Suppose U and W are subspaces of a vector space V over a field F. We define $U + W = \{u + w : u \in U, w \in W\}$.
Prove that $U + W$ is a subspace of V.

Problem Solving - What are the terms/strategies I may need? What do I know?

Proof by contradiction:

Assume that the result is false and arrive at a contradiction in the conditions.

Definition of a vector space over a field F:

V is a *vector space* over F if the following axioms are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $b, c \in F$

(A1) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.

(A2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(A3) There exists an element $\mathbf{0} \in V$ such that for all $\mathbf{u} \in V$ we have $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

(A4) For each $\mathbf{u} \in V$, there exists an element $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

(S1) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(S2) $(b + c)\mathbf{u} = b\mathbf{u} + c\mathbf{u}$

(S3) $(bc)\mathbf{u} = b(c\mathbf{u})$

(S4) $1\mathbf{u} = \mathbf{u}$

Theorems: (Subspace Test) If V is a vector space and $U \subseteq V$. Then U is a subspace of V if and only if U satisfies:

(a) The set is non-empty (usually easiest to show that there is the zero vector)

(b) If $u_1, u_2 \in U$, then $u_1 + u_2 \in U$

(c) If $c \in F$ and $u \in U$, then $cu \in U$

Suppose U and W are subspaces of a vector space $\boxed{V}$ over a field F. We define $U + W = \{u + w : u \in U, w \in W\}$.
Prove that $U + W$ is a subspace of V.

$0_V + 0_V = 0_V$

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

To show that $U + W$ is a subspace we apply the subspace test and show that

(a) The set is non-empty (usually easiest to show that there is the zero vector)

(b) If $u_1, u_2 \in U+W$, then $u_1 + u_2 \in U+W$

(c) If $c \in F$ and $u \in U+W$, then $cu \in U+W$

(a) To show that the vector space is non-empty, we show that it contains the zero vector:

Since U is a subspace of V it is a vector space. Thus it contains the zero vector $\boxed{0_V} \in U$

Since W is a subspace of V it is a vector space. Thus it also contains the zero vector $0_V \in W$

Thus $(0_V + 0_V) \in U + W$ $\quad\Rightarrow 0_V \in U + W \quad\Rightarrow U + W$ is non-empty.

Suppose U and W are subspaces of a vector space V over a field F. We define $U + W = \{u + w : u \in U, w \in W\}$.
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X_1 X_2

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

(b) Next lets suppose that $x_1, x_2 \in U + W$. By the definition of $U + W$ it follows that $x_1 = u_1 + w_1$, $x_2 = u_2 + w_2$ for some $u_1, u_2 \in U$, $w_1, w_2 \in W$. To see that $x_1 + x_2 \in U + W$ observe that:

$(x_1) \quad + (x_2)$ $\qquad \in V$ $\qquad$ Vector

$$(u_1+w_1) + (u_2+w_2) = (u_1+(w_1 + u_2)) + w_2 \qquad \text{(By Addition Associativity)}$$

$$= (u_1+(u_2+w_1)) + w_2 \qquad \text{(By Addition Commutativity)}$$

$$= \underbrace{(u_1+u_2)}_{U} + \underbrace{(w_1 + w_2)}_{W} \qquad \text{(By Addition Associativity)}$$

$\in U + W$, since U and W being subspaces implies that $u_1 + u_2 \in U$ and $w_1 + w_2 \in W$ by property (b) of the subspace test.

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This shows that $x_1 + x_2 \in U + W$ and hence proves that $U + W$ satisfies property (b) of the subspace test.

Suppose U and W are subspaces of a vector space V over a field F. We define $U + W = \{u + w : u \in U, w \in W\}$.
Prove that $U + W$ is a subspace of V.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

(c) Next suppose that $c \in F$ and $x \in (U + W)$. To show that $U + W$ satisfies property (c) of the subspace test, we must show that $cx \in U + W$,

Because $x \in U + W$, we know that $x = u + w$ for some $u \in U, w \in W$.
From this we observe that:

$$cx = c(u + w)$$

$$= cu + cw \qquad \text{(by Scalar Distribution)}$$

$$\in U + W, \quad \text{since } U \text{ and } W \text{ being subspaces implies that } cu \in U \text{ and } cw \in W \text{ by property (c) of the subspace test.}$$

This shows that $U + W$ satisfies property (c) of the subspace test.
Therefore $U + W$ satisfies all three properties of the subspace test and thus is a subspace.

Suppose U and W are subspaces of a vector space V over a field F. We define $U + W = \{u + w : u \in U, w \in W\}$.
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Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Where did we use that U and W are subspaces?

Did we use all the properties that U and W have from being subspaces to prove that $U + W$ is a subspace?

For Video Please click the link below:

Video