Hurewicz Theorem

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0.1 Introduction

Both the homotopy groups and the homology groups provide important insights into the properties of a space, especially so if the space happens to be path-coonected. Homology groups are rather easy to compute, using methods like excision and Mayer-Vietoris sequences, whereas in case of homotopy groups, only π_1 involves less computation, thanks to Van-Kampen's theorem etc. The higher homotopy groups are much tedius to compute. So if there is a relation between the homology groups and homotopy groups of a space under some conditions on the nature of the space, we can directly compute the homotopy groups.
It turns out that given $n > 1$, for a (n-1) connected space, H_i is 0 $\forall i < n$, and the n-th homotopy group is isomorphic to the n-th homology group. But homology groups are necessarily abelian, and the fundamental group of a space need not always be abelian, hence we cannot have such an isomorphism at the n = 1 level. Nevertheless, there is a relation for n = 1 as well. We would typically examine the abelianization of the fundamental group as a possible candidate for the isomorphism.

0.2 Objective

In this article, our goal is to show that the abelianization of π_1, i.e. the quotient group $\pi_1/[\pi_1, \pi_1]$ is isomorphic to H_1 for any path-connected space.

0.3 Proof

0.3.1 Definitions and Concepts Required

Definition: A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \to X$. We define $C_n(X)$ as the free abelian group generated by all such singular n-simplices in X. So the elements of $C_n(X)$, called **singular n-chains**, are finite formal sums $\sum_i n_i\sigma_i$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$.

Define the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_n]}$$

It is easy to check that Im $\partial_{n+1} \subset$ Ker ∂_n. An element of Ker ∂_n is called a **cycle** and an element of Im ∂_n is called a **boundary**. We define the **n-th singular homology group** $H_n(X)$ as Ker ∂_n/ Im ∂_{n+1}.

0.3.2 *Elements of* $H_1(X)$

$C_1(X)$ is generated by singular 1-simplices. A 1-simplex being just one line segment with end points v_0 and v_1, with a given orientation as in the diagram,

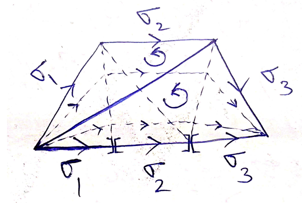

Figure 1: Constructing 2-simplices for k=3

we can identify a 1-simplex with the unit interval I = [0,1]. Consequently a singular 1-simplex can be identified with a path in X.

For any $\sigma \in$ Ker ∂_1, we can then identify σ with a closed oriented loop, since $\partial_1(\sigma) = \sigma|_{[v_1]}$ - $\sigma|_{[v_0]} = 0$, so that the corresponding path starts and ends at the same point in X, and the orientation of the loop is as induced from the 1-simplex $[v_0, v_1]$.

Claim: The formal sum of finitely many singular 1-simplices which has image 0 under ∂_1, i.e. the element $\sum_{i=1}^{k} \sigma_i$ (repititions of the same singular 1-simplex allowed) of Ker ∂_1 such that $\sigma_{i-1}|_{[v_1]} = \sigma_i|_{[v_0]}$ $\forall i \geq 2, i \leq k$ and $\sigma_1|_{[v_0]} = \sigma_k|_{[v_1]}$ upto signs and a suitable rearrangement of indices, **can be represented by a loop in X**.

Proof: Since the endpoints of the paths corresponding to σ_is in X match, we can talk about concatenation of such paths. First let us assume $\sigma_1|_{[v_0]}$ = $\sigma_k|_{[v_1]} = x_0$. Representing the path in X corresponding to σ_i by σ_i, we have $(\sigma_1 * \sigma_2 * ... * \sigma_k)$ $-\sigma_1 - \sigma_2 - ... - \sigma_k \in$ Im ∂_2, i.e. $[\sigma_1 * ... * \sigma_k]_H = \sum_{i=1}^{k}[\sigma_i]_H$. [Fig. 1 and 2 show the construction of the suitable 2-simplices for the cases k = 3 and 4 respectively, the boundary of the 2-simplices having sums $(\sigma_1 * \sigma_2 * \sigma_3) - \sigma_1 - \sigma_2 - \sigma_3$ in Fig. 1 and $(\sigma_1 * \sigma_2 * \sigma_3 * \sigma_4) - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4$ in Fig. 2. We can construct such 2-simplices out of a $(n+1)$-gon for k = n in a similar way.]

