Data Structures + Graph Algorithms

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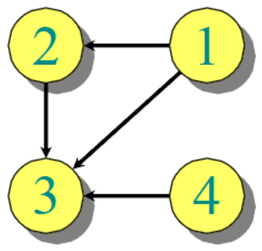
Quick Refresh of Graphs

- A Graph G = (V, E) is an ordered pair consisting of a set V of vertices (nodes) and a set of edges (links) $E \subseteq V \times V$. An edge set E consists of pair of vertices [the pair is unordered for undirected graphs and ordered for directed graphs]. In either case, |E| O(V^2); moreover if G is connected $|E| \geq |V| - 1$.
- **Adjacency matrix** of G = (V, E), where V = {1, 2, …, n} is given by

$$A[i, j] = \begin{cases} 1 \text{ if } (i, j) \in E \\ 0 \text{ if } (i, j) \notin E \end{cases} \quad \text{This takes } O(V^2) \text{ storage}$$

- **Adjacency-List representation**

An ***adjacency list*** of a vertex $v \in V$ is the list $Adj[v]$ of vertices adjacent to v.



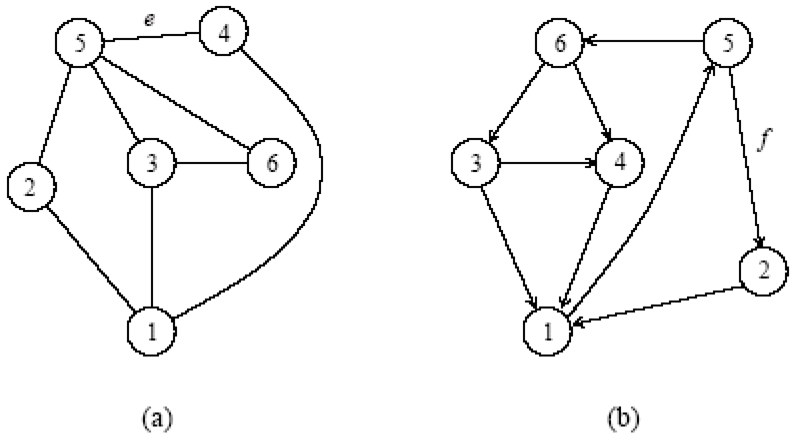
$Adj[1] = \{2, 3\}$
$Adj[2] = \{3\}$
$Adj[3] = \{\}$
$Adj[4] = \{3\}$

For undirected graphs, $|Adj[v]| = degree(v)$.
For digraphs, $|Adj[v]| = out\text{-}degree(v)$.

- Adjacency lists need O(|V| + |E|) storage – the so-called sparse representation vs. dense representation using adjacency matrix.
- When to use which one – depends on application

Definitions and Representation

- An undirected graph G is a pair (V, E), where V is a finite set of points called vertices and E is a finite set of edges.
- An edge e ∈ E is an unordered pair (u,v), where u,v ∈ V.
- In a directed graph, the edge e is an ordered pair (u,v). An edge (u,v) is incident from vertex u and is incident to vertex v.
- A path from a vertex v to a vertex u is a sequence $<v_0,v_1,v_2,\ldots,v_k>$ of vertices where $v_0 = v$, $v_k = u$, and $(v_i, v_{i+1}) \in E$ for I = 0, 1,…, k-1.
- The length of a path is defined as the number of edges in the path, for an weighted graph while sum of the weights of the edges in an weighted graph.
- n = |V| (number of vertices), m = |E| (number of edges), Edges can even have attributes (i.e. semantic graphs).



Data structures: graph representation

Static case

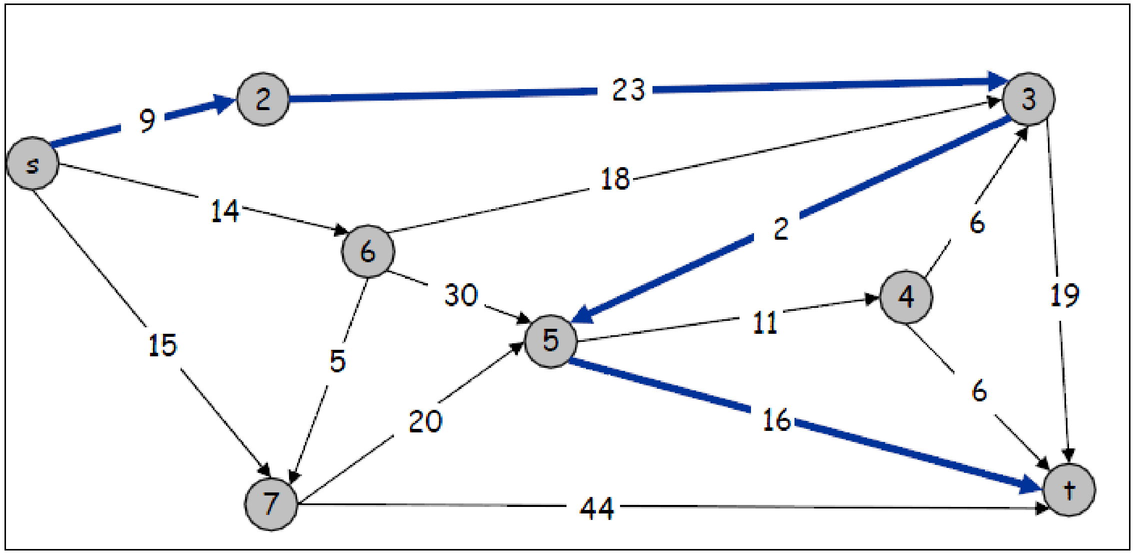
- Dense graphs ($m = O(n^2)$): adjacency matrix commonly used.
- Sparse graphs: adjacency lists

Dynamic

- representation depends on common-case query
- Edge insertions or deletions? Vertex insertions or deletions? Edge weight updates?
- Graph update rate
- Queries: connectivity, paths, flow, etc.
- Optimizing for locality a key design consideration.
- **Distributed Graph representation**
  - Each processor stores the entire graph (“full replication”)
  - Each processor stores n/p vertices and all adjacencies out of these vertices (“1D partitioning”)
  - How to create these “p” vertex partitions?
    - Graph partitioning algorithms: recursively optimize for conductance (edge cut/size of smaller partition)
    - Randomly shuffling the vertex identifiers ensures that edge count/processor are roughly the same

Shortest Path(s) in a Graph

- Assume a that each edge of a graph is associated with a weight function w: E → R^+ (only non-negative weights). For simplicity, assume that all edge weights are distinct (this is not necessary).
- Shortest Path Problem: Given a source node, say s, and a destination node, say t, find the shortest (cost of the path is the sum of the costs of all edges lying on the path) path from s to t.
- Example:

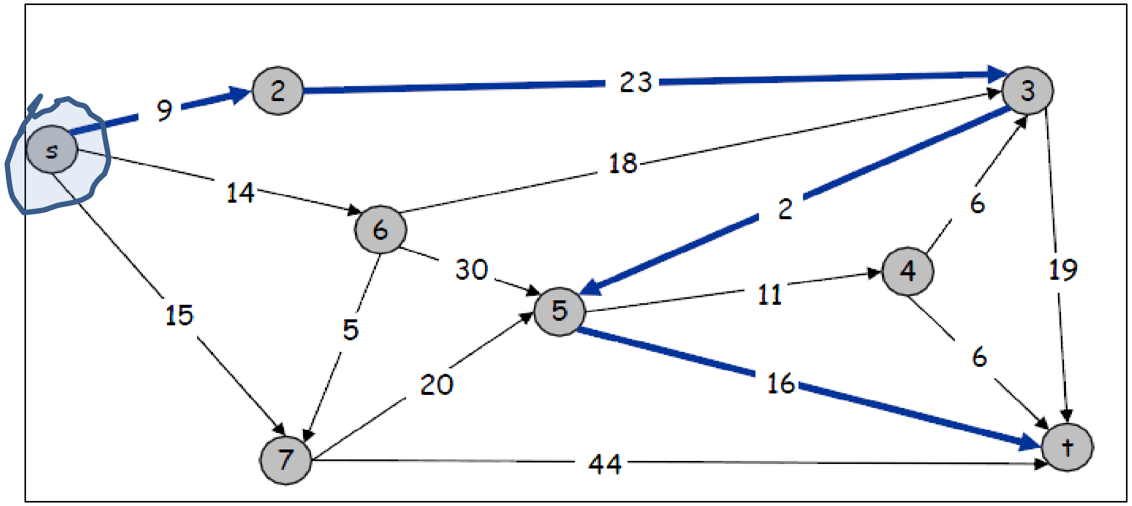


- Cost of the shortest path P from s to t = w(P) = 9 + 23 + 2 +16 = 50, where P = (s→2 →3 → 5 → t) .
- It is easy to see that there may exist more than one shortest paths.
- We are interested to find one such shortest path from s to t.

