Let $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Prove that $A^{-1} = A^T$. You may use the formula for the inverse of 2×2 matrices from class.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definition: The transpose of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$ is defined as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

Theorems:

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2$ be an invertible matrix. Then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

For every $\theta \in \mathbb{R}$, we have the equality $\cos^2\theta + \sin^2\theta = 1$

Let $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Prove that $A^{-1} = A^T$. You may use the formula for the inverse of 2×2 matrices from class.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Let $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

Recall the formula for the inverse of a 2x2 matrix from class: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

From this it follows that

$$A^{-1} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} = \frac{1}{\cos^2(\theta) + \sin^2(\theta)} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}^T$$

$$= A^T$$

This concludes the proof.

Let $A = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$. Prove that $A^{-1} = A^T$. You may use the formula for the inverse of 2×2 matrices from class.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Thinking about what rotation matrices do geometrically to vectors in $\mathbb{R}^2$, why is this result not surprising?

For Video Please click the link below:

Video