Notes on SHT

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1 Spectra

Definition 1 (The Spectra Category)**.** A *spectrum* X is a family of based topoogical spaces $\{X_n \, n \geq 0\}$ together with structure maps

$$\varepsilon_n : \Sigma X_n \to X_{n+1},$$

in which Σ is the based suspension functor.

A map between to spectra, $f : X \to Y$, is a family of maps of based topological spaces, $\{f_n : X_n \to Y_n : n \geq 0\}$, that agrees with the structure maps, that is, for each n we have a commutative square:

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\varepsilon"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma Y_n \arrow[r, "\varepsilon"] & Y_{n+1}
\end{tikzcd}.$$

Having objects and maps we can define a category of prespectra, which we will denote as **pSp**.

Let $X = (X_n : n \geq 0)$ be a spectrum. For each, n we have homotopy groups, $\pi_k(X_n)$. Notice that the structure maps induce a map

$$\varepsilon_n^* : \pi_k(\Sigma X_n) \to \pi_k(X_{n+1}),$$

and by the definition of suspension, for each map

$$\alpha : S^k \to X_n,$$

we have a suspension

$$\Sigma\alpha : S^{k+1} \to \Sigma X_n.$$

Therefore, Σ induce maps

$$\Sigma^* : \pi_k(X_n) \to \pi_{k+1}(\Sigma X_n).$$

Adding the above information together, for each $k \in \mathbb{Z}$, we get a sequence

$$\pi_k(X_0) \longrightarrow \cdots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\Sigma^*} \pi_{k+n+1}(\Sigma X_n) \xrightarrow{\varepsilon_n^*} \pi_{k+n+1}(X_{n+1}) \longrightarrow \cdots$$

The colimit of this sequence will be called the k-th homotopy group of the spectrum X, and denoted $\pi_k(X)$.
The book asserts that for $k < 0$ the sequence is defined from $n \geq |k|$. I would correct it as such: The sequence is defined for $n \geq \max\{2-k, 0\}$, for in that case all groups are abelian groups. There is no problem in not starting at 0 as this definition is meant to capture the asymptotical homotopical behavior.
Would this be the same as taking the homotopy colimit of

$$\cdots \longrightarrow X_n \xrightarrow{\Sigma} \Sigma X_n \xrightarrow{\varepsilon} X_{n+1} \longrightarrow \cdots \qquad (1)$$

If we were working with CW-complexes, this would be the moment we would discuss cells, but as we are not, we approximate this but a discussion of "elements".

Definition 2. Given $X = (X_n : n \geq 0)$ a spectrum, a k dimensional element of X is the set of maps

$$\sigma : S^{k+n} X_n \to X_n,$$

quotiented by the relation generated by

$$\sigma_i \sim \sigma_j, \text{ if } \Sigma\sigma_i = \sigma_j.$$

So now $\pi_k(X)$ is the set of k dimensional elements of X up to homotopy.

Remark 1. The suspension is given by smash product by S^1 on the right, i.e.

$$\Sigma X = X \wedge S^1.$$

Now we define a important type of spectrum.

Definition 3 (Suspension Spectra)**.** Let X be a based topological space. De suspension spectra over X, $\Sigma^\infty X$, is defined levelwise as

$$\Sigma^\infty X_n = \Sigma^n X = X \wedge S^n,$$

with structure maps the canonical homeomorphisms

$$X \wedge S^n \wedge S^1 \cong X \wedge S^{n+1}.$$