Entanglement Hamiltonians in quantum many-body physics

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work with
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TU Graz

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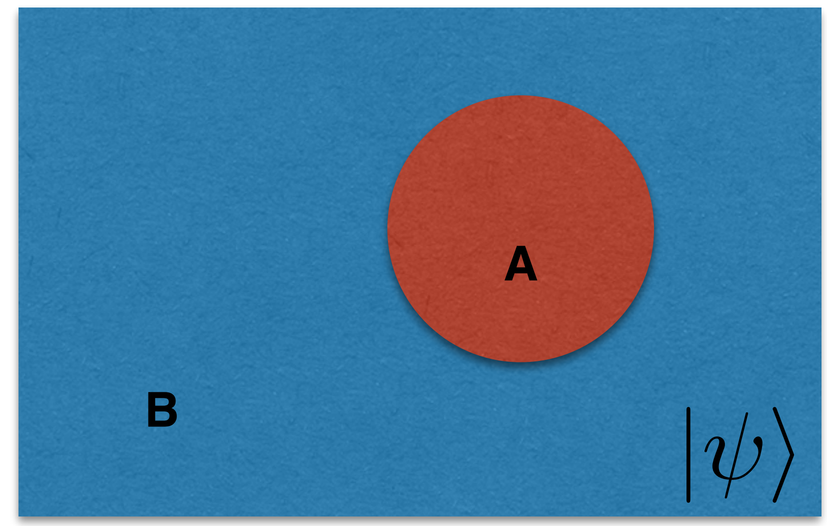
Outline

- Introduction
- EH in field theory
- EH on the lattice
- Further aspects

Dalmonte, Eisler, Falconi, Vermersch
Ann. Phys. (Berlin) **534**, 2200064 (2022)

Entanglement

- Bipartition of a large system



- Reduced density matrix

$$\rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\, \rho = \mathrm{Tr}_{\mathrm{B}} \left|\Psi\right\rangle \left\langle\Psi\right|$$

- RDM becomes a mixed state in general $\quad |\Psi\rangle \neq |\Psi_A\rangle|\Psi_B\rangle$

- von Neumann / entanglement entropy

$$S = -\mathrm{Tr}[\rho_{\mathrm{A}} \ln \rho_{\mathrm{A}}]$$

Entanglement entropy

- characterization of many-body ground states
- area law behaviour
- universal behaviour at quantum criticality
- can be used to distinguish phases of matter
- detects also topological order
- BUT fails to describe the spatial structure of entanglement

Entanglement Hamiltonian

- Entanglement content of many-body **ground states**
- Characterize reduced state via thermodynamic analogies
- Quantum correlations/fluctuations instead of thermal ones

Finite T

- Thermodynamic entropy:

Volume law

- $\rho_{\mathrm{A}} = \frac{1}{Z} e^{-\beta \hat{H}}$

Boltzmann-Gibbs

T=0

- Entanglement entropy:

Area law

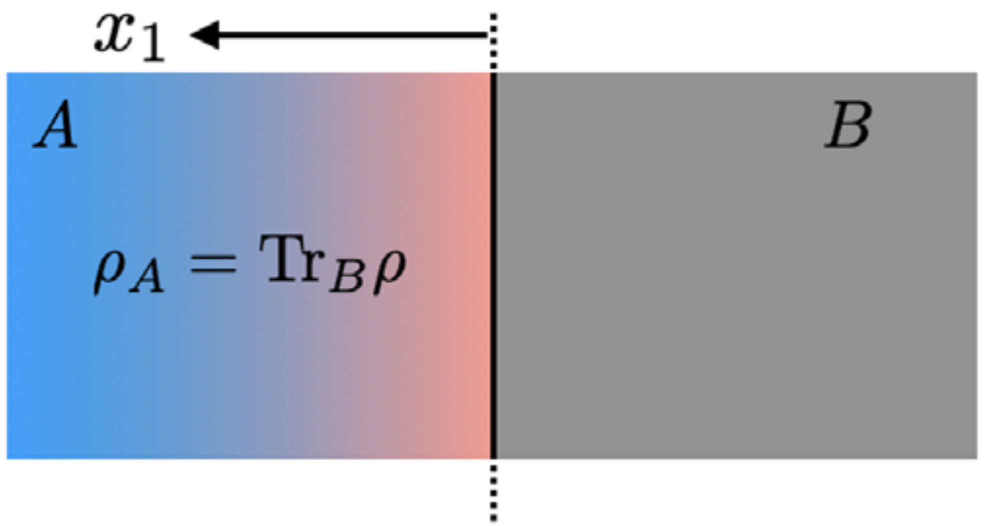
- $\rho_{\mathrm{A}} = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}}$

