Show that the following set is linearly independent: $\{1, 2^x, 3^x, 4^x\}$

Problem Solving - What are the terms/strategies I may need? What do I know?

Definition of two functions being equal:

For each x in the domain, $f(x) = g(x)$

Corollary 1.18

A set of vectors is linearly independent if and only if no vector in the set can be expressed as a linear combination of vectors before it.

Show that the following set is linearly independent: $\{1, 2^x, 3^x, 4^x\}$

Steps & Process

We will show that this set is linearly independent by showing that every element cannot be expressed as a linear combination of elements before it.

Clearly, $2^x = c_1(1)$ cannot hold true for any x, as if $x = 1$ we get $c_1 = 2$, but if $x = 2$ then $c_1 = 4$. Thus the constants do not match, and 2^x is not a combination of previous terms.

If we attempt to express $3^x = c_1(1) + c_2(2^x)$ we see that :
$x = 0$ gives $1 = c_1 + c_2$
$x = 1$ gives $3 = c_1 + 2c_2$

If we solve this system, we see that $c_1 = -1$ and $c_2 = 2$ is a unique solution. However, when $x = 2$ we get: $9 = c_1 + 4c_2$ will not hold true as $9 \neq 7$. Thus 3^x is not a combination of previous terms.

If we attempt to express $4^x = c_1(1) + c_2(2^x) + c_3(3^x)$ we see that :
$x = 0$ gives $1 = c_1 + c_2 + c_3$
$x = 1$ gives $4 = c_1 + 2c_2 + 3c_3$
$x = 2$ gives $16 = c_1 + 4c_2 + 9c$

If we solve this system, we see that $c_1 = 1$ and $c_2 = -3$, and $c_3 = 3$ is a unique solution. However, when $x = 3$ we get: $64 = c_1 + 8c_2 + 27c_3$ will not hold true as $64 \neq 58$. Thus 4^x is not a combination of previous terms.

Since no term can be expressed as a combination of previous terms, this set is linearly independent.

Show that the following set is linearly independent: $\{1, 2^x, 3^x, 4^x\}$

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why does Corollary 1.18 work (can you prove it)?

What do you think this would indicate for the infinite set of exponential equations:

$$\{1, 2^x, 3^x, 4^x, \ldots, n^x, \ldots\}$$

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