Life insurance reinvented: A cross-national analysis on annuity payments

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Abstract

Background: The insurance industry exhibits dynamic interactions and imperfect information, which often hinder companies from acquiring comprehensive data on their policyholders. This complexity requires an understanding of numerous risk factors, and insurers can leverage population insights to formulate policies and support national health strategies.

Objective: This study introduces a mathematical model for creating fair annuity values in life insurance, ensuring sufficient cash flow for beneficiaries upon a policyholder's death. The model employs the Gompertz-Makeham (GM) mortality formula to determine age-related death probabilities and uses commutation functions for streamlined computations. Additionally, a unique cash flow function considers each policyholder's expenses, income, and assets, maximizing their utility.

Data and Methods: Using data from the Human Mortality Database (HMD) and the Organisation for Economic Co-operation and Development (OECD), we calibrated and tested our model across 41 countries between 1959 and 2020. The results validate the model's applicability for calculating annuity payments, accounting for individual circumstances, life expectancy variability, and fluctuating annual income. This approach assures the insured's family of the expected cash flow in the event of the insured's death, thereby enhancing the robustness and attractiveness of life insurance policies.

Results: With the exception of Belarus, Greece, and Ukraine or where data were omitted, most major countries investigated have observed a positive correlation ($r^2 \geq 0.9$) between life expectancy and annual average income data from the OECD. Furthermore, the study employed SciPy to track life expectancy, which revealed a strong positive correlation over the years (approximately 1960 through 2021) in the life table. This supports the robustness of the GM model and the linearity model in modeling the life insurance industry. Lastly, an interest-based 3D model of the annuity was presented, which factors in the average life table parameters, and showed that the annuity is easy to adjust.

Policy Implication: In addition to the model's immediate utility, this study briefly examines the societal aspects of issuing insurance policies. As more individuals acquire insurance coverage, social behavior and economic dynamics are influenced mutually. Investigating the infomediary role in purchasing insurance and the interrelationships between insurance policies and societal impact highlights the concept of network effects. This research has the potential to glean insights extending beyond traditional risk management.

1 Introduction

Life insurance serves as a vital instrument in providing financial security to families upon the insured individual's death [9]. A key challenge in establishing the appropriate annuity payment for a life insurance policy lies in the uncertainties surrounding life expectancy and the variability in financial requirements. As life expectancy is a random variable function of age, and the desired cash flow upon death

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might differ based on factors such as family expenses, income, and assets, it is necessary to develop an effective mathematical model to address these concerns.

In this study, the Gompertz-Makeham model is adopted to estimate age-specific death probabilities, thereby capturing the exponentially increasing risk of death with age and accounting for age-independent factors contributing to mortality. Commutation functions are employed to streamline complex calculations in life insurance mathematics, bridging the gap between the mortality model and financial aspects. Furthermore, a cash flow function is designed that takes into account a range of factors, including family expenses, immediate expenses, future expenses, current annual income, and existing assets.

The comprehensive approach enables the calculation of the fair annuity value based on an individual's age and specific financial circumstances, ultimately ensuring that the insured's family receives the desired cash flow upon their death. By accounting for the randomness of life expectancy and the unique financial situation of each individual, the model offers a professional and reliable method for determining appropriate annuity payments in the context of life insurance policies.

1.1 Assumption

There are several assumptions that must be made in order to form a model:

1. We assume that the mortality rates are based on natural deaths and do not account for external factors such as accidents, war, or pandemics. Meaning that the uncertainty in the model comes only from the randomness of the age at death.

2. We assume that the interest rates are constant over the entire period, which simplifies the calculations but may not reflect real-world fluctuations in interest rates.

3. We assume that the Gompertz-Makeham law of mortality accurately represents the age-specific probability of death for the population being considered.

4. We assume that the present value of cash flow Y that the family will receive upon the policyholder's death does not vary with the policyholder's age. This simplification may not capture changes in the family's financial needs as the policyholder ages.

5. We assume that the policyholders' behavior, such as their willingness to pay annuities or the financial needs of their families, remains constant over time. That is the policyholder will continue to pay the annuity until their death, with no lapses or surrenders of the policy.

6. We assume that there are no taxes or expenses associated with the annuity payments or the cash flow received by the family.

2 Method

In this model, We approach this life insurance annuity problem as a potential consumer. Recognizing that there is imperfect competition with asymmetric information. We decide to divide the modeling process into three main components: mortality modeling, financial aspects, and cash flow considerations.

