Consider the subspace $V = \left\{ \begin{pmatrix} a \\ 2a \\ b \\ -a \\ 3b \end{pmatrix} : a, b, \in \mathbb{R} \right\}$ of $\mathbb{R}^5$. Find a basis for V.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definitions: Let V be a vector space.

(i) A set $\{v_1, \dots, v_n\}$ is linearly independent in V if and only if $c_1 v_1 + \cdots + c_n v_v = 0$
$\Rightarrow \quad \mathrm{c}_1 = \cdots = c_n = 0.$

(ii) A set $\{v_1, \dots, v_n\}$ is a basis for V if and only if it is linearly independent and spans V

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Steps & Process – Try to answer the question writing in many steps to avoid small errors.

We first place the space into a span:

$$V = \left\{ \begin{pmatrix} a \\ 2a \\ b \\ -a \\ 3b \end{pmatrix} : a, b, \in \mathbb{R} \right\} = \left\{ a \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} : a, b, \in \mathbb{R} \right\},$$

$$= \text{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

This show that the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ spans V.

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Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Next we show that the set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ is linearly independent.

By comparing the first and third entries we observe that

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad c_1 = c_2 = 0.$$

Therefore this set is linearly independent and spans V, and hence is a basis for V.

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Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

To solve this question we took the definition of the subspace and used it to find a spanning set, and then showed that this spanning set is linearly independent. Is there a subspace in which, when we do this we find a spanning set which isn't linearly independent?

For Video Please click the link below:

Video