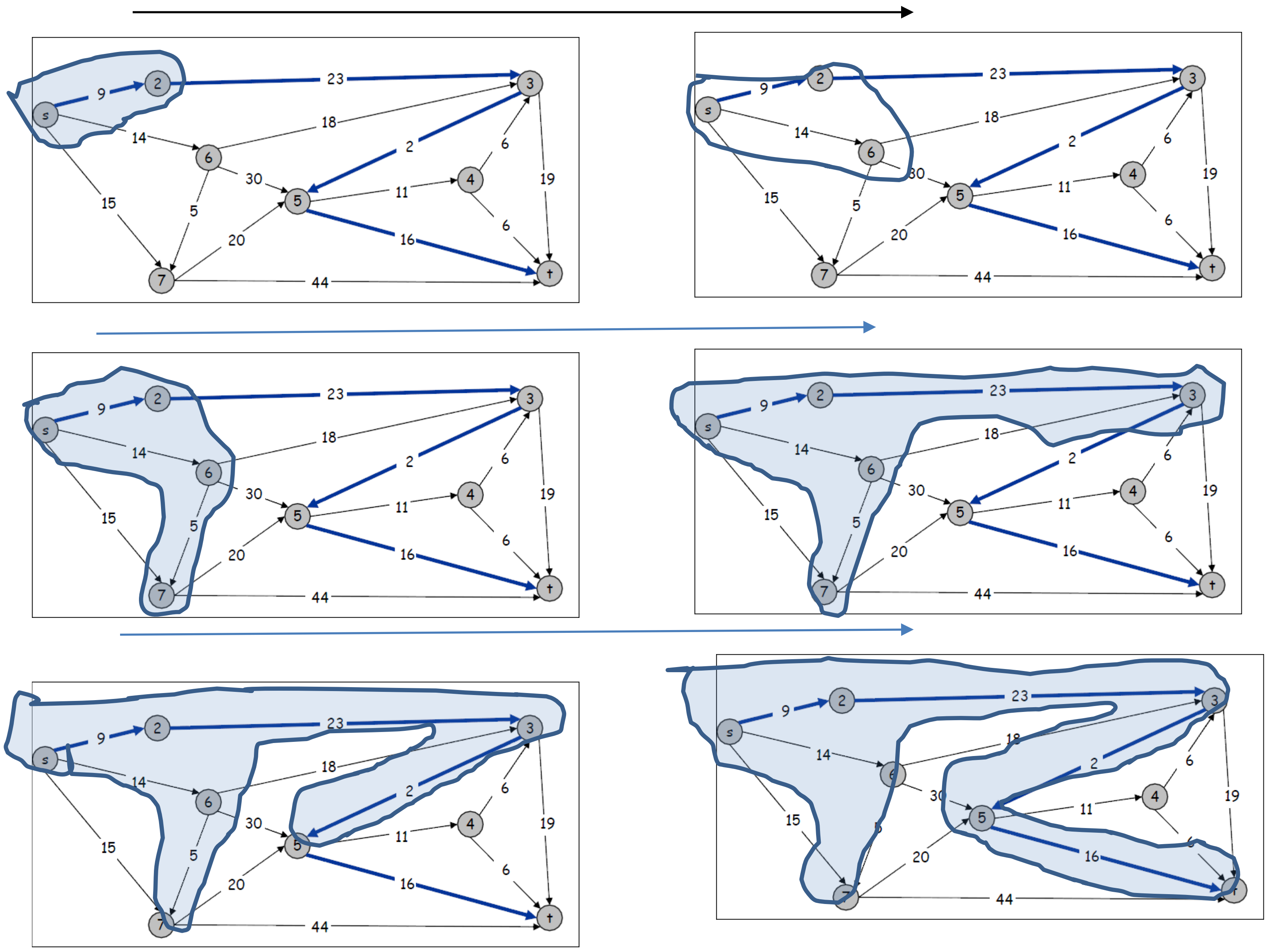
- **Approach**: We start from source s; we know the nodes directly reachable from s; none of them is the destination t in our example. Think of 2 things: (1) we have not seen t yet; the path from s to t must pass through one of the neighbors of s; (2) we sure need to explore the entire graph to compute the shortest path. [Note that the discussion applies equally well to both directed and undirected graphs with minimal changes].

Dijkstra's Algorithm

- The algorithm maintains a set *S* of vertices *u* for which we have determined a shortest-path distance *d(u)* from *s*; this is the “explored” part of the graph. Initially *S* = {*s*}, and *d(s)* = 0. Initialize S = { s }, d(s) = 0.
- while $S \neq V$
  - Select a node $v \notin S$ with at least one edge from *S;* compute **$[\pi(v) = \min_{e=(u,v):\, u \in S} d(u) + w(e)]$ and** set $S = S \cup \{v\}$ + w(e) and $d(v) = \pi(v)$. [w(e) is the weight of the edge (u, v)]
- To produce the s-u paths corresponding to the distances found, we simply record the edge (u, v) on which it achieved the value $\pi(v) = \min_{e=(u,v):\, u \in S} d(u) + w(e)$ [simple recursive backtracking will do]. Consider the previous graph; initially, the situation is as follows [the shaded area is S].

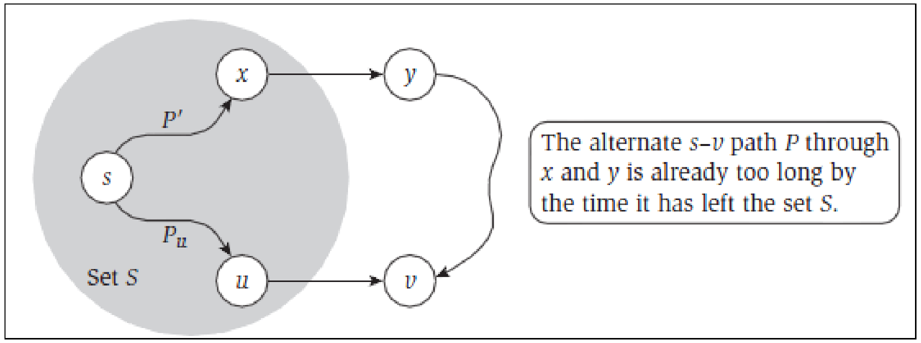


Dijkstra's Algorithm: Example Execution



Dijkstra's Algorithm: Proof of Correctness

- The Invariant is: For each node u ∈ S, d(u) is the length of the shortest s-u path [s is the given source node].
- We will prove by induction on |S|.
- Base case: Initially, S = {s}, or |S| = 1; the claim is trivially true.
- Inductive hypothesis: Assume true for |S| = k ≥ 1
  - Let v be next node added to S and let u-v be the chosen edge.
  - The shortest s-u path plus (u, v) is an s-v path of length **π(v)**.
  - Consider any s-v path P. We'll see that it's no shorter than π(v).
  - Let x-y be the first edge in P that leaves S, and let P' be the subpath to x; P is already too long as soon as it leaves S.


x
y
P′
s
P_u
Set S
u
v
The alternate s–v path P through x and y is already too long by the time it has left the set S.

$$w(P) \underset{\text{non-negative wts.}}{\geq} w(P') + w(x,y) \underset{\text{ind. hypothesis}}{\geq} d(x)+w(x,y) \underset{\text{defn. of } \pi}{\geq} \pi(y) \underset{\text{v is chosen over y}}{\geq} \pi(v).$$

Dijkstra's Algorithm: Implementation

- There are n − 1 iterations of the while loop for a graph with n nodes, as each iteration adds a new node v to S.
- We explicitly maintain the values of the minima $\pi(v) = \min_{e=(u,v):\, u \in S} d(u) + w(e)$ for each node $v \in V - S$; we maintain this information in a min-binary_heap [we have seen this data structure before]; select_min operation is done in O(log n) worst case time, n is the number of elements in the heap.
- Next node to explore = node with minimum $\pi(v)$.
- When exploring v, for each incident edge e = (v, w), update $\pi(w) = \min \{\pi(w), \pi(v)+w(e)\}$
- We get an implementation with O(m log n) worst case time.
- We can get a better amortized time of $O(m \log_{m/n} n)$ if we use a Fibonacci heap.