For the mortality modeling, since we are considering the life expectancy as a random variable function of the age, we primarily focused on the Gompertz-Makeham model. The Gompertz-Makeham law of mortality describes age-specific death probabilities, and this law has been used by actuaries and economics to model and forecast mortality rates [4]. In a protected environment where external causes of death are rare, the Gompertz-Makeham model emphasizes on the natural causes where the age-independent mortality component is often negligible [5]. The Gompertz-Makeham law provides a flexible and adaptable way to model the age-specific probability of death, using its parameters (A, B, and C) to adjust the mortality rates according to the population being considered.

As for the financial aspects, we considered the nominal interest rate and the corresponding discount factor 'δ' for the purpose of present value. We are able to access to nominal interest rate from the Federal Reserve website and then calculate the discount factor[1].

Regarding cash flow considerations, which is also known as the coverage in insurance. We aimed to create a function that represents the desired cash flow for the family upon the insured's death [2]. This function incorporates various factors, and can be personalized by weighting these factors accordingly. Note that, in our calculation, we are calculating our desired cash flow in present value, and the real future cash flow after the family receive after death should be calculated separately by considering the inflation.

In order to determine the correct annuity to pay for a life insurance policy, we combined these components into a unified model. We used commutation functions D_x and N_x to simplify calculations and connect the mortality model with the financial aspects and cash flow function [6]. In this model, the ratio between D_x and N_x is used to determine the fair annuity payment for a life insurance policy.

By following this approach, we were able to create a comprehensive model that accounts for the randomness of life expectancy, financial aspects, and the desired cash flow for the family in the event of the insured's death.

3 Fair Annuity Calculation Model

3.1 Variables

- x: Age of the insured individual (years).
- t: Number of years since the policyholder's age x (years).
- ω: Maximum attainable age (years).
- i: Nominal interest rate (percentage or decimal).
- δ: Discount factor, calculated as $\delta = \frac{1}{1+i}$.
- ${}_tq_x$: Probability of death within t years for a life aged x.
- ${}_tp_x$: Probability of survival for t years for a life aged x.
- A_x: Fair annuity value for a life insurance policy based on age x and the desired cash flow function B (USD).
- D_x: Present value of a sum assured payable immediately upon the death of a life aged x (USD).
- N_x: Present value of a whole life annuity-due of 1 per year payable at the beginning of each year of life to a life aged x (USD).
- Y: Desired cash flow for the insured's family upon death, which can be a function of family expenses, immediate expenses, future expenses, current annual income, and existing assets (USD).
- A: Constant parameter in the Gompertz-Makeham mortality model.
- B: Constant parameter in the Gompertz-Makeham mortality model.
- C: Constant parameter in the Gompertz-Makeham mortality model.
- $\mu(x)$: Force of mortality (hazard rate) at age x in the Gompertz-Makeham mortality model.

3.2 The Gompertz-Makeham Mortality Model

The Gompertz-Makeham model is a widely used mortality model that defines the force of mortality (hazard rate) at age x as:

$$\mu(x) = Ae^{Bx} + C$$

where A, B, and C are constants. And according to Missov and Lenart(2013), $A \approx 0.00046$, $B \approx 0.094$, and $C \approx 0.0007$[7]. Since today in 2023, we are experiencing a technology shock and great increase in medical related services, We are also expecting the value of $\mu(x)$ to drop. This could lead to the decrease in the parameters. And we assumed that Gompertz-Makeham parameters in these scenarios are $A = 0.0003$, $B = 0.094$ and

$C = 0.0005$.The Gompertz-Makeham model is used to estimate age-specific death probabilities, which are essential for life insurance and annuity calculations.

3.3 Commutation Functions

Commutation functions are used in life insurance mathematics to simplify calculations by connecting mortality models with financial aspects. In this study, we focus on two main commutation functions: D_x and N_x. We also discuss the role of interest rates in these functions[10].

3.3.1 Interest Rate and Discount Factor

The interest rate plays a crucial role in life insurance calculations, as it determines the present value of money and highly related to the purchasing power. The nominal interest rate fluctuates around 0.05 over times, and we can calculate the discount factor δ by:

$$\delta = \frac{1}{1+i} \tag{1}$$

3.3.2 Commutation Function D_x

D_x represents the present value of a sum assured payable immediately upon the death of a life aged x. It is calculated as follows:

$$D_x = \sum_{t=0}^{\omega-x} {}_tq_x \delta^t \tag{2}$$

where ${}_tq_x$ is the probability of death within t years for a life aged x, this can be gained using the cumulative hazard rates μ, δ is the discount factor based on the nominal interest rate, and ω represents the maximum attainable age [8].

