CSE548 & AMS542: Analysis of Algorithms, Fall 2017 **Lecturer:** Rezaul A. Chowdhury
Topic: Akra-Bazzi Recurrences, Generating Functions **Date:** 10/04/2017
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10.1 Background

Consider the following recurrence,

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum\limits_{i=1}^{k} a_i T(b_i x) + g(x), & \text{if } x > x_0; \end{cases} \tag{10.1.1}$$

Here,

1. $k \geq 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \leq i \leq k$
   If it is 0 then no there will be no recursive calls. Unlike master theorem, a_i can be any real number.
3. $b_i \in (0, 1)$ is a constant for $1 \leq i \leq k$
   In Akra-Bazzi, we assume that instead of recursing on just one problem size, there can be multiple problem sizes. For example, a_1 recursive calls for size $\frac{n}{b_1}$, a_2 recursive calls for size $\frac{n}{b_2}$, etc.
   $b_i \in (0, 1)$, because the problem size should decrease with each recurrence. Each b_i is $\frac{1}{b_i}$, if we compare to Master's theorem.
4. x ≥ 1 is a real number
5. $x_0 \geq max\{\frac{1}{b_i}, \frac{1}{1-b_i}\}, for 1 \leq i \leq k$
   x_0 is the base case size and is a constant.
6. g(x) is a non-negative function that satisfies a polynomial growth condition.
   g(x) is the cost of division and merging. It must satisfy a polynomial growth condition, i.e. it must be well-behaved. We start with an input of size x and in the recursion, the size is $b_i x$. If we take a value of g(x) between $b_i x$ and x, the value of g(x) should not change very abruptly.
   This is necessary because we want to integrate over 1 to x, instead of adding up discrete points.

   Thus, we can consider a number line: (in figure 10.1.1)

10.1.1 Solution of Akra-Bazzi Recurrence

For the recurrence,

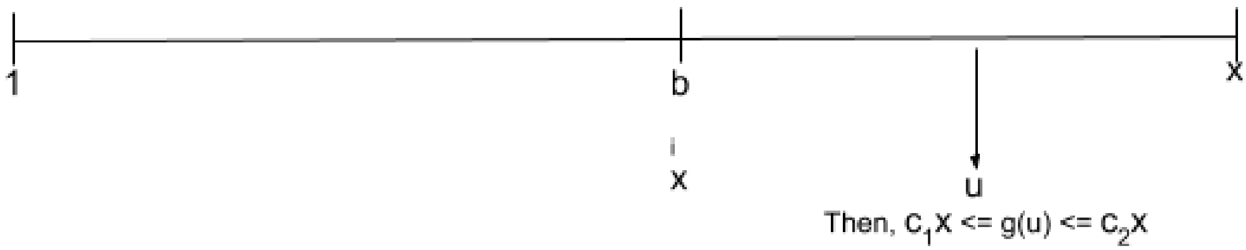

Figure 10.1.1: Consider recursion at two points x and $b_i x$. If we take any point u between x and $b_i x$, the value of g(u) will not change abruptly, i.e. $c_1 g(x) \leq g(u) \leq c_2 g(x)$

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum\limits_{i=1}^{k} a_i T(b_i x) + g(x), & \text{if } x > x_0; \end{cases} \tag{10.1.2}$$

The solution is,

$$T(x) = \Theta\left(x^p\left(1 + \int\limits_1^x \frac{g(u)}{u^{p+1}} du\right)\right) \tag{10.1.3}$$

where we assume p to be the unique real number for which $\sum\limits_{i=1}^{k} a_i b_i^p = 1$.

x^p is the number of leaves and x^p times the integral (i.e g(x)) is the cost of division and merging.

10.2 A Helping Lemma

This lemma will be used in proving the solution to Akra-Bazzi recurrence, which is a more general version of the Master theorem.

