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CONVERGENCE OF VORTEX METHODS FOR EULER'S EQUATIONS, III*

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Abstract. We prove the convergence of a large class of vortex methods for two-dimensional incompressible, inviscid flows with Hölder continuous initial data. We present several infinite order methods and establish fourth order rate of convergence in the time variable when the ordinary differential equations in the vortex method are solved by the classical Runge-Kutta method.

Key words. convergence, vorticity, Euler, Hölder, Runge-Kutta

AMS(MOS) subject classifications. 65M99, 76C05

Introduction. The purpose of this paper is threefold. First we prove the convergence of almost all vortex methods that have been considered (and some that have not). Our initial data have compact support and can be assumed to be merely Hölder continuous. Secondly we present a class of infinite order vortex methods. The rate of convergence depends solely on the smoothness of the underlying flow and can be arbitrarily high, a property which is obviously very desirable. Finally we prove that the rate of convergence in the time variable is of fourth order if the ordinary differential equations that arise in the vortex method are solved by the classical Runge-Kutta method, and if the flow is sufficiently smooth.

The vortex method is a numerical technique for approximating the flow of an incompressible fluid and we consider the two-dimensional inviscid case. The basic idea is to approximate the vorticity distribution by a collection of radially symmetric vortex blobs of fixed shape and let the center of the blobs be transported by the velocity field induced by the approximate vorticity distribution (see Chorin [8]). The vortex blobs are translates and multiples of a particular approximation to the delta function. Beale and Majda [3] call this approximation the cutoff function and the accuracy of the vortex method depends upon the choice of this function. The strength of a vortex blob is the total amount of vorticity it carries. It is assigned either by pointwise evaluation of the initial vorticity (Beale and Majda [4]) or by an average of the initial vorticity in a neighborhood of the center of the blob (Del Prete and Hald [11]). Cottet [9] has pointed out that the last choice restricts the possible rates of convergence, and this has been checked numerically by Perlman [24].

In the first part of the paper we prove the convergence of the vortex method under weaker assumptions than previously used. The smoothness of the vorticity field has been reduced from the previous three times continuous differentiability to just Hölder continuity. It should be mentioned that Del Prete and Hald [11] also assumed Hölder continuity, but they did not prove convergence for long time intervals. A second extension concerns the cutoffs. Our present class of cutoff functions includes all those that have been suggested so far—with one exception, namely, the infinite order cutoff of Leonard [19]. In particular, we get convergence for Chorin's [8] cutoff which is singular, and for the cutoff used by Milinazzo and Saffman [21] which is bounded but discontinuous. These two cutoffs are not covered by the theories by Hald [15], Beale and Majda [4], Cottet [9] or Anderson and Greengard [2], but are treated by Del

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Prete and Hald [11]. We also include cutoffs that decay slowly at infinity. These cutoffs occur naturally in connection with infinite order methods, but Leonard's cutoff decays too slowly to be included.

It is not enough that the spatial accuracy of the vortex method is satisfactory. We must also solve the resulting ordinary differential equations, but cannot expect high order convergence if the flow is not smooth. We consider the same second order Runge–Kutta method as Anderson and Greengard [2] and obtain quadratic convergence for flows that are only once differentiable in space and time. This is a very weak assumption, and it is surprising that it suffices. (Anderson and Greengard assumed the flow was three times continuously differentiable.) Our proof is based on a new property of the solutions of Euler's equations, namely that the Lagrangian representation of the flow is infinitely differentiable in the time variable (see Hald [17]). In other words, if we look at a particle that moves with the flow, then its x and y coordinates are smooth functions of time. Finally we consider two options for prescribing the strength of the blobs and prove that the computed vorticity field converges uniformly if we just delete at each point a single vortex blob. This is unexpected, since the remaining vortex blobs may be singular.

In the second part of the paper we present a method that will work for all flows and give a high rate of convergence for smooth flows. This leads to a new class of cutoff functions and to several examples of cutoffs (given in terms of Bessel functions). In these examples it is not the cutoff function that restricts the rate of convergence, but the smoothness of the flow. These methods can therefore properly be called infinite order methods, and our theory for this class is analogous to the theory of Fourier methods applied to Euler's equations (see Hald [16]). The possibility of constructing infinite order vortex methods was pointed out by Beale and Majda [3]. However the explicit cutoff function that we will present will not be a member of the Fourier symbol class which they considered. The theories of Beale and Majda [4], Cottet [9] and Anderson and Greengard [2] are no longer directly applicable and our whole theory is necessary to prove convergence.

One must also solve the ordinary differential equations numerically. We choose to investigate the classical fourth order Runge–Kutta method because (1) we are most interested in smooth flows and (2) Anderson and Greengard [2] have established high order convergence for a class of multistep methods but could not obtain similar results for the higher order Runge–Kutta methods. It turns out that the classical Runge–Kutta method can be viewed as a linear combination of first and second order accurate approximations to the solution at intermediate times. The intermediate steps represent flows which are not area preserving, and this has forced us to study consistency and stability for such flows. Our convergence proof is based on an asymptotic expansion of the solution at the intermediate steps, and entails a loss of a derivative at each step. It can therefore only be used for smooth flows.

The outline of the paper is as follows. In § 1 we present the Lagrangian formulation of Euler's equations as well as the vortex method. We introduce two specific cutoffs, one singular and one very smooth. These cutoffs belong to two different classes, which we define. In § 2 we state the assumptions about the smoothness of the flow. We can approximate the flow by taking one step of Euler's method instead of solving the ordinary differential equations in the Lagrangian representation of the flow. It turns out that if the time step is sufficiently small, then the approximate flow is invertible and as smooth as the real flow.

In §§ 3 and 4 we consider lower order methods and higher order methods, respectively. In both cases we rely on the fact that consistency plus stability imply

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convergence. Consistency of the lower order methods is analyzed in § 3.1. The argument is an updated version of Del Prete and Hald [11]. Our main technical tool is a strengthening of the stability result due to Beale and Majda [4]. It is presented in § 3.2, where we admit singular cutoffs and cutoffs that decay slowly at infinity. Moreover we establish stability for flows which are not area preserving and therefore not incompressible. This step seems unavoidable if one wants to prove convergence for higher order Runge-Kutta methods. In § 3.3 we estimate the truncation error in the time direction for a second order Runge-Kutta method. The proof follows Anderson and Greengard [2], but our result is sharper because we use the smoothness of the flow map as a function of time. Finally, in § 3.4 we prove that the computed trajectory of a particle is close to the corresponding trajectory in the exact flow. We assume that the initial vorticity is Hölder continuous and has compact support. Our class of cutoff functions contains many higher order cutoffs in addition to the singular ones; however, even with the infinite order cutoffs the rate of convergence is low because the underlying flow is not smooth.

It is not difficult to show that the computed velocity field converges once the trajectories of the particles converge. Such proofs have been given by Del Prete and Hald [11], Beale and Majda [4] and Cottet [9] and we shall not repeat the arguments. The convergence of the vorticity computed with a singular cutoff has not been proved before, and we will show that the computed vorticity field converges uniformly.

Infinite order methods are treated in §§ 4.1 to 4.4. In § 4.1 we establish the consistency of the infinite order methods by modifying the proof of Anderson and Greengard [2]. Part of their proof can be traced back to Cottet and Raviart [10]. In § 4.3 we estimate the truncation error of the classical fourth order Runge-Kutta method. This is the hardest part of the proof. The flows in the intermediate steps are not area preserving, and this leads to unexpected terms in the truncation error. If one does not take these terms into account one gets only second order convergence as in Anderson and Greengard [2], but fortunately the extra terms cancel eventually. The technique should also be applicable to other Runge-Kutta methods, but apart from Nyström's third order method I have not checked the details. Finally in § 4.4 we prove the convergence of the infinite order methods for flows with a finite number of derivatives. We also present, but do not prove, a convergence result for vortex methods with finite order cutoffs and where the ordinary differential equations are solved by the classical Runge-Kutta method.

This paper does not contain numerical experiments that support the theory. The spatial accuracy of the vortex method has been investigated by Del Prete and Hald [11], Perlman [24] and Beale and Majda [5]. In all cases the cutoff functions have finite order. The discretization in the time direction has not been viewed as a source of difficulty. Chorin [8] and Nakamura, Leonard and Spalart [22] have used Euler's method, Milinazzo and Saffman [21] and Sethian [25] have used second order Runge-Kutta methods, and the fourth order Runge-Kutta method has been used by Del Prete and Hald [11], Nakamura, Leonard and Spalart [22], Anderson [1] and Beale and Majda [5]. In § 3.3 we present an example that shows that if the velocity field is given, then a fourth order Runge-Kutta method may not converge faster than a second order method. In the example the velocity field is Lipschitz continuous, but not differentiable. It is difficult in practice to estimate the rate of convergence for the vortex method in the time direction.

The outcome of this paper is the following set of recommendations: If the initial vorticity is not smooth, then there are two options. One can use either an arbitrary cutoff and determine the strength of the vortices by local averages, or one can use a

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smooth cutoff and determine the strength by pointwise evaluation of the initial vorticity. In both cases one should use a second order Runge-Kutta method to solve the ordinary differential equations. Our theory indicates that the first algorithm converges faster than the second, but this may fail to be true in practice. If the initial vorticity is smooth we use a higher order cutoff, compute the strength of the vortices by pointwise evaluation, and solve the ordinary differential equations by a higher order Runge-Kutta method.

So what's new? We have proved the convergence for a larger class of vortex methods than previously considered and under weaker assumptions on the initial data. We have found simple infinite order methods and a convergence proof for higher order Runge-Kutta methods.

1. The basic equations. The vortex method is a technique for solving the uncountably many ordinary differential equations

$$\frac{d}{dt}\Phi(\alpha, t) = \int K(\Phi(\alpha, t) - \Phi(\alpha', t))\omega(\alpha')\, d\alpha', \qquad \Phi(\alpha, 0) = \alpha. \tag{1.1}$$

Here $\alpha = (\alpha_1, \alpha_2)^T$, ω is a function with compact support and $K(x) = (2\pi r^2)^{-1}(-x_2, x_1)^T$ where $r^2 = x_1^2 + x_2^2$ and ^{T} means transpose. Equation (1.1) is the Lagrangian formulation of Euler's equations for a two-dimensional incompressible inviscid fluid. We interpret $\Phi(\alpha, t)$ as the position of a particle starting at α at time $t = 0$. The particle moves by the velocity field u corresponding to the vorticity distribution ω where

$$u(x, t) = \int K(x - \Phi(\alpha', t))\omega(\alpha')\, d\alpha', \tag{1.2}$$

$$\omega(x, t) = \int \delta(x - \Phi(\alpha', t))\omega(\alpha')\, d\alpha', \tag{1.3}$$

where δ denotes the delta function. To solve (1.1), we cover the α plane by nonoverlapping squares Q_j with side length h and centered at $\alpha_j = jh$ where $j = (j_1, j_2)$. The vortex method is given by

$$\frac{d}{dt}\tilde{\Phi}(\alpha_i, t) = \sum_j K_\delta(\tilde{\Phi}(\alpha_i, t) - \tilde{\Phi}(\alpha_j, t))\kappa_j, \qquad \tilde{\Phi}(\alpha_i, 0) = \alpha_i, \tag{1.4}$$

where $K_\delta = K * \zeta_\delta$ and ζ_δ is an approximation to the delta function. We set $K_\delta(0) = 0$. The approximate velocity field and the approximate vorticity distribution are given by

$$\tilde{u}(x, t) = \sum_j K_\delta(x - \tilde{\Phi}(\alpha_j, t))\kappa_j, \tag{1.5}$$

$$\tilde{\omega}(x, t) = \sum_j \zeta_\delta(x - \tilde{\Phi}(\alpha_j, t))\kappa_j. \tag{1.6}$$

The constants κ_j are evaluated at $t = 0$ and we consider two choices, namely $\int_{Q_j} \omega\, dx$ and $\omega_j h^2$. Here $\omega_j = \omega(\alpha_j)$. The approximate delta function is given by $\zeta_\delta(x) = \delta^{-2}\zeta(|x|/\delta)$ where δ is a parameter and

$$\zeta(r) = \begin{cases} (2\pi r)^{-1}, & 0 < r \leqq 1, \\ 0, & 1 < r < \infty, \end{cases} \tag{1.7}$$

$$\zeta(r) = \frac{1}{3\pi r^2}[4J_2(2r) - J_2(r)] \tag{1.8}$$

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are two opposite extreme but representative choices. Here J_2 is the Bessel function of order 2. The first cutoff is unbounded and discontinuous and has compact support. The second cutoff is bounded and smooth and its support is unbounded. The cutoff function (1.7) is due to Chorin [8], while the cutoff (1.8) is new and gives infinite order rate of convergence for smooth flows. Our interest is focused on these two cutoffs. However, for the sake of generality, in the first part of the paper we use

Assumption 1. ζ is continuously differentiable in $(0, 1)$ and $(1, \infty)$, and $\zeta(r)$ and $\zeta'(r)$ approach finite limits as r tends to 1.

$$\text{(i)} \quad |\zeta^{(\nu)}(r)| \leqq L_0 r^{-1-\nu}, \qquad 0 < r \leqq 1, \quad \nu = 0, 1,$$

$$\text{(ii)} \quad |\zeta^{(\nu)}(r)| \leqq L_0 r^{-5/2}, \qquad 1 < r < \infty, \quad \nu = 0, 1,$$

$$\text{(iii)} \quad \left| 2\pi \int_0^r s\zeta(s)\, ds - 1 \right| \leqq L_0 r^{-3/2}, \quad 0 < r < \infty.$$

The cutoff functions (1.7) and (1.8) satisfy (i), (ii) and (iii). These three properties suffice to prove the convergence of the vortex method provided the initial vorticity is Hölder continuous and $\kappa_j = \int_{Q_j} \omega dx$. If instead $\kappa_j = \omega_j h^2$ we need that ω be differentiable and that $(\partial/\partial x_i)\omega$ be Hölder continuous. However, our rate of convergence is low. To obtain higher order convergence we consider the Fourier transform $\hat{\zeta}$ of ζ. Since ζ is radially symmetric, $\hat{\zeta}$ is also. In the second part of the paper we use

Assumption 2. $\hat{\zeta}$ is a real continuous function, equal to $(2\pi)^{-2}$ for $0 \leqq t \leqq 1$ and vanishing for $t \geqq b$. In the interval $1 \leqq t \leqq b$, $\hat{\zeta}$ is continuously differentiable and $\hat{\zeta}'$ is continuously differentiable except at a finite number of points where $\hat{\zeta}''$ has right- and left-hand limits. Let $L_1 = \operatorname{ess\,sup} |\hat{\zeta}^{(\nu)}(t)|$ for $\nu = 0, 1, 2$ and $1 < t < b$.

We will show in § 4.1 that the infinite order cutoff (1.8) satisfies Assumption 2 and that this assumption implies (i), (ii) and (iii). By using Assumption 2 and $\kappa_j = \omega_j h^2$ we obtain high order convergence for smooth flows.

2. Analysis of the flow. To prove the convergence of the vortex method, we need some assumptions about the smoothness of the flow. In the first part of the paper we use

Assumption 3. There exists a unique solution $\{\omega, u, \Phi\}$ such that

(i) The vorticity $\omega(x, t)$ has compact support, is continuous in t and uniformly Hölder continuous in x.

(ii) The velocity $u(x, t)$ is continuously differentiable in x and t, $\operatorname{div} u \equiv 0$ and ∇u is uniformly Hölder continuous in x.

(iii) The flow map $\Phi(x, t)$ is infinitely differentiable in t, $\partial_t^\nu \Phi(x, t)$ is once uniformly Hölder continuously differentiable in x for all ν and $\det \Phi' \equiv 1$.

Assumption 3 describes some properties of the flow and is not the minimal assumption. Wolibner's investigation [27] shows that if the initial vorticity has compact support and is uniformly Hölder continuous then there exists a classical solution of Euler's equations, and this solution satisfies (i) and (ii) in Assumption 3. This can also be proved by modifying Kato's arguments (see [18]). It also follows from McGrath's paper [20], except that he assumes that the initial vorticity is continuously differentiable. We note that $\omega(x, t) = 1 - |x|$ for $|x| < 1$ and zero otherwise corresponds to a classical solution of Euler's equations, but it is not a classical solution of the vorticity equation, which can be derived from Euler's equations, only a weak solution. Property (iii), that the flow map is a smooth function of t, is both new and surprising. The proof will be presented elsewhere [17] as it is tangential to our main investigation.

In the remainder of this section we shall study the smoothness of ω, u and Φ in the spatial directions and suppress the time dependence. To be explicit we let the

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vorticity in Assumption 3 be Hölder continuous with exponent λ with $0<\lambda<1$ and introduce the norm

$$H_\lambda(\omega)=\sup_{x\neq y}\frac{|\omega(x)-\omega(y)|}{|x-y|^\lambda}$$

where $|x|^2=x_1^2+x_2^2$. Let D be larger than or equal to the diameter of the support of ω. Then $D^\lambda H_\lambda(\omega)$ is an increasing function of λ. To estimate ω, u and Φ, we set $\|\omega\|=\max|\omega(x)|$, where the maximum is taken over all x in the plane and introduce

$$\|\omega\|_{C^\lambda(D)}=\|\omega\|+D^\lambda H_\lambda(\omega),$$
$$\|u\|_{C^{1+\lambda}(D)}=\|u\|+D\max_{j=1,2}\|\partial_j u\|+D^{1+\lambda}\max_{j=1,2}H_\lambda(\partial_j u),$$
$$|\Phi|_{C^{1+\lambda}(D)}=\max_{j=1,2}\|\partial_j\Phi\|+D^\lambda\max_{j=1,2}H_\lambda(\partial_j\Phi)$$

where $\partial_j u=(\partial/\partial x_j)u$. Note that the first two expressions are norms, but that the third is a seminorm. This choice is natural, as Φ is unbounded. Observe also that the norms are increasing functions of D.

To analyze the flow we need the Hölder-Korn-Lichtenstein-Giraud inequality. In his dissertation from 1882, Hölder proved that if the right-hand side of Poisson's equation is Hölder continuous then the potential is twice differentiable. Actually, the second derivatives are Hölder continuous. This result is due to Korn (1907). I have not been able to determine Lichtenstein's contribution to the subject, but Giraud extended Korn's theory to singular integrals in 1934. Let $u=K*\omega$. Then

$$\partial_j u(x)=\frac{1}{2}\begin{pmatrix}-\delta_{2j}\\ \delta_{1j}\end{pmatrix}\omega(x)+\int\partial_j K(x-y)[\omega(y)-\omega(x)]\,dy$$

where we integrate either over the whole plane or over a circular disk which contains the support of ω. By following the proof of the Hölder-Korn-Lichtenstein-Giraud inequality on page 244 in Bers, John and Schechter [6], we find that

$$H_\lambda(\partial_j u)\leqq\frac{34}{\lambda(1-\lambda)}H_\lambda(\omega)$$

for $j=1,2$. If D is the diameter of the support of ω, then $\|u\|$ is less than $D\|\omega\|$ and $\|\partial_j u\|$ is less than $\frac{1}{2}\|\omega\|+2^\lambda D^\lambda\lambda^{-1}H_\lambda(\omega)$. Hence,

$$\|u\|_{C^{1+\lambda}(D)}\leqq\frac{36D}{\lambda(1-\lambda)}\|\omega\|_{C^\lambda(D)}. \tag{2.1}$$

We can now reformulate

Assumption 3. There exists a unique solution $\{\omega, u, \Phi\}$ and constants C, D, λ and $\{\lambda_\nu\}$ such that

(i) The vorticity $\omega(x,t)\in C^{\lambda,1}(\mathbb{R}^2\times[0,T])$ and $\|\omega(\cdot,t)\|_{C^\lambda(D)}\leqq C$ for $0\leqq t\leqq T$ where $0<\lambda<1$ and the diameter of the support of $\omega(x,t)$ is less than D;

(ii) The velocity $u(x,t)\in C^{1+\lambda,1}(\mathbb{R}^2\times[0,T])$ and $\operatorname{div} u\equiv 0$;

(iii) The flow map $\Phi(x,t)\in C^{1+\lambda_\nu,\nu}(\mathbb{R}^2\times[0,T])$ for $\nu=0,1,\cdots$ and $1>\lambda_0>\lambda_1>\cdots>0$ and $|\Phi(\cdot,t)|_{C^{1+\lambda}(D)}\leqq\frac{1}{2}C$ for $0\leqq t\leqq T$.

By combining (1.1) and (1.2) we see that $\Phi^\cdot=u(\Phi,t)$ where $^\cdot$ means the derivative with respect to time. We will solve this equation by various Runge-Kutta methods. If we disregard the discretization in the spatial direction, then a typical intermediate step looks like

$$\tilde\Psi(\alpha)=\Phi(\alpha)+kv(\Psi(\alpha)),\qquad v(x)=\int K(x-\Psi(\alpha'))\omega(\alpha')\,d\alpha' \tag{2.2}$$

where $k>0$ and the first choice of Ψ is Φ. Note that v depends on Ψ. To emphasize this dependence we shall, on occasion, write $v[\Psi]$ instead of $v\circ\Psi$. It is not obvious that the iterates have the same smoothness as Φ and map the plane onto itself in a 1-1 manner. In fact, if $\Psi=\Phi$, then the Jacobian of $\tilde{\Psi}$ can vanish for k sufficiently large. However, we expect that $\tilde{\Psi}$ is approximately area preserving if the time step k is sufficiently small. This is the content of

LEMMA 1. *Let Assumption* 3 *be satisfied and let* $\tilde{\Psi}$ *be determined by* (2.2). *Assume that*

$$|\Psi|_{C^{1+\lambda}(D)}\leqq C \quad \textit{and} \quad \det\Psi'(\alpha)\geqq\tfrac{1}{2}. \tag{2.3}$$

There exists a constant k_0 *such that* $\tilde{\Psi}$ *satisfies* (2.3) *for all* $k\leqq k_0$. *The constant* k_0 *depends on* C *and* λ.

Proof. Hadamard's version of the inverse function theorem says that the map Ψ is invertible if Ψ' is nonsingular and the inverse of Ψ' is uniformly bounded (see [23, p. 137]). These two conditions are satisfied because of (2.3). By using the change of variables $\alpha'=\Psi^{-1}(y)$ in (2.2) we see that $v=K*f$, where

$$f(y)=\omega(\Psi^{-1}(y))\det\Psi^{-1\prime}(y). \tag{2.4}$$

We will show that f is Hölder continuous. This implies that v is in $C^{1+\lambda}$ and, as $\partial_j\Psi$ is in C^{λ}, we find that $\partial_j(v\circ\Psi)$ is too. Hence $k|v\circ\Psi|_{C^{1+\lambda}}\leqq C/2$ for k sufficiently small and the proof is complete.

