Homotopy Theory of Wedge of Spheres

A Project Report Submitted
in Partial Fulfilment of the Requirements
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by

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DECLARATION

I, **Arundhati Roy (Roll No: 18MS103)**, hereby declare that, this report entitled **“Homotopy Theory of Wedge of Spheres”** submitted to Indian Institute of Science Education and Research Kolkata towards partial requirement of **BS-MS Dual Degree**, **Major in Mathematics** is an original work carried out by me under the supervision of **Dr. Somnath Basu** and has not formed the basis for the award of any degree or diploma, in this or any other institution or university. I have sincerely tried to uphold the academic ethics and honesty. Whenever an external information or statement or result is used then, that have been duly acknowledged and cited.

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CERTIFICATE

This is to certify that the work contained in this project report entitled **"Homotopy Theory of Wedge of Spheres"** submitted by **Arundhati Roy** (**Roll No: 18MS103**) to Indian Institute of Science Education and Research Kolkata towards partial requirement of **BS-MS dual degree** majoring in Mathematics has been carried out by her under my supervision and that it has not been submitted elsewhere for the award of any degree.

IIISER Kolkata | **Dr. Somnath Basu**

May 2023 | Project Supervisor

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ABSTRACT

In the first three chapters of this report I have discussed basic definitions, properties and some important theorems along with examples related to homotopy groups and CW-complexes. The fourth chapter is about homology groups of spaces and how they are related to homotopy groups of the same space. Then I have described the Whitehead products along with a few examples and properties, proceeding to their usage in the main theorem of this project, Hilton's theorem, using which I have computed some homotopy groups of wedges of spheres.

Contents

Notations

D^n the unit disk of dimension n.

I the unit interval $[0,1]$ in $\mathbb{R}$.

S^n the unit sphere of dimension n.

$\cong$ isomorphic to, or in different context, homeomorphic to.

$\mathbb{R}$ the set of real numbers.

$\mathbb{Z}$ the additive group of integers.

∂X boundary of a space X.

$\simeq$ homotopic to.

$f \circ g$ composition of the two maps f and g.

$\mathbf{Im} f$ Image set of the map f.

$\mathbf{Ker} f$ Kernel of the map f.

Chapter 1

Homotopy Theory : Basic Concepts

1.1 Homotopy Groups

1.1.1 Introduction

Homotopy gives an equivalence relation between maps of spaces, and collection of such equivalence classes of maps from unit n-cubes to a topological space that takes boundary of the n-cube to a basepoint depends only on the topological properties of the space, and thus gives important insight into the structure of the space.

In this chapter, we will discuss how such collections of equivalence classes turn out to have group structures when equipped with some very specific operations (this group is what is called a *homotopy group*), some of the basic facts about such groups and over the next chapters, we shall proceed to talk about how these groups are connected to other invariants of the same space. We shall see how such groups 'look' in case of spheres, which is one of the most interesting among the simplest topological spaces. Not all such groups for arbitrary spheres are easily describable, but we can compute some of them without much trouble, and we shall assume the knowledge of the rest for the next part.

In homotopy theory, new spaces constructed through operations like **suspension**, **wedge**, **join**, **smash** are often examined in terms of their homotopy groups to understand the properties of the new space thus constructed. Now in case of spheres, **wedge** is the only operation among these which constructs a space which does not 'look like' a sphere of any finite dimension even though the input spaces were spheres. But computing homotopy groups of wedges of spheres is quite a non-trivial task. Our final goal therefore, is to discuss a result that would possibly help in computing such groups in terms of homotopy groups of spheres, which we have assumed to be known.

Definition 1.1.1. $\pi_n(X, x_0)$ is the set of homotopy classes of maps $(I^n, \partial I^n) \longrightarrow (X, x_0)$ (or equivalently $(S^n, s_0) \longrightarrow (X, x_0)$), through homotopies of the same form. Henceforth we denote an element of $\pi_n(X, x_0)$ by $[f]$.

For example, $\pi_1(X, x_0)$, the set of based homotopy classes of loops at x_0. We may extend the definition to the case $n = 0$ by considering I^0 to be a point and ∂I^0 to be empty, so that $\pi_0(X, x_0)$ is just the set of path components of X.

1.1.2 The Group structure

: Given two paths f, g in X, we define the **composition** of f and g, denoted by '$*$' as

$$(f * g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \tag{1.1}$$

$\pi_1(X, x_0)$ assumes a group structure with the operation '$*$' given by

$$[f] * [g] = [f * g]$$

This group is known as the **fundamental group** of X with the basepoint x_0. (Ref. [5] Chapter 1)

Now, we can generalise this operation to $n \geq 2$. We consider the operation '$+$' given by

$$(f + g)(s_1, s_2, ..., s_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, ..., s_n) & \text{if } \frac{1}{2} \leq s_1 \leq 1 \end{cases} \tag{1.2}$$

By a similar argument as in case of π_1, we can show that this operation is well-defined with respect to homotopy classes and gives π_n a group structure. Here, the identity element of the group is the constant map from I^n to $x_0 \in X$, and the inverse of $[f]$ is the homotopy class of $\bar{f}$, which is defined by $\bar{f}(s_1, s_2, ..., s_n) = f(1 - s_1, s_2, ..., s_n)$. In fact, we can show that similar properties hold for all operations '$+_i$', $2 \leq i \leq n$ where

$$(f +_i g)(s_1, ..., s_i, ..., s_n) = \begin{cases} f(s_1, ..., 2s_i, ..., s_n) & \text{if } 0 \leq s_i \leq \frac{1}{2} \\ g(s_1, ..., 2s_i - 1, ..., s_n) & \text{if } \frac{1}{2} \leq s_i \leq 1 \end{cases} \tag{1.3}$$

the identity element being the same for all i. The following lemma then leads to a very important property of homotopy groups through what is called the **Eckmann-Hilton argument**. (Ref. [4])

Lemma 1.1.2. *Let '$\times$' and '$\otimes$' be two binary operations with identity on the same set X with the property*

$$(a \times b) \otimes (c \times d) = (a \otimes c) \times (b \otimes d)$$

for all $a, b, c, d \in X$. Then '$\times$' and '$\otimes$' are the same operation, which is commutative and associative as well.

Proof. (i) '$\times$' and '$\otimes$' have the same identity : We denote the respective identities by $1_\times$ and $1_\otimes$. Then

$$1_\times = 1_\times \times 1_\times = (1_\times \otimes 1_\otimes) \times (1_\otimes \otimes 1_\times) = (1_\times \times 1_\otimes) \otimes (1_\otimes \times 1_\times) = 1_\otimes \otimes 1_\otimes = 1_\otimes$$

We will, henceforth, denote this identity element of '$\times$' and '$\otimes$' by 1.

(ii) $\times$ and $\otimes$ are the same operation : For any $a, b \in X$,

$$(a \times b) = (a \otimes 1) \times (1 \otimes b) = (a \times 1) \otimes (1 \times b) = (a \otimes b)$$

(iii) The operation is commutative : For any $a, b \in X$,

$$(a \times b) = (1 \otimes a) \times (b \otimes 1) = (1 \times b) \otimes (a \times 1) = (b \otimes a) = (b \times a)$$

(iv) The operation is associative : For any $a, b, c \in X$,

$$(a \times b) \times c = (a \times b) \times (1 \times c) = (a \times b) \otimes (1 \times c) = (a \otimes 1) \times (b \otimes c) = a \times (b \times c)$$

□

Proposition 1.1.3. *For any* $[f], [g], [h], [k] \in \pi_n(X, x_0)$ *,* $n \geq 2$*,*

$$([f] +_2 [g]) + ([h] +_2 [k]) = ([f] + [g]) +_2 ([h] + [k])$$

Proof.

