Solution of System of Linear Equations

(Gauss-Jacobi and Gauss-Seidel methods)

Theorem (Diagonally Dominant):

- A square matrix $\boldsymbol{A}$ is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|\,,\ i = 1,2,\dots,n.$$

Where, a_{ij} denotes the entry in the ith row and jth column of the matrix.

- A strictly diagonal dominant matrix (or an irreducibly diagonal dominant matrix) is non-singular.
- The Gauss-Jacobi and Gauss-Seidel methods for solving a linear system of equations converge if the matrix is strictly diagonally dominant. It converges for any initial approximation $\boldsymbol{x}^{(\mathbf{0})}$. Generally, $\boldsymbol{x}^{(\mathbf{0})} = \mathbf{0}$ is taken in the absence of any better initial approximation.

Iterative Methods:

- To solve the system of linear equations, we use two types of method. i.e., Direct method and Iterative method. Direct method includes Gauss-Elimination method and Gauss-Jordan method.
- Iterative method includes Gauss-Jacobi method and Gauss-Seidel method.

(1) Gauss-Jacobi Method:

- This is an iterative method and also known as "Method of simultaneous displacement".
- Here each diagonal element is solved and an approximate value is obtained. The process is then iterated until it converges.

Algorithm for Gauss-Jacobi method:

Step-1:

Consider a square matrix $\boldsymbol{A}$ of n-linear system of equations as $\boldsymbol{Ax} = \boldsymbol{b}.$

Where, $\boldsymbol{A} = [a_{ij}]_{n\times n}$,

$\boldsymbol{x} = [x_1\ \ x_2\ \ x_3\ \dots\ x_n]^T$ and

$\boldsymbol{b} = [b_1\ \ b_2\ \ b_3\ \dots\ b_n]^T.$

And the diagonal elements $a_{ii} \neq 0$ for $i = 1,2,\dots,n$. If any of $a_{ii} = 0\ \forall\ i$, then rearrange the above system of equations in such a way that the above conditions hold.

Step-2:

Rewrite the system of equations as,

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n].$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n].$$

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$x_{n-1} = \frac{1}{a_{n-1,n-1}}\left[b_{n-1} - a_{n-1,1}x_1 - a_{n-2,2}x_2 - \ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n\right].$

$x_n = \frac{1}{a_{nn}}[b_1 - a_{n1}x_1 - a_{n2}x_2 - \ldots - a_{n,n-1}x_{n-1}].$

In general, $x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j\right], i = 1,2,\ldots,n,\ a_{ii} \neq 0.$

Step-3:

Generate the iteration scheme $\boldsymbol{x}^{(\boldsymbol{k+1})}$ from $\boldsymbol{x}^{(\boldsymbol{k})}$ for $k \geq 0$ as

$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right], i = 1,2,\ldots,n$ and $a_{ii} \neq 0\ \forall\ i.$

Example 1:

Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by

$10x_1 - x_2 + 2x_3 = 6$

$-x_1 + 11x_2 - x_3 + 3x_4 = 25$

$2x_1 - x_2 + \mathbf{10}x_3 - x_4 = -\mathbf{11}$

$\mathbf{3}x_2 - x_3 + \mathbf{8}x_4 = \mathbf{15}$

by Gauss-Jacobi method rounded up to four decimal places.

Solution:

Here, $\boldsymbol{A}$ is strictly diagonally dominant since $10 > 1 + 2$, $11 > 1 + 1 + 3$, $10 > 2 + 1 + 1$ and $8 > 3 + 1$.

Now letting $\boldsymbol{x}^{(\mathbf{0})} = [0\ \ 0\ \ 0\ \ 0]^T$, we get

$\boldsymbol{x}^{(\mathbf{1})} = [0.6000 \quad 2.2727 \quad -1.1000 \quad 1.8750]^T$

$\boldsymbol{x}^{(\mathbf{2})} = [1.0473 \quad 1.7159 \quad -0.8052 \quad 0.8852]^T$ and

$\boldsymbol{x}^{(\mathbf{3})} = [0.9326 \quad 2.0533 \quad -1.0493 \quad 1.1309]^T$.

Proceeding similarly one can obtain,

$\boldsymbol{x}^{(\mathbf{5})} = [0.9890 \quad 2.0114 \quad -1.0103 \quad 1.0214]^T$ and

$\boldsymbol{x}^{(\mathbf{10})} = [1.0001 \quad 1.9998 \quad -0.9998 \quad 0.9998]^T$.

The solution is $\boldsymbol{x} = [1 \quad 2 \quad -1 \quad 1]^T$. You may note that $\boldsymbol{x}^{(\mathbf{10})}$ is a good approximation to the exact solution compared to $\boldsymbol{x}^{(\mathbf{5})}$.

(2) Gauss-Seidel Method:

- This is the modification of Gauss-Jacobi iteration method.
- This method is also known as “Method of successive displacement” since one must use the recent guesses to do the iterations.