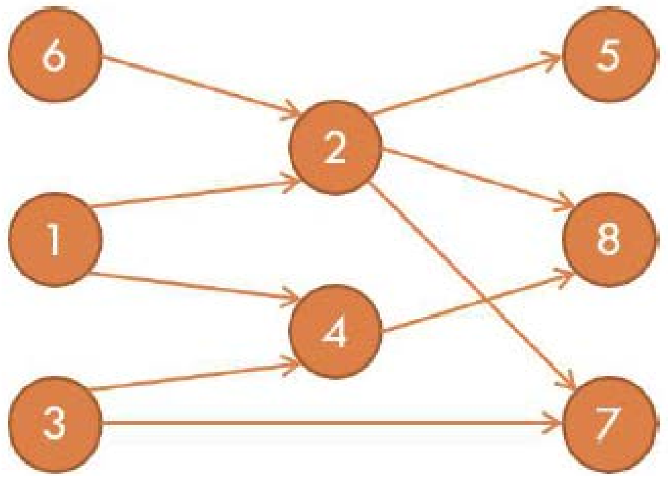
```
DIJKSTRA (V, E, s)
Create an empty priority queue.
FOR EACH v ≠ s : d(v) ← ∞; d(s) ← 0.
FOR EACH v ∈ V : insert v with key d(v) into priority queue.
WHILE (the priority queue is not empty)
    u ← delete-min from priority queue.
    FOR EACH edge (u, v) ∈ E leaving u:
        IF d(v) > d(u) + w(u, v)
        decrease-key of v to d(u) + w(u, v) in priority queue.
        d(v) ← d(u) + w(u, v).
```

Graph Traversal

- The most basic graph algorithm that visits nodes of a graph in certain order
- Used as a subroutine in many other algorithms
- Two Simple Traversals: (Assume symmetric or undirected graphs)
  - Depth-First Search (DFS): uses recursion (stack)
  - Breadth-First Search (BFS): uses queue
- **DFS(v):** visits all the nodes reachable from the given node v in depth-first order
  - Mark v as visited (how?)
  - For each edge v → u: If u is not visited, call DFS(u).
  - **Note**: non-recursive version replaces recursive calls with a stack (user maintained)
- **BFS(v):** visits all the nodes reachable from v in breadth-first order
  - Initialize a queue Q
  - Mark v as visited and push it to Q
  - While Q is not empty:
    - Take the front element of Q and call it w
    - For each edge w → u: If u is not visited, mark it as visited and push it to Q

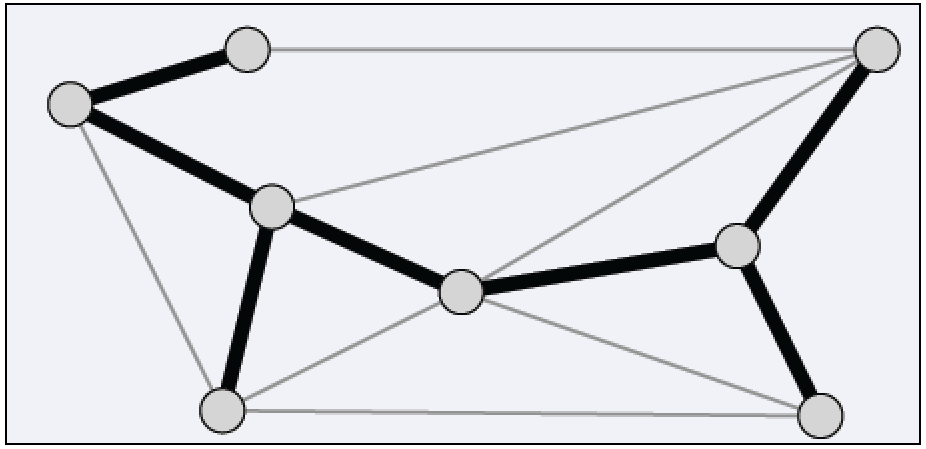
Topological Sort

- **Input**: a DAG G = (V, E)
- **Output**: an ordering of nodes such that for each edge u → v, u comes before v
- Explain Precedence Graph.
- **Note**: There can be multiple answers, e.g., both {6, 1, 3, 2, 7, 4, 5, 8} and {1, 6, 2, 3, 4, 5, 7, 8} (and others) are valid orderings for the graph below



Spanning Tree & Properties

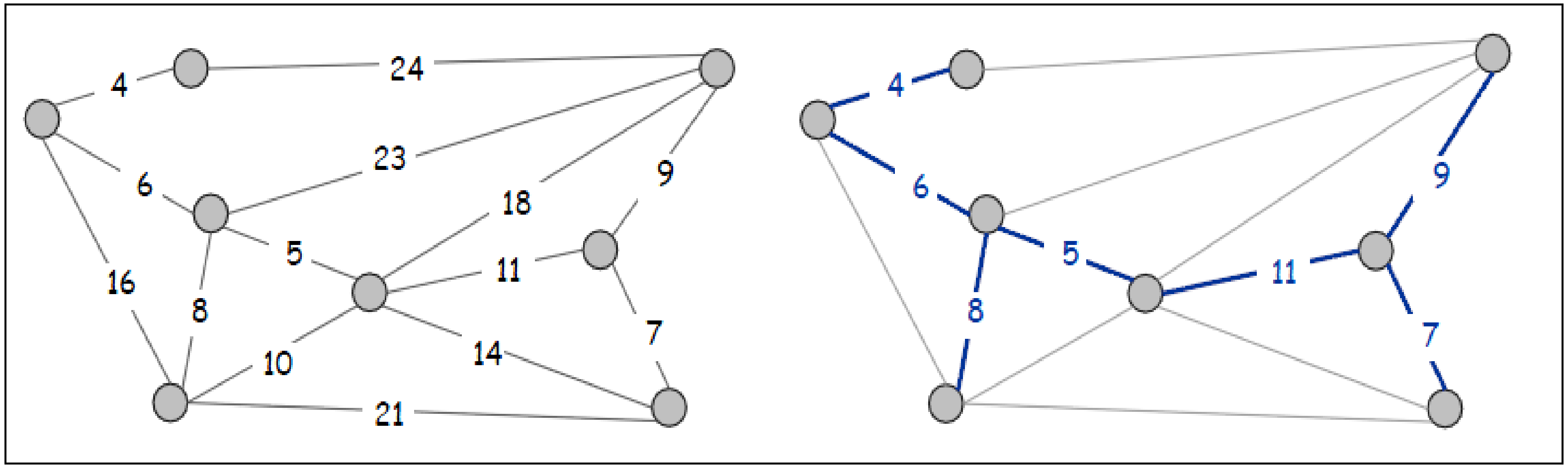
- Consider a connected, undirected graph G = (V, E); Let T = (V, F), F $\subseteq$ E, be a subgraph of G = (V, E). All the following statements must be true:
  - T is a spanning tree of G.
  - T is acyclic and connected.
  - T is connected and has n – 1 edges.
  - T is acyclic and has n – 1 edges.
  - T is minimally connected: removal of any edge disconnects it.
  - T is maximally acyclic: addition of any edge creates a cycle.
  - T has a unique simple path between every pair of nodes.



The black edges constitute the spanning tree. Satisfy yourself that all those statements are true [some are equivalent]

Minimal Spanning Tree

- A connected, undirected graph G = (V, E) with weight function w: E⟶ $\Re$ (set of reals); we'll assume all edge weights are distinct [Explain briefly the general case when duplicates are allowed]
- A connected acyclic subgraph T of G that spans all vertices is called a spanning tree of the graph. A minimal spanning tree (MST) is a spanning tree of minimum weight, i.e., a spanning tree whose sum of edge weights is minimized.
- Example:



- **Note:** Cayley's theorem. There are n^{n-2} spanning trees of K_n; thus, not possible to find MST by exhaustive enumeration.