If $\sigma_1|_{[v_0]} = \sigma_k|_{[v_1]} = x_1 \neq x_0$, we can choose a path γ from x_0 to x_1 and consider the 1-simplices $\gamma * \sigma_1$ and $\sigma_k * \bar{\gamma}$ instead of σ_1 and σ_k respectively. $[\gamma * \sigma_1 * ... * \sigma_k * \bar{\gamma}]_H = [\sigma_1 * ... * \sigma_k]_H$ by the argument demonstrated by Fig. 5.

Claim: Any element of Ker ∂_1 can be written as a finite sum of such loops.
Proof: Let $(n_1\sigma_1 + n_2\sigma_2) \in$ Ker ∂_1, i.e. $\partial_1(n_1\sigma_1 + n_2\sigma_2) = (n_1\sigma_1|_{[v_1]} + n_2\sigma_2|_{[v_1]})$ - $(n_1\sigma_1|_{[v_0]} + n_2\sigma_2|_{[v_0]}) = 0$, and none of σ_1 and σ_2 is a cycle.
$\therefore$ Either $n_1\sigma_1|_{[v_1]} = -n_2\sigma_2|_{[v_1]}$ or $n_1\sigma_1|_{[v_1]} = n_2\sigma_2|_{[v_0]}$. In the first case, $n_1 = -n_2$ and $\sigma_1|_{[v_1]} = \sigma_2|_{[v_1]}$, also $\sigma_1|_{[v_0]} = \sigma_2|_{[v_0]}$. Then $(n_1\sigma_1 + n_2\sigma_2) = n_1(\sigma_1 - \sigma_2)$, $(\sigma_1 - \sigma_2)$ is a 1-cycle, i.e. a loop. In the second case, $n_1 = n_2$, $\sigma_1|_{[v_1]} = \sigma_2|_{[v_0]}$, and $\sigma_1|_{[v_0]} = \sigma_2|_{[v_1]}$, so that $(n_1\sigma_1 + n_2\sigma_2) = n_1(\sigma_1 + \sigma_2)$,

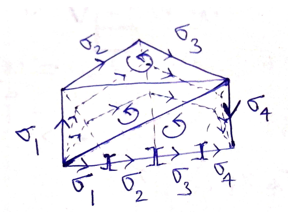

Figure 2: Constructing 2-simplices for k=4

$(\sigma_1 + \sigma_2)$ is a 1-cycle, i.e. a loop.
Now we use induction on the number of singular 1-simplices in the finite formal sum. Let us assume that any element of Ker ∂_1 that is written as formal sum of at most k singular 1-simplices with integer coefficients, none of which are cycles themselves, can be expressed as a finite sum of 1-cycles. We will show that the same holds for $\sum_{i=1}^{k+1} n_i\sigma_i \in$ Ker ∂_1, where none of the σ_i's are cycles themselves. Since $\partial_1(\sum_{i=1}^{k+1} n_i\sigma_i) = (\sum_{i=1}^{k+1} n_i\sigma_i|_{[v_1]}) - (\sum_{i=1}^{k+1} n_i\sigma_i|_{[v_0]}) = 0$, $\exists\ j_1 \in \{2, ..., k+1\}$ such that either $\sigma_{j_1}|_{[v_0]} = \sigma_1|_{[v_1]}$, or $\sigma_{j_1}|_{[v_1]} = \sigma_1|_{[v_0]}$. In the first case, we can again find a $j_2(\neq j_1)$ in the index set such that either $\sigma_{j_1}|_{[v_1]} = \sigma_{j_2}|_{[v_0]}$ or $\sigma_{j_1}|_{[v_1]} = \sigma_{j_2}|_{[v_1]}$, and similarly a choice of σ_i exists in the second case as well. This process would end after a finite number of steps, i.e. $\exists\ \sigma_{j_n} \in \{\sigma_2, ..., \sigma_{k+1}\}$, $j_n \notin \{j_1, j_2, ..., j_{n-1}\}$ chosen inductively by the same algorithm such that either $\sigma_{j_n}|_{[v_0]} = \sigma_1|_{[v_0]}$ or $\sigma_{j_n}|_{[v_1]} = \sigma_1|_{[v_0]}$, since the sum is a finite sum. Hence we have a 1-cycle $(\sigma_1 \pm \sigma_{j_1} \pm ... \pm \sigma_{j_n})$, the signs depending on which endpoint of the latter matches the image of v_1 under the former. The remaining part of the sum contains $(n_1 - 1)$ copies of σ_1, so we can perform this process $(n_1 - 1)$ more times to get, in total, a decomposition of the original cycle into n_1 such 1-cycles containing σ_1 and a formal sum of at most k singular 1-simplices with integer coefficients. This remaining part of the sum can again be decomposed into 1-cycles by our induction hypothesis, thus we have a complete decomposition of $\sum_{i=1}^{k+1} n_i\sigma_i$ into finite sum of loops in X.

