Consider the sets $S = \{u, v, w\}$ and $R = \{u + 2v + w, u - v, -u + v + w\}$ then show:

i) If S is linearly independent, then R is linearly independent.

ii) If R is linearly independent, then S is linearly independent. iii) Span(S) = Span(R)

Problem Solving - What are the terms/strategies I may need? What do I know?

Exchange Lemma:

Consider a basis for a space V given by $B = \{b_1, \dots, b_n\}$

Let $v = c_1 b_1 + \cdots + c_n b_n$ where $c_i \neq 0$. Then $B' = \{b_1, \dots, b_{i-1}, v, b_{i+1}, \dots, b_n\}$ is a basis for V.

Definition for a basis

A set of vectors that span the space, and the set of vectors are linearly independent.

Consider the sets $S = \{u, v, w\}$ and $R = \{u + 2v + w, u - v, -u + v + w\}$ then show:

i) If S is linearly independent, then R is linearly independent.

ii) If R is linearly independent, then S is linearly independent. iii) Span(S) = Span(R)

Steps & Process

We will prove i first:

Let us consider the exchange lemma in this case:

If S is linearly independent then

Let $B = S = \{u, v, w\}$ be a basis for Span (S).

By exchange lemma, we see that if $x = (1)u + (2)v + (1)w = u + 2v + w$ then since $c_1 \neq 0$, we can replace u with x and get:
$B_1 = \{u + 2v + w, v, w\}$ will be a basis for V

By exchange lemma, we see that if $y = (1)(u + 2v + w) + (-3)v + (-1)w = u - v$, then since $c_2 \neq 0$ we can replace v with y and get:
$B_2 = \{u + 2v + w, u - v, w\}$ will be a basis for V

By exchange lemma, we see that if $z = (0)(u + 2v + w) + (-1)(u - v) + (1)w = -u + v + w$, then since $c_3 \neq 0$ we can replace w with z and get:
$B_3 = R = \{u + 2v + w, u - v, -u + v + w\}$ will be a basis for V

Since R is a basis for a space, it is linearly independent.

(Note that this also proves 3, as this means that they span the same space, thus have the same span, so we get iii for free ☺)

Consider the sets $S = \{u, v, w\}$ and $R = \{u + 2v + w, u - v, -u + v + w\}$ then show:

i) If S is linearly independent, then R is linearly independent.

ii) If R is linearly independent, then S is linearly independent. iii) Span(S) = Span(R)

Steps & Process

We will prove ii next:

Let us also consider the exchange lemma in this case:

If R is linearly independent then

Let $B = R = \{u + 2v + w, u - v, -u + v + w\}$ be a basis for Span (R).

By exchange lemma, we see that if $x = (0)(u + 2v + w) + (1)(u - v) + (1)(-u + v + w) = w$ then since $c_3 \neq 0$, we can replace x_3 with w and get:

$B_1 = \{u + 2v + w, u - v, w\}$ will be a basis for V

By exchange lemma, we see that if $y = (\frac{1}{3})(u + 2v + w) + \left(-\frac{1}{3}\right)(u - v) + \left(-\frac{1}{3}\right)w = v$, then since $c_2 \neq 0$ we can replace x_2 with v and get:

$B_2 = \{u + 2v + w, v, w\}$ will be a basis for V

By exchange lemma, we see that if $z = (1)(u + 2v + w) + (-2)(v) + (-1)w = u$, then since $c_1 \neq 0$ we can replace x_1 with u and get:

$B_3 = S = \{u, v, w\}$ will be a basis for V

Since S is a basis for a space, it is linearly independent.

Consider the sets $S = \{u, v, w\}$ and $R = \{u + 2v + w, u - v, -u + v + w\}$ then show:

i) If S is linearly independent, then R is linearly independent.

ii) If R is linearly independent, then S is linearly independent.

iii) Span(S) = Span(R)

Solidify Understanding – Explain why the steps makes sense by connecting to math you know.

Why does the exchange lemma help in this case?

Can you find another way to prove these facts?

For Video Please click the link below:

Video