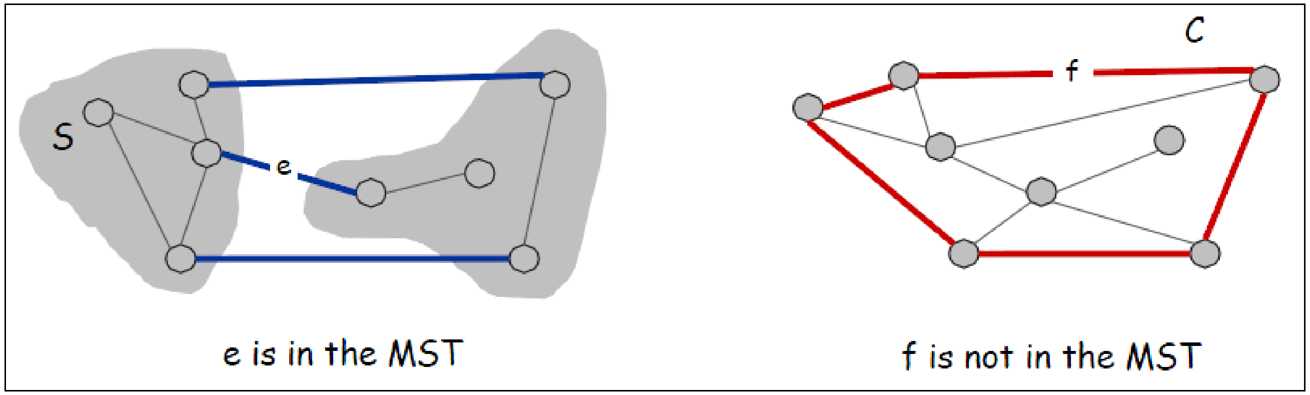
Applications

- MST is fundamental problem with diverse applications.
- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Greedy Algorithms for MST

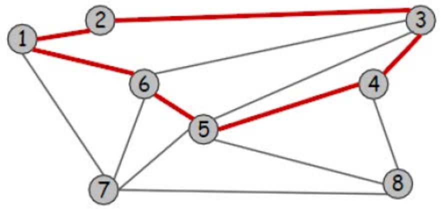
1. **Kruskal's algorithm**. Start with T = ϕ. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.
2. **Reverse-Delete algorithm**. Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.
3. **Prim's algorithm**. Start with some root node s and greedily grow a tree T from s outward. At each step, add the least weight edge e to T that has exactly one endpoint in T.

- We'll do 1 and 3. The greedy structure is the same; implementations are different, one uses the min-heaps and the other uses Union-Find data structure. Both works in O(m log n) time.
- **Cut property**. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST contains e [cheapest way to connect S and V – S].
- **Cycle property**. Let C be any cycle, and let f be the max cost edge belonging to C. Then the MST does not contain f. [Why? Assume f belongs to an MST T1 ⇒ deleting f breaks T1 in two subtrees with the two ends of f in different subtrees ⇒ remainder of C reconnects the subtrees, hence there is an edge e of C with ends in different subtrees, i.e., it reconnects the subtrees into a tree T2 with weight less than that of T1, because the weight of e is less than the weight of f.]



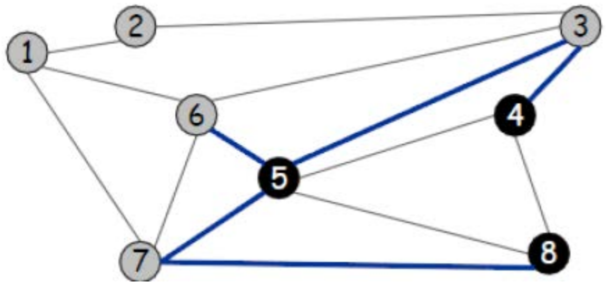
Cycles and Cuts

- **Cycle**: Consider an undirected graph G. A cycle is a simple (no node is visited more than once) path (sequence of edges) from any node back to itself.



Examples: (1) (1, 2, 3, 4, 5, 6, 1), (2) (1, 6, 7, 1), (3) (3, 4, 5, 6, 3), so on.

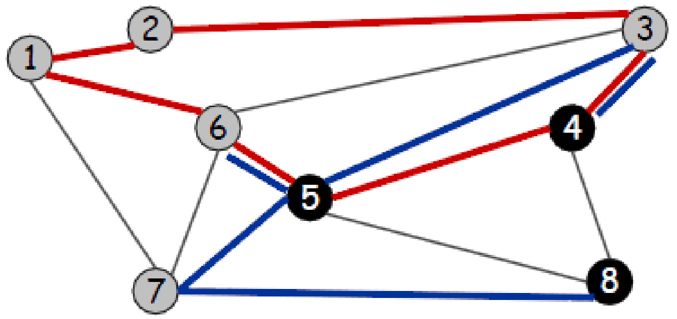
- A **cut** C = (S, T) is a partition of V of a graph G=(V, E) into two subsets S and T. The **cut-set** of a cut C=(S,T) is the set {(u,v) ϵ E | u ϵ S, v ϵ T} of edges that have one endpoint in S and the other endpoint in T.



Example: Cut S = {4, 5, 8}, T = V – S ={1, 2, 3, 6, 7}.
Cut-set = {5-6, 5-7, 3-4, 3-5, 7-8}

Cycle-Cut Intersection

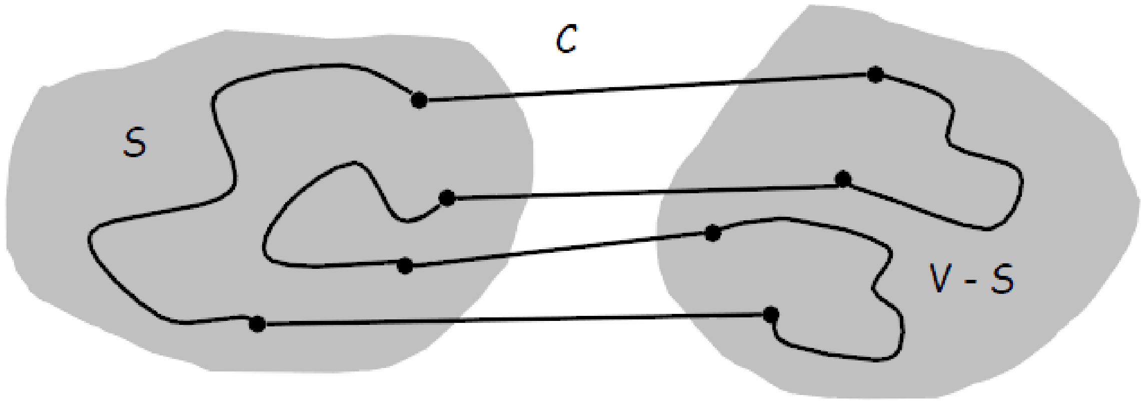
Claim. A cycle and a cutset intersect in an even number of edges.



Cycle C = 1-2, 2-3, 3-4, 4-5, 5-6, 6-1
Cutset D = 3-4, 3-5, 5-6, 5-7, 7-8
Intersection = 3-4, 5-6

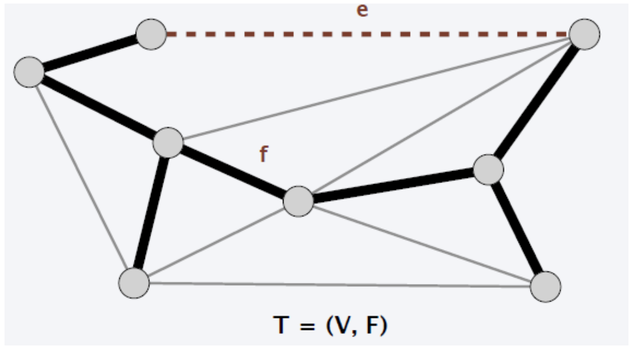
y

Pf. (by picture)



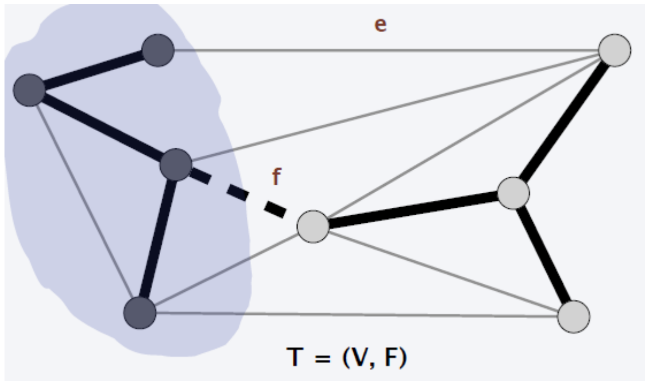
Fundamental Cycle and Fundamental Cutset

- Fundamental cycle.
  - Adding any non-tree edge e to a spanning tree T forms unique cycle C.
  - Deleting any edge f ∈ C from T ∪ { e } results in new spanning tree.



