The Wallpaper Groups

Classification and Orbifold Notation

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August 28, 2021

Definition

- A topologically discrete subgroup G of E_2 is called a Wallpaper group if its translation subgroup is generated by two independent translations of **minimum length**.
- There are 17 such groups upto isomorphism.

Figure 1: Ottoman Turkish dish

Figure 2: Ceiling of Egyptian tomb

- **"Topologically discrete group"** : G does not have any arbitrarily small rotations or translations.
- Subgroup of a discrete group is discrete.
- $$L_G \;:=\; \{\, a \in \mathbb{R}^2 \;|\; t_a \in G \,\}$$

  $L_G \cong T \cap G$ [the map that takes $a \longmapsto t_a$ is an isomorphism]

  $\Rightarrow$L$_G$ is discrete, so $\exists\ \epsilon$ such that L_G contains no vector of length less than ϵ except 0 .

Some properties of discrete subgroups of $\mathbb{R}^2$

Let L be a discrete subgroup of $\mathbb{R}^2$.

- Any bounded subset S of $\mathbb{R}^2$ contains only finitely many elements of L.

Proof.

If S is bounded, so is $L \cap S$. Now, a bounded set which is infinite cannot be discrete, i.e. we must have some point with any of its neighbourhoods containing infinitely many points of the set, so that the distance between the points are arbitrarily small.
$\Rightarrow L \cap S$ is finite. $\square$

Some properties of discrete subgroups of $\mathbb{R}^2$ (Contd.)

- If L $\neq \{0\}$, L contains a nonzero vector of minimal length.

Proof.

Let b $\in L$ be any nonzero vector. We consider a disc of radius $|b|$ around the origin.

Now, this disc is a bounded set. So it has only finitely many elements of L, including b. We choose the vector of minimal nonzero length among these finite number of vectors and denote it by a.

a is our required vector of minimal length in L. $\square$

- Every discrete subgroup L has one of these three forms:
  1. $\mathrm{L} = \{0\}$
  2. $\mathrm{L} = \{ma | m \in \mathbb{Z}\}$ for some nonzero vector a.
  3. $\mathrm{L} = \{ma + nb | m, n \in \mathbb{Z}\}$ where a and b are linearly independent.

Proof.

Let $\mathrm{L} \neq \{0\}$. Then we can choose a vector $\mathrm{a} \in \mathrm{L}$ of minimal length.

Case I : All vectors in L lie on one line ℓ through the origin.

Claim : $\mathrm{L} = \{ma | m \in \mathbb{Z}\}$

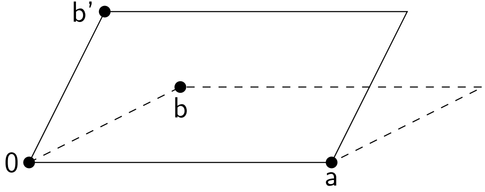
Proof : Let $\mathrm{v} \in \mathrm{L}$. Then $\mathrm{v} = \mathrm{ra}$ for some $\mathrm{r} \in \mathbb{R}$. We write $\mathrm{r} = \mathrm{n} + r_0$ where $\mathrm{n} \in \mathbb{Z}$ and $0 \leq r_0 < 1$. Then $r_0\mathrm{a} = \mathrm{v} - \mathrm{na} \in \mathrm{L}$, and has length less than a.

$\Rightarrow \mathrm{r}_0 = 0$.

□

Proof(Contd.)

Case II : The elements of L $\subset \mathbb{R}^2$ does not lie on a line.
L contains two linearly independent vectors a', b'. First we replace the vector a' by a shortest nonzero vector a on the line ℓ which a' spans. Then a generates $\ell \cap$ L.
Next, we consider the parallelogram P'$\subset \mathbb{R}^2$ with vertices 0, a, b', a+b'. P' being bounded, contains only finitely many elements of L. We choose b $\in$ P' $\cap$ L such that b has shortest nonzero distance from l among points of P' $\cap$ L, and replace b' with b. Let P be the parallelogram with vertices 0, a, b, a+b.



Proof(Contd.)

P contains no points of L except its vertices.
Let v $\in$ L. $\{a, b\}$ is a basis of $\mathbb{R}^2$. So, v = ra + sb for some r,s $\in \mathbb{R}$.
We write r = m + r_0, s = n + s_0, where m,n $\in \mathbb{Z}$ and $0 \leq r_0, s_0 < 1$.
Let $v_0 = r_0 a + s_0 b$ = v - ma - nb. Then v_0 lies in the parallelogram P and $v_0 \in$ L.
$\Rightarrow v_0$ is a vertex of P.
Since $r_0, s_0 < 1$, v_0 must be the origin.
$\Rightarrow v = ma + nb$.

Lattice

- Groups of the last type are called **Plane Lattices** and the generating set a,b is called the **lattice basis**.
  Translation groups of wallpaper groups are such *plane lattices*.

- There are only five types of plane lattices:
  (a) Parallelogram
  (b) Rhombic
  (c) Rectangular
  (d) Square
  (e) Hexagonal

Point groups

There is a group homomorphism $\varphi :$ M $\to$ **O** with kernel T. If we restrict φ to G, we obtain a homomorphism $\varphi|_G$: G $\to$ **O** with kernel T $\cap$ G [$\cong L_G$].

The *point group* $\bar{G}$ is defined as the image of G in **O** under φ.

