Contents

1 Introduction

1.1 Background

Life insurance serves as a vital instrument in providing financial security to families upon the insured individual's death [9]. A key challenge in establishing the appropriate annuity payment for a life insurance policy lies in the uncertainties surrounding life expectancy and the variability in financial requirements. As life expectancy is a random variable function of age, and the desired cash flow upon death might differ based on factors such as family expenses, income, and assets, it is necessary to develop an effective mathematical model to address these concerns.

In this study, the Gompertz-Makeham model is adopted to estimate age-specific death probabilities, thereby capturing the exponentially increasing risk of death with age and accounting for age-independent factors contributing to mortality. Commutation functions are employed to streamline complex calculations in life insurance mathematics, bridging the gap between the mortality model and financial aspects. Furthermore, a cash flow function is designed that takes into account a range of factors, including family expenses, immediate expenses, future expenses, current annual income, and existing assets.

The comprehensive approach enables the calculation of the fair annuity value based on an individual's age and specific financial circumstances, ultimately ensuring that the insured's family receives the desired cash flow upon their death. By accounting for the randomness of life expectancy and the unique financial situation of each individual, the model offers a professional and reliable method for determining appropriate annuity payments in the context of life insurance policies.

1.2 Assumption

There are several assumptions that must be made in order to form a model:

1. We assume that the mortality rates are based on natural deaths and do not account for external factors such as accidents, war, or pandemics. Meaning that the uncertainty in the model comes only from the randomness of the age at death.

2. We assume that the interest rates are constant over the entire period, which simplifies the calculations but may not reflect real-world fluctuations in interest rates.

3. We assume that the Gompertz-Makeham law of mortality accurately represents the age-specific probability of death for the population being considered.

4. We assume that the present value of cash flow Y that the family will receive upon the policyholder's death does not vary with the policyholder's age. This simplification may not capture changes in the family's financial needs as the policyholder ages.

5. We assume that the policyholders' behavior, such as their willingness to pay annuities or the financial needs of their families, remains constant over time. That is the policyholder will continue to pay the annuity until their death, with no lapses or surrenders of the policy.

6. We assume that there are no taxes or expenses associated with the annuity payments or the cash flow received by the family.

2 Method

In this model, We approach this life insurance annuity problem as a potential consumer. Recognizing that there is imperfect competition with asymmetric information. We decide to divide the modeling process into three main components: mortality modeling, financial aspects, and cash flow considerations.

For the mortality modeling, since we are considering the life expectancy as a random variable function of the age, we primarily focused on the Gompertz-Makeham model. The Gompertz-Makeham law of mortality describes age-specific death probabilities, and this law has been used by actuaries and economics to model and forecast mortality rates [4]. In a protected environment where external causes of death are rare, the Gompertz-Makeham model emphasizes on the natural causes where the age-independent mortality component is often negligible [5]. The Gompertz-Makeham law provides a flexible and adaptable way to model the age-specific probability of death, using its parameters (A, B, and C) to adjust the mortality rates according to the population being considered.

As for the financial aspects, we considered the nominal interest rate and the corresponding discount factor 'δ' for the purpose of present value. We are able to access to nominal interest rate from the Federal Reserve website and then calculate the discount factor[1].

Regarding cash flow considerations, which is also known as the coverage in insurance. We aimed to create a function that represents the desired cash flow for the family upon the insured's death [2]. This function incorporates various factors, and can be personalized by weighting these factors accordingly. Note that, in our calculation, we are calculating our desired cash flow in present value, and the real future cash flow after the family receive after death should be calculated separately by considering the inflation.

In order to determine the correct annuity to pay for a life insurance policy, we combined these components into a unified model. We used commutation functions D_x and N_x to simplify calculations and connect the mortality model with the financial aspects and cash flow function [6]. In this model, the ratio between D_x and N_x is used to determine the fair annuity payment for a life insurance policy.

By following this approach, we were able to create a comprehensive model that accounts for the randomness of life expectancy, financial aspects, and the desired cash flow for the family in the event of the insured's death.