We will first investigate Ψ. Let $\Psi_{,j}=\partial_j\Psi$ and $\Psi'=(\psi_{i,j})$. By differentiating both sides of the equation $\Psi(\Psi^{-1}(y))=y$ we get

$$\Psi^{-1\prime}(y)=[\Psi']^{-1}=\frac{\begin{pmatrix}\psi_{2,2} & -\psi_{1,2}\\ -\psi_{2,1} & \psi_{1,1}\end{pmatrix}}{\psi_{1,1}\psi_{2,2}-\psi_{1,2}\psi_{2,1}}$$

where $\psi_{i,j}$ are evaluated at $\Psi^{-1}(y)$. Thus, $\det\Psi^{-1\prime}(y)=1/\det\Psi'(\Psi^{-1}(y))$ and $|\Psi^{-1\prime}(y)|\leqq 2\sqrt{2}\,C$ for all y. Note that (2.3) implies that $\frac{1}{2}\leqq C^2$. By using the fundamental theorem of calculus we obtain

$$\begin{aligned}|\Psi(\alpha)-\Psi(\bar{\alpha})|&=\left|\int_0^1\Psi'(\bar{\alpha}+s(\alpha-\bar{\alpha}))\cdot(\alpha-\bar{\alpha})\,ds\right|\leqq\sqrt{2}\,C|\alpha-\bar{\alpha}|,\\ |\Psi^{-1}(y)-\Psi^{-1}(\bar{y})|&=\left|\int_0^1\Psi^{-1\prime}(\bar{y}+s(y-\bar{y}))\cdot(y-\bar{y})\,ds\right|\leqq 2\sqrt{2}\,C|y-\bar{y}|.\end{aligned} \tag{2.5}$$

These estimates are also valid for Φ, but since this flow is incompressible the constants in (2.5) can be replaced by $C/\sqrt{2}$. Our next calculation shows that $\det\Psi^{-1\prime}(y)$ is Hölder continuous. Let $\bar{\Psi}_{,j}=\Psi_{,j}(\Psi^{-1}(\bar{y}))$. By using the last inequality in (2.5) we have

$$\begin{aligned}|\det\Psi^{-1\prime}(y)-\det\Psi^{-1\prime}(\bar{y})|&\leqq\frac{|\Psi_{,1}-\bar{\Psi}_{,1}||\bar{\Psi}_{,2}|+|\Psi_{,2}-\bar{\Psi}_{,2}||\Psi_{,1}|}{\det\Psi'(\Psi^{-1}(y))\cdot\det\Psi'(\Psi^{-1}(\bar{y}))}\\ &\leqq 8C^2D^{-\lambda}(2\sqrt{2}\,C)^{\lambda}|y-\bar{y}|^{\lambda}.\end{aligned}$$

We can now show that the function f in (2.4) is Hölder continuous. By adding and subtracting the term $-\omega(\Psi^{-1}(\bar{y}))\det\Psi^{-1}(y)$, using the assumptions in the lemma and the results for Ψ we get

$$|f(y)-f(\bar{y})|\leqq[2\cdot CD^{-\lambda}(2\sqrt{2}\,C)^{\lambda}+C\cdot 8C^2D^{-\lambda}(2\sqrt{2}\,C)^{\lambda}]|y-\bar{y}|^{\lambda}. \tag{2.6}$$

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Thus $D^\lambda H_\lambda(f) \leqq C'$ where C' depends only on C and λ. It also follows from (2.4) that $\|f\| \leqq 2C$. Since $\operatorname{supp}(\omega \circ \Psi^{-1}) = \Psi(\operatorname{supp} \omega)$ we find that the diameter of the support of f is less than $D_1 = \sqrt{2}\, CD$. Observe that $D \leqq D_1$ as $1 \leqq \sqrt{2}\, C$. The bounds for f show that $\|f\|_{C^{1+\lambda}(D_1)}$ is less than $2C + C'$ and we can therefore conclude from (2.1) that

$$\|v\|_{C^{1+\lambda}(D)} \leqq \frac{36 D_1}{\lambda(1-\lambda)}(2C + C') = C''D. \tag{2.7}$$

Finally we will study the function $w = v \circ \Psi$. By using (2.7) we get

$$\begin{aligned} |w_{,j}(\alpha)| &= \left| \sum_{i=1}^{2} v_{,i}(\Psi(\alpha))\psi_{i,j}(\alpha) \right| \\ &\leqq C''\sqrt{2}\, C \end{aligned}$$

for $j = 1, 2$. The functions $w_{,j}$ are also Hölder continuous. By adding and subtracting the term $-v_{,i}(\Psi(\bar{\alpha}))\psi_{i,j}(\alpha)$ and using the inequalities (2.3), (2.5) and (2.7) we see that

$$\begin{aligned} |w_{,j}(\alpha) - w_{,j}(\bar{\alpha})| &\leqq C''D^{-\lambda}|\Psi - \bar{\Psi}|^\lambda \sqrt{2}|\Psi_{,j}| + C''\sqrt{2}|\Psi_{,j} - \bar{\Psi}_{,j}| \\ &\leqq C''D^{-\lambda}(\sqrt{2}\, C|\alpha - \bar{\alpha}|)^\lambda \sqrt{2}\, C + C''\sqrt{2}\, CD^{-\lambda}|\alpha - \bar{\alpha}|^\lambda, \end{aligned}$$

for $j = 1, 2$. The bounds for $w_{,j}$ shows that $|w|_{C^{1+\lambda}(D)}$ is less than C''' where C''' depends only on C and λ. Let us choose the constant k_0 such that $4k_0 CC''' = 1$. Since $1 \leqq 2C^2$ and $w = v \circ \Psi$ it follows from (2.2) that

$$|\tilde{\Psi}|_{C^{1+\lambda}(D)} \leqq \tfrac{1}{2}C + k_0 C''' \leqq C.$$

Thus $\tilde{\Psi}$ satisfies the first inequality in (2.3). To establish the second inequality in (2.3), we use the fact that $\tilde{\Psi}' = \Phi' + kw'$ and that the flow Φ is incompressible. Thus

$$\begin{aligned} \det \tilde{\Psi}' &= 1 + k[\det(\Phi_{,1}|w_{,2}) + \det(w_{,1}|\Phi_{,2})] + k^2 \det(w') \\ &\geqq 1 - k_0 CC''' - (k_0 C''')^2 \geqq \tfrac{1}{2}. \end{aligned}$$

This completes the proof.

Lemma 1 says that the approximate flow map is as smooth as the real flow map when the time step is sufficiently small. This is enough to establish the convergence of the vortex method, but to prove high rate of convergence, we need more smoothness. Thus we introduce the norms and seminorms

$$\|\omega\|_{C^{m+\lambda}(D)} = \sum_{\nu=0}^{m} D^\nu \max_{|\gamma|=\nu} \|\partial^\gamma \omega\| + D^{m+\lambda} \max_{|\gamma|=m} H_\lambda(\partial^\gamma \omega),$$

$$\|u\|_{C^{m+1+\lambda}(D)} = \sum_{\nu=0}^{m+1} D^\nu \max_{|\gamma|=\nu} \|\partial^\gamma u\| + D^{m+1+\lambda} \max_{|\gamma|=m+1} H_\lambda(\partial^\gamma u),$$

$$|\Phi|_{C^{m+1+\lambda}(D)} = \sum_{\nu=1}^{m+1} D^{\nu-1} \max_{|\gamma|=\nu} \|\partial^\gamma \Phi\| + D^{m+\lambda} \max_{|\gamma|=m+1} H_\lambda(\partial^\gamma \Phi).$$

Here we use $\partial^\gamma = \partial_1^{\gamma_1}\partial_2^{\gamma_2}$ to escape conflicting notation. We can now formulate

Assumption 4. There exists a unique solution $\{\omega, u, \Phi\}$ and constants M, D, λ, m and $\{\lambda_\nu\}$ such that

(i) The vorticity $\omega(x, t) \in C^{m+\lambda,1}(\mathbb{R}^2 \times [0, T])$ and $\|\omega(\cdot, t)\|_{C^{m+\lambda}(D)} \leqq M$ for $0 \leqq t \leqq T$ where $0 < \lambda < 1$ and the diameter of the support of $\omega(x, t)$ is less than D.

(ii) The velocity $u(x, t) \in C^{m+1+\lambda,1}(\mathbb{R}^2 \times [0, T])$ and $\operatorname{div} u \equiv 0$.

(iii) The flow map $\Phi(x, t) \in C^{m+1+\lambda_\nu, \nu}(\mathbb{R}^2 \times [0, T])$ for $\nu = 0, 1, \cdots$ and $1 > \lambda_0 > \lambda_1 > \cdots > 0$ and $|\partial_t^\nu \Phi(\cdot, t)|_{C^{m+1-\nu+\lambda}(D)} \leqq \frac{1}{2}M$ for $0 \leqq t \leqq T$ and $\nu = 0, 1, \cdots, m+1$.

If $m = 0$ we get Assumption 3. We can now present

COROLLARY 1. *Let Assumption* 4 *be satisfied and let* $\tilde{\Psi}$ *be determined by* (2.2). *Assume that*

$$|\Psi|_{C^{m+1+\lambda}(D)} \leqq M \quad \textit{and} \quad \det \Psi'(\alpha) \geqq \tfrac{1}{2}. \tag{2.8}$$

There exists a constant k_1 *such that* $\tilde{\Psi}$ *satisfies* (2.8) *for all* $k \leqq k_1$. *The constant* k_1 *depends on* M, λ *and* m.

Proof. Since $v = K * f$ where f is given by (2.4) it follows that $f \in C^{m+\lambda}$ and that $\partial^\gamma v = K * \partial^\gamma f$ for all $|\gamma| \leqq m$. The diameter of the support of f is less than $D_1 = \sqrt{2}\, MD$. By using the arguments leading up to (2.1) we find that

$$\|v\|_{C^{m+1+\lambda}(D_1)} \leqq \frac{36 D_1}{\lambda(1-\lambda)} \|f\|_{C^{m+\lambda}(D_1)}.$$

To estimate the norm of f, we note that $f \circ \Psi$ is equal to $\omega(\det \Psi')^{-1}$ and hence in $C^{m+\lambda}$ and since Ψ^{-1} is in $C^{m+1+\lambda}$, it follows that $\|f\|_{C^{m+\lambda}(D_1)}$ can be bounded by const $\cdot \|\omega\|_{C^{m+\lambda}(D_1)}$ where the constant depends on M and m. More precisely, it depends on the bound for the first $m+1$ derivatives of Ψ and Ψ^{-1} and on the Hölder continuity of $\partial^{m+1}\Psi$. Since our norms are monotone functions of D and $1 \leqq \sqrt{2}\, M$ we see that $\|v\|_{C^{m+1+\lambda}(D)}$ is less than $C'D$ where C' depends on M, m and λ. But v and Ψ are in $C^{m+1+\lambda}$ and thus $\|v \circ \Psi\|_{C^{m+1+\lambda}(D)}$ is less than a new constant $C''D$. If $k_1 = (4MC'')^{-1}$ then it follows from (2.2) that $|\tilde{\Psi}|_{C^{m+1+\lambda}(D)}$ is less than M. We can now use the final arguments in Lemma 1 and this completes the proof.

3.1. Low order consistency. In this section we will establish the consistency of the vortex method for flows that are not incompressible. Because our assumptions are general, the results can be improved in many cases. In § 4.1, we consider a particularly interesting case.

We begin by studying the kernel in the vortex method. It follows from Assumption 1 that the function ζ is absolutely integrable over $\mathbb{R}^2$. Thus $|x-y|^{-1}|\zeta_\delta(y)|$ is integrable with respect to y if $x \neq 0$, and we conclude that $K_\delta(x) = K * \zeta_\delta(x)$ exists for all $x \neq 0$. We can evaluate the kernel explicitly by switching to polar coordinates and get

$$\begin{aligned} K_\delta(x) &= K(x) \int_{|y|<r} \zeta_\delta(y)\, dy \\ &= \frac{1}{r^2} \int_0^r s\zeta_\delta(s)\, ds \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \end{aligned} \tag{3.1}$$

where $r = |x|$. The integral $\int_{|y|<r} \zeta_\delta$ is called the shape factor. If the cutoff function is bounded then the kernel is continuous and vanishes at the origin. The kernels corresponding to Chorin's cutoff (1.7) and the infinite order cutoff (1.8) are

$$K_\delta(x) = \begin{Bmatrix} (2\pi\delta r)^{-1} & 0 < r \leqq \delta \\ (2\pi r^2)^{-1} & \delta < r < \infty \end{Bmatrix} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \tag{3.2}$$

$$K_\delta(x) = (2\pi r^2)^{-1} \left[1 - \frac{8}{3} \frac{J_1(2r/\delta)}{2r/\delta} + \frac{2}{3} \frac{J_1(r/\delta)}{r/\delta} \right] \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{3.3}$$

Here $J_1(r)$ is the Bessel function of order 1. Note that Chorin's kernel is not continuous at the origin. Still, we set $K_\delta(0) = 0$. By using the properties of the cutoff function we can estimate the partial derivatives of the kernel in the vortex method.

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LEMMA 2. *Let Assumption* 1 *be satisfied and set* $L_2 = 36L_0$. *Let* K_δ *be given by* (3.1). *Then* $K_\delta(x)$ *is continuous for* $x \neq 0$ *and satisfies*

$$|D^\gamma K_\delta(x)| \leqq \begin{cases} L_2\delta^{-1}r^{-|\gamma|}, & 0<r<\delta, \quad |\gamma|=0,1,2, \\ L_2 r^{-1-|\gamma|}, & \delta<r<\infty, \quad |\gamma|=0,1, \\ L_2\delta^{-1/2}r^{-5/2}, & \delta<r<\infty, \quad |\gamma|=2. \end{cases}$$

$$|D^\gamma K_\delta(x) - D^\gamma K(x)| \leqq L_2\delta^{3/2-|\gamma|}r^{-5/2}, \quad \delta<r<\infty, \quad |\gamma|=0,1.$$

Remark. Here we use Schwartz' notation $D^\gamma = D_1^{\gamma_1}D_2^{\gamma_2}$ and $|\gamma| = \gamma_1+\gamma_2$ where D_i is equal $\partial/\partial x_i$. Our bounds for the derivatives of the kernel are slightly weaker than the bounds assumed by Del Prete and Hald [11], and much weaker than those used by Hald [15], Beale and Majda [4], Cottet [9], and Anderson and Greengard [2]. The decay of $D_iD_jK_\delta$ at infinity is slower than the decay of D_iD_jK. This complicates our theory, but is necessary in order to include the infinite order cutoff (1.8). The examples (3.2) and (3.3) show that our bounds are sharp.

Proof. The continuity of the kernel follows from (3.1). A straightforward calculation shows that

$$D_iK_\delta(x) = x_i\left[\frac{\zeta_\delta(r)}{r^2} - \frac{2}{r^4}\int_0^r s\zeta_\delta(s)\,ds\right]\begin{pmatrix}-x_2\\ x_1\end{pmatrix} + \frac{1}{r^2}\int_0^r s\zeta_\delta(s)\,ds\begin{pmatrix}-\delta_{2i}\\ \delta_{1i}\end{pmatrix},$$

$$D_iD_jK_\delta(x) = \left[\frac{\zeta_\delta(r)}{r^2} - \frac{2}{r^4}\int_0^r s\zeta_\delta(s)\,ds\right]\left[x_i\begin{pmatrix}-\delta_{2j}\\ \delta_{1j}\end{pmatrix} + x_j\begin{pmatrix}-\delta_{2i}\\ \delta_{1i}\end{pmatrix} + \delta_{ij}\begin{pmatrix}-x_2\\ x_1\end{pmatrix}\right]$$
$$+ x_ix_j\left[\frac{\zeta_\delta'(r)}{r^3} - \frac{4\zeta_\delta(r)}{r^4} + \frac{8}{r^6}\int_0^r s\zeta_\delta(s)\,ds\right]\begin{pmatrix}-x_2\\ x_1\end{pmatrix}.$$

The expressions are valid for $x \neq 0$ and $|x| \neq \delta$. Since $\zeta_\delta(r) = \delta^{-2}\zeta(r/\delta)$, it follows from Assumption 1 that $\zeta_\delta^{(\nu)}(r)$ is less than $L_0\delta^{-1}r^{-1-\nu}$ for $r \leqq \delta$, and less than $L_0\delta^{1/2-\nu}r^{-5/2}$ if $r > \delta$. Thus, the integral $\int_0^r s\zeta_\delta$ can be bounded by $L_0 r/\delta$ if $r \leqq \delta$ and by $3L_0$ otherwise. By using these estimates for the approximate delta function and the expressions for the partial derivatives of the kernel, we get the first three statements in the lemma. To get the last result we combine (3.1) with condition (iii) in Assumption 1. This completes the proof.

The consistency of the vortex method can be established in several ways. The most elementary one uses Taylor's formula (see Dushane [13], Del Prete and Hald [11] and Hald [15]). Beale and Majda [4] obtained higher order consistency results for a restricted class of methods. They use the Fourier transform technique and the theory of pseudodifferential operators, and study primarily the errors of the approximate vorticity distribution. Cottet [9] studies the approximate velocity field and does not use the Fourier transform. His approach is related to the derivation of the Euler–Maclaurin formula and goes back to Cottet and Raviart [10]. Finally, Anderson and Greengard [2] interpret one component of the error in the approximate velocity field as the error in the trapezoidal rule and use the Fourier transform technique to estimate it.

LEMMA 3. *Let Assumption* 3 *be satisfied. Let* $v(x)$ *be determined as in* (2.2) *and assume that the inequalities in* (2.3) *are satisfied. Set* $\varepsilon = \lambda/5$, $\delta = h^{4/(4+\lambda)}$ *and* $\kappa_j = \int_{Q_j}\omega\,dx$. *Let Assumption* 1 *be satisfied. Then there exist two constants* C_0 *and* h_0 *such that*

$$\max_x \left| v(x) - \sum_j K_\delta(x-\Psi(\alpha_j))\kappa_j \right| \leqq C_0 h^{1+\varepsilon}$$

for all $h \leqq h_0$. *The constants* C_0 *and* h_0 *depend on* C, D, L_2 *and* λ.

Remark. The result is weaker than the estimate by Del Prete and Hald [11], but applies to a larger class of vortex methods.

Proof. By using the change of variables $\alpha' = \Psi^{-1}(y)$ in (2.2) we see that $v = K * f$ where f is given by (2.4). Lemma 2 shows that $K - K_\delta$ is absolutely integrable and this implies that

$$v(x) - \sum_j K_\delta(x - \Psi(\alpha_j))\kappa_j = \int [K(x-y) - K_\delta(x-y)][f(y) - f(x)]\, dy$$
$$+ \sum_j \int_{Q_j} [K_\delta(x - \Psi(\alpha)) - K_\delta(x - \Psi(\alpha_j))]\omega(\alpha)\, d\alpha.$$

The expression on the left-hand side is called the consistency error, while the first term on the right-hand side is the moment error and the second the discretization error. We denote their absolute values by e_c, e_m and e_d.

In the proof of Lemma 1 we found that the function f is Hölder continuous with exponent λ (see (2.6)). Since the diameter of the support of f is less than $D_1 = \sqrt{2}\, CD$ and $D_1^\lambda H_\lambda(f)$ is an increasing function of λ, we conclude that f is Hölder continuous with exponent $\lambda/2$ and that $H_{\lambda/2}(f)$ can be bounded in terms of C, D and λ. Since $\int_{\mathbb{R}^2} \zeta = 1$, it follows from Assumption 1 that $6\pi L_0 \geqq 1$. The constant L_2 in Lemma 2 is therefore greater than 1. We estimate the moment error by using polar coordinates and get

$$e_m \leqq 2\pi \int_{r<\delta} \frac{2L_2}{r} H_{\lambda/2}(f) r^{\lambda/2} \cdot r\, dr + 2\pi \int_{r>\delta} \frac{L_2 \delta^{3/2}}{r^{5/2}} H_{\lambda/2}(f) r^{\lambda/2} \cdot r\, dr$$
$$\leqq 2\pi \cdot 2L_2 \cdot H_{\lambda/2}(f) \frac{2}{1-(\lambda/2)^2} \delta^{1+\lambda/2}.$$

To estimate the discretization error, we separate the indices into five groups

$$J = \{j\colon Q_j \cap \operatorname{supp}(\omega) \neq \phi\},$$
$$J_1 = \{j\colon |x - \Psi(\alpha_j)| < r_1\},$$
$$J_2 = \{j\colon r_1 \leqq |x - \Psi(\alpha_j)| < \delta - r_1\},$$
$$J_3 = \{j\colon \delta - r_1 \leqq |x - \Psi(\alpha_j)| < \delta + r_1\},$$
$$J_4 = \{j\colon \delta + r_1 \leqq |x - \Psi(\alpha_j)|\},$$

where $r_1 = (1+2C)h$ and we assume that $2r_1 < \delta < 1$. This condition will be satisfied if $h \leqq h_0$ and h_0 is less than $(2+4C)^{-1-4/\lambda}$. Note that the J_i are disjoint. Let S be the union of all Q_j with index in J, and let S_i be the union of those Q_j for which j is in J_i. Then the S_i are distinct and their union covers the plane, and in particular the support of ω and S. Finally we write the discretization error as a sum of four terms $E_1, \cdots, E_4$ where

$$E_i = \sum_{j \in J_i} \int_{Q_j} [K_\delta(x - \Psi(\alpha)) - K_\delta(x - \Psi(\alpha_j))]\omega(\alpha)\, d\alpha. \tag{3.4}$$

We observe first that if α is in Q_j then $|\alpha - \alpha_j| \leqq (h/2)\sqrt{2}$. Thus if $\alpha \in Q_j$ and $j \in J_1$, it follows from the first inequality in (2.5) that

$$|x - \Psi(\alpha)| \leqq |x - \Psi(\alpha_j)| + |\Psi(\alpha_j) - \Psi(\alpha)| \leqq r_1 + Ch. \tag{3.5}$$

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Since $2Ch < r_1$ and $2r_1 < \delta$ we conclude from (3.5) that $|x-\Psi(\alpha)| < \delta$. Thus it follows from Lemma 2 that

$$|E_1| \leqq \sum_{j\in J_1} \int_{Q_j} \left(\frac{L_2}{\delta}+\frac{L_2}{\delta}\right)|\omega(\alpha)|\, d\alpha \leqq \frac{2L_2}{\delta}\|\omega\| \int_{S_1} d\alpha.$$

Integrals like $\int_{S_1} d\alpha$ occur repeatedly in this paper. To estimate them, we use a change of variables, namely $\alpha = \Psi^{-1}(y)$. If $y = \Psi(\alpha)$ with α in S_1 then it follows from (3.5) that $|x-y|$ is less than $(1+3C)h$. This shows that $\Psi(S_1)$ is covered by the disk A_1 with center at x and with radius $(1+3C)h$. Hence,

$$\int_{S_1} d\alpha = \int_{\Psi(S_1)} \det \Psi^{-1\prime}(y)\, dy \leqq 2\int_{A_1} dy = 2\pi(1+3C)^2h^2$$

where we have used the inequality (2.3). By combining this result with our bound for E_1 we see that $|E_1|$ is less than const $\cdot\, h^2/\delta$ where the constant depends on C and L_2.

It is not so easy to bound E_2. We shall use the notation $\Psi = \Psi(\alpha)$, $\bar\alpha = \alpha_j$ and $\bar\Psi = \Psi_j = \Psi(\bar\alpha)$. Let $\alpha \in Q_j$ and $j \in J_2$. Then it follows from the triangle inequality and (2.5) that

$$h \leqq |x-\bar\Psi| - 2Ch \leqq |x-\Psi| - Ch \leqq |x-\bar\Psi+\tau(\bar\Psi-\Psi)| \leqq |x-\bar\Psi| + Ch < \delta - Ch \tag{3.6}$$

for all $0 \leqq \tau \leqq 1$. We can therefore use Taylor's theorem with remainder and get

$$K_\delta(x-\Psi) - K_\delta(x-\bar\Psi) = \nabla K_\delta(x-\bar\Psi)\cdot(\bar\Psi-\Psi) + R, \tag{3.7}$$

$$R = \int_0^1 (1-\tau) \sum_{|\gamma|=2} D^\gamma K_\delta(x-\bar\Psi+\tau(\bar\Psi-\Psi))\cdot(\bar\Psi-\Psi)^\gamma\, d\tau.$$

Here $\nabla = (D_1, D_2)$ and $x^\gamma = x_1^{\gamma_1}x_2^{\gamma_2}$. The sum consists of four terms. By using Lemma 2 and the first inequality in (2.5) we find that

$$|R| \leqq \int_0^1 (1-\tau)\cdot 4\cdot \frac{L_2(\sqrt{2}\, C|\alpha_j-\alpha|)^2}{\delta(|x-\Psi(\alpha)|-Ch)^2}\, d\tau \leqq 2L_2C^2\frac{h^2}{\delta}(|x-\Psi(\alpha)|-Ch)^{-2}$$

for all α in Q_j. Let A_2 be the annulus with center at x, inner radius $h+Ch$ and outer radius δ. The inequalities in (3.6) show that $\Psi(S_2)$ is a subset of A_2. We can therefore estimate the contribution from the remainder in (3.7) to the discretization error E_2 by

$$\left|\sum_{j\in J_2}\int_{Q_j} R\omega\, d\alpha\right| \leqq 2L_2C^2\frac{h^2}{\delta}\|\omega\| \int_{S_2} \frac{1}{(|x-\Psi(\alpha)|-Ch)^2}\, d\alpha$$

$$\leqq 2L_2C^2\frac{h^2}{\delta}\|\omega\| \int_{A_2} \frac{\det \Psi^{-1\prime}(y)}{(|x-y|-Ch)^2}\, dy.$$

Here we have used the change of variables $\alpha = \Psi^{-1}(y)$. The last integral is less than $4\pi(1+C)\cdot\log(\delta/h)$. Hence the contribution from the remainder is less than const $\cdot\,(h^2/\delta)\log(\delta/h)$, where the constant depends on C and L_2.