$$(f +_2 g) + (h +_2 k)(t_1, t_2, ..., t_n) = \begin{cases} f(2t_1, 2t_2, t_3, ..., t_n) & \text{if } t_1 \leq \frac{1}{2}, t_2 \leq \frac{1}{2} \\ g(2t_1, 2t_2 - 1, t_3, ..., t_n) & \text{if } t_1 \leq \frac{1}{2}, t_2 \geq \frac{1}{2} \\ h(2t_1 - 1, 2t_2, t_3, ..., t_n) & \text{if } t_1 \geq \frac{1}{2}, t_2 \leq \frac{1}{2} \\ k(2t_1 - 1, 2t_2 - 1, t_3, ..., t_n) & \text{if } t_1 \geq \frac{1}{2}, t_2 \geq \frac{1}{2} \end{cases} \tag{1.4}$$

$$(f + g) +_2 (g + k)(t_1, t_2, ..., t_n) = \begin{cases} f(2t_1, 2t_2, t_3, ..., t_n) & \text{if } t_1 \leq \frac{1}{2}, t_2 \leq \frac{1}{2} \\ h(2t_1 - 1, 2t_2, t_3, ..., t_n) & \text{if } t_1 \geq \frac{1}{2}, t_2 \leq \frac{1}{2} \\ g(2t_1, 2t_2 - 1, t_3, ..., t_n) & \text{if } t_1 \leq \frac{1}{2}, t_2 \geq \frac{1}{2} \\ k(2t_1 - 1, 2t_2 - 1, t_3, ..., t_n) & \text{if } t_1 \geq \frac{1}{2}, t_2 \geq \frac{1}{2} \end{cases} \tag{1.5}$$

(1.4) and (1.5) are in fact the same as maps from I^n to X, hence they represent the same elements at the level of homotopy classes. □

Using Lemma 1.1.2, we can now conclude the following :

Theorem 1.1.4. *'+' is commutative, so* $\pi_n(X, x_0)$ *is abelian for any* $n \geq 2$*.*

This argument is in fact, not applicable in case of $n = 1$, since we cannot define '$+_2$' in π_1. Fig gives an intuitive illustration of this. Clearly, the homotopy shown in the figure cannot be defined for $n = 1$. This is, of course, consistent with the fact that π_1 is not necessarily abelian for arbitrary topological spaces, e.g. the figure eight space ($S^1 \vee S^1$) has $\mathbb{Z} * \mathbb{Z}$ as fundamental group.

Next we show that π_n for $n \geq 2$ is 'independent of basepoint' for path-connected spaces, just like π_1.

Proposition 1.1.5. *Let* X *be a path-connected space. Then for any two points* x_0*,* x_1 *in* X*,* $\pi_n(X, x_0)$ *and* $\pi_n(X, x_1)$ *are isomorphic.*

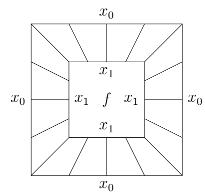

Figure 1.1: Defining γf

Proof. Given a path $\gamma : I \to X$ from x_0 to x_1, we can associate a map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ to each map $f : (I^n, \partial I^n) \to (X, x_1)$ by shrinking the domain of f to a smaller concentric cube contained in I^n, then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n, as shown in Fig 1.1.
A homotopy of γ or f through maps fixing ∂I or ∂I^n respectively, gives a homotopy of γf through maps $(I^n, \partial I^n) \to (X, x_0)$. Also, $\gamma(f+g) \simeq \gamma f + \gamma g$. To see this, consider the homotopy indicated in Fig 1.2, through shrinking the middle slab of $\gamma(f+0) + \gamma(0+g)$ until we have $\gamma(f+g)$. An explicit formula for this homotopy could be

$$h_t(s_1, s_2, ..., s_n) = \begin{cases} \gamma(f+0)((2-t)s_1, s_2, ..., s_n) & \text{if } s_1 \in [0, \frac{1}{2}] \\ \gamma(0+g)((2-t)s_1 + (t-1), s_2, s_3, ..., s_n) & \text{if } s_1 \in [\frac{1}{2}, 1] \end{cases} \tag{1.6}$$

Also, we have two more homotopies which are clear from the definition: $(\gamma\eta)f \simeq \gamma(\eta f)$ and $0f \simeq f$ where 0 in each case denotes the constant map at x_1. Using all these, we conclude that the change-of-basepoint transformation $\beta_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ defined by $\beta_\gamma([f]) = [\gamma f]$ is a group isomorphism, with inverse map being $\beta_{\bar{\gamma}}$ where $\bar{\gamma}$ is the inverse path of γ.

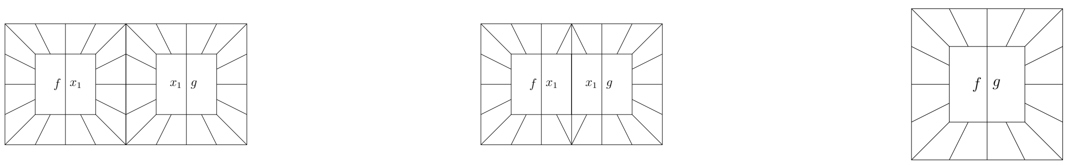

Figure 1.2: Homotopy of $\gamma f + \gamma g$ to $\gamma(f+g)$

For $n = 1$ the notation is somewhat ambiguous as γf by this definition should be the concatenation of three paths, γ, f, and $\bar{\gamma}$ consecutively, but the notation resembles that for concatenation of γ with f. $\square$

Henceforth, we may denote the n-th homotopy group of X by $\pi_n(X)$ without explicitly mentioning the basepoint whenever the space is path-connected, which will always be the case in the rest of this text unless otherwise mentioned.

Action of π_1 ***on*** π_n**:** Consider any loop γ at the basepoint x_0. Since $(\gamma\eta)f \simeq \gamma(\eta f)$, we have $\beta_{\gamma\eta} = \beta_\gamma\beta_\eta$. Hence the association $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1(X, x_0)$ to $Aut(\pi_n(X, x_0))$, the group of automorphisms of $\pi_n(X, x_0)$. This is called the 'action of π_1 on π_n'.

For $n > 1$, the action makes the abelian group $\pi_n(X, x_0)$ into a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Elements of $\mathbb{Z}[\pi_1]$ are $\sum_i n_i\gamma_i$ with $n_i \in \mathbb{Z}$ and $\gamma_i \in \pi_1(X, x_0)$, multiplication on the group ring being defined by distributivity and multiplication in π_1. The module structure on π_n is given by $(\sum_i n_i\gamma_i)\alpha = \sum_i n_i(\gamma_i\alpha)$ for $\alpha \in \pi_n$. For brevity we may say that π_n is a 'π_1-module' rather than $\mathbb{Z}[\pi_1]$-module.

Maps induced on π_n ***from the topological space:*** Any map $\varphi : (X, x_0) \to (Y, y_0)$ induces a map $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by $\varphi_*([f]) = [\varphi \circ f]$. Well-definedness of this map is clear from the definition. Also, this is a homomorphism for $n \geq 1$.

Next we state two very important properties of π_n :

Proposition 1.1.6. *Based homotopy equivalence between two spaces induce isomorphism between n-th homotopy groups for all n.*

The fundamental group homomorphisms induced by covering maps are not necessarily always isomorphisms. But such induced homomorphisms of π_n behave nicely for $n \geq 2$.

Proposition 1.1.7. *A covering map* $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ *induces isomorphisms* $p_* : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ *for* $n \geq 2$.

Proof. By the lifting criterion, every map $(S^n, s_0) \to (X, x_0)$ lifts to $(\tilde{X}, \tilde{x_0})$ for $n \geq 2$, since S^n is simply-connected for $n \geq 2$. Therefore p_* is surjective. Injectivity follows by similar argument. $\square$

In particular, $\pi_n(X, x_0) = 0$ for $n \geq 2$ whenever X has a contractible cover. For example, S^1, which has $\mathbb{R}$ as its universal cover. Also, consider the n-dimensional torus T^n, which is the product of n copies of S^1, has universal cover $\mathbb{R}^n$, hence $\pi_i(T^n) = 0$ for $i > 1$; whereas in case of homology groups, $H_i(T^n)$ are nonzero for all $i \leq n$. Spaces with $\pi_n = 0$ for all $n \geq 2$ are called **aspherical**.

The behaviour of π_n in case of product spaces is simple.

Proposition 1.1.8. *For an arbitrary collection $\{X_\alpha\}_{\alpha \in I}$ of path-connected spaces, $\pi_n(\prod_\alpha X_\alpha) \approx \prod_\alpha \pi_n(X_\alpha)$*

We end this chapter with the definition of an important term which we shall use in a theorem of utmost importance in our text.

Definition 1.1.9. A space X with basepoint x_0 is said to be n-connected if $\pi_i(X, x_0) = 0 \ \forall i \leq n$.

So 0-connected means path-connected and 1-connected means simply connected. Note that by definition, n-connected implies 0-connected as well, so the choice of the basepoint x_0 does not matter.