3.3.3 Commutation Function N_x

N_x represents the present value of a whole life annuity-due of 1 per year payable at the beginning of each year of life to a life aged x. It is calculated as follows:

$$N_x = \sum_{t=0}^{\omega-x} {}_tp_x \delta^t \tag{3}$$

Similarly, ${}_tp_x$ is the probability of survival for t years for a life aged x, this can be calculated using $1 - {}_tq_x$, and δ is the discount factor based on the nominal interest rate, and ω represents the maximum attainable age [8].

3.4 Fair Annuity Model

The fair annuity value A_x for an individual aged x can be calculated using the following equation:

$$A_x = \frac{D_x Y}{N_x}$$

In this model, the fair annuity value A_x is based on the individual's age x, the discounted expected number of people alive at age x (D_x), the discounted expected number of people alive at age $x+1$ (N_x), and the death benefit (Y). The death benefit function, Y, represents the desired cash flow for the insured's family upon their death, accounting for various financial factors:

$$Y = w_F \times F \times n + w_M \times M + w_E \times E - w_I \times I - w_S \times S$$

where F represents the family's annual living expenses, M represents immediate expenses, E represents future expenses, I represents current annual income, S represents existing assets, and w_F, w_M, w_E, w_I, and w_S are the corresponding weights assigned to each factor. And n represents the number of years family are going to live upon the death benefit. Note that they are all in present values.

4 Explanations

In this section, We want to provide a comprehensive explanation of each model involved in our analysis. The purpose is to offer a thorough understanding of the underlying concepts, assumptions, and calculations that constitute the foundation of our approach. Graphs are also included in the analysis for better understanding.

4.1 Calculating Gompertz-Makeham Death Probabilities

First, we need to calculate the age-specific death probabilities using the Gompertz-Makeham mortality model. Using the provided parameters, we can compute,and also describe the force of mortality $\mu(x)$ for each age x:

$$\mu(x) = Ae^{Bx} + C \tag{4}$$

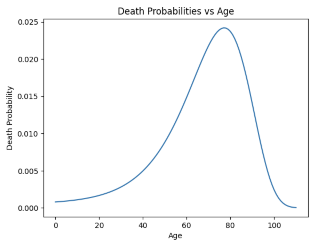


Figure 1: Death Probabilities vs Age

4.2 Computing Commutation Functions D_x and N_x

Next, we compute the commutation functions D_x and N_x using the calculated death probabilities and the nominal interest rate. The discount factor δ is calculated as follows:

$$\delta = \frac{1}{1+i} = \frac{1}{1+0.05} \approx 0.9524 \tag{5}$$

Then, we calculate D_x and N_x using the respective summation formulas:

$$D_x = \sum_{t=0}^{\omega-x} {}_tq_x\delta^t \tag{6}$$

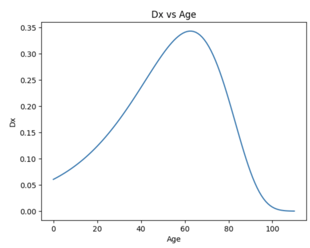


Figure 2: Dx vs Age

The D_x plot illustrates the present value of the expected number of deaths across ages, considering the mortality rates derived from the Gompertz-Makeham model. The time value of money is relevant in the calculation of D_x, as it takes into account the discount factor δ. This discount factor adjusts the expected number of deaths at different ages to account for the time value of money, acknowledging that a death occurring later has a lower present value than one occurring earlier.

$$N_x = \sum_{t=0}^{\omega-x} {}_tp_x\delta^t \tag{7}$$

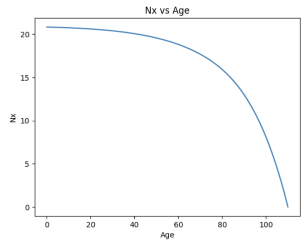


Figure 3: Nx vs Age

On the other hand, N_x represents the expected present value of the number of people alive at age x. It takes into account the survival probabilities at each age and discounts them to the present value using the discount factor δ. The N_x plot illustrates how the expected present value of the number of people alive changes across different ages, considering the survival probabilities derived from the Gompertz-Makeham model.