We will now study the contribution to the discretization error from the leading term in (3.7). By using Taylor's theorem with remainder we get

$$\int_{Q_j} [\bar\Psi-\Psi]\omega\, d\alpha = \int_{Q_j} [\bar\Psi-\Psi][\omega-\bar\omega]\, d\alpha - \bar\omega\Psi'(\bar\alpha)\int_{Q_j}(\alpha-\bar\alpha)\, d\alpha$$

$$-\bar\omega\int_{Q_j}\int_0^1 [\Psi'(\bar\alpha+\tau(\alpha-\bar\alpha))-\Psi'(\bar\alpha)](\alpha-\bar\alpha)\, d\tau\, d\alpha.$$

The integral $\int_{Q_j}(\alpha-\bar{\alpha})\,d\alpha$ is zero because $\bar{\alpha}$ is the center of mass for the square. Since Ψ is differentiable and $\partial_i\Psi$ and ω are Hölder continuous we find that

$$\left|\int_{Q_j}[\bar{\Psi}-\Psi]\omega\,d\alpha\right| \leqq \sqrt{2}[\max_{i=1,2}\|\partial_i\Psi\|H_\lambda(\omega)+\|\omega\|\max_{i=1,2}H_\lambda(\partial_i\Psi)]\int_{Q_j}|\alpha-\bar{\alpha}|^{1+\lambda}\,d\alpha$$

$$\leqq C^2D^{-\lambda}h^{1+\lambda}\int_{Q_j}d\alpha. \tag{3.8}$$

After multiplying the leading term in (3.7) by ω, integrating with respect to α and using the inequalities (3.8) and (3.6) with $\tau=0$ we obtain

$$\left|\sum_{j\in J_2}\nabla K_\delta(x-\Psi_j)\int_{Q_j}[\Psi_j-\Psi]\omega\,d\alpha\right| \leqq \sum_{j\in J_2}C^2D^{-\lambda}h^{1+\lambda}\int_{Q_j}\sqrt{2}\max_{i=1,2}|\partial_iK_\delta(x-\Psi_j)|\,d\alpha$$

$$\leqq C^2D^{-\lambda}h^{1+\lambda}\frac{\sqrt{2}\,L_2}{\delta}\int_{S_2}\frac{1}{|x-\Psi(\alpha)|-Ch}\,d\alpha.$$

Since $\Psi(S_2)$ is contained in the annulus A_2 we find after using a change of variables that the last integral is less than $4\pi(1+C)\delta$. Thus the contribution from the leading term in (3.7) to the discretization error E_2 is less than const $\cdot\, h^{1+\lambda}$ where the constant depends on C, D, L_2 and λ.

The difficulty in estimating E_3 is that $K_\delta(x-\Psi(\alpha))$ may not be differentiable for certain values of α. The line connecting $\Psi(\alpha_j)$ and $\Psi(\alpha)$ can intersect the circle with center at x and radius δ in up to two points. Let $\alpha\in Q_j$ with $j\in J_3$ and set $\bar{\alpha}=\alpha_j$. By using the triangle inequality, (2.5) and $2r_1<\delta$ we obtain

$$\begin{aligned}\delta/4<\delta-(1+3C)h &\leqq |x-\bar{\Psi}|-Ch\leqq|x-\bar{\Psi}+\tau(\bar{\Psi}-\Psi)|\\ &\leqq|x-\bar{\Psi}|+Ch\leqq\delta+(1+3C)h\end{aligned} \tag{3.9}$$

for $0\leqq\tau\leqq1$. Let us assume that $|x-\bar{\Psi}+\tau(\bar{\Psi}-\Psi)|$ is equal to δ for $\tau=\tau_1$ and $\tau=\tau_2$ where $0<\tau_1<\tau_2<1$. Set $\tau_0=0$ and $\tau_3=1$. The cases with one intersection or no intersection are similar but simpler. By using the fundamental theorem of calculus we get

$$K_\delta(x-\Psi)-K_\delta(x-\bar{\Psi})=\sum_{l=1}^{3}\int_0^1\nabla K_\delta(x-\bar{\Psi}+\{\tau\tau_l+(1-\tau)\tau_{l-1}\}[\bar{\Psi}-\Psi])$$
$$(\tau_l-\tau_{l-1})\cdot[\bar{\Psi}-\Psi]\,d\tau.$$

The kernel is evaluated at points outside the circle with center at the origin and radius $\delta/4$. Thus it follows from Lemma 2 that $|K_\delta(x-\Psi)-K_\delta(x-\bar{\Psi})|$ is less than $4\sqrt{2}\,L_2\delta^{-2}|\Psi-\bar{\Psi}|$. By combining this result with (3.4) and using (2.5) we see that

$$|E_3|\leqq\sum_{j\in J_3}\int_{Q_j}\frac{4\sqrt{2}\,L_2}{\delta^2}|\Psi(\alpha)-\Psi(\alpha_j)|\,|\omega(\alpha)|\,d\alpha\leqq\frac{4\sqrt{2}\,L_2}{\delta^2}Ch\|\omega\|\int_{S_3}d\alpha.$$

Let A_3 be the annulus with center at x, inner radius $\delta-(1+3C)h$ and outer radius $\delta+(1+3C)h$. The inequalities in (3.9) with $\tau=1$ imply that $\Psi(S_3)$ is a subset of A_3. The integral $\int_{S_3}d\alpha$ is therefore less than $8\pi(1+3C)h\delta$ and we conclude that $|E_3|$ is less than const $\cdot\, h^2/\delta$ where the constant depends on C and L_2.

To estimate E_4, we use the same approach as for E_2. Let $\alpha\in Q_j$ with $j\in J_4$. Then it follows from the triangle inequality and (2.5) that

$$\delta<|x-\bar{\Psi}|-2Ch\leqq|x-\Psi|-Ch\leqq|x-\bar{\Psi}+\tau(\bar{\Psi}-\Psi)| \tag{3.10}$$

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for all $0 \leqq \tau \leqq 1$. Thus (3.7) holds, and we can estimate the remainder pointwise by using Lemma 2, and find that

$$|R| \leqq \int_0^1 (1-\tau) \cdot 4 \frac{L_2(\sqrt{2}\, C|\alpha_j - \alpha|)^2}{\sqrt{\delta}(|x-\Psi(\alpha)|-Ch)^{5/2}}\, d\tau \leqq 2L_2C^2 \frac{h^2}{\sqrt{\delta}}(|x-\Psi(\alpha)|-Ch)^{-5/2}$$

for all α in Q_j. Let A_4 be the annulus with center at x, inner radius $\delta + Ch$ and stretching to infinity. It follows from (3.10) with $\tau = 1$ that $\Psi(S_4)$ is contained in A_4. We can therefore estimate the contribution from the remainder in (3.7) to the discretization error E_4 by

$$\left| \sum_{j \in J_4} \int_{Q_j} R\omega\, d\alpha \right| \leqq 2L_2C^2 \frac{h^2}{\sqrt{\delta}} \int_{A_4} \frac{\det \Psi^{-1\prime}(y)}{(|x-y|-Ch)^{5/2}}\, dy.$$

Since $4Ch < \delta$, the last integral is less than $10\pi\delta^{-1/2}$. Hence the contribution from the remainder is less than const $\cdot\, h^2/\delta$ where the constant depends on C and L_2.

Finally we consider the contribution from the leading term in (3.7) to the discretization error E_4. Here the difficulty is that $|D_iK_\delta|$ is not integrable over the whole plane. Hence we must use the fact that ω has compact support. We note that if the index j is not in J then the corresponding integral in (3.4) vanishes. We can therefore restrict the sum in (3.4) to $J_i \cap J$. By using the inequalities (3.8) and (3.10) with $\tau = 0$ plus Lemma 2 we get

$$\begin{aligned} &\left| \sum_{j \in J_4 \cap J} \nabla K_\delta(x-\Psi_j) \int_{Q_j} [\Psi_j - \Psi]\omega\, d\alpha \right| \\ &\quad \leqq C^2 D^{-\lambda} h^{1+\lambda} \sqrt{2}\, L_2 \int_{S_4 \cap S} \frac{1}{(|x-\Psi(\alpha)|-Ch)^2}\, d\alpha. \end{aligned} \tag{3.11}$$

We shall use the change of variables $\alpha = \Psi^{-1}(y)$. It follows from (3.10) that the distance d from x to $\Psi(S_4 \cap S)$ is greater than $\delta + Ch$. The definition of S shows that the diameter of S is less than $2\sqrt{2}\, h + D$. By using the bound for h_0 and the first inequality in (2.5) we find that the diameter of $\Psi(S_4 \cap S)$ is less than $D_2 = \sqrt{2}\, C(1+D)$. This implies that $\Psi(S_4 \cap S)$ is contained in the annulus A_5 with center at x, inner radius d and outer radius $d + D_2$. Since $4Ch < \delta \leqq d - Ch$ we can estimate the last integral in (3.11) by

$$\int_{A_5} \frac{\det \Psi^{-1\prime}(y)}{(|x-y|-Ch)^2}\, dy \leqq 2 \cdot 2\pi \int_{d-Ch}^{d-Ch+D_2} \frac{r+Ch}{r^2}\, dt \leqq 5\pi \log\left(\frac{d-Ch+D_2}{d-Ch}\right).$$

As $\log(1 + D_2/r)$ is a decreasing function of r we conclude that the contribution from the leading term in (3.7) to the discretization error E_4 is less than const $\cdot\, h^{1+\lambda} \log(1 + D_2/\delta)$ where the constant and D_2 depends on C, D, L_2 and λ.

We can now collect our results. Let $a = 1 + D_2$. Then δ/h and $1 + D_2/\delta$ are less than a/h. Hence

$$e_d \leqq \text{const} \cdot \left(\frac{h^2}{\delta} + \frac{h^2}{\delta} \log\left(\frac{a}{h}\right) + h^{1+\lambda} + h^{1+\lambda} \log\left(\frac{a}{h}\right) \right).$$

Since $h < 1$ and $\delta = h^{4/(4+\lambda)}$ we see that $h^{1+\lambda} < h^2/\delta$. It is therefore enough to estimate $(h^2/\delta)\log(a/h)$. It follows from the definition of ε that $h^2/\delta = h^{1+\varepsilon} \cdot h^b$ where $b = \lambda(1-\lambda)/(20+5\lambda)$. As $h^b \log(a/h)$ is less than $a^b(be)^{-1}$ for all $h < a$ and in particular for $h < 1$ we find that $(h^2/\delta)\log(a/h)$ can be bounded by const $\cdot\, h^{1+\varepsilon}$ where the constant depends on C, D, L_2 and λ. The discretization error can therefore be bounded in the same manner. Finally we observe that $\delta^{1+\lambda/2} = h^2/\delta$. The moment error e_m can therefore also be bounded by const $\cdot\, h^{1+\varepsilon}$. This completes the proof.

So far we have defined the constants κ_j by local averages of the initial vorticity. The alternative is to use point values, i.e. $\kappa_j = \omega_j h^2$. We can then interpret the discretization error as the error in the trapezoidal rule applied to the function $K_\delta(x - \Psi(\alpha)) \cdot \omega(\alpha)$ (see Anderson and Greengard [2]). We will show that the consistency error is less than const $\cdot\, h^\lambda$ if ω is Hölder continuous with exponent λ. We observe first that

$$\left|\kappa_j - \int_{Q_j} \omega\, d\alpha\right| \leqq \int_{Q_j} |\omega(\alpha_j) - \omega(\alpha)|\, d\alpha \leqq H_\lambda(\omega) h^\lambda \int_{Q_j} d\alpha. \tag{3.12}$$

The additional contribution to the discretization error can therefore be estimated by

$$\begin{aligned}\left|\sum_{j\in J} K_\delta(x-\Psi_j) \int_{Q_j} [\omega(\alpha_j) - \omega(\alpha)]\, d\alpha\right| &\leqq \sum_{j\in J_1\cup J_2\cup J_3} + \sum_{j\in J_4\cap J} \int_{Q_j} H_\lambda(\omega) h^\lambda |K_\delta(x-\Psi_j)|\, d\alpha \\ &\leqq H_\lambda(\omega) h^\lambda L_2 \left[\int_{\Psi(S_1\cup S_2\cup S_3)} \frac{\det \Psi^{-1\prime}(y)}{\delta}\, dy + \int_{\Psi(S_4\cap S)} \frac{\det \Psi^{-1\prime}(y)}{|x-y| - Ch}\, dy\right].\end{aligned}$$

It follows from (3.9) that $\Psi(S_1 \cup S_2 \cup S_3)$ is contained in a disk with radius 2δ and the corresponding integral is less than $8\pi\delta$. Since $\Psi(S_4 \cap S)$ is covered by the annulus A_5, we can estimate the last integral by $5\pi D_2$. The additional contribution to the discretization error is therefore less than const $\cdot\, h^\lambda$ where the constant depends on C, D, L_2 and λ. Thus the vortex method is consistent even with pointwise evaluation of the initial vorticity. However, we cannot prove the convergence because the basic assumption in our stability lemma will not be satisfied.

It is unlikely that the consistency result for pointwise evaluation can be improved. In fact, in one dimension it is possible to construct a Hölder continuous function with exponent λ such that the error in the trapezoidal rule is proportional to h^λ. Thus we have two alternatives. Either we restrict the class of cutoff functions or we require more smoothness of the initial vorticity. If the partial derivatives of the initial vorticity are Hölder continuous with exponent λ then the bound (3.12) can be sharpened. Since $\int_{Q_j} (\alpha - \alpha_j)\, d\alpha$ is zero we find by using Taylor's theorem with remainder that

$$\begin{aligned}\left|\int_{Q_j} [\omega(\alpha) - \omega(\alpha_j)]\, d\alpha\right| &= \left|\int_{Q_j}\int_0^1 [\nabla\omega(\alpha_j + \tau(\alpha-\alpha_j)) - \nabla\omega(\alpha_j)] \cdot (\alpha - \alpha_j)\, d\tau\, d\alpha\right| \\ &\leqq \max_{i=1,2} H_\lambda(\partial_i\omega) h^{1+\lambda} \int_{Q_j} d\alpha.\end{aligned}$$

Thus with pointwise evaluation of the initial vorticity and a stronger smoothness assumption the consistency error in the vortex method is less than const $\cdot\, h^{1+\varepsilon}$ where the constant depends on C, D, L_2 and λ.

3.2. Stability. In this section we will establish the stability of the vortex method for flows that are not incompressible. We admit singular cutoffs and cutoffs that decay slowly at infinity and assume that the initial vorticity is Hölder continuous and has compact support.

To estimate the error, we introduce the discrete L_p norm

$$\|e\|_p = \left(\sum_{j\in J} |e_j|^p h^2\right)^{1/p}. \tag{3.13}$$

Note that we include only those indices for which Q_j intersect the support of ω. Let V be the approximate velocity field corresponding to the flow Ψ. We denote the values

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of V at the points Ψ_i by

$$V_i = \sum_{j \in J} K_\delta(\Psi_i - \Psi_j)\kappa_j$$

(cf. (1.5) and Lemma 3). On occasion we shall emphasize the dependence on Ψ and write $V[\Psi]$ instead of V. We can now present

LEMMA 4. *Let Assumptions* 1 *and* 3 *be satisfied and assume that the inequalities in* (2.3) *are satisfied. Let* $\kappa_j = \int_{Q_j} \omega\, dx$ *or* $\omega_j h^2$ *and set* $c = (4\sqrt{2}\, C)^{-1}$. *There exist two constants* C_1 *and* h_1 *such that if* h/δ *and* h *are less than* h_1 *and*

$$\max_i \{|\tilde{\Psi}_i - \Psi_i|, |\bar{\Psi}_i - \Psi_i|\} \leqq \tfrac{1}{2}ch$$

then

$$\| V[\tilde{\Psi}] - V[\bar{\Psi}]\|_p \leqq C_1 \|\tilde{\Psi} - \bar{\Psi}\|_p$$

where $1 < p < \infty$. *The constant* C_1 *and* h_1 *depend on* C, D, L_2, λ *and* p.

Remark. This result is a generalization of the stability lemma due to Beale and Majda [4] and our proof is based on theirs. They consider a smaller class of cutoff functions, but use δ instead of $\frac{1}{2}ch$. As we allow singular cutoff functions, the lemma is false if we replace $\frac{1}{2}ch$ even by h. By using a different approach Cottet [9] has established the stability for a large class of cutoff functions and for flows where the vorticity does not have compact support. Finally Lemma 4 neither implies nor is it implied by the stability result for the maximum norm by Del Prete and Hald [11].

Proof. We use the flow Ψ to partition the indices in several groups

$$J = \{j\colon Q_j \cap \operatorname{supp} \omega \neq \phi\},$$

$$J_1 = \{(i,j)\colon |\Psi_i - \Psi_j| < r_1\} \cap J \times J,$$

$$J_2 = \{(i,j)\colon r_1 \leqq |\Psi_i - \Psi_j| < \delta - r_1\} \cap J \times J,$$

$$J_3 = \{(i,j)\colon \delta - r_1 \leqq |\Psi_i - \Psi_j| < \delta + r_1\} \cap J \times J,$$

$$J_4 = \{(i,j)\colon \delta + r_1 \leqq |\Psi_i - \Psi_j|\} \cap J \times J,$$

where $\Psi_i = \Psi(\alpha_i)$ and $r_1 = (1 + 2C + c)h$. We assume that $2r_1 < \delta < 1$. This condition will be satisfied for all $h \leqq h_1$ if h/δ and h_1 are less than $(2 + 4C + 2c)^{-1}$. For all i in J we let $J_k(i)$ consist of the indices j with (i,j) in J_k. If i is fixed, then the $J_k(i)$ are disjoint and their union is J. Let S be the union of all Q_j with index in J and let $S_k(i)$ be the union of those Q_j for which j is in $J_k(i)$. If i is fixed, then the $S_k(i)$ are disjoint and their union is S. To cover the two choices of κ_j at the same time, we introduce a function $\xi(x)$ such that $\kappa_j = h^2 \xi_j$. If κ_j is obtained by pointwise evaluation of the initial vorticity we set $\xi(x) = \omega(x)$; otherwise we let

$$\xi(x) = \frac{1}{h^2} \int_{Q_0} \omega(x+y)\, dy.$$

In both cases ξ is Hölder continuous with exponent λ. The support of ξ may be slightly larger than S, but the diameter of the support will be less than $2\sqrt{2}\, h + D$ and $\|\xi\|_{C^\lambda(D)} \leqq C$.

We observe first that all points on the line segment between $\tilde{\Psi}_i - \tilde{\Psi}_j$ and $\bar{\Psi}_i - \bar{\Psi}_j$ are bounded away from zero. Let $0 \leqq \vartheta \leqq 1$ and define

$$\chi_{ij}(\vartheta) = \vartheta[\tilde{\Psi}_i - \Psi_i - (\tilde{\Psi}_j - \Psi_j)] + (1 - \vartheta)[\bar{\Psi}_i - \Psi_i - (\bar{\Psi}_j - \Psi_j)].$$

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Note that $\chi_{ji} = -\chi_{ij}$. Since $|\chi_{ij}| \leqq ch$ it follows from (2.5) that

$$|\vartheta(\tilde{\Psi}_i - \tilde{\Psi}_j) + (1-\vartheta)(\bar{\Psi}_i - \bar{\Psi}_j)| = |\Psi_i - \Psi_j + \chi_{ij}(\vartheta)| \geqq \frac{|\alpha_i - \alpha_j|}{2\sqrt{2}\,C} - ch \geqq ch$$

for $0 \leqq \vartheta \leqq 1$. We can therefore use the mean value theorem and find that

$$\begin{aligned} V[\tilde{\Psi}]_i - V[\bar{\Psi}]_i &= \sum_{j \in J} [K_\delta(\tilde{\Psi}_i - \tilde{\Psi}_j) - K_\delta(\bar{\Psi}_i - \bar{\Psi}_j)]\xi_j h^2 \\ &= \sum_{k=1}^{4} \sum_{j \in J_k(i)} \int_0^1 \nabla K_\delta(\Psi_i - \Psi_j + \chi_{ij}(\vartheta))\, d\vartheta \cdot (e_i - e_j)\xi_j h^2 \end{aligned} \tag{3.14}$$

where $e_i = \tilde{\Psi}_i - \bar{\Psi}_i$. Strictly speaking, (3.14) is not correct. The line between $\tilde{\Psi}_i - \tilde{\Psi}_j$ and $\bar{\Psi}_i - \bar{\Psi}_j$ may intersect the circle with center at the origin and radius δ. In such a case we replace ∇K_δ by a weighted average of ∇K_δ evaluated just inside and outside the circle, in the same way as in § 3.1. This can only happen for (i, j) in J_3. We interpret the right-hand side of (3.14) as a sum of four diagonal matrices D_k minus the sum of four band matrices E_k all multiplied by the "error" vector $e = (e_j)$. We can also regard the error as a function $e(x)$ which is equal to e_j for all x in Q_j and $j \in J$ and zero otherwise. We can then rewrite (3.13) as

$$\|e\|_p = \left(\int_S |e(x)|^p\, dx \right)^{1/p}. \tag{3.15}$$

Sometimes we will emphasize that the norm depends on the domain by writing $L^p(S)$ instead of p. To estimate the error, we use that the p norm of a matrix is either less than the 1 norm or less than the maximum norm. The corresponding result for integral equations is

$$\begin{aligned} &\mathscr{L}f(\alpha) = \int_S \mathscr{L}(\alpha, \beta) f(\beta)\, d\beta, \\ &\|\mathscr{L}f\|_{L^p(S)} \leqq \max\left\{ \sup_{\beta \in S} \int_S |\mathscr{L}(\alpha, \beta)|\, d\alpha, \sup_{\alpha \in S} \int_S |\mathscr{L}(\alpha, \beta)|\, d\beta \right\} \|f\|_{L^p(S)}. \end{aligned} \tag{3.16}$$

Here $1 \leqq p \leqq \infty$ and both results follow from Hölder's inequality (see Folland [14]).

