MT1001 Introductory Mathematics
Integration Lecture Notes [1]

Tom Coleman

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Contents

Chapter 1

Definite integrals

1.1 What is integration?

You may have seen that **differential calculus** is the study of the rate of change of quantities. In differential calculus, the **derivative** $f'(x)$ of a function $f(x)$ is a function that measures the rate of change of $f(x)$. The derivative of a function has an application in geometry: evaluating $f'(x)$ at a point a allows you to find the gradient of the tangent to the curve of $f(x)$ at a.

This part of the course is concerned with **integration** of functions. There are two types of integration on a function $f(x)$:

- the **indefinite integral** of $f(x)$ with respect to x, written as

$$\int f(x)\,\mathrm{d}x$$

- the **definite integral** of $f(x)$ between limits a and b with respect to x, written as

$$\int_a^b f(x)\,\mathrm{d}x$$

  (It will be explained what this notation means shortly!)

The difference between the two types of integral is given by their uses. The indefinite integral is precisely the **reverse process** of differentiation; this result is known as the **Fundamental Theorem of Calculus** and is covered in Chapter 2 of the course. The definite integral

function	derivative w.r.t x
$f(x) = a$	$f'(x) = 0$
$f(x) = ax^n$	$f'(x) = nax^{n-1}$
$f(x) = a\sin(kx)$	$f'(x) = ka\cos(kx)$
$f(x) = a\cos(kx)$	$f'(x) = -ka\sin(kx)$
$f(x) = ae^{kx}$	$f'(x) = ake^{kx}$
$f(x) = a\ln(kx)$	$f'(x) = \frac{a}{x}$

Table 1.1: Some derivatives, with both a, k constants

represents (amongst other things) the signed area bounded by the curve $f(x)$ and the lines $x = a$ and $x = b$.

Before learning about how integration works, you should make sure that you know (and can find) the derivatives of the functions given in Table 1.1. Here, a, k are both constants.

1.2 Definite integrals as areas

Let's begin with an initial definition of a definite integral.

Definition 1.2.1. Let $f(x)$ be a positive (so for all $a \leq x \leq b$, then $f(x) \geq 0$) and continuous (informally, f can be drawn without taking your pen off the paper) function, defined on the interval $[a, b]$. Let A be the area bounded by $f(x)$, the lines $x = a$ and $x = b$ and the x-axis (see Figure 1.1). Then A equals the **definite integral** of $f(x)$ between the limits a and b with respect to x; in symbols,

$$A = \int_a^b f(x)\,\mathrm{d}x$$

Remark. Here, it's very important to note that this is only true if the function is positive. This condition can be removed, but the definition is **not** the same: see Definition 1.2.4.

Terminology. There are four pieces of notation that make up a definite integral, and each of them have different names.

- The $\int$ symbol is called the **integral sign**. This is adapted from a letter S for sum; as you will see later, an integral is actually a sum of an infinite amount of terms.

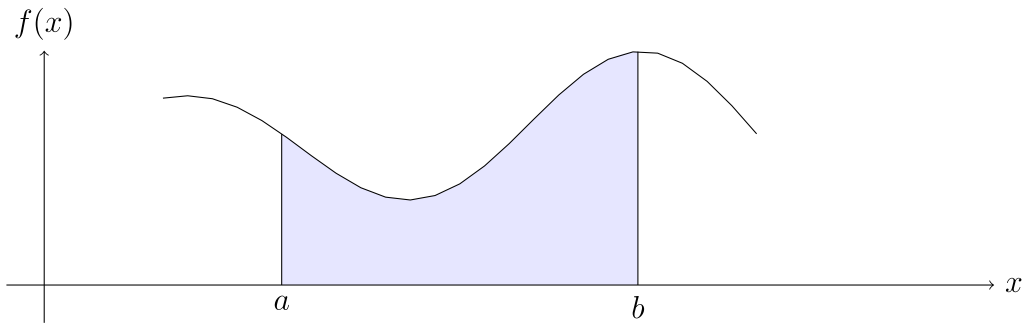

Figure 1.1: The area of the shaded region is equal to the integral $\int_a^b f(x)\,\mathrm{d}x$.

- The value a is called the **lower limit** and the value b is known as the **upper limit**.
- The function $f(x)$ is called the **integrand**.
- The $\mathrm{d}x$ denotes the variable you are integrating with respect to. Here, the variable x doesn't have to be called x; for instance it can be called t or u. The important thing to remember is that the integral doesn't change its value if you change the name of the variable throughout.

It is important to remember that in every integral, $f(x)$ and $\mathrm{d}x$ are *multiplied together*.

In **every** definite integral you write, you should include **every** one of these pieces of notation.

Example 1.2.2. Consider the area bounded by $f(x) = 2$, the x-axis, and the lines $x = 1$ and $x = 3$ (see Figure 1.2).

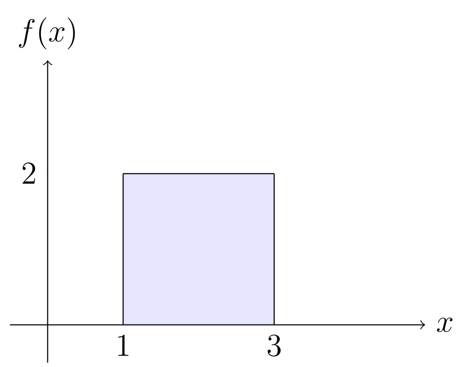

Figure 1.2: The area of the shaded region is equal to the integral $\int_1^3 2\,\mathrm{d}x$

As the shaded region is a square with side length 2, the area of this region is $A = 2 \cdot 2 = 4$. You can use Definition 1.2.1 to say that

$$\int_1^3 2\,\mathrm{d}x = 4$$

Example 1.2.3. You are given the area bounded by $f(x) = x$, the x-axis, and the lines $x = 0$ and $x = 2$ (see Figure 1.2).

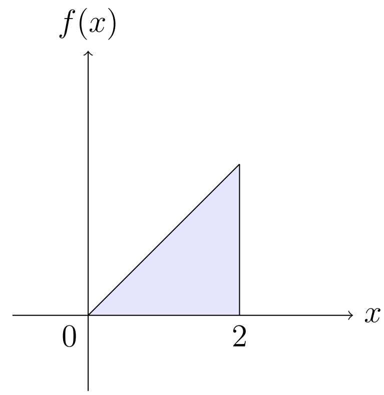

Figure 1.3: The area of the shaded region is equal to the integral $\int_0^2 x \, \mathrm{d}x$.

As the shaded region is a triangle with base and height both 2, the area is $A = (2 \cdot 2)/2 = 2$. You can use Definition 1.2.1 to say that

$$\int_0^2 x \, \mathrm{d}x = 2$$

However, evaluating areas gets more complicated if your curve is not a straight line. For instance, how would you work out the area bounded by $f(x) = x^2$, the limits $x = 0$ and $x = 1$ and the x-axis, as seen in Figure 1.4? In other words, what is the value of the definite integral of $f(x) = x^2$ between 0 and 1 with respect to x?

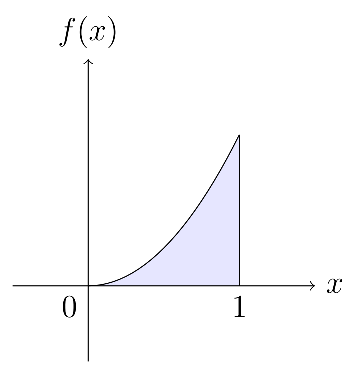

Figure 1.4: What is $\int_0^1 x^2 \, \mathrm{d}x$?

The idea is to approximate the area underneath the curve $f(x) = x^2$ by dividing it into n rectangular strips, each of width $1/n$. So for $i = 1, \ldots, n$, the ith strip is a rectangle with width $1/n$ and height given by $f((i-1)/n) = [(i-1)/n]^2$. This means that the area of the ith strip is $(1/n) \cdot [(i-1)/n]^2$ (see Figure 1.5).

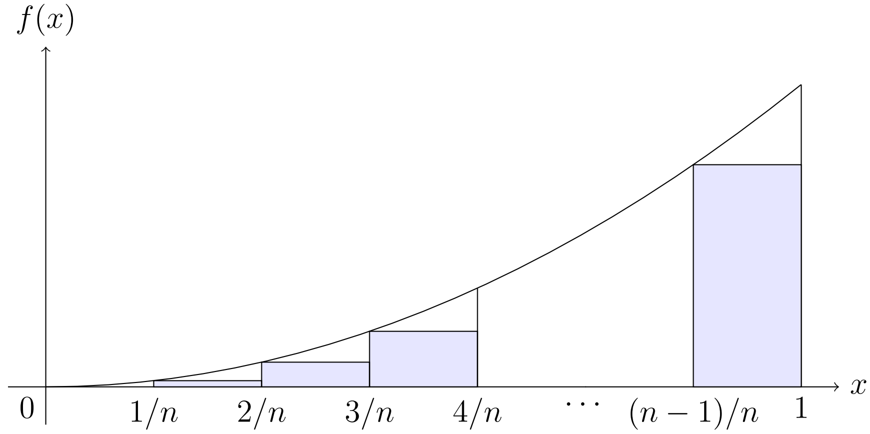

Figure 1.5: Approximating $\int_0^1 x^2 \, \mathrm{d}x$ with n strips.

You can add all of these strips together to approximate the area underneath the curve. Here, the sum of all the ith strips for $i = 1, \ldots, n$ is given by

$$S = \underbrace{\frac{1}{n} \cdot \frac{0^2}{n^2}}_{i=1} + \underbrace{\frac{1}{n} \cdot \frac{1^2}{n^2}}_{i=2} + \underbrace{\frac{1}{n} \cdot \frac{2^2}{n^2}}_{i=3} + \ldots + \underbrace{\frac{1}{n} \cdot \frac{(n-1)^2}{n^2}}_{i=n}$$

and you can factorise to get

$$S = \frac{1}{n^3} \left(1^2 + 2^2 + \ldots + (n-1)^2 \right)$$

The problem is that this is just an approximation. In order to find a value for the definite integral $\int_0^1 x^2 \, \mathrm{d}x$, you need to divide the area into **as many strips as possible**. This is achieved by letting the number of strips n tend to infinity. When you do this, the approximation S gets closer and closer to the true value of $\int_0^1 x^2 \, \mathrm{d}x$. You can write this mathematically by

$$\int_0^1 x^2 \, \mathrm{d}x = \lim_{n \to \infty} \frac{1}{n^3} \left(1^2 + 2^2 + \ldots + (n-1)^2 \right)$$

You can work this limit out by using the result

$$\sum_{r=1}^{k} r^2 = 1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

By setting $k = (n-1)$ in this result, you can see that

$$\int_0^1 x^2 \,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{n^3} \frac{(n-1)(n)(2n-1)}{6}$$

You can then reduce this in the same way you would any limit to get

$$\int_0^1 x^2 \,\mathrm{d}x = \lim_{n\to\infty} \frac{(1-1/n)(2-1/n)}{6} = \frac{1}{3}$$

However, this is just **one** example. It would be extremely useful to work out the definite integral of **any** positive, continuous function $f(x)$ between **any** limits a and b. This can be done like the process above: by dividing the interval $[a, b]$ into n strips, working out the area of each strip, adding up the area of all the strips and then letting n tend to infinity and working out the limit.

Definition 1.2.4. Let $f(x)$ be a positive and continuous function, defined on the interval $[a, b]$. Set $x_0 = a$ and $x_k = a + k(b-a)/n$ for all $k = 1, 2, \ldots, n$. Then the **definite integral** of $f(x)$ between the limits a and b with respect to x is given by

$$\int_a^b f(x) \,\mathrm{d}x = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$

where

- $(b-a)/n$ is the width of every strip;
- $f(x_k)$ is the height of the kth strip (the strip between x_k and x_{k+1}), and;
- $\frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$ is the sum of the areas of all n strips.

See Figure 1.6 for a diagram.

Example 1.2.5. You are asked to evaluate the definite integral $\int_0^1 e^x \,\mathrm{d}x$.

Using the fact that $a = 0$ and $b = 1$ and $f(x) = e^x$, and setting $x_k = k/n$, the sum in Definition 1.2.4 becomes

$$S = \frac{1}{n} \sum_{k=1}^{n} e^{k/n}$$

This is a geometric series. Using the formula for the sum of a geometric series

$$r + r^2 + \ldots + r^n = r\frac{r^n - 1}{r - 1}$$

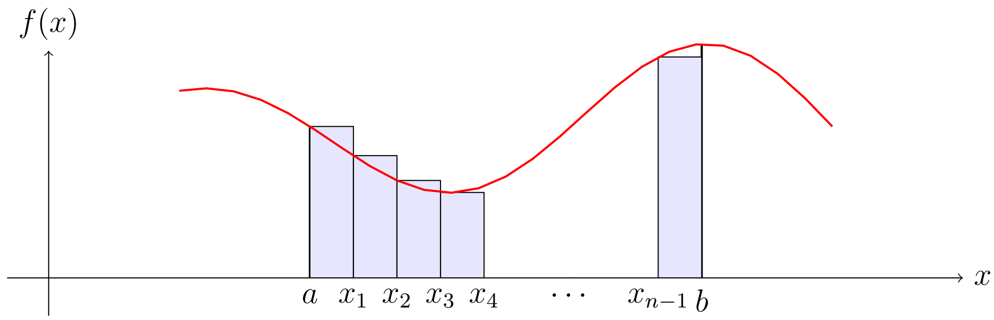

Figure 1.6: Partitioning the interval $[a, b]$ into n pieces to find the definite integral $\int_a^b f(x)\,\mathrm{d}x$. Not a very good approximation!

with $r = e^{1/n}$ you can obtain

$$S = \frac{1}{n} e^{1/n} \frac{e-1}{e^{1/n}-1}$$

The goal now is to work out the limit of this expression as n tends to infinity. Here, you can get

$$\int_0^1 e^x \,\mathrm{d}x = \lim_{n\to\infty} S = \lim_{n\to\infty} \frac{e^{1/n}}{n} \frac{e-1}{e^{1/n}-1}$$

You can take $(e-1)$ out of the limit and rearrange and, using the expression

$$\lim_{n\to\infty} n\left(e^{1/n}-1\right) = 1$$

gives

$$\int_0^1 e^x \,\mathrm{d}x = (e-1) \lim_{n\to\infty} \frac{e^{1/n}}{n\left(e^{1/n}-1\right)} = e-1$$

1.3 Properties of definite integrals

You can use Definition 1.2.4 and the properties of sums and limits to show the following properties of definite integrals.

Proposition 1.3.1. *Let $\int_a^b f(x)\,\mathrm{d}x$ be a definite integral. Then the following properties hold:*

(1) For c such that $a < c < b$, then:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

(2) If k is some constant, then:

$$\int_a^b kf(x)\,\mathrm{d}x = k\int_a^b f(x)\,\mathrm{d}x$$

(3) For a function $g(x)$ where the definite integral exists on the interval $[a,b]$:

$$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

(4) If $f(x) \leq g(x)$ for all x between a and b, then:

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

(5) If a is some real number, then:

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

(6) Finally if $a < b$, then:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Using properties (1) and (2) of Proposition 1.3.1, you can define a definite integral for functions that are negative. For instance, let $f(x)$ be a function where $f(x) \geq 0$ for all x between a and c, and $f(x) \leq 0$ for all x between c and b.

You can use property (1) of Proposition 1.3.1 to get:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

As $f(x) \leq 0$, it follows that $-|f(x)| \leq 0$ between c and b. So using property (2) gives:

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \int_a^c f(x)\,\mathrm{d}x + \int_c^b -|f(x)|\,\mathrm{d}x \\ &= \int_a^c f(x)\,\mathrm{d}x - \int_c^b |f(x)|\,\mathrm{d}x\end{aligned}$$

This shows that the definite integral is the area above the axis *minus* the area below the axis. So regions where $f(x) < 0$ correspond to negative areas (see Figure 1.7).

Example 1.3.2. Let $f(x) = \sin(x)$ be defined between $-\pi$ and π. Using property (1) of

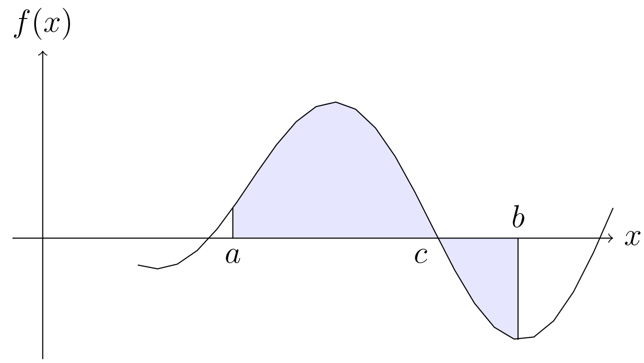

Figure 1.7: The curve $f(x)$ crosses the x-axis at $x = c$, so the shaded region below the x-axis has a negative area.

Proposition 1.3.1 gives:

$$\int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x = \int_{-\pi}^{0} \sin(x)\,\mathrm{d}x + \int_{0}^{\pi} \sin(x)\,\mathrm{d}x \tag{1.1}$$

As $\sin(x)$ is an odd function, then $\sin(x) = -\sin(x)$. You can use this, along with property (2) of Proposition 1.3.1 to see that

$$\int_{-\pi}^{0} \sin(x)\,\mathrm{d}x = \int_{-\pi}^{0} -\sin(-x)\,\mathrm{d}x = -\int_{-\pi}^{0} \sin(-x)\,\mathrm{d}x$$

Here, $-x$ is contained in the interval $-\pi < -x < 0$. Multiplying through by -1 gives $\pi > x > 0$. So

$$-\int_{-\pi}^{0} \sin(-x)\,\mathrm{d}x = -\int_{0}^{\pi} \sin(x)\,\mathrm{d}x$$

Putting this in to Equation 1.1 gives

$$\int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x = -\int_{0}^{\pi} \sin(x)\,\mathrm{d}x + \int_{0}^{\pi} \sin(x)\,\mathrm{d}x = 0$$

This result is illustrated in Figure 1.8; you can see that the areas are equal in size, but on opposite sides of the x-axis.

