Numerical Analysis
Homework Five

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Methods for Solving a parabolic non homogeneous PDE

→ Given PDE:

$$u_t - u_{xx} = 2$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = sin(\pi x) + x(1-x)$$

→ for $0 \leq x \leq 1$,$t \geq 0$

In This assignment, the focus will be on solving the PDE through finite difference methods. A second order central difference formula will be used for the space derivative in the PDE. But different finite difference formulas (different methods within FDM's) on the time derivative will be explored.

1 Finite Difference Methods

For FDM, need to start out with the discretization of the domain to obtain the mesh points.

$\rightarrow t_j = jk$ represents the jth step into time using step size k. For this problem, k = 0.01.

$\rightarrow x_i = ih$ represents the ith step in space using step size h. For this problem, h = 0.1. Since $0 \leq i \leq n+1$, then

$$h = \frac{1}{n+1} \longrightarrow n = 9$$

1.1 Explicit Method (Forward Euler in time)

Discretization of the problem using a forward finite difference approximation for the time derivative.

$$\frac{1}{k}[u(x, t+k) - u(x,t)] - \frac{1}{h^2}[u(x+h,t) - 2u(x,t) + u(x-h,t)] = 2$$

The problem in terms of its mesh points:

$$\frac{1}{k}[u_{i,j+1} - u_{i,j}] - \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] = 2$$

Need the system whose solution vector holds the discrete approximations of the solution to this PDE.

$$\frac{1}{k}[u_{i,j+1} - u_{i,j}] = 2 + \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$[u_{i,j+1} - u_{i,j}] = 2k + \frac{k}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

let $s = \frac{k}{h^2}$, so

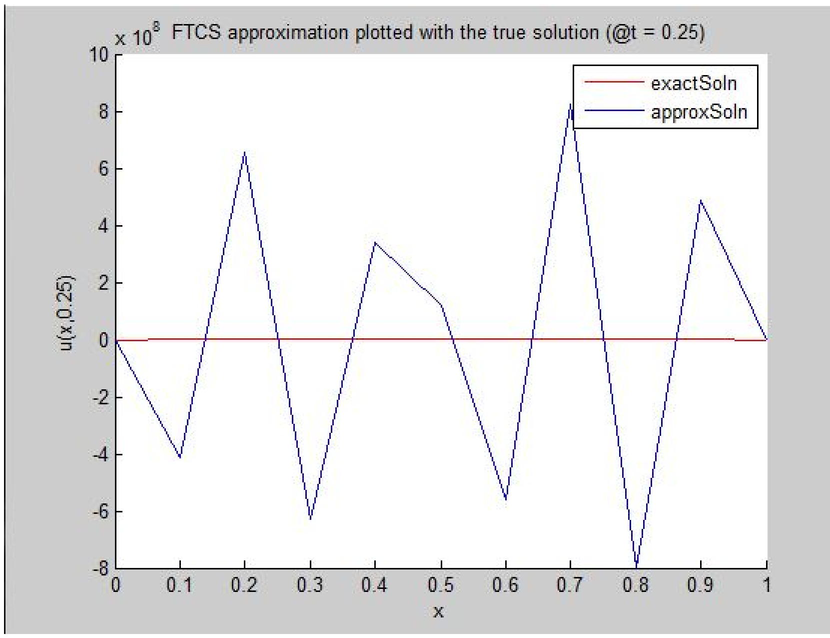
$$u_{i,j+1} - u_{i,j} = 2k + s[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$u_{i,j+1} = 2k + s[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + u_{i,j}$$

The forward time, centered space (FTCS) approximation to this PDE then becomes

$$u_{i,j+1} = su_{i+1,j} + (1-2s)u_{i,j} + su_{i-1,j} + 2k$$

Solving this recurrence relation yields the following solution:



The reason why this approximation blows up is because the given step sizes in this assignment do not satisfy the stability criteria for the explicit FD method. The method is stable when

$$\frac{k}{h^2} = s \leq \frac{1}{2}$$

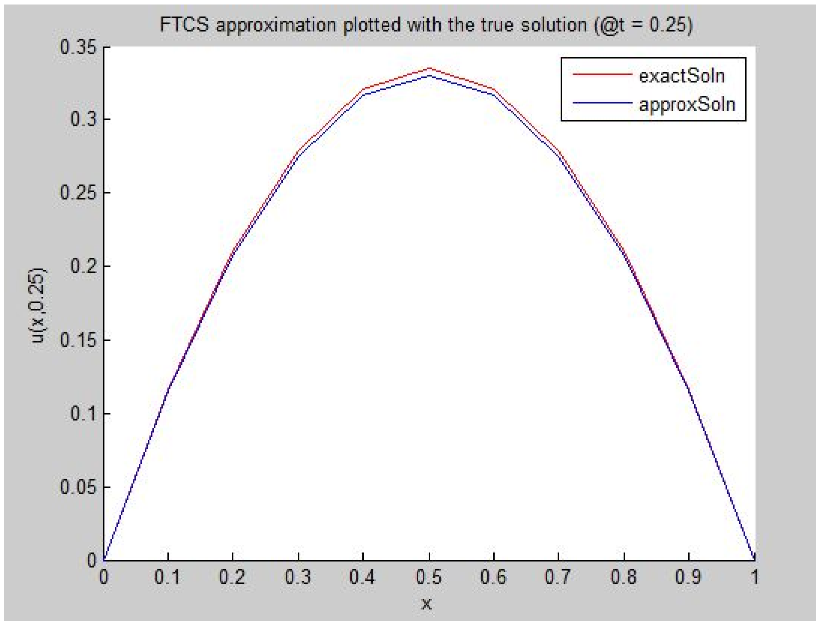
but in this case

$$\frac{0.01}{0.1^2} = \frac{0.01}{0.01} = 1 \not\leq \frac{1}{2}$$

But I verified that my FTCS algorithm worked by picking h to still be 0.1, but k to be 0.001. This does satisfy the the stability for this method since

$$\frac{0.001}{0.1^2} = \frac{0.001}{0.01} = 0.1 \leq \frac{1}{2} \checkmark$$

here is the new FTCS with the better step sizes




```
FTCS chart for each spacial step at time t = 0.25
trueSol   approx absError
0.000000 0.000000 0.000000
0.116206 0.114805 0.001401
0.209847 0.207180 0.002668
0.278609 0.274930 0.003678
0.320654 0.316320 0.004334
0.334805 0.330236 0.004569
0.320654 0.316298 0.004356
0.278609 0.274895 0.003714
0.209847 0.207144 0.002703
0.116206 0.114783 0.001423
0.000000 0.000000 0.000000
EDU>>
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1.2 Implicit Method (Backward Euler in time)

Discretization of the problem using a backward finite difference approximation for the time derivative.

$$\frac{1}{k}[u(x,t) - u(x,t-k)] - \frac{1}{h^2}[u(x+h,t) - 2u(x,t) + u(x-h,t)] = 2$$

The problem in terms of its mesh points:

$$\frac{1}{k}[u_{i,j} - u_{i,j-1}] - \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] = 2$$

Need the system whose solution vector holds the discrete approximations of the solution to this PDE.

$$\frac{1}{k}[u_{i,j} - u_{i,j-1}] = 2 + \frac{1}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$[u_{i,j} - u_{i,j-1}] = 2k + \frac{k}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$-u_{i,j-1} = 2k + \frac{k}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] - u_{i,j}$$

$$u_{i,j-1} = u_{i,j} - \frac{k}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] - 2k$$

let $s = \frac{k}{h^2}$, so

$$u_{i,j-1} = u_{i,j} - s[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] - 2k$$

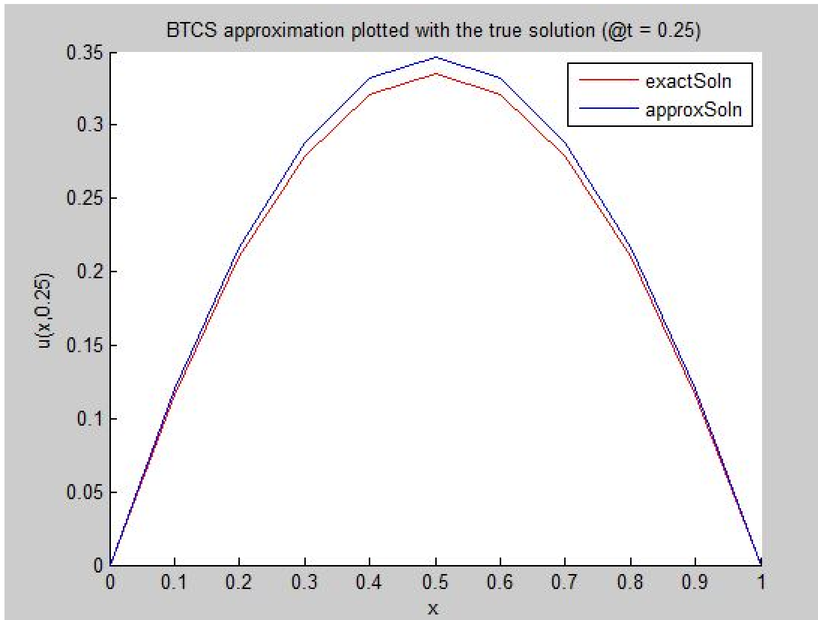
The backward time, centered space (BTCS) approximation to this PDE then becomes

$$u_{i,j-1} = -su_{i+1,j} + (1+2s)u_{i,j} - su_{i-1,j} - 2k$$

Since I can figure out all of $u_{i,j-1}$ before hand, I can make a linear system out of this system.

$$u_{i,j-1} + 2k = -su_{i+1,j} + (1+2s)u_{i,j} - su_{i-1,j}$$

So instead of having to solve a recurrence relation, can just solve the linear system:




```
BTCS chart for each spacial step at time t = 0.25
trueSol   approx absError
0.000000 0.000000 0.000000
0.116206 0.119926 0.003719
0.209847 0.216922 0.007075
0.278609 0.288346 0.009737
0.320654 0.332101 0.011447
0.334805 0.346841 0.012036
0.320654 0.332101 0.011447
0.278609 0.288346 0.009737
0.209847 0.216922 0.007075
0.116206 0.119926 0.003719
0.000000 0.000000 0.000000
EDU>>
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1.3 Crank-Nicolson Method

Discretization of the problem using a central finite difference approximation for the time derivative. The problem in terms of its mesh points is:

$$\frac{1}{k}[u_{i,j} - u_{i,j-1}] - \frac{1}{2}[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2}] = 2$$

$$\frac{1}{k}[u_{i,j} - u_{i,j-1}] = 2 + \frac{1}{2}[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2}]$$

$$[u_{i,j} - u_{i,j-1}] = 2k + \frac{k}{2}[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2}]$$

$$[u_{i,j} - u_{i,j-1}] = 2k + \frac{k}{2h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}]$$

$$2[u_{i,j} - u_{i,j-1}] = 4k + \frac{k}{h^2}[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}]$$

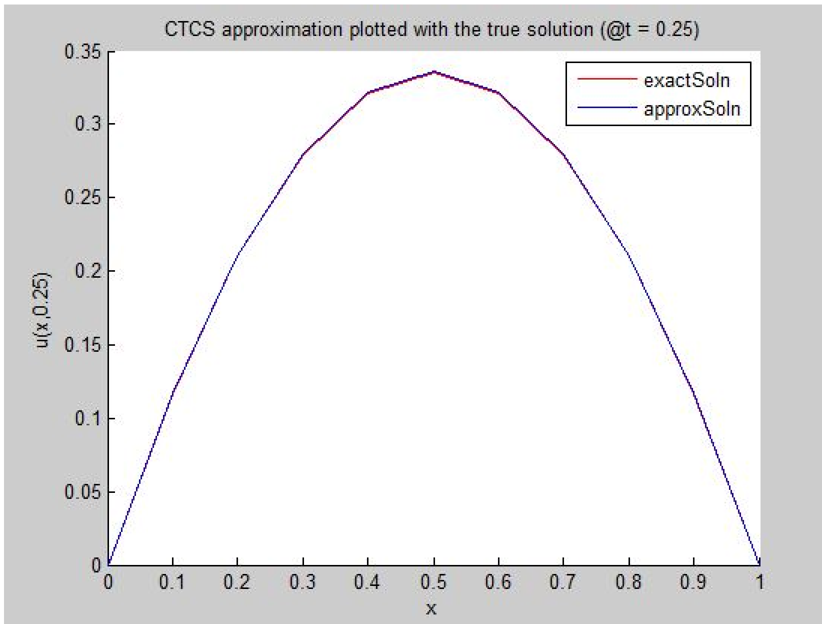
let $s = \frac{k}{h^2}$, so

$$2[u_{i,j} - u_{i,j-1}] = 4k + s[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}]$$