We begin with the band matrices in (3.14), i.e.

$$\begin{aligned} (E_k e)_i &= \sum_{j \in J_k(i)} \int_0^1 \nabla K_\delta(\Psi_i - \Psi_j + \chi_{ij}(\vartheta))\, d\vartheta \cdot e_j \xi_j h^2 \\ &= \int_S \mathscr{L}_k(\alpha_i, \beta) f(\beta)\, d\beta = (\mathscr{L}_k f)(\alpha_i) \end{aligned} \tag{3.17}$$

where $\mathscr{L}_k(\alpha, \beta) = \int_0^1 \nabla K_\delta(\Psi_i - \Psi_j + \chi_{ij}(\vartheta))\, d\vartheta$ for all $\alpha \in Q_i$, $\beta \in Q_j$ and $(i, j) \in J_k$, and zero otherwise and $f(\beta) = e_j \xi_j$ for all $\beta \in Q_j$ and $j \in J$, and zero otherwise. Since $\nabla K_\delta(x)$ is an even function of x it follows that $\mathscr{L}_k(\alpha, \beta) = \mathscr{L}_k(\beta, \alpha)$. This does not imply that the matrices E_k are symmetric. Note that if we fix one of the coordinates α and β and consider $\mathscr{L}_k$ as functions of the other coordinate, then their supports are disjoint. Let $\mathscr{L}_k f(\alpha) = \mathscr{L}_k f(\alpha_i)$ for all α in Q_i. By using (3.15), (3.16) and (3.17) we obtain

$$\begin{aligned} \|E_k e\|_p = \|\mathscr{L}_k f\|_{L^p(S)} &\leqq \sup_\alpha \int_S |\mathscr{L}_k(\alpha, \beta)|\, d\beta \|f\|_{L^p(S)} \\ &\leqq \max_i \int_{S_k(i)} |\mathscr{L}_k(\alpha_i, \beta)|\, d\beta \|\xi\| \cdot \|e\|_p. \end{aligned} \tag{3.18}$$

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Now let $k=1$ and fix i. It follows from Lemma 2 and the inequality preceding (3.14) that

$$\int_{S_1(i)} |\mathscr{L}_1(\alpha_i, \beta)|\, d\beta \leqq \int_{S_1(i)} \frac{\sqrt{2}\, L_2}{\delta \cdot ch}\, d\beta = \frac{\sqrt{2}\, L_2}{\delta ch} \int_{\Psi(S_1(i))} \det \Psi^{-1\prime}(y)\, dy.$$

Here we have used the change of variables $\beta = \Psi^{-1}(y)$ as in § 3.1. Let $y = \Psi(\beta)$ be a point in $S_1(i)$. If $x = \Psi_i$ then the inequalities in (3.5) show that $\Psi(S_1(i))$ is contained in a disk with center at Ψ_i and radius $(1+3C+c)h$. Our last integral is therefore less than $2\pi(1+3C+c)^2h^2$, where we have used (2.3). Since $\|\xi\| \leqq C$ we conclude from (3.18) that

$$\|E_1 e\|_p \leqq 2\pi \frac{C}{c}(1+3C+c)^2 \cdot \sqrt{2}\, L_2 \frac{h}{\delta} \|e\|_p.$$

Next let $k>1$ and fix i. If (i,j) is in J_k and β is in Q_j then

$$|\Psi_i - \Psi_j| - (2C+c)h \leqq |\Psi_i - \Psi(\beta)| - (C+c)h \leqq |\Psi_i - \Psi_j + \chi_{ij}| \leqq |\Psi_i - \Psi_j| + ch. \tag{3.19}$$

By using the definition of J_2, $S_2(i)$ and $\mathscr{L}_2(\alpha, \beta)$ plus (3.19) and Lemma 2 we get

$$\int_{S_2(i)} |\mathscr{L}_2(\alpha_i, \beta)|\, d\beta = \sum_{j \in J_2(i)} \int_{Q_j} |\mathscr{L}_2|\, d\beta \leqq \int_{S_2(i)} \frac{\sqrt{2}\, L_2}{\delta[|\Psi_i - \Psi(\beta)| - (C+c)h]}\, d\beta.$$

We have encountered a similar integral in § 3.1. Let A_2 be the annulus with center at Ψ_i, inner radius $(1+C+c)h$ and outer radius δ. Since $\Psi(S_2(i))$ is a subset of A_2 we can estimate the last integral by using a change of variables and a switch to polar coordinates. By combining the result with (3.18) we get

$$\|E_2 e\|_p \leqq 2 \cdot 2\pi(1+C+c)C \cdot \sqrt{2}\, L_2 \frac{\delta - (1+C+c)h}{\delta} \|e\|_p.$$

To estimate the maximum norm of E_3, we must know its elements more precisely. Assume that $|\Psi_i - \Psi_j + \chi_{ij}(\vartheta)| = \delta$ for two values $\vartheta_1 < \vartheta_2$ and set $\vartheta_0 = 0$ and $\vartheta_3 = 1$. Then

$$\mathscr{L}_3(\alpha, \beta) = \sum_{l=1}^{3} (\vartheta_l - \vartheta_{l-1}) \int_0^1 \nabla K_\delta(\Psi_i - \Psi_j + \vartheta \chi_{ij}(\vartheta_l) + (1-\vartheta)\chi_{ij}(\vartheta_{l-1}))\, d\vartheta$$

for all $\alpha \in Q_i$, $\beta \in Q_j$ and $(i,j) \in J_3$. With a suitable interpretation this formula is also valid if the line connecting $\tilde{\Psi}_i - \tilde{\Psi}_j$ and $\bar{\Psi}_i - \bar{\Psi}_j$ intersects the circle with center at Ψ_i and radius δ once or not at all. Since $c \leqq C$ and $2r_1 < \delta$, we find in all three cases that ∇K_δ is evaluated outside the disk with center at the origin and radius $\delta/4$. In addition, $\Psi(S_3(i))$ is contained in the annulus A_3 with center at Ψ_i, inner radius $\delta - (1+3C+c)h$ and outer radius $\delta + (1+3C+c)h$. Thus by inserting the expression for $\mathscr{L}_3$ in (3.18), using Lemma 2 and a change of variables we get

$$\begin{aligned} \|E_3 e\|_p &\leqq \int_{\Psi(S_3(i))} \frac{\sqrt{2}\, L_2 \det \Psi^{-1\prime}(y)}{(\delta/4)^2}\, dy \cdot C\|e\|_p \\ &\leqq 32 \cdot 4\pi(1+3C+c)C \cdot \sqrt{2}\, L_2 \frac{h}{\delta} \|e\|_p. \end{aligned}$$

Our approach fails for E_4 because $\int_S |\nabla K_\delta|$ is not bounded as δ tends to zero. Instead we follow Beale and Majda [4] and estimate the continuous analogue of $E_4 e$

by using the Calderon-Zygmund inequality. Let

$$g(\Psi(\alpha)) = \int_S \nabla K_\delta(\Psi(\alpha) - \Psi(\beta)) f(\beta)\, d\beta.$$

Equation (3.18) and the triangle inequality imply that

$$\|E_4 e\|_p \leqq \|g \circ \Psi\|_{L^p(S)} + \|\mathscr{L}_4 f - g \circ \Psi\|_{L^p(S)}. \tag{3.20}$$

By using the change of variables $\alpha = \Psi^{-1}(x)$ we get

$$\|g \circ \Psi\|_{L^p(S)} \leqq \left(\int_{\mathbb{R}^2} |g(x)|^p \det \Psi^{-1\prime}(x)\, dx \right)^{1/p} \leqq 2^{1/p} \|g\|_{L^p(\mathbb{R}^2)}.$$

As f vanishes outside S and $K_\delta = K * \zeta_\delta$ it follows from the definition of g plus several changes of variables and Fubini's theorem that

$$\begin{aligned} g(x) &= \int_{\mathbb{R}^2} \nabla_x K_\delta(x-y) f(\Psi^{-1}(y)) \det \Psi^{-1\prime}(y)\, dy \\ &= \nabla_x [K * (\zeta_\delta * \tilde{f})](x) \\ &= \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \zeta_\delta * \tilde{f}(x) + \int_{\mathbb{R}^2} \nabla_x K(x-y) \zeta_\delta * \tilde{f}(y)\, dy \end{aligned}$$

almost everywhere (see Calderon and Zygmund [7, p. 130]). Here $\tilde{f} = f \circ \Psi^{-1} \det \Psi^{-1\prime}$. The last integral is singular and must be evaluated in the principal value sense. We can now use the Calderon-Zygmund inequality (see [7, pp. 97 and 108]). It says that $\|(\nabla K) * f\|_{L^p} \leqq C' \|f\|_{L^p}$ for all f in L^p where $1 < p < \infty$ and C' depends only on p and the kernel ∇K, which is fixed in our case. Hence

$$\begin{aligned} \|g \circ \Psi\|_{L^p(S)} &\leqq 2^{1/p}(\tfrac{1}{2} + C') \|\zeta_\delta\|_{L^1(\mathbb{R}^2)} \|\tilde{f}\|_{L^p(\mathbb{R}^2)} \\ &\leqq 2^{1/p}(\tfrac{1}{2} + C') 6\pi L_0 2C^{2/p} \|f\|_{L^p(S)} \\ &\leqq 2(2C^2)^{1/p}(\tfrac{1}{2} + C') 6\pi L_0 C \|e\|_p \end{aligned} \tag{3.21}$$

where we have used Young's inequality and the bound for $\int |\zeta_\delta|$ from the proof of Lemma 2 plus a change of variables and (2.3) and (3.15).

We now consider the last term in (3.20). Let $\alpha \in S$ with $\alpha \in Q_i$. It follows from the definition of g and (3.17) that

$$\begin{aligned} g(\Psi(\alpha)) - \mathscr{L}_4 f(\alpha) &= \int_{S_1 \cup S_2 \cup S_3(i)} \nabla K_\delta(\Psi(\alpha) - \Psi(\beta)) f(\beta)\, d\beta \\ &+ \sum_{j \in J_4(i)} \int_{Q_j} \int_0^1 [\nabla K_\delta(\Psi(\alpha) - \Psi(\beta)) - \nabla K_\delta(\Psi_i - \Psi_j + \chi_{ij}(\vartheta))]\, d\vartheta f(\beta)\, d\beta. \end{aligned} \tag{3.22}$$

Let $\mathscr{L}(\alpha, \beta) = \nabla K_\delta(\Psi(\alpha) - \Psi(\beta))$ if $\alpha \in Q_i$, $\beta \in Q_j$ and $(i,j) \in J_1 \cup J_2 \cup J_3$ and zero otherwise. Then $\mathscr{L}(\alpha, \beta) = \mathscr{L}(\beta, \alpha)$. Since $\Psi(S_1 \cup S_2 \cup S_3(i))$ is contained in a disk with center at $\Psi(\alpha)$ and radius 2δ, we conclude from (3.16) and Lemma 2 that the p norm of the first expression in (3.22) is less than

$$\int_{|\Psi(\alpha) - y| < 2\delta} \frac{\sqrt{2}\, L_2 \det \Psi^{-1\prime}(y)}{\delta |\Psi(\alpha) - y|}\, dy\, \|f\|_{L^p(S)} \leqq 8\pi\sqrt{2}\, L_2 C \|e\|_p.$$

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By using the mean value theorem and Lemma 2 we find that the norm of the expression $[\cdots]$ in (3.22) equals

$$\left|\int_0^1 \sum_{k=1}^{2} D_k \nabla K_\delta(\Psi_i - \Psi(\beta) + \tau(\Psi(\alpha) - \Psi_i) + (1-\tau)(\Psi(\beta) - \Psi_j + \chi_{ij})) \cdot (\Psi(\alpha) - \Psi_i - (\Psi(\beta) - \Psi_j) - \chi_{ij})_k \, d\tau\right|$$

$$\leqq \frac{2L_2(2C+c)h}{\sqrt{\delta}(|\Psi_i - \Psi(\beta)| - (C+c)h)^{5/2}}.$$

Since $\Psi(S_4(i))$ is a subset of the annulus A_4 with center at Ψ_i, inner radius $\delta + (C+c)h$ and stretching to infinity, it follows from (3.16) that the p norm over the domain S of the last expression in (3.22) is less than

$$\int_{A_4} \frac{2L_2(2C+c)h \det \Psi^{-1\prime}(y)}{\sqrt{\delta}(|\Psi_i - y| - (C+c)h)^{5/2}} \, dy \, \|f\|_{L^p(S)} \leqq 4\pi(2C+c)2L_2 \frac{3h}{\delta} C \|e\|_p.$$

By combining our bounds for the terms on the right-hand side of (3.22) with (3.21) we conclude from (3.20) that $\|E_4 e\|_p$ is less than const $\cdot \|e\|_p$ where the constant depends on C, L_2 and p.

We now return to (3.14) and will study the matrices D_k. They consist of 2×2 blocks along the diagonal. More precisely

$$(D_k e)_i = \int_S \mathscr{L}_k(\alpha_i, \beta)\bar{\xi}(\beta) \, d\beta \cdot e_i = (\mathscr{L}_k \bar{\xi})(\alpha_i) e_i \tag{3.23}$$

where the kernel $\mathscr{L}_k(\alpha, \beta)$ is defined as in (3.17) and $\bar{\xi}(\beta) = \xi_j$ for all β in Q_j and j in J and zero otherwise. Thus the support of $\bar{\xi}$ is contained in the closure of S. It follows from (3.23) that

$$|\mathscr{L}_k \bar{\xi}(\alpha_i)| \leqq \max_i \int_{S_k(i)} |\mathscr{L}_k(\alpha_i, \beta)| \, d\beta \, \|\bar{\xi}\|.$$

We estimated the integrals with $k \leqq 3$ when we studied the E_k matrices, and they are bounded by a constant which depends on C and L_2 but is independent of h. By combining this observation with (3.13) we conclude that $\|D_k e\|_p$ is less than const $\cdot \|e\|_p$ for $k = 1$, 2 and 3.

Finally we will estimate $D_4 e$. Here our arguments are more complicated than Beale and Majda's because we assume less smoothness, namely that ξ is Hölder continuous. Another difficulty lies in that the support of ξ may not be contained in S. Thus we let $J_4[i]$ consist of $J_4(i)$ plus all the indices j for which $\delta + r_1 \leqq |\Psi_i - \Psi_j|$ and $\xi(\beta)$ is different from zero for some point β in Q_j. Note that $\xi_j = 0$ for all $j \in J_4[i] - J_4(i)$. Let $S_4[i]$ be the union of all Q_j with index in $J_4[i]$. By using (3.23) we get

$$\begin{aligned}
\mathscr{L}_4 \bar{\xi}(\alpha_i) = &\int_{S_4[i]} \nabla K(\Psi_i - \Psi(\beta)) \xi(\beta) \, d\beta \\
&+ \sum_{j \in J_4[i]} \int_{Q_j} \nabla K(\Psi_i - \Psi(\beta))[\xi_j - \xi(\beta)] \, d\beta \\
&+ \int_{S_4(i)} [\nabla K_\delta(\Psi_i - \Psi(\beta)) - \nabla K(\Psi_i - \Psi(\beta))] \bar{\xi}(\beta) \, d\beta \\
&+ \sum_{j \in J_4(i)} \int_{Q_j} \int_0^1 [\nabla K_\delta(\Psi_i - \Psi_j + \chi_{ij}(\vartheta)) - \nabla K_\delta(\Psi_i - \Psi(\beta))] \, d\vartheta \, \bar{\xi}(\beta) \, d\beta.
\end{aligned} \tag{3.24}$$

The last expression in (3.24) is almost identical to the last expression in (3.22). It is therefore less than

$$4\pi(2C+c)2L_2\frac{3h}{\delta}C.$$

Next, it follows from Lemma 2 and the usual change of variables that the third expression in (3.24) is less than

$$\int_{S_4(i)} \frac{\sqrt{2}\,L_2\sqrt{\delta}}{|\Psi_i-\Psi(\beta)|^{5/2}}|\bar{\xi}(\beta)|\,d\beta \leqq 4\pi\sqrt{2}\,L_2\frac{2\sqrt{\delta}}{\sqrt{\delta+(C+c)h}}C$$

where we have used that $\Psi(S_4(i))$ is a subset of A_4. As ξ is Hölder continuous and $\Psi(S_4[i])$ is contained in the annulus A_5 with center at Ψ_i, inner radius δ and outer radius $\sqrt{2}\,C(1+D)$, we can bound the second term in (3.24) by

$$\int_{S_4[i]} \frac{H_\lambda(\xi)h^\lambda}{2\pi|\Psi_i-\Psi(\beta)|^2}\,d\beta \leqq 2CD^{-\lambda}h^\lambda \log\left(\frac{\sqrt{2}\,C(1+D)}{\delta}\right).$$

Now $h^\lambda \log(a/h)$ is less than $a^\lambda/(\lambda e)$ for all $h<a$ and the second term in (3.24) is therefore less than $C(1+D)^\lambda D^{-\lambda}\lambda^{-1}$ for all $h \leqq h_1$. To estimate the remaining term in (3.24), we use a trick. Let

$$\tilde{\xi}(y)=\xi(\Psi^{-1}(y))\det \Psi^{-1\prime}(y)$$

(cf. (2.4)). Since ξ is Hölder continuous with exponent λ, it follows as in § 2 that $\tilde{\xi}$ is Hölder continuous with the same exponent (see (2.6)). More precisely, $\|\tilde{\xi}\|_{C^\lambda(D)} \leqq C''$ where C'' depends on C and λ. As $\Psi(S_4[i])$ is a subset of A_5 we find, after a change of variables, that the first term in (3.24) is equal to

$$\int_{A_5} \nabla K(\Psi_i-y)[\tilde{\xi}(y)-\tilde{\xi}(\Psi_i)]\,dy-\int_{A_5-\Psi(S_4[i])} \nabla K(\Psi_i-y)\tilde{\xi}(y)\,dy. \tag{3.25}$$

Here we have used one of the properties of a Calderon–Zygmund kernel, namely that the average of the kernel over any sphere centered at the origin is zero. Hence $\int_{A_5}\nabla K$ vanishes. Since $\tilde{\xi}$ is Hölder continuous we conclude that the first expression in (3.25) is less than $C''(\sqrt{2}\,C(1+D)/D)^\lambda/\lambda$.

We will now show that the nonzero contribution in the last expression in (3.25) must occur near $|y|=\delta$. Let $y \in A_5-\Psi(S_4[i])$ and assume that $\tilde{\xi}(y)$ is different from zero. Set $\beta=\Psi^{-1}(y)$. Then $\xi(\beta)\neq 0$ and $\beta \in Q_j$ for some j and suppose that $\delta+r_1 \leqq |\Psi_i-\Psi_j|$. This implies that $j \in J_4[i]$ and $\beta \in S_4[i]$. But this is a contradiction. Hence

$$\delta \leqq |\Psi_i-y| \leqq |\Psi_i-\Psi_j|+|\Psi_j-\Psi(\beta)| \leqq \delta+r_1+Ch=\delta+(1+3C+c)h.$$

Thus the nonzero contribution in the last integral in (3.25) must occur in the annulus with center at Ψ_i, inner radius δ and outer radius $\delta+(1+3C+c)h$, and the integral is therefore less than $2C(1+3C+c)h/\delta$. By combining our results we find from (3.24) that $|\mathscr{L}_4\bar{\xi}(\alpha_i)|$ is bounded by a constant which depends on C, D, L_2, λ and p. By using (3.13) and (3.23) we conclude that $\|D_4 e\|_p$ is less than const $\cdot\,\|e\|_p$. This completes the proof.

3.3. Low order truncation. In this section we will prove that if the differential equation for the flow is solved by a second order Runge–Kutta method, then the truncation error is less than const $\cdot\,k^3$. This is surprising because the right-hand side of (1.1) is not even Lipschitz continuous in the maximum norm. It is log-Lipschitz. Thus the usual arguments would at best give a truncation error of order $k^2 \log k$. The

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reason for the sharper result is that the flow map is a smooth function of time (see Assumption 3 or [17]). Our estimate is not optimal. However, suppose we place a particle in a stationary flow for which we know the velocity field, and determine the particle path by using a fourth order Runge–Kutta method. Then numerical examples show that the rate of convergence depends on the smoothness of the flow and can be as low as second order. This corresponds to a local truncation error of order k^3 and is comparable to our estimate in Lemma 5.

For simplicity, and in order to provide a comparison with the work of Anderson and Greengard [2], we consider the midpoint rule

$$\tilde{\Psi} = \tilde{\Phi}(t) + \tfrac{1}{2}kV[\tilde{\Phi}], \qquad \tilde{\Phi}(t+k) = \tilde{\Phi}(t) + kV[\tilde{\Psi}] \tag{3.26}$$

with initial conditions $\tilde{\Phi}(\alpha_i, 0) = \alpha_i$. Here $V[\ \ ]$ is the discrete approximation to the velocity field and defined in § 3.2, and the computed solution $\tilde{\Phi}(t)$ is obtained at time $t = nk$. The intermediate result $\tilde{\Psi}$ is a first order approximation to the solution at time $t + \frac{1}{2}k$. To define the truncation error τ, we use the continuous analogue of (3.26), namely

$$\Psi = \Phi(t) + \tfrac{1}{2}kv[\Phi], \qquad \Phi(t+k) = \Phi(t) + kv[\Psi] + \tau. \tag{3.27}$$

Here $\Phi(\alpha, t)$ is the solution of $\dot{\Phi} = v[\Phi]$ with $\Phi(\alpha, 0) = \alpha$ where v is given by (2.2), and $\dot{}$ means the derivative with respect to time. We can also determine the flow map by $\dot{\Phi} = u(\Phi, t)$, where u is the velocity field corresponding to the vorticity distribution ω at time t. Note that the truncation error is not defined by using this equation, as $u(\Psi, t + \frac{1}{2}k)$ is not the same as $v[\Psi]$. It is this difference that makes the estimate of the truncation error nontrivial, in particular for higher order methods.

LEMMA 5. *Let Assumption* 3 *be satisfied. There exist two constants C_2 and k_2 such that if k is less than k_2 then*

$$|\tau(\alpha, t)| \leqq C_2 k^3$$

for all α in $\mathbb{R}^2$ and $0 \leqq t \leqq T - k$. The constants C_2 and k_2 depend on C, D and λ.

Remark. If the initial vorticity is constant inside a circle and vanishes outside, and we solve $\dot{\Phi} = v[\Phi]$ by the classical fourth order Runge–Kutta method then the truncation error is $O(k^5)$. I have been unable to show this for Hölder continuous vorticity distributions with compact support, but higher order truncation can be established for smooth flows. This is proved in § 4.3. Our technique can also be used to prove quadratic convergence for autonomous systems with piecewise smooth solutions, even in the presence of a finite number of singularities.

Proof. We suppress the dependence on α or t if it is not needed in the arguments. It follows from (3.27) that

$$\tau = \Phi(t+k) - \Phi(t) - k\partial_t\Phi(t + \tfrac{1}{2}k) + k\{v[\Phi(t + \tfrac{1}{2}k)] - v[\Psi]\} = I_1 + k\{I_2\}.$$

Since the flow map is a smooth function of t we can use Taylor's formula with remainder and obtain

$$|I_1| = \frac{k^3}{24}|\partial_t^3\Phi(t + \vartheta_1 k)| \leqq \frac{k^3 C'}{24}$$

where $0 \leqq \vartheta_1 \leqq 1$ and C' is an upper bound for $\|\partial_t^3\Phi\|$ and $\|\partial_j\partial_t^2\Phi\|$. It can be shown that C' depends only on C, D and λ (see [17]). By using (2.2) and the mean value theorem we see that

$$I_2 = \int_{\mathbb{R}^2}\int_0^1 \nabla K(\vartheta_2[\Phi(\alpha) - \Phi(\beta)] + (1 - \vartheta_2)[\Psi(\alpha) - \Psi(\beta)])\, d\vartheta_2$$
$$\cdot [\Phi(\alpha) - \Phi(\beta) - \Psi(\alpha) + \Psi(\beta)]\omega(\beta)\, d\beta$$

where Φ is evaluated at $t+\frac{1}{2}k$. It follows from (3.27) that

$$\begin{aligned}\Phi\left(t+\frac{1}{2}k\right)-\Psi &= \Phi\left(t+\frac{1}{2}k\right)-\Phi(t)-\frac{1}{2}k\partial_t\Phi(t)\\ &= \frac{k^2}{4}\int_0^1 (1-\vartheta)\partial_t^2\Phi\left(t+\vartheta\frac{1}{2}k\right)d\vartheta.\end{aligned} \tag{3.28}$$

By using this identity we can rewrite the argument of ∇K as

$$\Phi(\alpha)-\Phi(\beta)-\frac{k^2}{4}(1-\vartheta_2)\int_0^1 (1-\vartheta)\left[\partial_t^2\Phi\left(\alpha, t+\vartheta\frac{1}{2}k\right)-\partial_t^2\Phi\left(\beta, t+\vartheta\frac{1}{2}k\right)\right]d\vartheta$$

and estimate it from below by

$$\frac{\sqrt{2}}{C}|\alpha-\beta|-\frac{k^2}{4}\frac{1}{2}\sqrt{2}\,C'|\alpha-\beta|$$

(see (2.5)). Set $c=(4\sqrt{2}\,C)^{-1}$ and choose k_2 such that $k_2^2\sqrt{2}\,C'=32c$. Then $|\nabla K|$ is less than $(2\pi)^{-1}(4c|\alpha-\beta|)^{-2}$ for all $k\leqq k_2$ and $0\leqq\vartheta_2\leqq 1$. By using (3.28) once more and writing Φ_α instead of $\Phi(\alpha, t)$, we have

$$|\Phi_\alpha-\Phi_\beta-\Psi_\alpha+\Psi_\beta|\leqq\frac{k^2}{4}\int_0^1(1-\vartheta)|\partial_t^2\Phi_\alpha-\partial_t^2\Phi_\beta|\,d\vartheta\leqq\frac{k^2}{4}\frac{1}{2}\sqrt{2}\,C'|\alpha-\beta|.$$

We can therefore estimate I_2 for all α by

$$|I_2|\leqq\int\frac{k^2\cdot\sqrt{2}\,C'|\alpha-\beta|}{8\cdot 2\pi(4c|\alpha-\beta|)^2}|\omega(\beta)|\,d\beta\leqq\frac{k^2\sqrt{2}\,C'}{128c^2}D\|\omega\|.$$

The proof is completed by combining this result with the bound for $|I_1|$ and choosing C_2 properly.

Just because the flow map is a smooth function of time it does not follow that we should use a higher order method to determine the particle paths for the flow. We will show this by an example. Set $r=|x|$ and consider the velocity field

$$u(x)=\frac{1}{r^2}[1-(1-\min(1, r^2))^{1+\lambda}]\begin{pmatrix}-x_2\\ x_1\end{pmatrix}.$$

The corresponding vorticity distribution is Hölder continuous with exponent λ. In fact $\omega(x)$ is equal to $2(1+\lambda)(1-r^2)^\lambda$ if $r<1$ and zero otherwise. If $\lambda=0$ then the vorticity is discontinuous and the velocity is no longer a classical solution of Euler's equations. We place a particle at $x=(1, 0)$ at time $t=0$ and solve the ordinary differential equation $\dot{x}=u(x)$ for $0\leqq t\leqq T$ where $T=\pi/2$. The particle remains on the unit circle and moves with speed 1. We use the classical fourth order Runge-Kutta method with time step $k=T/N$ where $N=10\cdot 4^i$. The errors at time T are presented in Table 1 for several

TABLE 1
Errors for the classical Runge-Kutta method.

