Given a Polynomial of Degree Bound 8 Find 8 Distinct Points to Efficiently Evaluate it at

 $A(x) = a_0 + a_1 \overline{x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7}$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

 $x_5 = -x_1$

 $x_6 = -x_2$

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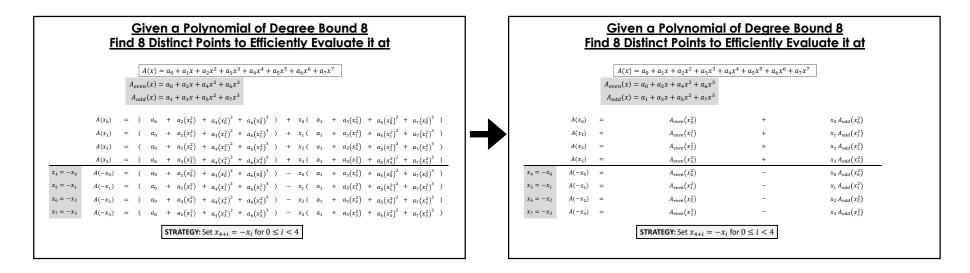
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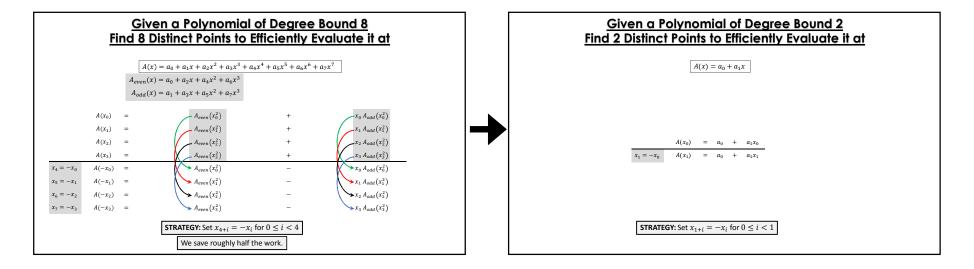
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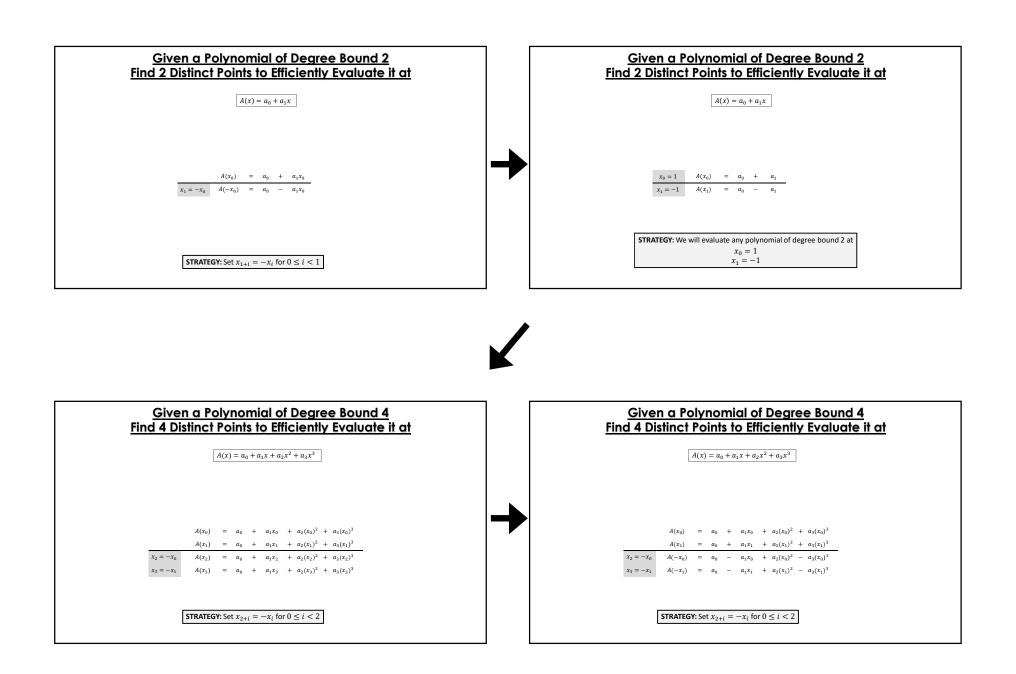
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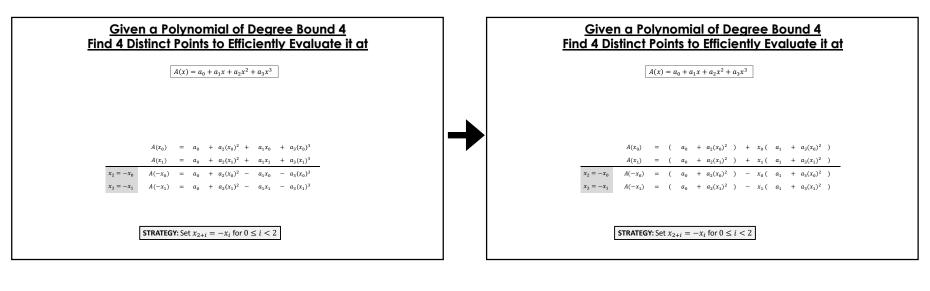
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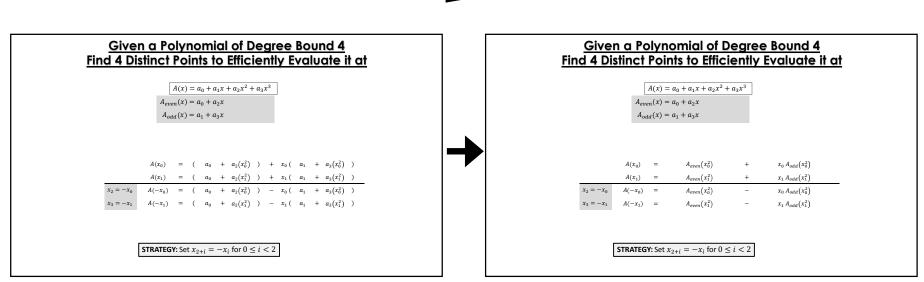


