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Problemas de equações diferenciais ordinárias e transformadas de Laplace

$\boxed{1.1}$ Calc. $(1-\cos x)\, y' = \operatorname{sen} x \cdot y$.

Resolução:

$$(1-\cos x)\, y' = \operatorname{sen} x \cdot y$$

$$(1-\cos x)\, \frac{dy}{dx} = \operatorname{sen} x \cdot y$$

$$\frac{1}{y} \quad \frac{dy}{} = \operatorname{sen} x \cdot \frac{1}{(1-\cos x)}$$

$$\ln|y| = \ln|1-\cos x| + \ln c$$

$$y = c(1-\cos x) \quad \checkmark$$

$\boxed{1.2}$ Det. curva c/ prop:

Resolução:

$$\frac{dy}{dx} = 4\,\frac{y}{x} \qquad \frac{dy}{y} = 4\,\frac{dx}{x} \qquad \int \frac{dy}{y} = \int 4\,\frac{dx}{x}$$

$$\ln|y| = \ln|x^4| + \ln c$$

$$y = cx^4 \quad \checkmark$$

$\boxed{1.3}$ Lei de Newton: rel. de arrefecimento de um corpo é proporcional à diferença entre a temp. T desse corpo e a temperatura amb. T_0

Dados: $T = 100°C$ → t_0 ambiente

$T = 80°C \rightarrow t = 2$ min

$T = 40°C \rightarrow t = ?$

Resolução:

$$\frac{dT}{dt} = k(T-20) \qquad \frac{dT}{(T-20)} = k\,dt$$

$$t=0 \Rightarrow T = 100$$

$$\ln|T-20| = kt + c_1 \qquad T-20 = c_1 e^{kt}$$

$$T = 20 + c_1 e^{kt} \qquad 100 = 20 + c_1 e^{k \cdot 0}$$

$$\boxed{c_1 = 80}$$

$$T(t) = 20 + 80 e^{kt}$$

$$T(2) = 20 + 80 e^{k2} \qquad e^{k2} = \frac{80-20}{80}$$

$$k = \frac{1}{2} \ln \frac{80-20}{80} = \frac{1}{2} \ln \frac{3}{4}$$

$$40 = 20 + 80 \exp\left[0{,}5 \ln\left(\frac{3t}{4}\right)\right]$$

$$\exp\left[0{,}5 \ln\left(\frac{3t}{4}\right)\right] = \frac{1}{4}$$

$$t = 9{,}638 \text{ min}$$

$\boxed{1.4}$ (a) $\boxed{y' = x^3/y^2}$

Resolução:

$$y^2 dy = x^3 dx \qquad \int y^2 dy = \int x^3 dx$$

$$\frac{y^3}{3} = \frac{x^4}{4} + c$$

(b) $\boxed{(x-1)\, y' = xy}$

Resolução:

$$(x-1)\, y' = xy \qquad (x-1)\frac{dy}{dx} = xy$$

$$\frac{dy}{y} = \frac{x}{(x-1)} dx$$

$$\ln|y| = \int \frac{x-1+1}{x-1} dx = \int \left(1 + \frac{1}{x-1}\right) dx$$

$$\ln|y| = x + \ln|x-1| + c$$

$$= \ln e^x + \ln|x-1| + \ln c$$

$$\ln|y| = \ln c(x-1)e^x$$

$$y = c(x-1)e^x$$

(c) $\boxed{(1+e^x)\, y y' = e^x}$

Resolução:

$$y\, dy = \frac{e^x}{(1+e^x)} dx \qquad \frac{y^2}{2} = \ln(1+e^x) + c$$

$$\frac{y^2}{2} = \ln(1+e^x) + c$$

(d) $x\sqrt{1+y^2} + yy'\sqrt{1+x^2} = 0$

Resolução:

$$x\sqrt{1+y^2} + y\frac{dy}{dx}\sqrt{1+x^2} = 0$$

$$y\frac{dy}{dx}\sqrt{1+x^2} = -x\sqrt{1+y^2}$$

$$\frac{y}{\sqrt{1+y^2}}\,dy = -\frac{x}{\sqrt{1+x^2}}\,dx$$

$$\sqrt{1+y^2} = -\sqrt{1+x^2} + c$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} = c$$