4.3 Determining the Fair Annuity Value A_x

Finally, we calculate the fair annuity value A_x using the commutation functions and the desired cash flow function Y:

$$A_x = \frac{D_xY}{N_x}. \tag{8}$$

When we calculate the fair annuity using the Dx and Nx values, we are essentially dividing the present value of the death benefits by the present value of the survival probabilities. This ratio gives us the fair annuity value for the insurance product, which represents the amount of money the policyholder should pay annually to receive the death benefits when they pass away.

5 Validation And Examples

This section provides an in-depth exploration of our model by applying it to a specific example. By doing so, we can determine the fair annuity value for a life insurance policy based on the individual's age, desired cash flow, and mortality assumptions.

5.1 Annuity Value Calculation

Understand that it's difficult to calculate everything by hand, I decide to use Python and to calculate the expected values of Nx, Dx, the cash flow Y and the annuity A through proper assumptions in the financial aspects and age.

Based on the code above, we can analyze a specific scenario for a typical Math professor at the age of 40, given the following parameters:

- Annual living expenses $F = 100000$
- Immediate expense $M = 40000$
- Future expenses $E = 800000$
- Current annual income $I = 200000$
- Existing assets $S = 1000000$
- Number of years $n = 2$
- Weights for each factor: $w_F = 0.1$, $w_M = 0.3$, $w_E = 0.3$, $w_I = 0.1$, $w_S = 0.2$

And we have:

- D_x at age 40: 0.371
- N_x at age 40: 19.939
- Death benefit today: 52000.0 dollars
- Future value of the death benefit (assuming the insured lives another 50 years): 227963.113 dollars
- Fair annuity value to be paid starting at age 40: 967.939 dollars

5.2 Annuity Value versus Age Graph

While we take the death benefit as given as we had in the prior section, we are able to present the graphical representation of the annuity value as a function of age as below:

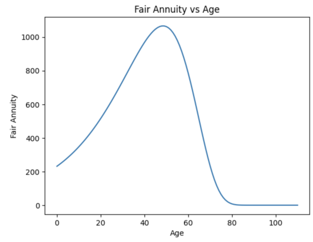


Figure 4: Death Probabilities vs Age

5.3 Insights and Interpretations

The Annuity Value versus Age Graph exhibits a bell-shaped trend, with annuity value initially increasing and then decreasing with age. However, this simplified model has limitations and is best suited for individuals within the approximate age range of 30 to 70. When applying this model, it is essential to consider each person's unique circumstances.

6 Validation on Real-world Data: A multi-national approach

6.1 Methodology

In our study, we employ `Python`, a popular programming language, for analyzing and visualizing the relationship between life expectancy and annual income across various countries. We utilize a range of `Python` libraries like `pandas`, `glob`, `matplotlib.pyplot`, `sklearn.linear_model`, `numpy`, and `scipy.stats` to perform these computations and generate insightful visuals.

Data Acquisition: Our script begins by loading demographic and mortality data available in files of format `*.bltper_1x1.txt`. Each file corresponds to a distinct country, and the country code is inferred directly from the file name. This data is read into individual `pandas` DataFrames and subsequently amalgamated into a single DataFrame for ease of manipulation.

Life Expectancy Computation: The life expectancy for each unique year is calculated by employing a custom function

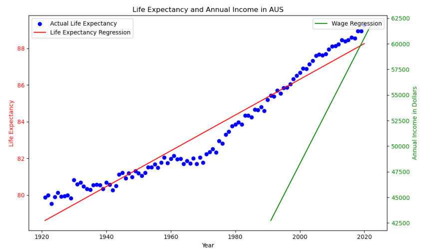
`calculate_life_expectancy()`. This function estimates life expectancy as the mean of the sum of the 'Age' and 'ex' columns in the data. In the event of non-numeric types or missing values, it gracefully defaults to computing the mean of the 'ex' column alone.

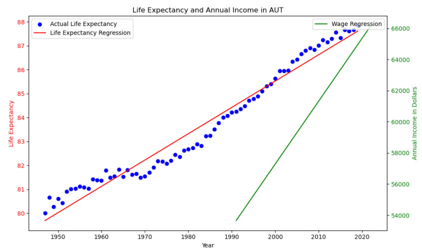
Life Expectancy Visualization: Grouping the DataFrame by 'Country', our script produces a scatter plot for each country to represent life expectancy against the progression of years. A linear regression model is then fitted to this data, and the resultant regression line is superimposed on the scatter plot to provide a graphical representation of the trend.

