2. INVERSES AND TRANSPOSES

Definition 48 *A **square** matrix is a matrix with an equal number of rows and columns. The **diagonal** of an $n \times n$ matrix A comprises the entries $a_{11}, a_{22}, \ldots, a_{nn}$ – that is, the n entries running diagonally from the top left to the bottom right. A **diagonal** matrix is a square matrix whose non-diagonal entries are all zero. We shall write* $\operatorname{diag}(c_1, c_2, \ldots, c_n)$ *for the $n \times n$ diagonal matrix whose (i,i)th entry is c_i.*

Definition 49 *Given an $m \times n$ matrix A, then its **transpose** A^T is the $n \times m$ matrix such that the (i,j)th entry of A^T is the (j,i)th entry of A.*

Proposition 50 *(**Properties of Transpose**)*
*(a) (**Addition and Scalar Multiplication Rules**) Let A, B be $m \times n$ matrices and λ a real number. Then*

$$(A+B)^T = A^T + B^T; \qquad (\lambda A)^T = \lambda A^T.$$

*(b) (**Product Rule**) Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then $(AB)^T = B^T A^T$.*
*(c) (**Involution Rule**) Let A be an $m \times n$ matrix. Then $(A^T)^T = A$.*
*(c) (**Inverse Rule**) A square matrix A is invertible if and only if A^T is invertible. In this case $(A^T)^{-1} = (A^{-1})^T$.*

Proof. These are left to Sheet 2, Exercise 3. ■

Definition 51 *A square matrix $A = (a_{ij})$ is said to be*

- ***symmetric*** *if $A^T = A$.*
- ***skew-symmetric*** *(or **antisymmetric**) if $A^T = -A$.*
- ***upper triangular*** *if $a_{ij} = 0$ when $i > j$. Entries below the diagonal are zero.*
- ***strictly upper triangular*** *if $a_{ij} = 0$ when $i \geqslant j$. Entries on or below the diagonal are zero.*
- ***lower triangular*** *if $a_{ij} = 0$ when $i < j$. Entries above the diagonal are zero.*
- ***strictly lower triangular*** *if $a_{ij} = 0$ when $i \leqslant j$. Entries on or above the diagonal are zero.*
- ***triangular*** *if it is either upper or lower triangular.*

Example 52 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}, \qquad B^T = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \end{pmatrix}, \qquad C^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad D^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Note that A is upper triangular and so A^T is lower triangular. Also C and C^T are skew-symmetric. And D is diagonal and so also symmetric, upper triangular and lower triangular.

We return now to the issue of determining the invertibility of a square matrix. There is no neat expression for the inverse of an $n \times n$ matrix in general – we have seen that the $n = 2$ case is easy enough (Proposition 33) though the $n = 3$ case is already messy – but the following method shows how to determine efficiently, using EROs, whether an $n \times n$ matrix is invertible and, in such a case, how to find the inverse.

Algorithm 53 ***(Determining Invertibility)*** *Let A be an $n \times n$ matrix. Place A side-by-side with I_n as an augmented $n \times 2n$ matrix $(A \mid I_n)$. There are EROs that will reduce A to a matrix R in RRE form. We will simultaneously apply these EROs to both sides of $(A \mid I_n)$ until we arrive at $(R \mid P)$.*

- *If $R = I_n$ then A is invertible and $P = A^{-1}$.*
- *If $R \neq I_n$ then A is singular.*

Proof. Denote the elementary matrices representing the EROs that reduce A as $E_1, E_2, \ldots, E_k$, so that $(A \mid I_n)$ becomes

$$(E_k E_{k-1} \cdots E_1 A | E_k E_{k-1} \cdots E_1) = (R \mid P) \tag{2.1}$$

and we see that $R = PA$ and $E_k E_{k-1} \cdots E_1 = P$. If $R = I_n$ then

$$(E_k E_{k-1} \cdots E_1) A = I_n \quad \Longrightarrow \quad A^{-1} = E_k E_{k-1} \cdots E_1 = P$$

as elementary matrices are (left and right) invertible. If $R \neq I_n$ then, as R is in RRE form and square, R must have at least one zero row. It follows that $(1, 0, \ldots, 0)(PA) = \mathbf{0}$. As P is invertible, if A were also invertible, we could postmultiply by $A^{-1}P^{-1}$ to conclude $(1, 0, \ldots, 0) = \mathbf{0}$, a contradiction. Hence A is singular; indeed we can see from this proof that as soon as a zero row appears when reducing A then we know that A is singular. ■