1.2 CW Complex

CW-complexes are a very important class of topological spaces on their own right. This structure was first introduced by J.H.C. Whitehead specially for the purpose of homotopy theory. The 'C' stands for closure-finite, while the 'W' stands for weak topology. (Ref. [3])

A space constructed inductively in the following manner is known as a **CW-complex** or **cell complex**.

1. Consider a discrete set X^0, call the points of X^0 the '0-cells'. X^0 is called the 0-skeleton of the space we are constructing.

2. We form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. So X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha D^n_\alpha$, where D^n_α are n-disks, under the identifications $x \sim \varphi(x)$ for $x \in \partial D^n_\alpha$. The maps φ_α are called the **attaching maps**.

3. We can either stop this inductive process at some finite stage, setting $X = X^n$ for some finite n, or the process can be continued indefinitely, so that $X = \bigcup_{n\in\mathbb{N}} X^n$.

In the latter case, X is given the weak topology: a subset A of X is open (or closed) if and only if $A \cap X^n$ is open (or closed) in X^n. Also, if $X = X^n$ for some n, X is called finite-dimensional and the smallest such n is called the **dimension of** X.

Example 1.2.1. 1. A graph X can be constructed as a 1-dimensional cell complex, i.e. $X = X^1$ consisting of vertices (elements of X^0) and edges (elements of X^1) such that the boundaries of edges are identified with vertices.

2. The n-sphere S^n has the structure of a cell complex with just two cells, one 0-cell e^0 and one n-cell e^n, the attaching map being the constant map $S^{n-1} \to e^0$. This is equivalent to regarding S^n as the quotient space $D^n/\partial D^n$.

3. S^n has another possible CW-structure. Consider two 0-cells. Attaching the boundaries of two 1-cells to these gives S^1. Then attaching the boundaries of

two 2-cells to this S^1 gives S^2, and so on. Thus we get a CW-structure on S^n consisting two cells of dimension i for each $i \in 0, 1, ..., n$. However, the previous structure will be used for most of our discussions.

Each cell in a cell complex has a **characteristic map** $\Phi_\alpha : D^n_\alpha \to X$ which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α. Namely, we can take Φ^n_α to be the composition $D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \to X^n \hookrightarrow X$ where the middle map is the quotient map defining X^n. For example, in the canonical cell structure on S^n given by one 0-cell and one n-cell, a characteristic map for the n-cell would be the quotient map $D^n \to S^n$ collapsing ∂D^n to a point.

A **subcomplex** of a cell complex is a closed subspace $A \subset X$ that is a union of cells of X. Since A is closed, the characteristic map of each cell has image contained in A. In particular the image of the attaching map of each cell is contained in A. Hence A is a cell complex on its own. A pair (X, A) consisting of a CW-complex X and some subcomplex A of X is called a **CW-pair**.

Example 1.2.2.

1. Each skeleton X^n of a cell complex X is a subcomplex of X.

2. In the cell structure of S^n given by two cells of each dimension upto n, S^k is a subcomplex of S^n for each $k \leq n$. But S^k is not a subcomplex of S^n when the other cell structure consisting of only one 0-cell and one n-cell is considered.

3. For all the cell complexes described above, i.e. graph and two different CW-structures on S^n, closure of a cell inside the whole cell complex is in fact a subcomplex.

1.3 A Few Theorems on Homotopy Groups

1.3.1 Cellular Approximation

Definition 1.3.1. Let X and Y be two CW-complexes. A map $f : X \to Y$ is called a ***cellular map*** if $f(X^n) \subset Y^n$ for all n.

The next theorem, in fact, states that arbitrary maps of CW-complexes can always be deformed to cellular maps, which has an interesting consequence.

Theorem 1.3.2. *(Cellular Approximation Theorem) Every map $f : X \to Y$ of CW-complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on A.*

Proof. [5] page 349-351. □

Corollary 1.3.3. $\pi_n(S^k) = 0$ *for* $n < k$.

Proof. We consider the CW-structure of S^n to be made up of one 0-cell and one n-cell. Similarly the CW-structure of S^k has one 0-cell and one k-cell. We choose the image of the 0-cells under the characteristic maps as the respective basepoints. Now, any element of $\pi_n(S^k)$ is represented by a map from S^n to S^k which preserves the basepoint. By Theorem 1.3.2, such a map is homotoped to a cellular map fixing the basepoint. But such a map must be constant as the given CW-structure on S^k does not contain any n-cell for $n < k$. □

1.3.2 Weak Homotopy Equivalence and CW-approximation

Definition 1.3.4. A map $f : X \to Y$ is called a weak homotopy equivalence if it induces isomorphisms $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all $n \geq 0$ and all basepoints x_0.

We denote the set of homotopy classes of maps from X to Y by $[X, Y]$ and the set of pointed homotopy classes of pointed maps from X to Y by $\langle X, Y \rangle$.

Proposition 1.3.5. *A weak homotopy equivalence $f : Y \to Z$ induces bijections $[X, Y] \to [X, Z]$ and $\langle X, Y \rangle \to \langle X, Z \rangle$ for all CW-complexes X.*

Proof. [5] page 357. □

For every space X, there is a CW-complex Z and a weak homotopy equivalence $f : Z \to X$. Such a map $f : Z \to X$ is called a **CW-approximation** to X. A weak homotopy equivalence induce isomorphisms on all homology groups as well, so CW-approximations allow many general statements to be reduced to the case of CW-complexes, which are usually much easier to work with than arbitrary spaces.

Definition 1.3.6. Given a pair (X, A) where the subspace $A \subset X$ is a nonempty CW-complex, an **n-connected CW-model** for (X, A) is an n-connected CW-pair (Z, A) and a map $f : Z \to X$ with $f|_A$ the identity, such that $f_* : \pi_i(Z) \to \pi_i(X)$ is an isomorphism for $i > n$ and an injection for $i = n$, for all choices of basepoint.

Proposition 1.3.7. *For every pair (X, A) where A is a non-empty CW-complex there exist n-connected CW-models $f : (Z, A) \to (X, A)$ for all $n \geq 0$ and these models can be chosen to have the additional property that Z is obtained from A by attaching cells of dimension greater than n.*

Proof. [5] page 353. □

This proposition gives a geometric interpretation of n-connectedness defined at the end of the first section :

Corollary 1.3.8. *If (X, A) is an n-connected CW-pair, then there exists a CW-pair $(Z, A) \simeq (X, A)$ rel A such that all cells of $Z - A$ have dimension greater than n.*

The following result states the uniqueness of CW-approximation for a given space upto homotopy equivalence:

Proposition 1.3.9. *An n-connected CW model for (X, A) is unique up to homotopy equivalence rel A. In particular, CW-approximations to spaces are unique up to homotopy equivalence.*

Proof. [5] page 355. □

CW-approximations are used in the proof of the **Hurewicz Theorem** ([5] page 366-367) stated later.

Chapter 2

Using Homology to Compute Homotopy Groups

2.1 Introduction

Both the homotopy groups and the homology groups provide important insights into the properties of a space, especially so if the space happens to be path-coonected. Homology groups of various spaces are rather easy to compute, using methods like excision and Mayer-Vietoris sequences, whereas in case of homotopy groups, only π_1 involves less computation, thanks to Seifert-Van Kampen theorem. The higher homotopy groups are much tedius to compute. So if there is a relation between the homology groups and homotopy groups of a space under some conditions on the nature of the space, we can directly compute the homotopy groups.

It turns out that given $n>1$, for a $(n-1)$ connected space, H_i is 0 $\forall\ i<n$, and the n-th homotopy group is isomorphic to the n-th homology group. But homology groups are necessarily abelian, and the fundamental group of a space need not always be abelian. Nevertheless, there is a relation for $n = 1$ as well. We would typically examine the abelianization of the fundamental group as a possible candidate for the isomorphism.

2.2 Definitions

Definition 2.2.1. A **singular n-simplex** in a space X is a continuous map $\sigma : \Delta^n \to X$. We define $C_n(X)$ as the free abelian group generated by all such singular

n-simplices in X. So the elements of $C_n(X)$, called **singular n-chains**, are finite formal sums $\sum_i n_i\sigma_i$ with $n_i \in \mathbb{Z}$ and $\sigma_i : \Delta^n \to X$.

