Let V be a set of ordered pairs (a, b) where $a, b \in R$. If $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$, which axioms of a vector space hold, and which ones fail?

Problem Solving - What are the terms/strategies I may need? What do I know?

Definition of a vector space over a field F:

V is a *vector space* over F if the following axioms are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $b, c \in F$

(A1) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(A2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(A3) There exists an element $\mathbf{0} \in V$ such that for all $\mathbf{u} \in V$ we have $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(A4) For each $\mathbf{u} \in V$, there exists an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(S1) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(S2) $(b + c)\mathbf{u} = b\mathbf{u} + c\mathbf{u}$

(S3) $(bc)\mathbf{u} = b(c\mathbf{u})$

(S4) $1\mathbf{u} = \mathbf{u}$

Let V be a set of ordered pairs (a, b) where $a, b \in R$. If $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$, which axioms of a vector space hold, and which ones fail?

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

For all axioms below, we will let $\boldsymbol{u} = (a, b), \boldsymbol{v} = (c, d), \boldsymbol{w} = (e, f)$

(A1) $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$

$$
\begin{aligned}
(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} &= \big((a, b) + (c, d)\big) + (e, f) \\
&= (a, b) + (e, f) \\
&= (a, b)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) &= (a, b) + ((c, d) + (e, f)) \\
&= (a, b) + (c, d) \\
&= (a, b)
\end{aligned}
$$

Thus this axiom holds true.

(A2) $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

$$
\begin{aligned}
\boldsymbol{u} + \boldsymbol{v} &= (a, b) + (c, d) \\
&= (a, b)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{v} + \boldsymbol{u} &= (c, d) + (a, b) \\
&= (c, d)
\end{aligned}
$$

Thus this axiom does not hold true (one counter example could be (2,2) and (1,1))

Let V be a set of ordered pairs (a, b) where $a, b \in R$. If $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$, which axioms of a vector space hold, and which ones fail?

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

(A3) There is a zero vector $\mathbf{0}$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$ for all $\boldsymbol{u} \in \boldsymbol{V}$
Thus this axiom holds true, in fact $\mathbf{0}$ could be any vector (that is it is not unique).

(A4) For each $\boldsymbol{u} \in \boldsymbol{V}$ there is a $-\boldsymbol{u} \in \boldsymbol{V}$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
This would be false as $\boldsymbol{u} + (-\boldsymbol{u}) = \boldsymbol{u}$ no matter which vector we choose to be $-\boldsymbol{u}$**.** This means if we choose any $\boldsymbol{u}$ to be our zero vector, then any other $\boldsymbol{v} \neq \boldsymbol{u}$ would give $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{v} \neq \boldsymbol{u}$**.**

(S1) $k(\boldsymbol{u} + \boldsymbol{v}) = k\boldsymbol{u} + k\boldsymbol{v}$

$$
\begin{aligned}
k(\boldsymbol{u} + \boldsymbol{v}) &= k((a, b) + (c, d)) \\
&= k(a, b) \\
&= (ka, kb)
\end{aligned}
$$

$$
\begin{aligned}
k\boldsymbol{u} + k\boldsymbol{v} &= k(a, b) + k(c, d) \\
&= (ka, kb) + (kc, kd) \\
&= (ka, kb)
\end{aligned}
$$

Thus this axiom does hold true.

Let V be a set of ordered pairs (a, b) where $a, b \in R$. If $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$, which axioms of a vector space hold, and which ones fail?

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

(S2) $(q + r)\boldsymbol{u} = q\boldsymbol{u} + r\boldsymbol{u}$

$$\begin{aligned} (q + r)\boldsymbol{u} &= (q + r)(a, b) \\ &= \big((q + r)a, (q + r)b\big) \end{aligned}$$

$$\begin{aligned} q\boldsymbol{u} + r\boldsymbol{u} &= q(a, b) + r(a, b) \\ &= (qa, qb) + (ra, rb) \\ &= (qa, qb) \end{aligned}$$

Thus this axiom does not hold. One counter example could be $(1,1)$ with $q = 1$ and $r = 2$

(S3) $(qr)\boldsymbol{u} = q(r\boldsymbol{u})$

$$\begin{aligned} (qr)\boldsymbol{u} &= (qr)(a, b) \\ &= ((qr)a, (qr)b) \\ &= \big(q(ra), q(rb)\big) && \text{(R is associative under mult)} \\ &= q(ra, rb) \\ &= q(r\boldsymbol{u}) \end{aligned}$$

Thus this axiom holds true.

(S4) $1\boldsymbol{u} = \boldsymbol{u}$

$$\begin{aligned} 1\boldsymbol{u} &= 1(a, b) \\ &= (1a, 1b) \\ &= (a, b) && \text{(1 is the multiplicative identity in R)} \\ &= \boldsymbol{u} \end{aligned}$$

Thus this axiom holds true.

Let V be a set of ordered pairs (a, b) where $a, b \in R$. If $(a, b) + (c, d) = (a, b)$ and $k(a, b) = (ka, kb)$, which axioms of a vector space hold, and which ones fail?

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

- Why does proving something is a vector space help?

For Video Please click the link below:

Video