Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

$$\begin{array}{lllllllllllllllll}
 & A(x_0) & = & a_0 & + & a_1x_0 & + & a_2(x_0)^2 & + & a_3(x_0)^3 & + & a_4(x_0)^4 & + & a_5(x_0)^5 & + & a_6(x_0)^6 & + a_7(x_0)^7 \\
 & A(x_1) & = & a_0 & + & a_1x_1 & + & a_2(x_1)^2 & + & a_3(x_1)^3 & + & a_4(x_1)^4 & + & a_5(x_1)^5 & + & a_6(x_1)^6 & + a_7(x_1)^7 \\
 & A(x_2) & = & a_0 & + & a_1x_2 & + & a_2(x_2)^2 & + & a_3(x_2)^3 & + & a_4(x_2)^4 & + & a_5(x_2)^5 & + & a_6(x_2)^6 & + a_7(x_2)^7 \\
 & A(x_3) & = & a_0 & + & a_1x_3 & + & a_2(x_3)^2 & + & a_3(x_3)^3 & + & a_4(x_3)^4 & + & a_5(x_3)^5 & + & a_6(x_3)^6 & + a_7(x_3)^7 \\
\hline
x_4 = -x_0 & A(x_4) & = & a_0 & + & a_1x_4 & + & a_2(x_4)^2 & + & a_3(x_4)^3 & + & a_4(x_4)^4 & + & a_5(x_4)^5 & + & a_6(x_4)^6 & + a_7(x_4)^7 \\
x_5 = -x_1 & A(x_5) & = & a_0 & + & a_1x_5 & + & a_2(x_5)^2 & + & a_3(x_5)^3 & + & a_4(x_5)^4 & + & a_5(x_5)^5 & + & a_6(x_5)^6 & + a_7(x_5)^7 \\
x_6 = -x_2 & A(x_6) & = & a_0 & + & a_1x_6 & + & a_2(x_6)^2 & + & a_3(x_6)^3 & + & a_4(x_6)^4 & + & a_5(x_6)^5 & + & a_6(x_6)^6 & + a_7(x_6)^7 \\
x_7 = -x_3 & A(x_7) & = & a_0 & + & a_1x_7 & + & a_2(x_7)^2 & + & a_3(x_7)^3 & + & a_4(x_7)^4 & + & a_5(x_7)^5 & + & a_6(x_7)^6 & + a_7(x_7)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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Find 8 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

$$\begin{array}{lllllllllllllllll}
 & A(x_0) & = & a_0 & + & a_1x_0 & + & a_2(x_0)^2 & + & a_3(x_0)^3 & + & a_4(x_0)^4 & + & a_5(x_0)^5 & + & a_6(x_0)^6 & + a_7(x_0)^7 \\
 & A(x_1) & = & a_0 & + & a_1x_1 & + & a_2(x_1)^2 & + & a_3(x_1)^3 & + & a_4(x_1)^4 & + & a_5(x_1)^5 & + & a_6(x_1)^6 & + a_7(x_1)^7 \\
 & A(x_2) & = & a_0 & + & a_1x_2 & + & a_2(x_2)^2 & + & a_3(x_2)^3 & + & a_4(x_2)^4 & + & a_5(x_2)^5 & + & a_6(x_2)^6 & + a_7(x_2)^7 \\
 & A(x_3) & = & a_0 & + & a_1x_3 & + & a_2(x_3)^2 & + & a_3(x_3)^3 & + & a_4(x_3)^4 & + & a_5(x_3)^5 & + & a_6(x_3)^6 & + a_7(x_3)^7 \\
\hline
x_4 = -x_0 & A(-x_0) & = & a_0 & - & a_1x_0 & + & a_2(x_0)^2 & - & a_3(x_0)^3 & + & a_4(x_0)^4 & - & a_5(x_0)^5 & + & a_6(x_0)^6 & - a_7(x_0)^7 \\
x_5 = -x_1 & A(-x_1) & = & a_0 & - & a_1x_1 & + & a_2(x_1)^2 & - & a_3(x_1)^3 & + & a_4(x_1)^4 & - & a_5(x_1)^5 & + & a_6(x_1)^6 & - a_7(x_1)^7 \\
x_6 = -x_2 & A(-x_2) & = & a_0 & - & a_1x_2 & + & a_2(x_2)^2 & - & a_3(x_2)^3 & + & a_4(x_2)^4 & - & a_5(x_2)^5 & + & a_6(x_2)^6 & - a_7(x_2)^7 \\
x_7 = -x_3 & A(-x_3) & = & a_0 & - & a_1x_3 & + & a_2(x_3)^2 & - & a_3(x_3)^3 & + & a_4(x_3)^4 & - & a_5(x_3)^5 & + & a_6(x_3)^6 & - a_7(x_3)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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Find 8 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