Entanglement Hamiltonian

Bisognano Wichmann result

- EH is **local** for a large family of models

$$\mathcal{H} = \int_{x \in A} dx\, \beta(x)\, \mathcal{H}(x)$$



- $\mathcal{H}(x) = T_{00}(x)$ energy density
- $\beta(x)$: local inverse temperature

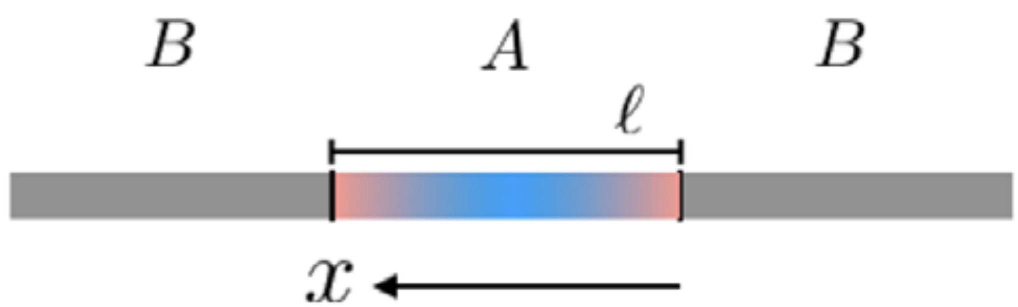
$$\beta(x) = 2\pi x$$

- Originally established in algebraic QFT
- Only for half-system geometry

Bisognano, Wichmann J. Math. Phys 1975, 1976

Generalizations of BW

- Generalizations to various geometries for CFTs
- Interval for 1D
- Ball for D>1
- Parabolic variation of $\beta(x)$



$$\beta(x) = 2\pi \frac{x(\ell - x)}{\ell}$$

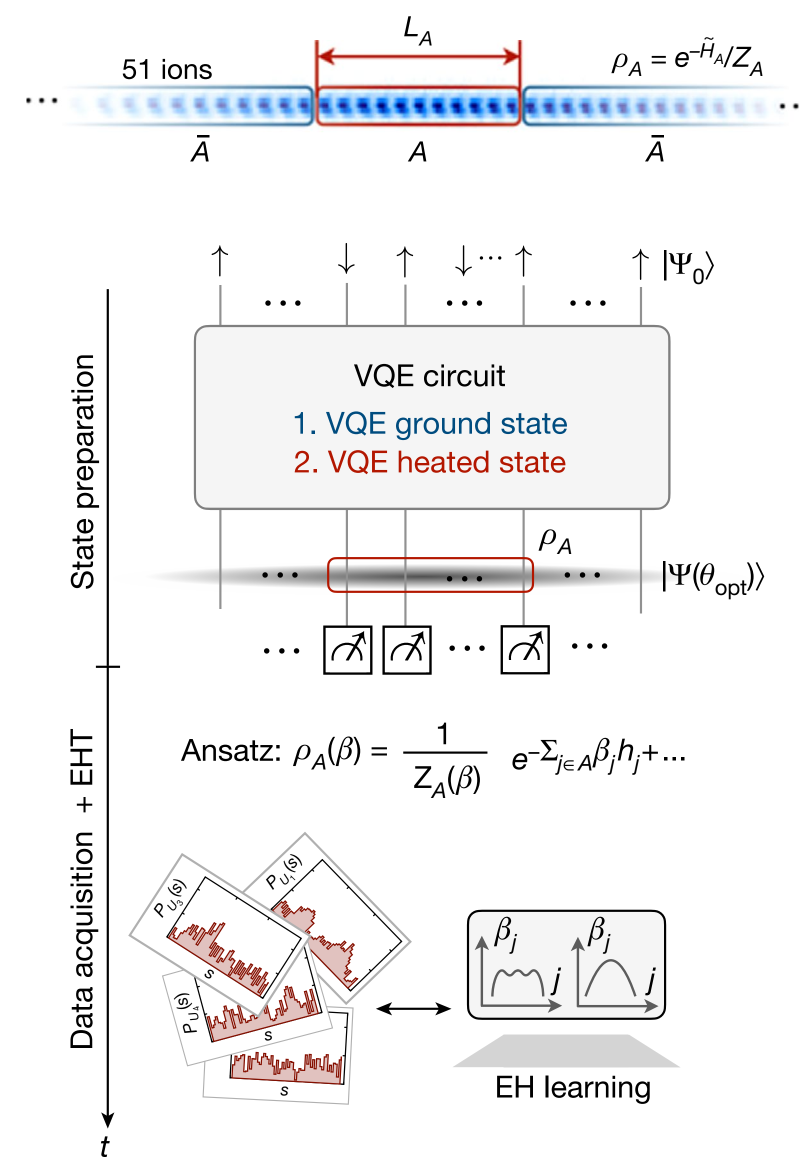
$$\hat{\mathcal{H}}_{\mathrm{CFT}} = \frac{\pi R}{v} \int_A \mathrm{d}^d\mathbf{x} \left(1 - \frac{|\mathbf{x}|^2}{R^2}\right) T_{00}(\mathbf{x})$$