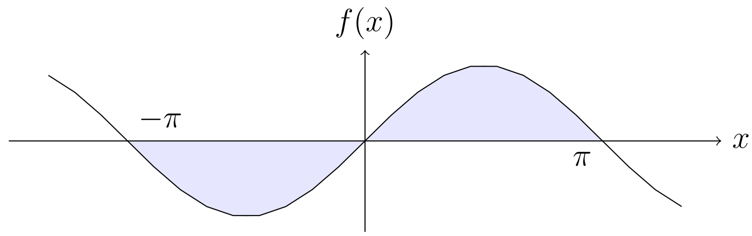

Figure 1.8: $\int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x$

Chapter 2

The Fundamental Theorem Of Calculus

2.1 From definite to indefinite

You may have wondered why the term 'definite' has been attached to integrals throughout Chapter 1. This is because the definite integral $\int_a^b f(x)\,\mathrm{d}x$ is an expression of a definite signed area bounded by $f(x)$, the x axis, the lower limit a and the upper limit b.

Now suppose instead of setting an upper limit b, this limit is allowed to vary. So here you can write:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

It is important to say that the variable in the integral has changed to t; this doesn't affect the value of the integral as mentioned in the Terminology section after Definition 1.2.1.

The function $F(x)$ represents the change of area expressed by the integral when x is allowed to change. This means that the area bounded by $f(t)$, the t-axis, the lower limit a and the upper limit x changes as x changes. In fact, you could say this area is *not definite*. Therefore, $F(x)$ is known as the **indefinite integral** of f.

In this chapter, the Fundamental Theorem of Calculus (Theorem 2.2.2) says that $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is the antiderivative of $f(x)$; that is, a function whose derivative is $f(x)$. This will show that integration is a reverse process of differentiation, and that you can evaluate definite integrals $\int_a^b f(x)\,\mathrm{d}x$ by considering the antiderivative of the integrand $f(x)$ and calculating the difference of the antiderivative at the two limits a and b.

2.2 The theorem

This section aims to show that the function $F(x)$ as stated above acts as the antiderivative of $f(x)$. You have seen from the part of the course on differentiation that the derivative of a function is defined to be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The aim is to demonstrate that $F'(x) = f(x)$ using this definition of derivative. So take

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

as above, representing the area under the curve $f(t)$ between the lines $t = a$ and $t = x$. It follows that

$$F(x+h) = \int_a^{x+h} f(t)\,\mathrm{d}t$$

(see Figure 2.1 for a picture).

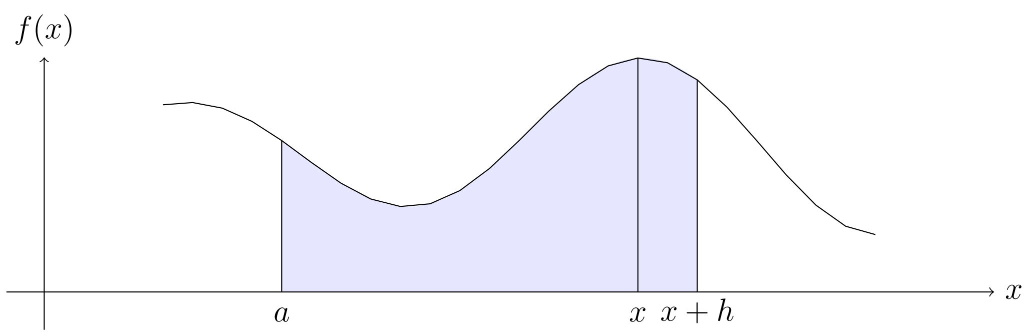

Figure 2.1: An illustration of the areas $\int_a^x f(t)\,\mathrm{d}t$ and $\int_a^{x+h} f(t)\,\mathrm{d}t$.

By Proposition 1.3.1 (1), you can write:

$$F(x+h) = \int_a^{x+h} f(t)\,\mathrm{d}t = \int_a^x f(t)\,\mathrm{d}t + \int_x^{x+h} f(t)\,\mathrm{d}t = F(x) + \int_x^{x+h} f(t)\,\mathrm{d}t$$

and so

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t$$

The idea now is to 'bound' the area underneath $f(t)$ between x and $x+h$ between two areas. To do this, set the minimum value of $f(t)$ between x and $x+h$ as m, and the similar maximum value by M. The minimum area of $\int_x^{x+h} f(t)\,\mathrm{d}t$ is given by the area of rectangle with side lengths h and m; and the maximum area is given by the area of the rectangle with side lengths h and M. You can see a diagram of this in Figure 2.2.

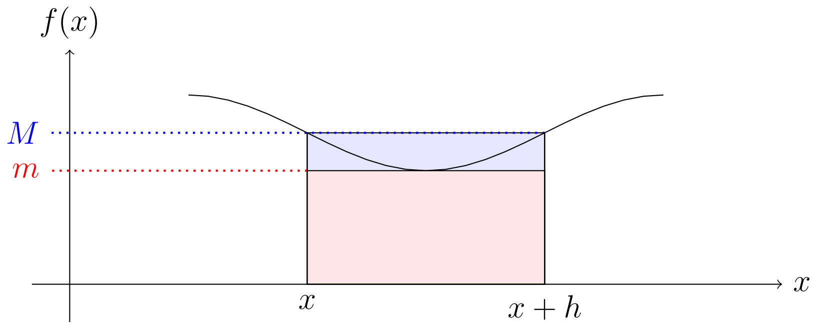

Figure 2.2: Describing the rectangles hm (red rectangles, minimum area) and hM (blue and red rectangles together, maximum area). The value of $\int_x^{x+h} f(t)\,\mathrm{d}t$ is somewhere between the two.

So you can write

$$hm \leq \int_x^{x+h} f(t)\,\mathrm{d}t \leq hM.$$

Dividing through each term by h gives

$$m \leq \frac{F(x+h) - F(x)}{h} \leq M.$$

As $f(t)$ is continuous, when h tends towards 0 both the upper and the lower bounds tend to the value $f(x)$. So, letting $h \to 0$ gives

$$f(x) \leq \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \leq f(x)$$

and so

$$f(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = F'(x)$$

This statement is the first of the two parts of the Fundamental Theorem of Calculus. Here is an example of how it can be used, illustrating the way forward for a proof of the second part.

Example 2.2.1. You are given the function

$$F(x) = \int_0^x e^t\,\mathrm{d}t \tag{2.1}$$

Using the above result, you can write

$$F'(x) = e^x$$

The idea now is to find $F(x)$ such that $F'(x) = e^x$. You can look at Table 1.1 to see that the derivative of e^x is e^x, which is on the right hand side. However, you can also see in Table 1.1 that the derivative of a constant C is 0. So when you are finding $F(x)$ such that $F'(x) = e^x$, you need to be careful to include a constant term! So you can take

$$F(x) = e^x + C$$

as a function whose derivative is e^x.

You can substitute this into Equation 2.1 to get

$$e^x + C = \int_0^x e^t \, \mathrm{d}t$$

You now have enough information to find the constant C. You can set $x = 0$ to see that

$$e^0 + C = 1 + C = \int_0^0 e^t \, \mathrm{d}t = 0$$

and so $C = -1$, giving $F(x) = e^x - 1$. Finally, evaluating $F(x)$ at $x = 1$ gives:

$$F(1) = \int_0^1 e^t \, \mathrm{d}t = e - 1$$

which is exactly the same result as you found in Example 1.2.5.

As has just been shown, the integral

$$F(x) = \int_a^x f(t) \, \mathrm{d}t$$

represents an antiderivative of $f(x)$. Following the discussion in Example 2.2.1, this antiderivative is **not** unique. In fact, $F(x) + C$, where C is some constant, is **also** an antiderivative of $f(x)$, as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x) + C\right) = F'(x) + 0 = f(x)$$

So you can write that

$$F(x) + C = \int_a^x f(t) \, \mathrm{d}t$$

To work out C, you can evaluate this function at $x = a$ to get that:

$$F(a) + C = \int_a^a f(t) \, \mathrm{d}t = 0$$

and so $C = -F(a)$. Therefore $F(x) - F(a) = \int_a^x f(t)\,\mathrm{d}t$; you can set $x = b$ to get

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$$

This is the second part of the Fundamental Theorem of Calculus.

Theorem 2.2.2. *[Fundamental Theorem of Calculus]*

Let $f(x)$ be a continuous function on the interval $[a, b]$. For x between a and b define $F(x)$ by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

Then $F(x)$ is differentiable on $[a, b]$ with derivative $F'(x) = f(x)$. Furthermore,

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$$

Both parts of the theorem are important:

- The first part says that finding the antiderivative of a function is a reverse process of differentiation.
- The second part says that you can find a definite integral by considering antiderivatives, rather than working through limits.

The Fundamental Theorem of Calculus also outlines the difference between definite and indefinite integrals. To evaluate a definite integral $\int_a^b f(x)\,\mathrm{d}x$, the first (and most important) step is to find an antiderivative $F(x)$ of $f(x)$, and the second involves finding the difference $F(b) - F(a)$. The first of these steps does not involve the limits a and b in any way. So here, you can use the notation

$$F(x) = \int f(x)\,\mathrm{d}x$$

to denote antiderivatives of $f(x)$. As stated in Section 2.1, this is the **indefinite integral** of $f(x)$.

There are two important differences between finding definite and indefinite integrals.

- When you work out a **indefinite integral** $\int f(x)\,\mathrm{d}x$, your final answer should be a **function** plus a **constant of integration** C. This expression represents all the possible antiderivatives of $f(x)$. You should always remember to add a $+C$ at the very end when finding an indefinite integral.

- When you work out a **definite integral** $\int_a^b f(x)\,\mathrm{d}x$, you could follow this method:

Step 1: Find an antiderivative $F(x)$ of $f(x)$. You do not need a $+C$.

Step 2: The result $F(x)$ can be put into square brackets, with the upper limit at the top right and the lower limit at the bottom right. So you can write:

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b$$

Step 3: Work out $F(x)$ at $x = b$ and then work out $F(x)$ at $x = a$. You can then subtract $F(a)$ from $F(b)$ to get

$$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$$

The number at the end is the answer.

Your final answer should be a **number**. This quantity represents the signed area underneath the curve of $f(x)$ between the lines $x = a$ and $x = b$. There should be **no** variables in your answer, and you do **not** need a $+C$ at the end.

2.3 Some antiderivatives

This section details some examples of finding antiderivatives. If you are unsure about derivatives of functions here, you could take the opportunity to re-familiarise yourself with Table 1.1 before continuing.

The idea of Table 1.1 is to provide a list of derivatives for common functions. As indefinite integration is the reverse process of differentiation, it makes sense to write a list of antiderivatives.

For instance, suppose you wanted to find the antiderivative of the function $f(x) = ax^n$, where a is a constant and $n \neq -1$. You can use the fact that integration is the opposite of differentiation to say that the solution will be a function $F(x)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = ax^n$$

Taking $F(x) = \frac{ax^{n+1}}{n+1}$, and using the power rule for differentiation gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{ax^{n+1}}{n+1}\right) = \frac{a}{n+1} \cdot (n+1)x^n = ax^n$$

which is $f(x)$. Whenever you differentiate a constant, it goes to 0; so you could take $F(x) = \frac{ax^{n+1}}{n+1} + C$, differentiate it with respect to x, and **still** get $f(x)$. So this means that you can write:

$$\boxed{\int ax^n \,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + C \qquad \text{for } n \neq -1} \tag{2.2}$$

A special case happens when $n = 0$; if this happens, then $f(x) = ax^0 = a$ is a constant. Integrating this using the formula above gives $\int a \,\mathrm{d}x = \int ax^0 \,\mathrm{d}x = \frac{ax^1}{1} + C = ax + C$ and so

$$\boxed{\int a \,\mathrm{d}x = ax + C} \tag{2.3}$$

You may have asked yourself already why doesn't this process work for $n = -1$? These are functions of the form $f(x) = \frac{a}{x}$. It is because if you do the process with $n = -1$, you end up dividing by 0; which is not good. However, you can look at Table 1.1 and see that

$$\frac{\mathrm{d}}{\mathrm{d}x}(a\ln(x)) = \frac{a}{x}$$

However, you need to make sure that $\ln(x)$ is defined on only positive x. You can get around this by writing $\ln|x|$ instead of $\ln(x)$; this ensures the value of the input is positive. This means you can write

$$\boxed{\int ax^{-1} \,\mathrm{d}x = a\ln|x| + C} \tag{2.4}$$

You have already seen in Example 2.2.1 that an antiderivative for $f(x) = e^x$ is $F(x) = e^x + C$. But what about $f(x) = ae^{kx}$? You are looking for a function $F(x)$ such that $\frac{\mathrm{d}}{\mathrm{d}x}(F(x)) = ae^{kx}$. Here, taking $F(x) = \frac{1}{k}ae^{kx} + C$, and differentiating with respect to x using the rules in Table 1.1 gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{k}ae^{kx} + C\right) = ae^{kx}$$

and so you can write

$$\boxed{\int ae^{kx} \,\mathrm{d}x = \frac{1}{k}ae^{kx} + C} \tag{2.5}$$

Finally, you can consider antiderivatives of $a\sin(kx)$ and $a\cos(kx)$. Using rules of differen-

function	antiderivative w.r.t x	notes
$f(x) = a$	$\int f(x)\,\mathrm{d}x = ax + C$	a constant (2.3)
$f(x) = ax^n$	$\int f(x)\,\mathrm{d}x = \frac{ax^{(n+1)}}{n+1} + C$	$n \neq -1$ (2.2)
$f(x) = ax^{-1}$	$\int f(x)\,\mathrm{d}x = a\ln\lvert x\rvert + C$	(2.4)
$f(x) = ae^{kx}$	$\int f(x)\,\mathrm{d}x = \frac{1}{k}ae^{kx} + C$	(2.5)
$f(x) = a\cos(kx)$	$\int f(x)\,\mathrm{d}x = \frac{1}{k}a\sin(kx) + C$	(2.6)
$f(x) = a\sin(kx)$	$\int f(x)\,\mathrm{d}x = -\frac{1}{k}a\cos(kx) + C$	(2.7)

Table 2.1: Some antiderivatives

tiation, you can say that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{k}a\cos kx\right) = a\sin(kx) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{k}a\sin kx\right) = a\cos(kx)$$

This means that you can write:

$$\boxed{\int a\cos(kx)\,\mathrm{d}x = \frac{1}{k}a\sin kx + C} \tag{2.6}$$

and

$$\boxed{\int a\sin(kx)\,\mathrm{d}x = -\frac{1}{k}a\cos kx + C} \tag{2.7}$$

Summarising the boxed results, Table 2.1 gives a list of common antiderivatives that will be useful throughout your mathematical career. Throughout, a, k, C are constants.

You can use the examples given in Table 2.1 to find some integrals that previously were harder to find.

Example 2.3.1. You are asked to find the definite integral $\int_0^2 x\,\mathrm{d}x$. To do this, you need to find an antiderivative $F(x)$ of $f(x) = x$, and then find out $F(b) - F(a)$ where $b = 2$

and $a = 0$. You can use Equation 2.2 with $a = 1$ and $n = 1$ to get

$$\int_0^2 x \,\mathrm{d}x = \left[\frac{1 \cdot x^{1+1}}{1+1}\right]_0^2 = \left[\frac{x^2}{2}\right]_0^2$$

Using $F(x) = x^2/2$, you can write that $F(2) = 4/2 = 2$ and $F(0) = 0$, so now you can evaluate the integral:

$$\int_0^2 x \,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} - 0 = 2$$

This is the same answer you got in Example 1.2.3.

It can be shown that properties (2) and (3) of Proposition 1.3.1 hold for indefinite integrals. So for any **constant** k (so nothing that involves a variable!) you can write that

$$\int kf(x) \,\mathrm{d}x = k \int f(x) \,\mathrm{d}x$$

and for a continuous function $g(x)$:

$$\int f(x) + g(x) \,\mathrm{d}x = \int f(x) \,\mathrm{d}x + \int g(x) \,\mathrm{d}x$$

Example 2.3.2. You are asked to find the indefinite integral $\int 16 - x^2 \,\mathrm{d}x$. Here, you can use the properties of indefinite integrals stated above (and a remark in Example 1.2.2) to write

$$\int 16 - x^2 \,\mathrm{d}x = \int 16 \,\mathrm{d}x + \int -x^2 \,\mathrm{d}x = 16 \int \,\mathrm{d}x - \int x^2 \,\mathrm{d}x$$

So the problem is reduced to working out antiderivatives of 1 and x^2. Using Equation 2.3 with $f(x) = 1$ gives an antiderivative of $1x = x$. Using Equation 2.2 with $n = 2$ gives $x^3/3$ as an antiderivative of x^2. So you can write:

$$\int 16 - x^2 \,\mathrm{d}x = 16 \int 1 \,\mathrm{d}x - \int x^2 \,\mathrm{d}x = 16x - \frac{x^3}{3} + C$$

You can notice here that you only need to add one constant of integration $+C$ at the very end of the working.