3 Fair Annuity Calculation Model

3.1 Variables

- x: Age of the insured individual (years).
- t: Number of years since the policyholder's age x (years).
- ω: Maximum attainable age (years).
- i: Nominal interest rate (percentage or decimal).
- δ: Discount factor, calculated as $\delta = \frac{1}{1+i}$.
- ${}_tq_x$: Probability of death within t years for a life aged x.
- ${}_tp_x$: Probability of survival for t years for a life aged x.
- A_x: Fair annuity value for a life insurance policy based on age x and the desired cash flow function B (USD).
- D_x: Present value of a sum assured payable immediately upon the death of a life aged x (USD).
- N_x: Present value of a whole life annuity-due of 1 per year payable at the beginning of each year of life to a life aged x (USD).
- Y: Desired cash flow for the insured's family upon death, which can be a function of family expenses, immediate expenses, future expenses, current annual income, and existing assets (USD).
- A: Constant parameter in the Gompertz-Makeham mortality model.
- B: Constant parameter in the Gompertz-Makeham mortality model.
- C: Constant parameter in the Gompertz-Makeham mortality model.
- $\mu(x)$: Force of mortality (hazard rate) at age x in the Gompertz-Makeham mortality model.

3.2 The Gompertz-Makeham Mortality Model

The Gompertz-Makeham model is a widely used mortality model that defines the force of mortality (hazard rate) at age x as:

$$\mu(x) = Ae^{Bx} + C$$

where A, B, and C are constants. And according to Missov and Lenart(2013), $A \approx 0.00046$, $B \approx 0.094$, and $C \approx 0.0007$[7]. Since today in 2023, we are experiencing a technology shock and great increase in medical related services, We are also expecting the value of $\mu(x)$ to drop. This could lead to the decrease in the parameters. And we assumed that Gompertz-Makeham parameters in these scenarios are $A = 0.0003$, $B = 0.094$ and $C = 0.0005$.The Gompertz-Makeham model is used to estimate age-specific death probabilities, which are essential for life insurance and annuity calculations.

3.3 Commutation Functions

Commutation functions are used in life insurance mathematics to simplify calculations by connecting mortality models with financial aspects. In this study, we focus on two main commutation functions: D_x and N_x. We also discuss the role of interest rates in these functions[10].

3.3.1 Interest Rate and Discount Factor

The interest rate plays a crucial role in life insurance calculations, as it determines the present value of money and highly related to the purchasing power. The nominal interest rate fluctuates around 0.05 over times, and we can calculate the discount factor δ by:

$$\delta = \frac{1}{1+i} \tag{1}$$

3.3.2 Commutation Function D_x

D_x represents the present value of a sum assured payable immediately upon the death of a life aged x. It is calculated as follows:

$$D_x = \sum_{t=0}^{\omega-x} {}_tq_x \delta^t \tag{2}$$

where ${}_tq_x$ is the probability of death within t years for a life aged x, this can be gained using the cumulative hazard rates μ, δ is the discount factor based on the nominal interest rate, and ω represents the maximum attainable age [8].

3.3.3 Commutation Function N_x

N_x represents the present value of a whole life annuity-due of 1 per year payable at the beginning of each year of life to a life aged x. It is calculated as follows:

$$N_x = \sum_{t=0}^{\omega-x} {}_tp_x \delta^t \tag{3}$$

Similarly, ${}_tp_x$ is the probability of survival for t years for a life aged x, this can be calculated using $1 - {}_tq_x$, and δ is the discount factor based on the nominal interest rate, and ω represents the maximum attainable age [8].

3.4 Fair Annuity Model

The fair annuity value A_x for an individual aged x can be calculated using the following equation:

$$A_x = \frac{D_x Y}{N_x}$$

In this model, the fair annuity value A_x is based on the individual's age x, the discounted expected number of people alive at age x (D_x), the discounted expected number of people alive at age $x + 1$ (N_x), and the death benefit (Y). The death benefit function, Y, represents the desired cash flow for the insured's family upon their death, accounting for various financial factors:

$$Y = w_F \times F \times n + w_M \times M + w_E \times E - w_I \times I - w_S \times S$$

where F represents the family's annual living expenses, M represents immediate expenses, E represents future expenses, I represents current annual income, S represents existing assets, and w_F, w_M, w_E, w_I, and w_S are the corresponding weights assigned to each factor. And n represents the number of years family are going to live upon the death benefit. Note that they are all in present values.

4 Explanations

In this section, We want to provide a comprehensive explanation of each model involved in our analysis. The purpose is to offer a thorough understanding of the underlying concepts, assumptions, and calculations that constitute the foundation of our approach. Graphs are also included in the analysis for better understanding.