$$\begin{array}{lllllllllllllllll}
 & A(x_0) & = & a_0 & + & a_2(x_0)^2 & + & a_4(x_0)^4 & + & a_6(x_0)^6 & + & a_1x_0 & + & a_3(x_0)^3 & + & a_5(x_0)^5 & + a_7(x_0)^7 \\
 & A(x_1) & = & a_0 & + & a_2(x_1)^2 & + & a_4(x_1)^4 & + & a_6(x_1)^6 & + & a_1x_1 & + & a_3(x_1)^3 & + & a_5(x_1)^5 & + a_7(x_1)^7 \\
 & A(x_2) & = & a_0 & + & a_2(x_2)^2 & + & a_4(x_2)^4 & + & a_6(x_2)^6 & + & a_1x_2 & + & a_3(x_2)^3 & + & a_5(x_2)^5 & + a_7(x_2)^7 \\
 & A(x_3) & = & a_0 & + & a_2(x_3)^2 & + & a_4(x_3)^4 & + & a_6(x_3)^6 & + & a_1x_3 & + & a_3(x_3)^3 & + & a_5(x_3)^5 & + a_7(x_3)^7 \\
\hline
x_4 = -x_0 & A(-x_0) & = & a_0 & + & a_2(x_0)^2 & + & a_4(x_0)^4 & + & a_6(x_0)^6 & - & a_1x_0 & - & a_3(x_0)^3 & - & a_5(x_0)^5 & - a_7(x_0)^7 \\
x_5 = -x_1 & A(-x_1) & = & a_0 & + & a_2(x_1)^2 & + & a_4(x_1)^4 & + & a_6(x_1)^6 & - & a_1x_1 & - & a_3(x_1)^3 & - & a_5(x_1)^5 & - a_7(x_1)^7 \\
x_6 = -x_2 & A(-x_2) & = & a_0 & + & a_2(x_2)^2 & + & a_4(x_2)^4 & + & a_6(x_2)^6 & - & a_1x_2 & - & a_3(x_2)^3 & - & a_5(x_2)^5 & - a_7(x_2)^7 \\
x_7 = -x_3 & A(-x_3) & = & a_0 & + & a_2(x_3)^2 & + & a_4(x_3)^4 & + & a_6(x_3)^6 & - & a_1x_3 & - & a_3(x_3)^3 & - & a_5(x_3)^5 & - a_7(x_3)^7
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

$$\begin{array}{llllll}
 & A(x_0) & = & (a_0 + a_2(x_0)^2 + a_4(x_0)^4 + a_6(x_0)^6) & + & x_0\,(a_1 + a_3(x_0)^2 + a_5(x_0)^4 + a_7(x_0)^6) \\
 & A(x_1) & = & (a_0 + a_2(x_1)^2 + a_4(x_1)^4 + a_6(x_1)^6) & + & x_1\,(a_1 + a_3(x_1)^2 + a_5(x_1)^4 + a_7(x_1)^6) \\
 & A(x_2) & = & (a_0 + a_2(x_2)^2 + a_4(x_2)^4 + a_6(x_2)^6) & + & x_2\,(a_1 + a_3(x_2)^2 + a_5(x_2)^4 + a_7(x_2)^6) \\
 & A(x_3) & = & (a_0 + a_2(x_3)^2 + a_4(x_3)^4 + a_6(x_3)^6) & + & x_3\,(a_1 + a_3(x_3)^2 + a_5(x_3)^4 + a_7(x_3)^6) \\
\hline
x_4 = -x_0 & A(-x_0) & = & (a_0 + a_2(x_0)^2 + a_4(x_0)^4 + a_6(x_0)^6) & - & x_0\,(a_1 + a_3(x_0)^2 + a_5(x_0)^4 + a_7(x_0)^6) \\
x_5 = -x_1 & A(-x_1) & = & (a_0 + a_2(x_1)^2 + a_4(x_1)^4 + a_6(x_1)^6) & - & x_1\,(a_1 + a_3(x_1)^2 + a_5(x_1)^4 + a_7(x_1)^6) \\
x_6 = -x_2 & A(-x_2) & = & (a_0 + a_2(x_2)^2 + a_4(x_2)^4 + a_6(x_2)^6) & - & x_2\,(a_1 + a_3(x_2)^2 + a_5(x_2)^4 + a_7(x_2)^6) \\
x_7 = -x_3 & A(-x_3) & = & (a_0 + a_2(x_3)^2 + a_4(x_3)^4 + a_6(x_3)^6) & - & x_3\,(a_1 + a_3(x_3)^2 + a_5(x_3)^4 + a_7(x_3)^6)
\end{array}$$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

