3. INTEGRATION TECHNIQUES

3.1. Introduction

In this chapter we are going to be looking at various integration techniques. There are a fair number of them and some will be easier than others. The point of the chapter is to teach you these new techniques and so this chapter assumes that you've got a fairly good working knowledge of basic integration as well as substitutions with integrals. Also, most of the integrals done in this chapter will be indefinite integrals. It is also assumed that once you can do the indefinite integrals you can also do the definite integrals and so to conserve space we concentrate mostly on indefinite integrals. There is one exception to this and that is the Trig Substitution section and in this case there are some subtleties involved with definite integrals that we're going to have to watch out for. Outside of that however, most sections will have at most one definite integral example and some sections will not have any definite integral examples.

Here is a list of topics that are covered in this chapter.

- ***Integration by substitution***: Here we look at functions that cannot be integrated by inspection with the help of the formulae discussed in earlier chapters.
- ***Integration by parts:*** Of all the integration techniques covered in this chapter this is probably the one that students are most likely to run into down the road in other classes.
- ***Integrals Involving Trig Functions***: In this section we look at integrating certain products and quotients of trig functions.
- ***Trig Substitutions:*** Here we will look using substitutions involving trig functions and how they can be used to simplify certain integrals.
- ***Partial Fractions:*** We will use partial fractions to allow us to do integrals involving some rational functions.
- ***Integrals Involving Roots*****:** We will take a look at a substitution that can, on occasion, be used with integrals involving roots.
- ***Integrals Involving Quadratics*****:** In this section we are going to look at some integrals that involve quadratics.
- ***Using Integral Tables*****:** Here we look at using Integral Tables as well as relating new integrals back to integrals that we already know how to do.

- ***Integration Strategy*:** We give a general set of guidelines for determining how to evaluate an integral.

3.2.Integration by Substitution

Because of the fundamental theorem of calculus, it is important to be able to find antiderivatives. But our anti differentiation formulas do not tell us how to evaluate integrals such as

$$\int 2x\sqrt{1+\ x^2}\,dx$$

To find this integral, we use the problem solving strategy of introducing something extra. Here the "something extra" is a new variable; we change from the variable x to a new variable u. Suppose that we let u be the quantity under the root sign.

That is,

$$u = 1+\ x^2 \Rightarrow \frac{du}{dx} = 2x \text{ or } dx = \frac{du}{2x}$$

In this case we have that;

$$\int 2x\sqrt{1+\ x^2}\,dx = \int 2x\,\sqrt{u}\,\frac{du}{2x}$$

$$= \int u^{1/2}\,du$$

$$= \frac{2}{3}u^{3/2} + c$$

$$= \boldsymbol{\frac{2}{3}(1+\ x^2) + c}$$

In general, this method works whenever we have an integral that can be written in the form

$$\int f(x)g'(x)\,dx$$

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\boldsymbol{\int f(g(x))g'(x)\,dx = \int f(u)du}$$

<u>Example 3.2.1.</u>

Evaluate each of the following integrals:

(a) $\int x^3\cos(x^4+2)\,dx$

(b) $\int \sqrt{2x+1}\,dx$

(c) $\int \frac{x}{\sqrt{1-4x^2}}\,dx$

(d) $\int e^{5x}\,dx$

(e) $\int \sqrt{1+x^2}\,x^5 dx$

3.3.Integration by parts

In this section, we study an important integration technique called integration by parts. This technique can be applied to a variety of functions and particularly useful for integrands involving products of algebraic and transcendental functions. For instance, integration by parts works well with integrals of the form such as the ones below:

$$\int x\ln x dx\,dx\,, \qquad \int x^2 e^x\,dx \qquad and \quad \int e^x \sin x\,dx$$

Integration by parts is based on the formula for the derivative of a product.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \;-------- (*)$$

Where $u\ and\ v$ are differentiable functions of x. If $u'and\ v'$, then we can integrate both sides of $(*)$ to obtain:

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\Rightarrow uv = \int u\,dv + \int v\,du$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

3.3.1. Guidelines for using integration by parts

1. Try letting dv be the most complicated portion of the integrand that fits a basic integration rule. Then u will be the remaining factor(s) of the integrand.