(e) $e^{-y}(1+y') = 1$

Resolução:

$$(1+y') = e^y$$

$$y' = e^y - 1 \qquad \frac{dy}{dx} = e^y - 1$$

$$\frac{dy}{(e^y-1)} = dx$$

(e)

$$\boxed{e^{-y}(1+y')=1}$$

$$\frac{\cancel{e^{-y}}}{\cancel{e^{-y}}}(1+y')=e^{y} \qquad \frac{dy}{\cancel{dx}}=(e^{y}-1)dx$$

$$\frac{dy}{(e^{y}-1)}=dx \qquad -\frac{1}{1-e^{y}}dy=dx$$

$$-\left(\frac{1-e^{y}+e^{y}}{1-e^{y}}\right)dy=dx \qquad -\left(\frac{1-e^{y}}{1-e^{y}}+\frac{e^{y}}{1-e^{y}}\right)dy=dx$$

$$-\left(1+\frac{e^{y}}{1-e^{y}}\right)dy=dx \qquad \left(-1-\frac{e^{y}}{1-e^{y}}\right)dy=dx$$

$$-y-\ln(1-e^{y})=x+c$$

$$e^{(-y-\ln(1-e^{y}))}=ce^{x}$$

$$e^{-y}e^{\ln(1-e^{y})}=ce^{x} \qquad e^{-y}(1-e^{y})=ce^{x}$$

$$\boxed{e^{-y}-1=ce^{x}}$$

✓

(f) $y' + 5x^4 y^2 = 0 \quad , \quad y(0) = 1$

Resolução:

$$\frac{dy}{dx} = -5x^4 y^2 \qquad \frac{dy}{y^2} = -5x^4 dx$$

$$\frac{y^{-1}}{-1} = -\frac{5x^5}{5} + c \qquad y^{-1} = x^5 + c$$

$$y = \frac{1}{x^5+c} \qquad y(0) = 1 = \frac{1}{c}, \boxed{c=1}$$

$$y = \frac{1}{x^5+1}$$ ✓

(g) $e^x \operatorname{sen}^3 y + (1+e^{2x}) \cos y \cdot y' = 0$

Resolução:

$$e^x \operatorname{sen}^3 y + (1+e^{2x}) \cos y \cdot \frac{dy}{dx} = 0$$

$$e^x + (1+e^{2x}) \frac{\cos y}{\operatorname{sen}^3 y} \frac{dy}{dx} = 0$$

$$(1+e^{2x}) \frac{\cos y}{\operatorname{sen}^3 y} dy = -e^x dx$$

$$\frac{\cos y}{\operatorname{sen}^3 y} dy = -\frac{e^x}{(1+e^{2x})} dx$$

$$\frac{1}{2 \operatorname{sen}^2 y} = \operatorname{arctg} e^x + c$$ ✓

1.6 $\quad f(x) = 2 + \int_1^x f(t)\,dt$

Resolução:

$$f'(x) = 0 + \left[\int_1^x f(t)\,dt\right]' = \left[F(x) - F(1)\right]'$$

$$f'(x) = f(x) - 0 = f(x)$$

$$\frac{dy}{dx} = y \qquad \frac{dy}{y} = dx$$

$$\ln|y| = x + c \qquad \boxed{y = ce^x}$$

$$f(x) = 2 + \int_1^x ce^t\,dt = 2 + \left[ce^x - ce^1\right]$$

$$f(x) = 2 + c\left[e^x - e^1\right]$$

$$f(1) = 2 + c\,[0] = 2$$

$$2 = ce^1 \implies \boxed{c = 2e^{-1}}$$

$$y = (2e^{-1})\,e^x = 2e^{x-1}$$

$$\boxed{y = 2e^{x-1}}$$ ✓

$\boxed{1.7}$ Det. $f(x)$ $\quad f'(x) + 2x\, e^{f(x)} = 0$ e $f(0)=0$

Resolução:

$y' + 2x\, e^{y} = 0$ $\qquad \dfrac{dy}{dx} = -2x\, e^{y}$

$$e^{-y} dy = -2x\, dx$$

$$-e^{-y} = -x^2 - c$$

$$e^{-y} = x^2 + c$$

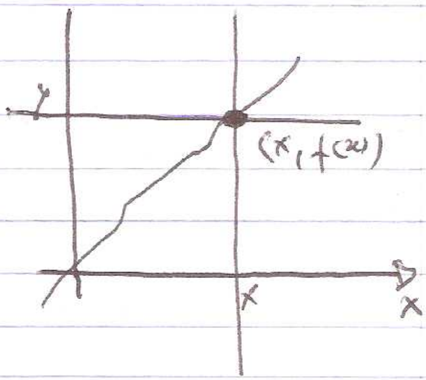
$$-y = \ln(x^2+c)$$

$f(0)=0 \longrightarrow \quad 0 = \ln(0+c) \quad \boxed{c=1}$

$$-y = \ln(x^2+1)$$

$$y = \ln\left(\frac{1}{x^2+1}\right)$$

$\boxed{1.8}$ $y = f(x)$, $x = 0$, $y = 0$



$a = u b$

ou

$b = u a$

área do rectângulo

=

$A_{\square} = x \cdot f(x)$

$$x \cdot f(x) = (1+u) \int_0^x f(x)\, dx$$

derivando

$$f(x) + x f'(x) = (1+u) f(x)$$

$$x f'(x) = u f(x)$$

$$x y' = u y \qquad y' - \frac{u}{x} y = 0$$

$$\frac{1}{y} dy - \frac{u}{x} dx = 0 \qquad \ln y - u \ln x = c$$

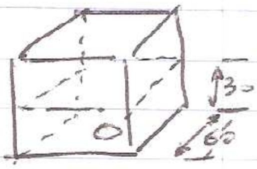
$$\ln \frac{y}{x^u} = c \qquad \frac{y}{x^u} = c \qquad \boxed{y = c x^u}$$

Se for $a = u b$

$$x \cdot f(x) = \left(1 + \frac{1}{u}\right) \int_0^x f(x)\, dx$$

$$\Rightarrow \boxed{y = c x^{1/u}}$$

$\boxed{1.9}$



Resol:

$$v_2 = \sqrt{2gh}$$

$h = 30\ cm = 0{,}3\ m$
$g = 9{,}8\ m/s^2$
$A = 10\ cm^2 =$
$= 10\ (10^{-2} m)^2 = 10^{-3}\ m^2$

$$v_2 = \sqrt{2 \times 9{,}8 \times 0{,}3}\ \sqrt{\frac{m}{s^2} \times m} = 2{,}42\ m/s$$

$$V_2 = v_2 A t = 2{,}42\ \frac{m}{s}\ 10^{-3} m^2\ t$$
$$= 2{,}42 \times 10^{-3}\ \frac{m^3}{s}\ t$$

$$V_1 = (60\ cm)^2 \times (30\ cm) = (60 \times 10^{-2} m)^2 (0{,}3 m)$$
$$= 0{,}6^2 \times 0{,}3\ m^3 = 0{,}108\ m^3$$

$$0{,}11\ \cancel{m^3} = 2{,}42 \times 10^{-3}\ \frac{\cancel{m^3}}{s}\ t$$

$$t = 0{,}11 / 2{,}42 \times 10^{-3} = 45{,}45 (45)\ s.$$