Assignment 03 :-

1. Evaluate the following integral analytically, by the trapezoidal rule, & by a simpson $\frac{1}{3}$ rule (take $n=4$).

$$I = \int_0^3 x e^{2x} dx.$$

(A) Analytical :-

Using integration by parts

$u = x \Rightarrow du = dx$

we know rule of antiderivative

$$\int e^{ax} = \frac{1}{a} e^{ax} + c$$

for $a=2$

$$\& \ \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2} \cdot 2e^{2x} = e^{2x}$$

$\therefore \ dv = e^{2x}dx \Rightarrow v = \frac{1}{2}e^{2x}.$

$$\int u \cdot dv = uv - \int v \cdot du.$$

$$\int_0^3 x \cdot e^{2x} dx = x \cdot \frac{1}{2}e^{2x} - \int_0^3 \frac{1}{2}e^{2x} \cdot dx \qquad \int_0^3 \frac{1}{2}e^{2x}$$

$$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + c$$

$$= \left[\left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} \cancel{+c}\right]_0^3$$

$$= \left[\left(\frac{3}{2} - \frac{1}{4}\right)e^{2\times 3}\right] - \left[\left(\frac{0}{2} - \frac{1}{4}\right)e^{2\times 0}\right]$$

$$= \frac{5}{4}e^6 - \left(-\frac{1}{4}\right)$$

$$I = \frac{5}{4}e^6 + \frac{1}{4} \Rightarrow \frac{5}{4} \times (403.4288) + \frac{1}{4} \Rightarrow 504.536$$

(B) By Trapezoidal rule :-

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$a = 0 \qquad b = 3 \qquad n = 4$

$$h = \frac{b-a}{n} = \frac{3-0}{4} = \frac{3}{4} \Rightarrow 0.75$$

$X_0 = a + ih.$

$$f(x) = x \cdot e^{2x}$$

$X_0 = 0 + 0 \times 0.75 = 0 \qquad f(x_0) = 0$

$X_1 = 0 + 1 \times 0.75 = 0.75 \qquad f(x_1) = 0.75 \cdot e^{1.5} = 3.3612$

$X_2 = 0 + 2 \times 0.75 = 1.5 \qquad f(x_2) = \cancel{0.75 \cdot e^{3}} = 1.5 e^{3} = 30.1283$

$X_3 = 0 + 3 \times 0.75 = 2.25 \qquad f(x_3) = \cancel{0.75\, e^{4.5}} = 2.25 \cdot e^{4.5} = 202.5385$

$X_4 = 0 + 4 \times 0.75 = 3 \qquad f(x_4) = \cancel{0.75\, e^{6}} = 3 \cdot e^{6} = 1210.2863$

Using Trapezoidal rule :-

$$T = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

$$= \frac{0.75}{2}\left[0 + 2\,(3.3612 + 30.1283 + 202.5385) + 1210.2863 \right]$$

$$= \frac{0.75}{2}\left[472.056 + 1210.2863 \right]$$

$$= \frac{0.75}{2}\left[1682.3423 \right]$$

$$= \underline{630.8783}$$

(c) Simpson $\frac{1}{3}$ rule :-

$$S = \frac{h}{3}\left[f(x_0) + 4\sum_{\text{odd } i} f(x_i) + 2\sum_{\text{even } i \neq 0, n} f(x_i) + f(x_n) \right]$$

$$= \frac{0.75}{3}\left[f(x_0) + 4\,(f(x_1) + f(x_3)) + 2\,(f(x_2)) + f(x_4) \right]$$

$$= \frac{0.75}{3}\left[0 + 4\,(3.3612 + 202.5385) + 2\,(30.1283) + 1210.2863 \right]$$

$$0.25\left[4\,(205.8997) + 2\,(30.1283) + 1210.2863 \right]$$

$= \cancel{\frac{0.75}{3}}$ ~~823.5988 + 1 60.2566 4 1210.2863~~

$= \cancel{0.25}$ ~~2094.1417~~

$= 0.25\left[4(205.8997) + 2(30.1283) + 1210.2363\right]$

$= 523.5354$

Hence, Exact $= 504.536$

Trapezoidal $= 630.8783$

Simpson $= 523.5354$.