2. Try letting u be the portion of the portion of the integrand whose derivative is a function simpler than u. Then dv will be the remaining factor(s) of the integrand. Note that dv always includes the dx of the original integrand.
3. For integrals of the form

$$\int x^n e^{ax}\, dx, \quad \int x^n \sin ax\, dx \;\; or \int x^n \cos ax\, dx$$

$$Let \;\; u = \; x^n and\; dv = \; e^{ax} dx, \quad \sin ax\; dx\; or \cos ax\; dx.$$

4. For integrals of the form

$$\int x^n \ln x\, dx, \quad \int x^n \arcsin ax\, dx, \quad or \int x^n \arctan ax\, dx$$

$$Let\; u = \ln x, \arcsin ax\; or \arctan ax\; and\; dv = \; x^n dx$$

5. For integrals of the form

$$\int e^{ax} \sin bx\, dx \;\; or \int e^{ax} \cos bx\, dx$$

$$Let\; u = \sin bx\; or \cos bx\; and\; let\; dv = \; e^{ax} dx$$

Example 3.3.2

1. Use integration by parts to evaluate the following integrals:

(a) $\int xe^x\, dx$ (b) $\int x^2 \ln x\, dx$ (c) $\int x^2 \sin x\, dx$ (d) $\int sec^3 x\, dx$ (e) $\int e^x \cos 2x\, dx$

2. Show that $(a) \int_0^1 \arcsin x\, dx = \frac{\pi - 2}{2}$ $(b) \int_0^1 \frac{xe^x}{(x+1)^2} dx = \frac{2-e}{2}$

3.3.3. **Tabular method**

In problems involving repeated applications of integration by parts, a tabular method can be useful.

Example 3.3.4

Evaluate the integral $\int x^2 \sin 4x\; dx$ using tabular method

Working

In this method, the choice of $u\; and\; dv$ depends on the guidelines discussed in section 3.3.1 above. However, instead of computing *du* and *v* we put these into the following table. We then differentiate down the column corresponding to *u* until we hit zero. In the column corresponding to *dv* we integrate once for each entry in the first column. There is also the first column which we will explain in a bit and it always starts with a "+" and then alternates signs as shown.

$$Let\ u = x^2\ and\ dv = \sin 4x$$

<u>*Alternate signs*</u>	<u>*u and its dervatives*</u>	<u>*dv and its antiderivatives*</u>
$+$	$\boldsymbol{x^2}$	$\boldsymbol{\sin 4x}$
$-$	$\boldsymbol{2x}$	$\boldsymbol{-\frac{1}{4}\cos 4x}$
$+$	$\boldsymbol{2}$	$\boldsymbol{-\frac{1}{16}\sin 4x}$
$-$	$\boldsymbol{0}$	$\boldsymbol{\frac{1}{64}\cos 4x}$

Now, multiply along the diagonals shown by arrows in the table. In front of each product put the sign in the first column that corresponds to the "*u*" term for that product. In this case this would give,

$$(x^2)\left(-\frac{1}{4}\cos 4x\right) + -(2x)\left(-\frac{1}{16}\sin 4x\right) + 2\left(\frac{1}{64}\cos 4x\right)$$

$$= \boldsymbol{-\frac{1}{4}x^2\cos 4x + \frac{1}{8}x\sin 4x + \frac{1}{32}\cos 4x}$$

Which is pretty much easier than doing integration by parts repeatedly.

So, in this section we've seen how to do integration by parts. In your later math classes this is liable to be one of the more frequent integration techniques that you'll encounter. It is important to not get too locked into patterns that you may think you've seen. In most cases, any pattern that you think you've seen can (and will be) violated at some point in time. Be careful! Also, don't forget the shorthand method for multiple applications of integration by parts problems. It can save you a fair amount of work on occasion.