This motivates a natural map from $\pi_1(X)$ to $H_1(X)$ when X is non-empty and path-connected, known as the Hurewicz map.

0.3.3 The Hurewicz Map

We choose a basepoint $x_0 \in X$. Define $h : \pi_1(X, x_0) \to H_1(X)$ as $h([\alpha]) = [\alpha]_H$ where α is a loop (therefore also a singular 1-simplex) at x_0, $[\alpha]$ is the homotopy class of α, and $[\alpha]_H$ is the homology class (i.e. the coset α + Im ∂_2) of the singular 1-simplex α.

h is well-defined: Let α and β be path-homotopic in X, i.e. $\exists$ a continuous map $H : I \times I \to X$ with H(t,0) = $\alpha(t)$, H(t,1) = $\beta(t)$, H(0,t) = H(1,t) = x_0.

We construct two 2-simplices and give them orientations as indicated in the

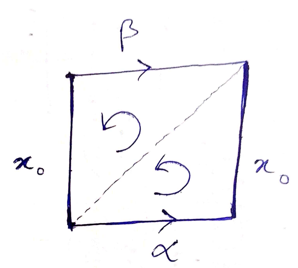

Figure 3: Constructing 2-simplices out of the homotopy

figure. Denote the 2-simplex containing the lower edge of $I \times I$ by A, and the other 2-simplex as B. Then $H|_A$ and $H|_B$ are singular 2-simplices, and $\partial_2(H|_A + H|_B) = \alpha - \beta$. So $\alpha - \beta \in \text{Im } \partial_2$, or $[\alpha]_H = [\beta]_H$.

h is a homomorphism: Let $[\alpha]$, $[\beta] \in \pi_1(X, x_0)$. By definition of h, $h([\alpha] * [\beta]) = [\alpha * \beta]_H$.

Claim: $[\alpha * \beta]_H = [\alpha]_H + [\beta]_H$, i.e. $(\alpha * \beta) - \alpha - \beta \in \text{Im } \partial_2$.
Proof: Taking a 2-simplex with orientation as indicated in Fig. 4, we construct a continuous map σ from this simplex to X which restricts to α in half of the base and to β in the other half, through appropriate reparametrisation, so that the restriction on the whole base is $\alpha * \beta$. Then, we take the map to be constant on the lines parallel to the height of this triangle, so that restriction of σ on one of the other two edges is α, and on the other one is β, as indicated in the figure. Thus we have a singular 2-simplex σ such that, taking into account the orientations in Fig 2, $\partial_2(\sigma) = (\alpha * \beta) - \alpha - \beta$.

h is surjective: Let us consider an arbitrary element of $H_1(X)$, and let

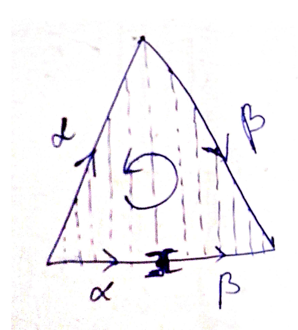

Figure 4: Constructing a 2-simplex with boundary $(\alpha * \beta) - \alpha - \beta$

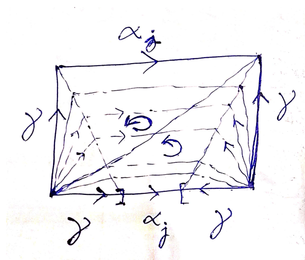

Figure 5: $\gamma * \alpha_j * \bar{\gamma}$ is homologous to α_j

$\sum_i n_i\sigma_i \in$ Ker ∂_1 be a representative of that homology class. According to our previous comment on the elements of $H_1(X)$, $\sum_i n_i\sigma_i$ can be written as a finite sum of loops in X, say $\sum_i n_i\sigma_i = \alpha_1 + \alpha_2 + ... + \alpha_n$ where each α_j is a closed oriented loop in X. So if α_j is based at x_0 $\forall j$, we have $[\sum_i n_i\sigma_i]_H = [\alpha_1]_H + [\alpha_2]_H + ... + [\alpha_n]_H = [\alpha_1 * ... * \alpha_n]_H$, so that $h([\alpha_1 * ... * \alpha_n]) = [\sum_i n_i\sigma_i]_H)$, since h is a homomorphism.