	$A(x_0)$	$= (a_0 + a_2(x_0^2) + a_4(x_0^2)^2 + a_6(x_0^2)^3) + x_0 (a_1 + a_3(x_0^2) + a_5(x_0^2)^2 + a_7(x_0^2)^3)$
	$A(x_1)$	$= (a_0 + a_2(x_1^2) + a_4(x_1^2)^2 + a_6(x_1^2)^3) + x_1 (a_1 + a_3(x_1^2) + a_5(x_1^2)^2 + a_7(x_1^2)^3)$
	$A(x_2)$	$= (a_0 + a_2(x_2^2) + a_4(x_2^2)^2 + a_6(x_2^2)^3) + x_2 (a_1 + a_3(x_2^2) + a_5(x_2^2)^2 + a_7(x_2^2)^3)$
	$A(x_3)$	$= (a_0 + a_2(x_3^2) + a_4(x_3^2)^2 + a_6(x_3^2)^3) + x_3 (a_1 + a_3(x_3^2) + a_5(x_3^2)^2 + a_7(x_3^2)^3)$
$x_4 = -x_0$	$A(-x_0)$	$= (a_0 + a_2(x_0^2) + a_4(x_0^2)^2 + a_6(x_0^2)^3) - x_0 (a_1 + a_3(x_0^2) + a_5(x_0^2)^2 + a_7(x_0^2)^3)$
$x_5 = -x_1$	$A(-x_1)$	$= (a_0 + a_2(x_1^2) + a_4(x_1^2)^2 + a_6(x_1^2)^3) - x_1 (a_1 + a_3(x_1^2) + a_5(x_1^2)^2 + a_7(x_1^2)^3)$
$x_6 = -x_2$	$A(-x_2)$	$= (a_0 + a_2(x_2^2) + a_4(x_2^2)^2 + a_6(x_2^2)^3) - x_2 (a_1 + a_3(x_2^2) + a_5(x_2^2)^2 + a_7(x_2^2)^3)$
$x_7 = -x_3$	$A(-x_3)$	$= (a_0 + a_2(x_3^2) + a_4(x_3^2)^2 + a_6(x_3^2)^3) - x_3 (a_1 + a_3(x_3^2) + a_5(x_3^2)^2 + a_7(x_3^2)^3)$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

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$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0 A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1 A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2 A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3 A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0 A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1 A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2 A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3 A_{odd}(x_3^2)$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7$$

$$A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$$

$$A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0 A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1 A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2 A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3 A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0 A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1 A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2 A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3 A_{odd}(x_3^2)$

STRATEGY: Set $x_{4+i} = -x_i$ for $0 \le i < 4$

We save roughly half the work.

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1 x$$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$
$x_1 = -x_0$	$A(x_1)$	$=$	a_0	$+$	$a_1 x_1$

STRATEGY: Set $x_{1+i} = -x_i$ for $0 \le i < 1$

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$
$x_1 = -x_0$	$A(-x_0)$	$=$	a_0	$-$	$a_1 x_0$

STRATEGY: Set $x_{1+i} = -x_i$ for $0 \le i < 1$

Given a Polynomial of Degree Bound 2
Find 2 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x}$

$x_0 = 1$	$A(x_0)$	$=$	a_0	$+$	a_1
$x_1 = -1$	$A(x_1)$	$=$	a_0	$-$	a_1

STRATEGY: We will evaluate any polynomial of degree bound 2 at
$x_0 = 1$
$x_1 = -1$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$	$+$	$a_2 (x_0)^2$	$+$	$a_3 (x_0)^3$
	$A(x_1)$	$=$	a_0	$+$	$a_1 x_1$	$+$	$a_2 (x_1)^2$	$+$	$a_3 (x_1)^3$
$x_2 = -x_0$	$A(x_2)$	$=$	a_0	$+$	$a_1 x_2$	$+$	$a_2 (x_2)^2$	$+$	$a_3 (x_2)^3$
$x_3 = -x_1$	$A(x_3)$	$=$	a_0	$+$	$a_1 x_3$	$+$	$a_2 (x_3)^2$	$+$	$a_3 (x_3)^3$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

