Is it possible for a basis of $P_3(\mathbb{R})$ (over $\mathbb{R}$) to consist of only polynomials of degree 3? If so, find such a basis and prove that your set is a basis. If not, disprove this assertion.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definitions:

A set $\{v_1, \ldots, v_n\}$ is linearly independent if and only if $c_1v_1 + \cdots + c_nv_v = 0$
$\Rightarrow \quad c_1 = \cdots = c_n = 0.$

Theorem:

If a vector space V has dimension n and $\{v_1, \ldots, v_n\}$ is a linearly independent set in V, then $\{v_1, \ldots, v_n\}$ is a basis for V.

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Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Yes, it is possible. Consider the set $S = \{x^3,\ x^3 + x^2,\ x^3 + x^2 + x,\ x^3 + x^2 + x + 1\}$.

S has 4 elements in it and $P_3(\mathbb{R})$ has dimension 4, so to show that S is a basis it suffices to show that S is a linearly independent set, by theorem ???

To see that S is linearly independent we observe that if

$$c_1 x^3 + c_2(x^3 + x^2) + c_3(x^3 + x^2 + x) + c_4(x^3 + x^2 + x + 1) = 0, \text{ then}$$

$$(c_1 + c_2 + c_3 + c_4)x^3 + (c_2 + c_3 + c_4)x^2 + (c_3 + c_4)x + c_4 = 0,$$

$$\Rightarrow \quad c_4 = 0,\ \ c_3 + c_4 = 0,\ c_2 + c_3 + c_4 = 0,\ c_1 + c_2 + c_3 + c_4 = 0$$

$$\Rightarrow \quad c_1 = c_2 = c_3 = c_4 = 0.$$

Therefore S is a linearly independent set and is thus a basis for $P_3(\mathbb{R})$.

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Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Is it possible to find a basis for $P_3(\mathbb{R})$ which consists of degree 2 polynomials? Why does the technique we applied to solve this question not work when trying to do solve this?

For Video Please click the link below:

Video