UNIT -2 Mathematical Logic

Propositional Logic

Propositional logic is a branch of mathematics that studies the logical relationships between propositions (or statements, sentences, assertions) taken as a whole, and connected via logical connectives.

Example:

1. *The sun rises in the East and sets in the West.*
2. *1 + 1 = 2*
3. *'b' is a vowel.*

Above 3 sentences are propositions, where the first two are Valid(True) and the third one is Invalid(False).

All of the above sentences are propositions, where the first two are Valid(True) and the third one is Invalid(False). Some sentences that do not have a truth value or may have more than one truth value are not propositions.

- It helps us figure out whether something makes sense or not.
- It's used in mathematics, philosophy, computer science, and in everyday life decisions.

Types of Propositions

In propositional logic, propositions are statements that can be evaluated as true or false. They are the building blocks of more complex logical statements. Here's a breakdown of the two main types of propositions:

- Atomic Propositions
- Compound Propositions

The first two sentences are not propositions because they have no truth value, and the third may be true or false. To represent propositions, we use propositional variables (like p, q, r, s). The branch of logic that studies propositions is called propositional logic or propositional calculus. It also deals with forming new propositions from existing ones. Propositions made from one or more propositions are called compound propositions, and they are joined using logical connectives (or operators).

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Connective Symbols	Words	Terms	Example
$\wedge$	AND	Conjunction	$A \wedge B$
$\vee$	OR	Disjunction	$A \vee B$
$\rightarrow$	Implies	Implication	$A \rightarrow B$
$\Leftrightarrow$	If and only If	Biconditional	$A \Leftrightarrow B$
$\neg$	Not	Negation	$\neg A$

Truth Table of Propositional Logic:

1. Negation

If p is a proposition, then the negation of p is denoted by ¬p $\neg p$, which when translated to simple English means- "It is not the case that p" or simply "not p". The truth value of -p is the opposite of the truth value of p. The truth table of -p is:

p	¬p
T	F
F	T

Example, Negation of "It is raining today", is "It is not the case that is raining today" or simply "It is not raining today"

2. Conjunction

For any two propositions p and q , their conjunction is denoted by $p \wedge q$, which means " p and q ". The conjunction $p \wedge q$ is True when both p and q are True, otherwise False. The truth table of $p \wedge q$ is:

p	q	p ∧ q
T	T	T

p	q	p ∧ q
T	F	F
F	T	F
F	F	F

Example:

Conjunction of the propositions p - "Today is Friday" and q - "It is raining today", $p \wedge q$ is "Today is Friday and it is raining today". This proposition is true only on rainy Fridays and is false on any other rainy day or on Fridays when it does not rain.

3. Disjunction

For any two propositions p and q, their disjunction is denoted by $p \vee q$, which means "p p or q q ". The disjunction $p \vee q$ is True when either p or q is True, otherwise False. The truth table of $p \vee q$ is:

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Example :

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Disjunction of the propositions p - "Today is Friday" and q - "It is raining today", $p\vee q$ is "Today is Friday or it is raining today". This proposition is true on any day that is a Friday or a rainy day(including rainy Fridays) and is false on any day other than Friday when it also does not rain.

4 . Implication

For any two propositions p and q , the statement "if p then q " is called an implication and it is denoted by $p\rightarrow q$. In the implication $p\rightarrow q$, p is called the hypothesis or antecedent or premise and q is called the conclusion or consequence. The implication is $p\rightarrow q$ is also called a conditional statement. The implication is false when p is true and q is false otherwise it is true. The truth table of $p\rightarrow q$ is**:**

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Example,:

"If it is Friday then it is raining today" is a proposition which is of the form $p\rightarrow q$. The above proposition is true if it is not Friday(premise is false) or if it is Friday and it is raining, and it is false when it is Friday but it is not raining.

5. Biconditional :

For any two propositions p and q , the statement " p if and only if(iff) q " is called a biconditional and it is denoted by $p\leftrightarrow q$. The statement $p\leftrightarrow q$ is also called a bi-implication. $p\leftrightarrow q$ has the same truth value as $(p\rightarrow q)\wedge(q\rightarrow p)$ The implication is true when p and q have same truth values, and is false otherwise. The truth table of $p\leftrightarrow q$ is:

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p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Example :

"It is raining today if and only if it is Friday today." is a proposition which is of the form p↔q $p \leftrightarrow q$. The above proposition is true if it is not Friday and it is not raining or if it is Friday and it is raining, and it is false when it is not Friday or it is not raining

Application of propostional Logic.

In the computer science field, propositional logic has a wide variety of applications and hence is very important. It is used in system specifications, circuit designing, logical puzzles, etc

1. Digital Circuits and Computer Hardware

- Logic gates in circuits are based on propositional logic.
- Each gate corresponds to a connective:
    - **AND gate** → p ∧ q
    - **OR gate** → p ∨ q
    - **NOT gate** → ¬p

2. Computer Programs & Algorithms

- Conditions in programs are written using logical statements.
- Example (in C/Java/Python style):

```
if (age >= 18 and citizen == True):
    print("Eligible to vote")
```

This is propositional logic:
p: age ≥ 18
q: citizen = True
Decision: $p \land q$

3. Database Systems

- Queries in SQL use logic.
- Example:

```
SELECT * FROM Students
WHERE Age > 18 AND Marks >= 50;
```

This means: $p \land q$

- p: Age > 18
- q: Marks ≥ 50

4. Artificial Intelligence & Expert Systems

- AI systems (like chatbots, medical diagnosis systems) use propositional logic to **infer knowledge**.
- Example:
    - Rule: If fever ∧ cough → flu.
    - Given: fever = True, cough = True
    - Inference: flu = True

5. Mathematical Proofs

- Used to prove theorems and logical arguments.
- Example:
  If $p \to q$ and $q \to r$, then $p \to r$.
  This helps build **rigorous proofs** in mathematics.