	$A(x_0)$	$=$	a_0	$+$	$a_1 x_0$	$+$	$a_2 (x_0)^2$	$+$	$a_3 (x_0)^3$
	$A(x_1)$	$=$	a_0	$+$	$a_1 x_1$	$+$	$a_2 (x_1)^2$	$+$	$a_3 (x_1)^3$
$x_2 = -x_0$	$A(-x_0)$	$=$	a_0	$-$	$a_1 x_0$	$+$	$a_2 (x_0)^2$	$-$	$a_3 (x_0)^3$
$x_3 = -x_1$	$A(-x_1)$	$=$	a_0	$-$	$a_1 x_1$	$+$	$a_2 (x_1)^2$	$-$	$a_3 (x_1)^3$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

$$
\begin{array}{c|ccccccc}
 & A(x_0) & = & a_0 & + \; a_2(x_0)^2 & + & a_1 x_0 & + \; a_3(x_0)^3 \\
 & A(x_1) & = & a_0 & + \; a_2(x_1)^2 & + & a_1 x_1 & + \; a_3(x_1)^3 \\
\hline
x_2 = -x_0 & A(-x_0) & = & a_0 & + \; a_2(x_0)^2 & - & a_1 x_0 & - \; a_3(x_0)^3 \\
x_3 = -x_1 & A(-x_1) & = & a_0 & + \; a_2(x_1)^2 & - & a_1 x_1 & - \; a_3(x_1)^3
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

$$
\begin{array}{c|ccl}
 & A(x_0) & = & (\; a_0 + a_2(x_0)^2 \;) + x_0 (\; a_1 + a_3(x_0)^2 \;) \\
 & A(x_1) & = & (\; a_0 + a_2(x_1)^2 \;) + x_1 (\; a_1 + a_3(x_1)^2 \;) \\
\hline
x_2 = -x_0 & A(-x_0) & = & (\; a_0 + a_2(x_0)^2 \;) - x_0 (\; a_1 + a_3(x_0)^2 \;) \\
x_3 = -x_1 & A(-x_1) & = & (\; a_0 + a_2(x_1)^2 \;) - x_1 (\; a_1 + a_3(x_1)^2 \;)
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

$$
\begin{array}{c|ccl}
 & A(x_0) & = & (\; a_0 + a_2(x_0^2) \;) + x_0 (\; a_1 + a_3(x_0^2) \;) \\
 & A(x_1) & = & (\; a_0 + a_2(x_1^2) \;) + x_1 (\; a_1 + a_3(x_1^2) \;) \\
\hline
x_2 = -x_0 & A(-x_0) & = & (\; a_0 + a_2(x_0^2) \;) - x_0 (\; a_1 + a_3(x_0^2) \;) \\
x_3 = -x_1 & A(-x_1) & = & (\; a_0 + a_2(x_1^2) \;) - x_1 (\; a_1 + a_3(x_1^2) \;)
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$\boxed{A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3}$

$A_{even}(x) = a_0 + a_2 x$

$A_{odd}(x) = a_1 + a_3 x$

$$
\begin{array}{c|ccccc}
 & A(x_0) & = & A_{even}(x_0^2) & + & x_0 \, A_{odd}(x_0^2) \\
 & A(x_1) & = & A_{even}(x_1^2) & + & x_1 \, A_{odd}(x_1^2) \\
\hline
x_2 = -x_0 & A(-x_0) & = & A_{even}(x_0^2) & - & x_0 \, A_{odd}(x_0^2) \\
x_3 = -x_1 & A(-x_1) & = & A_{even}(x_1^2) & - & x_1 \, A_{odd}(x_1^2)
\end{array}
$$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$A_{even}(x) = a_0 + a_2x$

$A_{odd}(x) = a_1 + a_3x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$

STRATEGY: Set $x_{2+i} = -x_i$ for $0 \le i < 2$

Given a Polynomial of Degree Bound 4
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$A_{odd}(x) = a_1 + a_3x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$

Observe that we evaluate both $A_{even}(x)$ and $A_{odd}(x)$ at $x = x_0^2$ and $x = x_1^2$.