Lemma 10.2.1 *If g(x) is a non-negative function that satisfies the polynomial growth condition, then there exist positive constants c_3 and c_4 such that for $1 \leq i \leq k$ and all $x \geq 1$,*

$$c_3 g(x) \leq x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \leq c_4 g(x) \tag{10.2.4}$$

10.2.1 Proof of the Lemma

Equation 10.1.3 can be written as,

$$T(x) = \Theta\left(x^p + x^p \int\limits_1^x \frac{g(u)}{u^{p+1}} du)\right) \tag{10.2.5}$$

Taking the second part from equation 10.2.5,

$$x^p \int_1^x \frac{g(u)}{u^{p+1}} du)) = x^p \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du)) + x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du)) \tag{10.2.6}$$

For the time being, we can ignore $x^p \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du))$, because it happens inside the recurrence and will be taken care of in the next steps for the recursion.

Considering only the second part,

$$x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du)) \tag{10.2.7}$$

we want to show that it is well-behaved and does not disrupt the entire solution.

The lemma says that equation 10.2.7 is $\theta(g(x))$ and thus will not disrupt the solution.

Let us try to prove this.

From equation 10.2.4, since x and $b_i x$ are the limits of the integral,

$$b_i x \le u \le x \tag{10.2.8}$$

Taking inverse,

$$\frac{1}{x} \le \frac{1}{u} \le \frac{1}{b_i x} \tag{10.2.9}$$

We want to get u^{p+1}, since it is inside the integral, however we cannot simply raise the terms in equation 10.2.8 to power (p + 1) because p can be less than 0 and the value will be decreasing and inequalities will change.

That is,

$$\begin{cases} (\frac{1}{x})^{p+1} \le (\frac{1}{u})^{p+1} \le (\frac{1}{b_i x})^{p+1}, & \text{if } p+1 \ge 0, \\ (\frac{1}{b_i x})^{p+1} \le (\frac{1}{u})^{p+1} \le (\frac{1}{x})^{p+1}, & \text{otherwise} \end{cases}$$

Therefore,

$$\frac{1}{max\{x^{p+1}, b_i x^{p+1}\}} \le \frac{1}{u^{p+1}} \le \frac{1}{min(x^{p+1}, b_i x^{p+1})} \tag{10.2.10}$$

The function 'max' is on the LHS because, we want the term on the LHS to be smaller and hence its denominator should be larger.

Using $u \in [b_i x, x], c_1 g(x) \le g(u) \le c_2 g(x)$, equation 10.2.10 becomes,

$$\frac{c_1 g(x)}{max(x^{p+1}, b_i x^{p+1})} \le \frac{g(u)}{u^{p+1}} \le \frac{c_2 g(x)}{min(x^{p+1}, b_i x^{p+1})}$$

Simplifying,

$$\frac{c_1 g(x)}{x^{p+1} max(1, b_i)} \leq \frac{g(u)}{u^{p+1}} \leq \frac{c_2 g(x)}{x^{p+1} min(1, b_i)}$$

Taking the integral from $b_i x$ to x,

$$\frac{c_1 g(x)}{x^{p+1} max(1, b_i)} \int_{b_i x}^{x} du \leq \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} \leq \frac{c_2 g(x)}{x^{p+1} min(1, b_i)} \int_{b_i x}^{x} du \tag{10.2.11}$$

All terms in this inequality other than g(u) are constant w.r.t to u, hence the integral is after the fraction for the left and right hand side.

We know that,

$$\int_{b_i x}^{x} du = x - b_i x = x(1 - b_i)$$

Therefore, equation 10.2.11 becomes,

$$\frac{c_1 g(x) x(1 - b_i)}{x^{p+1} max(1, b_i)} \leq \int_{b_i x}^{x} \frac{g(u) du}{u^{p+1}} \leq \frac{c_2 g(x) x(1 - b_i)}{x^{p+1} min(1, b_i)}$$

Cancelling out the x,

$$\frac{c_1 g(x)(1 - b_i)}{x^{p} max(1, b_i)} \leq \int_{b_i x}^{x} \frac{g(u) du}{u^{p+1}} \leq \frac{c_2 g(x)(1 - b_i)}{x^{p} min(1, b_i)}$$

Multiplying throughout by x^p,

$$\frac{c_1 g(x)(1 - b_i)}{max(1, b_i)} \leq x^p \int_{b_i x}^{x} \frac{g(u) du}{u^{p+1}} \leq \frac{c_2 g(x)(1 - b_i)}{min(1, b_i)} \tag{10.2.12}$$

Noticing that all terms on the LHS and RHS except g(x) are constants, we can write equation 10.2.12 as,

$$c_3 g(x) \leq x^p \int_{b_i x}^{x} \frac{g(u) du}{u^{p+1}} \leq c_4 g(x) \tag{10.2.13}$$

Thus, we have proved the lemma.

