum_{n=1}^{\infty} t_n z^n \frac{1}{(z-1)}$$

$$= z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right) + \frac{2}{(z-1)}\sum_{n=1}^{\infty} t_n z^n$$

$$= z\frac{d^2}{dz^2}\left(\frac{1}{(z-1)} - 1 - z\right) + \frac{2}{(z-1)}T(z)$$

Therefore,

$$T'(z) = \frac{2z}{(z-1)^3} + \frac{2}{(z-1)}T(z) \tag{1.3.5}$$

Multiplying throughout by $(1-z)^2$ we get,

$$(1-z)^2 T'(z) = \frac{2z}{(z-1)} + 2(z-1)T(z)$$

$$(1-z)^2T'(z) - +2(z-1)T(z) = \frac{2z}{(z-1)}$$

The above equation is nothing but,

$$\frac{d}{dx}((1-z)^2 + z) = \frac{d}{dx}(-2\log(1-z) - 2z)$$

Thus after integrating we get,

$$(1-z)^2T(z) = -2\log(1-z) - 2z + c$$

We have apply certain modification to above equation to get closed form of generating function for Quick sort