Let F be a field, $a \in F$ and $S = \left\{ \begin{pmatrix} 1+a \\ 1-a \end{pmatrix}, \begin{pmatrix} 1-a \\ 1+a \end{pmatrix} \right\} \subseteq F^2$

(a) If $F = \mathbb{C}$. Determine the conditions a must satisfy for S to be linearly independent over $\mathbb{C}$.

(b) Let $F = \mathbb{Z}_2$. Prove that S is linearly dependent over $\mathbb{Z}_2$.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definition:

Let V be a vector space over a field F. A set $\{v_1, \ldots, v_n\} \subseteq V$ is linearly independent if and only if $c_1 v_1 + \cdots + c_n v_n = 0 \quad \Rightarrow \quad c_1 = \cdots = c_n = 0$, where $c_1, \ldots, c_n \in F$.

Theorem

Let F^n be a vector space over a field F and $\{v_1, \ldots, v_n\} \subseteq F^n$. Then the following are equivalent:

(a) The vectors $\{v_1, \ldots, v_n\}$ are linearly independent

(b) The matrix $(v_1 \cdots v_n)$ is non-singular

Let F be a field, $a \in F$ and $S = \left\{\begin{pmatrix}1+a\\1-a\end{pmatrix}, \begin{pmatrix}1-a\\1+a\end{pmatrix}\right\} \subseteq F^2$

(a) If $F = \mathbb{C}$. Determine the conditions a must satisfy for S to be linearly independent over $\mathbb{C}$.

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Steps & Process

The set $\left\{\begin{pmatrix}1+a\\1-a\end{pmatrix}, \begin{pmatrix}1-a\\1+a\end{pmatrix}\right\}$ is linearly independent iff $c_1\begin{pmatrix}1+a\\1-a\end{pmatrix} + c_2\begin{pmatrix}1-a\\1+a\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0$, for $c_1, c_2 \in F$.

However if $c_1\begin{pmatrix}1+a\\1-a\end{pmatrix} + c_2\begin{pmatrix}1-a\\1+a\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0$, means $\begin{pmatrix}1+a & 1-a\\1-a & 1+a\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow c_1 = c_2 = 0.$

This is equivalent to the matrix $\begin{pmatrix}1+a & 1-a\\1-a & 1+a\end{pmatrix}$ being non-singular. If we row reduce this matrix to RREF over $\mathbb{C}$, we get:

$\begin{pmatrix}1+a & 1-a\\1-a & 1+a\end{pmatrix}$ $\quad \mathrm{R}_2 \to R_2 + R_1$

$\begin{pmatrix}1+a & 1-a\\2 & 2\end{pmatrix}$ $\quad R_2 \div 2$ and $R_1 \leftrightarrow R_2$

$\begin{pmatrix}1 & 1\\1+a & 1-a\end{pmatrix}$ $\quad R_2 \to R_2 - R_1$

$\begin{pmatrix}1 & 1\\a & -a\end{pmatrix}$ $\quad R_2 \div a \quad$ (Assume for now $a \neq 0$)

$\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$ $\quad R_2 \to R_2 - R_1$

$\begin{pmatrix}1 & 1\\0 & -2\end{pmatrix}$

If a $= 0$ we get

$\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ $\quad \mathrm{R}_2 \to R_2 - R_1$

$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$

Which does not have a pivot in the second column, thus is singular and so the set is linearly dependent.

Since there is a pivot in every column, this matrix is non-singular. Thus a $\neq 0$ will allow for the set to be linearly independent.

Let F be a field, $a \in F$ and $S = \left\{ \begin{pmatrix} 1+a \\ 1-a \end{pmatrix}, \begin{pmatrix} 1-a \\ 1+a \end{pmatrix} \right\} \subseteq F^2$

(a) If $F = \mathbb{C}$. Determine the conditions a must satisfy for S to be linearly independent over $\mathbb{C}$.

(b) Let $F = \mathbb{Z}_2$. Prove that S is linearly dependent over $\mathbb{Z}_2$.

Steps & Process

Over $\mathbb{Z}_2$ we want to show when $\begin{pmatrix} 1+a & 1-a \\ 1-a & 1+a \end{pmatrix}$ would be non-singular. If we row reduce this matrix to RREF over $\mathbb{Z}_2$, we get:

$\begin{pmatrix} 1+a & 1-a \\ 1-a & 1+a \end{pmatrix}$ $\quad \mathrm{R}_2 \to R_2 + R_1$

$\begin{pmatrix} 1+a & 1-a \\ 0 & 0 \end{pmatrix}$

Since there is no pivot in the last row no matter which a we choose, the set is always linearly dependent.

Let F be a field, $a \in F$ and $S = \left\{ \begin{pmatrix} 1+a \\ 1-a \end{pmatrix}, \begin{pmatrix} 1-a \\ 1+a \end{pmatrix} \right\} \subseteq F^2$.

(a) If $F = \mathbb{C}$. Determine the conditions a must satisfy for S to be linearly independent over $\mathbb{C}$.

(b) Let $F = \mathbb{Z}_2$. Prove that S is linearly dependent over $\mathbb{Z}_2$.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

If $F = \mathbb{R}$ then does the same argument as we did for part (a) show that S is LI if and only if $a \neq 0$?

Draw some elements of S in $\mathbb{R}^2$ for different values of a and explain geometrically why when $a \neq 0$ the elements of S are LI.

For Video Please click the link below:

Video