CSE548 & AMS542: Analysis of Algorithms, Fall 2017 **Lecturer:** Rezaul Chowdhury
Topic: Divide-and-Conquer Algorithms: Akra-Bazzi Recurrences **Date:** September 27, 2017
Scribe: Akshay Satam

10.1 Time-Complexity of Deterministic Select

The lecture started with the Time-Complexity of deterministic select algorithm:
$T(n) = \Theta(1)$ if $n < 140$
$T(n) = T(n/5) + T(7n/10 + 6) + \Theta(n)$ if $n >= 140$

Professor mentioned that though he had found the time-complexity of the Deterministic Algorithm in the previous lecture, we still had to calculate the upper bound for the running time of the algorithm. But the upper-bound for this time-complexity $T(n)$ cannot be found by Master Theorem.

Hence we had to modify $T(n)$ to $T'(n)$ so that calculating upper bound becomes easy. Here $T'(n)$ is:
$T'(n) = \Theta(1)$ if $n < 140$
$T'(n) = T'(n/5) + T'(4n/5) + \Theta(n)$ if $n >= 140$
Here, we substituted $7n/10 + 6$ with $8n/10$ in $T(n)$, which equates to $4n/5$. $4n/5$ will be upper-bound for $7n/10 + 6$ for all $n >= 60$.

However, we needed to find a more tighter bound, which is larger than $7n/10 + 6$ but smaller than $8n/10$. This is to ensure that we get a lower upper-bound for T(n). Hence we also proposed another upper bound $T''(n)$ given by:
$T''(n) = \Theta(1)$ if $n < 140$
$T''(n) = T''(n/5) + T''(3n/4) + \Theta(n)$ if $n >= 140$
Here, we substituted $7n/10 + 6$ with $7.5n/10$ in $T(n)$, which equates to $3n/4$. $3n/4$ will be upper-bound for $7n/10 + 6$ for all $n >= 120$.

In the previous lecture, we have proved that, if $T(n)$ is of the form :
$F(n) = a_1F(n/b_1) + a_2F(n/b_2) + a_3F(n/b_3)$,
then $F(n) = \Theta(n^p)$
where $p = \log_b a$.

Consider the recurrence F(n) mentioned above:
If upper-bound of $F(n)$ is given by n^p, then upper-bound of $a_1 * F(n/b_1)$ will be given by $a_1 * (n/b_1)^p$.
Similarly for $F(n/b_2)$ and $F(n/b_3)$. Putting these values in equation for $F(n)$, we get:
$F(n) = \Theta(a_1 * (n/b_1)^p + a_2 * (n/b_2)^p + a_3 * (n/b_3)^p)$
$F(n) = \Theta(n^p * (a_1/b_1^p + a_2/b_2^p + a_3/b_3^p))$

But $F(n) = \Theta(n^p)$. Hence, $(a_1/b_1^p + a_2/b_2^p + a_3/b_3^p) = 1$.
We can calculate the upper-bound of $F(n)$, which is n^p by simply using the equation

$(a_1/b_1^p + a_2/b_2^p + a_3/b_3^p) = 1$.
Since a_1, a_2, a_3 and a_1, a_2, a_3 are positive, we will get a unique value of p.

The intention of recalling $F(n)$ was to reiterate the fact that we can find upper-bound of $F(n)$ if it is of the form: $F(n) = a_1F(n/b_1) + a_2F(n/b_2) + a_3F(n/b_3)$. However, $T'(n)$ is of the form:
$T'(n) = T'(n/5) + T'(4n/5) + \Theta(n)$ if $n >= 140$
Similarly, $T''(n)$ is of the form: $T''(n) = T''(n/5) + T''(3n/4) + \Theta(n)$ if $n >= 140$
Both of them have an extra term, $\Theta(n)$. Hence, as per current knowledge, we cannot calculate upper-bound for $T(n)$ because the $\Theta(n)$ exists in the equation.

10.2 Master Theorem Revisited

Professor restated Master Theorem as:
If $T(n) = \Theta(1)$ for $n = 1$
and $T(n) = a * T(n/b) + f(n)$ otherwise,
then $T(n) = \Theta(n^{\log_b a} + \sum_{j=0}^{\log_b n-1} a^j * f(n/b^j))$ – equation 1

Consider the following diagram, which shows the different points m takes in above equation.