Algorithm for Gauss-Jacobi method:

Step-1:

Consider a square matrix $\boldsymbol{A}$ of n-linear equations and n-unknowns as $\boldsymbol{Ax} = \boldsymbol{b}.$

Where, $\boldsymbol{A} = [a_{ij}]_{n\times n}$,

$\boldsymbol{x} = [x_1 \;\; x_2 \;\; x_3 \; \dots \; x_n]^T$ and

$\boldsymbol{b} = [b_1 \;\; b_2 \;\; b_3 \; \dots \; b_n]^T$.

i.e.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$\begin{array}{ccccc} . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{array}$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

And the diagonal elements $a_{ii} \neq 0$ for $i = 1,2,\dots,n$.

Step-2:

If any of $a_{ii} = 0 \; \forall \; i$, then rearrange the above system of equations in such a way that the above conditions hold. i.e. the first equation is rewritten with x_1 on the left-hand side and the second equation is rewritten with x_2 on the left-hand side and so on. i.e.,

Rewrite the system of equations as,

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n].$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n].$$

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$$x_{n-1} = \frac{1}{a_{n-1,n-1}}\left[b_{n-1} - a_{n-1,1}x_1 - a_{n-2,2}x_2 - \dots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n\right].$$

$$x_n = \frac{1}{a_{nn}}[b_1 - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{nn-1}x_{n-1}].$$

In general, $x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j\right], i = 1,2,\dots,n, \;\; a_{ii} \neq 0.$

Step-3:

Generate the iteration scheme $\boldsymbol{x}^{(\boldsymbol{k}+\boldsymbol{1})}$ from $\boldsymbol{x}^{(\boldsymbol{k})}$ for $k \geq 0$ as

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{\substack{j=1 \\ j\neq i}}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{\substack{j=i+1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)}\right], i = 1,2,\dots,n \text{ and } a_{ii} \neq 0 \; \forall \; i.$$

Step-4:

Now to find x_i's, one must assume an initial guess for the x_i's and then use the rewritten equations to calculate the new guesses. Remember, one always uses the most recent guesses to calculate x_i.

Step-5:

At the end of each iteration, one calculates the absolute relative approximate error for each x_i as

$$|\varepsilon_a|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| * 100$$

Where x_i^{new} is the recently obtained value of x_i , and x_i^{old} is the previous value of x_i.

When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

Example 1:

Solve the given system of equations by Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + \mathbf{7}x_2 + \mathbf{13}x_3 = \mathbf{76}$$

Given $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as the initial guess.

Solution:

The coefficient matrix, $\boldsymbol{A} = \begin{bmatrix} \mathbf{12} & \mathbf{3} & \mathbf{-5} \\ \mathbf{1} & \mathbf{5} & \mathbf{3} \\ \mathbf{3} & \mathbf{7} & \mathbf{13} \end{bmatrix}$ is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

And the inequality is strictly greater than for at least one row. Hence the solution should converge using Gauss-Seidel method.

Rewriting the equations, we get,

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Given initial guess is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Iteration 1:

$$x_1 = \frac{1-3(0)+5(1)}{12} = 0.5000$$

$$x_2 = \frac{28-(0.5)-3(1)}{5} = 4.9000$$

$$x_3 = \frac{76-3(0.5000)-7(4.9000)}{13} = 3.0923$$

The absolute relative approximate error at the end of first iteration is

$$|\varepsilon_a|_1 = \left|\frac{0.5000-1.0000}{0.5000}\right| * 100 = 67.662\%$$

$$|\varepsilon_a|_2 = \left|\frac{4.9000-0}{4.9000}\right| * 100 = 100.000\%$$

$$|\varepsilon_a|_3 = \left|\frac{3.0923-1.0000}{3.0923}\right| * 100 = 67.662\%$$

The maximum absolute relative approximate error is 100.000%.

Iteration 2:

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(4.9000)}{13} = 3.8118$$

The absolute relative approximate error at the end of second iteration is

$$|\varepsilon_a|_1 = \left|\frac{0.14679-0.5000}{0.14679}\right| * 100 = 240.62\%$$

$$|\varepsilon_a|_2 = \left|\frac{3.7153-4.9000}{3.7153}\right| * 100 = 31.887\%$$

$$|\varepsilon_a|_3 = \left|\frac{3.8118-3.0923}{3.8118}\right| * 100 = 18.876\%$$

The maximum absolute relative approximate error is 240.62%. This is greater than the value of 67.612% we obtained in the first iteration. As we conduct more iterations, the solution converges as follows.

Iteration	a_1	$\|\varepsilon_a\|_1$	a_2	$\|\varepsilon_a\|_1$	a_3	$\|\varepsilon_a\|_1$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

Thus, the exact solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

Q. Why Gauss-Seidel method is more preferable than Gauss-Jacobi method?

Ans: The convergence rate of Gauss-Seidel method is faster than Gauss-Jacobi method as the rate of convergence of Gauss-Seidel method is almost twice than Gauss-Jacobi method. Hence, Gauss-Seidel method is more preferable than Gauss-Jacobi method.