Observe: If w(e) < w(f), then T is not an MST

- Fundamental cutset.
  - Deleting any tree edge f from a spanning tree T divide nodes into two connected components. Let D be cutset.
  - Adding any edge e ∈ D to T – { f } results in new spanning tree.



Observe: If w(e) < w(f), then T is not an MST

Prim's Sequential MST Algorithm

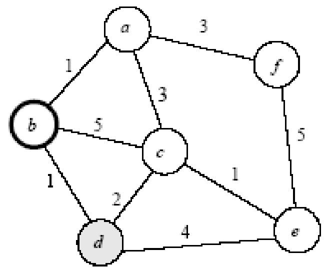
```
1.  procedure PRIM_MST($V, E, w, r$)
2.  begin
3.      $V_T := \{r\}$;
4.      $d[r] := 0$;
5.      for all $v \in (V - V_T)$ do
6.          if edge $(r, v)$ exists set $d[v] := w(r, v)$;
7.          else set $d[v] := \infty$;
8.      while $V_T \neq V$ do
9.      begin
10.         find a vertex $u$ such that $d[u] := \min\{d[v] | v \in (V - V_T)\}$;
11.         $V_T := V_T \cup \{u\}$;
12.         for all $v \in (V - V_T)$ do
13.             $d[v] := \min\{d[v], w(u, v)\}$;
14.     endwhile
15. end PRIM_MST
```

w is the weight vector; r is the starting node.

MST – Prim's Algorithm

- Prim's algorithm for finding an MST is a greedy algorithm. Start by selecting an arbitrary vertex, include it into the current MST. Grow the current MST by inserting into it the vertex closest to one of the vertices already in current MST.
- Note that we will use this Prim's algorithm to develop the PRAM algorithm.
- We want to see the same (almost) parallel algorithm in a different setting.

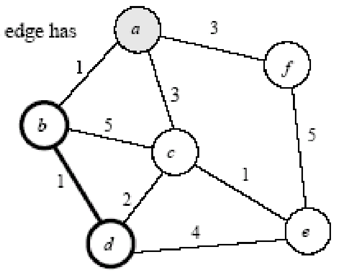
(a) Original graph



$$d[\,]\ \begin{array}{|c|c|c|c|c|c|} a&b&c&d&e&f\\ \hline 1&0&5&1&\infty&\infty\\ \hline \end{array}$$

$$\begin{array}{c|cccccc|} a&0&1&3&\infty&\infty&3\\ b&1&0&5&1&\infty&\infty\\ c&3&5&0&2&1&\infty\\ d&\infty&1&2&0&4&\infty\\ e&\infty&\infty&1&4&0&5\\ f&2&\infty&\infty&\infty&5&0 \end{array}$$

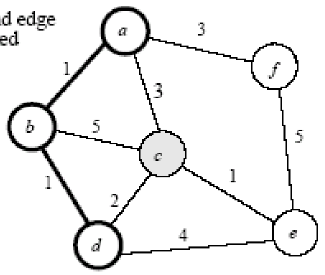
(b) After the first edge has been selected



$$d[\,]\ \begin{array}{|c|c|c|c|c|c|} a&b&c&d&e&f\\ \hline 1&0&2&1&4&\infty\\ \hline \end{array}$$

$$\begin{array}{c|cccccc|} a&0&1&3&\infty&\infty&3\\ b&1&0&5&1&\infty&\infty\\ c&3&5&0&2&1&\infty\\ d&\infty&1&2&0&4&\infty\\ e&\infty&\infty&1&4&0&5\\ f&2&\infty&\infty&\infty&5&0 \end{array}$$

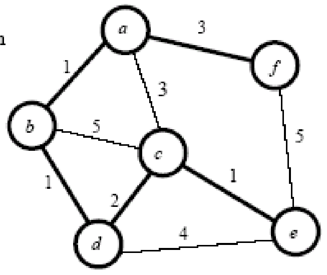
(c) After the second edge has been selected



$$d[\,]\ \begin{array}{|c|c|c|c|c|c|} a&b&c&d&e&f\\ \hline 1&0&2&1&4&3\\ \hline \end{array}$$

$$\begin{array}{c|cccccc|} a&0&1&3&\infty&\infty&3\\ b&1&0&5&1&\infty&\infty\\ c&3&5&0&2&1&\infty\\ d&\infty&1&2&0&4&\infty\\ e&\infty&\infty&1&4&0&5\\ f&2&\infty&\infty&\infty&5&0 \end{array}$$

(d) Final minimum spanning tree



$$d[\,]\ \begin{array}{|c|c|c|c|c|c|} a&b&c&d&e&f\\ \hline 1&0&2&1&1&3\\ \hline \end{array}$$

$$\begin{array}{c|cccccc|} a&0&1&3&\infty&\infty&3\\ b&1&0&5&1&\infty&\infty\\ c&3&5&0&2&1&\infty\\ d&\infty&1&2&0&4&\infty\\ e&\infty&\infty&1&4&0&5\\ f&2&\infty&\infty&\infty&5&0 \end{array}$$

A Simple Greedy Algorithm

- There are two simple rules: Apply the red and blue rules (in any sequence) until n – 1 edges are colored blue. The blue edges form an MST.
  - Red Rule: Select a cycle C in G with no red edges; Select an uncolored edge of C of max weight and color it red.
  - Blue Rule: Select a cut-set with no blue edges; Select an uncolored edge in D of min weight and color it blue.

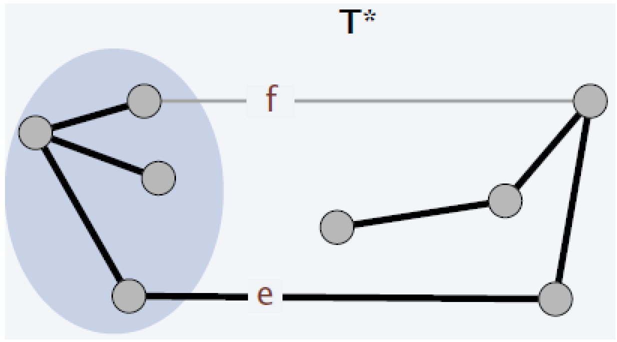
- **Color invariant.** There exists an MST T* containing all the blue edges and none of the red edges.
- We will prove by induction on the number of iterations;
  - **Base case**. No edges colored ⇒ every MST satisfies invariant.
  - We consider the two rules separately to show that the invariant is true.

Proof of Correctness

- **Induction step (red rule).** Suppose color invariant true before red rule.
  - let C be chosen cycle, and let e be edge colored red.
  - if e ∉ T*, T* still satisfies invariant.
  - Otherwise, consider fundamental cut-set D by deleting e from T*.
  - let f ∈ D be another edge in C.
  - f is uncolored and $w(e) \geq w(f)$ since (1) f ∉ T* ⇒ f not blue, (2) red rule ⇒ f not red and $w(e) \geq w(f)$
  - Thus, T* ∪ { f } – { e } satisfies invariant.

