$\boxed{2.1}$ $x^2 y''' + x y'' = 1$

$z = y''$

$$x^2 z' + x z = 1$$

$$z' + \frac{1}{x} z = \frac{1}{x^2}$$

$$z = e^{-\int \frac{1}{x} dx} \left[\int \frac{1}{x^2} e^{\int \frac{1}{x} dx} dx + c \right] =$$

$$= e^{\ln x^{-1}} \left[\int \frac{1}{x^2} e^{\ln x} dx + c \right] = x^{-1} \left[\int \frac{1}{x^2} x \, dx + c \right]$$

$$= x^{-1} \left[\int \frac{1}{x} dx + c \right] = \frac{1}{x} \left[\ln x + c \right]$$

$$z = \frac{\ln|x|}{x} + \frac{c_1}{x}$$

$$y'' = \frac{\ln|x|}{x} + \frac{c_1}{x}$$

$$y' = \frac{1}{2} \ln^2|x| + c_1 \ln x + c_2$$

$$y = \frac{1}{2} x \ln^2 x - c_1 (x \ln x - x) + c_2 x + c_3$$

12.27 $\quad yy'' - y'(1+y') = 0$

Resolução:

$\frac{dy}{dx} \to z$

$$yz\frac{dz}{dy} - z(1+z) = 0$$

$$\frac{dz}{1+z} = \frac{dy}{y} \longrightarrow z = c_1 y - 1$$

$z \to \frac{dy}{dx}$

$$\frac{dy}{dx} = c_1 y - 1$$

$$\frac{1}{c_1} \ln|c_1 y - 1| = x + c_2$$

$$y = \frac{1}{c_1}\left(1 + e^{c_1 x + c_2}\right)$$

$$y = \frac{1}{c_1} + c_2 e^{c_1 x}$$

$\boxed{2.3}$ $\quad x y'' + y' = 1 + x^2$

Resolução

$z = y'$

$x z' + z = 1 + x^2$

$z' + \frac{1}{x} z = \frac{1+x^2}{x}$ EDO 1ª ordem.

$z' + \frac{1}{x} z = 0$ $\qquad \frac{dz}{dx} + \frac{1}{x} z = 0$

$\frac{1}{z} dz = -\frac{1}{x} dx$

$\ln|z| = -\ln|x| + \ln c$

$z = \frac{c}{x}$

$x\left(\frac{c'(x)x - c(x)}{x^2}\right) + \frac{c}{x} = 1 + x^2$

$c'(x) = 1 + x^2$ $\qquad c = \int (1+x^2) dx$ $\quad c = x + \frac{x^3}{3} + c_1$

$z = \frac{c}{x} = \left(x + \frac{x^3}{3} + c_1\right)\frac{1}{x} = 1 + \frac{1}{3}x^2 + \frac{c_1}{x}$

$y = \int y' = \int z = \int \left(1 + \frac{1}{3}x^2 + \frac{c_1}{x}\right) dx = x + \frac{1}{9}x^3 + c_1 \ln x + c_2$

$y = x + \frac{1}{9}x^3 + c_1 \ln|x| + c_2$ ✓

$\boxed{2.4}$ $$xy'' = y' + x^2$$

Resolução:

$xy'' - y' = x^2$

$z = y'$

$xz' - z = x^2 \longrightarrow xz' - z = 0$ eq. homogénea

$x\,dz - z\,dx = 0$

$\frac{1}{z}dz - \frac{1}{x}dx = 0 \longrightarrow \ln|z| - \ln|x| = c$

$\ln\frac{z}{x} = c$

$\frac{z}{x} = e^c \longrightarrow z = xe^c$

$z = c(x)\,x \longrightarrow z' = c'(x)\,x + c(x)$

$x\left[c'(x)\,x + c(x)\right] - c(x)\,x = x^2 \quad c'x^2 = x^2 \quad c' = 1$

$c' = 1 \longrightarrow c = x + c_1 \longrightarrow z = (x + c_1)\,x$

$z = y' = x^2 + c_1 x$

$y = \frac{x^3}{3} + c_1\frac{x^2}{2} + c_2$

$y = \frac{x^3}{3} + c_1 x^2 + c_2$ ✓

$\boxed{2.5}$ $y'' + e^{2y}(y')^3 = 0$

Resolução:

$y' = z$

$z' + e^{2y} z^3 = 0$

$z \dfrac{dz}{dy} + e^{2y} z^3 = 0$

$\dfrac{dz}{dy} + e^{2y} z^2 = 0$

$dz + e^{2y} z^2 dy = 0$

$\dfrac{dz}{z^2} + e^{2y} dy = 0$

$e^{2y} dy = -\dfrac{dz}{z^2}$

$\dfrac{1}{2} e^{2y} + c = \dfrac{1}{z} = \dfrac{dx}{dy}$

$\int \left(\dfrac{1}{2} e^{2y} + c\right) dy = x$

$x = \dfrac{1}{4} e^{2y} + c_1 y + c_2$ ✓

$\boxed{2.7}$ $y'(1+(y')^2) = a\,y''$

Resolución:

$$y' = z$$

$$z\,(1+z^2) = a\,z\,\frac{dz}{dy}$$

$$\frac{dy}{a} = \frac{dz}{(1+z^2)}$$

$$\frac{1}{a}\,y + c = \operatorname{artg} z$$

$$z = \operatorname{tg}\left(\frac{1}{a}\,y + c\right)$$

$$y' = \operatorname{tg}\left(\frac{1}{a}\,y + c\right)$$

$$\cancel{y = a \ln\left|\cos\left(\frac{y}{a} + c\right)\right|}$$

$$\frac{dy}{dx} = \operatorname{tg}\left(\frac{1}{a}\,y + c\right) \qquad \operatorname{cotg}\left(\frac{1}{a}y + c\right) dy = dx$$

$$x + C_2 = -\ln\left|\operatorname{sen}\left(\frac{y}{a} + C_1\right)\right|$$

$\boxed{2.8}$ $$y y'' + (y')^2 - (y')^3 \ln y = 0$$

R:

$$y' = z$$

$$y'' = \frac{dy'}{dx} = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy} z$$

$$y y'' + z^2 + z^3 \ln y = 0$$

$$y'' = \frac{dz}{dy} z$$

$$y'' + z^2(1+z)\frac{\ln y}{y} = 0 \qquad \left(z\frac{dz}{dy}\right) + z^2(1+z)\frac{\ln y}{y} = 0$$

$$\frac{z}{z^2(1+z)}\,dz + \frac{\ln y}{y}\,dy = 0$$

$$\int \frac{1}{z(1+z)}\,dz + \int \frac{\ln y}{y}\,dy = C_1$$

$$\int\left(\frac{A}{z} + \frac{B}{(1+z)}\right)dz + \int \frac{\ln y}{y}\,dy = c_1$$

$$\ln z - \ln(z+1) + \frac{(\ln y)^2}{2} = c_1 \qquad \ln\frac{z}{z+1} = -\frac{\ln^2 y}{2} + c_1$$

$$\frac{z}{z+1} = c_1 \exp\left[-\frac{\ln^2 y}{2}\right] \qquad z = c_1 \frac{\exp[-(\ln^2 y)/2]}{1 - c_1 \exp[-(\ln^2 y)/2]}$$

$$z = c_1 y^{-\log\sqrt{y}} / (1 - c_1 y^{-\log\sqrt{y}}) = y' = dy/dx$$

$$dy\,[(c_1(y^{\log\sqrt{y}} - 1)] = dx$$

$\boxed{210}$ $y'' + y' = x e^{x}$

Resolução:

$y' = z$

$z' + z = x e^{x}$

$z_h' + z_h = 0$

$\frac{dz_h}{dx} + z_h = 0$ $\frac{dz_h}{dx} = -z_h$ $\frac{dz_h}{z_h} = -dx$

$\ln|z_h| = -x + c$

$z_h = c e^{-x}$

$z = c(x) e^{-x}$

$z' = c'(x) e^{-x} - c(x) e^{-x}$

$c'(x) e^{-x} - \cancel{c(x)} e^{-x} + \cancel{c(x)} e^{-x} = x e^{x}$

$c'(x) = x \frac{e^{x}}{e^{-x}} = x e^{2x}$

$c(x) = \int \underset{g}{x} \underset{f'}{e^{2x}} dx = x \frac{e^{2x}}{2} - \int 1 \frac{e^{2x}}{2} =$

$= \frac{1}{2}\left(x e^{2x} - \frac{1}{2} e^{2x}\right) = \frac{1}{2} e^{2x}\left(x - \frac{1}{2}\right) + c$