N	$\lambda=0$	$\lambda=.25$	$\lambda=.50$	$\lambda=.75$	$\lambda=1$
10	$.322_{10}-2$	$.921_{10}-3$	$.278_{10}-3$	$.986_{10}-4$	$.483_{10}-4$
40	$.202_{10}-3$	$.284_{10}-4$	$.407_{10}-5$	$.665_{10}-6$	$.187_{10}-6$
160	$.126_{10}-4$	$.884_{10}-6$	$.624_{10}-7$	$.477_{10}-8$	$.730_{10}-9$
640	$.789_{10}-6$	$.276_{10}-7$	$.969_{10}-9$	$.356_{10}-10$	$.285_{10}-11$

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choices of λ. The rate of convergence is $k^{2+2\lambda}$. The reason for the loss of accuracy is that in the intermediate steps, the Runge-Kutta method samples the values of the velocity from both sides of the unit circle.

We have also experimented with $-1<\lambda<0$. In these cases the vorticity is unbounded and the velocity is only a weak solution of Euler's equations. The solution of $\dot{x}=u(x)$ exists and is unique, even though u is not Lipschitz continuous. We have observed that the rate of convergence is also equal to $k^{2+2\lambda}$ for these values of λ.

3.4. Low order convergence. In this section we establish the convergence of the vortex method for long time intervals. We assume that the initial vorticity is Hölder continuous and admit singular and discontinuous cutoff's like Chorin's [8] and the one used by Milinazzo and Saffman [21]. The strengths of the vortex blobs are determined by local averages of the initial vorticity. The method can therefore not converge faster than quadratically. This observation is due to Cottet [9]. We do not even get quadratic convergence. This is probably caused by a limitation of our technique, as numerical experiments by Del Prete and Hald [11] and Perlman [24] indicate quadratic convergence in these cases.

In the second part of this section we show that the computed vorticity converges uniformly if we, at each point, delete just a single blob. This is surprising because the nearby vortices may be singular. Del Prete and Hald [11] have proved uniform convergence if the vorticity is evaluated with the cutoff used by Milinazzo and Saffman [21]. Our result is comparable to theirs, but applies to a larger class of cutoff functions.

THEOREM 1. *Let Assumptions* 1 *and* 3 *be satisfied and set* $\varepsilon=\lambda/5$, $\delta=h^{4/(4+\lambda)}$ *and* $\kappa_j=\int_{Q_j}\omega\,dx$. *Solve the differential equation* (1.4) *by the midpoint rule* (3.26) *with* $k\leqq h^{(1+\gamma)/2}$, $\gamma>0$. *Let* $1\leqq p<\infty$. *There exist two constants* C_3 *and* h_2 *such that*

$$\|\tilde{\Phi}(t)-\Phi(t)\|_p\leqq C_3(h^{1+\varepsilon}+k^2)$$

for all $h\leqq h_2$ *and* $t=nk\leqq T$. *The constants* C_3 *and* h_2 *depend on* C, D, L_2, T, λ, γ *and* p.

Remark. Here $\Phi(\alpha_i,t)$ and $\tilde{\Phi}(\alpha_i,t)$ are the exact position and the computed position of a particle starting at α_i at time $t=0$. This theorem extends the convergence results due to Beale and Majda [4], Cottet [9], Anderson and Greengard [2], Del Prete and Hald [11] and Hald [15] to a larger class of vortex methods and to a large class of initial data. The rate of convergence can be improved by using more restrictive assumptions. In Cottet's theory [9] the vorticity need not have compact support. We follow Beale and Majda [4] and work in L^p in order to get convergence for nonnegative cutoff functions. Our treatment of the time discretization follows Anderson and Greengard [2]. They do not define the truncation error in the time variable, but estimate it implicitly by using the stability of the vortex method. Note that we can use $h=k$. Theorem 1, Lemma 3 and Lemma 4 imply that $\|V[\tilde{\Phi}]-v[\Phi]\|_p\leqq\text{const}\cdot(h^{1+\varepsilon}+k^4)$.

Proof. Since the diameter of $\bigcup_{j\in J}Q_j$ is less than $2\sqrt{2}\,h+D$ it follows from (3.13) that

$$\|\tilde{\Phi}-\Phi\|_r\leqq(\textstyle\sum h^2)^{1/r-1/s}\|\tilde{\Phi}-\Phi\|_s\leqq(1+D)^2\|\tilde{\Phi}-\Phi\|_s$$

if $1\leqq r<s$ and $h<1/(2\sqrt{2})$. It is therefore enough to establish Theorem 1 for p sufficiently large. Let λ and γ be given and choose p such that $2/p$ is less than $\varepsilon/2$ and $\gamma/2$. We shall use induction and prove that

$$\|\tilde{\Phi}^n-\Phi^n\|_p\leqq\frac{\alpha^n-1}{\alpha-1}k\|1\|_p b \tag{3.29}$$

where $\alpha = 1+2C_1k$, $b = 2C_0h^{1+\varepsilon} + C_2k^2$ and $\Phi_j^n = \Phi(\alpha_j, nk)$. The constants C_0, C_1 and C_2 were determined in Lemmas 3, 4 and 5. Remember that C_1 depends upon p. We choose $h_2 < 1$ such that (1): h_2 is less than h_0 and h_1, (2): h_2^ε is less than h_1 and (3): $h_2^{1+\gamma}$ is less than k_0^2, k_2^2 and $C_1^{-1/2}$. The constants h_0, h_1, k_0 and k_2 were determined in Lemmas 3, 4, 1 and 5. Finally we set $c = (4\sqrt{2}\,C)^{-1}$ and assume that

$$\frac{e^{2C_1T}-1}{2C_1}(1+D)^2(2C_0h_2^{\varepsilon/2} + C_2h_2^{\gamma/2}) \leqq \frac{1}{2}c. \tag{3.30}$$

Let $h \leqq h_2$. At time $t = 0$ we have $\tilde{\Phi}_j = \Phi_j$ and (3.29) is therefore satisfied for $n = 0$. Next, assume that (3.29) holds for $nk \leqq T - k$. Since $|\tilde{\Phi}_j^n - \Phi_j^n|$ is less than $h^{-2/p}\|\tilde{\Phi}^n - \Phi^n\|_p$ for all j in J and $k^2 \leqq h^{1+\gamma}$, (3.29) and (3.30) yield

$$\begin{aligned}|\tilde{\Phi}_i^n - \Phi_i^n| &\leqq \frac{e^{2C_1nk}-1}{2C_1}(1+D)^2[2C_0h^{\varepsilon/2}h^{\varepsilon/2-2/p} + C_2h^{\gamma/2}h^{\gamma/2-2/p}]h \\ &\leqq \frac{1}{2}ch\end{aligned}$$

for all i in J. The assumptions in Lemma 4 are therefore satisfied and we get

$$\|V[\tilde{\Phi}^n] - V[\Phi^n]\|_p \leqq C_1\|\tilde{\Phi}^n - \Phi^n\|_p.$$

To estimate the error in the intermediate step of the Runge-Kutta method, we compare $\tilde{\Psi}$ and Ψ as defined by (3.26) and (3.27). By using the last inequality and the consistency of the vortex method we obtain

$$\begin{aligned}\|\tilde{\Psi} - \Psi\|_p &\leqq \|\tilde{\Phi}^n - \Phi^n\|_p + \tfrac{1}{2}k\|V[\tilde{\Phi}^n] - V[\Phi^n]\|_p + \tfrac{1}{2}k\|V[\Phi^n] - v[\Phi^n]\|_p \\ &\leqq (1+\tfrac{1}{2}kC_1)\|\tilde{\Phi}^n - \Phi^n\|_p + \tfrac{1}{2}k\|1\|_pC_0h^{1+\varepsilon}.\end{aligned} \tag{3.31}$$

Since $1+\frac{1}{2}kC_1 < \alpha$ and $C_0h^{1+\varepsilon} < b$ we conclude from (3.29) and (3.31) that

$$\begin{aligned}|\tilde{\Psi}_i - \Psi_i| &\leqq h^{-2/p}\left[a\frac{a^n-1}{a-1}k\|1\|_pb + k\|1\|_pb\right] \\ &\leqq \frac{e^{2C_1T}-1}{2C_1}(1+D)^2[2C_0h^{\varepsilon/2} + C_2h^{\gamma/2}]h \leqq \frac{1}{2}ch\end{aligned}$$

for all i in J. It follows from Lemma 1 that the map Ψ satisfies the inequalities in (2.3). The assumptions in Lemma 4 are therefore satisfied and we have

$$\|V[\tilde{\Psi}] - V[\Psi]\|_p \leqq C_1\|\tilde{\Psi} - \Psi\|_p. \tag{3.32}$$

We can now estimate the error in the $(n+1)$st step. By combining (3.26) and (3.27) and using (3.31) and (3.32) we obtain

$$\begin{aligned}\|\tilde{\Phi}^{n+1} - \Phi^{n+1}\|_p &\leqq \|\tilde{\Phi}^n - \Phi^n\|_p + k\|V[\tilde{\Psi}] - V[\Psi]\|_p + k\|V[\Psi] - v[\Psi]\|_p + \|\tau\|_p \\ &\leqq [1 + kC_1 + \tfrac{1}{2}(kC_1)^2]\|\tilde{\Phi}^n - \Phi^n\|_p + k\|1\|_p[(1+\tfrac{1}{2}kC_1)C_0h^{1+\varepsilon} + C_2k^2] \\ &\leqq \frac{a^{n+1}-1}{a-1}k\|1\|_pb.\end{aligned}$$

Here we have used the consistency of the initial vortex method (Lemma 2) and the estimate of the truncation error (Lemma 5) plus the induction hypothesis (3.29) and the fact that $kC_1 \leqq 1$. This completes the proof.

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So far we have determined the strength of the vortices by local averages of the initial vorticity. If the strengths are determined by pointwise evaluation, i.e. $\kappa_j = \omega(\alpha_j)h^2$ then the consistency error is of order h^λ and our proof fails, because we cannot use Lemma 4. There are two ways out. If the initial vorticity is once Hölder continuously differentiable, then the consistency error is less than const $\cdot\, h^{1+\varepsilon}$. This was shown in the discussion following the proof of Lemma 3. Our stability result is therefore applicable and we obtain convergence. If, however, the initial vorticity is just Hölder continuous, I can only prove convergence by restricting the class of cutoff functions, and modifying the consistency lemma and the stability lemma (using δ instead of ch). It can be shown that if we replace (i) in Assumption 1 by $|\zeta^{(\nu)}(r)| \leqq L_0$ for $0 \leqq r < \delta$ and $\nu = 0, 1$ and set $\delta = h^{2\lambda/(2+\lambda)}$, $\kappa_j = \omega_j h^2$ and $k^2 \leqq h^\lambda$ then the p-norm of the error in the vortex method is less than const $\cdot\, h^\lambda$ for h sufficiently small. Thus we obtain convergence if the strength of the vortices are determined by pointwise evaluation of the initial vorticity. Our estimate of the rate of convergence is not as good as if the strengths are determined by local averages. Note also that (1.7) is the only commonly used cutoff, which is singular.

The convergence of the computed velocity to the solution of Euler's equations follows from the convergence of the vortex method viewed as a particle method. Thus Del Prete and Hald [11] have shown uniform convergence over short time intervals for singular and discontinuous cutoffs with compact support. Their proof depends only on the error in the computed position of the vortices and can be extended to long time intervals. Hald [15] obtained quadratic convergence for smooth cutoffs and smooth solutions, while Beale and Majda [4] proved higher order convergence if the strengths of the vortices are determined by pointwise evaluation of the initial vorticity. Cottet [9] has generalized these results to the case where the initial vorticity does not have compact support, but decays at infinity. If $\gamma \geqq \varepsilon$ and the assumptions in Theorem 1 are satisfied, then the error in the computed velocity is less than const $\cdot\, h^{1+\varepsilon/2}$ uniformly. This can be proved by modifying the technique of Lemma 3. If instead we determine the strengths of the vortices by pointwise evaluation of the initial vorticity, and the assumptions in the previous paragraph hold, then the computed velocity differs from the exact solution by, at most, const $\cdot\, h^{\lambda/2}$. The decrease in the rate of convergence is caused by using the p-norm to estimate the maximum norm.

It is more difficult to prove uniform convergence for the computed vorticity. Del Prete and Hald [11] have shown uniform convergence over short time intervals for the discontinuous cutoff function used by Milinazzo and Saffman [21] and for Hölder continuous initial data. Hald [15] obtained linear convergence for smooth cutoffs and smooth solutions while Anderson [1] proved higher order convergence if the strengths of the vortices are determined by pointwise evaluation of the initial vorticity.

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied with* $\gamma \geqq \varepsilon$. *There exist two constants* C_4 *and* h_3 *such that*

$$\left| \omega(x, t) - \sum_{j \neq i} \zeta_\delta(x - \tilde{\Phi}_j)\kappa_j \right| \leqq C_4 h^\varepsilon$$

for $h \leqq h_3$, *all* x *and* $t = nk \leqq T$. *The constants* C_4 *and* h_3 *depend on* C, D, L_2, T *and* λ. *The index* $i \in J$ *is chosen such that*

$$|x - \tilde{\Phi}_i| = \min_{j \in J} |x - \tilde{\Phi}_j|.$$

Remark. Note that we delete the blob closest to x. Otherwise we cannot prove uniform convergence for singular cutoff functions. It is not clear how the computed

vorticity should be defined, and the present interpretation is unsuitable for the higher order cutoffs (see Anderson [1]).

Proof. Since $\Phi(x, t)$ is an area preserving map of $\mathbb{R}^2$ onto itself and $\omega(x, t)$ is equal to $\omega(\Phi^{-1}(x, t))$ where ω is the initial vorticity, a change of variables shows that

$$\omega(x, t) = \int \zeta_\delta(y)[\omega(x, t) - \omega(x-y, t)]\, dy + \int \zeta_\delta(x - \Phi(\alpha, t))\omega(\alpha)\, d\alpha. \tag{3.33}$$

Here $\zeta_\delta(x) = \delta^{-2}\zeta(|x|/\delta)$. Since $D^\lambda H_\lambda(\omega)$ is an increasing function of λ, it follows from (2.5) that $\omega(\cdot, t)$ is Hölder continuous with exponent $\lambda/2$ and that $H_{\lambda/2}(\omega(\cdot, t))$ can be bounded in terms of C, D and λ. By using Assumption 1 and switching to polar coordinates, we can estimate the first integral in (3.33) by

$$\begin{aligned} &2\pi \int_0^\delta \frac{L_0}{\delta r} H_{\lambda/2}(\omega(\cdot, t)) r^{\lambda/2} \cdot r\, dr + 2\pi \int_\delta^\infty \frac{L_0 \delta^{1/2}}{r^{5/2}} H_{\lambda/2}(\omega(\cdot, t)) r^{\lambda/2} \cdot r\, dr \\ &\qquad \leqq 2\pi L_0 H_{\lambda/2}(\omega(t)) \frac{3}{1-\lambda} \delta^{\lambda/2}. \end{aligned} \tag{3.34}$$

The last integral in (3.33) is equal to

$$\begin{aligned} &\sum_{j \neq i} \int_{Q_j} [\zeta_\delta(x - \Phi(\alpha)) - \zeta_\delta(x - \tilde{\Phi}_j)]\omega(\alpha)\, d\alpha \\ &\quad + \int_{Q_i} \zeta_\delta(x - \Phi(\alpha))\omega(\alpha)\, d\alpha + \sum_{j \neq i} \zeta_\delta(x - \tilde{\Phi}_j)\kappa_j. \end{aligned} \tag{3.35}$$

To complete the proof, we must estimate the first two expressions uniformly. We use the same separation of the indices as in the proof of Lemma 3, but with Φ instead of Ψ, and write the first term in (3.35) as a sum of four terms $E_1, \cdots, E_4$. The vortex blobs near x require the most delicate treatment. Since $k^2 \leqq h^{1+\varepsilon}$ and $\tilde{\Phi}_i$ is the center of the vortex blob closest to x, it follows from Theorem 1 and the last inequality in (2.5) that

$$\begin{aligned} \frac{\sqrt{2}\, h}{C} &\leqq |\Phi_i - \Phi_j| \leqq |\Phi_i - \tilde{\Phi}_i| + |\tilde{\Phi}_i - x| + |x - \tilde{\Phi}_j| + |\tilde{\Phi}_j - \Phi_j| \\ &\leqq 2|x - \tilde{\Phi}_j| + 4C_3 h^{1+\varepsilon-2/p} \end{aligned}$$

if $j \neq i$ and $j \in J$. We determine $h_3 \leqq h_2$ such that $4C_3 h^{\varepsilon-2/p}$ is less than $(\sqrt{2}-1)C^{-1}$ for all $h \leqq h_3$. Hence $|x - \tilde{\Phi}_j|$ is bounded from below by $h/(2C)$ for all $j \neq i$. Using Assumption 1, we obtain

$$|E_1| \leqq \int_{S_1} \left[\frac{L_0}{\delta |x - \Phi(\alpha)|} + \frac{L_0}{\delta \cdot h/(2C)} \right] C\, d\alpha.$$

Here S_1 is defined as in the proof of Lemma 3 and contains all Q_j with $j \in J_1 \cap J$ and $j \neq i$. It follows from (3.5) that $\Phi(S_1)$ is covered by the disk with center at x and radius $(1+3C)h$. Thus, we conclude after a change of variables, that E_1 is less than const $\cdot\, h/\delta$, where the constant depends on C and L_0.

Now we return to E_2. Since $2C_3 h^{1+\varepsilon-2/p}$ is less than $\frac{1}{2}Ch$ it follows from Theorem 1 and (2.5) that

$$h \leqq |x - \Phi_j| - 2Ch \leqq |x - \Phi(\alpha)| - Ch \leqq |x - \tilde{\Phi}_j + \tau[\tilde{\Phi}_j - \Phi(\alpha)]| \leqq |x - \Phi_j| + Ch < \delta - Ch$$

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for $0 \leqq \tau \leqq 1$. This is the analogue of (3.6). By using the mean value theorem and Assumption 1 we get

$$
\begin{aligned}
|E_2| &= \left| \sum_{\substack{j \in J_2 \cap J \\ j \neq i}} \int_{Q_j} \int_0^1 \nabla \zeta_\delta(x - \tilde{\Phi}_j + \tau[\tilde{\Phi}_j - \Phi(\alpha)]) \cdot [\tilde{\Phi}_j - \Phi(\alpha)] \, d\tau \, \omega(\alpha) \, d\alpha \right| \\
&\leqq \frac{\sqrt{2}\, L_0}{\delta} Ch \|\omega\| \int_{S_2} \frac{1}{(|x - \Phi(\alpha)| - Ch)^{5/2}} \, d\alpha.
\end{aligned} \tag{3.36}
$$

We encountered the last integral in § 3.1, and it is less than $4\pi(1+C)\log(\delta/h)$. E_2 can therefore be bounded by const $\cdot (h/\delta)\log(h^{-1})$, where the constant depends on C, L_0 and λ.

To estimate E_3, we observe that if α is a point in Q_j and j is in J_3 then

$$|x - \Phi(\alpha)| \geqq |x - \Phi_j| - |\Phi_j - \Phi(\alpha)| \geqq \delta - r_1 - Ch \geqq \delta/4.$$

These inequalities also hold with $\tilde{\Phi}_j$ instead of $\Phi(\alpha)$ if $j \in J_3 \cap J$. Thus it follows from Assumption 1 that

$$|E_3| \leqq \sum_{\substack{j \in J_3 \cap J \\ j \neq i}} \int_{Q_j} \frac{2L_0}{\delta \cdot \delta/4} |\omega(\alpha)| \, d\alpha \leqq \frac{8L_0}{\delta^2} \|\omega\| \int_{S_3} d\alpha.$$

Since the last integral is less than $8\pi(1+3C)h\delta$, we conclude that $|E_3|$ is less than const $\cdot (h/\delta)$, where the constant depends on C and L_0.

To bound E_4, we use the same approach as for E_2. Let $\alpha \in Q_j$ with $j \in J_4 \cap J$. Then it follows from Theorem 1 and (2.5) that

$$\delta < |x - \Phi_j| - 2Ch \leqq |x - \Phi(\alpha)| - Ch \leqq |x - \tilde{\Phi}_j + \tau[\tilde{\Phi}_j - \Phi(\alpha)]|$$

for $0 \leqq \tau \leqq 1$. Using Assumption 1 and the same arguments as in (3.36), we arrive at

$$|E_4| \leqq \frac{\sqrt{2}\, L_0}{\sqrt{\delta}} Ch \|\omega\| \int_{S_2} \frac{1}{(|x - \Phi(\alpha)| - Ch)^{5/2}} \, d\alpha.$$

In the proof of Lemma 3 we found that the last integral is less than $10\pi\delta^{-1/2}$. This implies that $|E_4|$ is less than const $\cdot\, h/\delta$ where the constant depends on C and L_0.

Finally we consider the second integral in the expression (3.35). Let d be the distance from x to $\Phi(Q_i)$. Then $\Phi(Q_i)$ is contained in the annulus with center at x, inner radius d and outer radius $d + Ch$. Using Assumption 1 and a change of variables, we get

$$\int_{Q_j} |\zeta_\delta(x - \Phi(\alpha))| \, d\alpha \leqq \int_{\Phi(Q_i)} \frac{L_0}{\delta |x - y|} \, dy \leqq 2\pi L_0 C \frac{h}{\delta}. \tag{3.37}$$

We can now collect our results. It follows from (3.34), the bounds for $E_1, \cdots, E_4$ and (3.37), that the error in the approximation of the vorticity is uniformly less than

$$\text{const}\left(\delta^{\lambda/2} + \frac{h}{\delta} + \frac{h}{\delta}\log\left(\frac{1}{h}\right)\right).$$

Since $\delta = h^{\lambda/(4+\lambda)}$ and $\varepsilon = \lambda/5$ we see that $\delta^{\lambda/2} < h^\varepsilon$ and that $h/\delta = h^\varepsilon h^b$ where $b = \lambda(1-\lambda)/(20+5\lambda)$. As $h^b \log h^{-1}$ is less than $(be)^{-1}$ we conclude that the error in our approximation of the vorticity is less than const $\cdot\, h^\varepsilon$. This completes the proof.

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The approximate vorticity may not be uniquely determined because several blobs can have the same distance to x. In this case we delete the blob with the maximum strength. Alternatively we can decide to keep all the blobs. In Theorem 2 the strengths of the vortices are determined by local averages. Suppose instead that the strengths are determined by pointwise evaluation of the initial vorticity, i.e. $\kappa_j = \omega_j h^2$. If the cutoff function and its first partial derivatives are bounded near the origin, $\delta = h^{2\lambda/(2+\lambda)}$ and $k^2 \leqq h^\lambda$, then the error in the computed vorticity is less than const $\cdot\, h^{\lambda^2/4}$. This can be shown by modifying the previous proof. Here it is not necessary to delete any blobs, because the cutoff function is not singular.

4.1. High order consistency. In this section we will study a new class of high order cutoffs. The advantage of these cutoffs is that the rate of convergence depends only on the smoothness of the flow. The disadvantage is that the cutoffs are costly to implement, as they are expressed in terms of Bessel functions. Leonard's [19] infinite order cutoff is not included in our class because it decays too slowly. It is not even absolutely integrable. We will first show that all cutoffs that satisfy Assumption 2 in § 1 also satisfy Assumption 1. We shall then give a recipe for constructing such cutoffs and show that for a fixed cutoff the order of the consistency error can be arbitrarily high. The convergence proof for the classical Runge–Kutta method is complicated and demands a fair amount of smoothness in the vorticity distribution, the velocity field and the flow map. I believe that the smoothness requirements can be relaxed a little.

LEMMA 6. *Assumption* 2 *implies Assumption* 1.

Remark. Some of the cutoffs that satisfy Assumption 2, and in particular the explicit ones in this section, fall outside the classes studied by Del Prete and Hald [11], Hald [15], Beale and Majda [4], Cottet [9] and Anderson and Greengard [2].

