Longest increasing subsequence

Longest increasing subsequence. Given a sequence of elements $c_1, c_2, \ldots, c_n$ from a totally-ordered universe, find the longest increasing subsequence.

Ex. 7 2 8 1 3 4 10 6 9 5.

Maximum Unique Match finder

Application. Part of MUMmer system for aligning entire genomes.

$O(n^2)$ dynamic programming solution. LIS is a special case of edit-distance.

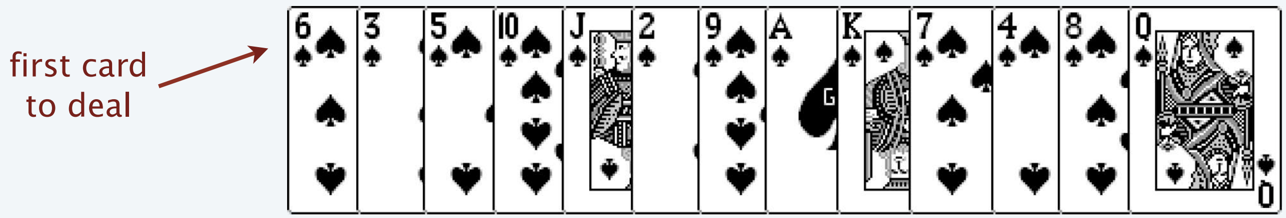
- $x = c_1 \, c_2 \cdots c_n$.
- y = sorted sequence of c_k, removing any duplicates.
- Mismatch penalty $= \infty$; gap penalty $= 1$.

Patience solitaire

Patience. Deal cards $c_1, c_2, \ldots, c_n$ into piles according to two rules:

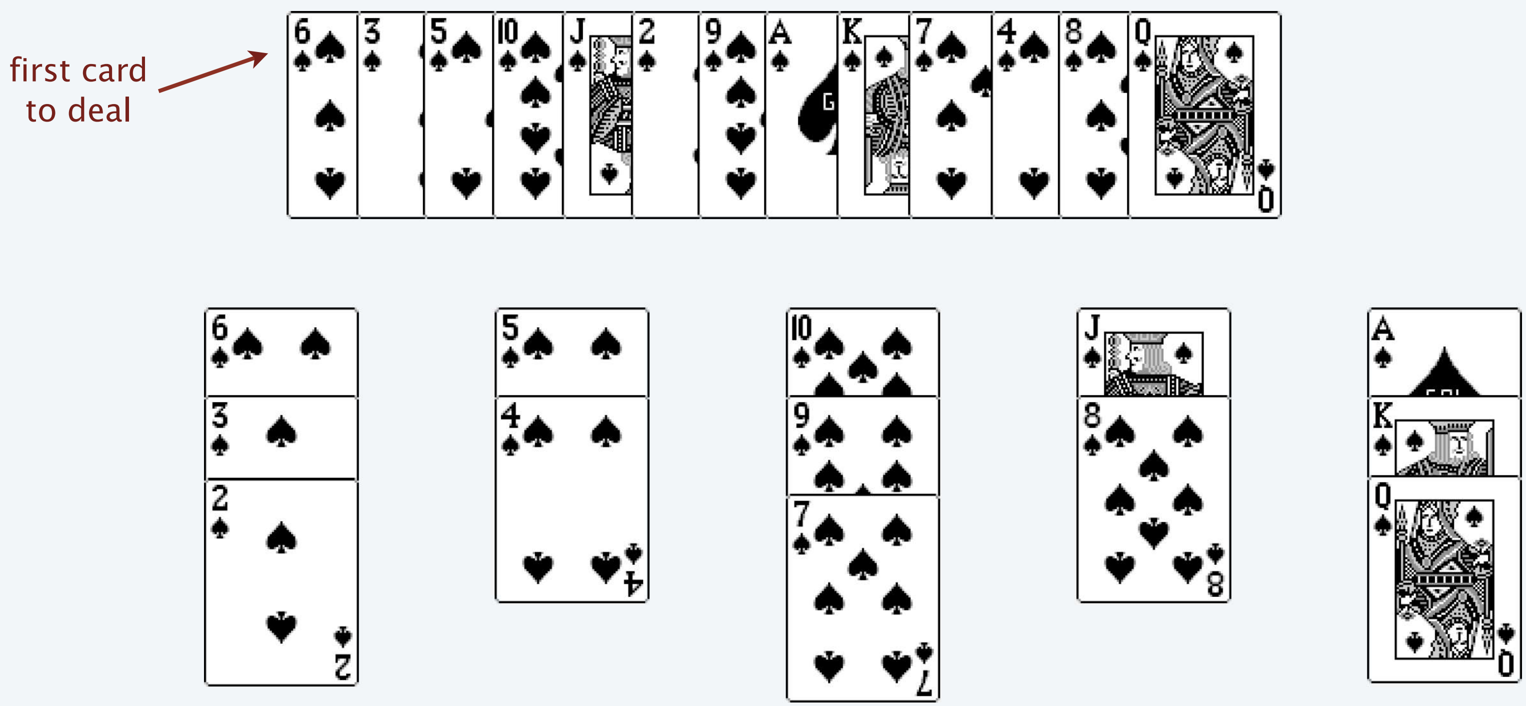
- Can't place a higher-valued card onto a lowered-valued card.
- Can form a new pile and put a card onto it.