Example 2.3.3. You are asked to find the definite integral $\int_{-\pi}^{\pi} \sin(2x) + 2\cos(x) \,\mathrm{d}x$. You can use Equation 2.7 with $a = 1$ and $k = 2$ to say that an antiderivative for $\sin(2x)$ is $-\frac{1}{2}\cos(2x)$. You can use Equation 2.6 with $a = 2$ and $k = 1$ to get that an antiderivative for $2\cos(x)$ is $2\sin(x)$. Therefore, you can write

$$\int_{-\pi}^{\pi} \sin(2x) + 2\cos(x)\,\mathrm{d}x = \left[-\frac{1}{2}\cos(2x) + 2\sin(x)\right]_{-\pi}^{\pi}$$

You can now evaluate the value of this antiderivative at the limits π and $-\pi$. Here

$$-\frac{1}{2}\cos(2\pi) + 2\sin(\pi) = -\frac{1}{2}\cdot(1) + 2\cdot 0 = -\frac{1}{2}$$

and

$$-\frac{1}{2}\cos(-2\pi) + 2\sin(-\pi) = -\frac{1}{2}\cdot(1) + 2\cdot 0 = -\frac{1}{2}$$

You can put this all together to get that

$$\int_{-\pi}^{\pi} \sin(2x) + 2\cos(x)\,\mathrm{d}x = \left[-\frac{1}{2}\cos(2x) + 2\sin(x)\right]_{-\pi}^{\pi} = \left[-\frac{1}{2}\right] - \left[-\frac{1}{2}\right] = 0$$

This example shows that you do not have to write $F(x)$ at any point when evaluating an integral, nor do you have to split up the integral of two functions added together into two parts before proceeding.

2.4 Summary of first two chapters

- The Fundamental Theorem of Calculus states that indefinite integration is the reverse process to differentiation, and that you can work out definite integrals (the limit of sums of areas) by using the antiderivative of the integrand.

- Working out both indefinite and definite integrals involve using antiderivatives; you can find these for common functions in Table 2.1.

- When you work out a **indefinite integral** $\int f(x)\,\mathrm{d}x$, your answer should be a **function** $F(x)$ plus a **constant of integration** C. This expression represents all the possible antiderivatives of $f(x)$. You should always remember to add a $+C$ at the end when working out an indefinite integral.

- When you work out a **definite integral**, your answer should be a **number**. This number represents the **signed** area underneath the curve of $f(x)$ between the lines $x = a$ and $x = b$. There should be **no** variables in your answer, and you do **not** need a $+C$.

Chapter 3

Integration by substitution

3.1 Techniques of calculus

You may have studied techniques for finding derivatives of composite functions using the *chain rule*, *the product rule* and the *quotient rule*. These rules are detailed in Table 3.1 for various operations on functions $f(x)$ and $g(x)$.

function	derivative w.r.t x	name
$y(x) = f(g(x))$	$y'(x) = \frac{\mathrm{d}}{\mathrm{d}g}\left(f(g(x))\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(g(x))$	chain rule
$y(x) = f(x) \cdot g(x)$	$y'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$	product rule
$y(x) = \frac{f(x)}{g(x)}$	$y'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$	quotient rule ($g \neq 0$)

Table 3.1: Helpful rules for differentiation

Similar techniques exist for when you want to find antiderivatives of composite functions. This chapter is on **integration by substitution**, which is similar to the technique of the chain rule. This technique is discussed further in Chapter 4, when considering trigonometric functions in more detail. Chapter 5 is on **integration by parts**, which is similar to the technique of the product rule. Both techniques are very important for finding antiderivatives of a variety of functions.

3.2 How integration by substitution works

Suppose you are asked to integrate a composite function $f(u(x))$. You can write the function $u(x)$ as a variable u here (see the terminology on page 4), and use the terms interchangeably; but you should always remember that u **is a function of** x. Finding this integral of $f(u(x)) = f(u)$ can be done in two different ways; one with respect to u and one with respect to x. Firstly, using the Fundamental Theorem of Calculus, you are looking for a function $F(u)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}u}(F(u)) = f(u)$$

and so you can write

$$F(u) = \int f(u)\,\mathrm{d}u$$

As u is a function of x, you can differentiate $F(u(x))$ with respect to x using the chain rule, to get

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(u(x))) = \frac{\mathrm{d}}{\mathrm{d}u}(F(u(x))) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(u(x)) = F'(u(x)) \cdot u'(x) = f(u(x)) \cdot u'(x)$$

You can see from this that $F(u(x))$ is an antiderivative (with respect to x) of $f(u(x)) \cdot u'(x)$. This means that you can write

$$F(u) = \int f(u) \cdot u'(x)\,\mathrm{d}x$$

This gives two expressions for $F(u)$, which you can compare to get:

$$\boxed{\int f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u} \tag{3.1}$$

Equation 3.1 demonstrates the principle of **integration by substitution** for indefinite integrals. Essentially, you can choose a function $u = u(x)$ of x, and 'substitute' this into the left hand side of Equation 3.1. This is then equal to the right hand side which, on the correct choice of $u(x)$, should be an easier integral to solve. You should consider using integration by substitution (with substitution $u = u(x)$) where the integrand is a composite function $f(u(x))$ of x multiplied by a term that 'looks like' the derivative $u'(x)$ of $u(x)$.

The difficulty of integration by substitution lies in the correct choice of $u = u(x)$. There is no general rule for choosing the 'right' substitution in a given integrand, so you will have to decide on a correct substitution for **every** integration by substitution that you do. You will know that you have made the correct substitution when there is no mention of x in your expression. A useful thing to remember is that the idea of this method is to make integration **easier**; so the integral on the right is something you know you can work out. This should motivate your choice for u.

One thing that you can notice is that integration by substitution at this stage is only used in finding antiderivatives of composite functions. Suppose that you are using integration by substitution to evaluate a **definite** integral $\int_a^b f(u(x))\,\mathrm{d}x$. You need to be careful here as the limits of the integral are defined in terms of x, and **not** in terms of u. To correct this, you need to evaluate the limits a and b in your chosen substitution $u = u(x)$, and then substitute these values in the correct place. In other words:

$$\boxed{\int_a^b f(u(x)) \cdot u'(x)\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\,\mathrm{d}u} \tag{3.2}$$

This is integration by substitution for definite integrals.

Another thing you can notice is that $f(u)$ is present in both integrands in Equation 3.1 (and Equation 3.2), and that this implies that $u'(x)\mathrm{d}x = \mathrm{d}u$. This expression can be rearranged; so you can write $\mathrm{d}x = \mathrm{d}u/u'(x)$. This technique is very useful; particularly if there is no $u'(x)$ term immediately visible in the integrand.

You can follow these steps whenever you would need to use integration by substitution. This method will be used throughout the examples given in Section 3.3.

3.2.1 Method for integration by substitution

Step 1: Choose a suitable $u = u(x)$. Your choice of substitution should **not** be a constant function $u(x) = a$.

Step 2: Work out $u'(x)$, and write down an expression for $\mathrm{d}x = \mathrm{d}u/u'(x)$. If you are considering a definite integral, work out $u(a)$ and $u(b)$ where a and b are the limits of the integral.

Step 3: Now, you should

- replace every instance of $u(x)$ with the letter u
- replace $\mathrm{d}x$ with $\mathrm{d}u/u'(x)$, and cancel;
- (for definite integrals only) replace a with the value $u(a)$ and b with $u(b)$.

Warning: At this stage, the integral should be solely in terms of u. If there are still terms containing x at this stage, stop and consider another choice of u.

Step 4: If you can, work out the integral. If you are considering a definite integral, then the method stops here with the answer. if you are considering an indefinite integral, don't forget the $+C$!

Step 5: (For indefinite integrals only) Your antiderivative should be in terms of u. Replace every instance of u with the original function $u(x)$. The method stops here for indefinite integrals.

3.3 Examples

Throughout these examples, the properties of integrals as outlined in Proposition 1.3.1 are used without reference.

Example 3.3.1. You are asked to evaluate $\int \sin(3x+9)\,\mathrm{d}x$. Since this is an indefinite integral, you have no limits to change but you must remember to do Step 5 of the method in Subsection 3.2.1.

Step 1: You can integrate $\sin$, so a good choice of substitution here would be $u = 3x+9$.

Step 2: Here, $u'(x) = 3$, and so $\mathrm{d}x = \mathrm{d}u/3$. As this is an indefinite integral, you have no limits to change.

Step 3: Replacing each instance of $u(x) = 3x+9$ with u, and replacing $\mathrm{d}x$ by $\mathrm{d}u/3$ gives

$$\int \sin(3x+9)\,\mathrm{d}x = \int \sin(u)\frac{1}{3}\,\mathrm{d}u = \frac{1}{3}\int \sin(u)\,\mathrm{d}u$$

Since there is no x left in the integral, you are OK to continue.

Step 4: Not forgetting the constant of integration, evaluating the integral using the appropriate rule from Table 2.1 gives

$$\frac{1}{3}\int \sin(u)\,\mathrm{d}u = -\frac{1}{3}\cos(u) + C$$

Step 5: You need to substitute in $3x+9 = u$ into your solution to get $\frac{1}{3}\cos(3x+9)$. So you can write

$$\int \sin(3x+9)\,\mathrm{d}x = -\frac{1}{3}\cos(3x+9) + C$$

which is the final answer.

Example 3.3.2. Say you are asked to evaluate $\int_0^4 \frac{1}{(x/2-4)^3}\,\mathrm{d}x$. Since this is a definite integral, you must remember to change the limits of integration, but you don't have to do Step 5 of the method in Subsection 3.2.1.

Step 1: You can integrate any function of the form x^n, so a good choice of substitution here would be $u = x/2 - 4$.

Step 2: Here, $u'(x) = 1/2$, and so $\mathrm{d}x = \mathrm{d}u/(1/2) = 2\mathrm{d}u$. As this is a definite integral, you must remember to change the limits. You can do this by evaluating u at $x = 0$ and $x = 4$. So here $u(0) = -4$ and $u(4) = -2$.

Step 3: Replacing each instance of $u(x) = x/2 - 4$ with u, replacing $\mathrm{d}x$ by $2\mathrm{d}u$, and replacing the lower limit 0 with -4 and the upper limit 4 with -2 gives

$$\int_0^4 \frac{1}{(x/2-4)^3}\,\mathrm{d}x = \int_{-4}^{-2} \frac{1}{u^3}\cdot 2\,\mathrm{d}u = 2\int_{-4}^{-2} u^{-3}\,\mathrm{d}u$$

As there are no terms involving x left in the integral, you are OK to continue and you can now evaluate the integral.

Step 4: Evaluating the integral using the corresponding rule from Table 2.1 gives

$$2\int_{-4}^{-2} u^{-3}\,\mathrm{d}u = 2\left[\frac{u^{-2}}{-2}\right]_{-4}^{-2} = \left[-u^{-2}\right]_{-4}^{-2}$$

So this means that

$$\left[-u^{-2}\right]_{-4}^{-2} = \left[-\frac{1}{(-2)^2}\right] - \left[-\frac{1}{(-4)^2}\right] = -\frac{1}{4} + \frac{1}{16} = -\frac{3}{16}$$

which is the final answer.

Example 3.3.3. You are asked to evaluate $\int \sin^3(x)\cos(x)\,\mathrm{d}x$. Since this is an indefinite integral, you have no limits to change but you must remember to do Step 5 of the method in Subsection 3.2.1.

Step 1: You can integrate u^n, so a good choice of substitution here would be $u = \sin(x)$.

Step 2: Here, $u'(x) = \cos(x)$, so you can write $\mathrm{d}x = \mathrm{d}u/\cos(x)$. Again, as this is an indefinite integral, you have no limits to change.

Step 3: Replacing each instance of $u(x) = \sin(x)$ with u, and replacing $\mathrm{d}x$ by $\mathrm{d}u/\cos(x)$ gives

$$\int \sin^3(x)\cos(x)\,\mathrm{d}x = \int u^3 \frac{\cos(x)}{\cos(x)}\,\mathrm{d}u = \int u^3\,\mathrm{d}u$$

Since there is no x left in the integral, you are OK to continue.

Step 4: Not forgetting the constant of integration, evaluating the integral using the corresponding rule from Table 2.1 gives

$$\int u^3\,\mathrm{d}u = \frac{u^4}{4} + C$$

Step 5: You need to substitute in $\sin(x) = u$ into your solution to get $\frac{1}{4}\sin^4(x)$. You can write

$$\int \sin(3x+9)\,\mathrm{d}x = -\frac{1}{3}\cos(3x+9) + C$$

which is the final answer.

Example 3.3.4. You are asked to evaluate $\int \frac{1}{x} + \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$. Since this is an indefinite integral, you have no limits to change but you must remember to do Step 5 of the method in Subsection 3.2.1. Here, you can integrate $1/x$ using a rule in Table 2.1 (giving $\ln|x|$, but you cannot integrate $e^{\sqrt{x}}/\sqrt{x}$ so easily. The idea here is to split the integrand into two, using integration by substitution on one of the parts. Using the techniques from Proposition 1.3.1 can make your life easier in these circumstances. So by saying

$$\int \frac{1}{x} + \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x + \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = \ln|x| + \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

you can focus on evaluating the remaining integral by substitution.

Step 1: You can integrate e^x, so a good choice of substitution here would be $u = \sqrt{x} = x^{1/2}$.

Step 2: Here, $u'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$. This means that

$$\mathrm{d}x = \mathrm{d}u/(1/2x^{1/2}) = 2x^{1/2}\mathrm{d}u$$

As $x^{1/2} = u$, you can write $\mathrm{d}x = 2u\mathrm{d}u$. Since this is an indefinite integral, you have no limits to change.

Step 3: Replacing each instance of $u(x) = \sqrt{x}$ with u, and replacing $\mathrm{d}x$ by $2u\mathrm{d}u$ gives

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \,\mathrm{d}x = \int \frac{e^u}{u} \cdot 2u \,\mathrm{d}u = 2 \int e^u \,\mathrm{d}u$$

Since there is no x left in the integral, you are OK to continue.

Step 4: Not forgetting the constant of integration, evaluating the integral using the corresponding rule from Table 2.1 gives

$$2 \int e^u \,\mathrm{d}u = 2e^u + C$$

Step 5: You need to substitute in $\sqrt{x} = u$ into your solution to get $2e^{\sqrt{x}}$. So you can write

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \,\mathrm{d}x = 2e^{\sqrt{x}} + C$$

which is the final answer.

So therefore, you can now integrate the entire expression. The final answer is

$$\int \frac{1}{x} + \frac{e^{\sqrt{x}}}{\sqrt{x}} \,\mathrm{d}x = \int \frac{1}{x} \,\mathrm{d}x + \int \frac{e^{\sqrt{x}}}{\sqrt{x}} \,\mathrm{d}x = \ln|x| + 2e^{\sqrt{x}} + C.$$

Example 3.3.5. Say you are asked to evaluate $\int_0^{\pi/4} x + \cos(2x)^2 \sin(2x) \,\mathrm{d}x$. This is a definite integral with two parts. Similar to the previous example, you can write

$$\int_0^{\pi/4} x + \cos^2(2x) \sin(2x) \,\mathrm{d}x = \underbrace{\int_0^{\pi/4} x \,\mathrm{d}x}_{(1)} + \underbrace{\int_0^{\pi/4} \cos^2(2x) \sin(2x) \,\mathrm{d}x}_{(2)}$$

and work out both integrals one at a time.

So for (1), you can evaluate this using the rules in Table 2.1. Here

$$\int_0^{\pi/4} x \,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^{\pi/4} = \frac{(\pi/4)^2}{2} - \frac{0}{2} = \frac{\pi^2}{32}$$

For part (2) however, you need to use integration by substitution. As this is a definite integral, you must remember to change the limits of integration.

Step 1: You can integrate any function of the form x^n, so a good choice of substitution here would be $u = \cos(2x)$.

Step 2: Here, $u'(x) = -2\sin(2x)$, and so $\mathrm{d}x = \mathrm{d}u/(-2\sin(x))$. As this is a definite integral, you must remember to change the limits. You can do this by evaluating u at $x = 0$ and $x = \pi/4$. So here $u(0) = \cos(0) = 1$ and $u(\pi/4) = \cos(\pi/2) = 0$.

Step 3: By replacing each instance of $\cos(2x)$ with u, the term $\mathrm{d}x$ with $\mathrm{d}u/(-2\sin(x))$, and the lower limit 0 with 1 and the upper limit $\pi/4$ with 0 gives

$$\int_0^{\pi/4} \cos^2(2x)\sin(2x)\,\mathrm{d}x = \int_1^0 \frac{u^2\sin(2x)}{-2\sin(2x)}\,\mathrm{d}u = -\frac{1}{2}\int_1^0 u^2\,\mathrm{d}u$$

As there are no terms involving x left in the integral, you are OK to continue and you can now evaluate the integral.

Step 4: Using Proposition 1.3.1 (6), you can see that

$$-\frac{1}{2}\int_1^0 u^2\,\mathrm{d}u = \frac{1}{2}\int_0^1 u^2\,\mathrm{d}u$$

Evaluating this integral using the corresponding rule from Table 2.1 gives

$$\frac{1}{2}\int_0^1 u^2\,\mathrm{d}u = \frac{1}{2}\left[\frac{u^3}{3}\right]_0^1 = \left[\frac{u^3}{6}\right]_0^1$$

So this means that

$$\left[\frac{u^3}{6}\right]_0^1 = \left[-\frac{1}{6}\right] - [0] = \frac{1}{6}$$

which is the final answer for (2).

Putting the results of (1) and (2) together give

$$\int_0^{\pi/4} x + \cos^2(2x)\sin(2x)\,\mathrm{d}x = \frac{\pi^2}{32} + \frac{1}{6}$$

Remark. In these examples, the tradition has been to substitute $\mathrm{d}x$ with $\mathrm{d}u/u'(x)$ and then cancel through. However, sometimes it is convenient to replace $u'(x)\mathrm{d}x$ with $\mathrm{d}u$. For instance, you could have used this technique in Example 3.3.3 and Example 3.3.5.