Define the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$$

It is easy to check that Im $\partial_{n+1} \subset$ Ker ∂_n. An element of Ker ∂_n is called a **cycle** and an element of Im ∂_n is called a **boundary**. We define the **n-th singular homology group** $H_n(X)$ as Ker $\partial_n/$ Im ∂_{n+1}.

Definition 2.2.2. Define the map $\varepsilon : C_0(X) \to \mathbb{Z}$ by $\sum_i n_i\sigma_i \mapsto \sum_i n_i$. Define the reduced homology groups of X,

$$\tilde{H}_n(X) := \begin{cases} Ker(\varepsilon)/Im(\partial_1) & \text{if } n = 0 \\ H_n(X) & \text{if } n \geq 1 \end{cases} \tag{2.1}$$

Proposition 2.2.3. *For any non-empty space X, $H_0(X) \cong \tilde{H}_0(X) \bigoplus \mathbb{Z}$.*

Proof. Ref. [5]. □

Proposition 2.2.4. *$H_n(S^n) \cong \mathbb{Z}$ for any $n \geq 1$.*

Proof. $\tilde{H}_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$ (from [5] page 114).

Now $\forall n \geq 1, \tilde{H}_n(X) = H_n(X)$, hence proved. □

2.3 Elements of $H_1(X)$

$C_1(X)$ is generated by singular 1-simplices. A 1-simplex being just one line segment with end points v_0 and v_1, with a given orientation as in Fig 2.1, we can identify a 1-simplex with the unit interval $I = [0, 1]$.

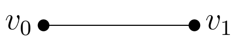

Figure 2.1: 1-simplex

Consequently a singular 1-simplex can be identified with a path in X.

For any $\sigma \in$ Ker ∂_1, we can then identify σ with a closed oriented loop, since $\partial_1(\sigma) = \sigma|_{[v_1]}$ - $\sigma|_{[v_0]} = 0$, so that the corresponding path starts and ends at the same point in X, and the orientation of the loop is as induced from the 1-simplex $[v_0, v_1]$.

Claim: The formal sum of finitely many singular 1-simplices which has image 0 under ∂_1, i.e., the element $\sum_{i=1}^{k} \sigma_i$ (repetitions of the same singular 1-simplex allowed) of Ker ∂_1 such that $\sigma_{i-1}|_{[v_1]} = \sigma_i|_{[v_0]}$ $\forall i \geq 2, i \leq k$ and $\sigma_1|_{[v_0]} = \sigma_k|_{[v_1]}$ upto signs and a suitable rearrangement of indices, **can be represented by a formal sum of loops in X.**

Proof: Since the endpoints of the paths corresponding to σ_i's in X match, we can talk about concatenation of such paths. First let us assume $\sigma_1|_{[v_0]} = \sigma_k|_{[v_1]} = x_0$. Representing the path in X corresponding to σ_i by σ_i, we have $(\sigma_1 * \sigma_2 * ... * \sigma_k)$ $-\sigma_1 - \sigma_2 - ... - \sigma_k \in$ Im ∂_2, i.e.

$$[\sigma_1 * ... * \sigma_k]_H = \sum_{i=1}^{k} [\sigma_i]_H$$

Fig. 2.2 and 2.3 show the construction of the suitable 2-simplices for the cases k = 3 and 4 respectively, the boundary of the 2-simplices having sums $(\sigma_1 * \sigma_2 * \sigma_3) - \sigma_1 - \sigma_2 - \sigma_3$ in Fig. 2.2 and $(\sigma_1 * \sigma_2 * \sigma_3 * \sigma_4) - \sigma_1 - \sigma_2 - \sigma_3 - \sigma_4$ in Fig. 2.3. We can construct such 2-simplices out of a $(n + 1)$-gon for k = n in a similar way.

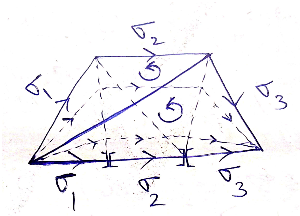

Figure 2.2: Constructing 2-simplices for k=3

If $\sigma_1|_{[v_0]} = \sigma_k|_{[v_1]} = x_1 \neq x_0$, we can choose a path γ from x_0 to x_1 and consider the 1-simplices $\gamma * \sigma_1$ and $\sigma_k * \bar{\gamma}$ instead of σ_1 and σ_k respectively. $[\gamma * \sigma_1 * ... * \sigma_k * \bar{\gamma}]_H$ $= [\sigma_1 * ... * \sigma_k]_H$ by the argument demonstrated by Fig. 5.

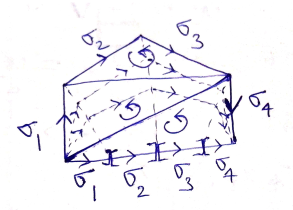

Figure 2.3: Constructing 2-simplices for k=4

Claim: Any element of Ker ∂_1 can be written as a finite sum of such loops.
Proof: Let $(n_1\sigma_1 + n_2\sigma_2) \in$ Ker ∂_1, i.e. $\partial_1(n_1\sigma_1 + n_2\sigma_2) = (n_1\sigma_1|_{[v_1]} + n_2\sigma_2|_{[v_1]})$ - $(n_1\sigma_1|_{[v_0]} + n_2\sigma_2|_{[v_0]}) = 0$, and none of σ_1 and σ_2 is a cycle.
$\therefore$ Either $n_1\sigma_1|_{[v_1]} = -n_2\sigma_2|_{[v_1]}$ or $n_1\sigma_1|_{[v_1]} = n_2\sigma_2|_{[v_0]}$. In the first case, $n_1 = -n_2$ and $\sigma_1|_{[v_1]} = \sigma_2|_{[v_1]}$, also $\sigma_1|_{[v_0]} = \sigma_2|_{[v_0]}$. Then $(n_1\sigma_1+n_2\sigma_2) = n_1(\sigma_1-\sigma_2)$, $(\sigma_1-\sigma_2)$ is a 1-cycle, i.e. a loop. In the second case, $n_1 = n_2$, $\sigma_1|_{[v_1]} = \sigma_2|_{[v_0]}$, and $\sigma_1|_{[v_0]} = \sigma_2|_{[v_1]}$, so that $(n_1\sigma_1 + n_2\sigma_2) = n_1(\sigma_1 + \sigma_2)$, $(\sigma_1 + \sigma_2)$ is a 1-cycle, i.e. a loop.

Now we use induction on the number of singular 1-simplices in the finite formal sum. Let us assume that any element of Ker ∂_1 that is written as formal sum of at most k singular 1-simplices with integer coefficients, none of which are cycles themselves, can be expressed as a finite sum of 1-cycles. We will show that the same holds for $\sum_{i=1}^{k+1} n_i\sigma_i \in$ Ker ∂_1, where none of the σ_i's are cycles themselves.
Since

$$\partial_1(\sum_{i=1}^{k+1} n_i\sigma_i) = (\sum_{i=1}^{k+1} n_i\sigma_i|_{[v_1]}) - (\sum_{i=1}^{k+1} n_i\sigma_i|_{[v_0]}) = 0$$

$\exists\ j_1 \in \{2, ..., k+1\}$ such that either $\sigma_{j_1}|_{[v_0]} = \sigma_1|_{[v_1]}$, or $\sigma_{j_1}|_{[v_1]} = \sigma_1|_{[v_0]}$.

In the first case, we can again find a $j_2(\neq j_1)$ in the index set such that either $\sigma_{j_1}|_{[v_1]} = \sigma_{j_2}|_{[v_0]}$ or $\sigma_{j_1}|_{[v_1]} = \sigma_{j_2}|_{[v_1]}$, and similarly a choice of σ_i exists in the second case as well. This process would end after a finite number of steps, i.e. $\exists$ $\sigma_{j_n} \in \{\sigma_2, ..., \sigma_{k+1}\}$, $j_n \notin \{j_1, j_2, ..., j_{n-1}\}$ chosen inductively by the same algorithm such that either $\sigma_{j_n}|_{[v_0]} = \sigma_1|_{[v_0]}$ or $\sigma_{j_n}|_{[v_1]} = \sigma_1|_{[v_0]}$, since the sum is a finite sum. Hence we have a 1-cycle $(\sigma_1 \pm \sigma_{j_1} \pm ... \pm \sigma_{j_n})$, the signs depending on which endpoint of the latter matches the image of v_1 under the former. The remaining part of the sum contains (n_1-1) copies of σ_1, so we can perform this process (n_1-1) more times to get, in total, a decomposition of the original cycle into n_1 such 1-cycles containing σ_1 and a formal sum of at most k singular 1-simplices with integer coefficients. This remaining part of the sum can again be decomposed into 1-cycles by our induction hypothesis, thus we have a complete decomposition of $\sum_{i=1}^{k+1} n_i\sigma_i$ into a finite sum of loops in X.