Goal. Form as few piles as possible.



Patience: greedy algorithm

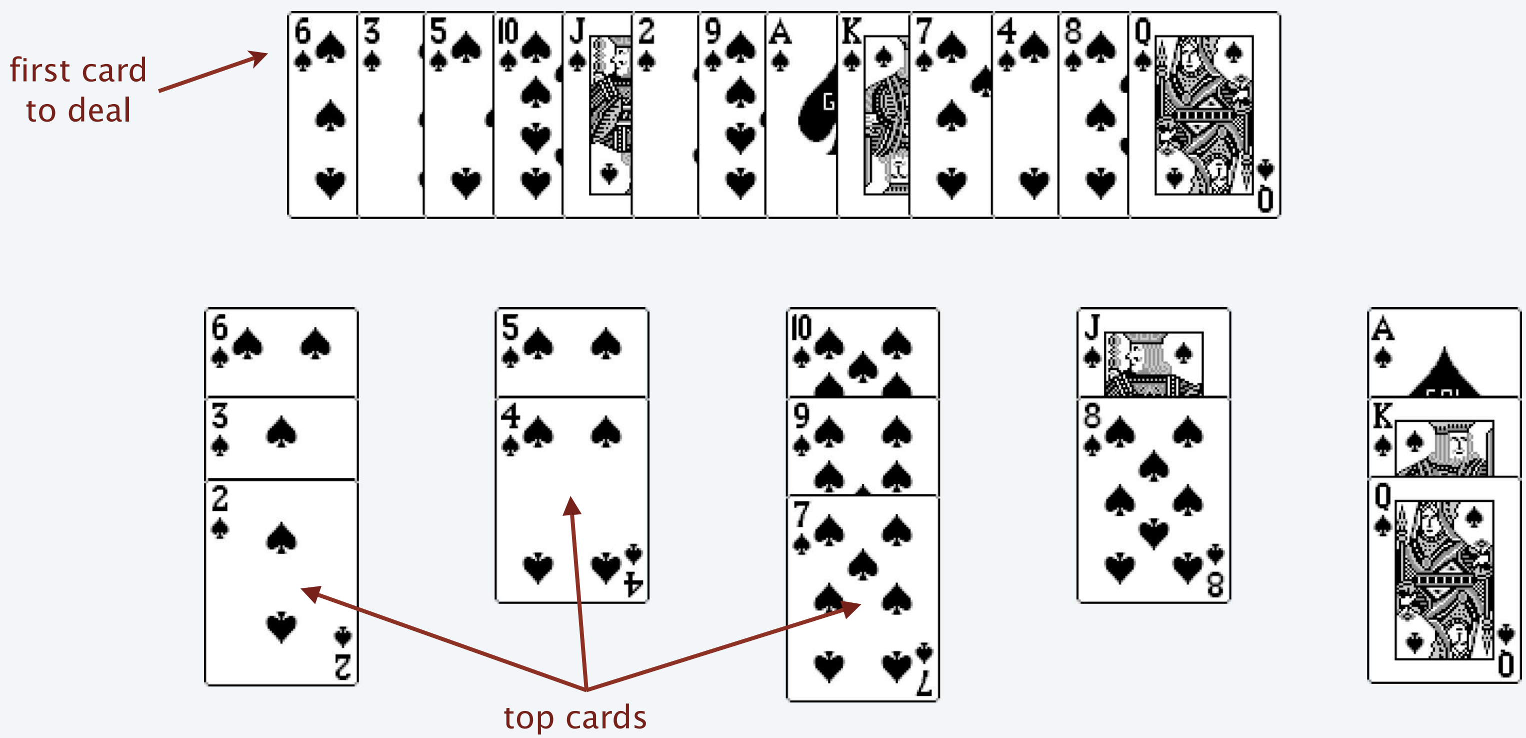
Greedy algorithm. Place each card on leftmost pile that fits.