3.4.Integrals involving trigonometric functions

In this section, we study the techniques for evaluating integrals of the form

$$\int sin^m x cos^n x\, dx \; and \; \int sec^m x tan^n x\, dx$$

Where either m or n is a positive integer. To find antiderivatives for these forms, we try to break them into combinations of trigonometric integrals to which we can apply power rule.

Let's begin with a couple of examples of integrals that can be evaluated by direct application of integration by substitution:

Example 3.4.1.

Evaluate each of the following integrals:

(a) $\int sin^5 x cox\, dx$ (b) $\int sec^4 x tan\, x\, dx$ (c) $\int \sec x\, dx$ (d) $\int \tan x\, dx$ (e) $\int \cot x\, dx$

To break up $\int sin^m x cos^n x\, dx$ into forms to which power rule can be applied, the following identities may be useful:

$$\boldsymbol{sin^2 x + cos^2 x = 1}$$

$$\boldsymbol{sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$\boldsymbol{cos^2 x = \frac{1 + \cos 2x}{2}}$$

$$\boldsymbol{sin\, x cos\, x = \frac{sin\, 2x}{2}}$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine. Let's begin with the integral of the form $\int sin^m x cos^n x\, dx$.

In this integral if the exponent on the sines (*m*) is odd we can strip out one sine, convert the rest to cosines using the above identities and then use the substitution $u = \cos x$. Likewise, if the exponent on the cosines (*n*) is odd we can strip out one cosine and convert the rest to sines and then use the substitution $u = \sin x$.

Of course, if both exponents are odd then we can use either method. However, in these cases it's usually easier to convert the term with the smaller exponent. The one case we haven't looked at is what happens if both of the exponents are even? In this case the technique we used in the first couple of examples simply won't work and in fact there really isn't any one set method for doing these integrals. Each integral is different and in some cases, there will be more than one way to do the integral. With that being said most, if not all, of integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the above identities to rewrite the integrand.

Let's now look at a couple of examples where such techniques can be applied.

Example 3.4.2

Evaluate each of the following integrals:

(a) $\int sin^5 x\, dx$ (b) $\int sin^6 x cos^3 x\, dx$ (c) $\int sin^2 x\, cos^2 x\, dx$

Okay, at this point we've covered pretty much all the possible cases involving products of sines and cosines. It's now time to look at integrals that involve products of secants and tangents. This time, let's do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int sec^m x tan^n x\, dx$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using the Pythagorean identity $sin^2 x + cos^2 x = 1$. From this identity, we can derive the following identities:

$$\boldsymbol{tan^2 x + 1 = sec^2 x}$$

$$\boldsymbol{1 + cot^2 x = cosec^2 x}$$

Now, we're going to want to deal with $\int sec^m x tan^n x\, dx$ similarly to how we dealt with $\int sin^m x cos^n x\, dx$. We'll want to eventually use one of the following substitutions.

$$u = \tan x \qquad dv = sec^2 x dx$$

$$u = \sec x \qquad dv = secxtanxdx$$

So, if we use the substitution $u = \tan x$ we will need two secants left for the substitution to work. This means that if the exponent on the secant (m) is even we can strip two out and then convert the remaining secants to tangents. Next, if we want to use the substitution $u = \sec x$ we will need one secant and one tangent left over in order to use the substitution. This means that if the exponent on the tangent (n) is odd and we have at least one secant in the integrand we can strip out one of the tangents along with one of the secants of course. The tangent will then have an even exponent and so we can use the above identities to convert the rest of the tangents to secants. Note that this method does require that we have at least one secant in the integral as well. If there aren't any secants, then we'll need to do something different.

If the exponent on the secant is even and the exponent on the tangent is odd, then we can use either case. Again, it will be easier to convert the term with the smallest exponent.

