Find a matrix A $\in M_{45}(\boldsymbol{C})$ that is not in RREF that has a null space equal to the solution set (of the below matrix). Justify your answer.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Problem Solving - What are the terms/strategies I may need? What do I know?

RREF Properties:

- Pivots are 1
- Pivots go down and to the right
- Everything above and below a pivot is 0
- Zero rows are at the bottom

Row Operations:

- Swap rows
- Scale a row by a non-zero number
- Row replace using $R_i \rightarrow R_i - k_j R_j$

Null Space:

The null space of a matrix A is the set of all solutions to the homogenous system $LS(A, \mathbf{0})$

Row Equivalent Matrices:

- Two matrices are Row Equivalent if one matrix can be obtained from the other by a sequence of row operations.

Theorem REMES:

-Two augmented matrices that are row equivalent are equivalent systems (have the same solution set).

Find a matrix A ∈ $M_{45}(\boldsymbol{C})$ that is not in RREF that has a null space equal to the solution set (of the below matrix). Justify your answer.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

We should note that we want to have a matrix A that has null space equal to the solution set given the one above. There are infinitely many answers, but we will find an answer taking advantage of Theorem REMES:

Thereon REMES

Two augmented matrices that are row equivalent are equivalent systems (have the same solution set).

This means we would want to find a matrix A ∈ $M_{45}(\boldsymbol{C})$ that has $LS(A, 0)$ row equivalent to the above matrix. Thus, we just need to find a matrix that starts with the matrix above and has at least one row operation and that also makes the matrix not in RREF (note that the matrix above is in RREF, so we need at least one row operation). Performing almost any row operation (other than adding combinations of the 0 row or multiplying a row by 1 will work) Thus you could take any row operation you like. Let us take an easy one:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow R1 = R1 \times\ 2 \rightarrow \left[\begin{array}{ccccc|c} 2 & 4 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see that the coefficient matrix is not in RREF (not all of the pivots are one), thus let A be that matrix. All that remains is for a bit of justification.

Find a matrix A ∈ $M_{45}(\boldsymbol{C})$ that is not in RREF that has a null space equal to the solution set (of the below matrix). Justify your answer.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Let A = $\begin{bmatrix} 2 & 4 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we note that A∈ $M_{45}(\boldsymbol{C})$ but is not in RREF as the pivot in the (1,1) entry is not 1.

Thus we get that the null space for A:

$$LS(A, \mathbf{0}) = \left[\begin{array}{ccccc|c} 2 & 4 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

When we perform one row operation we get:

$$\left[\begin{array}{ccccc|c} 2 & 4 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \to R1 = R1 \times \tfrac{1}{2} \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $LS(A, \mathbf{0})$ is row equivalent to the starting matrix. By Theorem REMES, they are equivalent systems and so have the same solution set.

Find a matrix A∈ $M_{45}(\boldsymbol{C})$ that is not in RREF that has a null space equal to the solution set (of the below matrix). Justify your answer.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why does Theorem REMES work (can you prove it?)

For Video Please click the link below:

Video