Finally in this chapter, you can use integration by substitution in a more general setting, and demonstrate that it works as the 'inverse' of the chain rule of differentiation. Consider the function $y = \ln(h(x))$, where h is some non-zero function of x. Using the chain rule in

Table 3.1 with $f = \ln(x)$ and $g = h(x)$ gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underbrace{\frac{1}{h(x)}}_{f'(g(x))} \cdot h'(x) = \frac{h'(x)}{h(x)}$$

Now, for some non-zero function h, you can think about solving

$$\int \frac{h'(x)}{h(x)} \,\mathrm{d}x$$

You can use integration by substitution to do this, with a choice of substitution $u = h(x)$. This means that $u'(x) = h'(x)$, and so $\mathrm{d}x = \mathrm{d}u/h'(x)$. Replacing $h(x)$ with u and $\mathrm{d}x$ with $\mathrm{d}u/h'(x)$ gives

$$\int \frac{h'(x)}{h(x)} \,\mathrm{d}x = \int \frac{h'(x)}{u \cdot h'(x)} \,\mathrm{d}u = \int \frac{1}{u} \,\mathrm{d}u$$

You can integrate this using the corresponding rule from Table 2.1 to get

$$\int \frac{1}{u} \,\mathrm{d}u = \ln|u| + C$$

Substituting $h(x)$ back in for u gives the following useful identity

$$\boxed{\int \frac{h'(x)}{h(x)} \,\mathrm{d}x = \ln|h(x)| + C} \tag{3.3}$$

You can notice that this reverses the differentiation performed above.

This equation is a valuable tool, as you can 'spot' the patterns of a function in the denominator and its derivative on the top. This will save you time in performing an integration by substitution. For instance, you can consider the indefinite integral

$$\int \frac{2x}{x^2+1} \,\mathrm{d}x$$

Here, you can see that the numerator $h'(x) = 2x$ is precisely the derivative of the denominator $h(x) = x^2 + 1$. Therefore, you can use Equation 3.3 to say that

$$\int \frac{2x}{x^2+1} \,\mathrm{d}x = \ln|x^2+1| + C.$$

Chapter 4

Trigonometry and integration

4.1 Previously in trigonometry...

So far in this portion on integration, you have seen the $\cos$ and the $\sin$ functions... but not much else. You may have seen at different points that there are **four** other trigonometric functions to consider; $\tan, \sec, \csc$ and $\cot$. The definitions of these functions are given in Table 4.1 along with their derivatives. You can find the derivatives on your own by using the chain rule on each of the definitions of the functions.

function	definition
$f(x) = \tan(x)$	$f(x) = \dfrac{\sin(x)}{\cos(x)}$
$f(x) = \sec(x)$	$f(x) = \dfrac{1}{\cos(x)}$
$f(x) = \csc(x)$	$f(x) = \dfrac{1}{\sin(x)}$
$f(x) = \cot(x)$	$f(x) = \dfrac{1}{\tan(x)} = \dfrac{\cos(x)}{\sin(x)}$

Table 4.1: Definitions of $\tan, \sec, \csc$ and $\cot$

You may have studied the derivatives of all of the functions in Table 4.1 in previous parts of the course. Table 4.2 provides the derivatives of the six trigonometric functions you have seen so far, as well as all the known antiderivatives.

function	derivative w.r.t x	$\int \mathrm{d}x$
$f(x) = a\cos(kx)$	$f'(x) = -ak\sin(kx)$	(2.6)
$f(x) = a\sin(kx)$	$f'(x) = ak\cos(kx)$	(2.7)
$f(x) = a\tan(kx)$	$f'(x) = \dfrac{ak}{\cos^2(kx)} = ak\sec^2(kx)$	(4.1)
$f(x) = a\sec(kx)$	$f'(x) = ak\dfrac{\sin(kx)}{\cos^2(kx)} = ak\sec(kx)\tan(kx)$	(6.3)
$f(x) = a\csc(kx)$	$f'(x) = -ak\dfrac{\cos(kx)}{\sin^2(kx)} = -ak\csc(kx)\cot(kx)$	(4.3)
$f(x) = a\cot(kx)$	$f'(x) = -\dfrac{ak}{\sin^2(kx)} = -ak\csc^2(kx)$	(4.2)

Table 4.2: Derivatives and antiderivatives of trigonometric functions

Here, the references given in the antiderivative column refers to the equation in which they are first presented.

The aim of this chapter is for you to expand your knowledge about integration of trigonometric functions, using trigonometric identities to calculate examples, and the use of trigonometric functions in integration by substitution.

To begin with, you should make sure that you know the derivative and antiderivative of both $\sin$ and $\cos$ functions (listed in Table 4.2).

4.2 Some more antiderivatives

This section is designed to find some more antiderivatives that you can use throughout your integration course. A good place to start would be to look at the functions outlined in Table 4.1, and work out antiderivatives for those.

Let's begin with $a\tan(kx)$. Here, you can write

$$\int a\tan(kx)\,\mathrm{d}x = \int a\frac{\sin(kx)}{\cos(kx)}\,\mathrm{d}x.$$

You can work this integral out using integration by substitution.

Here, you can integrate this expression using integration by substitution, with $u = \cos(kx)$. It follows that $u'(x) = -k\sin(kx)$ and so $\mathrm{d}x = \mathrm{d}u/-k\sin(kx)$. Making the substitution gives

$$\int a\frac{\sin(kx)}{\cos(kx)}\,\mathrm{d}x = \int -a\frac{\sin(kx)}{u\cdot k\sin(kx)}\,\mathrm{d}u = \int -\frac{a}{ku}\,\mathrm{d}u = -\frac{a}{k}\int\frac{1}{u}\,\mathrm{d}u$$

You can integrate this expression using Equation 2.4 to get

$$-\frac{a}{k}\int\frac{1}{u}\,\mathrm{d}u = -\frac{a}{k}\ln|u| + C$$

Substituting $u = \cos(kx)$ back into this antiderivative, and using the laws of logarithms together with the fact that $\sec(kx) = \frac{1}{\cos(kx)}$ gives

$$\int a\tan(kx)\,\mathrm{d}x = -\frac{a}{k}\ln|\cos(kx)| + C = \frac{a}{k}\ln\left|\frac{1}{\cos(kx)}\right| + C = \frac{a}{k}\ln|\sec(kx)| + C$$

and so you can write

$$\boxed{\int a\tan(kx)\,\mathrm{d}x = \frac{a}{k}\ln|\sec(kx)| + C} \tag{4.1}$$

There is another way in which you can obtain this result. By Table 4.2, the derivative of $\cos(kx)$ is $-\sin(kx)$, which is nearly equal to the numerator. So here, you can use the fact that $-a/k\cdot -k = a$, together with Proposition 1.3.1 (2) to write that

$$\int a\frac{\sin(kx)}{\cos(kx)}\,\mathrm{d}x = \int(-ak)\frac{\sin(kx)}{-k\cos(kx)}\,\mathrm{d}x = -\frac{a}{k}\int\frac{-k\sin(kx)}{\cos(kx)}\,\mathrm{d}x$$

You can recognise that this integrand is a fraction with a function $h(x)$ on the bottom and its derivative $h'(x)$ on the top. Therefore, you can use Equation 3.3 to write

$$-\frac{a}{k}\int\frac{-k\sin(kx)}{\cos(kx)}\,\mathrm{d}x = -\frac{a}{k}\ln|\cos(kx)| + C = \frac{a}{k}\ln|\sec(kx)| + C$$

and recover the result from Equation 4.1.

You can use a similar method to work out the antiderivative of $a\cot(kx)$. Here, you can use Table 4.1 to write that

$$a\cot(kx) = a\frac{\cos(kx)}{\sin(kx)}$$

You can integrate this by substitution, using $u = \sin(kx)$ as your choice of substitution.

This means that $u'(x) = k\cos(kx)$ and so $\mathrm{d}x = \mathrm{d}u/k\cos(kx)$. Substituting these in and cancelling gives

$$\int a\frac{\cos(kx)}{\sin(kx)}\,\mathrm{d}x = \int a\frac{\cos(kx)}{u \cdot k\cos(kx)}\,\mathrm{d}u = \frac{a}{k}\int \frac{1}{u}\,\mathrm{d}u$$

You can integrate this using Equation 2.4 to write

$$\frac{a}{k}\int \frac{1}{u}\,\mathrm{d}u = \frac{a}{k}\ln|u| + C$$

Finally, you can substitute $u = \sin(kx)$ into this to write

$$\boxed{\int a\cot(kx)\,\mathrm{d}x = \frac{a}{k}\ln|\sin(kx)| + C} \tag{4.2}$$

Remark. While the antiderivatives of $\tan$ and $\cot$ have been worked out with techniques that you have already studied up to this point, finding the indefinite integrals of $\sec(x)$ and $\csc(x)$ with respect to x need some more work.

You can use the Fundamental Theorem of Calculus (Theorem 2.2.2) to state the antiderivatives for those functions given in the second column of Table 4.2. These results are collected in Table 4.3. You can check these antiderivatives by differentiating them and getting $f(x)$ as an answer.

function	antiderivative w.r.t x
$f(x) = a\sec^2(kx)$	$\int f(x)\,\mathrm{d}x = \frac{a}{k}\tan(kx) + C$
$f(x) = a\sec(kx)\tan(kx)$	$\int f(x)\,\mathrm{d}x = \frac{a}{k}\sec(kx) + C$
$f(x) = a\csc(kx)\cot(kx)$	$\int f(x)\,\mathrm{d}x = -\frac{a}{k}\csc(kx) + C$
$f(x) = a\csc^2(kx)$	$\int f(x)\,\mathrm{d}x = -\frac{a}{k}\cot(kx) + C$

Table 4.3: Antiderivatives of some derivatives found in Table 4.2

Example 4.2.1. You are asked to integrate

$$\int -4\tan(2x) + 4\sec^2(x)\,\mathrm{d}x$$

Here, you can use Proposition 1.3.1 (3) to say that

$$\int -4\tan(2x) + 4\sec^2(x)\,\mathrm{d}x = \int -4\tan(2x)\,\mathrm{d}x + \int 4\sec^2(x)\,\mathrm{d}x$$

Using Equation 4.1 with $a = -4$ and $k = 2$ for the first integral, and using the result in Table 4.3 with $a = 4$ and $k = 1$ for the second integral, you can write

$$\begin{aligned}\int -4\tan(2x) + 4\sec^2(x)\,\mathrm{d}x =& -\frac{-4}{2}\ln|\sec(2x)| + \frac{4}{1}\tan(x) + C\\ =& 2\ln|\sec(2x)| + 4\tan(x) + C\\ =& \ln|\sec^2(2x)| + 4\tan(x) + C\end{aligned}$$

4.3 Using trigonometric identities

In your mathematical career, you may have come across a range of identities that relate two trigonometric functions together. These trigonometric identities prove to be very useful in integrating some functions that cannot be integrated directly or by substitution. Techniques to do this are discussed in the next two sections.

The first set of these pairs together trigonometric functions in the **square identities** (or **Pythagorean identities**; see Table 4.4):

functions	identity
$\sin,\ \cos$	$\cos^2(x) + \sin^2(x) = 1$
$\tan,\ \sec$	$1 + \tan^2(x) = \sec^2(x)$
$\cot,\ \csc$	$\cot^2(x) + 1 = \csc^2(x)$

Table 4.4: Square identities of trigonometric functions

These are so called because you can derive the first of these from Pythagoras' Theorem, and the second and third of these identities from the first by dividing through by $\cos^2(x)$ and $\sin^2(x)$ respectively.

The second set of these are known as **angle sum rules**, and are particularly useful for dealing with powers of trigonometric functions. These are very useful when you take the angles A and B to be the same angle, giving the related **double angle formulas** (see Table 4.5):

Sum rule	double angle formula
$\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$	$\sin(2A) = 2\sin(A)\cos(A)$
$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$	$\cos(2A) = \cos^2(A) - \sin^2(A)$
$\tan(A \pm B) = \dfrac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$	$\tan(2A) = \dfrac{2\tan(A)}{1 - \tan^2(A)}$

Table 4.5: Sum rules of trigonometric functions

You can choose the angles A and B carefully in these rules in order to find expressions to help you solve integrals. For instance, taking $A = 2x$ and $B = x$ in the sum rules yield the **triple angle formulas**. Here, for instance, is the derivation of the formula for $\cos^3(x)$, using the double angle formulas and the fact that $\sin^2(x) = 1 - \cos^2(x)$:

$$
\begin{aligned}
\cos(3x) = \cos(2x + x) &= \cos(2x)\cos(x) - \sin(2x)\sin(x) \\
&= (\cos^2(x) - \sin^2(x))\cos(x) - (2\sin(x)\cos(x))\sin(x) \\
&= \cos^3(x) - \sin^2(x)\cos(x) - 2\sin^2(x)\cos(x) \\
&= \cos^3(x) - 3\sin^2(x)\cos(x) \\
&= \cos^3(x) - 3(1 - \cos^2(x))\cos(x) \\
&= 4\cos^3(x) - 3\cos(x). \qquad (*)
\end{aligned}
$$

You should know, and be confident with manipulating the identities in Tables 4.4 and 4.5 before continuing.

4.3.1 Examples

Example 4.3.1. You are asked to find $\int_0^{\pi/4} \tan^2(x)\,\mathrm{d}x$. You can notice that you cannot integrate this using substitution, as it is not of the form $f(u(x)) \cdot u'(x)$ for some function $u(x)$. However, you can see that you can write down the antiderivative of $\sec^2(x)$ from Table 4.3, and that $\tan^2(x) = \sec^2(x) - 1$ from Table 4.4. This means you can write

$$\int_0^{\pi/4} \tan^2(x)\,\mathrm{d}x = \int_0^{\pi/4} \sec^2(x) - 1\,\mathrm{d}x$$

You can integrate this to get

$$\begin{aligned}\int_0^{\pi/4} \sec^2(x) - 1\,\mathrm{d}x &= [\tan(x) - x]_0^{\pi/4} \\ &= \left(\tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4}\right) - (0 + 0) \\ &= 1 - \frac{\pi}{4}\end{aligned}$$

Example 4.3.2. You are asked to find $\int \sin^2(x)\,\mathrm{d}x$. Again, you cannot integrate this using substitution, and this is not something you can write down the antiderivative for. Here, you need to use trigonometric identities to find a way to write $\sin^2(x)$ in a form you know you can integrate.

Here, you can notice that $\sin^2(x)$ appears in the double angle formula for $\cos(2x)$, which is $\cos^2(x) - \sin^2(x)$. Furthermore, $\sin^2(x)$ appears in the square identity $\cos^2(x) + \sin^2(x) = 1$, which you can rewrite as $\cos^2(x) = 1 - \sin^2(x)$. Combining these gives

$$\begin{aligned}\cos(2x) &= \cos^2(x) - \sin^2(x) \\ &= (1 - \sin^2(x)) - \sin^2(x) \\ &= 1 - 2\sin^2(x)\end{aligned}$$

Rearranging to make $\sin^2(x)$ the subject gives

$$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

and the right hand side of this equation is something that you can integrate. So now, you can write

$$\int \sin^2(x)\,\mathrm{d}x = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\,\mathrm{d}x$$

and integrate using Equation 2.3 and Equation 2.6 to get

$$\int \sin^2(x)\,\mathrm{d}x = \int \frac{1}{2} - \frac{1}{2}\cos(2x)\,\mathrm{d}x = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

Example 4.3.3. Now, you are asked to find $\int 2\cos^3(x)\,\mathrm{d}x$. Once again, you cannot use substitution or integrate directly to find the answer to this problem. This integrand is a little harder to represent as something you can integrate directly as there is no trigonometric identity in Tables 4.4 and 4.5 to use with $\cos^3(x)$ in it. However, at the start of the section, it was demonstrated in Table * that

$$\cos(3x) = 4\cos^3(x) - 3\cos(x)$$

You can rearrange this to say that

$$\cos^3(x) = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)$$

which is something that you can integrate. Replacing the integrand gives

$$\int 2\cos^3(x)\,\mathrm{d}x = \int \frac{1}{2}\cos(3x) + \frac{3}{2}\cos(x)\,\mathrm{d}x$$

and you can integrate this using Equation 2.6 to get

$$\int 2\cos^3(x)\,\mathrm{d}x = \int \frac{1}{2}\cos(3x) + \frac{3}{2}\cos(x)\,\mathrm{d}x = \frac{1}{6}\sin(3x) + \frac{3}{2}\sin(x) + C$$

Finally in this section, nou can use trigonometric identities, together with some previous results, to prove one of the final two antiderivatives of the functions from Table 4.1 not yet considered.

Example 4.3.4. You are asked to find $\int a\csc(kx)\,\mathrm{d}x$. You can use Table 4.1 and Proposition 1.3.1 (2) to write the integral as

$$\int a\csc(kx)\,\mathrm{d}x = \int \frac{a}{\sin(kx)}\,\mathrm{d}x$$

Now, you can use the sum rule for $\sin$ in Table 4.5 with $A = B = kx/2$ to write that

$$\frac{1}{\sin(kx)} = \frac{1}{2\sin(kx/2)\cos(kx/2)}$$

Next, you can use the fact that $1 = \cos^2(kx/2) + \sin^2(kx/2)$, and cancelling gives to write

$$\frac{1}{2\sin(kx/2)\cos(kx/2)} = \frac{\cos^2(kx/2) + \sin^2(kx/2)}{2\sin(kx/2)\cos(kx/2)} = \frac{\cos(kx/2)}{2\sin(kx/2)} + \frac{\sin(kx/2)}{2\cos(kx/2)}$$

Finally, you can use Table 4.1 to write that

$$\frac{1}{\sin(kx)} = \frac{1}{2}\cot(kx/2) + \frac{1}{2}\tan(kx/2)$$

You know how to integrate $\cot$ and $\tan$ by Equations (4.2) and (4.1) respectively. So replacing the integrand gives

$$\int \frac{a}{\sin(kx)}\,\mathrm{d}x = \int \frac{a}{2}\cot(kx/2) + \frac{a}{2}\tan(kx/2)\,\mathrm{d}x$$

and integrating using Equation 4.2 and Equation 4.1 gives

$$\int \frac{a}{2}\cot(kx/2) + \frac{a}{2}\tan(kx/2)\,\mathrm{d}x = \frac{2a}{2k}\ln|\sin(kx/2)| + \frac{2a}{2k}\ln|\sec(kx/2)| + C$$

Cancelling the 2's in each term, and using the laws of logarithms together with the fact that $\sin(kx)\sec(kx) = \tan(kx)$ gives:

$$\boxed{\int a\csc(kx)\,\mathrm{d}x = \frac{a}{k}\ln|\tan(kx/2)| + C} \tag{4.3}$$

Remark. This technique does not work in the case of integrating $a\sec(kx)$. This is because using the double angle formula on $\cos(kx)$ in the same fashion as in Example 4.3.4 causes the denominator to become $\cos^2(kx/2) - \sin^2(kx/2)$; an expression that you cannot cancel. The technique to integrate $a\sec(kx)$ will be explored later.

