Prove the following is a subspace of $\mathrm{P}_3\ (\mathrm{C})$: $\mathrm{S} = \{ax + bx^2 |\ a, b \in C\}$

Problem Solving - What are the terms/strategies I may need? What do I know?

Subspace Test:

The space is a subset of a known vector space.

The zero vector is in the space.

The space is closed under addition.

The space is closed under scalar multiplication

Properties of polynomials and complex numbers that are vector spaces:

Distributive	(a(b+c) = ab + ac)
Commutative	(a + b = b + a)
Associative	([a + b] + c = a + [b + c])

Prove the following is a subspace of $\mathrm{P}_3\ (\mathrm{C})$: $\mathrm{S} = \{ax + bx^2 \mid a, b \in C\}$

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

The space is a subset of a known vector space.

We know $\mathrm{P}_3\ (\mathrm{C})$ is a vector space, and we know that the set S are polynomials of degree 2 or less, which is inside a set of polynomials of degree 3 or less (the coefficients are complex in both sets as well).

The zero vector is in the space.

The zero vector in $\mathrm{P}_3\ (\mathrm{C})$ is $0 + 0x + 0x^2 + 0x^3$

We note that if we chose the complex numbers a = 0 and b = 0 we get the element $0x + 0x^2$

which is equal to $0 + 0x + 0x^2 + 0x^3$ due to the zero property.

The space is closed under addition.

Let $v_1 = a_1x + b_1x^2$ and $v_2 = a_2x + b_2x^2$ Then using the addition from $\mathrm{P}_3\ (\mathrm{C})$ we get:

$$
\begin{aligned}
v_1 + v_2 &= (a_1x + b_1x^2) + (a_2x + b_2x^2) \\
&= (a_1x + a_2x) + (b_1x^2 + b_2x^2) && \text{(Associative and Commutative)} \\
&= (a_1 + a_2)x + (b_1 + b_2)x^2 && \text{(Distributive)}
\end{aligned}
$$

Since we know that adding complex numbers results in a complex number (as C is a vector space), we know that $a = a_1 + a_2,\ b = (b_1 + b_2) \in C$ thus it is closed under addition.

The space is closed under scalar multiplication.

Let $v_1 = a_1x + b_1x^2$ and $k \in C$ Then using the scalar multiplication from $\mathrm{P}_3\ (\mathrm{C})$ we get:

$$
\begin{aligned}
\mathrm{k}v_1 &= \mathrm{k}(a_1x + b_1x^2) \\
&= (ka_1)x + (kb_1)x^2 && \text{(Distributive)}
\end{aligned}
$$

Since we know that multiplying complex numbers results in a complex number (as C is a vector space), we know that $a = ka_1,\ b = kb_1 \in C$ thus it is closed under scalar multiplication.

Thus we have S is a subspace of $\mathrm{P}_3\ (\mathrm{C})$ by the subspace test.

Prove the following is a subspace of $\mathrm{P}_3\ (\mathrm{C})$: $\mathrm{S} = \{ax + bx^2 \mid a, b \in C\}$

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why does the subspace test work?

Why are C and P vector spaces?

For Video Please click the link below:

Video