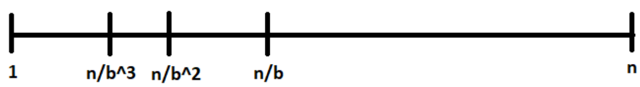

Fig. 1 Different values of m for input of size n

Let us assume that:
For any $m \in [n/b, n]$, $f(m)$ is within a constant factor of $f(n)$, i.e., $f(m) = \Theta(f(n))$.
If $f(m) = \Theta(f(n))$ for any $m \in [n/b, n]$, then for all intervals in the above diagram, $f(m) = \Theta(f(n))$
Suppose $p = log_b a$.
Then, equation 1 can be written as
$T(n) = \Theta(n^p + n^p * (\sum_{m=1}^{n} f(m)/m^{p+1}))$
Hence, $T(n) = \Theta(n^p * (1 + \sum_{m=1}^{n} f(m)/m^{p+1}))$

10.2.1 Why did we write Master's Theorem in this form?

Akra-Bazzi theorem teaches us how to find the upper-bounds to those recurrences, which cannot be solved by Master Theorem. We write Master Theorem in this form so that it closely matches the equation in Akra-Bazzi theorem, and thereby help us finding solution to the recurrences represented in Master Theorem form.

Also, we wanted to demonstrate that Master Theorem is a specialized case of Akra-Bazzi theorem.

10.2.2 Why did we write Master's theorem in terms of m ?

We wrote Master's Theorem in terms of m, so that we get rid of the constant b. Because if there are multiple recursions in $T(n)$, then there will be multiple constant terms like b_1, b_2, b_3 etc and could complicate the equation.

10.2.2.1 Q/A

Question: How did you create a continuous graph from the discrete one?
Answer: We did not create a continuous graph. The graph is still discrete but it is in terms of m and not b.

Question: We cannot generalize that $f(m)$ is well behaved (which means within a constant factor of $f(n)$) for the entire range. Right ?
Answer: No. we can generalize. Since, $f(m)$ is well-behaved for any $m \in [n/b, n]$, it will be well-behaved for any $m \in [n/b^2, n/b]$ and so on.

Question: In the new range, m will be in range 1 to n. Right?
Answer: Yes. It will be in the range 1 to n.

10.3 General form of Master Theorem

Consider the following recurrence:
$T(x) = \Theta(1)$ if $1 <= x <= x_0$,
$T(x) = \sum_{i=1}^{k} a_i * T(b_i * x) + g(x)$ if $x > x_0$
where,
1. $k >= 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 <= i <= k$
3. $b_i \in 0, 1$ is a constant for $1 <= i <= k$
4. $x >= 1$ is a real number
5. $x_0 >= max(1/b_i, 1/1 - b_i)$ is a constant for $1 <= i <= k$
6. $g(x)$ is a non-negative function that satisfies a polynomial-growth condition

10.3.1 What is Polynomial-growth function ?

We say that g(x) satisfies the polynomial-growth condition if there exist positive constants c_1 and c_2 such that for all $x >= 1$, for all $1 <= i <= k$, and for all $u \in [b_i x, x_1]$,
$c_1 g(x) <= g(u) <= c_2 g(x)$,
where x, k, b_i and $g(x)$ are as defined above.

Now consider the statement: $c_1 g(x) <= g(u) <= c_2 g(x)$.
It means that $g(u)$ is within a constant factor of $g(x)$.This is similar to the $f(m)$ function described in above section, where $f(m)$ was withing a constant factor of $f(n)$.

10.3.1.1 Q/A

Question: Any particular reason why we have real number here but integer in Master theorem?
Answer: Because this is a generalized form of Master theorem.

Question: If we take integer here, will this theorem become Master Theorem?
Answer: Yes. It will be the same as Master Theorem.

