9.1 Motivation for Akra-Bazzi Recurrence

9.1.1 Deterministic Select

The following recurrence is for the worst case running time of the deterministic selection algorithm :

$$T(n) = \begin{cases} \Theta(1) & if n < 140 \\ T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10} + 6) + \Theta(n) & if n \geq 140 \end{cases}$$

If we drop the ceiling for simplicity and observe that $\frac{7n}{10} + 6 \leq \frac{8n}{10}$ $\forall n \geq 60$, we obtain the following upper bound :

$$T'(n) = \begin{cases} \Theta(1) : & if n < 140 \\ T'(\frac{n}{5}) + T'(\frac{4n}{5}) + \Theta(n) & if n \geq 140 \end{cases}$$

But, by using $\frac{8n}{10}$, we might be overestimating, so instead we will select $\frac{7.5n}{10}$. Using $\frac{7.5n}{10}(= \frac{3n}{4})$ for all $n \geq 120$, we obtain the following upper bound.

$$T''(n) = \begin{cases} \Theta(1) : & if n < 140 \\ T''(\frac{n}{5}) + T''(\frac{3n}{4}) + \Theta(n) & if n \geq 140 \end{cases}$$

9.1.2 Master theorem

$$T(n) = \begin{cases} \Theta(1) : & if n \leq 1 \\ aT(\frac{n}{b}) + f(n), & otherwise (a \geq 1, b \geq 1) \end{cases}$$

Let's write the recurrence relation in different form as below :

$$T(n) = n^{\log_b a} + \sum_{j=0}^{\log_b n - 1} a^j f(\frac{n}{b^j})$$

now assume $p = \log_b a$ and $n_j = \frac{n}{b^j}$, then

$$T(n) = n^p + \sum_{j=0}^{\log_b n - 1} a^j f(n_j)$$

Since, $p = \log_b a \Rightarrow a = b^p \Rightarrow a^j = (b^j)^p = (\frac{n}{n_j})^p$
Hence,

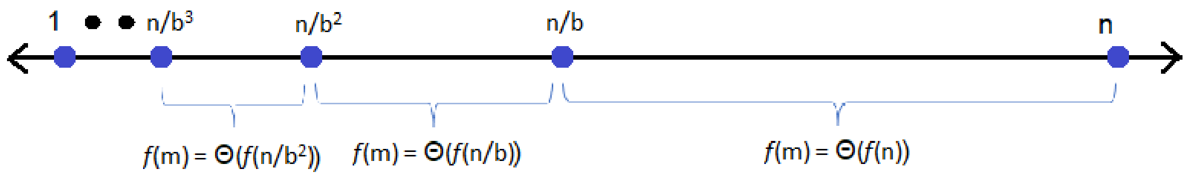

Figure 9.1.1: Shows $f(m)$ is in constant factor of $f(n)$

$$\sum_{j=0}^{\log_b n-1} a^j f(\frac{n}{b^j}) = \sum_{j=0}^{\log_b n-1} (\frac{n}{n_j})^p f(n_j)$$

The above equation considers discrete points, viz, n, $\frac{n}{b}$, $\frac{n}{b^2}$ and so on, but if we want to take more points on the real line into account, we have to modify the equation. Let m be the set of discrete points $(n, \frac{n}{b}, \frac{n}{b^2}....)$, then

$$\sum_{j=0}^{\log_b n-1} (\frac{n}{n_j})^p f(n_j) = n^p \sum_{m\in(n,\frac{n}{b},\frac{n}{b^2}....)} (\frac{f(m)}{m^p})$$

$$n^p \sum_{m\in(n,\frac{n}{b},\frac{n}{b^2}....)} (\frac{f(m)}{m^p}) \text{ becomes } \Theta(n^p \sum_{m=1}^{n} \frac{f(m)}{m^{p+1}})$$

$$\text{Hence, } T(n) = n^p + n^p \sum_{m=0}^{n} \frac{f(m)}{m^{p+1}}$$

$$\text{provided, } c_1 f(n) <= f(m) <= c_2 f(n)$$

$$\text{and } f(n) = n^\alpha \log^p n$$

9.1.3 Why do we need general form

With Master theorem, we can represent recurrences within polylog factor for n as shown in figure 10.1.1, Other points are not covered. Hence, we need general form for covering all the points.