Proof. We define the cutoff function ζ via its Fourier transform by

$$\zeta(x) = \int e^{ix\cdot\xi}\hat{\zeta}(\xi)\, d\xi, \qquad \hat{\zeta}(\xi) = (2\pi)^{-2}\int e^{-ix\cdot\xi}\zeta(x)\, dx \tag{4.1}$$

where $x\cdot\xi = x_1\xi_1 + x_2\xi_2$ and $d\xi = d\xi_1\, d\xi_2$. Since $\hat{\zeta}$ has compact support it follows that ζ is an entire function of x and of exponential type. Condition (i) in Assumption 1 is therefore satisfied. We will next study the decay of ζ and ζ'. Since ζ and $\hat{\zeta}$ are radially symmetric, we find by switching to polar coordinates and using the integral representation of the Bessel function J_0 (see [26, p. 19]) that

$$\zeta(r) = 2\pi\int_0^b J_0(rt)t\hat{\zeta}(t)\, dt \tag{4.2}$$

where $r = |x|$ and $\zeta(x) = \zeta(r)$. To simplify the following arguments we will assume that $\hat{\zeta}$ is twice continuously differentiable in $1 \leqq t \leqq b$. We remember now that

$$J_0'(z) = -J_1(z) \quad \text{and} \quad zJ_0(z) = (zJ_1(z))' \tag{4.3}$$

(see [26, p. 18]). Since $\hat{\zeta}$ is continuous and equal to $(2\pi)^{-2}$ for $0 \leqq t \leqq 1$ we find by integrating by parts in (4.2) and using (4.3) twice that

$$\zeta(r) = 2\pi\frac{J_0(rt)}{r^2}t\hat{\zeta}'(t)\Big|_{t=1}^{b} - 2\pi\int_1^b \frac{J_1(rt)}{r^2}(t\hat{\zeta}'(t))'\, dt.$$

Let $r > 1$. Since $J_0(r)$ and $J_1(r)$ can be bounded by const $\cdot\, r^{-1/2}$ (see [26, p. 206]) and $b > 1$, we conclude that condition (ii) in Assumption 1 is satisfied. To establish condition

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(iii), we use equation (4.2) and find after interchanging the order of integration, integrating by parts and using the properties of $\hat{\zeta}$ plus (4.3) that

$$2\pi \int_0^r s\zeta(s)\, ds - 1 = (2\pi)^2 \frac{J_1(rt)}{r} \hat{\zeta}'(t) \Big|_{t=1}^{b} - (2\pi)^2 \int_1^b \frac{J_1(rt)}{r} t(t^{-1}\hat{\zeta}'(t))'\, dt.$$

Since $t \geqq 1$ it follows that $J_1(rt)$ is less than const $\cdot\, r^{-1/2}$. This completes the proof.

We have already presented one cutoff function that satisfies Assumption 2, namely (1.8). Here $\hat{\zeta} = (2\pi)^{-2}(4-t^2)/3$ for $1 \leqq t \leqq 2$ and the corresponding shape function is given in equation (3.3). More generally we express $\hat{\zeta}$ as a linear combination of 1, t^2, $t^4, \cdots$ for $1 \leqq t \leqq b$. If $b=2$ and $\hat{\zeta}$ is a cubic polynomial in t^2 and $\hat{\zeta}'$ is continuous at $t=1$ and 2 then the cutoff function and shape function are

$$\zeta_\delta(r) = \frac{2}{9\pi(r/\delta)^3}\left[16J_3\left(\frac{2r}{\delta}\right) - 32J_5\left(\frac{2r}{\delta}\right) + 7J_3\left(\frac{r}{\delta}\right) + J_5\left(\frac{r}{\delta}\right)\right],$$

$$\int_{|x|<r} \zeta_\delta(x)\, dx = 1 - \frac{8}{27(r/\delta)^2}\left[4J_2\left(\frac{2r}{\delta}\right) - 32J_4\left(\frac{2r}{\delta}\right) + 11J_2\left(\frac{r}{\delta}\right) + 2J_4\left(\frac{r}{\delta}\right)\right].$$

Note that the cutoff function decays like $O(r^{-7/2})$. However, the calculation of the kernel K_δ requires four calls to a subroutine for Bessel functions. It is possible to obtain the same rate of decay by setting $\hat{\zeta} = (2\pi)^{-2}(44+2t^2-t^4)/45$ for $1 \leqq t \leqq 2$ and equal to $(2\pi)^{-2}(256-32t^2+t^4)/180$ for $2 \leqq t \leqq 4$. In this case the cutoff function and the shape function are

$$\zeta_\delta(r) = \frac{4}{45\pi(r/\delta)^3}\left[16J_3\left(\frac{4r}{\delta}\right) - 10J_3\left(\frac{2r}{\delta}\right) + J_3\left(\frac{r}{\delta}\right)\right],$$

$$\int_{|x|<r} \zeta_\delta(x)\, dx = 1 - \frac{8}{45(r/\delta)^2}\left[4J_2\left(\frac{4r}{\delta}\right) - 5J_2\left(\frac{2r}{\delta}\right) + J_2\left(\frac{r}{\delta}\right)\right]$$

and we need only three function calls to compute K_δ. Note that it is impossible to find a cutoff function with infinite order and compact support. The reason is that $\hat{\zeta}(\xi)$ would be an entire function of ξ and $D^\alpha \hat{\zeta}(0) = 0$ for all $\alpha \neq 0$. This would imply that $\hat{\zeta}$ is a constant or, equivalently, that ζ is the delta function times a constant. We must therefore accept that the computed vorticity distribution cannot have compact support if we use an infinite order cutoff.

Lemma 6 and the comments following the proofs of Theorems 1 and 2 imply that the vortex methods converge if the cutoff function satisfies Assumption 2 and the initial vorticity is Hölder continuous. However, the rate of convergence is low. To prove higher order convergence, we assume that the initial vorticity is smooth and determine the strengths of the vortices by pointwise evaluation. We begin with

LEMMA 7. *Let Assumption* 4 *be satisfied with* $m \geqq 1$. *Let* $v(x)$ *be determined as in* (2.2) *and assume that the inequalities in* (2.8) *are satisfied. Let* $K_\delta = K * \zeta_\delta$ *where* ζ *satisfies Assumption* 2. *Then there exists a constant* C_5 *such that*

$$\max_x \left| v(x) - \int K_\delta(x - \Psi(\alpha'))\omega(\alpha')\, d\alpha' \right| \leqq C_5 \delta^{m+\lambda-1}.$$

The constant C_5 *depends on* M, D, L_1, λ *and* m.

Remark. This result is called the moment lemma, but in our case the second moment of ζ_δ may not even exist. By using Taylor series techniques, Hald [15] obtained a stronger result for the cutoff functions with compact support and $\omega \in C^3$. That approach was also used in the proof of Lemma 3. The proof in this section follows Beale and Majda [4] and makes use of the Fourier transform.

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Proof. Let $e = F * f$ where $F = K * \zeta_\delta - K$ and f is given by (2.4). It follows from Assumption 2 and (3.1) that F is absolutely integrable and its Fourier transform is therefore continuous. We compute $\hat{F}$ as the limit of $\hat{F}_n$ where $F_n = F$ in the annulus $n^{-1} < |x| < n$ and zero otherwise. Using the factorization (3.1), we find after a long calculation that

$$\hat{F}_n(\xi) = i|\xi|^{-2}\left[\hat{\zeta}(\delta\xi) - \frac{1}{(2\pi)^2}\right]\begin{pmatrix}\xi_2\\ -\xi_1\end{pmatrix}$$
$$+ i(2\pi)^{-2}|\xi|^{-2}\left[\int_{|y|<n^{-1}} \{J_0(|\xi|n^{-1}) - J_0(|\xi||y|)\}\zeta_\delta(y)\,dy\right.$$
$$\left. - J_0(|\xi|n)\int_{|y|<n}\zeta_\delta(y)\,dy - \int_{|y|>n} J_0(|\xi||y|)\zeta_\delta(y)\,dy\right]\begin{pmatrix}\xi_2\\ -\xi_1\end{pmatrix}.$$

Let ξ be fixed. Since $J_0(|\xi|n)$ is bounded by one and tends to zero as n tends to infinity, we conclude from Assumption 2 and Lemma 6 that $\hat{F} \equiv 0$ for $|\xi| \leqq \delta^{-1}$ and less than $2\|\hat{\zeta}\|\,|\xi|^{-1}$ for $|\xi| > \delta^{-1}$.

In the proof of Corollary 1 we found that $f \in C^{m+\lambda}$ and that the diameter of its support is less than $D_1 = \sqrt{2}\,MD$. Using (4.1), we obtain, after integration by parts, that

$$\begin{aligned}|\xi_j|^m|\xi|^\lambda|\hat{f}(\xi)| &\leqq \frac{\pi^\lambda}{2(2\pi)^2}\int \frac{|\partial_j^m f(x+a) - \partial_j^m f(x)|}{|a|^\lambda}\,dx \\ &\leqq \tfrac{1}{8}D_1^{2-m-\lambda}\|f\|_{C^{m+\lambda}(D_1)}\end{aligned} \tag{4.4}$$

where $j = 1, 2$ and $a = \pi\xi|\xi|^{-2}$. The norm of f can be bounded by const $\cdot\ \|\omega\|_{C^{m+\lambda}(D)}$ and this implies that $|\hat{f}| \leqq C'|\xi|^{-m-\lambda}$ where C' depends on M, D, m and λ. Since $\hat{e} = \hat{F}\cdot\hat{f}$ we conclude that

$$|e(x)| \leqq \int_{|\xi|>\delta^{-1}} 2\|\hat{\zeta}\|\,|\xi|^{-1}C'|\xi|^{-m-\lambda}\,d\xi \leqq \text{const}\cdot\delta^{m+\lambda-1}$$

where the constant depends upon M, D, λ, m and $\|\hat{\zeta}\|$. As $\|\hat{\zeta}\|$ is less than the constant L_1 from Assumption 2, this completes the proof.

To estimate the discretization error in the vortex method, we need a bound for the error in the trapezoidal rule. Here we use an integral analogue of Hölder continuity.

LEMMA 8. *Assume that $g \in C^{m+\lambda}$ and has compact support. If $m \geqq 2$ and $0 < \lambda < 1$ then*

$$\left|\int g(x)\,dx - \sum_j g(jh)h^2\right| \leqq Ch^{m+\lambda}\max_{i=1,2}\sup_a \int \frac{|\partial_i^m g(x+a) - \partial_i^m g(x)|}{|a|^\lambda}\,dx$$

where the constant C is less than $4(2\pi)^{-m}(m+\lambda-1)/(m+\lambda-2)$.

Remark. Our proof of Lemma 8 is based on Poisson's summation formula and (4.4) and follows the arguments of Anderson and Greengard [2] so closely that we shall not present it. Cottet [9] has proved a similar result by using the Bramble–Hilbert lemma. I do not know whether his technique can be extended to Hölder continuous functions. Note that C is less than $1/30$ if $m \geqq 3$. By using the fundamental theorem of calculus and considering the cases $|a| < d$ and $|a| \geqq d$ separately we obtain the useful inequality

$$\int \frac{|g(x+a) - g(x)|}{|a|^\lambda}\,dx \leqq d^{1-\lambda}\int |\partial_1 g| + |\partial_2 g| + 2d^{-\lambda}\int |g|. \tag{4.5}$$

We shall apply (4.5) to derivatives of the kernel K_δ. Thus we need

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LEMMA 9. *Let $K = K * \zeta_\delta$ where ζ satisfies Assumption 2. Let A be bounded. If $m \geqq 3$ and $\delta < 1$ then*

$$\max_{|\alpha| \leqq m} \int_A |D^\alpha K_\delta(x)|\, dx \leqq C\delta^{1-m}.$$

The constant C depends on L_1, m and the diameter of A.

Remark. Anderson and Greengard [2] present a similar result for kernels that behave like K at infinity. The basic observation—from our point of view—is that $D^\alpha K_\delta$ is absolutely integrable if $|\alpha|$ is big enough.

Proof. Since K is a homogeneous function of degree -1 and $\zeta_\delta(x) = \delta^{-1}\zeta(x/\delta)$ it follows that $K * \zeta_\delta(x) = \delta^{-1} K * \zeta(x/\delta)$ and thus

$$\int_A |D^\alpha K_\delta(x)|\, dx = \delta^{1-|\alpha|} \int_{\delta^{-1}A} |D^\alpha K * \zeta(x)|\, dx.$$

If $D^\alpha K * \zeta$ is absolutely integrable, then the last integral is bounded no matter what A is. Otherwise, assume that $D^\alpha K * \zeta$ is bounded by C'. Then

$$\delta^{1-|\alpha|} \int_{\delta^{-1}A} |D^\alpha K * \zeta(x)|\, dx \leqq \delta^{1-|\alpha|} C' \pi (\tfrac{1}{2}\delta^{-1} \operatorname{diam} A)^2 \leqq C\delta^{-1-|\alpha|}.$$

To complete the proof, we will show that $D^\alpha K * \zeta$ is absolutely integrable for $|\alpha| > 1$ and bounded for $|\alpha| = 0, 1$. By rewriting the expression for the shape factor in the proof of Lemma 6 and using (3.1) we have

$$K * \zeta(x) = \left[G(rt) t^3 \hat{\zeta}'(t) \Big|_{t=1}^{b} - \int_1^b G(rt) t^4 (t^{-1}\hat{\zeta}'(t))'\, dt \right] \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

$$G(r) = \frac{2\pi}{r^2}\left(\frac{J_1(r)}{r} - \frac{1}{2}\right) = -\frac{\pi}{4} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (r/2)^{2\nu}}{(\nu+1)!(\nu+2)!}$$

where $r = |x|$. Since G is a power series in r^2 and converges for all r, it follows that $D^\alpha K * \zeta$ is bounded for $|x| \leqq 1$ and $|\alpha| \leqq m$. For $|x| > 1$ and $|\alpha| = 0, 1$ we use Lemma 2 and Lemma 6. Finally, we will prove that $D^\alpha K * \zeta = O(r^{-5/2})$ for $r > 1$ and $|\alpha| \geqq 2$. For $|\alpha| = 2$ it follows from Lemma 2 with $\delta = 1$. We can therefore assume that $|\alpha| > 2$. Since $D^\alpha(G \cdot x_i)$ is equal to $D^\alpha G \cdot x_i$ if $\alpha_i = 0$ and $D^\alpha G \cdot x_i + \alpha_i D^{\alpha - e_i} G$ otherwise, it is enough to show that $D^\gamma G = O(r^{-7/2})$ for $|\gamma| \geqq 2$. To estimate the partial derivatives of G for large values of r we employ

$$D^\gamma G(|x|) = \sum_{\nu=1}^{|\gamma|} G^{(\nu)}(r) r^{-(|\gamma|-\nu)} P_{\gamma,\nu}\left(\frac{x_1}{r}, \frac{x_2}{r}\right). \tag{4.6}$$

Here $P_{\gamma,\nu}$ is a polynomial in two variables. Its degree is at most $|\gamma|$ and $P_{\gamma,|\gamma|}(x_1, x_2) = x_1^{\gamma_1} x_2^{\gamma_2}$. This can be proved by induction. We will show that $G^{(\nu)}(r) = O(r^{-7/2})$ for $\nu \geqq 2$. The Leibnitz formula yields

$$G^{(\nu)}(r) = 2\pi \frac{\nu!}{r^2} \sum_{n=1}^{\nu} \frac{(-1)^{\nu-n}(\nu+1-n)}{n!\, r^{\nu-n}} \left(\frac{J_1}{r}\right)^{(n)} - 2\pi \frac{(-1)^\nu (\nu+1)!}{r^{2+\nu}} \left(\frac{1}{2} - \frac{J_1}{r}\right).$$

The last term is less than $2\pi(\nu+1)!\, r^{-2-\nu}$ because $|r^{-1}J_1| \leqq \frac{1}{2}$. Using the identity $J_n' = (n/r)J_n - J_{n+1}$, we find by induction that

$$\left(\frac{J_1}{r}\right)^{(n)} = \sum_{k=0}^{[n/2]} a_{nk} \frac{J_{n+1-k}}{r^{1+k}}$$

where $[n/2]$ is the largest integer less than or equal to $n/2$. Since $J_n = O(r^{-1/2})$ for all n we conclude that $(r^{-1}J_1)^{(n)} = O(r^{-3/2})$. This implies that $G^{(\nu)}$ is less than a const $\cdot\, r^{-7/2}$

for $\nu \geqq 2$ and $r > 1$ and a direct calculation shows that $G' = O(r^{-3})$. By combining these results with (4.6) we find that $D^\gamma G$ is $O(r^{-7/2})$ for $|\gamma| \geqq 2$. This completes the proof.

Our next result is similar to Lemma 3. It shows that the rate of convergence for the infinite order cutoff depends on the smoothness of the flow.

LEMMA 10. *Let Assumption 4 be satisfied with $m \geqq 3$. Let $v(x)$ be determined as in (2.2) and assume that the inequalities in (2.8) are satisfied. Let $K_\delta = K * \zeta_\delta$ where ζ satisfies Assumption 2. Set $\delta = h^q$ with $q = \frac{1}{2}(m+\lambda)/(m+\lambda-1)$ and $\kappa_j = \omega(\alpha_j)h^2$. Then there exist two constants C_6 and h_4 such that*

$$\max_x \left| v(x) - \sum_j K_\delta(x - \Psi(\alpha_j))\kappa_j \right| \leqq C_6 h^{(m+\lambda)/2}$$

for all $h \leqq h_4$. The constants C_6 and h_4 depend on M, D, L_1, λ and m.

Remark. In Beale and Majda's [4] presentation, and also in Cottet's [9], the flow is so smooth that full advantage can be taken of the order of the cutoff. This is impossible in our case. Our proof follows Anderson and Greengard [2], but is more complicated because we exploit the Hölder continuity.

Proof. Lemma 7 shows that the moment error is $O(h^{(m+\lambda)/2})$ if $h \leqq h_4 < 1$. It is therefore enough to study the discretization error

$$e_d = \left| \int F(\alpha')\, d\alpha' - \sum F(\alpha_j) h^2 \right|,$$

$$F(\alpha') = K_\delta(x - \Psi(\alpha'))\omega(\alpha') = k(\alpha')\omega(\alpha') \tag{4.7}$$

where x is fixed and $\alpha_j = jh$. We observe first that $K_\delta \in C^\infty$, $\Psi \in C^{m+\lambda+1}$ and that $\omega \in C^{m+\lambda}$ and has compact support. Thus it follows from Lemma 8 that e_d can be bounded by $C' h^{m+\lambda}$. We will show that C' is less than const $\cdot\, \delta^{1-m-\lambda}$. Leibnitz' formula applied to (4.7) yields

$$\partial_i^m F = k\partial_i^m \omega + \partial_i^m k\, \omega + \sum_{\nu=1}^{m-1} \binom{m}{\nu} \partial_i^\nu k \partial_i^{m-\nu} \omega \tag{4.8}$$

where $i = 1, 2$ and $\partial_i = \partial/\partial\alpha_i'$. We fix i and a and denote the integrals of the Hölder quotients for the three terms in (4.8) by I_1, I_2 and I_3. Let $\Omega = \operatorname{supp} \omega$. Since $\partial^\gamma k$ is a sum of $D^\alpha K_\delta(x - \Psi(\alpha'))$ times derivatives of Ψ of order less than or equal to $|\gamma|$, we conclude from Lemma 8, and after a change of variables, that

$$\int_\Omega |\partial^\gamma k(\alpha')|\, d\alpha' \leqq \text{const} \sum_{1 \leqq |\alpha| \leqq |\gamma|} \int_{x - \Psi(\Omega)} |D^\alpha K_\delta(x')|\, dx' \leqq \text{const} \cdot \delta^{1-m} \tag{4.9}$$

for $|\gamma| \leqq m$. The last constant depends on M, m and the bound D for the diameter of Ω. We observe now that the last term in (4.8), i.e. the sum, is a differentiable function of α'. Thus it follows from (4.5) with $d = 1$ and (4.9) that I_3 is less than const $\cdot\, \delta^{1-m}$, where the constant depends on M, D and m. To estimate the integral of the Hölder quotient of the second term in (4.8), we note that because $\Psi \in C^{m+1}$, the inequality (4.9) holds with m replaced by $m+1$ and $|\gamma| = m+1$. By using (4.5) with $d = \delta$ we find that I_2 is less than const $\cdot\, \delta^{1-m-\lambda}$. To estimate I_1, we must modify the proof of (4.5). Assume that $|a| \leqq 1$ and let B be the unit sphere. By using the fundamental theorem of calculus and (4.9) we obtain

$$\begin{aligned} I_1 &\leqq \int |k(\alpha')| \frac{|\partial_i^m \omega(\alpha') - \partial_i^m \omega(\alpha' - a)|}{|a|^\lambda}\, d\alpha' + \int_0^1 ds \int \frac{|\nabla k(\alpha') \cdot a|}{|a|^\lambda} |\partial_i^m \omega(\alpha' - sa)|\, d\alpha' \\ &\leqq \int_{\Omega+B} |k(\alpha')|\, d\alpha'\, H_\lambda(\partial_i^m \omega) + \int_{\Omega+B} |\partial_1 k(\alpha')| + |\partial_2 k(\alpha')|\, d\alpha'\, \|\partial_i^m \omega\| \\ &\leqq \text{const} \cdot \delta^{1-m} \end{aligned}$$

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where the constant depends on M, D, m and λ. If $|a|>1$ then I_1 is less than $2\int |k\partial_i^m\omega|$ and can be bounded by const $\cdot\ \delta^{1-m}$. By combining the estimates for I_1, I_2 and I_3 we conclude from Lemma 8 that the discretization error e_d is less than const $\cdot\ (h/\delta)^{m+\lambda}\delta$, which is the same as const $\cdot\ h^{(m+\lambda)/2}$. This completes the proof.

Observation 1. It is not necessary to choose the optimal value of q. If $\delta = h^q$ with $q<1$ then it follows from Lemma 7 and the proof of Lemma 10 that

$$\max_x \left| v(x) - \sum_j K_\delta(x-\Psi(\alpha_j))\kappa_j \right| \leqq \text{const} \cdot (h^{q(m+\lambda-1)} + h^{q+(m+\lambda)(1-q)}).$$

4.2. High order stability. If the cutoff function satisfies Assumption 2, then the constant ch in Lemma 4 can be replaced by const $\cdot\ \delta$ (see Beale and Majda [4]). This extension does not affect the optimal rate of convergence, but gives convergence for a wider range of parameters. However, we shall use Lemma 4 in its original form even for smooth cutoffs.

4.3. High order truncation. In this section we will show that the truncation error for the classical Runge-Kutta method is less than const $\cdot\ k^5$ where k is the time step. This is not surprising, except that here we solve a nonlinear differential integral equation with a singular kernel, and that the technique of Anderson and Greengard [2] yields a lower rate of convergence. To keep the notation simple, we let Φ', Φ'' and Φ''' denote the first, second and third derivative of Φ. In general, $\Phi^{(n)}$ is a n-linear mapping and

$$\Phi^{(n)} \cdot (a_1, \cdots, a_n) = \sum \partial_{j_1} \cdots \partial_{j_n} \Phi\, a_{j_1 1} \cdots a_{j_n n} \tag{4.10}$$

where a_{jn} is the jth component of the vector a_n (see [12, p. 177]). The sum is extended over all 2^n distinct n-tuples $(j_1, \cdots, j_n)$. We can now express the time derivatives of the flow map concisely.

LEMMA 11. *Let Φ be the solution of* (1.1). *Assume that $\omega \in C^{3+\lambda}$ and has compact support. Then*

$$\partial_t^2\Phi(\alpha, t) = \int K(\Phi(\alpha, t) - \Phi(\beta, t))[\omega' \cdot a]\, d\beta, \tag{4.11}$$

$$\partial_t^3\Phi(\alpha, t) = \int K(\Phi(\alpha, t) - \Phi(\beta, t))[\omega' \cdot b + \omega'' \cdot (a, a)]\, d\beta, \tag{4.12}$$

$$\partial_t^4\Phi(\alpha, t) = \int K(\Phi(\alpha, t) - \Phi(\beta, t))[\omega' \cdot c + 3\omega'' \cdot (a, b) + \omega'''(a, a, a)]\, d\beta \tag{4.13}$$

where ω', ω'' and ω''' are evaluated at β. The vectors a, b and c depend on α, β and t and are determined recursively by

$$\Phi' \cdot a + \partial_t\Phi = \partial_t\Phi(\alpha, t), \tag{4.14}$$

$$\Phi' \cdot b + \Phi'' \cdot (a, a) + 2\partial_t\Phi' \cdot a + \partial_t^2\Phi = \partial_t^2\Phi(\alpha, t), \tag{4.15}$$

$$\begin{aligned}\Phi' \cdot c + 3\Phi'' \cdot (a, b) + \Phi''' \cdot (a, a, a) + 3\partial_t\Phi' \cdot b \\ + 3\partial_t\Phi'' \cdot (a, a) + 3\partial_t^2\Phi' \cdot a + \partial_t^3\Phi = \partial_t^3\Phi(\alpha, t).\end{aligned} \tag{4.16}$$

The derivatives of Φ on the left-hand sides are evaluated at (β, t).

Remark. If ω is Hölder continuous, then Φ is infinitely differentiable with respect to t (see [17]), but this is not the way to prove it.