This motivates a natural map from $\pi_1(X)$ to $H_1(X)$ when X is non-empty and path-connected, known as the Hurewicz map.

2.4 The Hurewicz Theorem for $n = 1$

We choose a basepoint $x_0 \in X$. Define

$$h : \pi_1(X, x_0) \to H_1(X)$$

$$h([\alpha]) = [\alpha]_H$$

where α is a loop (therefore also a singular 1-simplex) at x_0, $[\alpha]$ is the homotopy class of α, and $[\alpha]_H$ is the homology class (i.e. the coset α + Im ∂_2) of the singular 1-simplex α.

The **Hurewicz Theorem** states that this map h induces an isomorphism from $\pi_1/[\pi_1, \pi_1]$ to H_1.

Proposition 2.4.1. *h is well-defined.*

Proof. Let α and β be path-homotopic in X, i.e. there exists a continuous map $H : I \times I \to X$ with $H(t, 0) = \alpha(t)$, $H(t, 1) = \beta(t)$, $H(0, t) = H(1, t) = x_0$.

We construct two 2-simplices and give them orientations as indicated in Fig 2.4.

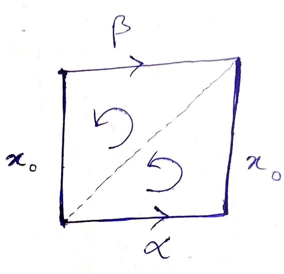

Figure 2.4: Constructing 2-simplices out of the homotopy

Denote the 2-simplex containing the lower edge of $I \times I$ by A, and the other 2-simplex as B. Then $H|_A$ and $H|_B$ are singular 2-simplices, and $\partial_2(H|_A + H|_B) = \alpha - \beta$. So $\alpha - \beta \in \text{Im } \partial_2$, or $[\alpha]_H = [\beta]_H$.

□

Proposition 2.4.2. *h is a homomorphism.*

Proof. Let $[\alpha], [\beta] \in \pi_1(X, x_0)$. By definition of h, $h([\alpha] * [\beta]) = [\alpha * \beta]_H$. We shall show that $[\alpha * \beta]_H = [\alpha]_H + [\beta]_H$, i.e. $(\alpha * \beta) - \alpha - \beta \in \text{Im } \partial_2$.

Taking a 2-simplex with orientation as indicated in Fig. 2.5, we construct a continu-

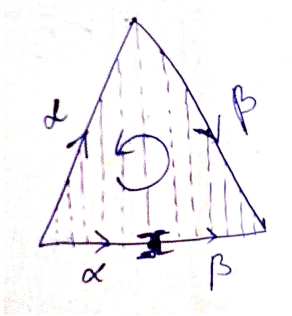

Figure 2.5: Constructing a 2-simplex with boundary $(\alpha * \beta) - \alpha - \beta$

ous map σ from this simplex to X which restricts to α in half of the base and to β in the other half, through appropriate reparametrisation, so that the restriction on the whole base is $\alpha * \beta$. Then, we take the map to be constant on the lines parallel to the height of this triangle, so that restriction of σ on one of the other two edges is α, and on the other one is β, as indicated in the figure. Thus we have a singular 2-simplex σ such that, taking into account the orientations in Fig 2.5, $\partial_2(\sigma) = (\alpha * \beta) - \alpha - \beta$.

□

Proposition 2.4.3. *h is surjective.*

Proof. Let us consider an arbitrary element of $H_1(X)$, and let $\sum_i n_i\sigma_i \in \text{Ker } \partial_1$ be a representative of that homology class. According to our previous comment on the elements of $H_1(X)$, $\sum_i n_i\sigma_i$ can be written as a finite sum of loops in X, say $\sum_i n_i\sigma_i = \alpha_1 + \alpha_2 + ... + \alpha_n$ where each α_j is a closed oriented loop in X. Then we have two possibilities.

1. **If α_j is based at x_0 $\forall j$,** we have

$$[\sum_i n_i\sigma_i]_H = [\alpha_1]_H + [\alpha_2]_H + ... + [\alpha_n]_H = [\alpha_1 * ... * \alpha_n]_H$$

so that $h([\alpha_1 * ... * \alpha_n]) = [\sum_i n_i\sigma_i]_H)$, since h is a homomorphism.

2. **If α_j is not a loop at x_0 for some j**: Let α_j be based at $x_1 \in X$. Since X is path-connected, we have a path γ from x_0 to x_1. Then $\gamma * \alpha_j * \bar{\gamma}$ is a loop based at x_0.

   We construct a continuous map from a square as in Fig 5 and divide the square into two 2-simplices by the diagonal along with the orientations as in the figure. Then we have two singular 2-simplices, say L and U, so that $\partial_2(L+U)$ $= (\gamma * \alpha_j * \bar{\gamma}) + \gamma - \alpha_j - \gamma = (\gamma * \alpha_j * \gamma) - \alpha_j$, so $[\gamma * \alpha_j * \bar{\gamma}]_H = [\alpha_j]_H$. So we can replace α_j by $\alpha_j' = \gamma * \alpha_j * \bar{\gamma}$ in the sum, and then write $[\sum_i n_i \sigma_i]_H$ $= [\alpha_1 * ... * \alpha_j' * ... * \alpha_n]_H$, which has pre-image $[\alpha_1 * ... * \alpha_j' * ... * \alpha_n]$ in $\pi_1(X)$.

□

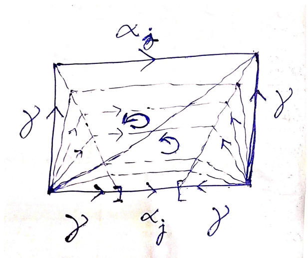

Figure 2.6: $\gamma * \alpha_j * \bar{\gamma}$ is homologous to α_j

Proposition 2.4.4. *Ker h = $[\pi_1, \pi_1]$.*

First, we observe that $[\pi_1, \pi_1] \subset$ Ker h. Let $[\gamma_1] * [\gamma_2] * [\bar{\gamma_1}] * [\bar{\gamma_2}] \in [\pi_1, \pi_1]$. Since h is a homomorphism,

$$h([\gamma_1 * \gamma_2 * \bar{\gamma_1} * \bar{\gamma_2}]) = [\gamma_1]_H + [\gamma_2]_H + [\bar{\gamma_1}]_H + [\bar{\gamma_2}]_H = [\gamma_1]_H + [\gamma_2]_H - [\gamma_1]_H - [\gamma_2]_H = 0$$

.

Next, we claim that Ker h $\subset [\pi_1, \pi_1]$.
Proof: Let $[\alpha] \in$ Ker h, i.e. $\exists \sum_i n_i \sigma_i \in C_2(X)$ such that $\partial_2(\sum_i n_i \sigma_i) = \alpha$, α being a loop at x_0. Allowing repetitions of σ_i's, we can take $n_i = \pm 1$ for each i.

First we shall discuss some specific cases and a geometric argument that gives an intuitive justification for the claim. There is a shorter algebraic argument as well, which we shall state at the end to formalise and conclude the proof.

We observe that for each singular 2-simplex σ_i in this sum, $\partial_2(\sigma_i) = \tau_{i0} - \tau_{i1} + \tau_{i2}$ for some singular 1-simplices τ_{ij}. Since $\sum_i n_i\sigma_i = \alpha$, we can group the τ_{ij}'s into cancelling pairs, leaving out only one τ, which is equal to α. This would imply that there has to be an odd number of non-trivial 2-simplices in the sum.

Let there be only one 2-simplex in the sum, i.e. $\exists\sigma : \Delta^2 \to X$ such that $\partial_2(\sigma) = \alpha$. σ sends the three faces of Δ^2 to τ, $(-\tau)$ and α in that case. α being a loop at x_0, τ has x_0 as one end-point. $\tau * (-\tau)$, or $\tau * \bar{\tau}$ is a loop at x_0. The image of σ in X would, then, form a cone (ref. Fig. 2.7) with base being the loop α and τ and $(-\tau)$ along the same path on the conical surface, but opposite directions.