If α_j is not a loop at x_0 for some j: Let α_j be based at $x_1 \in X$. Since X is path-connected, we have a path γ from x_0 to x_1. Then $\gamma * \alpha_j * \bar{\gamma}$ is a loop based at x_0.
We construct a continuous map from a square as in Fig 5 and divide the square into two 2-simplices by the diagonal along with the orientations as in the figure. Then we have two singular 2-simplices, say L and U, so that $\partial_2(L + U) = (\gamma * \alpha_j * \bar{\gamma}) + \gamma - \alpha_j - \gamma = (\gamma * \alpha_j * \gamma) - \alpha_j$, so $[\gamma * \alpha_j * \bar{\gamma}]_H = [\alpha_j]_H$. So we can replace α_j by $\alpha_j^{'} = \gamma * \alpha_j * \bar{\gamma}$ in the sum, and then write $[\sum_i n_i\sigma_i]_H$ $= [\alpha_1 * ... * \alpha_j^{'} * ... * \alpha_n]_H$, which has pre-image $[\alpha_1 * ... * \alpha_j^{'} * ... * \alpha_n]$ in $\pi_1(X)$.

$\underline{\textit{Ker } h = [\pi_1, \pi_1]}$: First, we observe that $[\pi_1, \pi_1] \subset$ Ker h. Let $[\gamma_1] * [\gamma_2] * [\bar{\gamma_1}] * [\bar{\gamma_2}] \in [\pi_1, \pi_1]$. Since h is a homomorphism, $h([\gamma_1 * \gamma_2 * \bar{\gamma_1} * \bar{\gamma_2}]) = [\gamma_1]_H + [\gamma_2]_H + [\bar{\gamma_1}]_H + [\bar{\gamma_2}]_H = [\gamma_1]_H + [\gamma_2]_H - [\gamma_1]_H - [\gamma_2]_H = 0$.

Next, we claim that Ker h $\subset [\pi_1, \pi_1]$.
Proof: Let $[\alpha] \in$ Ker h, i.e. $\exists \sum_i n_i\sigma_i \in C_2(X)$ such that $\partial_2(\sum_i n_i\sigma_i) = \alpha$, α being a loop at x_0. Allowing repetitions of σ_i's, we can take $n_i = \pm 1$ for each i.

Now, for each singular 2-simplex σ_i in this sum, $\partial_2(\sigma_i) = \tau_{i0} - \tau_{i1} + \tau_{i2}$ for some singular 1-simplices τ_{ij}. Since $\sum_i n_i\sigma_i = \alpha$, we can group the τ_{ij}'s into cancelling pairs, leaving out only one τ, which is equal to α. This would imply that there has to be an odd number of non-trivial 2-simplices in the sum.

Let there be only one 2-simplex in the sum, i.e. $\exists \sigma : \Delta^2 \to X$ such that $\partial_2(\sigma) = \alpha$. σ sends the three faces of Δ^2 to τ, $(-\tau)$ and α in that case. α

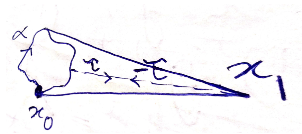

Figure 6: Construction with only one 2-simplex

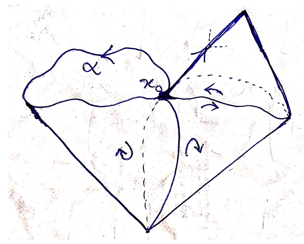

Figure 7: Case 1

being a loop at x_0, τ has x_0 as one end-point. $\tau * (-\tau)$, or $\tau * \bar{\tau}$ is a loop at x_0. The image of σ in X would, then, form a cone (ref. Fig. 6) with base being the loop α and τ and $(-\tau)$ along the same path on the conical surface, but opposite directions.

The cone is contractible, so its base α is nullhomotopic, therefore $[\alpha] \in [\pi_1, \pi_1]$.

Let's examine the case when there are 3 non-trivial 2-simplices in the sum. We can say that one edge of a 2-simplex is mapped to the loop α through the corresponding singular 2-simplex, and the other two edges do not cancel each other. Each of these two edges are cancelled in the sum by some edge of the remaining 2-simplices. Here we have 2 possibilities:

1. The cancelling edges both belong to the same 2-simplex (ref. Fig 7)
   We have three cones, two glued along the cancelling edges and the other glued to the cone not having α as base along the base.