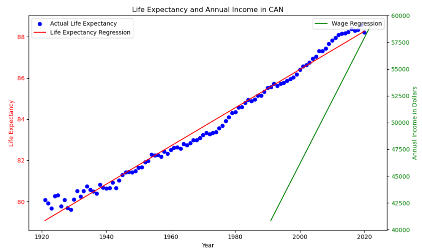
Income Analysis: To incorporate the financial aspect, our script reads another CSV file that is expected to contain information about annual income in different locations. This data is grouped by location and a linear regression is performed on the 'Value' (presumably income) column against the 'TIME' column for each group. The output of the regression – slope, intercept, and the R-squared value – are meticulously stored for further analysis.

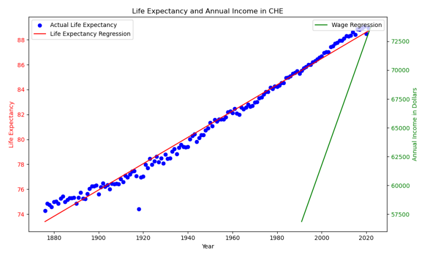
Life Expectancy vs. Annual Income Visualization: In the final stage, scatter plots of life expectancy against years are overlaid with the linear regression line to visualize the trends and correlations. Concurrently, the annual income regression line is graphed on a secondary y-axis, facilitating the direct comparison of trends in life expectancy and annual income for each country.

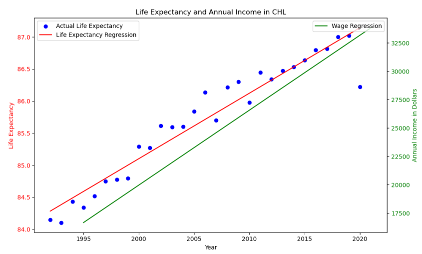
This procedural method not only explores but also graphically elucidates the relationship between life expectancy and annual income across various countries over time. However, one must interpret the findings considering the scope of the available data and any computational limitations. These findings are instrumental in highlighting temporal trends, disparities between countries, and the correlation between life expectancy and annual income. An important caveat to note is that the analysis inherently assumes the quality and completeness of the data, which if compromised, could affect the outcomes. Further studies could involve more variables and data from additional sources to provide a more holistic understanding of global life expectancy trends.