Let's now take a look at a couple of examples:

Example 3.4.3

Evaluate each of the following integrals:

(a) $\int sec^9 x tan^5 x\, dx$

(b) $\int sec^4 x tan^6 x\, dx$

(c) $\int tan^3 x\, dx$

(d) $\int \frac{sin^7 x}{cos^4 x} dx$

In this section (i.e. section 3.4), we've looked at products of sines and cosines as well as products of secants and tangents. It is also important to acknowledge that because we can relate cosecants and cotangents by

$$1 + cot^2 x = cosec^2 x$$

all of the work that we did for products of secants and tangents will also work for products of cosecants and cotangents. We'll leave it to you to verify that. However, the methods used to do these integrals can also be used on some quotients involving sines and cosines and quotients involving secants and tangents (and hence quotients involving cosecants and cotangents). One such example is what we have seen in 3.4.3 (d).

At this point, I am confident that readers of this module can confidently handle integrals involving trigonometric functions.

The following section introduces us to the technique of integration by trigonometric substitutions:

3.5.Trigonometric Substitutions

Before looking at integrals involving trigonometric substitutions, let us take a quick look at the derivatives of inverse trigonometric functions:

In chapter 2, we saw that the derivative of a transcendental function $f(x) = \ln x$ is the algebraic function $f'(x) = \frac{1}{x}$. We shall now see that the derivatives of the inverse trigonometric functions are also algebraic. The following theorem lists the derivatives of the six trigonometric functions. Also, note that the derivatives of $\arccos x, arc\cot x \; and \; arccosec\, x$ are negatives of the derivatives of $\arcsin x, \arctan x \; and \; \text{arcsec}\, x$.

<u>Theorem 3.5.1</u>

Let u be a differentiable function of x, then

1. $\frac{d}{dx}(\arcsin u) = \frac{u\prime}{\sqrt{1-u^2}} \Rightarrow \frac{d}{dx}(\arccos u) = \frac{-u\prime}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}(\tan u) = \frac{u\prime}{1+u^2} \Rightarrow \frac{d}{dx}(\text{arccot}\, u) = \frac{-u\prime}{1+u^2}$

3. $\frac{d}{dx}(\text{arcsec}\, u) = \frac{u\prime}{u\sqrt{u^2-1}} \Rightarrow \frac{d}{dx}(\text{arccosec}\, u) = \frac{-u\prime}{u\sqrt{u^2-1}}$

Let's look at the proof for 1 and the rest can be done using the same approach.

$$Let\ y = \arcsin u, the \sin y \ = u - - - - (*)$$

Differentiating (*) implicitly wrt x, we obtain:

$$\cos y \frac{dy}{dx} = \frac{1du}{dx}, \;\; Note\ that \frac{du}{dx} = u'$$

$$\Rightarrow \;\; \frac{dy}{dx} = u'/\cos y$$

Remember that $cos^2 y + \ sin^2 y = 1 \Rightarrow \cos y = \ \sqrt{1 - \ sin^2 y}$. Using this relation, we obtain:

$$\frac{dy}{dx} = \frac{u'}{\sqrt{1 - \ sin^2 y}} \; but \sin y = u$$

$$Thus, \frac{dy}{dx} = \frac{u'}{\sqrt{1 - \ u^2}} \quad QED$$

<u>Example 3.5.2.</u>

Find $\frac{dy}{dx}$ for each of the following:

(a) $y = \arcsin 2x$ (b) $y = \arctan 3x$ (c) $y = \text{arcsec}\, e^{2x}$ (d) $y = \arcsin x + x\sqrt{1 - x^2}$

In (d) we have that $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$ and by product rule, $\frac{d}{dx}\left(x\sqrt{1 - x^2}\right) = \frac{1-2x^2}{\sqrt{1-x^2}}$

$$Thus, \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}} + \frac{1 - 2x^2}{\sqrt{1 - x^2}} = \mathbf{2\sqrt{1 - x^2}}$$

Since integration is the reverse of differentiation, it follows that

$$\int 2\sqrt{1 - x^2}\, dx = sin^{-1}x + x\sqrt{1 - x^2} + c$$

Note that the notation $arcsinx$ can also be written as $sin^{-1}x$

For us to evaluate an integral such as $\int 2\sqrt{1 - x^2}\, dx$, we need to use a trigonometric substitution.