The center time, centered space (CTCS) approximation to this PDE then becomes

$$-su_{i-1,j} + (2 + 2s)u_{i,j} - su_{i+1,j} = 4k + su_{i+1,j-1} + (2 - 2s)u_{i,j-1} + su_{i-1,j-1}$$

This approximation can also be obtained by solving a linear system.




```
CTCS chart for each spacial step at time t = 0.25
trueSol   approx absError
0.000000 0.000000 0.000000
0.116206 0.116689 0.000483
0.209847 0.210766 0.000919
0.278609 0.279874 0.001265
0.320654 0.322141 0.001487
0.334805 0.336369 0.001564
0.320654 0.322141 0.001487
0.278609 0.279874 0.001265
0.209847 0.210766 0.000919
0.116206 0.116689 0.000483
0.000000 0.000000 0.000000
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2 Stability and Accuracy of Crank-Nicolson for the Homogeneous Parabolic Model

The Homogeneous Parabolic Model

$$u_t = u_{xx}$$

The CTCS for the the Homogeneous Parabolic Model:

$$-su_{i-1,j} + (2+2s)u_{i,j} - su_{i+1,j} = su_{i+1,j-1} + (2-2s)u_{i,j-1} + su_{i-1,j-1}$$

2.1 Accuracy

Need to make the CTCS to represent the exact solution. So including the associated errors with each finite difference formula will make the CTCS equation represent the true solution instead of the approximate solution.

$$\frac{1}{k}[u_{i,j} - u_{i,j-1}] + O(k^3)$$

$$-\frac{1}{2}[\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + O(h^5) + \frac{u_{i+1,j-1} - 2u_{i,j-1} + u_{i-1,j-1}}{h^2} + O(h^5)] = 2$$

which makes the global error $O(k^2) + O(h^2)$

2.2 Stability

By the Fourier method

$$u_{j,n} = e^{ij\beta h}e^{n\lambda k}$$

so the CTCS gets rewritten as

$$-se^{i(j-1)\beta h}e^{n\lambda k} + (2+2s)e^{ij\beta h}e^{n\lambda k} - se^{i(j+1)\beta h}e^{n\lambda k}$$

$$= se^{i(j+1)\beta h}e^{(n-1)\lambda k} + (2-2s)e^{ij\beta h}e^{(n-1)\lambda k} + se^{i(j-1)\beta h}e^{(n-1)\lambda k}$$

Removing the factor $e^{ij\beta h}e^{n\lambda k}$ gets

$$-se^{-i\beta h} + (2+2s) - se^{i\beta h} = se^{i\beta h}e^{-\lambda k} + (2-2s)e^{-\lambda k} + se^{-i\beta h}e^{-\lambda k}$$

$$-se^{-i\beta h} - se^{i\beta h} + (2+2s) = se^{i\beta h - \lambda k} + (2-2s)e^{-\lambda k} + se^{-(i\beta h + \lambda k)}$$

$$-s(e^{-i\beta h} + e^{i\beta h}) + (2+2s) = s(e^{i\beta h - \lambda k} + e^{-(i\beta h + \lambda k)}) + (2-2s)e^{-\lambda k}$$

Using Euler's identity

$$-2scos(\beta h) + (2+2s) = se^{-\lambda k}(e^{i\beta h} + e^{-i\beta h}) + (2-2s)e^{-\lambda k}$$

$$-2scos(\beta h) + (2+2s) = 2se^{-\lambda k}(cos(\beta h)) + (2-2s)e^{-\lambda k}$$

$$-2scos(\beta h) + (2+2s) = e^{-\lambda k}(2s(cos(\beta h)) + (2-2s))$$

$$\frac{-2scos(\beta h) + (2 + 2s)}{2scos(\beta h) + (2 - 2s)} = e^{-\lambda k}$$

$$e^{-\lambda k} = \frac{s(1 - cos(\beta h) + 1)}{s(cos(\beta h) - 1) + 1}$$

This equation holds for all values of s (all possible step sizes) which means that the Crank-Nicolson method for this problem is unconditionally stable.