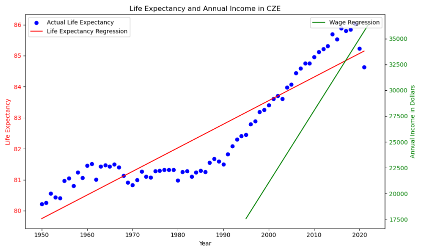








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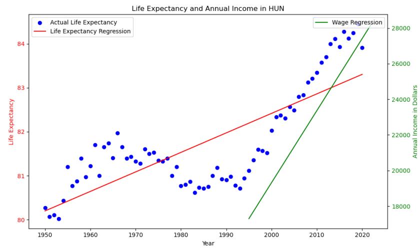
Life Expectancy Regression
Wage Regression
Life Expectancy
Annual Income in Dollars
Year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Life Expectancy
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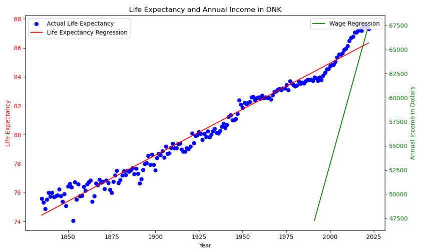
Year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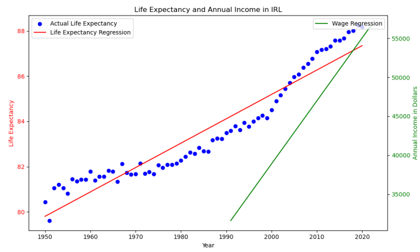
Year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Life Expectancy
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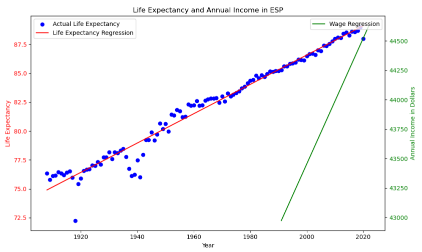
Year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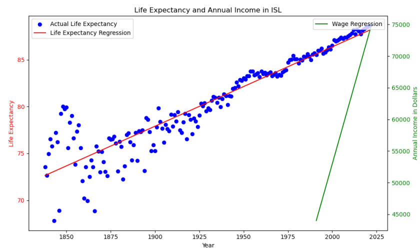
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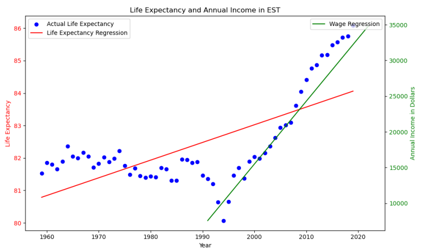
Annual Income in Dollars
Year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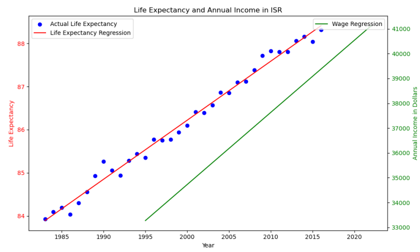
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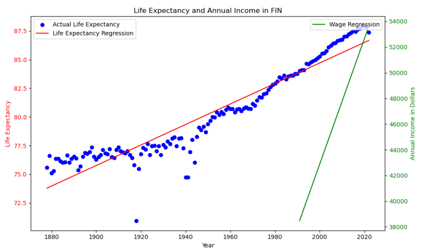
Year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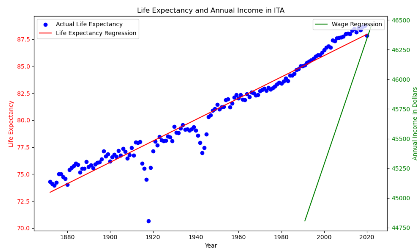
Year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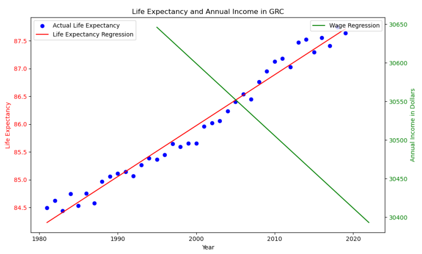
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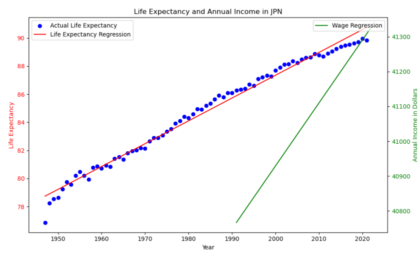
Annual Income in Dollars
Year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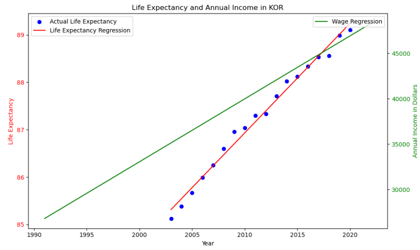
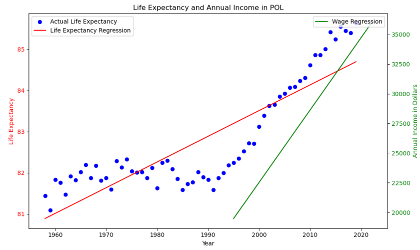
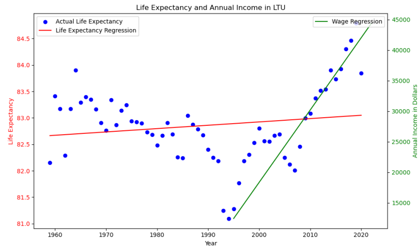
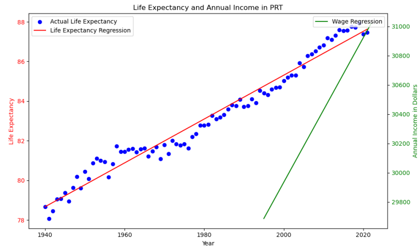
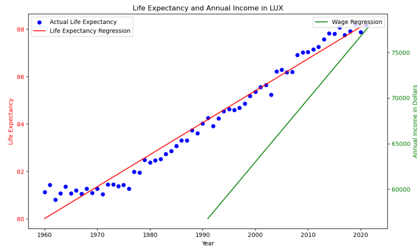
Wage Regression
Life Expectancy
Annual Income in Dollars
Year

