Predicate Logic

CSE 505 – Computing with Logic

Stony Brook University

http://www.cs.stonybrook.edu/~cse505

The alphabet of predicate logic

- Variables
- Constants (identifiers, numbers etc.)
- Functors (identifiers with arity > 0; e.g. date/3).
- Predicate symbols (identifiers with arity >= 0; e.g. append/3).
- Connectives:
  - $\wedge$ (conjunction),
  - $\vee$ (disjunction),
  - $\neg$ (negation),
  - $\leftrightarrow$ (logical equivalence),
  - $\rightarrow$ (implication).
- Quantifiers: $\forall$ (universal), $\exists$ (existential).
- Auxilliary symbols such as parentheses and comma.

Predicate Logic Formulas

- Terms (T) over an alphabet A is the smallest set such that:
  - Every constant c ∈ A is also c ∈ T .
  - Every variable X ∈ A is also X ∈ T .
  - If f /n ∈ A and t1, t2, . . . tn ∈ T then f (t1,t2,…, tn) ∈ T .
- Well-formed formulas (*wffs*, denoted by *F*) over alphabet A is the smallest set such that:
  - If p/n is a predicate symbol in A and t1, t2,…, tn ∈ T then p(t1, t2, …, tn) ∈ *F*.
  - If F,G ∈ *F* then so are (¬F), (F ∧ G), (F ∨ G), (F → G) and (F ↔ G)
  - If F ∈ *F* and X is a variable in A then (∀X F) and (∃X F) ∈ *F*.

Bound and Free Variables

- A variable X is *bound* in formula F if (∀X G) or (∃X G) is a subformula of F.
- A variable that occurs in F, but is not bound in F is said to be *free* in F.
- A formula F is *closed* if it has no free variables.
- Let X1,X2, . . . Xn be all the free variables in F. Then
  - (∀X1 (. . . (∀Xn F) . . .)) is the *universal closure* of F, and is denoted by ∀`F.
  - (∃X1 (. . . (∃Xn F) . . .)) is the *existential closure* of F, and is denoted by ∃F.

Interpretation

- An ***interpretation*** I of an alphabet is
  - a non-empty domain D, and
  - a mapping that associates:
    - each constant $c \in A$ with an element $cI \in D$
    - each n-ary functor $f \in A$ with an function $fI : D^n \rightarrow D$
    - each n-ary predicate symbol $p \in A$ with an relation $pI \subseteq D^n$
- For instance, one interpretation of the symbols in our "relations" program is that bob, pam et. al. are people in some set, and parent/2 is the parent-of relation, etc.
- Another interpretation could be that bob, pam etc are natural numbers, parent/2 is the greater-than relation, etc.

Valuation

- Given an interpretation I, the semantics of a variable-free (a.k.a. ***ground***) term is clear from I itself:

  $$I(f\,(t1, t2, \ldots, tn)) = fI(I(t1),I(t2), \ldots,I(tn))$$

- But to attach a meaning to terms with variables, we must first give a meaning to its variables
- This is done by a ***valuation***: which is a mapping from variables to the domain D of an interpretation.

  $$\varphi = \{X1 \rightarrow d1, X2 \rightarrow d2, \ldots, Xn \rightarrow dn\}$$

  $\varphi[X \rightarrow d]$ is identical to φ except that it maps X to d

Semantics of terms

- Terms are given a meaning with respect to a ***valuation:***
  - Given an interpretation I and valuation φ, the meaning of a term t, denoted by φI(t) is defined as:
    - if t is a constant c then φI(t) = cI
    - if t is a variable X then φI(t) = φX
    - if t is a structure f (t1, t2, …, tn) then

      φI(t) = fI(φI(t1), φI(t2), …, φI(tn))

Example

- Let A be an alphabet containing constant zero, a unary functor s and a binary functor plus.
- I, defined as follows, is an interpretation with N (the set of natural numbers) as its domain:
  - zeroI = 0
  - sI(x) = 1 + x
  - plusI(x, y) = x + y
- Now, if $\varphi = \{X \rightarrow 1\}$, then

$$\begin{aligned}\varphi I(plus(s(zero),X)) &= \varphi I(s(zero)) + \varphi I(X) \\ &= (1 + \varphi I(zero)) + \varphi(X) \\ &= (1 + 0) + 1 = 2\end{aligned}$$

Semantics of Well-Formed Formulae

- A formula's meaning is given w.r.t. an interpretation I and valuation φ
  - I |= φ p(t1, t2, …, tn) iff (φI(t1),φI(t2), …, φI(tn)) ∈ pI
  - I |=φ ¬F iff I |≠φ F
  - I |=φ F ∧ G iff I |=φ F and I |=φ G
  - I |=φ F ∨ G iff I |=φ F or I |=φ G (or both)
  - I |=φ F → G iff I |=φ G whenever I |=φ F
  - I |=φ F ↔ G iff I |=φ F → G and I |=φ G → F
  - I |=φ ∀X F iff I[X → d] |= φ F for every d ∈ | I |
  - I |=φ ∃X F iff I[X → d] |= φ F for some d ∈ | I |

Example 1.

- Consider the language with zero as the lone constant, s/1 as the only functor symbol, and a predicate symbol p/1.
- Consider an interpretation I with $| I | = N$, the set of natural numbers, $zeroI = 0$ and $sI(x) = 1 + x$
- Now consider the formula:

$$F1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

- Find an interpretation for p/1 such that $I \models F1$.

- Given a set of closed formulas P, an interpretation I is said to be a *model* of P iff every formula of P is true in I

Example 2.

- Recall Example 1:

$$F1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

- Consider extending the previous example with another predicate symbol q/1, and consider the formula:

$$F2 = q(s(zero)) \land (\forall X\ q(s(s(X))) \leftrightarrow q(X))$$

- Now extend the previous interpretation such that

$$I \models F1 \land F2$$

Example 3.

- Recall Example 2:

  $$F1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

  $$F2 = q(s(zero)) \land (\forall X\ q(s(s(X))) \leftrightarrow q(X))$$

- In the previous example, consider a new formula:

  $$F3 = (\forall X\ q(s(X)) \leftrightarrow p(X))$$

- Now extend the previous interpretation such that

  $$I \models F1 \land F2 \land F3$$

Interpretations and Consequences

- Is there any interpretation I such that $I \models F1 \wedge F2$, but $I \not\models F3$?

Logical Consequence

- Let P and F be closed formulas.
  - F is a *logical consequence* of P (denoted by $P \models F$) iff
  - F is true in every model of P.

Logical Consequence: An Example

1) $(\forall X\ (\forall Y\ (mother(X) \land child(Y, X)) \rightarrow loves(X, Y)))$
2) $mother(mary) \land child(tom, mary)$

- Is loves(mary , tom) a logical consequence of the above two statements?
  - For 1) to be true in some interpretation I:

    $$I \models_{\varphi} (mother(X) \land child(Y, X)) \rightarrow loves(X, Y)$$

    must hold for any valuation φ.
  - Specifically, for $\varphi = [X \rightarrow mary, Y \rightarrow tom]$

    $$I \models_{\varphi} (mother(mary) \land child(tom, mary)) \rightarrow loves(mary, tom)$$
  - Hence loves(mary,tom) is true in I if 2) above is true in I.