Hislop, Longo Comm. Math. Phys. 1982
Casini, Huerta, Myers, JHEP 2011

Motivation

- Use local structure of EH as ansatz
- EH tomography: reduce number of measurements drastically
- quantum variational learning
- EH can be “measured” in a quantum simulator
- experiments with ion traps

Kokail et.al. Nature Physics 2021
Joshi et. al. Nature 2023



Models

- Free fermi gas: $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} - \mu$

- Hopping chain: $\hat{H} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \cos q_F \sum_{n=-\infty}^{\infty} c_n^\dagger c_n$

- GS is a Fermi sea / Slater determinant

- Nonrelativistic models!

- Low energy description given by the massless Dirac fermion

- How does BW carry over?

EH for free fermions

- Wick's theorem implies that EH is also free fermion!

$$\mathcal{H} = \sum_{m,n\in A} H_{m,n} c^\dagger_m c_n \,,$$

- Characterized by the reduced correlation matrix

$$C_{m,n} = \langle c^\dagger_m c_n \rangle \qquad\qquad C = \frac{1}{e^H + 1}$$

Peschel J.Phys.A 2003

- Integral operator for continuum systems

$$(\hat{\mathcal{K}}_A\, \psi)(x) = \int_A dx'\, K(x, x')\, \psi(x') \qquad \hat{\mathcal{K}}_A = \frac{1}{e^{\hat{\mathcal{H}}} + 1},$$

- Task: solve the eigenvalue problem!