1. For integrals involving $\sqrt{a^2 - u^2}$, $let\ u = a\sin\theta\ , where\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
2. For integrals involving $\sqrt{a^2 + u^2}$, $let\ u = a\tan\theta\ where\ -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
3. For integrals involving $\sqrt{u^2 - a^2}$, $let\ u = a\sec\theta\ where\ 0 < \theta \leq \frac{\pi}{2}\ or\ \frac{\pi}{2} < \theta \leq \pi$

<u>Example 3.5.3</u>

Evaluate each of the following integrals using the appropriate substitution:

(a) $\int 2\sqrt{1 - x^2}$ (b) $\int \frac{dx}{x^2\sqrt{9-x^2}}dx$ (c) $\int \frac{dx}{\sqrt{x^2-1}}$ (d) $\int \frac{dx}{\sqrt{4x^2+1}}$ (e) $\int e^{4x}\sqrt{1 + e^{2x}}\, dx$

3.6. Integration by partial fraction method

In this section, we are going to take a look at integrals of rational expressions of polynomials and once again let’s start this section out with an integral that we can already do so that we can contrast it with the integrals that we’ll be doing in this section.

Suppose we want to evaluate the integral $\int \frac{2x-1}{x^2-x+6}dx$. In this particular case, we notice that the numerator is a derivative of the denominator and so simple method of integration by substitution which we have already discussed in section 3.2 can work.

$$Let\ u = x^2 - x + 6 \Rightarrow dx = \frac{du}{2x-1}$$

$$So \int \frac{2x-1}{x^2-x+6}dx = \int \frac{1}{u}du = \ln u + c = ln|x^2 - x + 6\,| + c$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator), doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral:

$$\int \frac{2x+11}{x^2-x+6}dx$$

In this case the numerator is definitely not the derivative of the denominator nor is it a constant multiple of the derivative of the denominator. Therefore, the simple substitution that we used above won't work. However, if we notice that the integrand can be decomposed into a partial sum like this one then we can do it as follows:

$$\frac{2x+11}{x^2-x+6} = \frac{2x+11}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2}$$

$$\Rightarrow 2x + 11 = A(x+2) + B(x-3)$$

$$Giving, A = 4\ and\ B = 1\ and\ so$$

$$\frac{2x+11}{x^2-x+6} = \frac{2x+11}{(x-3)(x+2)} \equiv \frac{A}{x-3} + \frac{B}{x+2} = \boldsymbol{\frac{4}{x-3}} - \boldsymbol{\frac{1}{x+2}}$$

This means that

$$\int \frac{2x+11}{x^2-x+6}dx = \int \left(\frac{4}{x-3} - \frac{1}{x+2}\right)dx$$

$$= 4\int \frac{1}{x-3}dx - \int \frac{1}{x+2}dx$$

$$= \boldsymbol{4\ln|x-3| - \ln|x+2| + c}$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both *P(x)* and *Q(x)* are polynomials and the degree of *P(x)* is smaller than the degree of *Q(x)*. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember. So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

	Factor in the denominator	**Term in the partial fraction decomposition**
Linear factor(s)	$ax + b$	$\frac{A}{ax+b}$
Repeated linear factor(s)	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
Quadratic factor(s)	$ax^2 + bx + c$	$\frac{Ax+B}{ax^2+bx+c}$
	$(ax^2+bx+c)^k$	

Repeated quadratic factor(s)		$\frac{A_1 + B_1}{ax^2 + bx + c} + \frac{A_2 + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_k + B_k}{(ax^2 + bx + c)^k}$

From the table above, we notice that the first and third cases are really special cases of the second and fourth cases respectively. There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.

