Show that the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverses, and does not have a left inverse.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definitions:

Suppose A is an $m\times n$ matrix. If B is an $n \times m$ matrix with $BA = I_n$, then we call B a left inverse of A.

If C is an $n \times m$ matrix with $AC = I_m$, then we call C a right inverse of A

Recall:

Two matrices are equal if and only all of their entries are equal

Show that the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverses, and does not have a left inverse.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

First lets show that $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has no left inverse.

To do this we suppose towards a contradiction that a left inverse $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ exists, which implies that

$$\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} a & b & a \\ c & d & c \\ e & f & e \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{cases} a = 1 \\ a = 0 \end{cases}, \text{ since entries (1,1) and (1,3) must be equal.}$$

This yields a contradiction, and thus no left inverse of A exists.

Show that the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverses, and does not have a left inverse.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Next lets show that $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverses.

To do this we'll show that the existence of a right inverse $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ is equivalent to finding solutions to a system of linear equations, and then show that this system has infinitely many solutions. Observe that

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \iff \begin{pmatrix} a+e & b+f \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\iff \begin{cases} a+e=1, \ b+f=0 \\ c=0, \ \ d=1 \end{cases}, \text{ since all the entries must be equal.}$$

$$\iff \begin{cases} a=1-e, \ b=-f \\ c=0, \ \ d=1 \end{cases}, \text{ since all the entries must be equal.}$$

$$\iff \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 1-e & -f \\ 0 & 1 \\ e & f \end{pmatrix}, \text{ where } e, f \text{ are free variables.}$$

Since any choice of e and f gives a valid right inverse of A, we conclude that A has infinitely many right inverses.

Show that the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverses, and does not have a left inverse.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

In this question we showed a non-square matrix can have infinitely many right inverses and no left inverses. Can a square matrix have a right inverse and not a left inverse?

Can a square matrix have infinitely many left (or right) inverses?

For Video Please click the link below:

Video