Research Statement

Toheeb A. Biala
PhD Candidate, Computational Science

Introduction

As a computational mathematician, my research cut across so many disciplines. In general, my primary research focuses on the development of novel analytical and computational methods for fractional partial differential equations with emphasis on applications arising in biology, physics, chemistry, physiology, and mathematical finance. A fractional partial differential equation (FPDE) is a class of partial differential equations that involves non-integer derivatives. Various class of FPDEs arise in several physical models which includes space-fractional PDEs (non-integer order space-derivative), time-fractional PDEs (non- integer order time derivative), time-space fractional PDEs (non-integer order time and space derivative), distributed-order fractional PDEs (time or space or both derivatives integrated over a given range). I study this class of PDEs with the development of numerical methods for the PDEs.

Past Work

1. **Numerical Methods for nonlinear time-fractional differential equations**: Prior to the commencement of my PhD program, I worked together with Dr. Samuel Jator on the development of a novel class of numerical methods for time-fractional equations [1]. The schemes are based on a continuous representation of the implicit Adams methods and are developed via the interpolation and collocation approach. The stability properties of the schemes are investigated and the schemes are implemented in a block-by-block fashion.
2. **Numerical Methods for nonlinear space fractional PDEs**: During the second year of my PhD program, I, my advisor Dr. A.Q.M. Khaliq, and other collaborators developed a class of numerical schemes for space-fractional PDEs [2, 3] based on Padé approximations ((0,1)-, (1,1)-, (0,2), (1,3) and (2,2)-Padé) to the exponential function. After some investigation on the developed schemes, we observed that the (m,m)-Padé, $m \in \mathbf{Z}^+$, approximations incurs oscillatory behavior for some time steps. This prompts us to provide a reliability estimate on the choice of the time step to avoid these unwanted oscillations. These concepts are fully discussed in the published article [2, 3].
3. **Parallel Algorithms**: One of the most interesting work in my dissertation is the development of a novel computational method for time-space fractional PDEs. The developed scheme [4] is a generalization of the Crank-Nicholson scheme for non-integer order time and space derivative. The schemes for time-space FPDEs are difficult to implement due to the nonlocality in space and memory dependencies in time. The solution at any point in the t-stencil is not only dependent on the solutions at neighboring points but on all the previous time-steps. This implies that the history or memory term has to be computed at each time-step. This is very time-consuming and uses a lot of memory in storing the history terms. We lessen this computational burden by implementing parallel versions of the developed algorithms. In particular, we implement the algorithms using the shared (OpenMP) and distributed (MPI) memory systems. Also, the theoretical and error analysis of the scheme are detailed in the paper [4].
4. **Numerical method for distributed space-fractional PDEs**: In this work [5], we developed schemes for a distributed-order space-fractional reaction-diffusion equations. Of particular importance is the generalization of the reliability estimate given in [2] to PDEs having distributed-order space-derivatives to avoid unwanted oscillations.

Present Work

1. **A Modified model of α-synuclein (α-syn) transport and aggregation in neurons**: The onset of the Parkinson's disease is characterized by the accumulation or aggregation of infectious α-syn and

loss of dopamine in the brain. Kuznetsov and Kuznetsov [6] discussed the conditions under which these infectious proteins aggregates by simulating the process using two compartments (the soma and synapse) of a cell body. Our question about the model includes the following:

(a) What is the effect of the transport of α-syn via the axon from an infectious cell body to a healthy cell body?

(b) Could there be axonal variations or effects on the transported proteins?

(c) In which of the compartments (soma, axon and synapse in our case) is the conversion of the healthy or monomeric α-syn to the infectious or polymeric α-syn fastest or slowest?

(d) Is the transport in the axon only dependent on the active transport (in which case we model the transport using an advection term) or by an interplay between active and diffusion-driven transport (in which case we model the transport using an advection and a fractional diffusion term)? This is in line with suggestions from published results that α-syn transport occurs either by active (motor-driven) transport [7] or by an interplay between diffusion and motor-driven transport [8, 9].

(e) Is the rate of change of the proteins in the different compartments same or different? We model this by incorporating the effect of a time-fractional derivative.

