Let F be a field. Suppose $\{v_1, \dots, v_n\}$ is a basis of F^n. Let $A \in M_{n\times n}(F)$ be a nonsingular matrix. Prove that $\{Av_1, \dots, Av_n\}$ is also a basis of F^n.

Problem Solving - What are the terms/strategies I may need? What do I know?

Definitions:

A set $\{v_1, \dots, v_n\}$ is linearly independent if and only if $c_1v_1 + \cdots + c_nv_v = 0$
$\Rightarrow \quad \mathrm{c}_1 = \cdots = c_n = 0.$

Theorem:

(i) If a vector space V has dimension n and $\{v_1, \dots, \mathrm{v}_n\}$ is a linearly independent set in V, then $\{v_1, \dots, \mathrm{v}_n\}$ is a basis for V.

(ii) $A \in M_{n\times n}(F)$ is a nonsingular matrix if and only if A is invertible.

Let F be a field. Suppose $\{v_1, \ldots, v_n\}$ is a basis of F^n. Let $A \in M_{n \times n}(F)$ be a nonsingular matrix. Prove that $\{Av_1, \ldots, Av_n\}$ is also a basis of F^n.

Steps & Process – Try to answer the question writing in many steps to avoid small errors.

Since the set $S = \{Av_1, \ldots, Av_n\}$ has n elements and F^n has dimension n, to show that S is a basis it sufficed to show that S is a linearly independent set, by theorem basis theorem.

To see that S is a linearly independent set observe that

$$c_1 A v_1 + \cdots + c_n A v_n = 0$$

$$\Rightarrow \quad A(c_1 v_1 + \cdots + c_n v_n) = 0,$$

$$\Rightarrow \quad A^{-1}A(c_1 v_1 + \cdots + c_n v_n) = A^{-1}0, \text{ since } A \text{ nonsingular iff } A \text{ is invertible}$$

$$\Rightarrow \quad c_1 v_1 + \cdots + c_n v_n = 0, \text{ since } A^{-1}A = I, \text{ and } A^{-1}0 = 0$$

$$\Rightarrow \quad c_1 = \cdots = c_n = 0, \text{ since } \{v_1, \ldots, v_n\} \text{ is a linearly independent set}$$

Therefore $\{Av_1, \ldots, Av_n\}$ is a linearly independent set and is hence a basis of F^n.

Let F be a field. Suppose $\{v_1, \dots, v_n\}$ is a basis of F^n. Let $A \in M_{n\times n}(F)$ be a nonsingular matrix. Prove that $\{Av_1, \dots, Av_n\}$ is also a basis of F^n.

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Is this still true if we remove the assumption that A is nonsingular?

For Video Please click the link below:

Video