Fermi gas

- Sine kernel

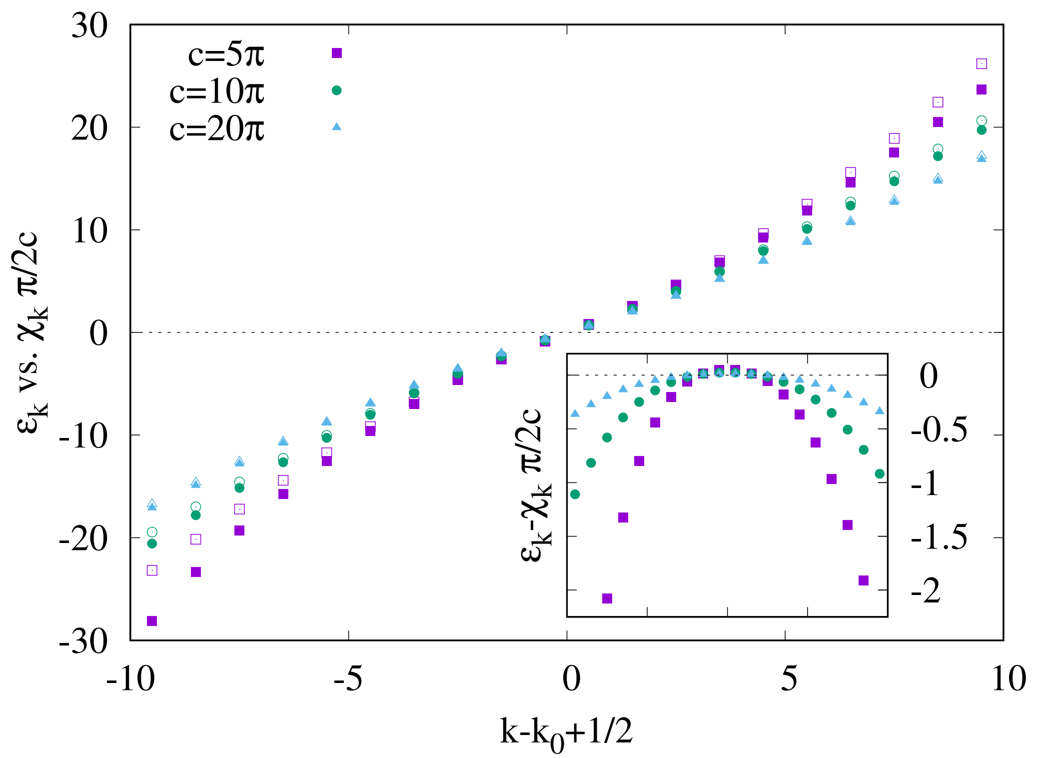
$$K(y,y') = \frac{\sin c(y-y')}{\pi(y-y')} \qquad (c = q_F R)$$

- Commuting diferential operator

$$\hat{D} = -\frac{\mathrm{d}}{\mathrm{d}y}(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y} - c^2(1-y^2)$$

- BW prediction reads

$$\hat{\mathcal{H}}_{\mathrm{CFT}} = \frac{\pi R}{v_F}\frac{\hat{D}}{2mR^2} = \frac{\pi}{2c}\hat{D}$$



$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathrm{CFT}} + \sum_{n=1}^{\infty}\frac{1}{c^n}P_{n+1}(\hat{\mathcal{H}}_{\mathrm{CFT}})$$

Slepian, Pollak Bell Syst. Tech. J. 1961
Eisler Phys. Rev. B 2024

Hopping chain

- Discrete sine kernel

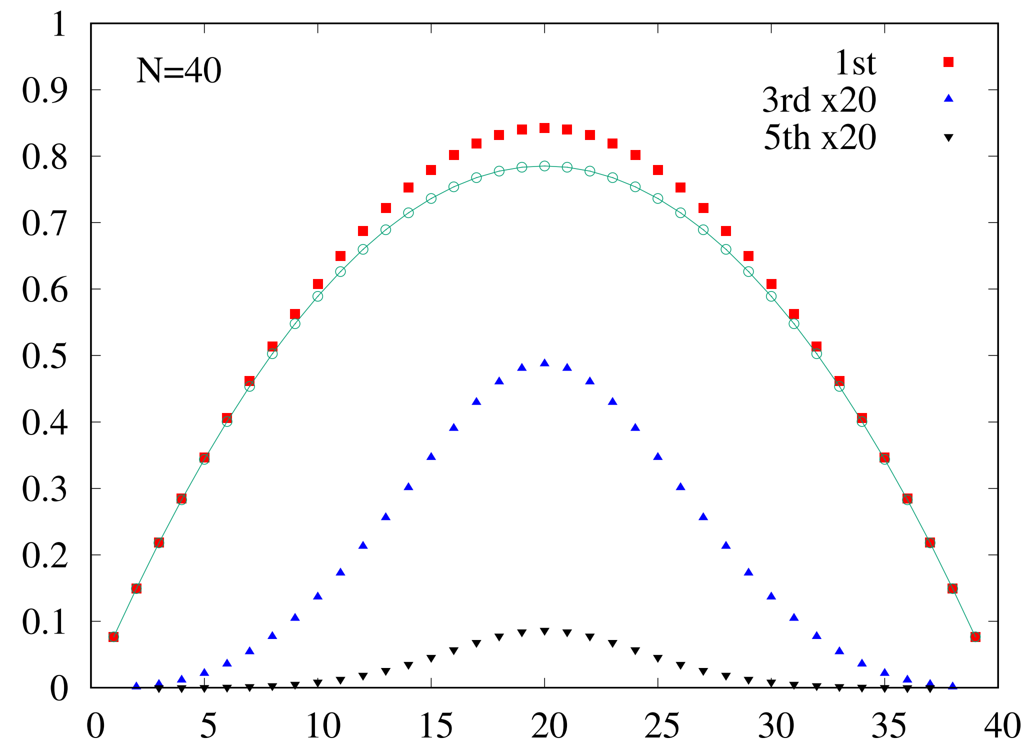
$$C_{nn'} = \frac{\sin q_F(n-n')}{\pi(n-n')}$$