4.4 Inverse functions: a different kind of substitution

Every trigonometric function has an inverse. For instance, the inverse of $\sin(x)$ is $\sin^{-1}(x)$. You may have seen in the differentiation part of the course that you can use implicit differentiation to find derivatives of these functions. A list of inverse trigonometric functions and their derivatives are given in Table 4.6; this means that you can find antiderivatives for each of these derivatives by the Fundamental Theorem of Calculus.

However, sometimes the integral you are considering does not come in the same form as

function	derivative w.r.t x	$\int \mathrm{d}x$
$f(x) = a\cos^{-1}(kx)$	$f'(x) = \dfrac{-ak}{\sqrt{1-k^2x^2}}$	
$f(x) = a\sin^{-1}(kx)$	$f'(x) = \dfrac{ak}{\sqrt{1-k^2x^2}}$	
$f(x) = a\tan^{-1}(kx)$	$f'(x) = \dfrac{ak}{1+k^2x^2}$	
$f(x) = a\sec^{-1}(kx)$	$f'(x) = \dfrac{a}{kx^2\sqrt{1-(kx)^{-2}}}$	
$f(x) = a\csc^{-1}(kx)$	$f'(x) = \dfrac{-a}{kx^2\sqrt{1-(kx)^{-2}}}$	
$f(x) = a\cot^{-1}(kx)$	$f'(x) = \dfrac{-ak}{1+k^2x^2}$	

Table 4.6: Derivatives and antiderivatives of inverse trigonometric functions

one of the derivatives in Table 4.6. For instance, how would you go about integrating $\int \frac{4}{4+9x^2}\,\mathrm{d}x$? Or integrating something like $\int x^2\sqrt{x^2-16}\,\mathrm{d}x$, which looks nothing like one of the derivatives above and can't be integrated by substitution? You can notice that the parts of the integrals that look like

$$\sqrt{x^2-a^2}, \quad \sqrt{a^2-x^2}, \quad a^2+x^2$$

can be rearranged to look like constant multiples of the square identities given in Table 4.4. To make use of these identities, you need to make a **trigonometric substitution** in terms of some angle θ.

This type of substitution is different to the integration by substitution that you already know. This is because instead of substituting u as a function of x, you are having to **substitute x with a function of some variable** θ. So here, your substitution will look something like $x = g(\theta)$, where g is some trigonometric function of θ.

For instance, what substitution could you consider for a function that contains $\sqrt{a^2-x^2}$? By substituting $x = a\sin(\theta)$, and knowing that $1-\sin^2(\theta) = \cos^2(\theta)$ from Table 4.4, you can see that

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2(\theta)}=\sqrt{a^2(1-\sin^2(\theta))}=a\cos(\theta)$$

Similarly, for functions that contain x^2+a^2, you could consider the substitution $a\tan(\theta)$. Then, using the fact that $1+\tan^2(\theta)=\sec^2(\theta)$, you can write

$$x^2+a^2=a^2\tan^2(\theta)+a^2=a^2(\tan^2(\theta)+1)=a^2\sec^2(\theta)$$

Finally, for functions that contain $\sqrt{x^2-a^2}$ you could consider the substitution $a\sec(\theta)$. Then, using the identity $\tan^2(\theta)=\sec^2(\theta)-1$, you can say that

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2(\theta)-a^2}=\sqrt{a^2(\sec^2(\theta)-1)}=a\tan(\theta)$$

Warning. Of course, there are functions containing these types of expressions that do not require a trigonometric substitution. For instance,

$$\int x^2+a^2\,\mathrm{d}x=\frac{x^3}{3}+a^2x+C$$

Some integrands containing these types of expression can be solved using a conventional substitution rather than a trigonometric one. For instance, using the substitution $u=x^2-16$ gives that

$$\int x\sqrt{x^2-16}\,\mathrm{d}x=\frac{(x^2-16)^{3/2}}{3}+C$$

Therefore, you should only use a trigonometric substitution unless you really have to! This is usually when one of the three expressions above is in the **denominator** of the integrand.

The choice of substitution is not the only thing you need to consider before making a trigonometric substitution. Because you are considering a function of θ, this has the effect of changing how you handle the $\mathrm{d}x$ term in the integral. Here, the correct substitution for $\mathrm{d}x$ is $\mathrm{d}x=g'(\theta)\mathrm{d}\theta$.

Furthermore, if you are considering a definite integral, you need to change the limits; again, this is slightly different to the usual process. As your limits a and b will be in terms of x, you need to change this into terms of θ. As $x=g(\theta)$, it follows that $g^{-1}(x)=\theta$, where g^{-1} is the inverse trigonometric function of g. This means that you should replace a with $g^{-1}(a)$ and replace b with $g^{-1}(b)$.

This leads to a modified method for integration via a trigonometric substitution. The principle is the same, with a few key differences.

4.4.1 Method for integration by trigonometric substitution

Step 1: Choose a suitable $x = g(\theta)$.

Step 2: Work out $g'(\theta)$ (the derivative of $g(\theta)$ with respect to θ, and write down the expression $\mathrm{d}x = g'(\theta)\mathrm{d}\theta$. If you are considering a definite integral, work out an expression for θ in terms of x (usually, this is the inverse function $g^{-1}(x) = \theta$), and then evaluate this expression at a and b where a and b are the limits of the integral.

Step 3: Now, you should

- replace every instance of x with the function $g(\theta)$
- replace $\mathrm{d}x$ with $g'(\theta)\mathrm{d}\theta$, and cancel if you need to;
- (for definite integrals only) replace a with the value $g^{-1}(a)$ and b with $g^{-1}(b)$.

Warning: At this stage, the integral should be solely in terms of θ. If there are still terms containing x at this stage, stop and consider another choice of $g(\theta)$.

Step 4: If you can, work out the integral. If you are considering a definite integral, then the method stops here with the answer. if you are considering an indefinite integral, don't forget the $+C$!

Step 5: (For indefinite integrals only) Your antiderivative should be in terms of θ. You can replace every instance of θ with the inverse function $g^{-1}(x)$, or find an expression for the antiderivative $h(\theta)$ in terms of x if the alternative is too difficult (see Example 4.4.2). The method stops here for indefinite integrals.

If you do require a trigonometric substitution, then the information about the choice you could make is summarised in Table 4.7. The reason why these substitutions are suggested is the fact that what results after the substitution $x = g(\theta)$ looks like a constant multiple of the derivative $g'(\theta)$ of $g(\theta)$.

4.4.2 Examples

Example 4.4.1. You are asked to find the integral $\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x$.

function contains...	suggested substitution	expression for $\mathrm{d}x$
$\sqrt{a^2 - x^2}$	$g(\theta) = a\sin(\theta)$	$\mathrm{d}x = a\cos(\theta)\mathrm{d}\theta$
$\sqrt{x^2 - a^2}$	$g(\theta) = a\sec(\theta)$	$\mathrm{d}x = a\sec(\theta)\tan(\theta)\mathrm{d}\theta$
$a^2 + x^2$	$g(\theta) = a\tan(\theta)$	$\mathrm{d}x = a\sec^2(\theta)\mathrm{d}\theta$

Table 4.7: Suggested trigonometric substitutions

Since this is a definite integral, you must remember to change the limits of integration, but you don't have to do Step 5 of the method in Subsection 4.4.1.

Step 1: Following the advice of Table 4.7, and the fact that $2^2 = 4$, a good choice of substitution here would be $x = g(\theta) = 2\sin(\theta)$.

Step 2: Here, $g'(\theta) = 2\cos(\theta)$, and so $\mathrm{d}x = 2\cos(\theta)\mathrm{d}\theta$. As this is a definite integral, you must remember to change the limits. You can do this by evaluating $g^{-1}(x)$ at $x = 1$ and $x = \sqrt{2}$. Now, as $x = 2\sin(\theta)$, this means that $\theta = \sin^{-1}(x/2) = g^{-1}(x)$. This means that

$$g^{-1}(1) = \sin^{-1}(1/2) = \pi/6$$

and

$$g^{-1}(\sqrt{2}) = \sin^{-1}(\sqrt{2}/2) = \pi/4$$

Step 3: Replacing each instance of x with $2\sin(\theta)$, replacing $\mathrm{d}x$ by $2\cos(\theta)\mathrm{d}\theta$, and replacing the lower limit 1 with $\pi/6$ and the upper limit $\sqrt{2}$ with $\pi/4$ gives

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x = \int_{\pi/6}^{\pi/4} \frac{2\cos(\theta)}{(2\sin(\theta))^2\sqrt{4-(2\sin(\theta))^2}}\,\mathrm{d}\theta$$

Using the fact that $\sqrt{4-(2\sin(\theta))^2} = 2\cos(\theta)$ you can simplify to get

$$\begin{aligned}\int_{\pi/6}^{\pi/4} \frac{2\cos(\theta)}{(2\sin(\theta))^2\sqrt{4-(2\sin(\theta))^2}}\,\mathrm{d}u &= \int_{\pi/6}^{\pi/4} \frac{2\cos(\theta)}{4\sin^2(\theta)\cdot 2\cos(\theta)}\,\mathrm{d}\theta \\ &= \int_{\pi/6}^{\pi/4} \frac{1}{4\sin^2(\theta)}\,\mathrm{d}\theta \\ &= \frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2(\theta)\,\mathrm{d}\theta\end{aligned}$$

As there are no terms involving x left in the integral, you are OK to continue and you can now evaluate the integral.

Step 4: You can evaluate the integral using the corresponding rule from Table 4.3, which gives

$$\frac{1}{4}\int_{\pi/6}^{\pi/4} \csc^2(\theta)\,\mathrm{d}\theta = \frac{1}{4}\left[-\cot(\theta)\right]_{\pi/6}^{\pi/4}$$

Using the fact that $\cot(\pi/6) = \sqrt{3}$ and $\cot(\pi/4) = 1$, you can evaluate this to get

$$\frac{1}{4}\left[-\cot(\theta)\right]_{\pi/6}^{\pi/4} = \frac{1}{4}\left[-1-(-\sqrt{3})\right] = \frac{\sqrt{3}-1}{4}$$

which is the final answer.

Example 4.4.2. You are asked to find the integral $\int \frac{4}{(x^2+9)^{3/2}}\,\mathrm{d}x$.

Since this is an indefinite integral, you have no limits to change but you must remember to do Step 5 of the method in Subsection 4.4.1.

Step 1: Following the advice of Table 4.7, and the fact that $3^2 = 9$, a good choice of substitution here would be $x = g(\theta) = 3\tan(\theta)$.

Step 2: Here, $g'(\theta) = 3\sec^2(\theta)$, and so $\mathrm{d}x = 3\sec^2(\theta)\mathrm{d}\theta$. As this is an indefinite integral, you have no limits to change.

Step 3: Replacing each instance of x with $3\tan(\theta)$, and replacing $\mathrm{d}x$ by $3\sec^2(\theta)\mathrm{d}\theta$ gives

$$\int \frac{4}{(x^2+9)^{3/2}}\,\mathrm{d}x = 4\int \frac{3\sec^2(\theta)}{(9\tan^2(\theta)+9)^{3/2}}\,\mathrm{d}\theta$$

You can use the fact that $\tan^2(x) + 1 = \sec^2(x)$ to simplify the integral to:

$$\begin{aligned}4\int \frac{3\sec^2(\theta)}{(9\tan^2(\theta)+9)^{3/2}}\,\mathrm{d}\theta =& 12\int \frac{\sec^2(\theta)}{9^{3/2}\cdot(\sec^2(\theta))^{3/2}}\,\mathrm{d}\theta \\ =& \frac{12}{27}\int \frac{1}{(\sec^2(\theta))^{1/2}}\,\mathrm{d}\theta \\ =& \frac{4}{9}\int \frac{1}{\sec(\theta)}\,\mathrm{d}\theta = \frac{4}{9}\int \cos(\theta)\,\mathrm{d}\theta\end{aligned}$$

Since there is no x left in the integral, you are OK to continue.

Step 4: Not forgetting the constant of integration, evaluating the integral using Equation 2.6 gives

$$\frac{4}{9}\int \cos(\theta)\,\mathrm{d}\theta = \frac{4}{9}\sin(\theta) + C \qquad (*)$$

Step 5: You now need to express your answer in terms of the original variable x. It is not wise to use the inverse function $g^{-1}(x) = \theta$ here, as although it is correct, putting this expression into a $\sin$ function will look less than pleasing. Here, you can use the fact that $x = 3\tan(\theta)$ to write that

$$3\sin(\theta) = x \cdot \cos(\theta)$$

Now, you can square this expression to get that

$$9\sin^2(\theta) = x^2\cos^2(\theta)$$

Using the fact that $\cos^2(\theta) = 1 - \sin^2(\theta)$, you can write that

$$9\sin^2(\theta) = x^2(1 - \sin^2(\theta))$$

Therefore, as $9\sin^2(\theta) = x^2 - x^2\sin^2(\theta)$, you can write that $(9 + x^2)\sin^2(\theta) = x^2$ and so

$$\sin^2(\theta) = \frac{x^2}{9 + x^2}$$

You can square root this equation to get $\sin(\theta) = x/\sqrt{9 + x^2}$ and substitute this in to the antiderivative in Equation (*) to say that

$$\int \frac{4}{(x^2 + 9)^{3/2}}\,\mathrm{d}x = \frac{4x}{9\sqrt{9 + x^2}} + C$$

and this is your final answer.

Finally, you can use trigonometric substitution to find expressions for antiderivatives for the functions

$$\frac{a}{\sqrt{b^2 - x^2}}, \quad \frac{a}{x\sqrt{x^2 - b^2}}, \quad \frac{a}{x^2 + b^2}$$

For an example, you can work out the antiderivative of $\frac{a}{x\sqrt{x^2-b^2}}$ by using trigonometric substitution.

Step 1: Following the advice of Table 4.7, a good choice of substitution here would be $x = g(\theta) = b\sec(\theta)$.

Step 2: Here, $g'(\theta) = b\sec(\theta)\tan(\theta)$ by Table 4.2, and so $\mathrm{d}x = b\sec(\theta)\tan(\theta)\mathrm{d}\theta$. As this is an indefinite integral, you have no limits to change.

Step 3: Replacing each instance of x with $b\sec(\theta)$, and replacing $\mathrm{d}x$ by $b\sec(\theta)\tan(\theta)\mathrm{d}\theta$ gives

$$\int \frac{a}{x\sqrt{x^2-b^2}}\,\mathrm{d}x = a\int \frac{b\sec(\theta)\tan(\theta)}{b\sec(\theta)\sqrt{b^2\sec^2(\theta)-b^2}}\,\mathrm{d}\theta$$

You can use the fact that $\sec^2(x)-1=\tan^2(x)$ to simplify the integral to:

$$\begin{aligned} a\int \frac{b\sec(\theta)\tan(\theta)}{b\sec(\theta)\sqrt{b^2\sec^2(\theta)-b^2}}\,\mathrm{d}\theta =& a\int \frac{\tan(\theta)}{\sqrt{b^2\tan^2(\theta)}}\,\mathrm{d}\theta \\ =& \frac{a}{b}\int \frac{\tan(\theta)}{\tan(\theta)}\,\mathrm{d}\theta \\ =& \frac{a}{b}\int \mathrm{d}\theta \end{aligned}$$

Since there is no x left in the integral, you are OK to continue.

Step 4: Not forgetting the constant of integration, evaluating the integral using Equation 2.3 gives

$$\frac{a}{b}\int \mathrm{d}\theta = \frac{a}{b}\theta + C$$

Step 5: You now need to express your answer in terms of the original variable x. Here, as you only have to find an expression for θ, finding $\theta = g^{-1}(x)$ will do. Here, you can use the fact that $x = b\sec(\theta)$ to write that

$$\theta = \sec^{-1}(x/b)$$

And so

$$\boxed{\int \frac{a}{x\sqrt{x^2-b^2}}\,\mathrm{d}x = \frac{a}{b}\sec^{-1}\left(\frac{x}{b}\right) + C} \tag{4.4}$$

You can use similar techniques to show that

$$\boxed{\int \frac{a}{\sqrt{b^2-x^2}}\,\mathrm{d}x = a\sin^{-1}\left(\frac{x}{b}\right) + C} \tag{4.5}$$

and

$$\boxed{\int \frac{a}{b^2+x^2}\,\mathrm{d}x = \frac{a}{b}\tan^{-1}\left(\frac{x}{b}\right) + C} \tag{4.6}$$

These expressions could also be found by using the Fundamental Theorem of Calculus on the derivatives in Table 4.6. Finally in this chapter, the information is summed up in Table 4.8.

function	antiderivative w.r.t x
$f(x) = \dfrac{a}{\sqrt{b^2 - x^2}}$	$\displaystyle\int f(x)\,\mathrm{d}x = a\sin^{-1}\left(\frac{x}{b}\right) + C$
$f(x) = \dfrac{a}{b^2 + x^2}$	$\displaystyle\int f(x)\,\mathrm{d}x = \frac{a}{b}\tan^{-1}\left(\frac{x}{b}\right) + C$
$f(x) = \dfrac{a}{x\sqrt{x^2 - b^2}}$	$\displaystyle\int f(x)\,\mathrm{d}x = -\frac{a}{b}\sec^{-1}\left(\frac{x}{b}\right) + C$

Table 4.8: Summary of Equations (4.4), (4.5), and (4.6)

Chapter 5

Integration by parts

5.1 What is integration by parts?

Say you are given the integration

$$\int xe^x \, \mathrm{d}x$$

How would you go about evaluating it? You can't use Equation 2.5 as x is not a constant. Since the derivative of x is 1, you can't use integration by substitution as x is not a constant multiple of 1. This means a new approach has to be developed.