Proof. We replace α' in (1.1) by β and consider the change of variables

$$\beta = g(\alpha, y, t) = \Phi^{-1}(\Phi(\alpha, t) - y, t) \tag{4.17}$$

where α is fixed. Just because Φ is infinitely differentiable with respect to t, it does not follow that the inverse is, too. A counterexample in one dimension is $y = x + x|x| + t$. We need a weaker result and argue as follows: Since $\Phi(\alpha, t)$ is four times continuously differentiable in α and t, $\|\partial_j \Phi\|$ is bounded and $\det \Phi' = 1$, it follows from the implicit function theorem that Φ^{-1} has the same properties. This implies that $g \in C^4$ with respect to α, x and t. Since Φ is an area preserving map, we conclude from (1.1) and (4.17) that

$$\partial_t \Phi(\alpha, t) = \int K(y)\omega(g(\alpha, y, t))\, dy.$$

Let $K_\delta = K * \zeta_\delta$ where $\zeta = \pi^{-1}$ if $r < 1$ and zero otherwise. If we replace K by K_δ, then we can differentiate under the integral sign with respect to t because $\omega \in C^{3+\lambda}$ and has compact support. Letting δ tend to zero and using that $\int |K - K_\delta| = \frac{2}{3}\delta$, we obtain

$$\partial_t^{\nu+1} \Phi(\alpha, t) = \int K(y)\partial_t^\nu \omega(g(\alpha, y, t))\, dy$$

for $\nu = 1$. The same arguments apply for $\nu = 2$ and 3. Using the chain rule and (4.10) repeatedly, we get

$$\begin{aligned}
&\partial_t \omega \circ g = \omega' \cdot \dot{g},\\
&\partial_t^2 \omega \circ g = \omega' \cdot \ddot{g} + \omega'' \cdot (\dot{g}, \dot{g}),\\
&\partial_t^3 \omega \circ g = \omega' \cdot \dddot{g} + 3\omega'' \cdot (\dot{g}, \ddot{g}) + \omega'''(\dot{g}, \dot{g}, \dot{g})
\end{aligned}$$

where ω', ω'' and ω''' are evaluated at $g(\alpha, y, t)$. To find $\dot{g}$, $\ddot{g}$ and $\dddot{g}$ we rewrite (4.17) as $\Phi(g(\alpha, y, t), t) = \Phi(\alpha, t) - y$ and differentiate both sides of this equation with respect to t. Thus

$$\begin{aligned}
&\Phi' \cdot \dot{g} + \partial_t \Phi = \partial_t \Phi(\alpha, t),\\
&\Phi' \cdot \ddot{g} + \Phi'' \cdot (\dot{g}, \dot{g}) + 2\partial_t \Phi' \cdot \dot{g} + \partial_t^2 \Phi = \partial_t^2 \Phi(\alpha, t),\\
&\Phi' \cdot \dddot{g} + 3\Phi'' \cdot (\dot{g}, \ddot{g}) + \Phi''' \cdot (\dot{g}, \dot{g}, \dot{g}) + 3\partial_t \Phi' \cdot \ddot{g} + 3\partial_t \Phi'' \cdot (\dot{g}, \dot{g})\\
&\qquad + 3\partial_t^2 \Phi' \cdot \dot{g} + \partial_t^3 \Phi = \partial_t^3 \Phi(\alpha, t)
\end{aligned}$$

where the $\partial_t^\nu \Phi^{(n)}$ on the left-hand sides are evaluated at $\beta = g(\alpha, y, t)$. Finally, we set $y = \Phi(\alpha, t) - \Phi(\beta, t)$ and denote $\dot{g}$, $\ddot{g}$ and $\dddot{g}$ by a, b and c. This completes the proof.

To solve the ordinary differential equations (1.4), we use the classical fourth order Runge-Kutta method. Let $\tilde{\Phi}(t)$ be the computed solution at time $t = nk$ and denote the intermediate steps by $\tilde{\Psi}^l$, $l = 0, 1, 2, 3$. Set $\tilde{\Psi}^0 = \tilde{\Phi}(t)$ and

$$\begin{aligned}
&\tilde{\Psi}^l = \tilde{\Phi}(t) + \vartheta_l \frac{k}{2} V[\tilde{\Psi}^{l-1}], \qquad l = 1, 2, 3,\\
&\tilde{\Phi}(t+k) = \tilde{\Phi}(t) + \frac{k}{6}(V[\tilde{\Psi}^0] + 2V[\tilde{\Psi}^1] + 2V[\tilde{\Psi}^2] + V[\tilde{\Psi}^3])
\end{aligned} \tag{4.18}$$

where $\vartheta_0, \cdots, \vartheta_3$ are 0, 1, 1, 2. The initial condition is $\tilde{\Phi}(\alpha_i, 0) = \alpha_i$ and V is the discrete approximation to the velocity field and defined in the beginning of § 3.2. We define the truncation error τ for the Runge-Kutta method by setting $\Psi^0 = \Phi(t)$ and using the continuous analogue of (4.18), namely

$$\begin{aligned}
&\Psi^l = \Phi(t) + \vartheta_l \frac{k}{2} v[\Psi^{l-1}], \qquad l = 1, 2, 3,\\
&\Phi(t+k) = \Phi(t) + \frac{k}{6}(v[\Psi^0] + 2v[\Psi^1] + 2v[\Psi^2] + v[\Psi^3]) + \tau.
\end{aligned} \tag{4.19}$$

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Here $\Phi(\alpha, t)$ is the solution of $\partial_t \Phi = v[\Phi]$ with initial condition $\Phi(\alpha, 0) = \alpha$, where v is given by (2.2). We shall estimate the truncation error in (4.19).

LEMMA 12. *Let Assumption* 4 *be satisfied with* $m \geqq 6$. *There exists two constants* C_7 *and* k_3 *such that if* k *is less than* k_3 *then*

$$|\tau(\alpha, t)| \leqq C_7 k^5$$

for all α *in* $\mathbb{R}^2$ *and* $0 \leqq t \leqq T - k$. *The constants* C_7 *and* k_3 *depend on* M, D *and* λ.

Remark. If we compare this result with Lemma 5, we see that we require three derivatives more than expected. This is due to our technique. The proof is based on several changes of variables, and each one entails a loss of derivatives. I have been unable to find a proof without this defect. Note that Lemma 12 does not depend on a particular cutoff.

Proof. The obvious idea is to expand the functions $\Psi^l(\alpha, k, t)$ in (4.19) in a Taylor series in k. The coefficients will depend on α and t. The difficulty lies in estimating the remainder. We use the fact that the remainder in Taylor's formula is a smooth function. This follows from its integral representation. We can therefore differentiate the remainder by differentiating under the integral sign and find that the derivatives are of the same order as the remainder. The proof consists of five major steps.

Step 1. Since $\Psi^0 = \Phi$ it follows from (1.1) and (4.19) that

$$v[\Psi^0] = \partial_t \Phi(\alpha, t), \qquad \Psi^1(\alpha, \kappa, t) = \Phi(\alpha, t) + \kappa \partial_t \Phi(\alpha, t) \tag{4.20}$$

where $\kappa = \frac{1}{2}k$. In most of this section we will use κ instead of $\frac{1}{2}k$ and suppress the t-dependence in Φ, Ψ and their derivatives. We denote all the remainders by the same symbol R, but keep track of the smoothness of R.

Step 2. We want to find a Taylor series like (4.20) for Ψ^2. The basic idea is to use a change of variables such that the kernel K in $v(\Psi^1(\alpha))$ is evaluated at $\Phi(\alpha) - \Phi(\gamma)$ instead of $\Psi^1(\alpha) - \Psi^1(\beta)$. We can then use (4.11) etc. to interpret the terms in the Taylor series. Throughout this step we denote Ψ^1 by Ψ. We assume that the time step k is less than the constant k_2 from Corollary 1 and let

$$\beta = \Psi^{-1}(\Phi(\gamma) + \Psi(\alpha) - \Phi(\alpha)) \tag{4.21}$$

where Ψ^{-1} is the inverse function of Ψ. Set $\kappa_2 = \frac{1}{2}k_2$. By using the change of variables $\beta = \beta(\alpha, \gamma)$ we get

$$\begin{aligned} v[\Psi](\alpha) &= \int K(\Psi(\alpha) - \Psi(\beta))\omega(\beta)\, d\beta \\ &= \int K(\Phi(\alpha) - \Phi(\gamma))\omega(\beta(\gamma)) \det \frac{\partial \beta}{\partial \gamma}\, d\gamma. \end{aligned} \tag{4.22}$$

Our first goal is to expand $\beta(\alpha, \gamma, \kappa)$, $\omega(\beta(\gamma))$ and $\det(\partial\beta/\partial\gamma)$ in Taylor series in κ. By using the mean value theorem we conclude from (4.21) that

$$\begin{aligned} \beta &= \gamma + \bar{a}\kappa + \bar{b}\frac{\kappa^2}{2!} + \bar{c}\frac{\kappa^3}{3!} + \int_0^1 \Psi^{-1\prime}\left(\gamma + \bar{a}\kappa + \bar{b}\frac{\kappa^2}{2!} + \bar{c}\frac{\kappa^3}{3!} + \tau E\right) d\tau \cdot E, \\ E &= \Phi(\gamma) + \Psi(\alpha) - \Phi(\alpha) - \Psi\left(\gamma + \bar{a}\kappa + \bar{b}\frac{\kappa^2}{2!} + \bar{c}\frac{k^3}{3!}\right) \end{aligned} \tag{4.23}$$

for all values of $\bar{a}$, $\bar{b}$ and $\bar{c}$. It follows from Corollary 1 and the inverse function theorem that $\Psi^{-1\prime}$ and its derivatives can be bounded independent of κ if $\kappa \leqq \kappa_2$. We will choose

$\bar{a}$, $\bar{b}$, $\bar{c}$ such that E is small. By using (4.20) and expanding Φ and $\partial_t\Phi$ in a Taylor series around γ, we find after collecting terms that

$$\begin{aligned} E = &-\kappa[\Phi'(\gamma)\cdot\bar{a}+\partial_t\Phi(\gamma)-\partial_t\Phi(\alpha)] \\ &-\frac{\kappa^2}{2!}[\Phi'(\gamma)\cdot\bar{b}+\Phi''(\gamma)\cdot(\bar{a},\bar{a})+2\partial_t\Phi'(\gamma)\cdot\bar{a}] \\ &-\frac{\kappa^3}{3!}[\Phi'(\gamma)\cdot\bar{c}+3\Phi''(\gamma)\cdot(\bar{a},\bar{b})+\Phi'''(\gamma) \\ &\qquad\cdot(\bar{a},\bar{a},\bar{a})+3\partial_t\Phi'(\gamma)\cdot\bar{b}+3\partial_t\Phi''(\gamma)\cdot(\bar{a},\bar{a})] \\ &+\kappa^4R(\alpha,\gamma,\kappa). \end{aligned}$$

We choose $\bar{a}$, $\bar{b}$ and $\bar{c}$ such that the first three terms vanish. The coefficients depend on α, γ and a finite number of derivatives of Φ. The Hölder continuity of $\Phi^{(m+1)}$ plays no role in this proof. Instead of $C^{m+1+\lambda}(D)$ we consider the continuous, uniformly bounded functions that have $m+1$ continuous, uniformly bounded derivatives. We denote this space by C^{m+1}. Since the inverse of $\Phi'(\gamma)$ is uniformly bounded, we see that $\bar{a}$, $\bar{b}$ and $\bar{c}$ are uniformly bounded and in C^m, C^{m-1} and C^{m-2} with respect to α and γ. The remainder R is in C^{m-3} uniformly in κ for κ small, i.e., $\partial_\alpha^\nu\partial_\gamma^\mu R$ is uniformly bounded if $|\nu|+|\mu|\leqq m-3$ and $\kappa\leqq\kappa_2$. Equation (4.14) shows that $\bar{a}=a$. If we set $\bar{b}=b+d$ then it follows from (4.15) and (4.16) that

$$-\Phi'(\gamma)\cdot d+\partial_t^2\Phi(\gamma)=\partial_t^2\Phi(\alpha), \tag{4.24}$$

$$\Phi'\cdot\bar{c}+3\Phi''\cdot(a,b+d)+\Phi'''\cdot(a,a,a)+3\partial_t\Phi'\cdot(b+d)+3\partial_t\Phi''\cdot(a,a)=0 \tag{4.25}$$

where the derivatives of Φ are evaluated at γ. We have therefore proved that

$$\beta=\gamma+a\kappa+(b+d)\frac{\kappa^2}{2!}+\bar{c}\frac{\kappa^3}{3!}+\kappa^4R(\alpha,\gamma,\kappa). \tag{4.26}$$

We can now compute $\omega(\beta)$. By using Taylor's formula with remainder we get

$$\begin{aligned} \omega(\beta)=\omega(\gamma)&+\kappa[\omega'\cdot a]+\frac{\kappa^2}{2!}[\omega'\cdot(b+d)+\omega''\cdot(a,a)] \\ &+\frac{\kappa^3}{3!}[\omega'\cdot\bar{c}+3\omega''\cdot(a,b+d)+\omega'''\cdot(a,a,a)]+\kappa^4R_c(\alpha,\gamma,\kappa) \end{aligned} \tag{4.27}$$

where the derivatives of ω are evaluated at γ. The function R_c is in C^{m-4} and has compact support. Hence the subscript c. Since a, b, $\bar{c}$, d and R are uniformly bounded, we can choose κ so small that each of the last four terms in (4.26) is less than $\frac{1}{8}D$. The diameter of the support of R_c in the γ variable is therefore less than $2D$ if κ is less than a constant κ_3 where $\kappa_3\leqq\kappa_2$. We could now compute $\det(\partial\beta/\partial\gamma)$ directly from (4.26) but prefer a different approach. By using the chain rule and that Φ is an area preserving mapping we conclude from (4.21) and (4.20) that

$$\det\frac{\partial\beta}{\partial\gamma}=\frac{1}{\det\Psi'(\beta(\gamma))}=\frac{1}{1+\kappa^2\det\partial_t\Phi'(\beta)}.$$

Finally we use (4.26) and expand $\partial_t\Phi'$ around γ. Thus

$$\det\frac{\partial\beta}{\partial\gamma}=\frac{1}{[illegible]}$$

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where the derivatives of Φ are evaluated at γ. The last R is uniformly bounded and in C^{m-3} if κ is sufficiently small, say $\kappa \leqq \kappa_4$ where $\kappa_4 \leqq \kappa_3$. By combining this result with the expansion of $\omega(\beta)$, we get

$$\omega(\beta) \det \frac{\partial \beta}{\partial \gamma} = \omega(\gamma) + A\kappa + (B+D)\frac{\kappa^2}{2!} + \bar{C}\frac{\kappa^3}{3!} + \kappa^4 R_c(\alpha, \gamma, \kappa) \tag{4.28}$$

where R_c is in C^{m-4} uniformly in κ for κ small and the diameter of its support in the γ variable is less than $2D$ and

$$\begin{aligned} A &= \omega' \cdot a, \\ B &= \omega' \cdot b + \omega'' \cdot (a, a), \\ D &= \omega' \cdot d - 2\omega \det \partial_t \Phi', \\ \bar{C} &= \omega' \cdot \bar{c} + 3\omega'' \cdot (a, b+d) + \omega''' \cdot (a, a, a) \\ &\qquad - 6\omega' \cdot a \det \partial_t \Phi' - 6\omega (\det \partial_t \Phi')' \cdot a. \end{aligned} \tag{4.29}$$

Note that A and B are the expressions in (4.11) and (4.12) and that D is used in two different contexts. This should not cause confusion. We now insert (4.28) in (4.22), use (4.11) and (4.12) and get

$$\begin{aligned} v[\Psi^1](\alpha) &= \partial_t \Phi + \kappa \partial_t^2 \Phi + \frac{\kappa^2}{2!}\partial_t^3 \Phi + \frac{\kappa^2}{2!} X + \frac{\kappa^3}{3!}\bar{\Upsilon} + \kappa^4 \bar{R}(\alpha, \kappa), \\ \Psi^2(\alpha) &= \Phi + \kappa \partial_t \Phi + \kappa^2 \partial_t^2 \Phi + \frac{\kappa^3}{2}\partial_t^3 \Phi + \frac{\kappa^3}{2} X + \kappa^4 R(\alpha, \kappa). \end{aligned} \tag{4.30}$$

Here Φ, X and $\bar{\Upsilon}$ and the derivatives of Φ are evaluated at α and

$$X(\alpha) = \int K(\Phi(\alpha) - \Phi(\gamma)) D(\alpha, \gamma)\, d\gamma,$$

$$\bar{\Upsilon}(\alpha) = \int K(\Phi(\alpha) - \Phi(\gamma)) \bar{C}(\alpha, \gamma)\, d\gamma,$$

$$\bar{R}(\alpha, \kappa) = \int K(\Phi(\alpha) - \Phi(\gamma)) R_c(\alpha, \gamma, \kappa)\, d\gamma$$

and $R = \frac{1}{6}\bar{\Upsilon} + \kappa \bar{R}$. The integrals converge because D, $\bar{C}$ and R_c have compact support in the γ variable. We will show that $X \in C^{m-1}$, $\bar{\Upsilon} \in C^{m-3}$ and $\bar{R} \in C^{m-4}$ uniformly for κ small. Take the last case. Since $\det \Phi' = 1$, we find after a change of variables that

$$\bar{R}(\alpha, \kappa) = \int K(y) R_c(\alpha, \Phi^{-1}(\Phi(\alpha) - y), \kappa)\, dy.$$

It follows from (2.5) that the diameter of the support of R_c as a function of y is less than $2\sqrt{2}MD$. The support is therefore contained in a circle with center y_0 and radius $\varepsilon = \sqrt{2}MD$. Let $r_0 = |y_0|$. Since K is absolutely integrable over compact sets, we can compute the partial derivatives of $\bar{R}$ by differentiating under the integral sign. By using the chain rule we see that the partial derivatives of R_c can be written as a sum of $\partial_\alpha^\nu \partial_\gamma^\mu R_c$ times derivatives of Φ^{-1} and Φ. The support of such an expression can be no larger than the support of R_c. It is either inside a circle with center at the origin and radius 2ε (if $r_0 \leqq \varepsilon$), or it is contained in an angular segment of an annulus as indicated in Fig. 1. Since the $\partial_\alpha^\nu \partial_\gamma^\mu R_c(\alpha, \gamma, \kappa)$ are uniformly bounded for $|\nu| + |\mu| \leqq m-4$ and

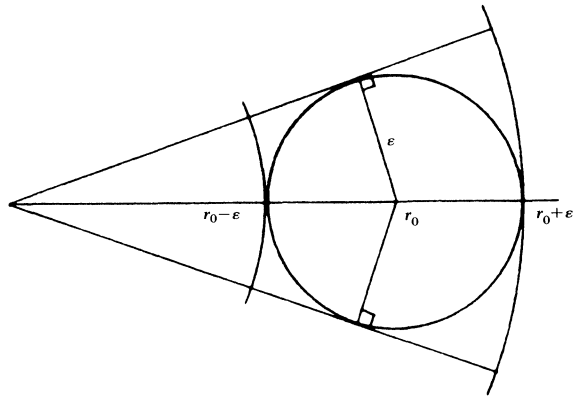

FIG. 1. *Inclusion of the support of the vorticity.*

$\kappa \leqq \kappa_4$, and since Φ is in C^{m-4}, we conclude that $|\partial_\alpha^\nu \bar{R}|$ is less than $4\pi\varepsilon \cdot \text{const}$ if $r_0 \leqq \varepsilon$ and $|\nu| \leqq m-4$. The constant can be bounded independent of κ if $\kappa \leqq \kappa_4$. If $r_0 > \varepsilon$ then

$$|\partial_\alpha^\nu \bar{R}(\alpha, \kappa)| \leqq \int_{-\arcsin(\varepsilon/r_0)}^{\arcsin(\varepsilon/r_0)} \int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{1}{2\pi r} \text{const} \cdot r\, dr\, d\vartheta$$

$$\leqq \pi^{-1} \arcsin(\varepsilon/r_0) 2\varepsilon \text{ const.}$$

This estimate shows that $\bar{R}$ and its derivatives tend to zero as α (or r_0) tends to infinity.

It follows from (4.24) and (4.25) that $d \in C^{m-1}$ and $\bar{c} \in C^{m-2}$. As ω is in C^m we see that $D \in C^{m-1}$ and $\bar{C} \in C^{m-3}$. By following the previous arguments for the differentiability of $\bar{R}$ we conclude that $X \in C^{m-1}$ and $\bar{Y} \in C^{m-3}$. The remainder R in (4.30) is therefore in C^{m-4} uniformly in κ if $\kappa \leqq \kappa_4$. This completes Step 2.

Step 3. We have now presented the main ideas, and in the next two steps we will only give the highlights. In this step we denote Ψ^2 by Ψ. Using the change of variables (4.21), we expand $\beta(\gamma)$ as in (4.23), but replace $\bar{a}$, $\bar{b}$, $\bar{c}$ by $\tilde{a}$, $\tilde{b}$, $\tilde{c}$. After using (4.30) and choosing $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ as in Step 2, we discover that $\tilde{a} = a$, $\tilde{b} = b - d$ and that $\tilde{c}$ satisfies

$$\begin{aligned} \Phi' \cdot \tilde{c} + 3\Phi'' \cdot (a, b-d) + \Phi''' \cdot (a, a, a) + 3\partial_t \Phi' \cdot (b-d) + 3\partial_t \Phi'' \cdot (a, a) \\ + 6\partial_t^2 \Phi' \cdot a + 3\partial_t^3 \Phi + 3X(\gamma) = 3\partial_t^3 \Phi(\alpha) + 3X(\alpha). \end{aligned} \tag{4.31}$$

Here we have used (4.11), (4.12) and (4.24). Since $\Psi^{-1\prime}$ and its derivatives can be bounded independent of κ if κ is small, it follows from (4.23) that

$$\beta = \gamma + a\kappa + (b-d)\frac{\kappa^2}{2!} + \tilde{c}\frac{\kappa^3}{3!} + \kappa^4 R(\alpha, \gamma, \kappa) \tag{4.32}$$

where R is in C^{m-4} uniformly in κ if $\kappa \leqq \kappa_4$. The expansion of $\omega(\beta)$ looks like (4.27) with d, $\bar{c}$ replaced by $-d$, $\tilde{c}$. The expansion of $\det(\partial\beta/\partial\gamma)$ is more interesting. Since $\det \Phi' = 1$, we find by using (4.30) that

$$\det \frac{\partial \beta}{\partial \gamma} = 1 \Big/ \left(1 - \kappa^2 \det(\partial_t \Phi') - \frac{\kappa^3}{2} \partial_t \det(\partial_t \Phi') + \frac{\kappa^3}{2} \Delta + \kappa^4 R\right),$$

$$\Delta = \det(\partial_1 \Phi, \partial_2 X) + \det(\partial_1 X, \partial_2 \Phi).$$

Here the derivatives of Φ and X are evaluated at $\beta(\gamma)$ and $R \in C^{m-5}$ uniformly in κ for κ small. It is at this point that we loose a derivative in R. By using the expansion (4.32) we obtain

$$\det \frac{\partial \beta}{\partial \gamma} = 1 + \kappa^2 \det \partial_t \Phi' + \kappa^3 (\det \partial_t \Phi')' \cdot a + \frac{\kappa^3}{2} \partial_t (\det \partial_t \Phi') - \frac{\kappa^3}{2} \Delta + \kappa^4 R(\alpha, \gamma, \kappa)$$

where Δ and the derivatives of Φ are evaluated at γ and $R \in C^{m-5}$ uniformly in κ for κ small. Combining the expressions for $\omega(\beta(\gamma))$ and $\det(\partial\beta/\partial\gamma)$, we get

$$\omega(\beta) \det \frac{\partial \beta}{\partial \gamma} = \omega(\gamma) + A\kappa + (B - D)\frac{\kappa^2}{2!} + \tilde{C}\frac{\kappa^3}{3!} + \kappa^4 R_c(\alpha, \gamma, \kappa) \tag{4.33}$$

where A, B and D are given in Step 2. The function R_c is in C^{m-5} uniformly in κ for κ small and has compact support and

$$\begin{aligned} \tilde{C} = {} & \omega' \cdot \tilde{c} + 3\omega'' \cdot (a, b - d) + \omega''' \cdot (a, a, a) + 6\omega' \cdot a \det \partial_t \Phi' \\ & + 6\omega (\det \partial_t \Phi')' \cdot a + 3\omega \partial_t (\det \partial_t \Phi') - 3\omega \Delta. \end{aligned} \tag{4.34}$$

We now insert (4.33) in (4.22) and get

$$\begin{aligned} v[\Psi^2](\alpha) &= \partial_t \Phi + \kappa \partial_t^2 \Phi + \frac{\kappa^2}{2!} \partial_t^3 \Phi - \frac{\kappa^2}{2!} X + \frac{\kappa^3}{3!} \tilde{Y} + \kappa^4 \tilde{R}(\alpha, \kappa), \\ \Psi^3(\alpha) &= \Phi + 2\kappa \partial_t \Phi + 2\kappa^2 \partial_t^2 \Phi + \kappa^3 \partial_t^3 \Phi - \kappa^3 X + \kappa^4 R(\alpha, \kappa). \end{aligned} \tag{4.35}$$

Here Φ, X and $\tilde{Y}$ and the derivatives of Φ are evaluated at α and

$$\tilde{Y}(\alpha) = \int K(\Phi(\alpha) - \Phi(\gamma)) \tilde{C}(\alpha, \gamma)\, d\gamma.$$

It follows from (4.31) and (4.34) that $\tilde{c} \in C^{m-2}$ and $\tilde{C} \in C^{m-3}$ with respect to α and γ. Thus the arguments from Step 2 shows that $\tilde{Y} \in C^{m-3}$ and that the remainders $\tilde{R}$ and R are in C^{m-5} uniformly in κ for $\kappa \leqq \kappa_5$, where $\kappa_5 \leqq \kappa_4$.