But we decided to always evaluate polynomials of degree bound 2 at $x = 1$ and $x = -1$.

So, $x_0^2 = 1 \Rightarrow x_0 = 1$ and $x_1^2 = -1 \Rightarrow x_1 = \sqrt{-1} = i$.

Given a Polynomial of Degree Bound 4
Find 4 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

$A_{even}(x) = a_0 + a_2x$

$A_{odd}(x) = a_1 + a_3x$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
$x_2 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_3 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$

So, we evaluate any polynomial of degree bound 4 at
$x_0 = 1,\ x_1 = i$
and
$x_2 = -x_0 = -1,\ x_3 = -x_1 = -i$

Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$

$A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$

$A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2\,A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3\,A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2\,A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3\,A_{odd}(x_3^2)$

Observe that we evaluate both $A_{even}(x)$ and $A_{odd}(x)$ at $x = x_0^2$, $x = x_1^2$, $x = x_2^2$ and $x = x_3^2$.

But we decided to always evaluate polynomials of degree bound 4 at $x = 1$, $x = i$, $x = -1$ and $x = -i$.

So, $x_0^2 = 1 \Rightarrow x_0 = 1$, $x_1^2 = i \Rightarrow x_1 = \frac{1+i}{\sqrt{2}}$, $x_2^2 = -1 \Rightarrow x_2 = i$, and $x_3^2 = -i \Rightarrow x_3 = \frac{-1+i}{\sqrt{2}}$.

Given a Polynomial of Degree Bound 8
Find 8 Distinct Points to Efficiently Evaluate it at

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7$$

$$A_{even}(x) = a_0 + a_2x + a_4x^2 + a_6x^3$$
$$A_{odd}(x) = a_1 + a_3x + a_5x^2 + a_7x^3$$

	$A(x_0)$	$=$	$A_{even}(x_0^2)$	$+$	$x_0\,A_{odd}(x_0^2)$
	$A(x_1)$	$=$	$A_{even}(x_1^2)$	$+$	$x_1\,A_{odd}(x_1^2)$
	$A(x_2)$	$=$	$A_{even}(x_2^2)$	$+$	$x_2\,A_{odd}(x_2^2)$
	$A(x_3)$	$=$	$A_{even}(x_3^2)$	$+$	$x_3\,A_{odd}(x_3^2)$
$x_4 = -x_0$	$A(-x_0)$	$=$	$A_{even}(x_0^2)$	$-$	$x_0\,A_{odd}(x_0^2)$
$x_5 = -x_1$	$A(-x_1)$	$=$	$A_{even}(x_1^2)$	$-$	$x_1\,A_{odd}(x_1^2)$
$x_6 = -x_2$	$A(-x_2)$	$=$	$A_{even}(x_2^2)$	$-$	$x_2\,A_{odd}(x_2^2)$
$x_7 = -x_3$	$A(-x_3)$	$=$	$A_{even}(x_3^2)$	$-$	$x_3\,A_{odd}(x_3^2)$

So, we evaluate any polynomial of degree bound 8 at

$$x_0 = 1,\ x_1 = \frac{1+i}{\sqrt{2}},\ x_2 = i,\ x_3 = \frac{-1+i}{\sqrt{2}}$$

and

$$x_4 = -x_0 = -1,\ x_5 = -x_1 = -\frac{1+i}{\sqrt{2}},\ x_6 = -x_2 = -i,\ x_7 = -x_3 = -\frac{-1+i}{\sqrt{2}}$$

Given a Polynomial of Degree Bound $n = 2^k$
Find $n = 2^k$ Distinct Points to Efficiently Evaluate it at

degree bound	how did we find the points to evaluate the polynomial at?	the points	point property
2^1	… … …	$1,\ -1$	all 2nd roots of unity
2^2	take positive and negative square roots of points used for degree bound 2^1 which are already the 2nd roots of unity	$1,\ i,\ -1,\ -i$	all 4th roots of unity
2^3	take positive and negative square roots of points used for degree bound 2^2 which are already the 4th roots of unity	$1,\ \frac{1+i}{\sqrt{2}},\ i,\ \frac{-1+i}{\sqrt{2}},\ -1,\ -\frac{1+i}{\sqrt{2}},\ -i,\ -\frac{-1+i}{\sqrt{2}}$	all 8th roots of unity
2^4	take positive and negative square roots of points used for degree bound 2^3 which are already the 8th roots of unity	$1,\ \frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2},\ \ldots,\ \ldots,\ -1,\ -\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}\right),\ \ldots,\ \ldots$	all 16th roots of unity
… … …	… … …	… … …	… … …
2^{k-1}	take positive and negative square roots of points used for degree bound 2^{k-2} which are already the 2^{k-2}th roots of unity	… … …	all 2^{k-1}th roots of unity
$n = 2^k$	take positive and negative square roots of points used for degree bound 2^{k-1} which are already the 2^{k-1}th roots of unity	… … …	all 2^kth roots of unity (i.e., n^{th} roots of unity)