$u.v - \int v\,du$

2) Evaluate the integral using simpson $\frac{3}{8}$ rule (take n=6)

$$\int_0^{2\pi} \cos^2 x \, dx.$$

Sol^n :-

$$I = \int_0^{2\pi} \cos^2 x\, dx$$

$$= \int_0^{2\pi} \frac{1+\cos 2x}{2}\, dx.$$

$$= \frac{1}{2}\int_0^{2\pi} dx + \frac{1}{2}\int_0^{2\pi} \cos 2x\, dx$$

$$= \frac{1}{2}[x]_0^{2\pi} + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\cancel{2}} \times \cancel{2}\pi + \frac{1}{4} \times (\underbrace{\sin 4\pi}_{0} - \underbrace{\sin 2(0)}_{0})$$

$$= \pi.$$

$a = 0 \quad b = 2\pi \quad h = \frac{b-a}{n} = \frac{2\pi - 0}{6} = \frac{2\pi}{6} = \frac{\pi}{3}$

$x_i = a + ih$

$x_0 = 0 + 0 \times \frac{\pi}{3} = \cancel{\frac{\pi}{3}}\ 0$

$x_1 = 0 + 1 \times \frac{\pi}{3} = \frac{\pi}{3}$

$x_2 = 0 + 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$

$x_3 = 0 + 3 \times \frac{\pi}{3} = \pi$

$x_4 = 0 + 4 \times \frac{\pi}{3} = \frac{4\pi}{3}$

$x_5 = 0 + 5 \times \frac{\pi}{3} = \frac{5\pi}{3}$

$x_6 = 0 + 6 \times \pi/3 = 2\pi$

$f(x_0) = \cos^2(0) = 1$

$f(x_1) = \cos^2\left(\frac{\pi}{3}\right) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$f(x_2) = \cos^2\left(\frac{2\pi}{3}\right) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$f(x_3) = \cos^2(\pi) = (-1)^2 = 1$

$f(x_4) = \cos^2(4\pi/3) = (-1/2)^2 = 1/4$

$f(x_5) = \cos^2(5\pi/3) = (1/2)^2 = 1/4$

$f(x_6) = \cos^2(2\pi) = 1$

0 1 2 3 4 5 6
× × o ✓ ×

Applying Simpson $\frac{3}{8}$ th rule :-

$$= \frac{3h}{8}\left[f(x_0) + 3\sum_{\substack{i \not\equiv 0 \\ \text{mod} 3}} f(x_i) + 2\sum_{\substack{i \equiv 0 \\ \text{mod} 3}} f(x_i) + f(x_n)\right]$$

$$= \frac{3(\pi/3)}{8}\left[f(x_0) + 3\left(f(x_1) + f(x_2) + f(x_4) + f(x_5)\right) + \cancel{f(x_6)}\ 2\left(f(x_3)\right) + f(x_6)\right]$$

$$= \frac{3(\pi/3)}{8}\left[1 + 3\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + 2(1) + 1\right]$$

$$= \frac{3\pi/3}{8}\left[1 + 3(1) + 2(1) + 1\right]$$

$$= \frac{3\pi/3}{8}\left[1 + 3 + 2 + 1\right]$$

$$= \frac{3\pi/3}{8}[7]$$

$$= \frac{\pi}{8} \times 7$$

$$= \frac{7\pi}{8}$$

$$= 2.7488$$

Hence,

Exact Analytical answer is $\Rightarrow \int_0^{2\pi} \cos^2 x\,dx = \pi \approx 3.1415$

Simpson $\frac{3}{8}$ th rule $\Rightarrow$ 2.7488

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3) Evaluate

$\int_0^{0.5} f(x)\,dx$ if

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	1	7	4	3	5	2

$n = 5$ $\quad h = \frac{b-a}{n} = \frac{0.5 - 0}{5} = 0.1$

① By Trapezoidal rule :

$$T = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2\left(f(x_1) + f(x_2) + f(x_3) + f(x_4)\right) + f(x_5)\right]$$

$$= \frac{h}{2}\left[1 + 2(7 + 4 + 3 + 5) + 2\right]$$

$\Rightarrow \frac{0.1}{2}[1 + 38 + 2]$

$\Rightarrow 0.05 \times 41$

$\Rightarrow \underline{2.05}$.

f_0 1 2 3 4 5 f_n

② By simpson $\frac{3}{8}$th rule :-

$= \frac{3h}{8}\left[f(x_0) + 3\sum_{\substack{i \not\equiv 0 \\ \text{mod } 3}} f(x_i) + 2\sum_{\substack{i \equiv 0 \\ \text{mod } 3}} f(x_i) + f(x_n)\right]$

$= \frac{3 \times 0.1}{8}\left[f(x_0) + 3\left(f(x_1) + f(x_2) + f(x_4)\right) + 2\left(f(x_3)\right) + f(x_5)\right]$

$= 0.0375\left[1 + 3(7 + 4 + 5) + 2(3) + 2\right]$ $\frac{3}{8} = 1.5375$

$= 0.0375[57] \Rightarrow \underline{2.1375}$. $\frac{1}{3} = 2.0333$

③ By simpson $\frac{1}{3}$rd rule :-

$= \frac{h}{3}\left[f(x_0) + 4\sum_{odd} f(x_i) + 2\sum_{even} f(x_i) + f(x_n)\right]$

$= \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2\left(f(x_2) + f(x_4)\right) + f(x_5)\right]$

$= \frac{0.1}{3}\left[1 + 4(7+3) + 2(4+5) + 2\right]$

$= 0.03333[61]$

$= \underline{2.033}$.