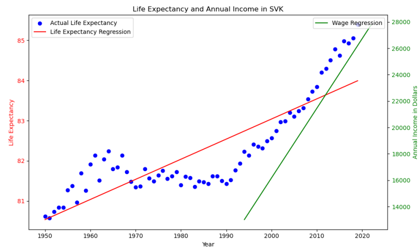


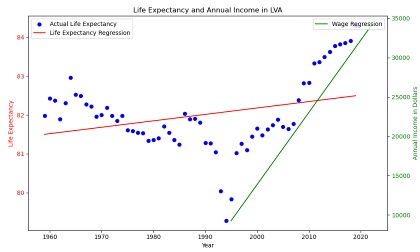


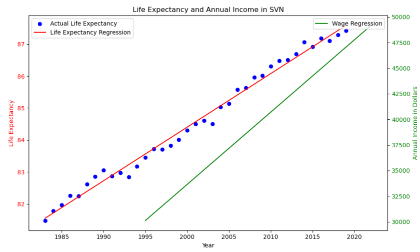


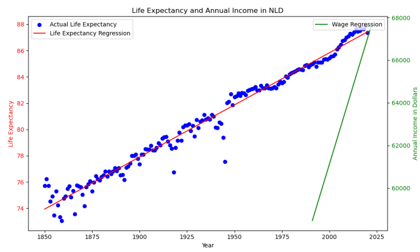


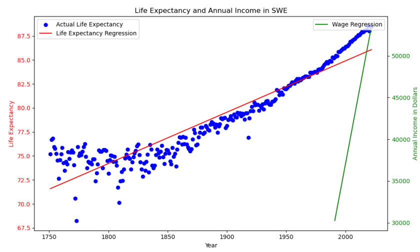


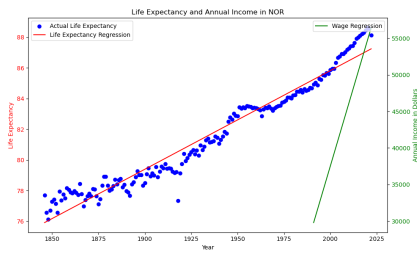












6.2 Data Table

In addition to the visualization, data table was also used to demonstrate possibility of applying life table and correlates with insurance, as shown below.

For each country, there are four different indicators given: 1) LES (Life Expectancy

Slope): This is likely indicating the rate of change of life expectancy over a certain time period. A higher value would suggest that life expectancy is increasing at a faster pace. 2) LER (Life Expectancy Regression Coefficient): The regression coefficient in the context of life expectancy probably indicates how well the life expectancy can be predicted by one or more other variables. A value close to 1 suggests a strong positive linear relationship. 3) WS (Wage Slope): This likely indicates the rate of change of wages over a certain time period. A higher value could suggest that wages are increasing at a faster pace. 4) WR (Wage Regression Coefficient): The regression coefficient in the context of wages likely indicates how well the wages can be predicted by one or more other variables. A value close to 1 suggests a strong positive linear relationship.

By analyzing this data, we could gain insights into how life expectancy and wages are changing in different countries, and how strongly these changes are associated with other factors (possibly factors not provided in this table). For example, most countries, like South Korea (KOR) and Slovenia (SVN) show strong linear relationships and significant positive slopes for both life expectancy and wage changes, suggesting strong growth and potential positive correlation between these two factors. There are also some outliers like Greece (GRC) with a negative WS value, indicating decreasing wages over the observed period, or Estonia (EST) and Latvia (LVA) with very low LER coefficients, suggesting weak or non-linear relationships between life expectancy and the other variables.

In evaluating the dataset provided, a significant trend is observable which suggests that, in the majority of countries included in this analysis, life expectancy (LE) appears to correlate positively with wage slope (WS).

Despite a few outlier nations that do not adhere to this pattern, the prevailing trend highlights a generally positive relationship between these two factors. This indicates that in nations where life expectancy increases, there is also a tendency for wages to rise over the same period. It is worth noting that while this is a general trend, the strength and consistency of this correlation vary considerably across the countries studied.

