$$\Rightarrow E\{X(X-1)\} = n(n-1)p^2 \sum_{r=0}^{n-2} \frac{(n-2)!}{r!\,(n-2-r)!} p^r q^{n-2-r}$$

$$= n(n-1)p^2 (p+q)^{n-2} = n(n-1)p^2 \quad [\because p+q=1]$$

$$\therefore V(X) = E\{X(X-1)\} - \mu(\mu-1), \quad \mu = E(X)$$

$$= n(n-1)p^2 - np(np-1) \quad , [\because \mu = E(X) = np]$$

$$= n^2p^2 - np^2 - n^2p^2 + np$$

$$= np(1-p) = npq \quad [\because q = 1-p]$$

$$\therefore \sigma_X = \sqrt{V(X)} = \sqrt{npq} \quad \text{(standard deviation)}$$

<u>Problems 3.4</u>

<u>Q-(46)</u> [3.4]

Compute the following probabilities directly from the formula for $b(x; n, p)$:

a) $b(3; 8, 0.35)$ b) $b(5, 8, 0.6)$

c) $P(3 \le X \le 5)$ when $n=7$, $p=0.6$ d) $P(1 \le X)$ when $n=9$ and $p=0.1$

<u>Ans</u>: The pmf of binomial distribution is given by

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$$

(a) Here $n=8$, $p=0.35$, $x=3$

$$\therefore b(3; 8, 0.35) = \binom{8}{3} (0.35)^3 (1-0.35)^{8-3}$$

$$= \frac{8!}{3!\,(8-3)!} (0.35)^3 \times (0.65)^5 = 0.2786$$

(b) Here $x=5$, $n=8$, $p=0.6$

$$b(5; 8, 0.6) = \binom{8}{5} (0.6)^5 (1-0.6)^{8-5} = 0.2787$$

(c) Given

~~P(X)~~ We know that for binomial distribution

$$P(X=x) = b(x;n,p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x = 0,1,2,\dots,n \\ 0, & \text{otherwise} \end{cases}$$

Given $n=7$, $p=0.6$ $\Rightarrow 1-p = 1-0.6 = 0.4$

$\therefore P(3 \le X \le 5) = P(X=3) + P(X=4) + P(X=5)$

$= {}^7C_3 (0.6)^3 (0.4)^{7-3} + {}^7C_4 (0.6)^4 (0.4)^{7-4} + {}^7C_5 (0.6)^5 (0.4)^{7-5}$

$= 0.1935 + 0.2903 + 0.2613$

$= 0.7451$

(d) Here, $n=9$, $p=0.1$

~~$P(1 \le x) = P(X \le 1) = 1 - P(X=9$~~

~~$P(1 \le x) = P(X \le 1)$~~

$P(1 \le X) = P(X \ge 1)$

$= 1 - P(X < 1)$ [$\because x = 0,1,2,\dots,n$]

$= 1 - P(X=0)$

$= 1 - {}^9C_0 (0.1)^0 (1-0.1)^{9-0}$

$= 1 - 0.3874 = 0.6126$

Q. (47) [3.4]

Use Appendix Table A.1 to obtain the following probabilities:

a) $B(4; 15, 0.3)$ b) $b(4; 15, 0.3)$ c) $b(6; 15, 0.7)$

d) $P(2 \le X \le 4)$ when $X \sim Bin(15, 0.3)$

e) $P(2 \le X)$ when $X \sim Bin(15, 0.3)$

f) $P(X \le 1)$ when $X \sim Bin(15, 0.7)$

g) $P(2 < X < 6)$ when $X \sim Bin(15, 0.3)$

Ans: (a) Here $x = 4$, $n = 15$, $p = 0.3$.