10.3.2 The Akra-Bazzi solution to recurrences

Consider the following recurrence presented above:
$T(x) = \Theta(1)$ if $1 <= x <= x_0$,
$T(x) = \sum_{i=1}^{k} a_i * T(b_i * x) + g(x)$ if $x > x_0$

The upper-bound to above recurrence is given by:
$T(x) = \Theta(x^p(1 + (\int_1^x (g(u)/u^{p+1})du)))$
where p is assumed to a real number for which $\sum_{i=1}^{k} a_i * b_i^p = 1$

The only difference between this solution and that of Master Theorem solution is that, here we are doing continuous summation whereas in Master Theorem, we are doing discrete summation.

10.3.3 Deterministic Select

The professor reverted back to the Deterministic Select and mentioned the equation:
$T'(n) = \Theta(1)$ if $n < 140$
$T'(n) = T'(n/5) + T'(4n/5) + \Theta(n)$ if $n >= 140$
Now that we know Akra-Bazzi Theorem will give solution to these types of recurrences, let us find solution to this recurrence.

Here $a_1 = 1, b_1 = 1/5, a_2 = 1, b_2 = 4/5$.
Now let us find the value of p, which will be given by : $\sum_{i=1}^{k} a_i * b_i^p = 1$
$\therefore a_1 * b_1^p + a_2 * b_2^p = 1$,
$\therefore 1 * 1/5^p + 1 * 4/5^p = 1$, which is possible for only one value of p, ie, $p = 1$
Hence, value of p is 1.

Now, the solution to the recurrence will be given by: $T(x) = \Theta(x^p(1 + (\int_1^x (g(u)/u^{p+1})du)))$
Here, $p = 1, g(u) = u$
$\therefore T(x) = \Theta(x^1(1 + (\int_1^x (u/u^{p+1})du)))$
$\therefore T(x) = \Theta(x(1 + (\int_1^x (u/u^{1+1})du)))$
$\therefore T(x) = \Theta(x(1 + (\int_1^x (1/u)du)))$

We know that, $\int (1/u)du = \log u$
Applying limits of the integrations, we get: $\int_1^x (1/u)du$ is $\log x - \log 1$
$\therefore \int_1^x (1/u)du$ is $\log x$ (Since, $\log 1$ is 0)

$\therefore T(x) = \Theta(x(1 + \log x))$
$\therefore T(x) = \Theta(x + x * \log x)$
Discarding lower order term, i.e. x, we get
$T(x) = \Theta(x * \log x)$

Now, let us consider the tighter recurrence to Deterministic Select:
$T''(n) = \Theta(1)$ if $n < 140$
$T''(n) = T''(n/5) + T''(3n/4) + \Theta(n)$ if $n >= 140$

According to Akra-Bazzi method, let us find solution to this theorem
Here $a_1 = 1, b_1 = 1/5, a_2 = 1, b_2 = 3/4$.
Now let us find the value of p, which will be given by :
$\sum_{i=1}^{k} a_i * b_i^p = 1$
$\therefore a_1 * b_1^p + a_2 * b_2^p = 1$
$\therefore 1 * 1/5^p + 1 * 3/4^p = 1$
This is possible only if value of p is less than 1.
Because, if $p = 1$, then $1 * 1/5 + 1 * 3/4 = 0.95$
If we increase p, the LHS of above equation will be reduced further. Hence, p should be decreased so that LHS becomes 1.
Hence, $p < 1$.

Now, the solution to the recurrence will be given by: $T(x) = \Theta(x^p(1 + (\int_1^x (g(u)/u^p * u)du)))$
Here, $p < 1$,
$g(u) = u$
$T(x) = \Theta(x^p(1 + (\int_1^x (u/u^{p*u})du)))$
$\therefore T(x) = \Theta(x^p(1 + (\int_1^x (1/u^p)du)))$
$\therefore T(x) = \Theta(x^p(1 + (\int_1^x (u^{-p})du)))$

We know that $\int (u^{-p})du$ is $u^{-p+1}/-p+1$
$\therefore \int_1^x (u^{-p})du$ is $(x^{-p+1}/-p+1) - (1^{-p+1}/{-p}+1)$

$\therefore T(x) = \Theta(x^p * ((1 + (x^{-p+1}/-p+1) - (1^{-p+1}/{-p}+1))))$
$\therefore T(x) = \Theta(x^p * ((1 + (x^{-p} * x/-p+1) - (1/{-p}+1))))$
$\therefore T(x) = \Theta(x^p + (x^p * x^{-p} * x/-p+1) - x^p * (1/{-p}+1)))$
$\therefore T(x) = \Theta(x^p + (x/-p+1) - x^p * (1/{-p}+1)))$
Here, we can ignore all terms including x^p as they are asymptotically smaller than x (Since, $p < 1$)
Hence, $T(x) = \Theta(x)$
This means running time of Deterministic Select algorithm after applying the tighter bound is $O(n)$.