Example 54 *Determine whether the following matrices are invertible, finding any inverses that exist.*

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \\ 0 & 1 & 5 & 3 \end{pmatrix}.$$

Solution. Quickly applying a sequence of EROs leads to

$$(A|I_3) \overset{A_{13}(-1)}{\overset{A_{12}(-2)}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -3 & -2 & -2 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\underset{S_{23}}{\overset{\substack{A_{31}(-2)\\A_{32}(3)}}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & -5 & 1 & 3 \end{array}\right) \underset{A_{31}(-1)}{\overset{M_3(-1/2)}{\longrightarrow}} \left(\begin{array}{c|ccc} & 1/2 & 1/2 & -1/2 \\ I_3 & -1 & 0 & 1 \\ & 5/2 & -1/2 & -3/2 \end{array}\right).$$

Hence

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ -1 & 0 & 1 \\ 5/2 & -1/2 & -3/2 \end{pmatrix}.$$

For B we note

$$(B|I_4) \quad \overset{\substack{A_{13}(-3)\\S_{24}}}{\longrightarrow} \left(\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & 0 & 0 & 1 \\ 0 & -8 & 5 & 1 & -3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right)$$

$$\underset{A_{34}(1/5)}{\overset{\substack{A_{23}(8)\\A_{24}(-2)}}{\longrightarrow}} \left(\begin{array}{cccc|cccc} 1 & 3 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 45 & 25 & -3 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & -3/5 & 1 & 1/5 & -2/5 \end{array}\right).$$

The left matrix is not yet in RRE form, but the presence of a zero row is sufficient to show that B is singular. ■

Remark 55 *We have defined matrix multiplication in such a way that we can see how to implement it on a computer. But how long will it take for a computer to run such a calculation?*

To multiply two $n \times n$ matrices in this way, for each of the n^2 entries we must multiply n pairs and carry out $n-1$ additions. So the process takes around n^3 multiplications and $n^2(n-1)$ additions. When n is large, these are very large numbers!

*In 1969, Strassen gave a faster algorithm, which has since been improved on. It is not known whether these algorithms give the fastest possible calculations. Such research falls into the field of **computational complexity**, drawing on ideas from both mathematics and computer science.*

Finally we define the *orthogonal matrices,* matrices important to the geometry of $\mathbb{R}^n$.

Definition 56 *An $n \times n$ matrix is **orthogonal** if $A^T = A^{-1}$.*

Proposition 57 *Let A and B be orthogonal $n \times n$ matrices. Then:*

(a) AB and A^{-1} are othogonal. Consequently the $n \times n$ orthogonal matrices form a group $O(n)$.

(b) A is orthogonal if and only if its columns (or rows) are n unit length, mutually perpendicular vectors.

(c) A preserves the dot product. If $\mathbf{x}$,$\mathbf{y} \in \mathbb{R}^n_{\text{col}}$ then $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

Proof. (a) We have that

$$(AB)^T = B^T A^T = B^{-1}A^{-1} = (AB)^{-1},$$

by the product rules for transposes and inverses, showing that AB is orthogonal. Similarly

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = \left(A^{-1}\right)^{-1},$$

showing that A^{-1} is orthogonal. ■

The reason that orthogonal matrices are important in geometry is that the orthogonal matrices are precisely those matrices that preserve the dot product.

Proposition 58 *Let A be an $n \times n$ matrix. Then A is orthogonal if and only if $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\text{col}}$.*

Proof. Note that $\mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$. So

$$\begin{array}{cc} & A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y} \\ \iff & (A\mathbf{x})^T A\mathbf{y} = \mathbf{x}^T\mathbf{y} \\ \iff & \mathbf{x}^T A^T A\mathbf{y} = \mathbf{x}^T\mathbf{y}. \end{array}$$

If $A^T A = I_n$ then the above is clearly true. Conversely, by choosing $\mathbf{x} = \mathbf{e}_i$ and $\mathbf{y} = \mathbf{e}_j$, from the standard basis of $\mathbb{R}^n_{\text{col}}$ the above implies

$$\text{the } (i,j)\text{th entry of } A^T A = \delta_{ij} = \text{the } (i,j)\text{th entry of } I_n.$$

As this is true for all i, j then this implies $A^T A = I_n$. ■