6. Networking and Internet Protocols

- Firewalls and routing rules use propositional logic.

- Example:
  Rule: Allow packet if (port = 80 ∨ port = 443) ∧ source ≠ blocked_IP.

7. Software Testing & Verification

- Test cases are written as logical conditions.
- Example:
  A login system must satisfy:
  (user_exists ∧ password_correct) → login_success

8. Problem Solving & Decision Making

- Logic helps in real-life reasoning.
- Example:
  "If it rains, I will carry an umbrella. If I don't carry an umbrella, I will get wet."
  This can be written as:
  $(p \rightarrow q), (\neg q \rightarrow r) \Rightarrow (p \rightarrow r)$.

Predicates and Quantifiers:

Predicates and Quantifiers are fundamental concepts in mathematical logic, essential for expressing statements and reasoning about the properties of objects within a domain. These concepts are widely used in computer science, engineering, and mathematics to formulate precise and logical statements.

Predicates

A predicate is a statement that contains variables and becomes a proposition when specific values are substituted for those variables. Predicates express properties or relations among objects. Example:

P(x) = "x is an even number"

- When x = 2, P(2) is True.
- When x = 3, P(3) is False.

Quantifiers

Quantifiers are used in logic to show how much a statement is true for a set of things. They show whether something is true for all members of the group, or only for some.

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Predicates vs Quantifiers

This table shows the key differences between predicates and quantifiers in logic.

Predicate	**Quantifier**
A statement containing variables	A symbol indicating the scope of the predicate
Describes a property or relation	Specifies the extent to which the predicate is true
P(x): "x is an even number."	∀: "For all" or ∃: "There exists"
No specific symbol	∀ (Universal), ∃ (Existential)
Used to form logical statements	Used to quantify logical statements
Alone or with quantifiers	Always used with predicates
P(x), Q(x, y)	∀x, ∃y

Solved Questions Predicates and Quantifiers

Question 1: Let P(x) be the predicate "x > 5" where x is a real number.

Solution:

P(7) is true because 7 > 5

P(3) is false because 3 is not > 5

Question 2: Let Q(x, y) be the predicate "x + y = 10" where x and y are integers.

Solution:

- *Q(3, 7) is true because 3 + 7 = 10*
- *Q(4, 5) is false because 4 + 5 ≠ 10*

Nested quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: "Every real number has an additive inverse" is translated as $\forall x\ \exists y(x + y = 0)$, where the domains of x and y are the real numbers.

Examples of Nested Quantifiers

Example 1:

∀x ∃y (x + y = 0)

- "For every x, there exists a y such that x + y = 0."
- True ✅ because if x = 5, then y = -5 works. If x = 7, then y = -7 works.

Example 2 (Real life):

- Domain: Students in a class.
- Predicate: P(x, y) = "x knows y."

1. ∀x ∃y P(x, y)
   - "Every student knows at least one student."
   - (True in most cases.)

2. ∃y ∀x P(x, y)
    - "There exists a student whom everyone knows."
    - (That student could be a very popular one

Rules of Inference :

Rules of inference are standard logical patterns that allow us to derive a conclusion from one or more given premises in a logically valid way.

Rules of inference are important because these are the building blocks for formal proofs in mathematics, computer science, and logic. They ensure that every step in your reasoning is valid, so your conclusion is guaranteed to be true if your premises are true.

1. Modus Ponens (Law of Detachment)

If a conditional statement ("if-then" statement) is true, and its antecedent (the "if" part) is true, then its consequent (the "then" part) must also be true.

Form: If $p \rightarrow q$ and p, then q.

Example:

- Premise: If it rains, the ground will be wet.
- Premise: It is raining.
- Conclusion: The ground is wet.

2. Modus Tollens (Law of Contrapositive)

If a conditional statement is true, and its consequent is false, then its antecedent must also be false.

Form: *If $p \rightarrow q$ and $\neg q$, then $\neg p$.*

Example:

- *Premise: If it rains, the ground will be wet.*
- *Premise: The ground is not wet.*
- *Conclusion: It is not raining.*

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3. Hypothetical Syllogism

If two conditional statements are true, where the consequent of the first is the antecedent of the second, then a third conditional statement combining the antecedent of the first and the consequent of the second is also true.

Form: *If* $p \to q$ *and* $q \to r$*, then* $p \to r$*.*

Example:

- *Premise: If it rains, the ground will be wet.*
- *Premise: If the ground is wet, the plants will grow.*
- *Conclusion: If it rains, the plants will grow.*

4. Disjunctive Syllogism

If a disjunction (an "or" statement) is true, and one of the disjuncts (the parts of the "or" statement) is false, then the other disjunct must be true.

Form: *If* $p \vee q$ *and* $\neg p$*, then* q*.*

Example:

- *Premise: It is either raining or sunny.*
- *Premise: It is not raining.*
- *Conclusion: It is sunny.*

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Mathematical Logic – Question Bank

Q1. Define propositional logic with an example.

Q2. Explain Truth Table of Propositional Logic.(Negation, Conjunction, Disjunction ,Implication, Biconditional)

Q3. Explain Applications of Propositional Logic.

Q4. Give differences between predicates and quantifiers in logic.

Q5. What is Modus Ponens? Give an example.

Q6. Explain Rules of Inference.

Q7.Explain Nested quantifiers.