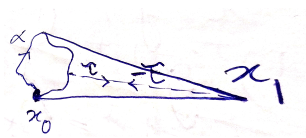

Figure 2.7: Construction with only one 2-simplex

The cone is contractible, so its base α is nullhomotopic, therefore $[\alpha] \in [\pi_1, \pi_1]$.

Let us examine the case when there are 3 non-trivial 2-simplices in the sum. We can say that one edge of a 2-simplex is mapped to the loop α through the corresponding singular 2-simplex, and the other two edges do not cancel each other. Each of these two edges are cancelled in the sum by some edge of the remaining 2-simplices. Here we have two possibilities:

1. The cancelling edges both belong to the same 2-simplex

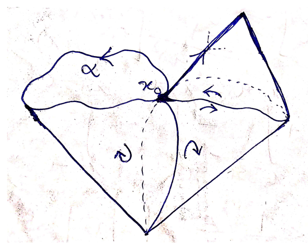

Figure 2.8: Case 1

We have three cones, two glued along the cancelling edges and the other glued to the cone not having α as base along the base(ref. Fig 2.8).

2. One edge from each remaining 2-simplex cancels each of these two edges γ_1 and $\tilde{\gamma_1}$ cancel each other, and so do γ_2 and $\tilde{\gamma_2}$.

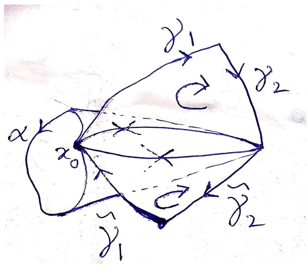

Figure 2.9: Case 2

So these are also identified in pairs, making the two open flaps in Fig 2.9 closed into one disk whose boundary is identified to the loop constructed by consecutively traversing the two remaining edges of the 2-simplex containing α. This space deformation retracts to a cone which is again contractible, so in this case α is nullhomotopic, therefore $[\alpha] \in [\pi_1, \pi_1]$.

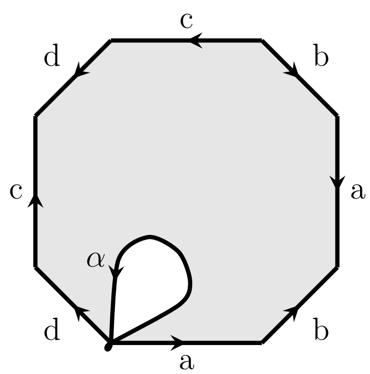

Figure 2.10: g = 2

Given $\sum_i n_i\sigma_i \in C_2(X)$ such that $\partial_2(\sum_i n_i\sigma_i) = \alpha$ where $[\alpha] \in$ Ker h, the singular 2-simplices clearly give us a Δ-complex structure, where the triangles given by each of the σ_i's are attached along their boundaries, and the boundary of the surface thus formed is α, since all the other boundary terms form cancelling pairs by the argument given in the above proof. Also this surface is compact and orientable. So by **Classification Theorem of Surfaces**, this surface is in fact homeomorphic to a closed orientable surface of some genus g with an open disk removed. Such a surface can be represented by identifying pairs of edges of a polygon with $4g$ edges as shown in Fig. 2.10 for the case g = 2. (Ref. [2]) Hence α would be homotopic to the product of g commutators, which intuitively justifies the fact that the kernel of h is contained in the commutator subgroup of π_1.

Now to complete the formal proof, we observe that for any such 2-simplex, if each edge of the simplex gives a loop at x_0 in X, then we can represent the boundary of the singular 2-simplex as concatenation of three loops at x_0.

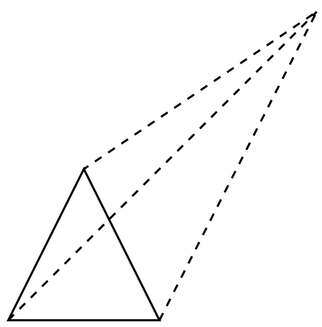

Figure 2.11: Boundary of each 2-simplex

Otherwise, since X is path-connected, we can choose paths from endpoints of each edge to x_0 as in Fig. 2.11, thereafter concatenating them to the images of each edge of the 2-simplex in X, so that we have actual loops at x_0, the homology classes still remaining the same as the original simplices by a construction similar to Fig. 2.6. So the boundary of each 2-simplex is represented by a concatenation of three loops at x_0, say $a*b*c$, which is nullhomotopic (since 2-simplex is homeomorphic to a disk).

In the abelianization of π_1, the boundary of the sum $\sum_i n_i\sigma_i$ can be represented by product of such nullhomotopic 3-products of loops at x_0, which is the identity element in $\pi_1/[\pi_1, \pi_1]$. On the other hand, since the boundary of $\sum_i n_i\sigma_i$ consists of cancelling pairs of 1-simplices and α, this product would be, in the abelianization of π_1, the class of $[\alpha]$. Hence the class of $[\alpha]$ is the identity element in $\pi_1/[\pi_1, \pi_1]$, therefore $[\alpha]$ is in $[\pi_1, \pi_1]$.

Hence, we have an isomorphism from $\pi_1(X, x_0)/[\pi_1, \pi_1]$ to $H_1(X)$ induced by h.

Example 2.4.5.

1. If π_1 is abelian, the commutator subgroup must be trivial. Then the abelianization of π_1 is isomorphic to π_1 itself, thereby giving an isomorphism between π_1 and H_1. A simple example would be S^1.

2. If X is a simply connected space, then by this result, $H_1(X) = 0$.

2.5 General Hurewicz Theorem

The general version of the Hurewicz theorem gives a relation between π_n and H_n for $n \geq 2$ under some conditions on the space. For $n > 1$ as well, there is a **Hurewicz map** similar to the $n = 1$ case, which is

$$h : \pi_n(X, x_0) \longrightarrow H_n(X)$$

defined by

$$h([f]) = f_*(\alpha)$$

where $f : (S^n, s_0) \to (X, x_0)$ and α is a chosen generator of $H_n(S^n)$.

Theorem 2.5.1. *If a space X is $(n-1)$-connected, $n \geq 2$, then $\tilde{H}_i(X) = 0$ for $i < n$ and $\pi_n(X) \cong H_n(X)$ where the isomorphism is given by the Hurewicz map.*

Proof. [5] page 366-367, page 369-370. □

Corollary 2.5.2. *$\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_n(S^n)$ is generated by the homotopy class of the identity map.*

Proof. By 1.3.3, S^n is $(n-1)$-connected. So $\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}$. Now to show that the homotopy class of the identity map $Id : S^n \to S^n$ generates $\pi_n(S^n)$, we use the Hurewicz map.
$H_n(S^n) \cong \mathbb{Z}$, so $f_*(\alpha) = d_f(\alpha)$ for some integer d_f. $Id_*(\alpha) = \alpha$, so

$$f_*(\alpha) = d_f Id_*(\alpha)$$

The Hurewicz map is an isomorphism, so for any $[f] \in \pi_n(S^n)$,
$[f] = h^{-1}(d_f Id_*(\alpha)) = d_f h^{-1}(Id_*(\alpha)) = d_f[Id]$. □

Chapter 3

Hilton's Theorem

3.1 Whitehead Product

Whitehead products give graded quasi-Lie algebra structures on homotopy groups of spaces. They also relate to many other significantly important facts in homotopy theory.

Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ be represented by maps $f : (D^p, \partial D^p) \to (X, x_0)$ and $g : (D^q, \partial D^q) \to (X, x_0)$ respectively, for some basepoint x_0.
Now $D^{p+q} \cong D^p \times D^q$ and $S^{p+q-1} \cong \partial D^{p+q} \cong (D^p \times \partial D^q) \bigcup (\partial D^p \times D^q)$. So we can define the map $h : (S^{p+q-1}, s_0) \to (X, x_0)$ given by

$$h(x,y) = \begin{cases} f(x) & \text{if } x \in D^p,\ y \in \partial D^q \\ g(y) & \text{if } x \in \partial D^p\ ,\ y \in D^q \end{cases} \tag{3.1}$$

which represents an element in $\pi_{p+q-1}(X)$, which we denote by $[\alpha, \beta]$.

