Let $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Determine a formula for A^n for all $\mathrm{n} \in N$ and prove this formula is true.

Problem Solving - What are the terms/strategies I may need? What do I know?

Proof by induction:

Prove a base case for $n = b$ (usually b = 0 or b = 1)

Assume the assertion holds true for $n = k \geq 1$

Prove that the assertion holds true for $n = k + 1$

Matrix products:

$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik} [B]_{kj}$

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Steps & Process – Try to answer the question writing in many steps to avoid small errors.

We first must uncover a pattern to identify A^n

We see that:

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & b+b \\ 0+0 & 0+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} A^3 = A^2 A &= \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & 2b+b \\ 0+0 & 0+1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3b \\ 0 & 1 \end{bmatrix} \end{aligned}$$

From here we may see that it seems that $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. We will prove this using induction.

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Steps & Process – Try to answer the question writing in many steps to avoid small errors.

$$A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$$

Proof by induction:

$Let\ n = 1$, we see that $A^n = \begin{bmatrix} 1 & (1)b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = A$ as required.

Inductive assumption:

Assume that when $n = k \geq 1$ we have $A^k = \begin{bmatrix} 1 & kb \\ 0 & 1 \end{bmatrix}$

Inductive step:

When $n = k + 1$ we have:

$$\begin{aligned} A^{k+1} &= AA^k \\ &= \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & kb \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+0 & kb+b \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & (k+1)b \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$\therefore$ we have proven the fact using induction.

Let $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Determine a formula for A^n for all $\mathrm{n} \in N$ and prove this formula is true.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why does induction help in this case?

For Video Please click the link below:

Video