You can notice here that the integrand is the product of two functions, $u(x) = x$ and $v(x) = e^x$. The deep connection between differentiation and integration implies that evaluating this integral may be related to the product rule.

Chapter 3 asserted that integration by substitution is the integral version of the chain rule. In that chapter, it was mentioned that the integral version of the product rule of differentiation (see Table 3.1) is known as **integration by parts**. This is the second main technique of finding antiderivatives of functions, and is the technique to use on $\int xe^x \, \mathrm{d}x$.

This chapter is dedicated to the technique of integration by parts: where it comes from, how to use it, and several examples detailing different ways to use it.

5.2 How integration by parts works

Let $u(x)$ and $v(x)$ be functions of x, and say that $f(x) = u(x)v(x)$, the product of $u(x)$ and $v(x)$. Table 3.1 says that the derivative of f with respect to x is given by the **product**

rule, which is

$$f'(x) = u(x)v'(x) + v(x)u'(x)$$

To save time, you can write $u = u(x)$, $u' = u'(x)$, $v = v(x)$ and $v' = v'(x)$, and so the product rule becomes

$$f'(x) = uv' + vu'$$

You can integrate both sides of this equation to say that

$$\int f'(x)\,\mathrm{d}x = \int uv'\,\mathrm{d}x + \int vu'\,\mathrm{d}x$$

By the Fundamental Theorem of Calculus (Theorem 2.2.2), it follows that

$$\int f'(x)\,\mathrm{d}x = f(x) = u(x)v(x) = uv$$

and so

$$uv = \int uv'\,\mathrm{d}x + \int vu'\,\mathrm{d}x$$

You can rearrange this to get

$$\boxed{\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x} \tag{5.1}$$

This equation is the principle of **integration by parts**.

When you do integration by parts, you can see that you are left with an integral $\int vu'\,\mathrm{d}x$ of a product of two functions; the general idea is that this integral is somehow "easier" to solve than $\int uv'\,\mathrm{d}x$.

If you cannot integrate this immediately, you can use either integration by substitution or integration by parts **again** until you find an integral you know you can solve. You can repeat this as many times as you need in order to find an integral you can solve. Alternatively, the remaining integral $\int vu'\,\mathrm{d}x$ may be a constant multiple of your original integral $\int uv'\,\mathrm{d}x$, in which case you can rearrange to find an expression for $\int uv'\,\mathrm{d}x$ in terms of anything you have found.

If you are evaluating a definite integral using integration by parts, you should evaluate the limits at the very end of your calculations of the antiderivative. This saves any confusion in the middle of your working.

5.2.1 How to use integration by parts effectively

The idea is to take the product function you are asked to evaluate, name one of the functions u and the other v', and then work through the formula to find $uv - \int vu' \, \mathrm{d}x$, where hopefully the integral $\int vu' \, \mathrm{d}x$ is easier to integrate than $\int uv' \, \mathrm{d}x$.

The difficulty in integration by parts is in the naming of the functions at the start; that is, which function u to differentiate and which one v' to integrate. As is the case with integration by substitution, there is no general rule for choosing the functions $u(x)$ and $v'(x)$. However, here are a few hints you can use when doing integration by parts.

- If the function you are trying to integrate is of the form $x^n g(x)$ (for $n \geq 1$), the idea is to take $u = x^n$ and $v' = g(x)$ (as in Examples 5.2.1, 5.3.1 and 5.3.2). This way, the integral $\int vu' \, \mathrm{d}x = \int x^{n-1} v \, \mathrm{d}x$, which is theoretically easier to evaluate than $\int uv' \, \mathrm{d}x$. There are exceptions to this case; particularly where you cannot integrate $v(x)$ easily (see Example 5.3.3). Sometimes, it is not easy to decide which of the functions you can integrate; however, integrating by parts can often yield a solution (see Example 5.3.5).

- Sometimes, you may be confronted with a function $f(x)$ that you cannot integrate at all, but you are able to differentiate it without much difficulty. In this case, you can try integrating by parts with $u = f(x)$ and $v' = 1$; this may make the integral easier to evaluate (see Example 5.3.4).

- If you find that you do not know what to do with the integral $\int vu' \, \mathrm{d}x$, or that it looks 'more complicated' than your original integral $\int uv' \, \mathrm{d}x$, it is best to stop immediately, and change your choices of u and v'. If this does not make the integration simpler, try another technique.

As with integration by substitution, there is a method that you can use to evaluate an integral using integration by parts.

Step 1: Identify your choice of $u(x) = u$ and $v'(x) = v'$ in the integrand.

Step 2: Work out u' by differentiating u with respect to x, and work out v by integrating v' with respect to x (you do not need a $+C$ for this integral).

Step 3: Write down

$$\int uv' \, \mathrm{d}x = uv - \int vu' \, \mathrm{d}x$$

with your values for u, u', v and v'. Simplify if you can.

Step 4: There are a number of cases for $\int vu' \, \mathrm{d}x$, which are listed below:

Case 1: **You can integrate $\int vu' \, \mathrm{d}x$ directly or by substitution:** In this case, do what you need to do to find $\int vu' \, \mathrm{d}x$. You can then write your answer and proceed to Step 5 (see Example 5.2.1).

Case 2: **You can integrate $\int vu' \, \mathrm{d}x$ by parts:** In this case, you need to start the process over with $\int vu' \, \mathrm{d}x$, from Step 1. Here, using different letters for these functions (such as f, g or a, b) at this stage is highly recommended. You can repeat this case as many times as you need (see Example 5.3.2).

(You will find that if you have to repeat integration by parts more than once, you will have a lot of minus signs in your working. **Take care to make sure that your signs are correct!**)

Case 3: **Some part of the integral $\int vu' \, \mathrm{d}x$ is a constant multiple of $\int uv' \, \mathrm{d}x$:** In this case, you can write

$$K \int uv' \, \mathrm{d} = [uv + \cdots]$$

where the number of terms on the left hand side corresponds to how many times you needed to integrate by parts. You can now proceed to Step 5 (see Example 5.3.5 and Example 5.3.6).

Case 4: **After trying everything, you do not know what to do with $\int vu' \, \mathrm{d}x$:** In this case, you can try reversing your choices of u and v', or use a different integration technique.

Remark. Sometimes, you are asked to do integration by parts on a function with an unknown power n, in order to find a **reduction formula**. In this case (and this case alone), you can leave an integral sign on the right hand side of the equation, almost always after an occurrence of Case 3.

Before continuing to Step 5 you should ensure that in any of these cases (apart from when you are finding a reduction formula), your answer on the right hand side **does not have an integral sign in it**.

Step 5: If you are finding an indefinite integral, you should add the constant C on the right hand side. If you are evaluating a definite integral between limits a and b, you should evaluate this integral as normal.

To illustrate this method, you can use integration by parts on the following example.

Example 5.2.1. Suppose you are asked to find the integral $\int_0^1 xe^x \,\mathrm{d}x$. This is a common example of an integral you can solve by integration by parts. This is a definite integral, so you should aim to find the complete antiderivative in Step 4 before evaluating the limits in Step 5.

Step 1: Here, you can follow the advice at the beginning of Subsection 5.2.1 and take $u = x$ and $v' = e^x$.

Step 2: Now, differentiating u with respect to x gives $u' = 1$. Integrating $v' = e^x$ with respect to x gives $v = e^x$; remember that you do not need the $+C$ here.

Step 3: You can now write that

$$\begin{aligned}\int_0^1 xe^x \,\mathrm{d}x &= \left[x \cdot e^x - \int 1 \cdot e^x \,\mathrm{d}x\right]_0^1 \\ &= \left[xe^x - \int e^x \,\mathrm{d}x\right]_0^1\end{aligned}$$

Step 4: You can integrate $\int e^x \,\mathrm{d}x$ directly (Case 1). Integrating this gives

$$\int e^x \,\mathrm{d}x = e^x$$

Substituting this into the above equation gives

$$\int_0^1 xe^x \,\mathrm{d}x = [xe^x - e^x]_0^1$$

which you can now evaluate as it does not have an integral sign in it.

Step 5: Here,

$$\begin{aligned}[x \cdot e^x - e^x]_0^1 =&(1 \cdot e^1 - e^1) - (0 \cdot e^0 - e^0) \\ =&1\end{aligned}$$

Therefore you can write

$$\int_0^1 xe^x \,\mathrm{d}x = 1$$

and this is your final answer.

Remark. Suppose that in Step 1 in Example 5.2.1 you picked $u = e^x$ and $v' = x$; which means that $u' = e^x$ and $v = x^2/2$. Putting these terms into the integration by parts formula

(Equation 5.1) gives:

$$\int_0^1 xe^x \, \mathrm{d}x = \left[\frac{x^2 e^x}{2} - \int \frac{x^2 e^x}{2} \, \mathrm{d}x \right]_0^1$$

But this integral is **harder** to evaluate; and so this is not a good choice for u and v'.

Table 5.1 gives some recommended choices for integration by parts questions.

integral	choice of u	choice of v
$\int ax^n \cos(kx) \, \mathrm{d}x$	$u = ax^n$	$v' = \cos(kx)$
$\int ax^n \sin(kx) \, \mathrm{d}x$	$u = ax^n$	$v' = \sin(kx)$
$\int ax^n e^{kx} \, \mathrm{d}x$	$u = ax^n$	$v' = e^{kx}$
$\int ax^n \ln(kx) \, \mathrm{d}x$	$u = \ln(kx)$	$v' = ax^n$

Table 5.1: Some recommended choices for u and v' for common integration by parts questions

5.3 Examples

The rest of this chapter is devoted to a range of different examples involving integration by parts.

Example 5.3.1. Suppose you are asked to evaluate $\int -x \csc^2(x) \, \mathrm{d}x$. This is a definite integral, so you should aim to find the complete antiderivative in Step 4 before evaluating the limits in Step 5.

Step 1: Here, you can follow the advice at the beginning of Subsection 5.2.1 and take $u = x$. You can integrate $-\csc^2(x)$ (see Table 4.3) and so you can take $v' = -\csc^2(x)$. It does not matter whether either u or v' contains the minus sign; as long as it is not forgotten!

Step 2: Now, differentiating u with respect to x gives $u' = 1$. Integrating $v' = -\csc^2(x)$

with respect to x gives $v = \cot(x)$ by Table 4.3; remember that you do not need the $+C$ here.

Step 3: You can now write that

$$\int -x\csc^2(x)\,\mathrm{d}x = \left[x\cot(x) - \int \cot(x)\,\mathrm{d}x\right]$$

Step 4: You can integrate $\cot(x)$ directly by Equation 4.2 with $a = k = 1$ (Case 1). Doing this gives

$$\int \cot(x)\,\mathrm{d}x = \ln|\sin(x)|$$

Substituting this into the above equation gives

$$\int -x\csc^2(x)\,\mathrm{d}x = [x\cot(x) - \ln|\sin(x)|]$$

which does not have an integral sign in it; so you can proceed to Step 5.

Step 5: All that is left to do add the $+C$ here; so you can write

$$\int -x\csc^2(x)\,\mathrm{d}x = x\cot(x) - \ln|\sin(x)| + C$$

and this is your final answer.

The next example details what happens when you need to integrate by parts more than once.

Example 5.3.2. Suppose you are asked to find the integral $\int_0^{\pi/4} 4x^2\cos(2x)\,\mathrm{d}x$. This is a definite integral, so you should aim to find the complete antiderivative in Step 4 before evaluating the limits in Step 5.

Step 1: Again you can follow the advice at the beginning of Subsection 5.2.1 and take $u = x^2$ and $v' = 4\cos(2x)$. Here, it does not matter whether the 4 goes in u or v'; as long as it goes in one of them.

Step 2: Now, differentiating u with respect to x gives $u' = 2x$. Using Equation 2.6 with $a = 4$ and $k = 2$, you can integrate $v' = 4\cos(2x)$ with respect to x to get $v = 2\sin(2x)$; remember that you do not need the $+C$ here.

Step 3: You can now write that

$$\int_0^{\pi/4} 4x^2\cos(2x)\,\mathrm{d}x = \left[x^2\cdot 2\sin(2x) - \int 2x\cdot 2\sin(2x)\,\mathrm{d}x\right]_0^{\pi/4}$$
$$= \left[2x^2\sin(2x) - \int 4x\sin(2x)\,\mathrm{d}x\right]_0^{\pi/4} \qquad (*)$$

Step 4: Here, you can notice that $\int 4x\sin(2x)\,\mathrm{d}x$ is only integrable by parts (Case 2); so this is what you need to do.

Step 1.1: Here, you are trying to integrate the indefinite integral $\int 4x\sin(2x)\,\mathrm{d}x$; you can take $a = x$ and $b' = 4\sin(2x)$. Notice here that you can't use u and v again in this working.

Step 1.2: Differentiating $a = x$ with respect to x gives $a' = 1$; integrating $b' = 4\sin(2x)$ with respect to x gives $b = -2\cos(2x)$. You should be careful with the minus signs here.

Step 1.3: Using the fact that $-\int -2\cos(2x)\,\mathrm{d}x = \int 2\cos(2x)\,\mathrm{d}x$, you can now write that

$$\int 4x\sin(2x)\,\mathrm{d}x = \left[x\cdot -2\cos(2x) - \int 1\cdot -2\cos(2x)\,\mathrm{d}x\right]$$
$$= \left[-2x\cos(2x) + \int 2\cos(2x)\,\mathrm{d}x\right]$$

Step 1.4: You can integrate $2\cos(2x)$ directly (Case 1). Doing this gives

$$\int 2\cos(2x)\,\mathrm{d}x = \sin(2x)$$

Substituting this into the above equation gives

$$\int 4x\sin(2x)\,\mathrm{d}x = [-2x\cos(2x) + \sin(2x)]$$

which does not have an integral sign in it. As you are finding this integral as part of another integral, you do not need a $+C$ here.

You can substitute your expression for $\int 4x\sin(2x)\,\mathrm{d}x$ into item * to get

$$\int_0^{\pi/4} 4x^2\cos(2x)\,\mathrm{d}x = \left[2x^2\sin(2x) - [-2x\cos(2x) + \sin(2x)]\right]_0^{\pi/4}$$
$$= \left[2x^2\sin(2x) + 2x\cos(2x) - \sin(2x)\right]_0^{\pi/4}$$

Finally, the right hand side of this integral does not have an integral sign in it; so you can proceed to Step 5.

Step 5: Using the fact that $2 \cdot \pi/4 = \pi/2$, you can evaluate the limits to get

$$\begin{aligned}\int_0^{\pi/4} 4x^2 \cos(2x)\,\mathrm{d}x &= \left[2\left(\frac{\pi}{4}\right)^2 \sin(\pi/2) + 2\left(\frac{\pi}{4}\right)\cos(\pi/2) - \sin(\pi/2)\right] - [\sin(0)] \\ &= \left[\frac{\pi^2}{8} + 0 - 1\right] - [0] \\ &= \frac{\pi^2}{8} - 1\end{aligned}$$

and this is your final answer.

The next two examples are slightly different, and it shows that integration by parts can be used in creative ways. They also provide antiderivatives for some common functions that have not been found yet.

Example 5.3.3. Suppose you are asked to find the integral $\int a\ln(kx)\,\mathrm{d}x$. You do not know to find the antiderivative of $\ln(x)$; but you know that the derivative of $\ln(kx)$ is $\frac{1}{x}$ from Table 1.1. You can use integration by parts to find the antiderivative of $a\ln(kx)$.

Step 1: As discussed above, you do not know how to integrate $\ln(kx)$; but you know how to differentiate it. This means that you are forced to take $u = \ln(kx)$; here, you can take $v' = a$. Notice here that if $a = 1$, then you can take $v' = 1$; this is perfectly acceptable!

Step 2: Now, differentiating u with respect to x gives $u' = \frac{1}{x}$ by Table 1.1. Integrating $v' = a$ with respect to x gives $v = ax$.

Step 3: You can now write that

$$\begin{aligned}\int a\ln(kx)\,\mathrm{d}x &= \left[ax\ln(kx) - \int ax \cdot \frac{1}{x}\,\mathrm{d}x\right] \\ &= \left[ax\ln(kx) - \int a\,\mathrm{d}x\right]\end{aligned}$$

Step 4: You can integrate $\int a\,\mathrm{d}x$ directly to get ax (Case 1). Substituting this into the above equation gives

$$\int a\ln(kx)\,\mathrm{d}x = [ax\ln(kx) - ax]$$

As this expression does not have an integral sign in it, you can proceed to Step 5.

Step 5: Finally, you can write that

$$\boxed{\int a\ln(kx)\,\mathrm{d}x = a(x\ln(kx) - x) + C} \tag{5.2}$$

Example 5.3.4. Suppose you are asked to find the integral $\int \sin^{-1}(x)\,\mathrm{d}x$. You do not know to find the antiderivative of $\sin^{-1}(x)$; but you know that the derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$ from Table 4.6. You can use integration by parts to find the antiderivative of $\sin^{-1}(x)$.

Step 1: As discussed above, you do not know how to integrate $\sin^{-1}(x)$; but you know how to differentiate it. This means that you are forced to take $u = \sin^{-1}(x)$; here, you can take $v' = 1$.

Step 2: Now, differentiating u with respect to x gives $u' = \frac{1}{\sqrt{1-x^2}}$ by Table 1.1. Integrating $v' = 1$ with respect to x gives $v = x$.

Step 3: You can now write that

$$\begin{aligned}\int \sin^{-1}(x)\,\mathrm{d}x &= \left[x\sin^{-1}(x) - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x\right]\\ &= \left[x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x\right]\end{aligned} \tag{**}$$

Step 4: You can integrate $\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$ using integration by substitution (Case 1).