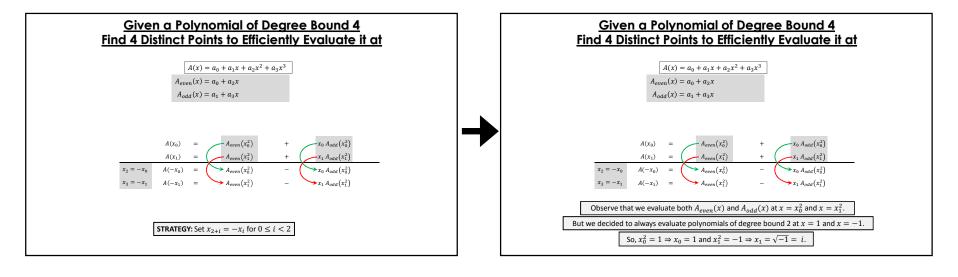




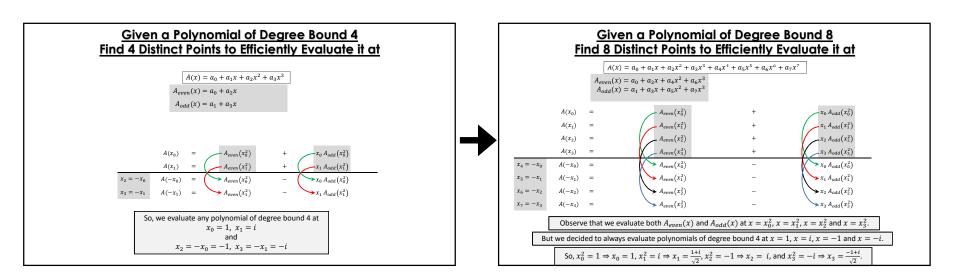


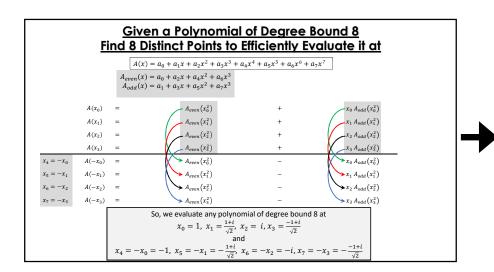












Given a Polynomial of Degree Bound $n = 2^k$ Find $n = 2^k$ Distinct Points to Efficiently Evaluate it at