$[\alpha, \beta]$ is known as the **Whitehead Product** of α and β (Ref. [11]). It is easy to see that this product is well-defined, in the sense that any other representatives of α or β, being homotopic to f or g respectively, will give a based map from S^{p+q-1} to X which is homotopic to h, and hence represents the same homotopy class as h. So the Whitehead Product gives a well-defined map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)$$

defined by

$$(\alpha, \beta) \longmapsto [\alpha, \beta]$$

Example 3.1.1. Let $p = q = 1$. Then the Whitehead Product gives a map

$$\pi_1(X) \times \pi_1(X) \longrightarrow \pi_1(X)$$

$$(\alpha, \beta) \longmapsto [\alpha, \beta]$$

where $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, since $\partial(D^1 \times D^1) = \partial(I \times I) \cong I \times \partial I \bigcup \partial I \times I$. For representatives f and g of α and β respectively, the representative of $[\alpha, \beta]$ is clearly $fg\bar{f}\bar{g}$ (ref. Fig 3.1).

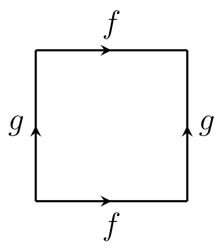

Figure 3.1: $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$

Proposition 3.1.2. *The Whitehead Product as defined above satisfies the following three properties:*

1. *Bilinearity :* $[\alpha + \beta, \gamma] = [\alpha, \gamma] + [\beta, \gamma]$ *and* $[\alpha, \beta + \gamma] = [\alpha, \beta] + [\alpha, \gamma]$
2. *Skew-symmetry :* $[\beta, \alpha] = (-1)^{pq}[\alpha, \beta]$
3. *Graded Jacobi Identity :*

$$(-1)^{pr}[[\alpha, \beta], \gamma] + (-1)^{pq}[[\beta, \gamma], \alpha] + (-1)^{qr}[[\gamma, \alpha], \beta] = 0$$

where $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$, $\gamma \in \pi_r(X)$.

Proof. 1. Follows from definition.

2. Ref. [10].

3. Ref. [7].

□

These three properties give a **Graded quasi-Lie algebra** structure where the grading is given by $L_k = \pi_{k+1}(X)$, $L = \bigoplus_k L_k$ and the Lie bracket is given by the Whitehead product.

Next, we discuss an important characterisation.

Theorem 3.1.3. *Let $\alpha \in \pi_p(X)$, $\beta \in \pi_q(X)$ be represented by f and g respectively. $[\alpha, \beta] = 0$ if and only if $f \vee g : S^p \vee S^q \to X$ extends to a map $S^p \times S^q \to X$.*

Proof. As a CW-complex, $S^p \times S^q$ can be obtained from $S^p \vee S^q$ by attaching a $(p+q)$-cell. Let the corresponding attaching map be $\varphi : S^{p+q-1} \to S^p \vee S^q$ and the characteristic map be $\Phi : D^{p+q} \to S^p \vee S^q$.

Now $[\alpha, \beta] = 0 \implies f \simeq 0$. By Homotopy Extension Property, this homotopy extends to a nullhomotopy on all of D^{p+q}, and therefore all of $S^p \times S^q$.

For the converse, let us assume there exists such an extension. $i_1 : S^p \hookrightarrow S^p \times S^q$, $i_2 : S^q \hookrightarrow S^p \times S^q$, $i = i_1 \vee i_2$, $j : S^{p+q-1} \hookrightarrow D^{p+q}$ be the corresponding inclusion maps. Now, $[\alpha, \beta]$ is the homotopy class of $(f \vee g) \circ \varphi : S^{p+q-1} \to X$. If F is an extension of this map to $S^p \times S^q$, then $f \vee g = F \circ i$, i.e.

$$(f \vee g) \circ \varphi = (F \circ i) \circ \varphi = F \circ (i \circ \varphi)$$

Since $i \circ \varphi = \Phi \circ j$ and D^{p+q} is contractible, so $[\alpha, \beta] = 0$. (ref. [9]) $\square$

An example of a space where the Whitehead Product is zero is any Lie Group, e.g. S^1. In case of S^2 though, Id_{S^2} represents the positive generator of $\pi_2(S^2)$, and $[Id_{S^2}, Id_{S^2}]$ is a non-trivial element of $\pi_3(S^2)$. (Ref. [8])

In the next section, we shall use the Whitehead Product to define some new objects that would finally help us look into the homotopy groups of wedges of spheres.

3.2 Basic Products and Computation of Homotopy Groups

In this chapter, we shall finally discuss the main theorem of this text, which will make computation of wedge of spheres simpler.

We consider the space $S^{m_1} \vee S^{m_2} \vee ... \vee S^{m_k}$ where $1 < m_1 \leq m_2 \leq ... \leq m_k$. Let $r_i = m_i - 1$, so that $r_i \geq 1$, and let α_j be the positive generator of $\pi_{m_j}(S^{m_j})$ (i.e. the homotopy class of the identity map). We call $\alpha_1, \alpha_2, ..., \alpha_k$ the **basic products** of **weight** 1.(Ref. [6])

We define the basic products of higher weights inductively, and define an order among them. Assume that the basic products of weight less than w have been defined and ordered. Basic products of order w are the Whitehead Products $[a, b]$ where a, b are basic products of weights u and v respectively, such that $u + v = w$ and $a < b$ in the chosen order of basic products.

Now if $b = [c, d]$ where c and d are basic products then $c \leq a$. Order the weight w elements among themselves and greater than all lower weight basic products. So we must have $u \leq v$.

Example 3.2.1. (Ref. [1]) Let $k = 3$. Then there are three basic products of weight 1, namely α_1, α_2 and α_3. We order them as : $\alpha_1 < \alpha_2 < \alpha_3$.
The weight 2 basic products are $[\alpha_1, \alpha_2], [\alpha_2, \alpha_3]$ and $[\alpha_1, \alpha_3]$. Note that Whitehead products like $[\alpha_1, \alpha_1]$ or $[\alpha_3, \alpha_1]$ are not basic products, because of the order constraints. We chose to extend the order as :

$$\alpha_1 < \alpha_2 < \alpha_3 < [\alpha_1, \alpha_2] < [\alpha_1, \alpha_3] < [\alpha_2, \alpha_3]$$

Similarly, the weight 3 basic products are ordered as $[\alpha_1, [\alpha_1, \alpha_2]] < [\alpha_1, [\alpha_1, \alpha_3]] < [\alpha_2, [\alpha_1, \alpha_2]] < [\alpha_2, [\alpha_1, \alpha_3]] < [\alpha_2, [\alpha_2, \alpha_3]] < [\alpha_3, [\alpha_1, \alpha_2]] < [\alpha_3, [\alpha_1, \alpha_3]] < [\alpha_3, [\alpha_2, \alpha_3]]$.

Observe that the ordering on the weight two basic products played no role in determining the basic products of weight 3. However, they play a role in determining the basic products of weight 4. For example, $[[\alpha_1, \alpha_2], [\alpha_1, \alpha_3]]$ is a basic product of weight 4 but $[[\alpha_1, \alpha_3], [\alpha_1, \alpha_2]]$ is not, since we chose an order in which $[\alpha_1, \alpha_2] < [\alpha_1, \alpha_3]$. Had we chosen to extend the order to basic products of weight two in such a way that $[\alpha_1, \alpha_3] < [\alpha_1, \alpha_2]$, then $[[\alpha_1, \alpha_3], [\alpha_1, \alpha_2]]$ would be a basic product of weight 4 but $[[\alpha_1, \alpha_2], [\alpha_1, \alpha_3]]$ would not be. In general, the ordering of the elements of weight k only affects the basic products of weight $2k$ and above.

Any such basic product p of weight w is a string of symbols $\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_w}$ suitably bracketed. Let w_j be the number of occurences of α_j in the string representing p.

Definition 3.2.2. The **height** of p is defined as

$$q := \sum_{i=1}^{k} r_i w_i$$

Let p_s be the sequence of all basic products written in increasing order. We denote the height of p_s by q_s. Hilton's theorem (Ref. [6]) then gives an isomorphism which makes computation of homotopy groups of wedge of spheres much easier.