Step 4. In this step we shall compute $v[\Psi^3]$ and denote Ψ^3 by Ψ. Using the change of variables (4.21) we expand $\beta(\gamma)$ as in (4.23) but replace $\bar{a}$, $\bar{b}$, $\bar{c}$, κ by $\hat{a}$, $\hat{b}$, $\hat{c}$, 2κ. After using (4.35) and choosing $\hat{a}$, $\hat{b}$, $\hat{c}$ as in Step 2, we discover that $\hat{a} = a$, $\hat{b} = b$ and that $\hat{c}$ satisfies

$$\begin{aligned} & \Phi' \cdot \hat{c} + 3\Phi'' \cdot (a, b) + \Phi''' \cdot (a, a, a) + 3\partial_t \Phi' \cdot b + 3\partial_t \Phi'' \cdot (a, a) \\ & \quad + 3\partial_t^2 \Phi' \cdot a + \tfrac{6}{8} \partial_t^3 \Phi(\gamma) - \tfrac{6}{8} X(\gamma) = \tfrac{6}{8} \partial_t^3 \Phi(\alpha) - \tfrac{6}{8} X(\alpha) \end{aligned} \tag{4.36}$$

where the derivatives of Φ are evaluated at γ. This implies that

$$\beta = \gamma + a 2\kappa + b \frac{(2\kappa)^2}{2!} + \hat{c} \frac{(2\kappa)^3}{3!} + \kappa^4 R(\alpha, \gamma, \kappa) \tag{4.37}$$

where $R \in C^{m-5}$ uniformly in κ for $\kappa \leqq \kappa_5$. The expansion of $\omega(\beta)$ is therefore

$$\begin{aligned} \omega(\beta) = {} & \omega(\gamma) + 2\kappa[\omega' \cdot a] + \frac{(2\kappa)^2}{2!}[\omega' \cdot b + \omega'' \cdot (a, a)] \\ & + \frac{(2\kappa)^3}{3!}[\omega' \cdot \hat{c} + 3\omega'' \cdot (a, b) + \omega''' \cdot (a, a, a)] + \kappa^4 R_c(\alpha, \gamma, \kappa). \end{aligned} \tag{4.38}$$

The derivatives of ω are evaluated at γ. It follows from (4.35) and (4.37) that

$$\begin{aligned} \det \frac{\partial \beta}{\partial \gamma} &= \frac{1}{1 + \kappa^3 \partial_t \det \partial_t \Phi' - \kappa^3 \Delta + \kappa^4 R} \\ &= 1 - \kappa^3 \partial_t \det \partial_t \Phi' + \kappa^3 \Delta + \kappa^4 R(\alpha, \gamma, \kappa) \end{aligned}$$

for κ sufficiently small. Here Φ' and Δ are evaluated at (γ, t) and we lose one more derivative, i.e. $R \in C^{m-6}$ uniformly in κ for κ small. Combining this result with (4.38), we get

$$\omega(\beta)\det\frac{\partial\beta}{\partial\gamma} = \omega(\gamma) + A(2\kappa) + B\frac{(2\kappa)^2}{2!} + \hat{C}\frac{(2\kappa)^3}{3!} + \kappa^4 R_c(\alpha, \gamma, \kappa) \tag{4.39}$$

where A and B are given in Step 2. The function $R_c \in C^{m-6}$ uniformly in κ for κ small and the diameter of its support in the γ variable can be made less than $2D$. Moreover

$$\hat{C} = \omega' \cdot \hat{c} + 3\omega'' \cdot (a, b) + \omega''' \cdot (a, a, a) - \tfrac{6}{8}\omega\partial_t \det \partial_t\Phi' + \tfrac{6}{8}\omega\Delta. \tag{4.40}$$

Inserting (4.39) in (4.22) and using (4.11) and (4.12), we obtain

$$v[\Psi^3](\alpha) = \partial_t\Phi + 2\kappa\partial_t^2\Phi + \frac{(2\kappa)^2}{2!}\partial_t^3\Phi + \frac{(2\kappa)^3}{3!}\hat{\Upsilon} + \kappa^4\hat{R}(\alpha, \kappa).$$

Here $\hat{\Upsilon}$ and the derivatives of Φ are evaluated at α and

$$\hat{\Upsilon}(\alpha) = \int K(\Phi(\alpha) - \Phi(\gamma))\hat{C}(\alpha, \gamma)\, d\gamma.$$

It follows from (4.36) and (4.40) that $\hat{c} \in C^{m-2}$ and $\hat{C} \in C^{m-3}$. This implies that $\hat{\Upsilon} \in C^{m-3}$. The remainder $\hat{R} \in C^{m-6}$ uniformly in κ if $\kappa \leqq \kappa_6$ where $\kappa_6 \leqq \kappa_5$. This completes Step 4.

Step 5. We can finally complete the proof of Lemma 12. Assume that $\kappa \leqq \kappa_6$ and set $k_3 = 2\kappa_6$. In Steps 1–4 we have computed the velocity fields for the intermediate steps of the Runge–Kutta method. By adding these we get

$$\begin{aligned} &\frac{1}{6}(v[\Psi^0] + 2v[\Psi^1] + 2v[\Psi^2] + v[\Psi^3]) \\ &= \partial_t\Phi + \frac{(2\kappa)}{2!}\partial_t^2\Phi + \frac{(2\kappa)^2}{3!}\partial_t^3\Phi + \frac{(2\kappa)^3}{4!}\left(\frac{1}{6}\bar{\Upsilon} + \frac{1}{6}\tilde{\Upsilon} + \frac{4}{6}\hat{\Upsilon}\right) + \kappa^4 R \end{aligned} \tag{4.41}$$

where $6R = 2\bar{R} + 2\tilde{R} + \hat{R}$ and $|R(\alpha, \kappa)|$ is less than a constant C' for all $\alpha \in \mathbb{R}^2$ and $\kappa \leqq \kappa_6$. We will show that $\bar{\Upsilon} + \tilde{\Upsilon} + 4\hat{\Upsilon} = 6\partial_t^4\Phi$. From (4.29), (4.34) and (4.40) follows that

$$\tfrac{1}{6}\bar{C} + \tfrac{1}{6}\tilde{C} + \tfrac{4}{6}\hat{C} = \omega' \cdot (\tfrac{1}{6}\bar{c} + \tfrac{1}{6}\tilde{c} + \tfrac{4}{6}\hat{c}) + 3\omega'' \cdot (a, b) + \omega''' \cdot (a, a, a). \tag{4.42}$$

On the other hand, (4.25), (4.31) and (4.36) yield

$$\begin{aligned} &\Phi' \cdot (\tfrac{1}{6}\bar{c} + \tfrac{1}{6}\tilde{c} + \tfrac{4}{6}\hat{c}) + 3\Phi'' \cdot (a, b) + \Phi''' \cdot (a, a, a) + 3\partial_t\Phi' \cdot b \\ &\qquad + 3\partial_t\Phi'' \cdot (a, a) + 3\partial_t^2\Phi' \cdot a + \partial_t^3\Phi(\gamma) = \partial_t^3\Phi(\alpha). \end{aligned}$$

The derivatives of Φ are evaluated at γ. Comparing this equation with (4.16), we see that $\bar{c} + \tilde{c} + 4\hat{c} = 6c$. The right-hand side of (4.42) is therefore equal to the factor $[\cdots]$ in the integral (4.13). This implies that $\bar{\Upsilon} + \tilde{\Upsilon} + 4\hat{\Upsilon} = 6\partial_t^4\Phi$. Thus it follows from (4.41) and (4.19) that

$$\Phi(t+k) = \Phi(t) + k\partial_t\Phi + \frac{k^2}{2!}\partial_t^2\Phi + \frac{k^3}{3!}\partial_t^3\Phi + \frac{k^4}{4!}\partial_t^4\Phi + \kappa^5 R + \tau.$$

By using Taylor's formula with remainder we see that $|\tau| \leqq C_7 k^5$ if $k \leqq k_3$ and the constant C_7 is larger than $(2\cdot 5!)^{-1}M + C'$. This completes the proof.

Anderson and Greengard [2] could not establish high accuracy for higher order Runge-Kutta methods. In their proof they compare the computed solution in the intermediate steps with the exact solution at intermediate times. This is a poor approximation and does not take into account that the intermediate flows are not area preserving (see (4.20) and (4.30)). Step 5 in our proof shows that the high accuracy of the classical Runge-Kutta method is due to a cancellation of errors.

4.4. High order convergence. In this section we will prove the convergence of the vortex method for long time intervals. We assume that the flow is at least six times differentiable and that the cutoff satisfies Assumption 2. The strengths of the vortex blobs are determined by pointwise evaluation of the initial vorticity. The cutoff is of infinite order, but the rate of convergence depends on the smoothness of the flow.

THEOREM 3. *Let Assumption* 2 *and Assumption* 4 *be satisfied with* $m \geqq 6$. *Set* $\delta = h^q$ *with* $q = \frac{1}{2}(m+\lambda)/(m+\lambda-1)$ *and* $\kappa_j = \omega(\alpha_j)h^2$. *Solve the differential equation* (1.4) *by the classical Runge-Kutta method* (4.18) *with* $k \leqq h^{(1+\gamma)/4}$, $\gamma > 0$. *Let* $1 \leqq p < \infty$. *There exist two constants* C_8 *and* h_5 *such that*

$$\|\tilde{\Phi}(t) - \Phi(t)\|_p \leqq C_8(h^{(m+\lambda)/2} + k^4)$$

for all $h \leqq h_5$ *and* $t = nk \leqq T$. *The constants* C_8 *and* h_5 *depend on* M, D, L_1, T, λ, γ, m *and* p.

Remark. The infinite order cutoff (1.8) satisfies Assumption 2, but falls outside the classes of higher order cutoffs studied by Beale and Majda [4], [5], Cottet [9] and Anderson and Greengard [2]. With our cutoffs it is not necessary to tailor the cutoffs to the flow and to construct progressively more complicated expressions for smoother flows. Theorem 3 implies higher order convergence for the velocity field and for the vorticity distribution. The proofs of Theorem 1 and Theorem 3 are similar. We replace Lemma 3 by Lemma 10 and Lemma 5 by Lemma 12. The main difference is that we have more intermediate steps in the classical Runge-Kutta method. Note that we can use $k = \sqrt{h}$.

Proof. It is enough to establish Theorem 3 for p sufficiently large. Choose p such that $2/p$ is less than $\lambda/2$ and $\gamma/2$. We shall use induction and prove that

$$\|\tilde{\Phi}^n - \Phi^n\|_p \leqq \frac{a^n - 1}{a - 1} k \|1\|_p b \tag{4.43}$$

where $a = 1 + 2C_1 k$, $b = 2C_6 h^{(m+\lambda)/2} + 2C_7 k^4$ and $\Phi_j^n = \Phi(\alpha_j, nk)$. The constants C_6 and C_7 were determined in Lemma 10 and Lemma 12. We will now explain how C_1 is determined. We observe first that $\|\omega\|_{C^\lambda(D)} \leqq 2\|\omega\|_{C^{m+\lambda}(D)}$ and $|\Phi|_{C^{1+\lambda}(D)} \leqq 3|\Phi|_{C^{m+1+\lambda}(D)}$. The assumptions of Lemma 1 are therefore satisfied if $C = 3M$. We denote the bound for the time step by k_0. If $k \leqq k_0$ and $C = 3M$ then the assumptions in Lemma 4 are satisfied, and the constant C_1 in the definition of a is the stability constant from Lemma 4. We now choose $h_5 < 1$ such that (1): h_5 is less than h_1 from Lemma 4 and less than h_4 from Lemma 10, (2): h_5^{1-q} is less than h_1 and (3): $h_5^{(1+\gamma)/4}$ is less than k_0 from Lemma 1, k_1 from Corollary 1, k_3 from Lemma 12 and $(4C_1)^{-1}$ from Lemma 4. Finally we set $c = (4\sqrt{2}C)^{-1}$ and assume that

$$2\frac{e^{2C_1 T} - 1}{2C_1}(1+D)^2(2C_6 h_5^{(m-2)/2} + 2C_7 h_5^{\gamma/2}) \leqq \frac{1}{2}c. \tag{4.44}$$

We will show by induction that the computed flows $\tilde{\Psi}^l$ in the intermediate steps of the Runge-Kutta method (4.18) are close to the intermediate flows Ψ^l of (4.18). Let $h \leqq h_5$. At time $t = 0$ we have $\tilde{\Phi}_j = \Phi_j$ and (4.43) is therefore satisfied for $n = 0$.

Assume next that (4.43) holds for $nk \leqq T-k$ and set $\tilde{\Psi}^0 = \tilde{\Phi}^n$ and $\Psi^0 = \Phi^n$. If $l=1$ then

$$\|\tilde{\Psi}^{l-1} - \Psi^{l-1}\|_p \leqq 2 \max\left(1, \frac{a^n-1}{a-1}\right) k\|1\|_p b. \tag{4.45}$$

Since $|\tilde{\Psi}_j^{l-1} - \Psi_j^{l-1}|$ is less than $h^{-2/p}\|\tilde{\Psi}^{l-1} - \Psi^{l-1}\|_p$ for all j in J and $k^4 \leqq h^{1+\gamma}$ we conclude from (4.44) and (4.45) that

$$\begin{aligned} |\tilde{\Psi}_i^{l-1} - \Psi_i^{l-1}| &\leqq 2\frac{e^{2C_1T}-1}{2C_1}(1+D)^2(2C_6h^{(m-2)/2}h^{\lambda/2-2/p} + 2C_7h^{\gamma/2}h^{\gamma/2-2/p})h \\ &\leqq \frac{1}{2}ch \end{aligned} \tag{4.46}$$

for all $i \in J$. (Remember that J is the set of all indices such that $Q_j \cap \operatorname{supp} \omega \neq \varnothing$ and that the diameter of $\bigcup_{j\in J} Q_j$ is less than $2\sqrt{2}h + D$. The area of $\bigcup_{j\in J} Q_j$ or the sum $\sum_{j\in J} h^2$ is therefore less than $(1+D)^2$.) Since $\vartheta_l \leqq 2$, it follows from (4.18) and (4.19) and the triangle inequality, that

$$\begin{aligned} \|\tilde{\Psi}^l - \Psi^l\|_p &\leqq \|\tilde{\Phi}^n - \Phi^n\|_p + k\|V[\tilde{\Psi}^{l-1}] - V[\Psi^{l-1}]\|_p + k\|V[\Psi^{l-1}] - v[\Psi^{l-1}]\|_p \\ &\leqq \frac{a^n-1}{a-1}k\|1\|_p b + kC_1\|\tilde{\Psi}^{l-1} - \Psi^{l-1}\|_p + kC_6h^{(m+\lambda)/2}\|1\|_p \\ &\leqq 2\max\left(1, \frac{a^n-1}{a-1}\right)k\|1\|_p b. \end{aligned}$$

The complete argument is more complicated. Since $\Psi^0 = \Phi^n$, $|\Phi|_{C^{1+\lambda}(D)} \leqq \frac{1}{2}C$ and $k \leqq k_0$ we see that Ψ^{l-1} satisfies the inequalities in (2.3). Our choice of h_5 insures that h/δ and h are less than h_1. Combining these facts with (4.46), we find that the assumptions in Lemma 4 are satisfied, and we can use the stability estimate, the induction hypothesis (4.45) and the inequality $kC_1 \leqq \frac{1}{4}$. Since $\Psi^0 = \Phi^n$, $|\Phi|_{C^{m+1+\lambda}(D)} \leqq \frac{1}{2}M$ and $k \leqq k_1$, it follows from Corollary 1 that Ψ^{l-1} satisfies (2.8). We can therefore use Lemma 10 to estimate the consistency error, first pointwise and then in the discrete L^p norm. Since $C_6h^{(m+\lambda)/2}$ is less than $\frac{1}{2}b$ we conclude that (4.45) holds for $l = 1, 2, 3, 4$.

We can now estimate the error in the $(n+1)$st step. Let $\eta_1 = \eta_4 = \frac{1}{6}$ and $\eta_2 = \eta_3 = \frac{2}{6}$ and note that $\sum \eta_l = 1$. Combining (4.18) and (4.19) and using the triangle inequality, we obtain

$$\begin{aligned} \|\tilde{\Phi}^{n+1} - \Phi^{n+1}\|_p \leqq \|\tilde{\Phi}^n - \Phi^n\|_p &+ k\sum_{l=1}^4 \eta_l\|V[\tilde{\Psi}^{l-1}] - V[\Psi^{l-1}]\|_p \\ &+ k\sum_{l=1}^4 \eta_l\|V[\Psi^{l-1}] - v[\Psi^{l-1}]\|_p + \|\tau\|_p. \end{aligned}$$

By using the stability of the vortex method (Lemma 4), and the high order consistency (Lemma 10), plus the estimate of the truncation error (Lemma 12), we conclude from (4.45) that

$$\begin{aligned} \|\tilde{\Phi}^{n+1} - \Phi^{n+1}\|_p \leqq \frac{a^n-1}{a-1}k\|1\|_p b &+ k\sum_{l=1}^4 \eta_l C_1 2\max\left(1, \frac{a^n-1}{a-1}\right)k\|1\|_p b \\ &+ k\sum_{l=1}^4 \eta_l C_6 h^{(m+\lambda)/2}\|1\|_p + kC_7k^4\|1\|_p. \end{aligned}$$

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The case $n=0$ is special. The first term vanishes and we see from the definition of b that $\|\tilde{\Phi}^1-\Phi^1\|_p$ is less than $k\|1\|_p b$. If $n \geqq 1$ then

$$\|\tilde{\Phi}^{n+1}-\Phi^{n+1}\|_p \leqq (1+2C_1k)\frac{a^n-1}{a-1}k\|1\|_p b + k\|1\|_p(C_6h^{(m+\lambda)/2}+C_7k^4)$$

$$\leqq \frac{a^{n+1}-1}{a-1}k\|1\|_p b.$$

This completes the proof.

It is not necessary to choose the optimal value of q. If $\delta = h^q$ where q is fixed and $(m+\lambda-1)^{-1} < q < 1$ then it follows from Observation 1 and the proof of Theorem 3 that

$$\|\tilde{\Phi}-\Phi\|_p \leqq \text{const} \cdot (h^{q(m+\lambda-1)}+h^{q+(m+\lambda)(1-q)}+k^4)$$

for h and k sufficiently small. $q=\frac{1}{2}$ is almost optimal.

In the proof of Theorem 3 we have used the consistency and the stability of the vortex method plus our estimate of the truncation error. The technique is general. We can therefore prove convergence of finite order cutoffs, provided we solve the differential equations by the classical Runge-Kutta method and the flow is sufficiently smooth. Here is an example. Let Assumption 4 be satisfied with $m \geqq 6$ and assume that the cutoff is in the class $M^{\mu,\nu}$ of Anderson and Greengard [2] with $\mu \geqq 3$ and $\nu \geqq 2$. This means, roughly speaking, that the cutoff is μ times continuously differentiable and that the moments vanish up to order $\nu-1$. Let $\delta = h^q$ with q fixed and $\nu^{-1} < q < 1$. If $m \geqq \mu$ and $m \geqq \nu+1$ and we solve the ordinary differential equations by the classical Runge-Kutta method then

$$\|\tilde{\Phi}-\Phi\|_p \leqq \text{const} \cdot (h^{q\nu}+h^{q+\mu(1-q)}+k^4)$$

for all h sufficiently small and $k \leqq h^{(1+\gamma)/4}$, $\gamma > 0$. Here we do not get convergence for $q=\frac{1}{2}$ and $\nu=2$. The restriction $q\nu > 1$ can be traced back to Lemma 4 and can be lifted by proving a special stability result for smooth cutoffs as mentioned in § 4.2.

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REFERENCES

[1] C. R. ANDERSON, *Vortex methods for flows with variable density*, Ph.D. thesis, Univ. California, Berkeley, 1983.

[2] C. ANDERSON AND C. GREENGARD, *On vortex methods*, this Journal, 22 (1985), pp. 413-440.

[3] J. T. BEALE AND A. MAJDA, *Vortex methods*, I: *Convergence in three dimensions*, Math. Comp., 39 (1982), pp. 1-27.

[4] ———, *Vortex methods*. II: *Higher order accuracy in two and three dimensions*, Math. Comp., 39 (1982), pp. 29-52.

[5] ———, *Higher order accurate vortex methods with explicit velocity kernels*, J. Comput. Phys., 58 (1985), pp. 188-208.

[6] L. BERS, F. JOHN AND M. SCHECHTER, *Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 1964.

[7] A. P. CALDERON AND A. ZYGMUND, *On the existence of certain singular integrals*, Acta Math., 88 (1952), pp. 85-139.

[8] A. J. CHORIN, *Numerical study of slightly viscous flow*, J. Fluid Mech., 57 (1973), pp. 785-796.

[9] G. H. COTTET, *Thèse de 3eme cycle*, l'Université Pierre et Marie Curie, Paris 6, 1982.

[10] G. H. COTTET AND P. A. RAVIART, *Particle methods for one-dimensional Vlasov Poisson equations*, this Journal, 21 (1984), pp. 52-76.

[11] V. M. DEL PRETE AND O. H. HALD, *Convergence of vortex methods for Euler's equations*, Math. Comp., 32 (1978), pp. 791-809.

[12] J. DIEUDONNÉ, *Foundations of Modern Analysis*, Academic Press, New York, 1960.

[13] T. E. DUSHANE, *Convergence for a vortex method for solving Euler's equation*, Math. Comp., 27 (1973), pp. 719-728.

[14] G. FOLLAND, *Introduction to Partial Differential Equations*, Princeton Univ. Press, Princeton, NJ, 1978.

[15] O. H. HALD, *Convergence of vortex methods for Euler's equations*, II, this Journal, 16 (1979), pp. 726-755.

[16] ———, *Convergence of Fourier methods for Navier-Stokes equations*, J. Comput. Phys., 40 (1981), pp. 305-317.

[17] ———, *The flowmap for Euler's equations*, to appear.

[18] T. KATO, *On classical solutions of the two-dimensional nonstationary Euler equation*, Arch. Rational Mech. Anal., 25 (1967), pp. 188-200.

[19] A. LEONARD, *Vortex methods for flow simulation*, J. Comput. Phys., 37 (1980), pp. 289-335.

[20] F. J. MCGRATH, *Nonstationary plane flow of viscous and ideal fluids*, Arch. Rational Mech. Anal., 27 (1986), pp. 329-348.

[21] F. MILINAZZO AND P. G. SAFFMAN, *The calculation of large Reynolds number two-dimensional flow using discrete vortices with random walk*, J. Comput. Phys., 23 (1977), pp. 380-392.

[22] Y. NAKAMURA, A. LEONARD AND P. SPALART, *Vortex simulation of an inviscid shear layer*, AIAA/ASME 3rd joint Thermophysics, Fluids, Plasma and Heat Transfer Conference, AIAA-82-0948, 1982.

[23] J. M. ORTEGA AND W. C. RHEINBOLDT, *Iterative Solution of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[24] M. B. PERLMAN, *On the accuracy of vortex methods*, J. Comp. Phys., 59 (1985), pp. 200-223.

[25] J. SETHIAN, *Turbulent combustion in open and closed vessels*, J. Comput. Phys., 54 (1984), pp. 425-456.

[26] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge Univ. Press, London, 1944.

[27] W. WOLIBNER, *Un theorème sur l'existence du mouvement plan d'un fluide parfait, homogene, incompressible, pendent un temps infiniment long*, Math. Z., 37 (1933), pp. 698-726.