How to Find all n^{th} Roots of Unity

The n^{th} roots of unity are: $1,\ \omega_n, (\omega_n)^2, (\omega_n)^3, \ldots\ldots\ldots, (\omega_n)^{n-1}$,

where $\omega_n = \cos\frac{2\Pi}{n} + i\sin\frac{2\Pi}{n} = e^{\frac{2\Pi i}{n}}$ is known as the primitive n^{th} roots of unity.

The result above can be derived using Euler's Formula.

Euler's Formula: For any real number α, $\cos\alpha + i\sin\alpha = e^{i\alpha}$

Euler's formula follows very easily from the following three power series each of which holds for $-\infty < \alpha < +\infty$:

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \frac{\alpha^8}{8!} - \cdots$$

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \frac{\alpha^7}{7!} + \frac{\alpha^9}{9!} - \cdots$$

$$e^\alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + \frac{\alpha^5}{5!} + \frac{\alpha^6}{6!} + \frac{\alpha^7}{7!} + \frac{\alpha^8}{8!} + \cdots$$

How to Find all n^{th} Roots of Unity

Observe that for (any) real numbers α and p,

$$(\cos\alpha + i\sin\alpha)^p = \left(e^{i\alpha}\right)^p = e^{i(p\alpha)} = \cos(p\alpha) + i\sin(p\alpha)$$

Also observe that for any integer k, $\cos(k\times 2\Pi) + i\sin(k\times 2\Pi) = 1 + i\times 0 = 1$

Then the n^{th} root of 1 (unity) is $= 1^{\frac{1}{n}} = (\cos(k\times 2\Pi) + i\sin(k\times 2\Pi))^{\frac{1}{n}} = \cos\left(k\times\frac{2\Pi}{n}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right)$

Observe that $\cos\left(k\times\frac{2\Pi}{n}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right)$ takes n distinct values for $0 \le k < n$, and then simply repeats those values for $k < 0$ and $k \ge n$.

When $k = 1$, we have $\cos\left(k\times\frac{2\Pi}{n}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = \cos\left(\frac{2\Pi}{n}\right) + i\sin\left(\frac{2\Pi}{n}\right) = \omega_n =$ primitive n^{th} root of 1.

Clearly, for any k, $\cos\left(k\times\frac{2\Pi}{n}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = \left(\cos\left(\frac{2\Pi}{n}\right) + i\sin\left(\frac{2\Pi}{n}\right)\right)^k = (\omega_n)^k$

Hence, $1^{\frac{1}{n}} = \cos\left(k\times\frac{2\Pi}{n}\right) + i\sin\left(k\times\frac{2\Pi}{n}\right) = (\omega_n)^k$, for $k = 0, 1, 2, \ldots, n-1$.

In other words, the n^{th} roots of 1 (unity) are: $1,\ \omega_n, (\omega_n)^2, (\omega_n)^3, \ldots\ldots\ldots, (\omega_n)^{n-1}$

Coefficient Form ⇒ Point-Value Form

```
Rec-FFT ( ( a_0, a_1, ..., a_{n-1} ) )    { n = 2^k for integer k ≥ 0 }
if n = 1 then
    return ( a_0 )
ω_n ← e^{2πi/n}
ω ← 1
y^even ← Rec-FFT ( ( a_0, a_2, ..., a_{n-2} ) )
y^odd ← Rec-FFT ( ( a_1, a_3, ..., a_{n-1} ) )
for j ← 0 to n/2 − 1 do
    y_j ← y_j^even + ω y_j^odd
    y_{n/2+j} ← y_j^even − ω y_j^odd
    ω ← ω ω_n
return y
```

Running time:

$$T(n) = \begin{cases} \Theta(1), & if\ n = 1, \\ 2T\left(\frac{n}{2}\right) + \Theta(n), & otherwise. \end{cases}$$

$$= \Theta(n \log n)$$