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4) Evaluate $I = \int_0^{3\pi/2} \sin(5x+1)\,dx$. using simpson 1/3 rule by taking $n=4$. find a bound on absolute error.

Soln :-

$I = \int_0^{3\pi/2} \sin(5x+1)\,dx \Rightarrow \int \cos($

$\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

$= u = (5x+1)$

$= du = 5dx$

$dx = \frac{du}{5}$

$\Rightarrow \left[-\frac{1}{5}\cos(5x+1)\right]_0^{3\pi/2} + c$

$\Rightarrow \left[-\frac{1}{5}\cos\left(5 \cdot \frac{3\pi}{2} + 1\right) + \frac{1}{5}\cos(1)\right]$

cosine shift identity

$\cos(\alpha + \pi/2) = -\sin\alpha$

$\cos(\alpha + \pi) = -\cos\alpha$

$= \frac{1}{5}\left[\cos\left(\frac{15\pi}{2} + 1\right) - \cos(1)\right]$ ← $\cos(\alpha + \frac{3\pi}{2}) = \sin\alpha$ ✓

$\cos(\alpha + 2\pi) = \cos\alpha$

$= \frac{1}{5}$ ~~cos [(sin(1)~~ $\therefore \cos(\alpha + \frac{3\pi}{2}) = \sin\alpha$

$\therefore \cos(1 + \frac{15\pi}{2}) = \sin 1$.

$I = \frac{1}{5}\left[\cos(1) - \sin(1)\right]$

$= \frac{1}{5} \times (-0.301168678)$

$= -0.060233735$ ← Exact.

$a = 0 \quad b = \frac{3\pi}{2} \quad n = 4 \quad h = \frac{b-a}{n} = \frac{3\pi/2 - 0}{4} \Rightarrow h = \frac{3\pi}{8}$

$x_i = a + ih$

$x_0 = 0 + 0 \times \frac{3\pi}{8} = 0$ $\qquad f(x_0) = \sin(5x+1) = \sin(5\times 0+1) = \sin(1) = 0.8414$

$x_1 = 0 + 1 \times \frac{3\pi}{8} = \frac{3\pi}{8}$ $\qquad f(x_1) = \sin(5 \times \frac{3\pi}{8} + 1) = \sin\left(\frac{15\pi}{8} + 1\right) = 0.5706$

$x_2 = 0 + 2 \times \frac{3\pi}{8} = \frac{6\pi}{8}$ $\qquad f(x_2) = \sin\left(\frac{30\pi}{8} + 1\right) = 0.2129$

$x_3 = 0 + 3 \times \frac{3\pi}{8} = \frac{9\pi}{8}$ $\qquad f(x_3) = \sin\left(\frac{45\pi}{8} + 1\right) = -0.1771$

$x_4 = 0 + 4 \times \frac{3\pi}{8} = \frac{12\pi}{8} = \frac{3\pi}{2}$ $\qquad f(x_4) = \sin\left(\frac{60\pi}{8} + 1\right) = -0.5403$

0 1 2 3 4

By using Simpson $\frac{1}{3}$rd Rule :-

$= \frac{h}{3}\left[f(x_0) + 4\sum_{odd} f(x_i) + 2\sum_{even} f(x_i) + f(x_n)\right]$

$= \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + f(x_3)\right) + 2\left(f(x_2)\right) + f(x_4)\right]$

$= \frac{3\pi/8}{3}\left[0.8414 + 4\left((0.5706) + (-0.1771)\right) + 2\left(0.2129\right) + (-0.5403)\right]$

$= \frac{3\pi/8}{3}\left[0.8414 + 1.574 + 0.4258 - 0.5403\right]$

$= \frac{3\pi/8}{3}\left[2.3009\right]$ $\qquad \Rightarrow \frac{3\pi/8}{3}\left[2.3009\right] \Rightarrow 0.903561717$

$= 0.903561317$

Actual error $= I_{simp} - I$

$= 0.903561317 - (-0.06023)$

$= 0.9639$

Error bound :-

By simpson $\frac{1}{3}$:-

$$|E| \le \frac{(b-a)h^4}{180} \max_{[a,b]} |f^{(4)}(x)|$$

$$|E| \le \frac{\frac{3\pi}{2} \cdot \left(\frac{3\pi}{8}\right)^4}{180} \cdot 625$$

$$= \frac{3375\pi^5}{32768}$$

$$|E| = 31.52$$

$f(x) = \sin(5x+1)$

$f'(x) = 5\cos(5x+1)$

$f''(x) = -\sin(5x+1)\,25$

$f'''(x) = -\cos(5x+1)\,125$

$f^4(x) = \sin(5x+1)\,625$

Simpson 1/3 approximation with $n=4$

$$I_{simp} = 0.9036$$

exact value :

$$I \approx -0.0602$$

Actual error $= 0.964$

Error bound

$$\underline{|E| \le 31.52}$$