<u>Example 3.6.1</u>

Evaluate each of the following integrals:

(a) $\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$ (b) $\int \frac{x}{(x-1)(x^2+4)}\,dx$ (c) $\int \frac{3x-5}{(x+1)^2(3x-2)}\,dx$

To this point we've only looked at rational expressions where the degree of the numerator was strictly less that the degree of the denominator. Of course, not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case. Before that let's run through the following explanation in brief:

If the degree of the denominator is less than or equal to the degree of the numerator, then we have an improper fraction. If $\frac{N(x)}{D(x)}$ is an improper fraction divide the denominator into the numerator to obtain:

$$\frac{N(x)}{D(x)} = \ a\ polynomial + \frac{P(x)}{D(x)}$$

In this case $\frac{P(x)}{Q(x)}$ is a proper fraction that can easily be split into a partial sum.

<u>Example 3.6.2</u>

Evaluate the integral $\int \frac{x^3}{x^3-6x^2+11x-6}\,dx$.

3.7. Some important guidelines for evaluating integrals

We’ve now seen a fair number of different integration techniques and so we should probably pause at this point and talk a little bit about a strategy to use for determining the correct technique to use when faced with an integral. Here are a couple of points that need to be made about this strategy.

First, it isn’t a hard and fast set of rules for determining the method that should be used. It is really nothing more than a general set of guidelines that will help us to identify techniques that may work. Some integrals can be done in more than one way and so depending on the path you take through the strategy you may end up with a different technique than somebody else who also went through this strategy.

Second, while the strategy is presented as a way to identify the technique that could be used on an integral also keep in mind that, for many integrals, it can also automatically exclude certain techniques as well. When going through the strategy keep two lists in mind. The first list is integration techniques that simply won’t work and the second list is techniques that look like they might work. After going through the strategy and the second list has only one entry then that is the technique to use. If, on the other hand, there are more than one possible technique to use we will then have to decide on which is liable to be the best for us to use. Unfortunately, there is no way to teach which technique is the best as that usually depends upon the person and which technique they find to be the easiest.

Third, don’t forget that many integrals can be evaluated in multiple ways and so more than one technique may be used on it. This has already been mentioned in each of the previous points, but is important enough to warrant a separate mention. Sometimes one technique will be significantly easier than the others and so don’t just stop at the first technique that appears to work. Always identify all possible techniques and then go back and determine which you feel will be the easiest for you to use.

Next, it’s entirely possible that you will need to use more than one method to completely do an integral. For instance, a substitution may lead to using integration by parts or partial fractions integral.

The following guidelines may be useful:

1. ***Simplify the integrand, if possible.*** This step is very important in the integration process. Many integrals can be taken from impossible or very difficult to very easy with a little simplification or manipulation. Don't forget basic trig and algebraic identities as these can often be used to simplify the integral.

   We used this idea when we were looking at integrals involving trig functions. For example, consider the following integral.

$$\int cos^2x\,dx$$

   This integral can't be done as it is however, simply by recalling the identity,

$$cos^2x = \frac{1}{2}(1 + \cos 2x)$$

   the integral becomes very easy to do.

   Note that this example also shows that simplification does not necessarily mean that we'll write the integrand in a "simpler" form. It only means that we'll write the integrand into a form that we can deal with and this is often longer and/or "messier" than the original integral.

2. ***See if a "simple" substitution will work.*** Look to see if a simple substitution can be used instead of the often more complicated methods. For example, consider both of the following integrals:

$$\int \frac{x}{(x-1)(x+1)}dx \qquad\qquad \int x\sqrt{x^2-1}\,dx$$

   Someone who looks at the first integral may think of partial fraction method and trigonometric substitution for the second integral. However, it would be easier to use integration by substitution in both cases. The denominator in the first integral is a difference of two squares and so it can be written as $\int \frac{x}{x^2-1}dx$ which can easily be done by simple substitution. So, both could also be evaluated using the substitution $u = x^2 - 1$ and the work involved in the substitution would be significantly less than the work involved in either partial fractions or trig substitution. So, always look for quick, simple substitutions before moving on to the more complicated Calculus II techniques.