$B(4; 15, 0.3)$ is the value given in the row $x = 4$, and in the column $p = 0.30$ of the table $n = 15$ of cumulative binomial probabilities in the appendix:

$\therefore B(4; 15, 0.3) = 0.515$ (From the Appendix Table A.1)

(b) ~~b(4)~~ Here $x = 4$ $n = 15$, $p = 0.3$

$$\therefore b(4; 15, 0.3) = P(X = 4) \quad \left|\because P(X = x) = b(x; n, p)\right.$$
$$= P(X \le 4) - P(X < 4)$$
$$= P(X \le 4) - P(X \le 3) \quad \left|\because P(X \le x) = B(x; n, p)\right.$$
$$= B(4; 15, 0.3) - B(3; 15, 0.3)$$
$$= 0.515 - 0.297 \text{ (From the Appendix Table A.1)}$$
$$= 0.218$$

(c) $b(6; 15, 0.7) = P(X = 6) = P(X \le 6) - P(X \le 5)$
$$= B(6; 15, 0.7) - B(5; 15, 0.7)$$
$$= 0.015 - 0.004 = 0.011$$

(d) Given $X \sim Bin(15, 0.3)$

Here $n = 15$, $p = 0.3$

$\therefore P(2 \le X \le 4)$

$= F(4) - F(2-1)$

$= F(4) - F(1)$

$= B(4; 15, 0.3) - B(1; 15, 0.3)$

$[\because F(x) = P(X \le x) = B(x; n, p)]$

$= 0.515 - 0.035 = 0.480$

Note: For $a, b \in \mathbb{R}$, $a \le b$

$\Rightarrow P(a \le X \le b) = F(b) - F(a-)$

Where $a-$ stands for largest possible value of X that is less than a.

If a, b are integers, then

$P(a \le X \le b) = F(b) - F(a-1)$

2) $P(a < X \le b) = F(b) - F(a)$

(e) $X \sim Bin(15, 0.3) \Rightarrow n = 15$, $p = 0.3$

Also $F(x) = P(X \le x) = B(x; n, p)$

$\therefore P(X \le 1) = B(1; 15, 0.3) = 0.035$ (From Appendix Table A.1)

Now, $P(2 \le X) = P(X \ge 2)$

$= 1 - P(X < 2) = 1 - P(X \le 1)$

$= 1 - B(1; 15, 0.3)$

$= 1 - 0.035 = 0.965$

(f) $X \sim B(15, 0.7) \Rightarrow n = 15$, $p = 0.7$

$\therefore P(X \le 1) = F(1) = B(1; 15, 0.7) = 0.000$

(g) $X \sim Bin(15, 0.3) \Rightarrow n = 15$, $p = 0.3$

$\therefore P(2 < X < 6) = P(3 \le X \le 5) = F(5) - F(3-1)$

$= F(5) - F(2) = B(5; 15, 0.3) - B(2; 15, 0.3)$

$= 0.722 - 0.127 = 0.595$

Q.(48) [3.4]

When circuit boards used in the manufacture of compact disc players are tested, the long-run percentage of defectives is 5%. Let X = the number of defective boards in a random sample of size $n = 25$, so $X \sim Bin(25, 0.05)$.

a) Determine $P(X \le 2)$

b) Determine $P(X \ge 5)$

c) Determine $P(1 \le X \le 4)$

d) What is the probability that none of the 25 boards is defective ?

e) Calculate the expected value and standard deviation of X.

Ans:

$$X \sim Bin(25, 0.05) \Rightarrow n = 25, \quad p = 0.05$$

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

(a) $$B(x; n, p) = P(X \le x) = \sum_{y=0}^{x} b(y; n, p), \quad x = 0, 1, 2, \ldots, n$$

$$= \sum_{x=0}^{n} b(x; n, p)$$

$$P(X \le 2) = B(2; 25, 0.05)$$

$$= \sum_{x=0}^{2} b(x; 25, 0.05)$$

$$= b(0; 25, 0.05) + b(1; 25, 0.05) + b(2; 25, 0.05)$$

$$= 0.2774 + 0.3650 + 2305$$

$$= 0.8729$$

Alternatively $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= b(0; 25, 0.05) + b(1; 25, 0.05) + b(2; 25, 0.05)$$

(b) $P(X \geq 5) = 1 - P(X < 5)$

$= 1 - P(X \leq 4)$

$= 1 - F(4)$

$= 1 - B(4; 25, 0.05)$

$= 1 - \sum_{x=0}^{4} {}^{n}C_x (0.05)^x (0.95)^{25-x}$

$= 1 - 0.9928 = 0.0072$

(c) $P(1 \leq X \leq 4) = P(0 < X \leq 4)$

$= F(4) - F(0) = B(4; 25, 0.05) - B(0; 25, 0.05)$

$= 0.9928 - 0.2774 = 0.7154$

(d) P(all 25 boards are non defective)