2. One edge from each remaining 2-simplex cancels each of these two edge (ref. Fig 8)
   γ_1 and $\tilde{\gamma_1}$ cancel each other, and so do γ_2 and $\tilde{\gamma_2}$. So these are also identified in pairs, making the two open flaps in Fig 8 closed into one disk whose boundary is identified to the loop constructed by consecutively traversing the two remaining edges of the 2-simplex containing α. This space deformation retracts to a cone which is again contractible, so in this case α is nullhomotopic, therefore $[\alpha] \in [\pi_1, \pi_1]$.

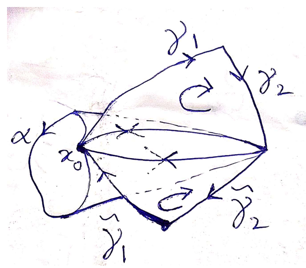

Figure 8: Case 2

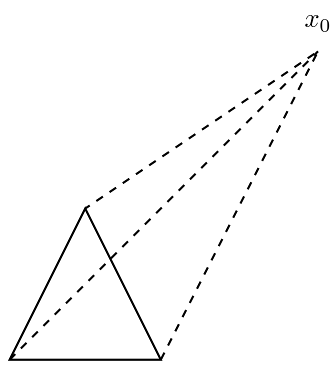

Figure 9: Boundary of each 2-simplex

Now, for any such 2-simplex, if each edge of the simplex gives a loop at x_0 in X, then we can represent the boundary of the singular 2-simplex as concatenation of three loops at x_0. Otherwise, since X is path-connected, we can choose paths from endpoints of each edge to x_0 as in Fig. 9, thereafter concatenating them to the images of each edge of the 2-simplex in X, so that we have actual loops at x_0, the homology classes still remaining the same as the original simplices by a construction similar to Fig. 5. So the boundary of each 2-simplex is represented by a concatenation of three loops at x_0, say $a * b * c$, which is nullhomotopic (since 2-simplex is homeomorphic to a disk).

In the abelianization of π_1, the boundary of the sum $\sum_i n_i\sigma_i$ can be represented by product of such nullhomotopic 3-products of loops at x_0, which is the identity element in $\pi_1/[\pi_1, \pi_1]$. On the other hand, since the boundary of $\sum_i n_i\sigma_i$ consists of cancelling pairs of 1-simplices and α, this product would be, in the abelianization of π_1, the class of $[\alpha]$. Hence the class of $[\alpha]$ is the identity element in $\pi_1/[\pi_1, \pi_1]$, therefore $[\alpha]$ is in $[\pi_1, \pi_1]$.

Hence, we have an isomorphism from $\pi_1(X, x_0)/[\pi_1, \pi_1]$ to $H_1(X)$ induced by h.

0.4 *Remarks*

Given $\sum_i n_i\sigma_i \in C_2(X)$ such that $\partial_2(\sum_i n_i\sigma_i) = \alpha$ where $[\alpha] \in$ Ker h, the singular 2-simplices clearly give us a Δ-complex structure, where the triangles given by each of the σ_i's are attached along their boundaries, and the boundary of the surface thus formed is α, since all the other boundary terms form cancelling pairs by the argument given in the above proof. Also this surface is compact and orientable. So by **Classification Theorem of Surfaces**, this surface is in fact homeomorphic to a closed orientable surface of some genus g with an open disk removed. Such a surface can be represented by identifying pairs of edges of a polygon with $4g$ edges as shown in Fig. 10 for the case g = 2. Hence α would be homotopic to the product of g commutators, which intuitively justifies the fact that the kernel of h is contained in the commutator subgroup of π_1.

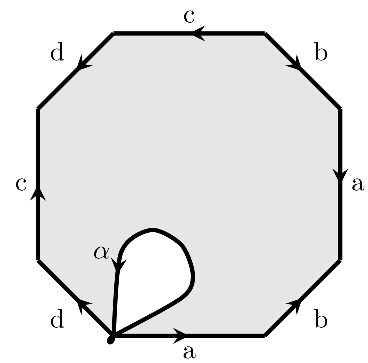

Figure 10: g = 2

0.5 Discussions

1) If π_1 is abelian, the commutator subgroup must be trivial. Then the abelianization of π_1 is isomorphic to π_1 itself, thereby giving an isomorphism between π_1 and H_1. A simple example would be S^1.

2) If X is a simply connected space, then by this result, $H_1(X) = 0$.

0.6 References and Notes

1) *Algebraic Topology*, A. Hatcher. Cambridge University Press, 2002.
2) *Basic Topology*, M. A. Armstrong.