Country	**LES**	**LER**	**WS**	**WR**
AUS	0.10	0.940	615.27	0.984
AUT	0.11	0.965	403.96	0.974
CAN	0.09	0.985	588.27	0.986
CHE	0.11	0.985	545.32	0.968
CHL	0.10	0.923	660.20	0.966
CZE	0.08	0.835	697.12	0.979
DNK	0.06	0.973	653.61	0.986
ESP	0.13	0.966	53.41	0.416
EST	0.05	0.453	880.88	0.981
FIN	0.09	0.890	488.36	0.962
GRC	0.09	0.976	-9.37	-0.019
HUN	0.04	0.615	404.45	0.895
IRL	0.11	0.916	809.03	0.956
ISL	0.08	0.859	993.39	0.855
ISR	0.14	0.989	290.82	0.760
ITA	0.10	0.944	52.99	0.456
JPN	0.16	0.986	18.07	0.294
KOR	0.23	0.989	694.78	0.986
LTU	0.01	0.025	1183.52	0.972
LUX	0.13	0.972	689.22	0.988
LVA	0.02	0.093	918.06	0.971
NLD	0.08	0.973	294.45	0.885
NOR	0.06	0.954	860.17	0.983
POL	0.06	0.744	609.77	0.961
PRT	0.11	0.969	49.39	0.374
SVK	0.05	0.723	527.56	0.989
SVN	0.17	0.990	706.47	0.974
SWE	0.05	0.874	752.03	0.990

6.3 Interpretation

7 Policy Implication

This study's findings have significant implications for insurance policy, societal behavior, and economic dynamics. The increase in insurance coverage among individuals has led to noticeable shifts in social behavior and economic activity. These shifts may be attributed to increased financial security among individuals and reduced stress associated with unexpected costs, thus improving overall economic health.

The results indicate that insuring more individuals has a knock-on effect on the wider society. Increased coverage can lead to healthier and more economically stable communities. Furthermore, this can lessen the burden on public services due to lower reliance in times of financial crises.

Secondly, the study highlights the crucial role of insurance providers and brokers as infomediaries in facilitating the purchase of insurance. These infomediaries assist individuals in navigating complex insurance markets and contribute to raising awareness and understanding about the role of insurance in risk management. Greater public understanding of insurance products can lead to increased insurance coverage, creating a positive feedback loop that promotes societal stability and economic development.

Thirdly, the study highlights the concept of network effects in insurance markets. Increased insurance coverage among individuals may lead to lower premiums due to the expanded risk pool, which would make insurance even more accessible to a wider population. This can improve societal resilience to unforeseen financial crises.

This research takes a step towards expanding the conventional perspective of insurance as only a risk management solution. Additionally, it presents a wider viewpoint that encompasses the societal consequences of broad insurance coverage. Understanding these impacts can enable policymakers to make more knowledgeable choices regarding regulations, subsidies, and public awareness campaigns that can enhance the benefits of insurance.

Further research could investigate the underlying mechanisms of these implications and provide more precise quantification of their impacts. Such research could enhance our understanding and result in more effective policy recommendations. Insurance can bring benefits beyond the individual level, and it is important for our policy to reflect this understanding.

8 Conclusion

In conclusion, discussing death can be uncomfortable due to its uncertainties. However, by purchasing insurance, we can minimize these uncertainties and ensure our loved ones receive financial support to continue living well [3]. This study presents a comprehensive mathematical model for calculating fair annuity payments for life insurance policies, considering life expectancy uncertainties and varying financial needs. By employing the Gompertz-Makeham mortality model to estimate age-specific death probabilities and utilizing commutation functions, we streamline complex actuarial calculations, making it easier to determine appropriate insurance coverage.

Central to our model is the equation for determining the fair annuity value:

$$A_x = \frac{D_x Y}{N_x}$$

This equation incorporates the cash flow function Y, which considers various factors such as family expenses, immediate expenses, and to mention only a few, allowing for the adaptation of the model to cater to an individual's specific financial situation.

Despite its strengths, the model has certain limitations, such as the assumptions of constant interest rates and the exclusion of deaths caused by external factors. Additionally, the Gompertz-Makeham model may not capture mortality patterns for all populations or account for improvements in life expectancy over time. Furthermore, it may only be suitable only for a specific age range, and the applicability of the model for each individual should be considered.

Nevertheless, the model's ability to calculate fair annuity values by incorporating individual financial circumstances and uncertainties in life expectancy makes it a promising approach for determining appropriate annuity payments in life insurance policies. Future research could address the model's limitations by incorporating more complex mortality models, time-varying interest rates, or other factors to improve its accuracy and applicability.

In summary, the proposed model offers a valuable method for calculating fair annuity payments in life insurance policies, accounting for uncertainties in life expectancy, financial needs, and individual circumstances.

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