Prove: $(b^{-1})^{-1} = b$ for any $b \neq 0$ in a field **F**.

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**Problem Solving** - What are the terms/strategies I may need? What do I know?

Field Axioms:

| name | addition | multiplication |
|---|---|---|
| associativity | $(a+b)+c=a+(b+c)$ | $(a\,b)\,c=a\,(b\,c)$ |
| commutativity | $a+b=b+a$ | $a\,b=b\,a$ |
| distributivity | $a\,(b+c)=a\,b+a\,c$ | $(a+b)\,c=a\,c+b\,c$ |
| identity | $a+0=a=0+a$ | $a\cdot 1=a=1\cdot a$ |
| inverses | $a+(-a)=0=(-a)+a$ | $a\,a^{-1}=1=a^{-1}\,a$ if $a\neq 0$ |

Prove: $(b^{-1})^{-1} = b$ for any $b \neq 0$ in a field **F**.

**Steps & Process** – Try to answer the question writing in many steps to avoid small errors.

Let $b \neq 0$ be in **F**

Since $b \neq 0$ and **F** is a field, we know there is $b^{-1}$ in **F** such that $b^{-1}b = bb^{-1} = 1$ (Multiplicative Inverse)

Let $a = b^{-1}$, we want to find $a^{-1} = (b^{-1})^{-1}$

That is, we want to find a c in **F** such that $ac = ca = 1$ (if one exists)

Thus we want to find a c in F such that $(b^{-1})c = c(b^{-1}) = 1$

But we know that $b^{-1}b = bb^{-1} = 1$, thus if we let $c = b$, we get $ac = ca = 1$, and thus $c = a^{-1}$.

Since $a^{-1} = (b^{-1})^{-1}$ and $c = b$, we have $(b^{-1})^{-1} = b$.

We should note that the multiplicative inverse is unique, thus $(b^{-1})^{-1} = b$ is the only possibility (there is only one inverse for $(b^{-1})$).

Prove: $(b^{-1})^{-1} = b$ for any $b \neq 0$ in a field **F**.

**Solidify Understanding** – Explain why the steps makes sense by connecting to math you know.

Why do we replace $a = b^{-1}$ to help us in this proof?

For Video Please click the link below:

Video