Here, you can take the substitution $u = 1-x^2$. This means that $u'(x) = -2x$, and so $\mathrm{d}x = \mathrm{d}u/-2x$. Replacing each instance of $u(x) = 1-x^2$ with u and replacing $\mathrm{d}x$ by $\mathrm{d}u/-2x$ in the integral gives

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x &= \int \frac{x}{\sqrt{u}\cdot -2x}\,\mathrm{d}u\\ &= \int \frac{1}{-2\sqrt{u}}\,\mathrm{d}u\end{aligned}$$

Integrating this expression using Equation 2.2 with $a = -1/2$ and $n = -1/2$ gives

$$\int \frac{1}{\sqrt{-2u}}\,\mathrm{d}u = -u^{1/2}$$

and substituting $u = 1 - x^2$ back into this gives the integral as

$$\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x = -\sqrt{1-x^2}$$

As you are finding this integral as part of another integral, you do not need a $+C$ here.

Finally, you can substitute this into item ** to get that

$$\int \sin^{-1}(x)\,\mathrm{d}x = \left[x\sin^{-1}(x) + \sqrt{1-x^2}\right]$$

As this expression does not have an integral sign in it, you can proceed to Step 5.

Step 5: Finally, you can write that

$$\boxed{\int \sin^{-1}(x)\,\mathrm{d}x = x\sin^{-1}(x) + \sqrt{1-x^2} + C} \tag{5.3}$$

Remark. You can use a similar method to show that

$$\boxed{\int \tan^{-1}(x)\,\mathrm{d}x = x\tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C} \tag{5.4}$$

The next example of integration by parts demonstrate the need for Case 3 in Step 4 of the method in Subsection 5.2.1

Example 5.3.5. Suppose you are asked to find the integral $\int e^x\cos(x)\,\mathrm{d}x$. Here, the choice of u and v' is less clear, as neither choice is going to make the integral a lot easier to find. However, you know that if you integrate $\cos(x)$ twice, you get to $-\cos(x)$; so perhaps you can find an expression for $\int e^x\cos(x)\,\mathrm{d}x$ after integrating by parts twice.

Step 1: It doesn't matter too much which function you take to be u and which function you take to be v' in this case. Here, you can take $u = e^x$ and $v' = \cos(x)$.

Step 2: Now, differentiating u with respect to x gives $u' = e^x$, and integrating $v' = \cos(x)$ with respect to x to get $v = \sin(x)$; remember that you do not need the $+C$ here.

Step 3: You can now write that

$$\int e^x\cos(x)\,\mathrm{d}x = \left[e^x\sin(x) - \int e^x\sin(x)\,\mathrm{d}x\right] \tag{†}$$

Step 4: Here, you can notice that $\int e^x \sin(x)\,\mathrm{d}x$ is only integrable by parts (Case 2); so this is what you need to do.

Step 1.1: You are trying to integrate the indefinite integral $\int e^x \sin(x)\,\mathrm{d}x$; so here you can take $a = e^x$ and $b' = \sin(x)$. Notice here that you can't use u and v again in this working. It is also best to be consistent with your choices; so this means differentiating e^x throughout and integrating the other term.

Step 1.2: Differentiating $a = e^x$ with respect to x gives $a' = e^x$; integrating $b' = \sin(x)$ with respect to x gives $b = -\cos(x)$. You should be careful with the minus signs here.

Step 1.3: Using the fact that $-\int -\cos(x)\,\mathrm{d}x = \int \cos(x)\,\mathrm{d}x$, you can now write that

$$\begin{aligned}\int e^x \sin(x)\,\mathrm{d}x &= \left[-e^x \cos(x) - \int -e^x \cos(x)\,\mathrm{d}x\right]\\ &= \left[-e^x \cos(x) + \int e^x \cos(x)\,\mathrm{d}x\right]\end{aligned}$$

Step 1.4: You can notice here that the integral $\int e^x \cos(x)\,\mathrm{d}x$ is a constant multiple of your original integral (Case 3). Therefore, you can return to the original integral.

You can substitute your expression for $\int e^x \sin(x)\,\mathrm{d}x$ into item † to get

$$\begin{aligned}\int e^x \cos(x)\,\mathrm{d}x =& \left[e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x)\,\mathrm{d}x\right]\right]\\ =& e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,\mathrm{d}x\end{aligned}$$

and so collecting the $\int e^x \cos(x)\,\mathrm{d}x$ terms together gives

$$2\int e^x \cos(x)\,\mathrm{d}x = e^x(\sin(x) + \cos(x))$$

There is no integral sign on the right hand side; so you can continue to Step 5.

Step 5: Dividing by 2 on both sides and adding the constant of integration gives

$$\int e^x \cos(x)\,\mathrm{d}x = \frac{e^x}{2}(\sin(x) + \cos(x)) + C$$

and this is your final answer.

There are some instances of integration by parts where you do not know how many times you need to do this. This is usually when the power of the function is some number n; for instance, in the functions $\sin^n(x)$ and $x^n e^x$. You can do integration by parts to find what is called a **reduction formula**; this formula provides an iterative process to find the integral.

Example 5.3.6. You are asked to find a reduction formula for the integral $\int \cos^n(x)\,\mathrm{d}x$. This involves doing integration by parts. Even though this is an indefinite integral, you do not need a $+C$ at the end unless you are working out the integral for a given n.

Step 1: You can follow the advice at the beginning of Subsection 5.2.1; as you don't know how to integrate $\cos^n(x)$, but you do know how to integrate $\cos(x)$, you should take $u = \cos^{n-1}(x)$ and $v' = \cos(x)$. You can notice here that $uv' = \cos^n(x)$.

Step 2: Now, you can use the chain rule to differentiate u with respect to x; this gives $u' = -(n-1)\cos^{n-2}(x)\sin(x)$. You can integrate $v' = \cos(x)$ with respect to x to get $v = \sin(x)$; remember that you do not need the $+C$ here.

Step 3: You can now write that

$$\begin{aligned}\int \cos^n(x)\,\mathrm{d}x &= \left[\cos^{n-1}(x)\cdot\sin(x) - \int -(n-1)\cos^{n-2}(x)\sin(x)\cdot\sin(x)\,\mathrm{d}x\right]\\ &= \left[\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\sin^2(x)\,\mathrm{d}x\right]\end{aligned}$$

Here, you can use the fact that $1-\cos^2(x) = \sin^2(x)$ to write that

$$\begin{aligned}\int \cos^n(x)\,\mathrm{d}x &= \left[\cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)(1-\cos^2(x))\,\mathrm{d}x\right]\\ &= \left[\cos^{n-1}(x)\sin(x) + (n-1)\left(\int \cos^{n-2}(x)\,\mathrm{d}x - \int \cos^n(x)\,\mathrm{d}x\right)\right]\end{aligned}$$

Step 4: After expanding the brackets, you can see that the integral $-(n-1)\int\cos^n(x)\,\mathrm{d}x$ is a constant multiple of your original integral (Case 3). You can take this term over to the left hand side and write that

$$n\int \cos^n(x)\,\mathrm{d}x = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)\,\mathrm{d}x$$

Since you are finding a reduction formula, it is OK to leave the integral sign on the right hand side and proceed to Step 5.

Step 5: Dividing through by n gives your final answer as

$$\int \cos^n(x)\,\mathrm{d}x = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,\mathrm{d}x$$

You can use this reduction formula with $n = 5$ (and again, with $n = 3$ for $\int \cos^3(x)\,\mathrm{d}x$) to say that

$$\begin{aligned}\int \cos^5(x)\,\mathrm{d}x =& \frac{1}{5}\cos^4(x)\sin(x) + \frac{4}{5}\int \cos^3(x)\,\mathrm{d}x\\ =& \frac{1}{5}\cos^4(x)\sin(x) + \frac{4}{5}\left[\frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\int \cos(x)\,\mathrm{d}x\right]\\ =& \frac{1}{5}\cos^4(x)\sin(x) + \frac{4}{15}\cos^2(x)\sin(x) + \frac{8}{15}\sin(x) + C\end{aligned}$$

Chapter 6

Integration using partial fractions

6.1 What are partial fractions?

In your mathematical education, you may have learned how to add and subtract algebraic fractions from each other. This is acheived by the technique of 'cross-multiplication'. For instance

$$\frac{2}{x-1}+\frac{3}{x+2}=\frac{2(x+2)+3(x-1)}{(x-1)(x+2)}=\frac{5x+1}{x^2+x-2}$$

Now, suppose you were asked to integrate this function. Which would you rather integrate:

$$\int \frac{2}{x-1}+\frac{3}{x+2}\,\mathrm{d}x \quad \text{or} \quad \int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x \ \ ?$$

They are the same function, and so the integrals are the same. However, the integral on the left seems easier to integrate than the one on the right; you can use integration by substitution on each of the terms, and your answer should involve $\ln$. In fact:

$$\int \frac{2}{x-1}+\frac{3}{x+2}\,\mathrm{d}x = 2\ln|x-1|+3\ln|x+2|+C$$

However, this answer would not seem at all plausible if you started with $\int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x$.

The idea is to develop a method to reverse the technique of cross-multiplying algebraic fractions. This way, if you are asked to integrate a function like $\frac{5x+1}{x^2+x-2}$, you can split it up into smaller fractions that are easier to integrate. This technique is called **partial fraction decomposition**. The final chapter in this course deals with the technique of **integration using partial fractions**.

6.2 Rational functions

You may remember from a previous course that a **polynomial in** x is a function $f(x)$ that is a finite sum of powers of x. For example,

$$f(x) = x^3 + 3x^2 - x + 6$$

is a polynomial. If $f(x)$ and $g(x)$ are two polynomials with $g(x) \neq 0$, then the function $h(x) = \frac{f(x)}{g(x)}$ is called a **rational function**. You should be confident with how to factorise and manipulate polynomial functions before continuing with this chapter of the course.

There are two types of polynomial that will appear as factors in the denominators of partial fractions.

- A **linear polynomial** is a function of the form $ax + b$, for some real numbers a, b where a is non zero.
- A **quadratic polynomial** is a function of the form ax^2+bx+c, for some real numbers a, b, c with a not equal to 0.

You may have been asked to find roots of quadratic polynomials in the past. If a quadratic polynomial $f(x)$ has a real root, then it can be written as $f(x) = (x - a)(x - b)$, where a and b are the real roots of the polynomial. If a quadratic polynomial $g(x)$ has **no** real roots (so it has complex roots), it cannot be written in this way. If this happens, you can call the quadratic equation $g(x)$ **irreducible**.

In this course, you will only be asked to consider partial fraction decomposition for rational functions $h(x) = \frac{f(x)}{g(x)}$ where the polynomial $g(x)$ in the denominator has a larger power in it than the polynomial $f(x)$ in the numerator. In addition to this, you will only be asked to perform partial fraction decomposition on partial fractions whose denominator can be factorised into one of the following three cases:

(1) **Fractions containing distinct, linear factors in the denominator**: These include functions like

$$\frac{3x-3}{(x-4)(x+2)} \quad \text{and} \quad \frac{-x^2-1}{(x-5)(x+1)(x-2)}$$

(2) **Fractions containing distinct, possibly repeated linear factors in the denominator**: These include functions like

$$\frac{2x+1}{(x-1)^2} \quad \text{and} \quad \frac{-5x+2}{(x-2)(x+1)^2}$$

(3) **Fractions containing distinct linear factors and a distinct or repeated irreducible quadratic factor**: These include fractions like

$$\frac{2x-1}{(x^2+1)(x-1)} \quad \text{and} \quad \frac{-4x+1}{(x^2+9)^2}$$

To split rational functions into partial fractions, you need to:

- write the denominators into separate fractions with unknown numerators, then;
- then solve a series of equations to find the numerators.

You need to be careful when writing the denominators into separate fractions, as you do not want to lose any information. To do this, you could follow the following guidelines.

- For **every linear factor, repeated n times** in your denominator, you need **one partial fraction for each power less than or equal to n** in your expansion; so

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n}$$

  This means that for **every distinct linear factor** in your denominator, you need a term that looks like

$$\frac{A}{(ax+b)}$$

- For **every irreducible factor, repeated n times** in your denominator, you need **one partial fraction for each power less than or equal to n** in your expansion; so

$$\frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

  This means that for **every distinct irreducible factor** in your denominator, you need a term that looks like

$$\frac{Ax+B}{(ax^2+bx+c)}$$

In this course, the factors will be repeated no more than twice.

Table 6.1 gives a brief list of common types of partial fractions you may encounter, and a suggested decomposition for them.

To find the unknown numerators of a partial fraction decomposition of a rational function $\frac{f(x)}{g(x)}$, you could follow these steps:

rational function looks like	partial fractions look like
$\dfrac{f(x)}{(ax+b)(cx+d)}$	$\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}$
$\dfrac{f(x)}{(ax+b)^2(cx+d)}$	$\dfrac{A_1}{(ax+b)}+\dfrac{A_2}{(ax+b)^2}+\dfrac{B}{(cx+d)}$
$\dfrac{f(x)}{(ax+b)(cx+d)(mx+n)}$	$\dfrac{A}{(ax+b)}+\dfrac{B}{(cx+d)}+\dfrac{C}{(mx+n)}$
$\dfrac{f(x)}{(ax^2+bx+c)(mx+n)}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{D}{(mx+n)}$
$\dfrac{f(x)}{(ax^2+bx+c)(mx+n)^2}$	$\dfrac{Ax+B}{(ax^2+bx+c)}+\dfrac{D_1}{(mx+n)}+\dfrac{D_2}{(mx+n)^2}$
$\dfrac{f(x)}{(ax^2+bx+c)^2(mx+n)}$	$\dfrac{A_1x+B_1}{(ax^2+bx+c)}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\dfrac{D}{(mx+n)}$

Table 6.1: Common layouts for partial fraction decomposition

Step 1: Add together the fractions in your partial fraction decomposition. If necessary, cancel terms on top and bottom so the denominator of the added fractions is the same as the original rational function $\frac{f(x)}{g(x)}$.

Step 2: As the denominators are the same and the fractions are equal, the numerators $f(x)$ and your expression for unknowns are the same. Write out these numerators again.

Step 3: For every $(x-a)$ factor on the expression involving unknowns, set $x = a$ and solve the resulting equation. This should give you a solution to the unknown. If you find all the unknowns, you can stop here.

Step 4: For each remaining unknown, you need to generate an equation for that unknown. For instance, two remaining unknowns require two equations.

You can generate these by letting $x = n_1, n_2, ...$ where each of these n's is not equal to some a used in Step 3. You should then get a system of equations, which you can solve to find the remaining unknowns.

Let's see how this works in practice.

Example 6.2.1. Suppose you are asked to decompose

$$\frac{-3x^2+8x-1}{(x-1)^2(x^2+1)}$$

into partial fractions. You can use the guidelines above to write your decomposition as

$$\frac{-3x^2+8x-1}{(x-1)^2(x^2+1)} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{Bx+D}{(x^2+1)}$$

in an attempt to split the left hand side into partial fractions. You now need to find the numerators A_1, A_2, B and B; you can do this using the method above.

Step 1: You need to add together the partial decomposition back into a fraction. This can be done by the usual method of adding fractions. So this becomes

$$\begin{aligned}\frac{-3x^2+8x-1}{(x-1)^2(x^2+1)} &= \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{Bx+D}{(x^2+1)} \\ &= \frac{A_1(x-1)^2(x^2+1) + A_2(x-1)(x^2+1) + (Bx+D)(x-1)^3}{(x-1)^3(x^2+1)}\end{aligned}$$

At this stage, you need to make sure the denominators are the same. You can notice there is an $(x-1)$ common to each of the terms, and an $(x-1)^3$ term on the bottom. This means that you can write

$$\begin{aligned}\frac{-3x^2+8x-1}{(x-1)^2(x^2+1)} &= \frac{A_1(x-1)^2(x^2+1) + A_2(x-1)(x^2+1) + (Bx+D)(x-1)^3}{(x-1)^3(x^2+1)} \\ &= \frac{A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+D)(x-1)^2}{(x-1)^2(x^2+1)}\end{aligned}$$

Step 2: Now the denominators are the same, the numerators must also be the same. So you can write

$$-3x^2+8x-1 = A_1(x-1)(x^2+1) + A_2(x^2+1) + (Bx+D)(x-1)^2$$

Step 3: To find some of the numerators, you can set each factor to equal 0 in turn. For instance, setting $x=1$ throughout this equation gives

$$-3+8-1 = 0 + A_2(2) + 0 + 0$$

and so $4 = 2A_2$, giving $A_2 = 2$. There are no more ways to set a linear factor equal to 0 here, so you should proceed to Step 4.

Step 4: You have three numerators left to find in this case, and so you need to generate three equations to find them. So setting $x = 0$, $x = -1$ and $x = 2$ (notice that you can't set $x = 1$ here) gives

$$\begin{aligned} -1 &= -A_1 + 2 + D && (x = 0) \\ -12 &= -4A_1 + 4 - 4B + 4D && (x = -1) \\ 3 &= 5A_1 + 10 + 2B + D && (x = 2) \end{aligned}$$

Solving this gives $A_1 = -1$, $B = 1$, $D = 4$. So this means that

$$\frac{-3x^2 + 8x - 1}{(x-1)^2(x^2+1)} = \frac{-1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{x-4}{(x^2+1)}$$

Don't worry; it is unlikely that you will be asked to find anything so complicated on a tutorial sheet or exam! This example is just to illustrate the ways in which you can find the numerators. You can find some easier examples in the integrations that follow in the next section.

6.3 Integration using partial fractions

The two most common types of partial fraction you are likely to integrate are of the form

$$\frac{a}{bx + c} \quad \text{and} \quad \frac{a}{(bx+c)^n}$$

where $n \neq 1$ in the second term.