Based on the questions above, we proposed the model [10]

$$\begin{aligned} V_s \frac{d^\gamma [A]_s}{dt^\gamma} &= V_s \left(Q_{[A]_s} - k_1 [A]_s - k_2 [A]_s [B]_s - \frac{[A]_s \ \ln(2)}{T_{[A]_s, \frac{1}{2}}} \right) - q_1, \\ V_s \frac{d^\gamma [B]_s}{dt^\gamma} &= V_s \left(Q_{[B]_s} h_1(t) + k_1 [A]_s + k_2 [A]_s [B]_s - \frac{[B]_s \ \ln(2)}{T_{[B]_s, \frac{1}{2}}} \right) - q_2, \end{aligned} \tag{1}$$

$$\begin{aligned} A_c \frac{d^\gamma [A]_a}{dt^\gamma} &= A_c \left(-d_3 (-\Delta)^{\frac{\beta}{2}} [A]_a - \nu_3 \nabla [A]_a - k_1 [A]_a - k_2 [A]_a [B]_a - \frac{[A]_a \ \ln(2)}{T_{[A]_a, \frac{1}{2}}} \right), \\ A_c \frac{d^\gamma [B]_a}{dt^\gamma} &= A_c \left(-d_4 (-\Delta)^{\frac{\beta}{2}} [B]_a - \nu_4 \nabla [B]_a + k_1 [A]_a + k_2 [A]_a [B]_a - \frac{[B]_a \ \ln(2)}{T_{[B]_a, \frac{1}{2}}} \right), \end{aligned} \tag{2}$$

$$\begin{aligned} V_{syn} \frac{d^\gamma [A]_{syn}}{dt^\gamma} &= V_{syn} \left(-k_1 [A]_{syn} - k_2 [A]_{syn} [B]_{syn} - \frac{[A]_{syn} \ \ln(2)}{T_{[A]_{syn}, \frac{1}{2}}} \right) + q_5 \\ V_{syn} \frac{d^\gamma [B]_{syn}}{dt^\gamma} &= V_{syn} \left(Q_{[B]_{syn}} h_2(t) + k_1 [A]_{syn} + k_2 [A]_{syn} [B]_{syn} - \frac{[B]_{syn} \ \ln(2)}{T_{[B]_{syn}, \frac{1}{2}}} \right) + q_6, \end{aligned} \tag{3}$$

where eqns. (1), (2) and (3) simulates the rate of conversion of the healthy or monomeric α-syn to infectious or polymeric α-syn in the soma, axon and synapse, respectively.

2. **An efficient GPU implementation of a novel numerical scheme for a fully distributed-order time-space FPDEs**: In this work, we develop a novel numerical scheme for a fully distributed-order time-space fractional PDEs and implement the schemes using GPUs due to the complexity involved in the class of PDEs. This work is currently in progress.

Future Work

1. **Simulating models arising from FPDEs using data-driven modeling algorithms**: In the near future, I would like to develop data-driven modeling algorithms to simulate mathematical models

arising from FPDEs. As the influx of data reaches the highest level ever seen, there have been recent advances in machine learning and data analytics techniques. This has resulted in transformative results and suggestions across several scientific disciplines, including natural language processing, imaging science, bioinformatics, cheminformatics, genomics among others. In my research, I would like to predict parameters (such as the order of the derivatives, diffusion or dispersion coefficients) of FPDEs from data that describes a certain phenomenona (known as the inverse problem) or provide solutions to the class of FPDEs (known as the forward problem) by developing deep learning algorithms. To further elaborate on the technique, we discuss the deep learning algorithm for the time-space fractional advection-diffusion equation

$$\begin{aligned} &{}_cD^{\gamma}_{0,t}u = -\kappa\,(-\Delta)^{\frac{\beta}{2}}\,u(x,t) - v\cdot\nabla u(x,t) + f(u),\ \text{in } \Omega\times(0,T],\\ &u(x,0) = g(x), \quad x\in\Omega\subset\mathbb{R}, \end{aligned} \tag{4}$$

with homogeneous Dirichlet boundary conditions where κ is the diffusion coefficient, v is the flow velocity and Ω is a subset of $\mathbb{R}^D$. ${}_cD^{\gamma}_{0,t}u$ is the Caputo derivative with respect to t, of order $0<\gamma\leq 1$, defined by

$${}_cD^{\gamma}_{0,t}u(x,t) = \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}\frac{\partial u(x,s)}{\partial s}ds.$$

$(-\Delta)^{\frac{\beta}{2}}$ is the fractional Laplacian of order $1<\beta\leq 2$, and $f(u)$ is a sufficiently smooth function.
The formulation of the forward problem may be stated as: Given the parameters $\gamma,\ \beta,\ \kappa,\ v$ of the FPDE (4) and the initial and boundary conditions, we would like to find approximate solutions of the concentration field $u(x,t)$. The steps involved are as follows:

(a) Approximate the solution $u(x,t)$ by $u_{_E}(x,t)$ using a neural network such that the approximant satisfies the initial and boundary conditions.

(b) Define a loss function

$$\ell(t,x) =_c D^{\gamma}_{0,t}u_{_E} + \kappa\,(-\Delta)^{\frac{\beta}{2}}\,u_{_E} + v\cdot\nabla u_{_E} - f(u_{_E}) = 0,$$

and minimize the mean squared error of the loss function using any appropriate optimitization algorithm like the Adams method [11].