Patience: greedy algorithm

Greedy algorithm. Place each card on leftmost pile that fits.

Observation. At any stage during greedy algorithm, top cards of piles increase from left to right.



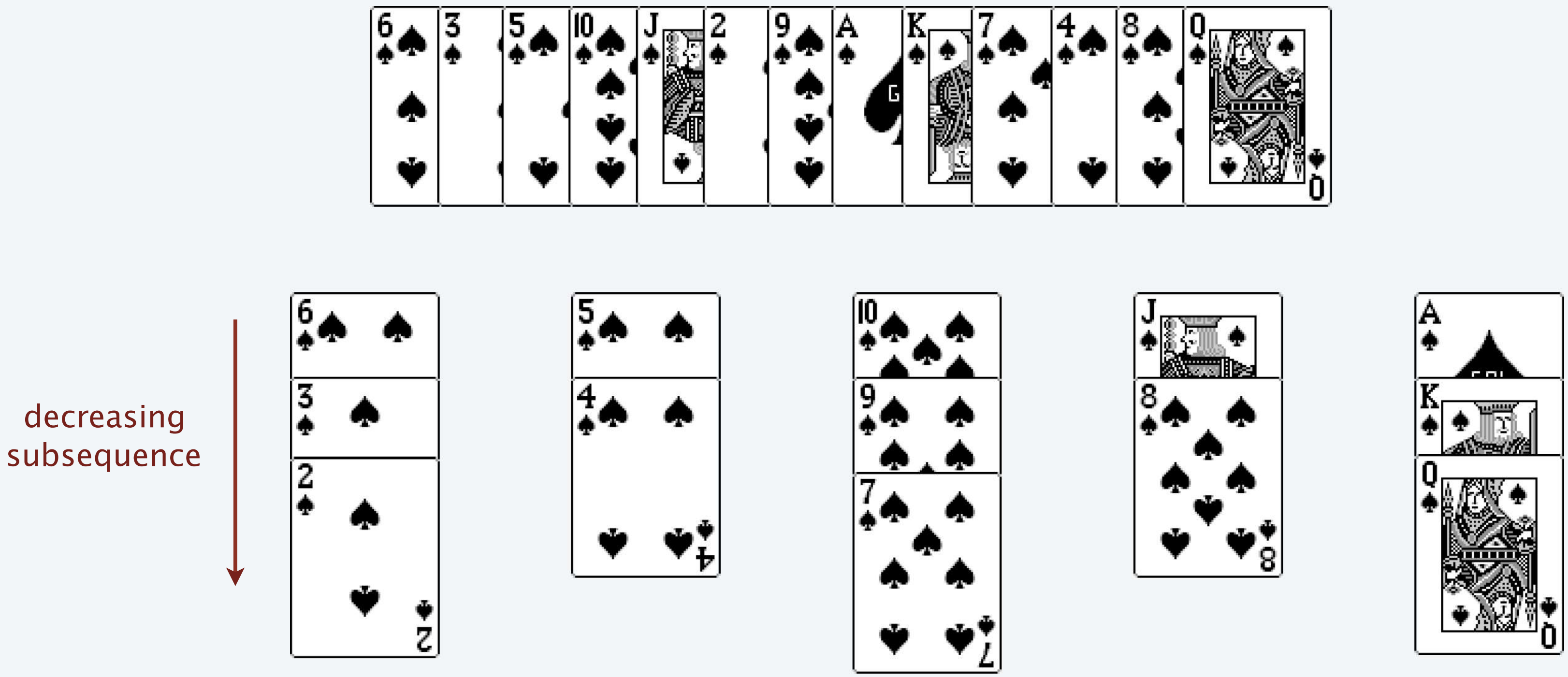
Patience-LIS: weak duality

Weak duality. In any legal game of patience, the number of piles ≥ length of any increasing subsequence.

选择一个递增序列时，一旦从某堆里选中一张牌，下一张不能选择这张牌底下的（因为底下的牌是原序列的左边），也不能选择这张牌上上面的（因为上面的牌比自己小）。这样，堆数一定大于递增序列的长度。

Pf.