4.1 Calculating Gompertz-Makeham Death Probabilities

First, we need to calculate the age-specific death probabilities using the Gompertz-Makeham mortality model. Using the provided parameters, we can compute,and also describe the force of mortality $\mu(x)$ for each age x:

$$\mu(x) = Ae^{Bx} + C \tag{4}$$

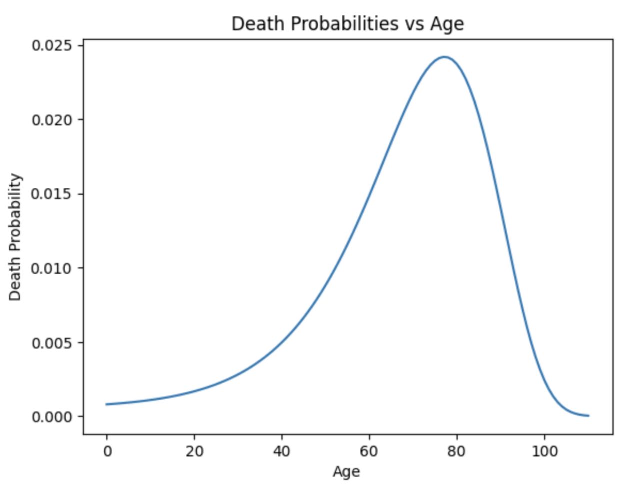

Figure 1: Death Probabilities vs Age

4.2 Computing Commutation Functions D_x and N_x

Next, we compute the commutation functions D_x and N_x using the calculated death probabilities and the nominal interest rate. The discount factor δ is calculated as follows:

$$\delta = \frac{1}{1+i} = \frac{1}{1+0.05} \approx 0.9524 \tag{5}$$

Then, we calculate D_x and N_x using the respective summation formulas:

$$D_x = \sum_{t=0}^{\omega - x} {}_tq_x \delta^t \tag{6}$$

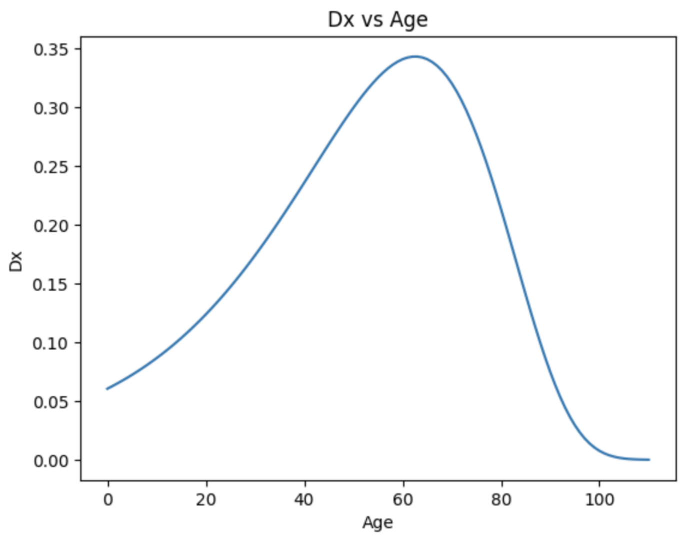

Figure 2: Dx vs Age

The D_x plot illustrates the present value of the expected number of deaths across ages, considering the mortality rates derived from the Gompertz-Makeham model. The time value of money is relevant in the calculation of D_x, as it takes into account the discount factor δ. This discount factor adjusts the expected number of deaths at different ages to account for the time value of money, acknowledging that a death occurring later has a lower present value than one occurring earlier.

$$N_x = \sum_{t=0}^{\omega - x} {}_tp_x \delta^t \tag{7}$$

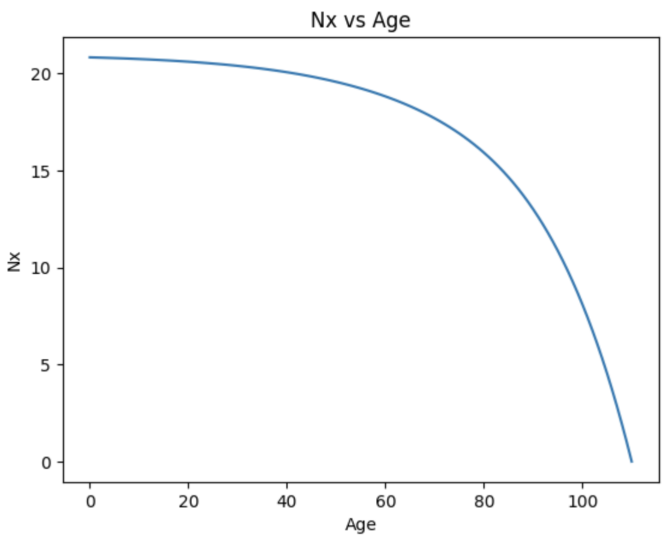

Figure 3: Nx vs Age

On the other hand, N_x represents the expected present value of the number of people alive at age x. It takes into account the survival probabilities at each age and discounts them to the present value using the discount factor δ. The N_x plot illustrates how the expected present value of the number of people alive changes across different ages, considering the survival probabilities derived from the Gompertz-Makeham model.