(c) Sample training points (t,x) from a distribution. This is a very key aspect as inappropriate sampled points may slow the training process or in some cases deflect the approximation $u_{_E}(x,t)$ to inaccurate results.

(d) Provide approximations for the fractional derivatives in (4). We would like to use the approximations provided in our earlier paper [4].

(e) Select the choice of your neural network. This involves the selection of the learning rate, starting weights and biases, the number of hidden layers, the number of neurons in each layer, among others. This step involves careful analysis as there is no clear cut as to what numbers of these parameters will give a faster rate of convergence to the exact solution. However, it has been shown in practice [12, 13] that the higher the number of layers and neurons, the more the algorithm learns faster. These higher numbers create more complexity and difficulty in the algorithm as there are hundreds of thousands (or even millions) of parameters to be minimized.

The inverse problem may also be formulated as: Given the solution $u(x,t)$ at the final time T, the initial and boundary conditions, we would like to obtain the parameters $\gamma,\ \beta,\ \kappa,\ v$ and the solution $u(x,t)$ at any given point in the domain Ω. The steps involved here is similar to that in the forward problem except that the new parameters here are added to the list of weights and biases to be minimized.

References

[1] T. A. Biala and S. N. Jator, "Block implicit Adams methods for fractional differential equations," *Chaos, Solitons and Fractals*, vol. 81, pp. 365 – 377, 2015.

[2] A. Q. M. Khaliq, T. A. Biala, S. S. Alzahrani, and K. M. Furati, "Linearly implicit predictor-corrector methods for space-fractional reaction-diffusion equations with non-smooth initial data," *Computers and Mathematics with Applications*, vol. 75, no. 8, pp. 2629 – 2657, 2018.

[3] S. S. Alzahrani, A. Q. M. Khaliq, T. A. Biala, and K. M. Furati, "Fourth-order time stepping methods with matrix transfer technique for space-fractiobal reaction-diffusion equations," *Applied Numerical Mathematics*, vol. 146, pp. 123–144, 2019.

[4] T. A. Biala and A. Q. M. Khaliq, "Parallel algorithms for nonlinear time-space fractional parabolic PDEs," *Journal of Computational Physics*, vol. 375, no. 6-7, pp. 135 – 154, 2018.

[5] T. A. Biala, "Second-order predictor-corrector schemes for nonlinear distributed-order space-fractional differential equations with non-smooth initial data," *International Journal of Computer Mathematics*, vol. 96, no. 9, pp. 1861–1878, 2019.

[6] I. A. Kuznetsov and A. V. Kuznetsov, "What can trigger the onset of parkinsons disease a modeling study based on a compartmental model of α-synuclein transport and aggregation in neurons," *Mathematical Biosciences*, vol. 278, pp. 22–29, 2016.

[7] A. Saha, J. Hill, M. Utton, A. Asuni, S. Ackerley, A. Grierson, C. Miller, A. Davies, V. Buchman, B. Anderton, and D. Hanger, "Parkinson's disease -synuclein mutations exhibit defective axonal transport in cultured neurons," *Journal of Cell Science*, vol. 117, no. 7, pp. 1017–1024, 2004.

[8] S. Roy, *The Paradoxical Cell Biology f α-Synuclein. Cell Biology of the Axon*, vol. 48, pp. 159–172. Berlin, Heidelberg: Springer Berlin Heidelberg, 2009.

[9] K. J. Spinelli, J. K. Taylor, V. R. Osterberg, M. J. Churchill, E. Pollock, C. Moore, C. K. Meshul, and V. K. Unni, "Presynaptic alpha-synuclein aggregation in a mouse model of parkinson's disease," *Journal of Neuroscience*, vol. 34, no. 6, pp. 2037–2050, 2014.

[10] T. A. Biala and A. Q. M. Khaliq, "A fractional ODE/PDE model of α-synuclein transport and aggregation in neurons with axonal transport: A factor that prompts the onset of the Parkinson's disease," *In preparation*, 2020.

[11] D. P. Kingma and J. Ba, "Adam: A method for stochastic optimization," *arXivpreprint arXiv:1412.6980*, 2014.

[12] M. Raissi, P. Perdikaris, and G. E. Karniadakis, "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations," *Journal of Computational Physics*, vol. 378, pp. 686 – 707, 2019.

[13] G. Pang, L. Lu, and G. E. Karniadakis, "fpinns: Fractional physics-informed neural networks," *SIAM Journal of Scientific Computing*, vol. 41, no. 4, pp. A2603 – A2626, 2019.