You can integrate both of these expressions this using the substitution $u = bx + c$. Differentiating this gives $u'(x) = b$, and so $\mathrm{d}x = \mathrm{d}u/b$.

Integrating the first term here gives

$$\int \frac{a}{bx+c}\,\mathrm{d}x = \int \frac{a}{b \cdot u}\,\mathrm{d}u = \frac{a}{b}\int \frac{1}{u}\,\mathrm{d}u = \frac{a}{b}\ln|u| + C$$

and so

$$\boxed{\int \frac{a}{bx+c}\,\mathrm{d}x = \frac{a}{b}\ln|bx+c| + C} \tag{6.1}$$

Now, integrating the second term here gives

$$\int \frac{a}{(bx+c)^n}\,\mathrm{d}x = \int \frac{a}{b\cdot u^n}\,\mathrm{d}u = \frac{a}{b}\int u^{-n}\,\mathrm{d}u = \frac{au^{-n+1}}{b(-n+1)} + C$$

and so

$$\boxed{\int \frac{a}{(bx+c)^n}\,\mathrm{d}x = \frac{a(bx+c)^{1-n}}{b(1-n)} + C} \tag{6.2}$$

for $n \neq 1$. You can use these results throughout the chapter. In addition to these, you also need to be able to integrate by substitution effectively; see Chapter 3 for more details.

Note: In this examples section, all integrals considered are **indefinite**. If you are asked to find a definite integral using partial fractions, you should evaluate the limits after calculating the antiderivative; in a similar fashion to integration by parts.

Example 6.3.1. You are asked to find the integral $\int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x$. Here, you can factorise the denominator to get

$$x^2 + x - 2 = (x+2)(x-1)$$

This means that the integral becomes

$$\int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x = \int \frac{5x+1}{(x+2)(x-1)}\,\mathrm{d}x$$

You can use the technique of partial fraction decomposition to find this integral. Using the recommended layout for the partial fraction decomposition as seen in Table 6.1, you can write

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

Adding these fractions gives

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A(x-1)+B(x+2)}{(x+2)(x-1)}$$

As the denominators are the same, the numerators are the same and so you can write

$$5x+1 = A(x-1) + B(x+2) \tag{*}$$

Here, you can set each factor to be equal to 0 in order to find A and B. Setting $x = 1$ and substituting it into Equation * gives $6 = 3B$, and so $B = 2$. You can now set $x = -2$ and substitute into Equation * to give $-9 = -3A$, and so $A = 3$. This means that

$$\frac{5x+1}{(x+2)(x-1)} = \frac{3}{x+2} + \frac{2}{x-1}$$

and this is the same answer as in the introduction. Following this, the integral becomes

$$\int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x = \int \frac{3}{x+2} + \frac{2}{x-1}\,\mathrm{d}x$$

You can see from the introduction that

$$\int \frac{5x+1}{x^2+x-2}\,\mathrm{d}x = \int \frac{3}{x+2} + \frac{2}{x-1}\,\mathrm{d}x = 3\ln|x+2| + 2\ln|x-1| + C$$

and this is your final answer.

Example 6.3.2. You are asked to find

$$\int \frac{-3x^2+8x-1}{(x-1)^2(x^2+1)}\,\mathrm{d}x$$

You can split the integrand into partial fractions... which you have already done in Example 6.2.1. So this means the integral is

$$\int \frac{-3x^2+8x-1}{(x-1)^2(x^2+1)}\,\mathrm{d}x = \int \frac{-1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{x-4}{(x^2+1)}\,\mathrm{d}x$$

You can integrate the first term using Equation 6.1 with $a = -1 = c$ and $b = 1$ to get

$$\int \frac{-1}{x-1}\,\mathrm{d}x = -\ln|x-1|$$

Integrating the second term using Equation 6.2 with $a = 2$, $b = 1$, $c = -1$ and $n = 2$ gives

$$\int \frac{2}{(x-1)^2}\,\mathrm{d}x = -\frac{2}{(x-1)}$$

You cannot integrate the third term directly. In this case, you have to split the integral into two separate parts. You can do this by saying that

$$\int \frac{x-4}{x^2+1}\,\mathrm{d}x = \int \frac{x}{x^2+1}\,\mathrm{d}x + \int \frac{-4}{x^2+1}\,\mathrm{d}x$$

You can integrate the first of these terms using integration by substitution. Taking $u =$

x^2+1, you can work through the method of integration by substitution to get

$$\int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln|x^2+1|$$

In the second of these integrals, you can use Equation 4.6 with $a=-4$ and $b=1$ to get

$$\int \frac{-4}{x^2+1}\,\mathrm{d}x = -4\tan^{-1}(x)$$

Collecting together these four antiderivatives, you can write that

$$\int \frac{-3x^2+8x-1}{(x-1)^2(x^2+1)}\,\mathrm{d}x = -\ln|x-1| - \frac{2}{(x-1)} + \frac{1}{2}\ln|x^2+1| - 4\tan^{-1}(x) + C$$

not forgetting the $+C$!

Example 6.3.3. You are asked to find

$$\int \frac{2x^2+x-7}{(x-2)(x+1)(x-1)}\,\mathrm{d}x$$

You can do this using partial fractions. Following the recommended layout for the partial fraction decomposition as seen in Table 6.1, you can write

$$\frac{2x^2+x-7}{(x-2)(x+1)(x-1)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{D}{x-1}$$

Adding these fractions gives

$$\frac{2x^2+x-7}{(x-2)(x+1)(x-1)} = \frac{A(x+1)(x-1)+B(x-2)(x-1)+D(x-2)(x+1)}{(x-2)(x+1)(x-1)}$$

As the denominators are the same, the numerators are the same and so you can write

$$2x^2+x-7 = A(x+1)(x-1)+B(x-2)(x-1)+D(x-2)(x+1) \tag{†}$$

Here, you can set each factor to be equal to 0 in order to find A, B and D. So, setting $x=2$ and substituting this into Equation † gives $3 = A(3)(1)$, and so $A=1$. You can now set $x=-1$ and substitute this into Equation † to get $-6 = B(-3)(-2) = 6B$, and so $B=-1$. Finally, you can set $x=1$ and substitute this into Equation † to get that $-4 = D(-1)(2) = -2D$, and so $D=2$. This means that

$$\frac{2x^2+x-7}{(x-2)(x+1)(x-1)} = \frac{1}{x-2} - \frac{1}{x+1} + \frac{2}{x-1}$$

Following this, the integral becomes

$$\int \frac{2x^2+x-7}{(x-2)(x+1)(x-1)}\,\mathrm{d}x = \int \frac{1}{x-2} - \frac{1}{x+1} + \frac{2}{x-1}\,\mathrm{d}x$$

You can use Equation 6.1 to integrate each of these terms to say that

$$\begin{aligned}\int \frac{2x^2+x-7}{(x-2)(x+1)(x-1)}\,\mathrm{d}x &= \int \frac{1}{x-2} - \frac{1}{x+1} + \frac{2}{x-1}\,\mathrm{d}x \\ &= \ln|x-2| - \ln|x+1| + 2\ln|x-1| + C\end{aligned}$$

Now, let's look at an example with a repeated linear factor.

Example 6.3.4. You are asked to evaluate

$$\int \frac{7x^2-25x-2}{(x-3)^2(2x+1)}\,\mathrm{d}x$$

You can do this using partial fractions. Following the recommended layout for the partial fraction decomposition as seen in Table 6.1, you can write

$$\frac{7x^2-25x-2}{(x-3)^2(2x+1)} = \frac{A_1}{(x-3)} + \frac{A_2}{(x-3)^2} + \frac{B}{2x+1}$$

Adding these fractions gives

$$\frac{7x^2-25x-2}{(x-3)^2(2x+1)} = \frac{A_1(x-3)^2(2x+1) + A_2(x-3)(2x+1) + B(x-3)^3}{(x-3)^3(2x+1)}$$

Here, the denominators are not the same. However, there is an $(x-3)$ term on the top and bottom of the right hand side, so you can cancel to get

$$\frac{7x^2-25x-2}{(x-3)^2(2x+1)} = \frac{A_1(x-3)(2x+1) + A_2(2x+1) + B(x-3)^2}{(x-3)^2(2x+1)}$$

So as the denominators are the same, the numerators are the same and therefore you can write

$$7x^2-25x-2 = A_1(x-3)(2x+1) + A_2(2x+1) + B(x-3)^2 \qquad (\ddagger)$$

Here, you can set each factor to be equal to 0 in order to find A_1, A_2 and B. So, setting $x=3$ and substituting this into Equation ‡ gives

$$63-75-2 = 0 + A_2(7) + 0$$

giving $-14 = 7A_2$; so $A_2 = -2$. You can now set $x = -1/2$ and substitute this into Equation ‡ to get

$$\frac{7}{4} + \frac{25}{2} - 2 = \frac{49}{4}B$$

Multiplying through by 4 on both sides gives

$$7 + 50 - 8 = 49B$$

giving $49 = 49B$; this means $B = 1$. There are no other factors you can set to be 0 in order to find A_1. However, you only need to generate one more equation. So here, you can set $x = 0$ and substitute this into Equation ‡ to get that $-2 = A_1(-3)(1) - 2 + 9$, and so $-9 = -3A_1$, giving $A_1 = 3$. This means that

$$\frac{7x^2 - 25x - 2}{(x-3)^2(2x+1)} = \frac{3}{(x-3)} - \frac{2}{(x-3)^2} + \frac{1}{2x+1}$$

Following this, the integral becomes

$$\int \frac{7x^2 - 25x - 2}{(x-3)^2(2x+1)}\,\mathrm{d}x = \int \frac{3}{(x-3)} - \frac{2}{(x-3)^2} + \frac{1}{2x+1}\,\mathrm{d}x$$

You can use Equation 6.1 to integrate the first and third of these terms, and Equation 6.2 to integrate the second to get

$$\begin{aligned}\int \frac{7x^2 - 25x - 2}{(x-3)^2(2x+1)}\,\mathrm{d}x =& \int \frac{3}{(x-3)} - \frac{2}{(x-3)^2} + \frac{1}{2x+1}\,\mathrm{d}x \\ =& 3\ln|x-3| + \frac{2}{x-3} + \frac{1}{2}\ln|2x+1| + C\end{aligned}$$

Example 6.3.5. Suppose you are asked to find the integral

$$\int \frac{x^2 + 3x + 2}{x(x^2+1)}\,\mathrm{d}x$$

You can use the technique of partial fraction decomposition to find this integral. Using the recommended layout for the partial fraction decomposition as seen in Table 6.1, you can write

$$\frac{x^2 + 3x + 2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx + D}{x^2+1}$$

Adding these fractions gives

$$\frac{x^2+3x+2}{x(x^2+1)} = \frac{A(x^2+1)+(Bx+D)x}{x(x^2+1)}$$

As the denominators are the same, the numerators are the same and so you can write

$$x^2+3x+2 = A(x^2+1)+(Bx+D)x \tag{$\bigstar$}$$

Here, you can set one factor to be equal to 0 in order to find A. Setting $x = 0$ and substituting it into Equation ★ gives $2 = A$. There are no other factors you can set to be 0 in order to find B and D. As you need to find two unknowns, you need to generate two equations. So taking $x = 1$ and $x = -1$ and substituting these into Equation ★ gives the simultaneous equations

$$\begin{aligned} 6 &= 4 + (B + D) && (x = 1) \\ 0 &= 4 - (-B + D) && (x = -1) \end{aligned}$$

Adding these two equations together cancels the D's to give $6 = 8+2B$; so $B = -1$. This means that $D = 3$. You can therefore write:

$$\frac{x^2+3x+2}{x(x^2+1)} = \frac{2}{x} + \frac{-x+3}{x^2+1}$$

So the integral becomes

$$\int \frac{x^2+3x+2}{x(x^2+1)}\,\mathrm{d}x = \int \frac{2}{x} + \frac{-x+3}{x^2+1}\,\mathrm{d}x$$

The first of these you can integrate directly by Equation 6.1 to get

$$\int \frac{2}{x}\,\mathrm{d}x = 2\ln|x|$$

However, the second term is not integrable directly. Like in Example 6.3.2, you can split the integral up into two parts to get

$$\int \frac{-x+3}{x^2+1}\,\mathrm{d}x = \int \frac{-x}{x^2+1}\,\mathrm{d}x + \int \frac{3}{x^2+1}\,\mathrm{d}x$$

You can integrate the first of these using integration by substitution (with $u = x^2+1$). The integrand of the second term is the derivative of $3\tan^{-1}(x)$ (see Equation 4.6); and so you can write (with some working omitted):

$$\int \frac{-x+3}{x^2+1}\,\mathrm{d}x = -\frac{1}{2}\ln|x^2+1| + 3\tan^{-1}(x)$$

At last, you can write your answer as

$$\int \frac{x^2+3x+2}{x(x^2+1)}\,\mathrm{d}x = 2\ln|x| - \frac{1}{2}\ln|x^2+1| + 3\tan^{-1}(x) + C$$

6.3.1 And finally...

You can use the technique of partial fractions to find the as yet unconsidered antiderivative of $a\sec(kx)$. You know that $\sec(x) = \frac{1}{\cos(x)}$ and so you can write

$$\int a\sec(kx)\,\mathrm{d}x = \int \frac{a}{\cos(kx)}\,\mathrm{d}x = a\int \frac{1}{\cos(kx)}\,\mathrm{d}x$$

Multiplying top and bottom of this fraction by $\cos(kx)$ gives

$$a\int \frac{\cos(kx)}{\cos^2(kx)}\,\mathrm{d}x$$

As $\cos^2(kx) = 1-\sin^2(kx)$, you can write

$$a\int \sec(kx)\,\mathrm{d}x = a\int \frac{\cos(kx)}{1-\sin^2(kx)}\,\mathrm{d}x$$

Now, you can integrate this by substitution by taking $u = \sin(kx)$. Using the chain rule, it follows that $u'(x) = k\cos(kx)$ and so $\mathrm{d}x = \mathrm{d}u/k\cos(kx)$. So the integral becomes

$$\begin{aligned} a\int \frac{\cos(kx)}{1-\sin^2(kx)}\,\mathrm{d}x =& a\int \frac{\cos(kx)}{(1-u^2)\cdot k\cos(kx)}\,\mathrm{d}u \\ =& \frac{a}{k}\int \frac{1}{1-u^2}\,\mathrm{d}u \end{aligned}$$

As $1-u^2 = (1-u)(1+u)$, you can integrate $\frac{1}{1-u^2}$ using partial fractions. Following the advice in Table 6.1, you can write

$$\frac{1}{1-u^2} = \frac{A}{1-u} + \frac{B}{1+u}$$

Adding these fractions together gives

$$\frac{1}{1-u^2} = \frac{A(1+u)+B(1-u)}{1-u^2}$$

As the denominators are the same, the numerators are the same and so you can write

$$1 = A(1+u) + B(1-u) \tag{$\blacklozenge$}$$

Here, you can set each factor to be equal to 0 in order to find A and B. Setting $x = 1$ and substituting it into Equation ♦ gives $1 = 2A$, and so $A = 1/2$. You can now set $x = -1$ and substitute into Equation ♦ to give $1 = -2B$, and so $B = -1/2$. This means that

$$\frac{1}{1-u^2} = \frac{1}{2(1-u)} - \frac{1}{2(1+u)}$$

Following this, the integral becomes

$$\frac{a}{k}\int \frac{1}{1-u^2}\,\mathrm{d}u = \frac{a}{k}\int \frac{1}{2(1-u)} - \frac{1}{2(1+u)}\,\mathrm{d}u = \frac{a}{k}\int \frac{1}{2(u+1)} - \frac{1}{2(u-1)}\,\mathrm{d}u$$

Integrating gives

$$\frac{a}{k}\int \frac{1}{1-u^2}\,\mathrm{d}u = \frac{a}{2k}\ln|u+1| - \frac{a}{2k}\ln|u-1| + C$$

Combining the two logarithms using the laws of logs gives

$$\frac{a}{k}\int \frac{1}{1-u^2}\,\mathrm{d}u = \frac{a}{2k}\ln\left|\frac{u+1}{u-1}\right| + C$$

Substituting $u = \sin(x)$ back into this equation gives the integral as

$$\int a\sec(kx)\,\mathrm{d}x = \frac{a}{2k}\ln\left|\frac{\sin(x)+1}{\sin(x)-1}\right| + C$$

While this is a perfectly valid answer, you could find a way to simplify this. Here, multiplying top and bottom of the term inside the natural log by $\sin(x)+1$ and simplifying gives

$$\frac{(\sin(x)+1)\cdot(\sin(x)+1)}{(\sin(x)-1)\cdot(\sin(x)+1)} = \frac{(\sin(x)+1)^2}{(\sin^2(x)-1)}$$

As $\sin^2(x) - 1 = -\cos^2(x)$, you can write

$$\frac{\sin(x)+1}{\sin(x)-1} = \frac{(\sin(x)+1)^2}{-\cos^2(x)}$$

Putting this expression into the natural log gives

$$\int a\sec(kx)\,\mathrm{d}x = \frac{a}{2k}\ln\left|\frac{(\sin(x)+1)^2}{-\cos^2(x)}\right| + C$$

You can use the properties of the absolute value to get rid of the minus sign in the denominator, giving that

$$\int a\sec(kx)\,\mathrm{d}x = \frac{a}{2k}\ln\left|\frac{(\sin(x)+1)^2}{\cos^2(x)}\right| + C$$

Using the laws of logarithms, you can cancel the $\frac{1}{2}$ by taking the square root of everything inside the logarithm, writing that

$$\int a\sec(kx)\,\mathrm{d}x = \frac{a}{k}\ln\left|\frac{(\sin(x)+1)}{\cos(x)}\right| + C$$

The final step in the course is to simplify the fraction to get

$$\boxed{\int a\sec(kx)\,\mathrm{d}x = \frac{a}{k}\ln|\tan(x)+\sec(x)| + C} \tag{6.3}$$