- Cards within a pile form a decreasing subsequence.
- Any increasing sequence can use at most one card from each pile. ▪



Patience-LIS: strong duality

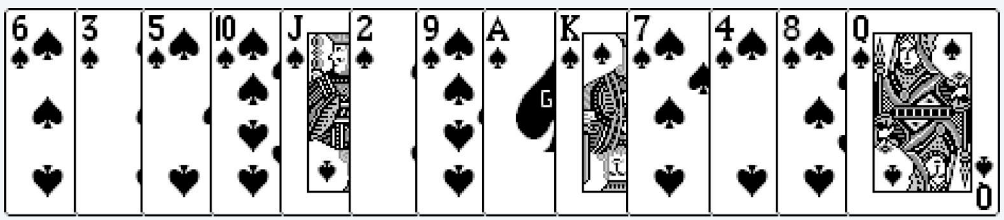
Theorem. [Hammersley 1972] Min number of piles = max length of an IS; moreover greedy algorithm finds both.

at time of insertion

Pf. Each card maintains a pointer to top card in previous pile.

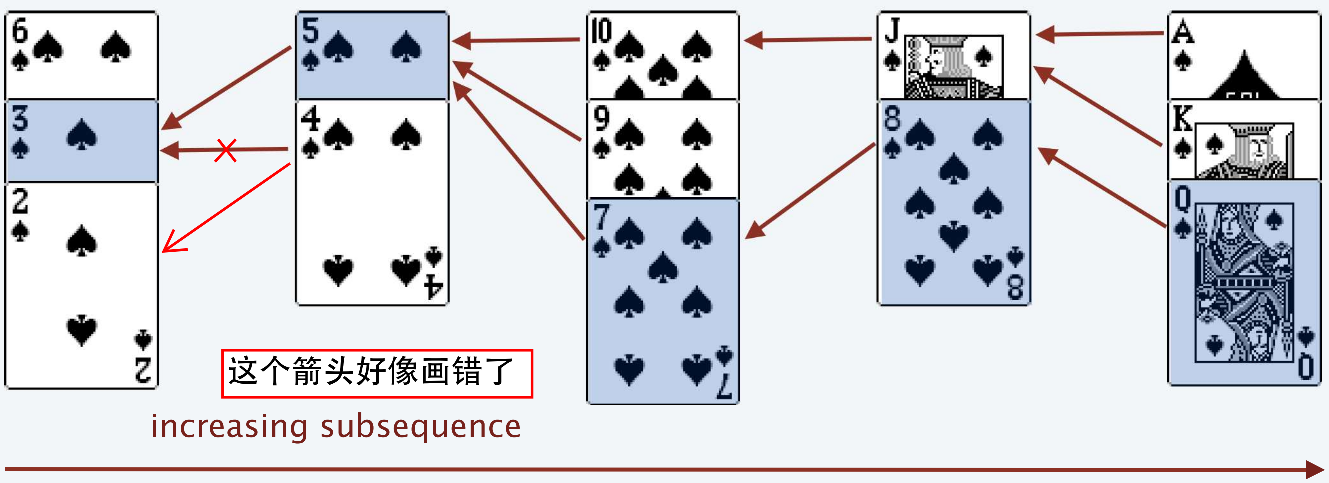
- Follow pointers to obtain IS whose length equals the number of piles.
- By weak duality, both are optimal.

1.这些箭头标明了牌在原序列中的前后顺序;
2.箭头起点的牌一定比终点的牌大（或等于），否则就可以放在前一堆了;



3.每新建一个堆都可以画一个箭头，所以一定可以沿着最右边的箭头回溯到最左边的堆。

综上，一定可以找到一条和长度等于堆数的路径，这个路径上就是我们要求的一个子序列。



Greedy algorithm: implementation

Theorem. The greedy algorithm can be implemented in $O(n \log n)$ time.

- Use n stacks to represent n piles.
- Use binary search to find leftmost legal pile.

PATIENCE $(n, c_1, c_2, \ldots, c_n)$

INITIALIZE an array of n empty stacks $S_1, S_2, \ldots, S_n$.

FOR $i = 1$ TO n

  $S_j \leftarrow$ binary search to find leftmost stack that fits c_i.

  PUSH (S_j, c_i).

  $pred[c_i] \leftarrow$ PEEK (S_{j-1}). ⟵ null if j = 1

RETURN sequence formed by following pointers from top card of rightmost nonempty stack.

Patience sorting

Patience sorting. Deal all cards using greedy algorithm; repeatedly remove smallest card.

Theorem. For uniformly random deck, the expected number of piles is approximately $2n^{1/2}$ and the standard deviation is approximately $n^{1/6}$.

Remark. An almost-trivial $O(n^{3/2})$ sorting algorithm.

Speculation. [Persi Diaconis] Patience sorting is the fastest way to sort a pile of cards by hand.

Bonus theorem

Theorem. [Erdös-Szekeres 1935] Any sequence of $n^2 + 1$ distinct real numbers either has an increasing or decreasing subsequence of size $n + 1$.

Pf. [by pigeonhole principle]