4.3 Determining the Fair Annuity Value A_x

Finally, we calculate the fair annuity value A_x using the commutation functions and the desired cash flow function Y:

$$A_x = \frac{D_x Y}{N_x}. \tag{8}$$

When we calculate the fair annuity using the Dx and Nx values, we are essentially dividing the present value of the death benefits by the present value of the survival probabilities. This ratio gives us the fair annuity value for the insurance product, which represents the amount of money the policyholder should pay annually to receive the death benefits when they pass away.

5 Validation And Examples

This section provides an in-depth exploration of our model by applying it to a specific example. By doing so, we can determine the fair annuity value for a life insurance policy based on the individual's age, desired cash flow, and mortality assumptions.

5.1 Annuity Value Calculation

Understand that it's difficult to calculate everything by hand, I decide to use Python and to calculate the expected values of Nx, Dx, the cash flow Y and the annuity A through proper assumptions in the financial aspects and age.

```
import numpy as np

def gompertz_makeham_death_probabilities(A, B, C, max_age):
    ages = np.arange(max_age + 1)
    hazard_rates = A * np.exp(B * ages) + C
    death_probabilities = hazard_rates * np.exp(-np.cumsum(hazard_rates))
    return death_probabilities

def calculate_Dx(death_probabilities, age, interest_rate, max_age):
    qx = death_probabilities[age:max_age]
    t = np.arange(max_age - age)
    delta_t = (1 + interest_rate) ** (-t)
    Dx = np.sum(qx * delta_t)
    return Dx

def calculate_Nx(death_probabilities, age, interest_rate, max_age):
    px = 1 - death_probabilities[age:max_age]
    t = np.arange(max_age - age)
    delta_t = (1 + interest_rate) ** (-t)
    Nx = np.sum(px * delta_t)
    return Nx

def death_benefit(F, M, E, I, S, n, wF, wM, wE, wI, wS):
    return (wF * F * n) + (wM * M) + (wE * E) - (wI * I) - (wS * S)

def fair_annuity(death_probabilities, age, cash_flow, interest_rate,
    max_age):
    Dx = calculate_Dx(death_probabilities, age, interest_rate, max_age)
    Nx = calculate_Nx(death_probabilities, age, interest_rate, max_age)

    if Nx == 0:
        return 0

    annuity = cash_flow * Dx / Nx
    return annuity

# Gompertz-Makeham parameters
A = 0.0003
B = 0.094
```

```
C = 0.0005
max_age = 110

death_probabilities = gompertz_makeham_death_probabilities(A, B, C, max_age)

# Death benefit (cash flow) calculation
F = 100000  # Family's annual living expenses
M = 40000  # Immediate expenses
E = 800000  # Future expenses
I = 200000  # Current annual income
S = 1000000  # Existing assets
n = 2  # Number of years the family is expected to rely on the death benefit

# Weights for each factor
wF = 0.1
wM = 0.3
wE = 0.3
wI = 0.1
wS = 0.2

cash_flow = death_benefit(F, M, E, I, S, n, wF, wM, wE, wI, wS)

# Example usage:
age = 40
interest_rate = 0.05
inflation_rate = 0.03

dx = calculate_Dx(death_probabilities, age, interest_rate, max_age)
nx = calculate_Nx(death_probabilities, age, interest_rate, max_age)

#To calculate the future cash flow, here we assume that you will die in 50 years
future_cash_flow = cash_flow*(1+inflation_rate)**50
print("Dx at age 40:", dx)
print("Nx at age 40:", nx)
print("Present cash flow", cash_flow)
print("Cash flow when you die:", future_cash_flow)

fair_annuity_value = fair_annuity(death_probabilities, age, cash_flow, interest_rate, max_age)
print("Fair annuity value at age 40:", fair_annuity_value)
```

Based on the code above, we can analyze a specific scenario for a typical Math professor at the age of 40, given the following parameters:

- Annual living expenses $F = 100000$
- Immediate expense $M = 40000$
- Future expenses $E = 800000$
- Current annual income $I = 200000$
- Existing assets $S = 1000000$
- Number of years $n = 2$

- Weights for each factor: $w_F = 0.1$, $w_M = 0.3$, $w_E = 0.3$, $w_I = 0.1$, $w_S = 0.2$

And we have:

- D_x at age 40: 0.371
- N_x at age 40: 19.939
- Death benefit today: 52000.0 dollars
- Future value of the death benefit (assuming the insured lives another 50 years): 227963.113 dollars
- Fair annuity value to be paid starting at age 40: 967.939 dollars

5.2 Annuity Value versus Age Graph

While we take the death benefit as given as we had in the prior section, we are able to present the graphical representation of the annuity value as a function of age as below:

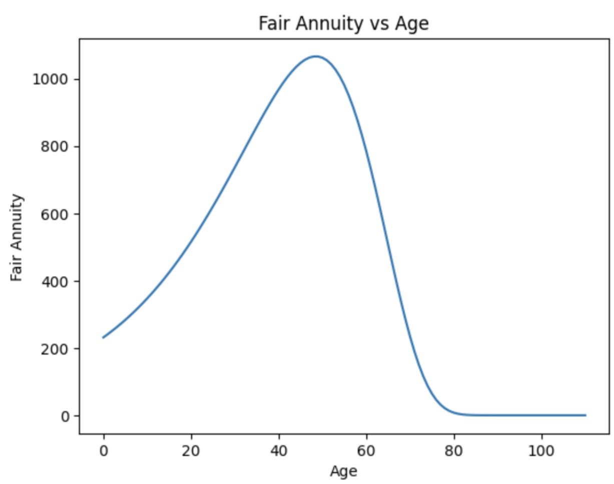

Figure 4: Death Probabilities vs Age

5.3 Insights and Interpretations

The Annuity Value versus Age Graph exhibits a bell-shaped trend, with annuity value initially increasing and then decreasing with age. However, this simplified model has limitations and is best suited for individuals within the approximate age range of 30 to 70. When applying this model, it is essential to consider each person's unique circumstances.

6 Conclusion

In conclusion, discussing death can be uncomfortable due to its uncertainties. However, by purchasing insurance, we can minimize these uncertainties and ensure our loved ones receive financial support to continue living well [3]. This study presents a comprehensive mathematical model for calculating fair annuity payments for life insurance policies, considering life expectancy uncertainties and varying financial needs. By employing the Gompertz-Makeham mortality model to estimate age-specific death probabilities and utilizing commutation functions, we streamline complex actuarial calculations, making it easier to determine appropriate insurance coverage.

Central to our model is the equation for determining the fair annuity value:

$$A_x = \frac{D_x Y}{N_x}$$

This equation incorporates the cash flow function Y, which considers various factors such as family expenses, immediate expenses, and to mention only a few, allowing for the adaptation of the model to cater to an individual's specific financial situation.

Despite its strengths, the model has certain limitations, such as the assumptions of constant interest rates and the exclusion of deaths caused by external factors. Additionally, the Gompertz-Makeham model may not capture mortality patterns for all populations or account for improvements in life expectancy over time. Furthermore, it may only be suitable only for a specific age range, and the applicability of the model for each individual should be considered.

Nevertheless, the model's ability to calculate fair annuity values by incorporating individual financial circumstances and uncertainties in life expectancy makes it a promising approach for determining appropriate annuity payments in life insurance policies. Future research could address the model's limitations by incorporating more complex mortality models, time-varying interest rates, or other factors to improve its accuracy and applicability.

In summary, the proposed model offers a valuable method for calculating fair annuity payments in life insurance policies, accounting for uncertainties in life expectancy, financial needs, and individual circumstances.