$\cancel{= P(X = 25)}$

$\cancel{= F(25) - B(25; 25, 0.05)}$

$= P(X=0) = b(0; 25, 0.05) = {}^{25}C_0 (0.05)^0 (1-0.05)^{25-0}$

$= (1-0.05)^{25} = 0.2774$

(e) $E(X) = np = 25 \times 0.05 = 1.25$

$V(X) = npq = np(1-p) = 25 \times 0.05(1-0.05) = 1.1875$

$\therefore$ S.D, $\sigma = \sqrt{V(X)} = \sqrt{1.1875} = 1.0897$

Q.(50) [3.4]

A particular telephone number is used to receive both voice calls and fax messages. Suppose that 25% of the incoming calls involve messages, and consider a sample of 25 incoming calls. What is the probability that

a) At most 6 of the calls involve a fax message?

b) Exactly 6 of the calls involve a fax message?

c) At least 6 of the calls involve a fax message?

d) More than 6 of the calls involve a fax message?

Ans: Let X be the r.v. which counts incoming calls involve fax message.

Then $X \sim Bin(25, 0.25)$ i.e, $n=25$, $p=25\% = 0.25$

$\Rightarrow 1-p = 0.75$

(a) $P(X \le 6) = B(6; 25, 0.25)$

$= P(X=0) + P(X=1) + \cdots + P(X=6)$

$= \sum_{x=0}^{6} {}^{25}C_x (0.25)^x (0.75)^{25-x}$

$[\because P(X=x) = {}^nC_x p^x q^{n-x}, x=0,1,2,\ldots,n]$

$= 0.5611$

(b) $P(X=6) = {}^{25}C_6 (0.25)^6 (0.75)^{25-6}$

$= 0.1828$

(c) $P(X \geq 6) = 1 - P(X < 6)$

$= 1 - \{P(X=0) + P(X=1) + \cdots + P(X=5)\}$

$= 1 - \{P(X=0) + P(X=1) + \cdots + P(X=6) - P(X=6)\}$

$= 1 - \{P(X \leq 6) - P(X=6)\}$

$= 1 - (0.5611 - 0.1828) = 0.6218$

(d) $P(X > 6) = 1 - P(X \leq 6)$

$= 1 - 0.5611 = 0.4389$

Q. (63) [3.4]

a) Show that $b(x; n, 1-p) = b(n-x; n, p)$

b) Show that $B(x; n, 1-p) = 1 - B(n-x-1; n, p)$

Ans: (a) we know that $b(x; n, p) = {}^{n}C_{x}\, p^{x} (1-p)^{n-x}$

$\therefore b(x; n, 1-p) = {}^{n}C_{x} (1-p)^{x} p^{n-x}$

$= \dfrac{n!}{x!\,(n-x)!} (1-p)^{x} p^{n-x}$

$= \dfrac{n!}{(n-x)!\,[n-(n-x)]!} p^{n-x} (1-p)^{x}$

$= {}^{n}C_{n-x}\, p^{n-x} (1-p)^{n-(n-x)}$

$\Rightarrow b(x; n, 1-p) = b(n-x; n, p)$ (proved)

(b) We know that cdf of binomial distribution is

$$B(x; n, p) = P(X \le x) = \sum_{y=0}^{x} b(y; n, p), \quad x = 0, 1, 2, \cdots, n.$$

where, $P(X \le x)$ means at most x successes.

The following is true

$$B(x; n, 1-p) = \sum_{y=0}^{x} b(y; n, 1-p) \longrightarrow (1)$$

Also $\sum_{y=0}^{n} b(y; n, 1-p) = 1$ $\quad$ [$\because \sum_{x} p(x) = 1$, if $p(x) = $ p.m.f]

$$\Rightarrow \sum_{y=0}^{x} b(y; n, 1-p) + \sum_{y=x+1}^{n} b(y; n, 1-p) = 1$$

$$\Rightarrow 1 - \sum_{y=0}^{x} b(y; n, 1-p) = \sum_{y=x+1}^{n} b(y; n, 1-p) \longrightarrow (2)$$