$$T(n) = a_1 T(\frac{n}{b_1}) + a_2 T(\frac{n}{b_2}) + a_3 T(\frac{n}{b_3})$$

Assume that $T(n) = n^p$, then

$$\begin{aligned} T(n) &= a_1 T(\frac{n}{b_1})^p + a_2 T(\frac{n}{b_2})^p + a_3 T(\frac{n}{b_3})^p \\ &= [\frac{a_1}{b_1{}^p} + \frac{a_2}{b_2{}^p} + \frac{a_3}{b_3{}^p}].n^p \end{aligned}$$

As per our initial assumption($T(n) = n^p$) the value of $[\frac{a_1}{b_1{}^p} + \frac{a_2}{b_2{}^p} + \frac{a_3}{b_3{}^p}]$ should be 1. We can find value of p by solving $[\frac{a_1}{b_1{}^p} + \frac{a_2}{b_2{}^p} + \frac{a_3}{b_3{}^p}] = 1$.

9.2 General form: Akra-bazzi recurrence

$$T(x) = \begin{cases} \Theta(1) & if 1 \leq x \leq x_0 \\ \sum_{i=i}^{k} a_i T(b_i x) + g(x), & if x > x_0 \end{cases}$$

where,

1. $k \geq 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \leq i \leq k$
3. $b_i \in$ (0,1) is a constant for $1 \leq i \leq k$
4. $x \geq 1$ is a real number
5. $x_0 \geq max\{\frac{1}{b_i}, \frac{1}{1-b_i}\}$
6. g(x) is a non-negative function that satisfies a polynomial-growth condition.

If can expand above recurrence as follows,
$T(x) = a_1 T(b_1 x) + a_2 T(b_2 x) + a_3 T(b_3 x) + ... + a_k T(b_k x) + g(x)$

9.2.1 Polynomial-Growth condition

We say that $g(x)$ satisfies the polynomial-growth condition if there exist positive constants c_1 and c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x, x]$

$$c_1 g(x) \leq g(u) \leq c_2 g(x)$$

$$i.e. \quad g(u) = \Theta(g(x))$$

9.2.2 The Akra-Bazzi Solution

Consider the recurrence given in 10.2

$$T(x) = \begin{cases} \Theta(1) & if 1 \leq x \leq x_0 \\ \sum_{i=i}^{k} a_i T(b_i x) + g(x), & if x > x_0 \end{cases}$$

Let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$ Then,

$$T(x) = \Theta(x^p(1 + \int_1^x \frac{g(u)}{u^p + 1} du))$$

9.3 Akra-Bazzi Solution

9.3.1 Deterministic Select

For following recurrence,

$$T'(n) = \begin{cases} \Theta(1) : & if n < 140 \\ T'(\frac{n}{5}) + T'(\frac{4n}{5}) + \Theta(n) & if n \geq 140 \end{cases}$$

$$\text{a}_1 = 1, b_1 = \frac{1}{5}, a_2 = 1, b_2 = \frac{4}{5}$$
$$\text{a}_1 b_1^p + a_2 b_2^p = 1$$