7 Appendix

```python
import numpy as np
import matplotlib.pyplot as plt

# The functions from the previous code snippets should be included here
def gompertz_makeham_death_probabilities(A, B, C, max_age):
    ages = np.arange(max_age + 1)
    hazard_rates = A * np.exp(B * ages) + C
    death_probabilities = hazard_rates * np.exp(-np.cumsum(hazard_rates))
    return death_probabilities

def calculate_Dx(death_probabilities, age, interest_rate, max_age):
    qx = death_probabilities[age:max_age]
    t = np.arange(max_age - age)
    delta_t = (1 + interest_rate) ** (-t)

    Dx = np.sum(qx * delta_t)
    return Dx

def calculate_Nx(death_probabilities, age, interest_rate, max_age):
    px = 1 - death_probabilities[age:max_age]
    t = np.arange(max_age - age)
    delta_t = (1 + interest_rate) ** (-t)

    Nx = np.sum(px * delta_t)
    return Nx

def death_benefit(F, M, E, I, S, n, wF, wM, wE, wI, wS):
    return (wF * F * n) + (wM * M) + (wE * E) - (wI * I) - (wS * S)

def fair_annuity(death_probabilities, age, cash_flow, interest_rate, max_age):
    Dx = calculate_Dx(death_probabilities, age, interest_rate, max_age)
    Nx = calculate_Nx(death_probabilities, age, interest_rate, max_age)

    if Nx == 0:
        return 0

    annuity = cash_flow * Dx / Nx
    return annuity

# Gompertz-Makeham parameters
A = 0.0003
B = 0.094
C = 0.0005
max_age = 110

death_probabilities = gompertz_makeham_death_probabilities(A, B, C, max_age)

# Death benefit (cash flow) calculation
F = 100000  # Family's annual living expenses
M = 40000  # Immediate expenses
E = 800000  # Future expenses
I = 200000  # Current annual income
S = 1000000  # Existing assets
```

```python
n = 2  # Number of years the family is expected to rely on the death benefit

# Weights for each factor
wF = 0.1
wM = 0.3
wE = 0.3
wI = 0.1
wS = 0.2

cash_flow = death_benefit(F, M, E, I, S, n, wF, wM, wE, wI, wS)

# Example usage:
age = 40
interest_rate = 0.05
inflation_rate = 0.03

# Calculate Dx, Nx, and fair annuity values for each age

ages = np.arange(max_age + 1)
dx_values = [calculate_Dx(death_probabilities, i, interest_rate, max_age)
    for i in ages]
nx_values = [calculate_Nx(death_probabilities, i, interest_rate, max_age)
    for i in ages]
fair_annuity_values = [fair_annuity(death_probabilities, i, cash_flow,
    interest_rate, max_age) for i in ages]

# Plot death probabilities
plt.figure()
plt.plot(ages, death_probabilities)
plt.xlabel("Age")
plt.ylabel("Death Probability")
plt.title("Death Probabilities vs Age")

# Plot Dx values
plt.figure()
plt.plot(ages, dx_values)
plt.xlabel("Age")
plt.ylabel("Dx")
plt.title("Dx vs Age")

# Plot Nx values
plt.figure()
plt.plot(ages, nx_values)
plt.xlabel("Age")
plt.ylabel("Nx")
plt.title("Nx vs Age")

# Plot fair annuity values
plt.figure()
plt.plot(ages, fair_annuity_values)
plt.xlabel("Age")
plt.ylabel("Fair Annuity")
plt.title("Fair Annuity vs Age")

# Show plots
plt.show()
```

References

[1] Interest rates, discount rate for united states, 2022. FRED, Federal Reserve Bank of St. Louis.

[2] Kristian Buchardt, Thomas Møller, and Kristian Bjerre Schmidt. Cash flows and policyholder behaviour in the semi-markov life insurance setup. *Scandinavian Actuarial Journal*, 2015(8):660–688, 2015.

[3] Ritchie A Campbell. The demand for life insurance: An application of the economics of uncertainty. *The Journal of Finance*, 35(5):1155–1172, 1980.

[4] Fredy Castellares, Silvio Patrício, and Artur J. Lemonte. On the Gompertz–Makeham law: A useful mortality model to deal with human mortality. *Brazilian Journal of Probability and Statistics*, 36(3):613 – 639, 2022.

[5] Thomas BL Kirkwood. Deciphering death: a commentary on gompertz (1825)'on the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies'. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 370(1666):20140379, 2015.

[6] Andreas Nordvall Lagerås. Commutation functions under gompertz–makeham mortality. *Scandinavian Actuarial Journal*, 2010(2):161–164, 2010.

[7] Trifon I. Missov and Adam Lenart. Gompertz–makeham life expectancies: Expressions and applications. *Theoretical Population Biology*, 90:29–35, 2013.

[8] MG Ogungbenle. A theoretical summation-integral scheme involving commutation function in life insurance business. *Ceylon Journal of Science*, 51(2):147–154, 2022.

[9] Sarah Quinn. The transformation of morals in markets: Death, benefits, and the exchange of life insurance policies. *American Journal of Sociology*, 114(3):738–780, 2008.

[10] Eric V Slud. *Actuarial mathematics and life-table statistics*. 2001.