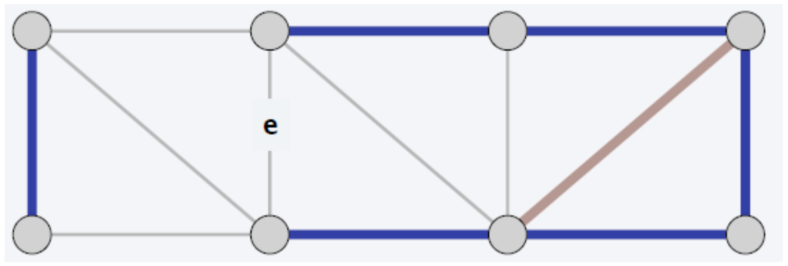
Induction step (blue rule). Suppose color invariant true before blue rule.

- let D be chosen cutset, and let f be edge colored blue.
- if f ∈ T*, T* still satisfies invariant.
- Otherwise, consider fundamental cycle C by adding f to T*
- let e ∈ C be another edge in D.
- e is uncolored and $w(e) \geq w(f)$ since (1) e ∈ T* ⇒ e not red (2) blue rule ⇒ e not blue and $c_e \geq c_f$
- Thus, T* ∪ { f } – { e } satisfies invariant.

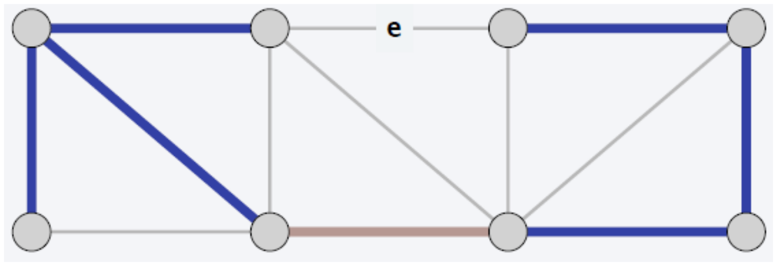


Proof of Termination

- **Theorem**. The greedy algorithm terminates. Blue edges form an MST.
- **Proof**. We need to show that either the red or blue rule (or both) applies.
  - Suppose edge e is left uncolored.
  - Blue edges form a forest.
  - Case 1: both endpoints of e are in same blue tree.
    ⇒ apply red rule to cycle formed by adding e to blue forest.
  - Case 2: both endpoints of e are in different blue trees.
    ⇒ apply blue rule to cutset induced by either of two blue trees.



Case 1



Case 2

Prim's Algorithm

- **Algorithm**: Initialize S = any node, Sort the edges in O(m log n) time, n = no. of nodes and m = no. of edges. Then repeat n – 1 times {Add to tree the min weight edge with one endpoint in S; Add new node to S.}
- **Theorem**. Prim's algorithm computes the MST. **Proof**: Special case of greedy algorithm (blue rule repeatedly applied to S).
- Implement: [d(v) = weight of cheapest known edge between v and S] The data structure and approach are almost identical to Dijkstra's algorithm for shortest path

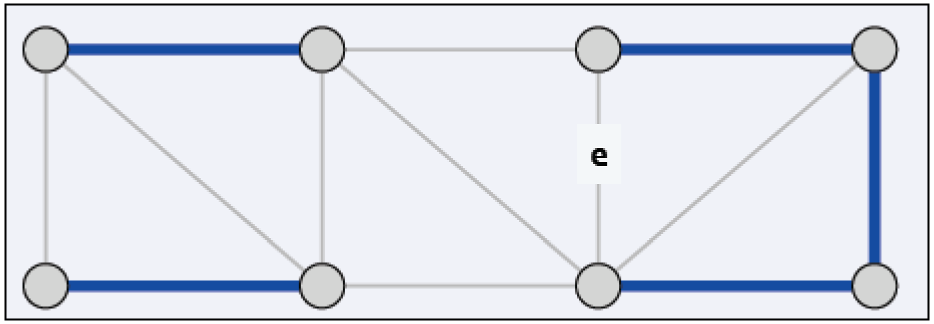
```
PRIM (V, E, w)
Create an empty priority queue.
s ← any node in V.
FOR EACH v ≠ s : d(v) ← ∞; d(s) ← 0.
FOR EACH v : insert v with key d(v) into priority queue.
REPEAT n – 1 times
    u ← delete-min from priority queue.
    FOR EACH edge (u, v) ∈ E incident to u:
        IF d(v) > w(u, v)
            decrease-key of v to w(u, v) in priority queue.
            d(v) ← w(u, v).
```

Kruskal's algorithm

- Consider edges in ascending order of weight:
- **Algorithm**: Add to tree unless it would create a cycle.
- **Theorem**. Kruskal's algorithm computes the MST.
- **Proof**. Special case of greedy algorithm.
  - Case 1: both endpoints of e in same blue tree.
    ⇒ color red by applying red rule to unique cycle.
  - Case 2. If both endpoints of e are in different blue trees.
    ⇒ color blue by applying blue rule to cut-set defined by either tree.

all other edges in cycle are blue

no edge in cut-set has smaller weight
(since Kruskal chose it first)



Kruskal's Algorithm: implementation

- Theorem. Kruskal's algorithm can be implemented in **O(m log m)** time.
- Algorithm:
  - Sort edges by weight.
  - Use union-find data structure to dynamically maintain connected components.

```
KRUSKAL ($V$, $E$, $w$)
SORT $m$ edges by weight so that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_m)$
$S \leftarrow \varphi$
FOREACH $v \in V$:  MAKESET($v$).
FOR $i$ = 1 TO $m$
    $(u, v) \leftarrow e_i$
    IF FINDSET($u$) $\neq$ FINDSET($v$)
        $S \leftarrow S \cup \{ e_i \}$
        UNION($u$, $v$).
RETURN $S$
```

Are u and v in the same component?

make u and v in
same component

Parallel MST☆

- Here is the idea:
    1. Place any vertex in the MST, vertex u for example. Let $C[v] \leftarrow u$, for $1 \leq v \leq n$. (At any stage in the algorithm, C[v] will contain a vertex already in the tree such that the edge (v, C[v]) is the minimum-weight edge)
    2. Include in the tree the closest vertex not yet in the tree.
    3. Update C[v] for every vertex v that is not included in the MST yet.
    4. Repeat steps 2 and 3 until all vertices of the graph are included in the MST.
- We use an EREW PRAM with p processors, where $1 < p \leq n$. Parallelism is achieved by assigning a distinct subsequence of the vertices to each processor. The vertices are divided among the processors such that each processor becomes responsible for an almost equal number of distinct vertices. We use Algorithm
- Broadcast_EREW to simulate concurrent read on EREW.

Data Structures Needed☆

- Array tree[1..n-1] is used for output. Each of its elements is an edge (x,y) that has been added to the MST.
- Array mark[1..n] is a boolean array used to determine whether a vertex has been added to the MST.
- Array C[1..n]. C[v] will contain a vertex already in the tree such that the edge (v, C[v]) is the minimum-weight edge.
- Array V[1..p] is used to store the sets of vertices that are assigned to the different processors. For example, V[i] contains the set of vertices for which processor Pi is responsible
- Array Q[1..p], where each of its elements is a 3-tuple (x, y, d). The component x is a node that is not in the MST yet, y = C[x] is a vertex that is already in the tree, and d is weight on the edge (x, y).
- Array W[1..n, 1..n] is the weight matrix. The element W[i,j] is equal to the weight of the edge (i,j). If there is no edge connecting i and j, W[i,j] = ∞. Also, W[i,i] = ∞.
- Variable x is used to store the node that should be added to the MST at each stage.

