T15† Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a set of vectors. Prove that

$$\{\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4, \ldots, \mathbf{v}_n - \mathbf{v}_1\}$$

is a linearly dependent set.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definition of Linearly Independent:

- A set V = $\{v_1, v_2, \ldots, v_n\}$ is linearly independent when the homogenous system:

$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ only has the trivial solution

Vector Space Properties:

- ACC Additive Closure, Column Vectors
  If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, *then* $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.
- SCC Scalar Closure, Column Vectors
  If $\alpha \in \mathbb{C}$ *and* $\mathbf{u} \in \mathbb{C}^m$, *then* $\alpha\mathbf{u} \in \mathbb{C}^m$.
- CC Commutativity, Column Vectors
  If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, *then* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
- AAC Additive Associativity, Column Vectors
  If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, *then* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- ZC Zero Vector, Column Vectors
  There is a vector, $\mathbf{0}$, *called the zero vector, such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$ *for all* $\mathbf{u} \in \mathbb{C}^m$.
- AIC Additive Inverses, Column Vectors
  If $\mathbf{u} \in \mathbb{C}^m$, *then there exists a vector* $-\mathbf{u} \in \mathbb{C}^m$ *so that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
- SMAC Scalar Multiplication Associativity, Column Vectors
  If $\alpha, \beta \in \mathbb{C}$ *and* $\mathbf{u} \in \mathbb{C}^m$, *then* $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.
- DVAC Distributivity across Vector Addition, Column Vectors
  If $\alpha \in \mathbb{C}$ *and* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, *then* $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.
- DSAC Distributivity across Scalar Addition, Column Vectors
  If $\alpha, \beta \in \mathbb{C}$ *and* $\mathbf{u} \in \mathbb{C}^m$, *then* $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.
- OC One, Column Vectors
  If $\mathbf{u} \in \mathbb{C}^m$, *then* $1\mathbf{u} = \mathbf{u}$.

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is a linearly dependent set.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Using the definition of Linear Independence we get:

$\{v_1 - v_2, v_2 - v_3, \ldots v_{n-1} - v_n, v_n - v_1\}$ is linearly dependent when we can find a non-trivial solution to:

$c_1(v_1 - v_2) + c_2(v_2 - v_3) + \ldots c_{n-1}(v_{n-1} - v_n) + c_n(v_n - v_1) = 0$

$c_1v_1 - c_1v_2 + c_2v_2 - c_2v_3 + \ldots c_{n-1}v_{n-1} - c_{n-1}v_n + c_nv_n - c_nv_1 = 0$ (Vectors are distributive)

$- c_nv_1 + c_1v_1 - c_1v_2 + c_2v_2 - c_2v_3 + \ldots c_{n-1}v_{n-1} - c_{n-1}v_n + c_nv_n = 0$ (Vectors are commutative)

$(- c_n+c_1)v_1 + (- c_1+c_2)v_2 + \ldots (- c_{n-2}+c_{n-1})v_{n-1} + (- c_{n-1}+c_n)v_n = 0$ (Vectors are distributive)

If we set the coefficients to 0, we could try to solve for the c's to see if we can get a non-trivial solution:

$$\left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & \ldots & -1 & 0 \\ -1 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & -1 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & 0 & -1 & 1 & 0 \end{array}\right].$$

We should notice that if we take $c_1 = c_2 = c_3 = \ldots = c_n = 1$, we get the equation holds true. Yet this is a non-trivial solution. Thus the original set is linearly dependent.

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is a linearly dependent set.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why do we rearrange the homogenous equation to help us in this question?

For Video Please click the link below:

Video