degree bound	how did we find the points to evaluate the polynomial at?	the points	point property
2 ¹		1, -1	all 2 nd roots of unity
22	take positive and negative square roots of points used for degree bound 2^1 which are already the $2^{\rm nd}$ roots of unity	1, i, -1, -i	all 4 th roots of unity
23	take positive and negative square roots of points used for degree bound 2^2 which are already the $4^{\rm th}$ roots of unity	1, $\frac{1+i}{\sqrt{2}}$, i , $\frac{-1+i}{\sqrt{2}}$, -1, $-\frac{1+i}{\sqrt{2}}$, $-i$, $-\frac{-1+i}{\sqrt{2}}$	all 8 th roots of unity
24	take positive and negative square roots of points used for degree bound 2^3 which are already the 8^{th} roots of unity	1, $\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}$,,,,,,,	all 16 th roots of unity
2 ^{k-1}	take positive and negative square roots of points used for degree bound 2^{k-2} which are already the 2^{k-2} th roots of unity		all 2^{k-1} th roots of unity
$n = 2^k$	take positive and negative square roots of points used for degree bound 2^{k-1} which are already the 2^{k-1} th roots of unity		all 2^k th roots of unity (i.e., n^{th} roots of unity)



How to Find all nth Roots of Unity

The n^{th} roots of unity are: 1, ω_{n} , $(\omega_{n})^{2}$, $(\omega_{n})^{3}$, , $(\omega_{n})^{n-1}$,

where
$$\omega_n=\cosrac{2\Pi}{n}+i\sinrac{2\Pi}{n}=e^{rac{2\Pi i}{n}}$$
 is known as the primitive $n^{ ext{th}}$ roots of unity.

The result above can be derived using Euler's Formula.

Euler's Formula: For any real number α , $\cos \alpha + i \sin \alpha = e^{i\alpha}$

Euler's formula follows very easily from the following three power series each of which holds for $-\infty < \alpha < +\infty$:

$$\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{9!} - \cdots$$

$$\sin \alpha = \alpha - \frac{\alpha^3}{2!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \frac{\alpha^9}{9!} - \cdots$$

$$e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} + \frac{\alpha^7}{7!} + \frac{\alpha^8}{8!} + \cdots$$

How to Find all n^{th} Roots of Unity

Observe that for (any) real numbers α and p,

$$(\cos \alpha + i \sin \alpha)^p = (e^{i\alpha})^p = e^{i(p\alpha)} = \cos(p\alpha) + i \sin(p\alpha)$$

Also observe that for any integer k, $\cos(k \times 2\Pi) + i\sin(k \times 2\Pi) = 1 + i \times 0 = 1$

Then the n^{th} root of 1 (unity) is $=1^{\frac{1}{n}}=(\cos(k\times 2\Pi)+i\sin(k\times 2\Pi))^{\frac{1}{n}}=\cos\left(k\times \frac{2\Pi}{n}\right)+i\sin\left(k\times \frac{2\Pi}{n}\right)$

Observe that $\cos\left(k \times \frac{2\Pi}{n}\right) + i\sin\left(k \times \frac{2\Pi}{n}\right)$ takes n distinct values for $0 \le k < n$, and then simply repeats those values for k < 0 and $k \ge n$.

When k=1, we have $\cos\left(k\times\frac{2\Pi}{n}\right)+i\sin\left(k\times\frac{2\Pi}{n}\right)=\cos\left(\frac{2\Pi}{n}\right)+i\sin\left(\frac{2\Pi}{n}\right)=\omega_n=\text{primitive }n^{\text{th}}\text{ root of }1.$

Clearly, for any k, $\cos\left(k \times \frac{2\Pi}{n}\right) + i\sin\left(k \times \frac{2\Pi}{n}\right) = \left(\cos\left(\frac{2\Pi}{n}\right) + i\sin\left(\frac{2\Pi}{n}\right)\right)^k = (\omega_n)^k$

Hence, $1^{\frac{1}{n}} = \cos\left(k \times \frac{2\Pi}{n}\right) + i\sin\left(k \times \frac{2\Pi}{n}\right) = (\omega_n)^k$, for $k = 0, 1, 2, \dots, n-1$.

In other words, the $n^{\rm th}$ roots of 1 (unity) are: 1, ω_n , $(\omega_n)^2$, $(\omega_n)^3$,, $(\omega_n)^{n-1}$