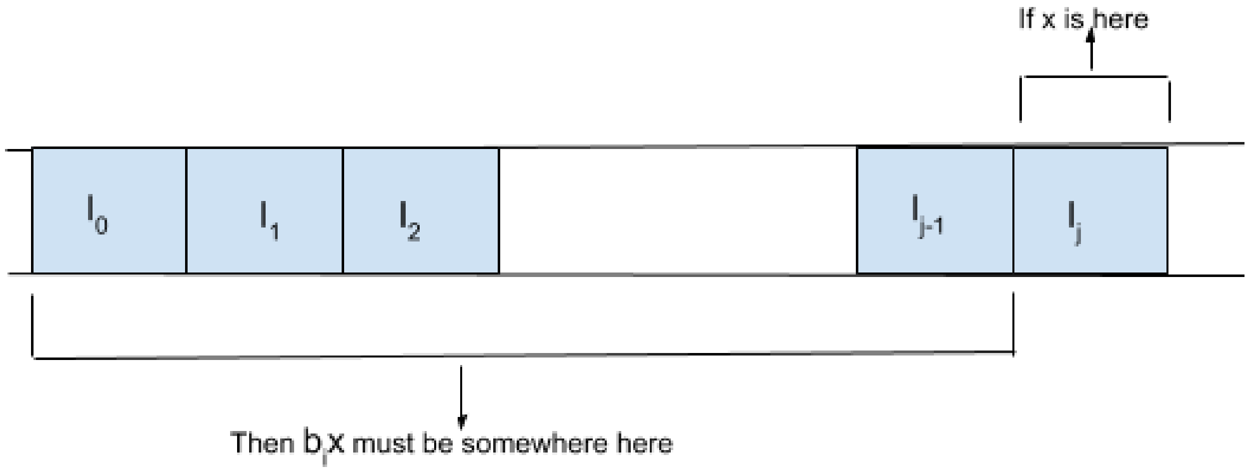

Figure 10.3.2: the first segment is of length x_0 and the rest of the partitions are of size 1 each. The last segment will include x.

10.3 Partitioning the domain of x

The original problem had size x. In the Akra-Bazzi recurrence, we divided the problem into sizes $b_i x$. In order to prove the Akra-Bazzi recurrence, we will show that the solution in equation 10.1.3 holds for size $b_i x$, and by induction will hold for a problem of size x.

We divide the domain of x into segments, as shown in figure 10.3.2.

We will show that if x lies in the last segment, $b_i x$ will lie in any of the segments before it.

Assume that x lies in some segment Ij. The length of Ij is $[x_0 + j - 1, x_0 + j]$ and

$$x_0 + j - 1 < x \leq x_0 + j, \text{for } j \geq 1 \tag{10.3.14}$$

Since, $b_i x$ lies in any segment before x, $b_i x$ is $1 < b_i x \leq x_0 + j - 1$. We want to prove this.

Using equation 10.3.14 and multiplying it by b_i,

$$b_i(x_0 + j - 1) < b_i x \leq b_i(x_0 + j) \tag{10.3.15}$$

Also $j \geq 1$, therefore j - 1, is 0 in the least, it cannot be negative. Hence, we can ignore the j - 1 on the LHS of the above inequality. The RHS of the inequality is $(b_i.x_0 + b_i.j)$, $b_i < 1$ and hence $b_i.j < j$. Thus, we can write $b_i.j$ as j.