Note : From the two recurrences to Deterministic Select, we see that we have to be very careful when approximating recurrences. The tighter the approximation, the lower the upper bound.

10.3.4 Examples of Akra-Bazzi Recurrences

10.3.4.1 Example 1: $T(x) = 2T(x/4) + 3T(x/6) + \Theta(x \log x)$

Here, $a_1 = 2, b_1 = 1/4, a_2 = 3, b_2 = 1/6$
$\therefore a_1 * b_1^p + a_2 * b_2^p = 1$
$\therefore 2 * 1/4^p + 3 * 1/6^p = 1$
If $p = 1$, then $2 * 1/4 + 3 * 1/6 = 1$
$\therefore p = 1$
Also, $g(x) = x * \log_n x$
$\therefore T(x) = \Theta(x(1 + \int_1^x ((u * \log u)/u^2) du))$
$\therefore T(x) = \Theta(x(1 + \int_1^x ((\log u)/u) du))$

Now, $\int ((\log u)/u) du$ should be done via Integration by parts:
$\int u * v$ dx = $u * \int v$dx $- \int du/dx * \int v$dx
where u is easily differentiable and v is easy integrable

$\therefore \int \log u * 1/u$ du = $\log u * \int 1/u$ du$- \int d \log u/du * \int 1/u$ du
$\therefore \int \log u * 1/u$ du = $\log u * \int 1/u$ du $- \int 1/u * \log u$
$\therefore 2 * \int \log u * 1/u$ du =$\log u * \int 1/u$ du
$\therefore 2 * \int \log u * 1/u$ du = $\log u * \int u^{-1}$ du
$\therefore 2 * \int \log u * 1/u$ du = $\log u * \log u$
$\therefore 2 * \int \log u * 1/u$ du = $(\log u)^2$
Applying limits, 1 to x
$2 * \int \log u * 1/u$ du = $(\log x)^2$
$\int \log u * 1/u$ du = $1/2 * (\log x)^2$

$\therefore T(x) = \Theta(x(1 + 1/2 * (\log x)^2))$
Discarding smaller order terms, we get
$T(x) = \Theta(x * (\log x)^2)$

Here, since $p = 1$, it corresponds to Master Theorem case 2.
Whenever, $p < 1$, it corresponds to Master Theorem case 3.
And when, $p > 1$, it corresponds to Master Theorem case 1.

10.3.4.2 Example 2: $T(x) = 2T(x/2) + 8/9T(3x/4) + \Theta(x^2 \log x)$

This equation cannot be solved by Master Theorem because $a_2 = 8/9 < 1$ but a should be grater than 1. Also, the function f(n) should be polynomially smaller than $n^{\log_b a}$, which is not in this case.

Solution by Akra-Bazzi method :
Here $p = 2$. $\therefore T(x) = \Theta(x^2(1 + \int_1^x ((u^2)/\log u * u^3)$ du)) = $\Theta(x^2 * \log \log x)$

10.3.4.3 Example 3: $T(x) = T(x/2) + \Theta(\log x)$

This is solvable by Master theorem also.

Solution by Akra-Bazzi method :
Here $p = 0$. $\therefore (x) = \Theta(1 + \int_1^x \log u/u$ du) = $\Theta((\log x)^2)$

10.3.4.4 Example 4: $T(x) = 1/2T(x/2) + \Theta(1/x)$

This cannot be solved by Master Theorem. Because $a_1 = 1/2$, which is less than 1. Also $f(n) is 1/x$, which is not within poly-log factor of $n^{\log_b a}$

Solution by Akra-Bazzi method:
Here $p = -1$. $\therefore T(x) = \Theta(1/x(1 + \int_1^x 1/u$ du)) = $\Theta(\log x/x)$