$z(x) = \frac{1}{2} e^{2x}\left(x - \frac{1}{2}\right) e^{-x} + c = \frac{1}{2} e^{x}\left(x - \frac{1}{2}\right) + c e^{-x} = y'$

$y = \int \frac{1}{2} e^{x}\left(x - \frac{1}{2}\right) dx + c e^{-x} = \frac{1}{2} e^{x}\left(x - \frac{3}{2}\right) - c_1 e^{-x} + c_2$

$\boxed{2.11}$

$$a \cdot v = K = 2\, cm^2 s^{-3}$$

$t = 9\, s$
$x = ?$
$v = ?$
$x_0 = 5\, cm$

Resol:

$$\frac{dv}{dt} v = K \qquad v\,dv = K\,dt$$

$$\frac{v^2}{2} = Kt \qquad v = \sqrt{2Kt} + c$$

$$v_0 = 0 = c \implies v = \sqrt{2Kt}$$

$$\frac{dx}{dt} = \sqrt{2Kt} \qquad dx = \sqrt{2Kt}\,dt \qquad x = \frac{(2Kt)^{3/2}}{(2K)\frac{3}{2}} + c_1$$

$$x = (2K)^{1/2} \cdot \frac{2}{3} t^{3/2} + c_1$$

$$x_0 = 5 = c_1 \implies x = (2K)^{1/2} \cdot \frac{2}{3} t^{3/2} + 5$$

$$x_9 = (2 \times 2)^{1/2} \frac{2}{3} \sqrt{9^3} + 5 = 2 \times \frac{2}{\cancel{3}} \times 9\sqrt{\cancel{9}} + 5$$

$$\boxed{x_9 = 41\, cm}$$

$$v_9 = \sqrt{2 \cdot 2 \cdot 9} = \sqrt{36} = 6\, m s^{-1}$$

$\boxed{2.14}$ $y'' - 4y = 0$

Resolução:

$r^2 - 4 = 0 \qquad r^2 = 4 \qquad r = \pm 2$

$y(x) = c_1 e^{2x} + c_2 e^{-2x}$ ✓

$\boxed{2.15}$ $y'' + y = 0$

Resolução:

$r^2 + 1 = 0 \qquad r^2 = -1 \qquad r = \pm\sqrt{-1} = \pm i$

$y(x) = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \operatorname{sen} \beta x$

$y(x) = c_1 e^{0x} \cos x + c_2 e^{0x} \operatorname{sen} x$

$y(x) = c_1 \cos x + c_2 \operatorname{sen} x$ ✓

$\boxed{2.16}$ $y'' - 5y' + 6y = 0$

Resolução:

$r^2 - 5r + 6 = 0 \qquad r_{1,2} = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = 3; 2$

$y(x) = c_1 e^{2x} + c_2 e^{3x}$ ✓

$\boxed{2.17}$ $y'' - y' = 0$

Resolução:

$r^2 - r = 0 \qquad r(r-1) = 0 \qquad r = 0 \vee r = 1$

$y(x) = c_1 + c_2 e^{x}$ ✓

$\boxed{2.18}$ $y'' + 2y' + y = 0$

Resolução:

$r^2 + 2r + 1 = 0$ $r_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = -1$ dupla

$y(x) = c_1 e^{-x} + c_2 x e^{-x}$ ✓

$\boxed{2.19}$ $y'' + 4y' + 13y = 0$

Resolução:

$r^2 + 4r + 13 = 0$ $r_{1,2} = \frac{-4 \pm \sqrt{16 - 52}}{2(1)} = -2 \pm 3i$

$y(x) = c_1 e^{-2x} \cos(3x) + c_2 e^{-2x} \operatorname{sen}(3x)$ ✓

$\boxed{2.20}$ $y'' + 2\sqrt{3}\, y' + 3y = 0$

Resolução:

$r^2 + 2\sqrt{3}\, r + 3 = 0$ $r_{1,2} = \frac{-2\sqrt{3} \pm \sqrt{(2\sqrt{3})^2 - 4 \cdot 1 \cdot 3}}{2(1)} = -\sqrt{3}$

$y(x) = c_1 e^{-\sqrt{3}x} + c_2 x e^{-\sqrt{3}x}$ ✓

$\boxed{2.21}$ $y''' - 13y'' + 12y' = 0$

Resolução:

$r^3 - 13r^2 + 12r = 0 \qquad r(r^2 - 13r + 12) = 0$

$r_1 = 0 \vee r_{2,3} = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(1)12}}{2(1)} = 12; 1$

$y(x) = c_1 + c_2 e^{x} + c_3 e^{12x}$ ✓

$\boxed{2.22}$ $y''' - y' = 0$

Resolução:

$r^3 - r = 0 \qquad r(r^2 - 1) \qquad r = 0 \vee r = \pm 1$

$y(x) = c_1 + c_2 e^{-x} + c_2 e^{+x}$

$\boxed{2.23}$ $y''' + y = 0$

Resolução:

$r^3 + 1 = 0 \qquad r^3 = -1 \qquad r = -1$

$$\begin{array}{l|l} r^3 + 1 & r+1 \\ \hline \underline{r^3 + r^2} & r^2 - r + 1 \\ -r^2 + 1 & \\ \underline{-r^2 - r} & \\ +r + 1 & \\ \underline{r + 1} & \\ 0 & \end{array}$$

$\longrightarrow r_{1,2} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$r_{1,2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$y(x) = c_1 e^{-x} + c_2 e^{x/2} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{x/2} \operatorname{sen}\left(\frac{\sqrt{3}}{2}x\right)$ ✓

$\boxed{2.24}$ $y''' - 3y'' + 3y' - y = 0$

Resolução:

$r^3 - 3r^2 + 3r - 1 = 0$

$$\begin{array}{l|l} r^3 - 3r^2 + 3r - 1 & r - 1 \\ \hline r^3 - r^2 & r^2 - 2r + 1 \\ \hline -2r^2 + 3r - 1 & \\ -2r^2 + 2r & \\ \hline r - 1 & \\ r - 1 & \\ \hline 0 & \end{array}$$

$r^2 - 2r + 1 = 0$ $\qquad r_{1,2} = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$

$r_{1,2} = 1$ (dupla)

raízes: 1 (tripla)

$y(x) = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$

$\boxed{2.25}$ $y''' - y'' + y' - y = 0$

Resolução:

$r^3 - r^2 + r - 1 = 0 \qquad (r-1)(r^2+1) = 0$

raízes $\equiv r_{1,2,3} = 1; \pm i$

$y(x) = c_1 e^x + c_2 \cos(x) + c_3 \operatorname{sen}(x)$

$\boxed{2.26}$ $$y^{(iv)} - 2y''' + y'' = 0$$

Resolução:

$$r^4 - 2r^3 + r^2 = 0$$

$$r^2 (r^2 - 2r + 1) = 0 \qquad r_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$$

$$r_{1,2} = 1 \ ; \ r_3 = 0 \quad \text{(duplas)}$$

$$y(x) = c_1 + c_2 x + c_3 e^x + c_4 x e^x$$

$\boxed{2.27}$ $$\boxed{y^{(iv)} + 4y = 0}$$

Resolução:

$$r^4 + 4 = 0$$

$$(x^2 + ax + 2)(x^2 + bx + 2) = 0$$

$$x^4 + bx^3 + 2x^2 + ax^3 + abx + 2ax + 2x^2 + 2bx + 4 = 0$$

$$x^4 + x^3[b+a] + x^2[ab+4] + x[2a+2b] + 4 = 0$$

$$\left.\begin{array}{l} b+a=0 \\ ab+4=0 \\ 2a+2b=0 \end{array}\right| \begin{array}{l} b=-a \\ ab=-4 \\ 2a+2b=0 \end{array} \left| -a^2=-4 \right| a=2 \left| \ b=-2 \right.$$

$$(x^2 + 2x + 2)(x^2 - 2x + 2) = 0$$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2(1)} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

b $$x_{1,2} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)2}}{2(1)} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\boxed{r_{1,2,3,4} = \pm 1 \pm i}$$

$$y = c_2 e^{-x} \sin(x) + c_3 e^x \sin(x) + c_1 e^{-x} \cos(x) + c_4 e^x \cos(x)$$

$\boxed{2.28}$ $y^{(iv)} + y'' - 12y = 0$

Resolução:

$r^4 + r^2 - 12 = 0$

$z = r^2$

$z^2 + z - 12 = 0$ $\qquad z_{1,2} = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-12)}}{2(1)}$