Thus equation 10.3.15 becomes,

$$b_i x_0 < b_i x \leq b_i x_0 + j \tag{10.3.16}$$

We also have, $x_0 \geq max\{\frac{1}{b_i}, \frac{1}{1-b_i}\}$

Hence, $x_0 \geq \frac{1}{b_i}$ and $x_0 b_i \geq 1$

Finally, equation 10.3.16 becomes,

$$1 < b_i.x \leq b_i x_0 + j \tag{10.3.17}$$

Considering the RHS of equation 10.3.17, we can write

$$b_i x_0 + j = x_0 + j - (1 - b_i)x_0 \tag{10.3.18}$$

Also from $x_0 \geq max\{\frac{1}{b_i}, \frac{1}{1-b_i}\}$, we have $x_0 \geq \frac{1}{1-b_i}$ and $x_0(1 - b_i) \geq 1$. Hence, $-x_0(1 - b_i) \leq -1$
Therefore, equation 10.3.18 becomes,

$$1 < b_i.x \leq x_0 + j - 1 \tag{10.3.19}$$

This proves that $b_i x$ lies in a segment before $x_0 + j$, that is the subproblem size does not lie in a segment in which x lies.

We assume that equation 10.2.5 holds for subproblems of size $b_i x$ and prove that it will also hold for problems of size x.

Equation 10.2.5 also means that, T(x) $\geq c_5$.(RHS of 10.2.5) and T(x) $\leq c_6$.(RHS of 10.2.5), since it is θ.

Let us first prove that,

$$T(x) \geq c_5 \left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}} du) \right) \tag{10.3.20}$$

By induction on the interval I_j containing x.

Base case: when $x \in I_0$, it is trivially true, since T(x) = $\theta(1)$

Using this, we can extend it to I_1, I_2, etc upto the segment before the one that contains x.

Consider the non-base case part of the original recurrence from equation 10.1.1. Our assumption is that equation 10.3.20 holds for the entire summation from equation 10.1.1.

Therefore, replacing T($b_i x$) of equation 10.1.1, with equation 10.3.20, we can write,

$$T(x) \geq c_5 \sum_{i=1}^{k} a_i \left((b_i x)^p + (b_i x)^p \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du \right) + g(x) \tag{10.3.21}$$

We have taken c_5 outside the sum because it is independant of i.

The interval from [1, $b_i x$] can be written as [1, x] - [$b_i x$, x].

Thus, equation 10.3.21 can be written as,

$$T(x) = c_5 \sum_{i=1}^{k} a_i b_i^p \left(x^p + x^p \int_{1}^{x} \frac{g(u)}{u^{p+1}} du - x^p \int_{b_i x}^{x} \frac{g(u)}{u^{p+1}} du \right) + g(x) \tag{10.3.22}$$

The reason for splitting the integral is that, the first part becomes the same as the original recursion and the second part has been proved in the lemma to be $\theta(g(x))$.

Mutliplying equation 10.2.13 throughout by -1, we get,

$$-c_3 g(x) \geq -x^p \int_{b_i x}^{x} \frac{g(u)du}{u^{p+1}} \geq -c_4 g(x) \tag{10.3.23}$$

Using 10.3.23, we can re-write 10.3.22 as, (the sign also changes from = to $\geq$)

$$T(x) \geq c_5 \sum_{i=1}^{k} a_i b_i^p \left(x^p + x^p \int_{1}^{x} \frac{g(u)}{u^{p+1}} du - c_4 g(x) \right) + g(x) \tag{10.3.24}$$

Now, taking the terms independant of i outside the sum, we get,

$$T(x) \geq c_5 \left(x^p + x^p \int_{1}^{x} \frac{g(u)}{u^{p+1}} du - c_4 g(x) \right) \sum_{i=1}^{k} a_i b_i^p + g(x) \tag{10.3.25}$$

Remember from equation 10.1.3, that $\sum\limits_{i=1}^{k} a_i b_i^p = 1$. Therefore the sum becomes 1 in equation 10.3.25.