$\rho_\theta \in \bar{G} \Leftrightarrow \exists$ a $\in L_G$ such that $t_a\rho_\theta \in$ G.

ρ_θr $\in \bar{G} \Leftrightarrow \exists$ a $\in L_G$ such that $t_a\rho_\theta$r $\in$ G.

$\Rightarrow$ $\bar{G}$ is a discrete subgroup of **O**.

Point groups

- A discrete subgroup of **O** is a finite group.

Proof.

A $\in$ **O** $\Leftrightarrow AA^T =$ I. So the entries of A satisfy n^2 polynomial equations.

$\therefore$ **O** is the subset of $\mathbb{R}^{n^2}$ which is the intersection of sets of solutions of these equations.

$\Rightarrow$ **O** $\subset \mathbb{R}^{n^2}$ is closed.

Now,if A $\in$ **O**, for any column vector v of A, we have $< v, v > = |v|^2 = 1$. So, for any entry $a_i j$ of A, $|a_i j| \leq 1$.

$\Rightarrow$ **O** $\subset \mathbb{R}^{n^2}$ is bounded.

$\therefore$ **O** is compact in $M_n(\mathbb{R})$.

$\Rightarrow$ Infinite subsets of **O** must have some limit point, hence cannot be discrete.

Hence proved. □

Point groups(Contd.)

- If G is a finite subgroup of $\mathbf{O}$, either G $= C_n$ or G $= D_n$.
- The point group $\bar{G}$ is, thus, of the form C_n or D_n.
- $\bar{G}$ is contained in the group of symmetries of L_G.
- If H $\subset \mathbf{O}$ be a finite subgroup of the group of symmetries of a *lattice* L, every rotation in H must have order 1,2,3,4 or 6.

  $\Rightarrow \bar{G} = C_n$ or D_n where n $=$ 1, 2, 3, 4 or 6.

  This condition is known as *Crystallographic Restriction*.

Conway's Orbifold notation

The notation system is based on William Thurston's concept of **orbifold**. This system can be used in describing discrete groups of isometries on any of the following three surfaces: the sphere, the Euclidean plane and the hyperbolic plane, all simply connected surfaces of constant Gaussian curvature.

Roughly, an *orbifold* is the quotient of a manifold by a discrete group acting on it. Therefore each orbit of the group on the manifold is identified to a point in this orbifold.
Geometrically, in case of surface patterns, this is described as "folding up" the pattern into a surface to bring the "corresponding" points together.

- ***** : "*" denotes a *mirror-point* in the group. The points of an orbifold that correspond to mirror-points are boundary points. An *ordinary* boundary point is the image of an **ordinary** mirror-point and a **type m-corner point** is the image of an *m-fold* mirror point. At a type m-corner point, the boundary has an angle of $\frac{\pi}{m}$.
  A boundary curve has type *ab...c means its corners have types a, b, ...,c reading around the curve in some consistent direction. Thus, the orbifold might have a boundary curve even though the original surface don't.

  e.g. *442 would denote an orbifold corresponding to a wallpaper group which contains 2 different "kinds" of 4-fold mirror points and a 2-fold mirror point. Whereas, type * would denote an oribold having a boundary curve without corners, as in **.

Examples

Figure 3: Indian metalwork, Great Exhibition, 1851

- The image of an m-fold **gyration point** in the orbifold is called a **cone-point** of order m. The angle around this point in the orbifold is $\frac{2\pi}{m}$ and not 2π.
  The digits A, B, ...,C *not* precededd by * denote orders of distinct cone-points on the orbifold. e.g. 2222, 333, 632, 442.
- $\circ$ in case of Wallpaper Groups, indicates that the group contains no other symmetry except lattice translations. The corresponding orbifold contains a handle e.g. a torus.
- $\times$ denotes a glide reflection, and corresponds to a crosscap in the orbifold.
- Classification theorem for 2-manifold tells us that we can obtain any connected compact 2-manifold by adding handles and crosscaps to a sphere and punching a hole for each boundary curve.

Signature	Gyration	Mirror	Glide Reflection	Point Gro
○	-	-	-	C_1
× ×	-	-	yes	D_1
*×	-	parallel	between mirrors	D_1
**	-	parallel	-	D_1
632	6,3,2	-	-	C_6
*632	yes	group	this.	D_6
333	3,3,3	-	-	C_3
*333	-	3,3,3	-	D_3
3*3	exists	-	exists	D_3
442	4,4,2	-	-	C_4
*442	4,4	-	yes	D_4
4*2	4,4,2	perpendicular	yes	D_4
2222	4 of order 2	-	-	C_2
22×	2,2	-	perpendicular	D_2
22*	2,2	one	yes	D_2
*2222	-	perpendicular	-	D_2
2*22	perpendicular	2	-	D_2

The **Euler characteristic** of the orbifold is 2, i.e. the orbifold when viewed as a polygon with faces, edges and vertices, $\chi =$ V - E + F.

Given the following assigned "costs", the total "cost" of orbifold signature of each Wallpaper Group comes out to be 2:

- A digit n without or before * counts as $\frac{(n-1)}{n}$.
- A digit n after * counts as $\frac{(n-1)}{2n}$.
- Both * and $\times$ count as 1.
- $\circ$ counts as 2.

A calculation using this also shows that there are exactly 17 Wallpaper groups.