$z_{1,2} = \dfrac{-1 \pm 7}{2} = -\dfrac{8}{2} ; \dfrac{6}{2} = -4 ; 3$

$r^2 = -4 ; 3$

$r = \pm\sqrt{-4} ; \pm\sqrt{3}$

$r = \pm 2i ; \pm\sqrt{3}$

$y(x) = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x} + c_3 \cos 2x + c_4 \operatorname{sen} 2x$

$\boxed{2.29}$ $y^{(iv)} + 10y'' + 25y = 0$

Resolução:

$r^4 + r^2 - 12 = 0$

$z = r^2$

$z^2 + z - 12 = 0$ $z_{1,2} = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-12)}}{2(1)}$

$z_{1,2} = \dfrac{-1 \pm 7}{2} = -4 ; 3$

$r = \pm\sqrt{z} = \pm\sqrt{-4} ; \pm\sqrt{3} = \pm 2i ; \pm\sqrt{3}$

$y(x) = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x} + c_3 \cos(2x) + c_4 \operatorname{sen}(2x)$

$\boxed{2.30}$ $y=u(x)$ $y''-4y'+29y=0$

$y=v(x)$ $y''+4y'+13y=0$

u e v se $u'(\pi/2)=1$; mesmo declive no origem

Resolução:

$r^2-4r+29=0$ $r_{1,2}=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)29}}{2(1)}=\frac{4\pm10i}{2}$

$r_{1,2}=2\pm5i$

$u(x)=c_1e^{2x}\cos(5x)+c_2e^{2x}\operatorname{sen}(5x)$

$r^2+4r+13=0$ $r_{1,2}=\frac{-4\pm\sqrt{4^2-4(1)13}}{2(1)}=\frac{-4\pm6i}{2}$

$r_{1,2}=-2\pm3i$

$v(x)=c_3e^{-2x}\cos(3x)+c_4e^{-2x}\operatorname{sen}(3x)$

intersectam na origem

$v(x)=u(x)=c_1e^0\cos(5\times0)+c_2e^0\cancel{\operatorname{sen}(5\times0)}^{\,0}$

$=c_3e^0\cos(3\times0)+c_4e^0\cancel{\operatorname{sen}(3\times0)}^{\,0}$

$c_1=c_3=0$

$u(x)=c_2e^{2x}\operatorname{sen}(5x)$

$u'(x)=c_2\left[2e^{2x}\operatorname{sen}(5x)+e^{2x}\cos(5x)\right]$

$u'\left(\frac{\pi}{2}\right)=c_2\left[2e^{\pi}\operatorname{sen}\left(\frac{5\pi}{2}\right)+e^{\pi}\cos\left(\frac{5\pi}{2}\right)\right]$

$=c_2\left[2e^{\pi}\cdot(1)+0\right]=1 \Rightarrow \boxed{c_2=\frac{1}{2e^{\pi}}}$

$$u(x) = \frac{1}{2} e^{2x-\pi} \operatorname{sen}(5x)$$

$$v(x) = C_4 \, e^{-2x} \operatorname{sen}(3x)$$

$$v'(x) = C_4 \left[-2 e^{-2x} \operatorname{sen}(3x) + 3 e^{-2x} \cos(3x) \right]$$

$$u'(0) = v'(0)$$

$$u'(0) = \frac{1}{2e^{\pi}} \left[2 \cancelto{1}{e^{2\times 0}} \operatorname{sen}\cancelto{0}{(5\times 0)} + 5 \cancelto{1}{e^{2\times 0}} \cancelto{1}{\cos(5\times 0)} \right]$$

$$= \frac{5}{2} e^{-\pi}$$

$$v'(0) = C_4 \left[-2 \cancelto{1}{e^{-2\times 0}} \operatorname{sen}\cancelto{0}{(3\times 0)} + 3 \cancelto{1}{e^{-2\times 0}} \cancelto{1}{\cos(3\times 0)} \right]$$

$$= C_4 \times 3$$

$$\boxed{C_4 = \frac{5}{6} e^{-\pi}}$$

$$u(x) = \frac{1}{2} e^{2x-\pi} \operatorname{sen}(5x)$$

$$v(x) = \frac{5}{6} e^{-(2x+\pi)} \operatorname{sen}(3x)$$