Thus, from equation 10.3.25,

$$T(x) \geq c_5 x^p \left(1 + \int_{1}^{x} \frac{g(u)}{u^{p+1}} du \right) - c_4 c_5 g(x) + g(x)$$

$$T(x) \geq c_5 x^p \left(1 + \int_{1}^{x} \frac{g(u)}{u^{p+1}} du \right) + (1 - c_4 c_5) g(x)$$

$$T(x) \geq c_5 x^p \left(1 + \int_{1}^{x} \frac{g(u)}{u^{p+1}} du \right) \text{, provided } (1 - c_4 c_5) g(x) \geq 0$$

Since g(x) is already greater than 0, it implies that $1 - c_4 c_5 \geq 0$.

Therefore,

$$T(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right) \text{, such that } c_4 c_5 \leq 1 \tag{10.3.26}$$

The proof for the other side is similar.

Therefore,

$$T(x) \geq c_5 \left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}} du)\right)$$

and

$$T(x) \leq c_6 \left(x^p + x^p \int_1^x \frac{g(u)}{u^{p+1}} du)\right)$$

Thus, we derived the solution to the Akra-Bazzi recurrence and showed that,

$$T(x) = \Theta\left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du\right)\right)$$

10.4 Generating Functions

Generating functions represent a sequence and every entry in the sequence is a co-efficient of some power of a variable. For example, we can represent the sequence $s_0, s_1, s_2, ...$ as:

$$S(z) = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + ... + s_n z^n + ... \tag{10.4.27}$$

The co-efficient of z^i the number of ways of doing i things.

10.4.1 An impossible counting problem

Consider an example of going to the grocery store and buying some fruits, under some constraints. We can write the different ways of taking some number of fruits based on a condition as a power series.

1. The store has only two apples left: one red and one green. So you cannot take more than 2 apples. This can be represented as a power series

   $$A(z) = 1 + 2z + z^2 = (1 + z)^2$$

   because there are 4 ways of taking a red and a green apple. We can take 0 apples, 1 red apple, 1 green apple or both apples. The series means we can take 0 apples in 1 way, 1 apple in 2 ways, and 2 apples in 1 way. The co-efficients for higher powers are zero because there are no other ways of taking apples.
   Thus, the co-efficient of z^i gives the number of ways of taking i apples.

2. All but 3 bananas are rotten. You do not like rotten bananas.

   We can choose from among 3 bananas, as the rest are rotten.

   $$B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}$$

   Here the co-efficient of each term is 1 because all bananas are alike, and taking 2 or more bananas can be done in only 1 way.

3. Figs are sold 6 per pack. You can take as many packs as you want.

   Here we can take 0 figs, or multiples of 6 number of figs at a time. Therefore,

   $$F(z) = 1 + z^6 + z^{12} + z^{18} + ... = \frac{1}{1 - z^6}$$

   because

   $$1 + x + x^2 + x^3 + ... = \frac{1}{1 - x}$$

4. Mangoes are sold in pairs. But you must not take more than a pair of pairs.

   $$M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}$$

   because we can take 0, 2 or 4 mangoes at a time. The co-efficients are 1 because all mangoes are similar.

5. They sell 4 peaches per pack. Take as many packs as you want.

   $$P(z) = 1 + z^4 + z^8 + ... = \frac{1}{1 - z^4}$$

   We can take 0 peaches or mulitples of 4 peaches, at a time.

In order to take n fruits from all these fruits, we multiply the series for each fruit.

Therefore,

$$S(z) = A(z)B(z)F(z)M(z)P(z)$$

$$S(z) = (1 + z)^2 . \frac{1 - z^4}{1 - z} . \frac{1}{1 - z^6} . \frac{1}{1 - z^6} . \frac{1}{1 - z^4}$$

$$S(z) = \frac{1 + z}{(1 - z)^2}$$

where,

$$\frac{1}{(1 - z)^2} = 1 + 2z + 3z^2 + 4z^3 + ... + (i + 1)z^i + ... \text{ Hence, } \frac{1}{(1 - z)^2} = \sum_{n=0}^{\infty} (n + 1)z^n$$

Therefore,

$$S(z) = (1+z)\sum_{n=0}^{\infty}(n+1)z^n$$

$$S(z) = \sum_{n=0}^{\infty}(2n+1)z^n$$

In this product, the co-efficient of z^n is the number of ways of taking n fruits. Thus, there are $2n+1$ ways of taking n fruits.