Algorithm MST_EREW ⋆

```
/* Step 1 */
mark[1] ← true
∀ Pj, where 1 ≤ j ≤ p pardo
    for each vertex k in the set V[j] do {C[k] ← 1};
/* Step 2 */
for i = 1 to n- 1 do
2.1   ∀ Pj, where 1 ≤ j ≤ p pardo
    Q[j] ← (l, C[l], W[l,C[l]]), where W[l,C[l]] = min{W[k,C[k]]}
     ∀ k ∈ V[j] and mark[k] = false
2.2   for z = 1 to log p do
    ∀ Pj, where 1 ≤ j ≤ p/2 pardo
        if (2j modulo 2z) = 0 then
                read (x1, y1, dist1) from Q[2j]
                read (x2, y2, dist2) from Q[2j - 2z-1]
                if dist1 < dist2 then
                        Q[2j] ← (x1, y1, dist1)
                else
                        Q[2j] ← (x2, y2, dist2);
```

Algorithm MST_EREW [contd.] ☆

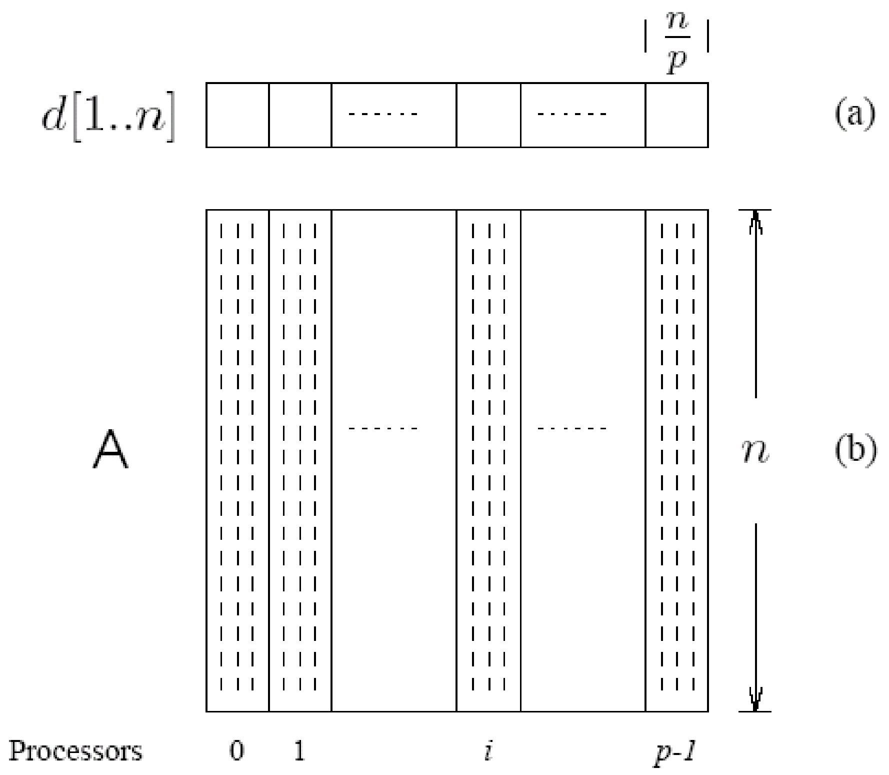
```
2.3     Processor Pp
            read (x, y, d) from Q[p]
            tree(i) ← (x,y)
2.4     Using Algorithm Broadcast_EREW
            x is made known to all processors
2.5     ∀ Pj, where 1 ≤ j ≤ p **pardo**
            if x ∈ V[j] then mark[x] ← true;
            for each vertex k in the set V[j] **do**
                **if** mark[k] = false **then**
                    **if** W[k,x] < W[k,C[k]] then C[k] ← x;
```

Analysis of MST_EREW ☆

- The total number of steps in the algorithm can be calculated as follows.
  - Since each processor accesses all the vertices assigned to it sequentially in step 1, it requires a number of assignments that is equal to the size of each set, which is n/p. Therefore, Step 1 takes O(n/p) time.
  - Similarly, step 2.1 takes O(n/p) time since each processor finds the minimum sequentially within the set of vertices assigned to it.
  - Step 2.2 finds the minimum of p elements in parallel using an idea similar to the one used in Algorithm Sum_EREW, which should take O(log p) time.
  - Step 2.3 takes constant time.
  - The broadcast operation in step 2.4 takes O(log p) time. The complexity of step 2.5 is clearly O(n/p).
  - Since the sequential loop of step 2 is executed n-1 times, the run time of the algorithm is $O(n^2/p)$.
- The complexity measures of Algorithm MST_EREW are summarized as follows: (1) Run time, $T(n) = O(n^2/p)$ (2) Number of processors, $P(n) = p$, (3) Cost, $C(n) = O(n^2)$. Note that this algorithm is cost optimal

Prim's Algorithm: Parallel Formulation

- The algorithm works in n outer iterations – it is hard to execute these iterations concurrently.
- The inner loop is relatively easy to parallelize. Let p be the number of processes, and let n be the number of vertices.
- The adjacency matrix is partitioned in a 1-D block fashion, with distance vector d partitioned accordingly.
- In each step, a processor selects the locally closest node, followed by a global reduction to select globally closest node. This node is inserted into MST, and the choice is broadcast to all processors; each processor updates its part of the d vector locally.



The partitioning of the distance array d and the adjacency matrix A among p processes.

Analysis

1. The cost to select the minimum entry is $O(n/p + \log p)$.
2. The cost of a broadcast is $O(\log p)$.
3. The cost of local updating of the d vector is $O(n/p)$.
4. The parallel time per iteration is $O(n/p + \log p)$.
5. The total parallel time is given by $O(n^2/p + n \log p)$.
6. The corresponding iso-efficiency is $O(p^2 \log^2 p)$.

Huffman Codes & Data Compression

- The problem is how to encode symbols using bits. Consider a small alphabet: 26 English letters, the space and 5 punctuation symbols: period, question mark, exclamation point, and apostrophe. We have 32 symbols, and we must have minimum 5 bits to encode each symbol (mapping of bit strings to symbols is arbitrary). We need to spend an average of 5 bits per symbol. Is it necessary?
- Note the letters in most human alphabets do not get used equally frequently. In English, for example, the letters e, t, a, o, i, and n get used much more frequently than q, j, x, and z (by more than an order of magnitude) – we will try to use *a small number of bits for the frequent letters, and a larger number of bits for the less frequent ones (***VLC – variable length codes***)*; the goal is to reduce the average number of bits per letter when we average over a long string of typical text – a fundamental problem in the area of data compression.
- We still need to solve the problem of ambiguity (there will exist pairs of letters where the bit string that encodes one letter is a *prefix* of the bit string that encodes another). We need to map letters to bit strings in such a way that no encoding is a prefix of any other (**prefix code**).
- Example: a → 0, b → 100, c → 101, d → 11. Observe, this is a valid prefix code on the alphabet {a, b, c, d}.
- Coding: aabddcaa → 00100111110100 [14 bits] **Prefix code ensures unique decodability.**

Optimal Prefix Code

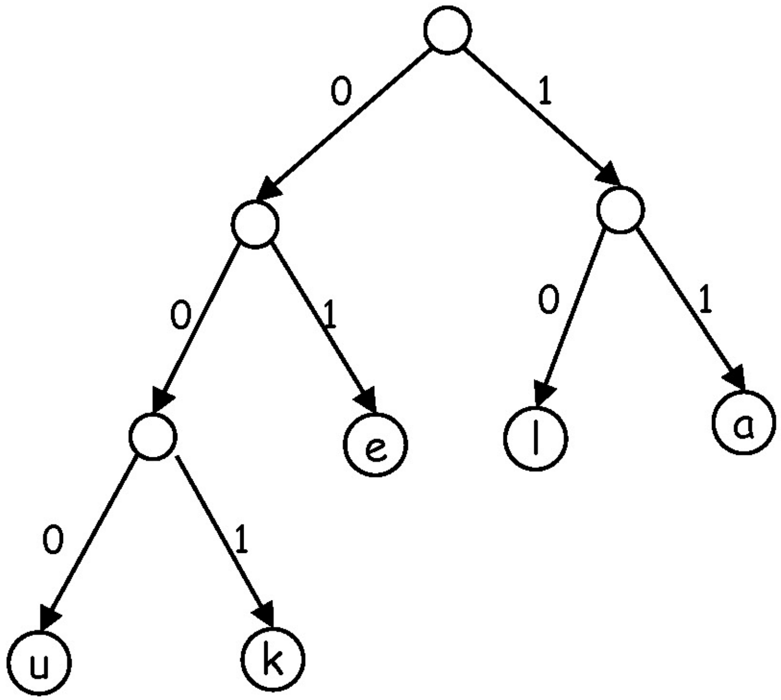
- Prefix property ensures quick and unique decode-ability. We need some notation to express the frequencies of letters to consider optimality.
- Consider a text S over an alphabet of size n; let f_x be the frequency of symbol $x \in S$; note that $\sum_{x \in S} f_x = 1$; total encoding length is $n \times \sum_{x \in S} f_x|\sigma(x)|$, where $|\sigma(x)|$ is the length of the bit string $\sigma(x)$ used to encode x. Thus, we get **average bit length, ABL, (of the mapping function σ_1) is $ABL(\sigma_1) = \sum_{x \in S} f_x|\sigma_1(x)|$.**
- **Example**: Assume text with the letters S = {a, b, c, d, e}, and their frequencies are as follows: $f_a = .32$, $f_b = .25$, $f_c = .20$, $f_d = .18$, $f_e = .05$ where σ_1 [a, b, c, d, e] = [11, 01, 001, 10, 000]. We get $ABL(\sigma_1) = 2.25$; better than 3, savings of 25 percent. Is σ_1 optimal? **No.**
  - Consider another mapping σ_2 [a, b, c, d, e] = [11, 10, 01, 001, 000] for the same problem. We get $ABL(\sigma_2) = 2.23 < ABL(\sigma_1)$.