3. ***Identify the type of integral.*** Note that any integral may fall into more than one of these types. Because of this fact, it's usually best to go all the way through the list and identify all possible types since one may be easier than the other and it's entirely possible that the easier type is listed lower in the list. Ask/criticize yourself as much as possible:
   (i) Is the integrand a rational expression (*i.e.* is the integrand a polynomial divided by a polynomial)? If so, then partial fractions may work on the integral.
   (ii) Is the integrand a polynomial times a trig function, exponential, or logarithm? If so, then integration by parts may work.
   (iii) Is the integrand a product of sines and cosines, secant and tangents, or cosecants and cotangents? If so, then employ techniques we learnt in section 3.4 (i.e. integrals involving trigonometric functions. Likewise, don't forget that some quotients involving these functions can also be done using these techniques
   (iv) Does the integrand involve $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$? If so, then a trig substitution might work nicely.
   (v) Does the integrand have a quadratic in it? If so, then completing the square on the quadratic might put it into a form that we can deal with easily.
4. ***Can we relate the integral to an integral we already know how to do?*** In other words, can we use a substitution or manipulation to write the integrand into a form that does fit into the forms we've looked at previously in this chapter? A typical example here is the following integral:

$$\int cos\, x\sqrt{1 + sin^2 x}\, dx$$

   This integral doesn't obviously fit into any of the forms we looked at in this chapter. However, with the substitution *u* = sin *x* we can reduce the integral to the form,

$$\int \sqrt{1 + u^2}\, dx$$

   Which can be done using trigonometric substitution.

5. ***Do we need to use multiple techniques?*** In this step, we need to ask ourselves if it is possible that we'll need to use multiple techniques. The example in the previous part is a good one for this question. Using a substitution didn't allow us to actually do the integral. All it did was put the integral and put it into a form that we could use a different technique on. Don't ever get locked into the idea that an integral will only require one step to completely evaluate it. Many will require more than one step.

6. ***Try again.*** If everything that you've tried to this point doesn't work, then go back through the process and try again. This time try a technique that you didn't use the first time around.

With the above guidelines in mind, try out the following set of problems:

Work sheet

1. Evaluate each of the following integrals using the appropriate technique:

(a) $\int xe^x\,dx$ (b) $\int(\ln x)^2\,dx$ (c) $\int \frac{xe^{2x}}{(2x+1)^2}dx$ (d) $\int x^2e^{x^3}\,dx$ (e) $\int \frac{(\ln x)^2}{x}dx$

(f) $\int x^2\ln x\,dx$ (g) $\int x^2\sin x\,dx$ (h) $\int sec^3x\,dx$ (i) $\int e^x\cos 2x\,dx$ (j) $\int \frac{\ln 2x}{x^2}dx$

2. Show that $(a)\int_0^1 \arcsin x\,dx = \frac{\pi-2}{2}$ $(b)\int_0^1 \frac{xe^x}{(x+1)^2}dx = \frac{2-e}{2}$

(a) $\int sin^3x cos\,x\,dx$ (b) $\int sin^2x\,cos^4x dx$ (c) $\int_0^{\pi/2} cos^4x\,dx$ (d) $\int sec^4 3x tan^3 3x\,dx$

(e) $\int_0^{\pi/4} tan^4x\,dx$ (f) $\int csc^4x cot^4x\,dx$ (g) $\int tan^5 2x sec^2 2x\,dx$

3. Evaluate each of the following integrals:

(a) $\int \frac{dx}{(x^2+1)^2}$ (b) $\int \sqrt{4+\ 9x^2}\,dx$ (c) $\int \frac{x^2}{\sqrt{25-x^2}}$ (d) $\int x^2\sqrt{x^2-4}\,dx$ (e) $\int \frac{\sqrt{x^2-4}}{x}dx$

4. Evaluate each of the following integrals:

(a) $\int \frac{5x^2+20x+6}{x^3+2x^2+x}dx$ (b) $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ (c) $\int \frac{8x^3+13x}{(x^2+2)^2}dx$ (d) $\int \frac{x^2}{(x^2+1)^2}dx$

(e) $\int \frac{\tan x}{sec^4x}dx$ (f) $\int \frac{x^2}{x^2-1}dx$ (g) $\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}dx$ (h) $\int cos\,x\sqrt{1+\ sin^2x}\,dx$