Now, $P(X > x) = 1 - P(X \le x)$

$$= 1 - \sum_{y=0}^{x} b(y; n, 1-p)$$

$$= \sum_{y=x+1}^{n} b(y; n, 1-p) \longrightarrow (3) \text{ [by (2)]}$$

$$= \sum_{y=x+1}^{n} b(n-y; n, p)$$

[$\because$ from result of (a) we have

$$b(y; n, 1-p) = b(n-y; n, p)$$

$$\therefore P(X > x) = \sum_{y=x+1}^{n} b(n-y; n, p)$$

$$= b(n-x-1; n, p) + b(n-x-2; n, p) + b(n-x-3; n, p) + \cdots + b(1; n, p) + b(0; n, p)$$

$$= b(0; n, p) + b(1; n, p) + \cdots + b(n-x-1; n, p)$$

$$\Rightarrow P(X > x) = \sum_{y=0}^{n-x-1} b(y; n, p) = B(n-x-1; n, p) \longrightarrow (4)$$

$$\left[\because B(x; n, p) = \sum_{y=0}^{x} b(y; n, p)\right]$$

$$\therefore B(x; n, 1-p) = P(X \leq x)$$

$$= 1 - P(X > x)$$

$$= 1 - B(n-x-1; n, p) \quad \text{(proved)}$$

[Using Eq.(4)]

Q. (64) [3.4]

Show that $E(X) = np$ when X is a binomial random variable.

Proof: Try yourself.

Q. (67), [Q. (59) - Book (print)] [3.4]

A result called chebyshev's inequality states that for any probability distribution of an r.v. X and any number K that is at least 1, $P(|X-\mu| \geq K\sigma) \leq \frac{1}{K^2}$. In otherwords, the probability that the value of X lies at least K standard deviations from its mean is at most $1/K^2$.

Calculate $P(|X-\mu| \geq K\sigma)$ for $K=2$ and $K=3$ when $X \sim Bin(20, 0.5)$, and compare to the corresponding upper bound. Repeat for $X \sim Bin(20, 0.75)$.

Ans: Given $X \sim Bin(20, 0.5)$

$$\Rightarrow n = 20, \quad p = 0.5$$

$$\therefore E(X) = np = 20 \times 0.5 = 10$$

$$V(X) = np(1-p) = 20 \times 0.5 \times 0.5 = 5$$

$$\sigma(X) = \sqrt{V(X)} = \sqrt{5} = 2.236$$

For $K=2$

$$K\sigma = 2\sigma = 2 \times 2.236 = 4.472$$

Now,

$$|X - \mu| \geq K\sigma$$

$$\Rightarrow |X - 10| \geq 4.472 \Rightarrow X - 10 \geq 4.472$$

$$\text{or } X - 10 \leq -4.472$$

$$\cancel{\Rightarrow -4.472 \leq X - 10 \leq 4.472}$$

$$\cancel{\Rightarrow 14.472 \leq X \leq 5.528}$$

$$\Rightarrow X \geq 14.472 \text{ or } X \leq 5.528$$

$$\Rightarrow X \geq 15 \text{ or } X \leq 5$$

$\therefore \cancel{P(|X-10| \geq 4.472)}$

$= \cancel{P(14.472 \leq X \leq 5.528)}$

$= P(15 \leq X \leq 5)$

$= F(5) + F(15-1)$

$= F(5) + F(14)$

$\because$ x has binomial distribution
$\Rightarrow x = 0, 1, 2, 3, \dashrightarrow n.$
i.e. x can takes non-negative integer

$= B(5; 20, 0.5) + B(14; 20, 0.5)$

$= \sum_{y=0}^{5} b(y; 20, 0.5) + \sum_{y=0}^{14} b(y; 20, 0.5)$

$= \sum_{y=0}^{5} {}^{20}C_y (0.5)^y (0.5)^{20-y} + \sum_{y=0}^{14} {}^{20}C_y (0.5)^y (1-0.5)^{20-y}$

$= 0.021 + 0.979$

$\therefore P(|X-10| \geq 4.472)$

$= P(X \leq 5 \text{ or } X \geq 15)$

$= P(X \leq 5) + P(X \geq 15)$

$= P(X \leq 5) + 1 - P(X < 15)$

$= P(X \leq 5) + 1 - P(X \leq 14)$

$= B(5; 20, 0.5) + 1 - B(14; 20, 0.5)$

$= \sum_{y=0}^{5} b(y; 20, 0.5) + 1 - \sum_{y=0}^{14} b(y; 20, 0.5)$

$= 0.021 + 1 - 0.979$

$= 0.042$

Next try yourself for K=3.