So, the problem is to minimize $ABL(\sigma)$ over all possible mapping σ, given an alphabet and a set of frequencies for the letters.

- We want to design a greedy algorithm to generate such an optimal mapping for a given alphabet and their frequencies. Note that if the frequencies change, the generated mapping may not be optimal anymore.
- We need to represent Prefix codes using binary trees.

Representing Prefix Codes using Binary Trees

Ex. c(a) = 11
c(e) = 01
c(k) = 001
c(l) = 10
c(u) = 000



Q. How does the tree of a prefix code look?
A. Only the leaves have a label.
Pf. An encoding of x is a prefix of an encoding of y if and only if the path of x is a prefix of the path of y.

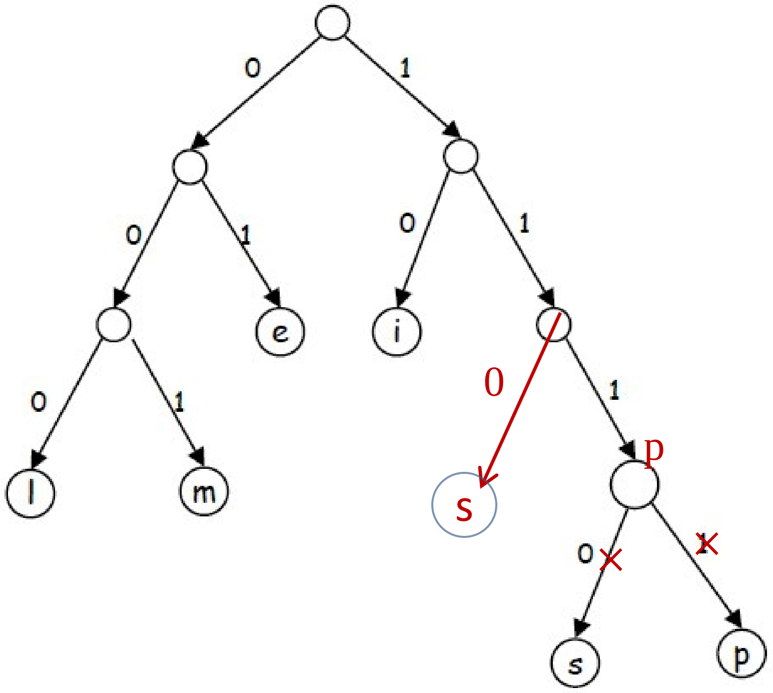
Representing Prefix Codes using Binary Trees

Q. What is the meaning of
111010001111101000 ?
A ‘Simpel’

Q. How can this prefix code be made more efficient?
A. Change encoding of p and s to a shorter one.
This tree is now full.

Note: Since the symbols are always at the leaves, the codes are uniquely decodable and the length of the encoding of any symbol is given by the depth of x in the tree T. So we can also write

$$ABL(T) = \sum_{x \in S} f_x \cdot \text{depth}_T(x)$$

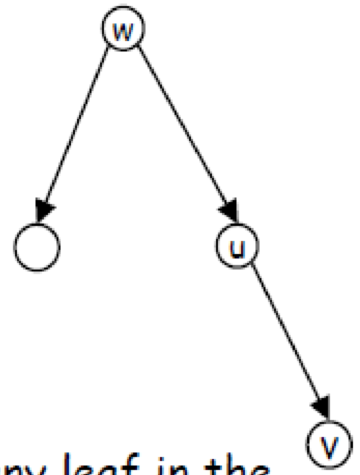


Representing Prefix Codes using Binary Trees

Definition. A tree is full if every node that is not a leaf has two children.

Claim. The binary tree corresponding to the optimal prefix code is full.
Pf. (by contradiction)

- Suppose T is binary tree of optimal prefix code and is not full.
- This means there is a node u with only one child v.
- *Case 1*: u is the root; delete u and use v as the root
- *Case 2*: u is not the root
  - let w be the parent of u
  - delete u and make v be a child of w in place of u
- In both cases the number of bits needed to encode any leaf in the subtree of v is decreased. The rest of the tree is not affected.
- Clearly this new tree T' has a smaller ABL than T. Contradiction.



Optimal Prefix Codes: Huffman Encoding

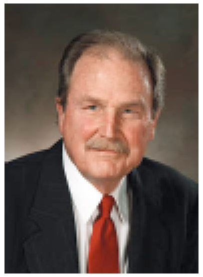
Observation. Lowest frequency items should be at the lowest level in tree of optimal prefix code.

Observation. For $n > 1$, the lowest level always contains at least two leaves.

Observation. The order in which items appear in a level does not matter.

Claim. There is an optimal prefix code with tree T* where the two lowest-frequency letters are assigned to leaves that are siblings in T*.

Greedy template. [Huffman, 1952] Create tree bottom-up.
Make two leaves for two lowest-frequency letters y and z.
Recursively build tree for the rest using a meta-letter for yz.



Optimal Prefix Codes: Huffman Encoding

```
Huffman(S) {
   if |S|=2 {
      return tree with root and 2 leaves
   } else {
      let y and z be lowest-frequency letters in S
      S' = S
      remove y and z from S'
      insert new letter ω in S' with $f_ω=f_y+f_z$
      T' = Huffman(S')
      T = add two children y and z to leaf ω from T'
      return T
   }
}
```

Q. What is the time complexity?

A. $T(n) = T(n-1) + O(n)$

so $O(n^2)$

Q. How to implement finding lowest-frequency letters efficiently?

A. Use priority queue for S: $T(n) = T(n-1) + O(\log n)$ so $O(n \log n)$

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.
Pf. by induction, based on optimality of T' (y and z removed, ω added) (see next page)

Claim. ABL(T')=ABL(T)-f_ω
Pf.

$$\begin{aligned}
\text{ABL}(T) &= \sum_{x\in S} f_x \cdot \text{depth}_T(x) \\
&= f_y \cdot \text{depth}_T(y) + f_z \cdot \text{depth}_T(z) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= \left(f_y + f_z\right) \cdot \left(1 + \text{depth}_T(\omega)\right) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega \cdot \left(1 + \text{depth}_T(\omega)\right) + \sum_{x\in S, x\neq y,z} f_x \cdot \text{depth}_T(x) \\
&= f_\omega + \sum_{x\in S'} f_x \cdot \text{depth}_{T'}(x) \\
&= f_\omega + \text{ABL}(T')
\end{aligned}$$

Huffman Encoding: Greedy Analysis

Claim. Huffman code for S achieves the minimum ABL of any prefix code.

Pf. (by induction)

Base: For n=2 there is no shorter code than root and two leaves.

Hypothesis: Suppose Huffman tree T' for S' with ω instead of y and z is optimal. (IH)

Step: (by contradiction)

- Suppose Huffman tree T for S is not optimal.
- So there is some tree Z such that ABL(Z) < ABL(T).
- Then there is also a tree Z for which leaves y and z exist that are siblings and have the lowest frequency (see observation).
- Let Z' be Z with y and z deleted, and their former parent labeled ω.
- Similar T' is derived from S' in our algorithm.
- We know that $ABL(Z')=ABL(Z)-f_{\omega}$, as well as $ABL(T')=ABL(T)-f_{\omega}$.
- But also ABL(Z) < ABL(T), so ABL(Z') < ABL(T').
- Contradiction with IH.

Test

- Write Something; Pradip Srimani MWrite Something; Pradip Srimani Monai, Rono