Notes on SHT

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1 Spectra

Definition 1 (The Spectra Category)**.** A *spectrum* X is a family of based topoogical spaces $\{X_n\, n \geq 0\}$ together with structure maps

$$\varepsilon_n : \Sigma X_n \to X_{n+1},$$

in which Σ is the based suspension functor.

A map between to spectra, $f : X \to Y$, is a family of maps of based topological spaces, $\{f_n : X_n \to Y_n : n \geq 0\}$, that agrees with the structure maps, that is, for each n we have a commutative square:

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\varepsilon_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma Y_n \arrow[r, "\varepsilon_n"] & Y_{n+1}
\end{tikzcd}.$$

Having objects and maps we can define a category of prespectra, which we will denote as **pSp**.

Let $X = (X_n : n \geq 0)$ be a spectrum. For each, n we have homotopy groups, $\pi_k(X_n)$. Notice that the structure maps induce a map

$$\varepsilon_{n*} : \pi_k(\Sigma X_n) \to \pi_k(X_{n+1}),$$

and by the definition of suspension, for each map

$$\alpha : S^k \to X_n,$$

we have a suspension

$$\Sigma\alpha : S^{k+1} \to \Sigma X_n.$$

Therefore, Σ induce maps

$$\Sigma_* : \pi_k(X_n) \to \pi_{k+1}(\Sigma X_n).$$

Adding the above information together, for each $k \in \mathbb{Z}$, we get a sequence

$$\pi_k(X_0) \longrightarrow \cdots \longrightarrow \pi_{k+n}(X_n) \xrightarrow{\Sigma^*} \pi_{k+n+1}(\Sigma X_n) \xrightarrow{\varepsilon_{n*}}$$

$$\xrightarrow{\varepsilon_{n*}} \pi_{k+n+1}(X_{n+1}) \longrightarrow \cdots$$

The colimit of this sequence will be called the k-th homotopy group of the spectrum X, and denoted $\pi_k(X)$.
The book asserts that for $k < 0$ the sequence is defined from $n \geq |k|$. I would correct it as such: The sequence is defined for $n \geq \max\{2-k, 0\}$, for in that case all groups are abelian groups. There is no problem in not starting at 0 as this definition is meant to capture the asymptotical homotopical behavior. Would this be the same as taking the homotopy colimit of

$$\cdots \longrightarrow X_n \xrightarrow{\Sigma} \Sigma X_n \xrightarrow{\varepsilon} X_{n+1} \longrightarrow \cdots \tag{1}$$

Remark 1. For each k, π_k is a functor from **pSp** to **Ab**, the category of abelian groups.

If we were working with CW-complexes, this would be the moment we would discuss cells, but as we are not, we approximate this but a discussion of "elements".

Definition 2. Given $X = (X_n : n \geq 0)$ a spectrum, a k dimensional element of X is the set of maps

$$\sigma : S^{k+n} X_n \to X_n,$$

quotiented by the relation generated by

$$\sigma_i \sim \sigma_j, \text{ if } \Sigma\sigma_i = \sigma_j.$$

So now $\pi_k(X)$ is the set of k dimensional elements of X up to homotopy.

Remark 2. The suspension is given by smash product by S^1 on the right, i.e.

$$\Sigma X = X \wedge S^1.$$

Now we define a important type of spectrum.

Definition 3 (Suspension Spectrum)**.** Let X be a based topological space. De suspension spectrum over X, $\Sigma^\infty X$, is defined levelwise as

$$\Sigma^\infty X_n = \Sigma^n X = X \wedge S^n,$$

with structure maps the canonical homeomorphisms

$$X \wedge S^n \wedge S^1 \cong X \wedge S^{n+1}.$$

We can "reverse" this process.

Definition 4 (Shift desuspension spectrum)**.** Given X a based topological space, and a natural number k, the *k free spectrum*, or *k shift desuspension spectrum* of X, F_kX, is given levelwise as

$$F_kX_n = \begin{cases} \Sigma^{n-k}X, \text{ for } n \geq d, \\ *, \text{ for } n < d, \end{cases}$$

with the same structure maps of the suspension spectrum.

Notice that
$$\pi_m(F_kX) = \pi_{m-k}(\Sigma^\infty X).$$
NOw let's talk about homotopy theory.

Definition 5 (homotopy of spectra)**.** Let $X = (X_n)$, $Y = (Y_n)$ be two spectra, and $f, g : X \rightrightarrows Y$ two maps of spectra. Let $\mathrm{Cyl}(X)$ be the spectrum defined levelwise as

$$\mathrm{Cyl}(X)_n = X_n \wedge I_+,$$

in which the structure maps are

$$\varepsilon'_n = \varepsilon_n \wedge 1_{I_+},$$

ε_n being the structure maps of X.

We say that f and g are homotopic, and write $f \sim g$, if there is a map of spectra
$$h : \mathrm{Cyl}(X) \to Y$$
such that for each $n \geq 0$,

$$h_n : X_n \wedge I_+ \to Y_n,$$

is a homotopy of based topological spaces.

Definition 6. Let $X = (X_n)$ and $Y = (Y_n)$ be two spectra, and $f : X \to Y$ be a map of spectra. We say that f is a *stable equivalence*, or a *π_*-isomorphism*, if, for each $k \in \mathbb{Z}$, $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism.

Also, f is called a *level equivalence* if, for each $n \geq 0$, $f_n : X_n \to Y_n$ is a weak homotopy equivalence.

Lastly, f is called a *homotopy equivalence* if there is a map $g : Y \to X$ such that
$$fg \sim 1_Y \text{ and } gf \sim 1_X,$$
where $\sim$ means that there is a homotopy of spectra.

And stable homotopy theory will focus, as the names suggests, on stable equivalences. Notice that if f is a homotopy equivalence, then it is a level equivalence. And if f is a level equivalence, then it is a stable equivalence.

We now have the language to better express the idea that the stable homotopy groups are meant to capture the asymptotical homotopy behaviour of spectra.

Definition 7. Let $X = (X_n)$ be a spectrum, and $k \geq 0$. We define the k eventual spectrum, X^k levelwise as

$$X_n^k = \begin{cases} X_n, & \text{if } n \geq k; \\ *, & \text{if } 0 \leq n < k. \end{cases}$$

Proposition 1.1. *Let X be a spectrum, and $k \geq 0$. The inclusion $X^k \subseteq X$ is a stable equivalence.*

Proof. Let's begin by proving that $\pi_m(X^k) = \pi_m(X)$ for all $m \in \mathbb{Z}$. Indeed, for each $n \geq 0$, we have maps

$$\pi_{m+n}(X_n) \to \pi_m(X),$$

that factor any map leaving $\pi_{m+n}(X_n)$. And, as

$$\pi_{m+n'}(X_{n'}^k) = \pi_{m+n'}(*) = *, n' < k$$

any map leaving $\pi_{m+n'}(X_n^k)$ factor trough

$$\pi_{m+n'}(X_{n'}^k) \to \pi_m(X).$$

That is, $\pi_m(X)$ is the colimit of the sequence that defines $\pi_m(X^k)$, and therefore they are isomorphic. NOw let's look at the inclusion

$$\iota : X^k \to X.$$

It is given levelwise as

$$\iota_n = \begin{cases} 0, & \text{if } n < k; \\ 1_{X_n}, & \text{if } n \geq k. \end{cases}$$

so that it induces the identity isomorphism, or the 0 morphism in each homotopy group. By the same reasoning about the 0 map, the identity on $\pi_m(X)$ is the desired colimit, and therefore the inclusion is a stable equivalence. $\square$