Putting values of a$_1, b_1, a_2, b_2$

$$1 * (\frac{1}{5})^p + 1 * (\frac{3}{4})^p = 1$$

$$(\frac{1}{5})^p + (\frac{4}{5})^p = 1$$

$$\text{p = 1}$$

putting value of p in Akra-Bazzi Solution,

$$T(n) = \Theta(n^1(1 + \int_1^n \frac{u}{u^2} du))$$

$$T(n) = \Theta(n(1 + \int_1^n \frac{1}{u} du))$$

$$T(n) = \Theta(n(1 + [\ln u]_1^n))$$

$$T(n) = \Theta(n(1 + (\ln n - \ln 1))$$

$$T(n) = \Theta(n \ln n)$$

9.3.2 Deterministic Select - 2

For following recurrence,

$$T'(n) = \begin{cases} \Theta(1) & if n < 140 \\ T'(\frac{n}{5}) + T'(\frac{3n}{4}) + \Theta(n) & if n \geq 140 \end{cases}$$

$$a_1 = 1, b_1 = \frac{1}{5}, a_2 = 1, b_2 = \frac{3}{4}$$

Putting values of a_1, b_1, a_2, b_2

$$1 * (\frac{1}{5})^p + 1 * (\frac{3}{4})^p = 1$$

$$p < 1$$

putting value of p in Akra-Bazzi Solution,

$$T''(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du))$$

$$T(n) = \Theta(n(1 + \int_1^n \frac{1}{u^{p+1}} du))$$

$$T(n) = \Theta(n(1 + [u^{\frac{-p+1)}{-p1+1}}]_1^n))$$

$$T(n) = \Theta((\frac{1}{1-p})^n - (\frac{p}{1-p})^n n^p)$$

$$T(n) = \Theta(n)$$

9.3.3 Examples of Akra-Bazzi recurrence

1. $T(x) = 2T(\frac{x}{4}) + 3T(\frac{3x}{6}) + \Theta(x \log x)$

$$a_1 = 2, b_1 = \frac{1}{4}, a_2 = 3, b_2 = \frac{1}{6}$$

Putting values of a_1, b_1, a_2, b_2

$$2 * (\frac{1}{2})^p + 3 * (\frac{1}{6})^p = 1$$

$$p = 1$$

putting value of p in Akra-Bazzi Solution,

$$T(x) = \Theta(x^1(1 + \int_1^x \frac{u \ln u}{u^2} du))$$

$$T(x) = \Theta(x(1 + \int_1^x \frac{\ln u}{u} du))$$

$$\int_1^u \frac{\ln u}{u} du = \ln u \int_1^u \frac{1}{u} du - \int_1^u \left(\frac{d}{du} \ln u\right) \int_1^u \frac{1}{u} du$$

$$2 \int_1^u \frac{\ln u}{u} du = (\ln u)^2$$

$$\int_1^u \frac{\ln u}{u} du = \frac{(\ln u)^2}{2}$$

$$T(x) = \Theta(x(1 + [(\ln u)^2]_1^x))$$

$$T(x) = \Theta(x(1 + [(\ln x)^2 - (\ln 1)^2]))$$

$$T(x) = \Theta(x \log^2 x)$$

Similarly, we can solve following recurrences,

2. $T(x) = 2T(\frac{x}{4}) + 3T(\frac{3x}{6}) + \Theta(x \log x)$

p = 2

$$T(x) = \Theta(x^2(1 + \int_1^x \frac{\frac{u^2}{\log u}}{u^3} du))$$

$$T(x) = \Theta(x^2 \log \log x)$$

3. $T(x) = T(\frac{x}{2}) + \Theta(\log x)$

p = 0

$$T(x) = \Theta(x^0(1 + \int_1^x \frac{\log u}{u} du))$$

$$T(x) = \Theta(log^2 x)$$

4. $T(x) = \frac{1}{2}T(\frac{x}{2}) + \Theta(\frac{1}{x})$

p = -1

$$T(x) = \Theta(\frac{1}{x}(1 + \int_1^x \frac{1}{u} du))$$

$$T(x) = \Theta(\frac{log x}{x})$$

9.4 Summary

1. Revisited Deterministic Select.
2. Revisited Master theorem.
3. Akra-Bazzi theorem is generalization of masters theorem.
4. We can map relation of p in Akra-Bazzi theorem and three cases in Master theorem as follows:
   - $p > 1$: Master theorem case 1 (might)
   - $p = 1$: Master theorem case 2
   - $p < 1$: Master theorem case 3