Theorem 3.2.3. ***(Hilton's Theorem)*** *For any $n > 0$,*

$$\pi_n(S^{m_1} \vee S^{m_2} \vee \ldots \vee S^{m_k}) \cong \bigoplus_{i=1}^{\infty} \pi_n(S^{q_i+1})$$

Proof. Ref. [6]. □

We can simplify the expression on the r.h.s. to a finite direct sum by using the fact that $\pi_n(S^k) = 0$ for $k > n$, and the heights of basic products form an unbounded increasing sequence (follows from definition).

Example 3.2.4. 1. $\pi_2(S^2 \vee S^2) \cong \pi_2(S^2) \oplus \pi_2(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

The only basic products of order 1 here are α_1 and α_2. There is only one basic product of order 2, $[\alpha_1, \alpha_2]$. Also, $r_1 = r_2 = 1$, so the heights of basic products are now $w_1 + w_2$. Now by Hilton's theorem,

$$\pi_2(S^2 \vee S^2) \cong \bigoplus_{i=1}^{\infty} \pi_2(S^{q_i+1})$$

Now by the above remark, the only dimensions of the spheres in the direct summands which might give a non-zero π_2 are 1 and 2, and $\pi_2(S^1) = 0$ as $\mathbb{R}$ is universal cover of S^1 and covering maps induce isomorphisms of the homotopy groups for $n > 1$. So the only non-zero direct summands on the r.h.s. are copies of $\pi_2(S^2)$.

To find the number of such summands, we find the number of non-negative integral solutions of the equation $w_1 + w_2 + 1 = 2$, or $w_1 + w_2 = 1$. There are only 2 solutions to this, namely $(1,0)$ and $(0,1)$. So there are two copies of $\pi_2(S^2)$ in the r.h.s., i.e. $\pi_2(S^2 \vee S^2) \cong \pi_2(S^2) \oplus \pi_2(S^2)$.

We can verify this result using Hurewicz theorem. By Van Kampen's theorem, $\pi_1(S^2 \vee S^2) \cong \pi_1(S^2) \oplus \pi_1(S^2)$, and S^2 is simply-connected, therefore $S^2 \vee S^2$ is also simply-connected. So by 2.5.1, we have $\pi_2(S^2 \vee S^2) \cong H_2(S^2 \vee S^2) \cong H_2(S^2) \oplus H_2(S^2) \cong \pi_2(S^2) \oplus \pi_2(S^2)$.

2. $\pi_3(S^2 \vee S^2) \cong \pi_3(S^2) \oplus \pi_3(S^2) \oplus \pi_3(S^3)$

   The possible non-zero direct summands are copies of $\pi_3(S^2)$ and $\pi_3(S^3)$, by the same argument as the previous example. To find the number of such copies in the sum, we consider the equations $w_1 + w_2 + 1 = 2$ and $w_1 + w_2 + 1 = 3$. The first one has 2 solutions while the second has 3 solutions, but only one of these three give a valid basic product according to the above scheme. So by Hilton's theorem, $\pi_3(S^2 \vee S^2)$ is isomorphic to the direct sum of two copies of $\pi_3(S^2)$ and one copy of $\pi_3(S^3)$.

   In this case though, Hurewicz theorem does not give us anything.

While defining the order of basic products, we had made choices in each step. Ideally, such choices should not have a significant effect on the statement of Hilton's theorem. It turns out that the change in choice of the ordering of basic products merely reorders the direct summands.

In fact, by a theorem of Witt (Ref. [12]), the number of basic products involving w_j copies of α_j which necessarily have weight $w = w_1 + w_2 + ... + w_k$ is given by

$$A(w_1, w_2, ..., w_k) = \frac{1}{w} \sum_{d|w_j} \frac{\mu(d)(\frac{w}{d})!}{(\frac{w_1}{d})!...(\frac{w_k}{d})!}$$

where μ denotes the Möbius function defined on the positive integers as

$$\mu(d) = \begin{cases} 1 & \text{if } d \text{ is square-free, has even number of prime factors} \\ -1 & \text{if } d \text{ is square-free, has odd number of prime factors} \\ 0 & \text{if } d \text{ has a squared prime factor} \end{cases} \tag{3.2}$$

Clearly, $A(w_{\sigma(1)}, w_{\sigma(2)}, ..., w_{\sigma(k)}) = A(w_1, w_2, ..., w_k)$ for all permutations σ of $\{1, 2, ..., k\}$.

Example 3.2.5. 1.

$$A(1,0,0) = \frac{\mu(1)1!}{1!0!0!} = 1$$

To verify this, note that there is only one basic product with 1 copy of α_1, and no copy of α_2 and α_3 for $k = 3$, which is α_1 itself.

2.

$$A(2,1,0) = \frac{1}{3}\frac{\mu(1)3!}{2!1!0!} = 1$$

To verify this, note that there is only one basic product with 2 copies of α_1, one copy of α_2 and no copies of α_3, according to the order chosen above in the example which is $[\alpha_1, [\alpha_1, \alpha_2]]$.

3.

$$A(3,0,0) = \frac{1}{3}[\frac{\mu(1)3!}{3!0!0!} + \frac{\mu(3)1!}{1!0!0!}] = \frac{1}{3}(1-1) = 0$$

Of course, we can see from the example discussed above that there is no basic product with 3 copies of α_1 and no copies of α_2 or α_3.

4.

$$A(1,0,1) = \frac{1}{2}\frac{\mu(1)2!}{1!0!1!} = 1$$

Observe that there is only one basic product with one copy of α_1 and α_3 each and none of α_2. This matches with the value the formula gave.

Remark 3.2.6. This formula will actually help us to simplify the expression in 3.2.3. For each q, define c_{q+1} to be the sum of all $A(w_1, w_2, ..., w_k)$ for which $\sum_{i=1}^{k} r_i w_i = q$. Observe that this is the number of direct summands of the form $\pi_n(S^{q+1})$. So we have

$$\pi_n(S^{m_1} \vee S^{m_2} \vee ... \vee S^{m_k}) \cong \bigoplus_{q=1}^{\infty} (\pi_n(S^{q+1}))^{\bigoplus c_{q+1}} = \bigoplus_{q=2}^{\infty} (\pi_n(S^q))^{c_q}$$

Using the remark after 3.2.3, we further simplify the expression to

$$\pi_n(S^{m_1} \vee S^{m_2} \vee ... \vee S^{m_k}) \cong \bigoplus_{q=2}^{n} \pi_n(S^q)^{c_q}$$

(Ref. [1])

Example 3.2.7. 1. $n < m_1 + m_2 - 1$

In this case,

$$\pi_n(S^{m_1} \vee S^{m_2} \vee ... \vee S^{m_k}) \cong \pi_n(S^{m_1}) \oplus \pi_n(S^{m_2}) \oplus ... \oplus \pi_n(S^{m_a})$$

where a is such that $m_a \le m_1 + m_2 - 1 < m_{a+1}$. (Ref. [1])
Also from the simplified version of Hilton's theorem, for $2 \le q < m_1 + m_2 - 1$, any solution of the equation

$$(m_1 - 1)w_1 + ... + (m_k - 1)w_k = q - 1$$

must be of the form $w_i = 1$ for some i with $m_i = q$ and zero for all other m_j. This solution corresponds to a unique basic product of weight 1, namely α_i. So we see that c_q is equal to the number of spheres in the wedge product of dimension q and hence arrive at the same result.

2. By the previous computation,

$$\pi_n(S^3 \vee S^4 \vee S^5) \cong \pi_n(S^3) \oplus \pi_n(S^4) \oplus \pi_n(S^5)$$

for all $n < 3 + 4 - 1$, i.e., $n < 6$.

3. For $n = 6$, we look at the solutions of the equation $2w_1 + 3w_2 + 4w_3 = 5$. The only solution is $(1, 1, 0)$, and $c_6 = A(1, 1, 0) = 1$, since there is only one basic product involving 1 copy of α_1 and 1 copy of α_2, which is $[\alpha_1, \alpha_2]$. So

$$\pi_6(S^3 \vee S^4 \vee S^5) \cong \pi_6(S^3) \oplus \pi_6(S^4) \oplus \pi_6(S^5) \oplus \pi_6(S^6)$$

4. Similarly, for $n = 7$, (Ref. [1])

$$\pi_7(S^3 \vee S^4 \vee S^5) \cong \pi_7(S^3) \oplus \pi_7(S^4) \oplus \pi_7(S^5) \oplus \pi_7(S^6) \oplus \pi_7(S^7)$$

Thus we can compute homotopy groups of increasingly higher dimensions using Hilton's theorem.

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