- Commuting tridiagonal matrix

$$T = \begin{pmatrix} d_1 & t_1 & & & \\ t_1 & d_2 & t_2 & & \\ & t_2 & d_3 & t_3 & \\ & & \ddots & \ddots & \\ & & & t_{L-1} & d_L \end{pmatrix}$$

$$t_n = \tfrac{1}{2}n(L-n)$$

$$d_i = -2\cos q_F \, \frac{2i-1}{2N}\left(1-\frac{2i-1}{2N}\right)$$



$$h_{i,j} = -\lim_{N\to\infty} \frac{H_{i,j}}{N}$$

$$h = \sum_{m=0}^{\infty} \alpha_m \beta_m T^{2m+1}$$

Slepian Bell Syst. Tech. J. 1978
Eisler, Peschel, J. Phys. A 2017

Continuum limit

- Why the discrepancy?
- Proper continuum limit needed!

$$c_i \longrightarrow \sqrt{s}\left(e^{\mathrm{i}q_F x}\,\psi_{\mathrm{R}}(x) + e^{-\mathrm{i}q_F x}\,\psi_{\mathrm{L}}(x)\right) \qquad x = is$$

- Renormalizes the Fermi velocity

$$v_F(x) = 2s\sum_{r=1}^{\infty} r\,\sin(r q_F s)\,h_{i,i+r}\,.$$

- One recovers the CFT result exactly!

Arias, Blanco, Casini, Huerta, PRD 2017
Eisler, Tonni, Peschel, J. Stat. Mech. 2019

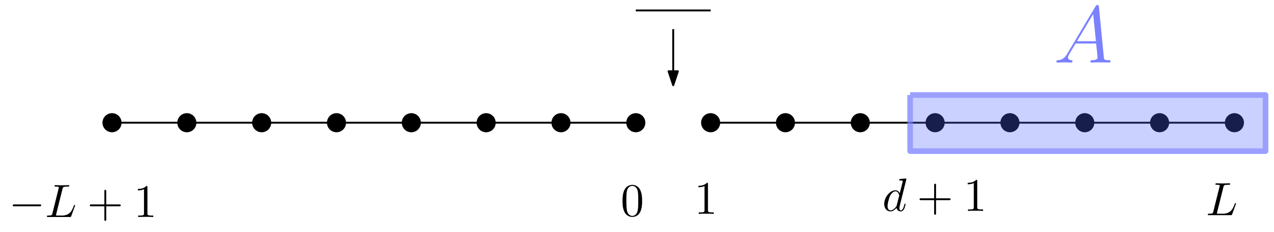
Further aspects

- treatment can be extended to inhomogeneous chains (e.g. slowly varying potentials and/or hoppings)

$$\widehat{H} = \sum_{n=0}^{N-2} J_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) - \sum_{n=0}^{N-1} B_n c_n^\dagger c_n$$

Tonni, Rodríguez-Laguna, Sierra JSTAT 2018
Bonsignori, Eisler J. Phys. A 2024
Bernard, Bonsignori, Eisler, Parez, Vinet Nucl. Phys. B 2025

- one can also discuss non-equilibrium situations (e.g. quench)



Bonsignori, Eisler arXiv:2508.19406

Conclusions

- CFT results for EH can be recovered on the lattice
- Inhomogeneous systems / out-of equilibrium scenario
- Many-body states with topological properties?
- Funding:

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“Entanglement Hamiltonians in quantum many-body physics”

“Entanglement structures and topology”

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