Random variable:

A random variable X on a sample space S is a function that assigns a real number to each sample point of the sample space S. i.e., $X: S \to \mathbb{R}$.

As an example, consider that a balanced coin is tossed three times. The sample space will be

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Now if we assign to each outcome a number X representing number of heads, then the values taken by the random variable X will be as follows:

Elements in the sample space	No. of heads X
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

A random variable is of two types: Discrete random variable and Continuous random variable.

- Discrete random variable
- Continuous random variable

Discrete random variable:

If a random variable is allowed to take on only limited number of values, which can be listed, it is called discrete random variable. In other words, a discrete random variable is a random variable whose possible values are either constitute a finite set or countably infinite.

Example 1.

Describe the sample space and random variable corresponding to the number of heads obtained in four tosses of a coin.

Solution:

Let S be the sample space. Then

$$S = \{HHHH, HHHT, THTT, HTTT, THHH, TTHH, HTHH, HHTH, THHT, TTTH,$$
$$HTTH, HTHT, THTH, HHTT, TTHT, TTTT\}.$$

$|S| = 16$.

Let X be the random variable corresponding to the number of heads obtained in four tosses of a coin.

So, $X = \{0,1,2,3,4\}$.

Example 2.

Suppose two dice are rolled and X be a random variable representing their sum. Find all possible values of the random variable.

Solution:

Let S be the sample space. Then

$$S = \{(1,1),(1,2),\cdots,(1,6),(2,1),(2,2),\cdots,(2,6),(3,1),(3,2),\cdots,(3,6),(4,1),(4,2),\cdots,(4,6),\\ (5,1),(5,2),\cdots,(5,6),(6,1),(6,2),\cdots,(6,6)\}$$

$$|S| = 36.$$

and $X = \{2,3,4,5,6,7,8,9,10,11,12\}$.

Example 3.

For each discrete random variable defined here, describe the set of possible values for the variable.

a) $X =$ the number of unbroken eggs in a randomly chosen standard egg carton.
b) $Y =$ the number of students on a class list for a particular course who are absent on the
first day of classes.
c) $Z =$ the number of times a duffer has to swing at a golf ball before hitting it.
d) $U =$ the amount of royalties earned from the sale of a first edition of 10,000 textbooks.

Solution:

a) A standard egg cartoon contains 12 eggs and thus the number of unbroken eggs in the cartoon can be integer from 0 to 12.
So, $X = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$.
b) $Y = \{0,1,2,\cdots,N\}$, where N is the number of students in the list.
c) $Z = \{0,1,2,3,\cdots\}$ is a countably infinite set.
d) $U = \{0,c,2c,3c,\cdots,10000c\}$, where c is the royalty per book.

Probability distribution:

The values of the random variable X together with their associated probability define the probability distribution of the random variable X.

Example 4.

A coin is tossed 2 times and X is a random variable representing No. of heads. Find probability distribution of X.

Solution:

No. of heads X can be 0, 1 or, 2. i.e., $X = \{0,1,2\}$.

X	0	1	2
$P(X)$	1/4	2/4	1/4

Example 5.

Suppose two dice are rolled and X be a random variable representing their sum. Find Probability distribution of X.

Solution:

Here, $X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Discrete Probability Distribution or Probability Mass Function (pmf):

Let X be discrete random variable, associated with a sample space S. Let $R_X = \{x_1, x_2, \cdots, x_n\}$ be the range space of X and each x_i is associated with a number $f_X(x_i) = P(X = x_i)$, then the number $f_X(x_i), i = 1, 2, \cdots, n$ satisfies the following properties:

1. $f_X(x_i) \geq 0$,
2. $\sum_{x_i \in R_X} f_X(x_i) = 1$,

is called probability mass function (pmf) of X.

Example 6.

Let X be the random variable corresponding to the number of heads obtained in four tosses of a coin. Find the pmf.

Solution:

Here, $X = \{0, 1, 2, 3, 4\}$.

Now, $f_X(0) = P(X = 0)$

$f_X(0) = P(X = 0)$ = Prob. of getting no head in four tosses of coin = 1/16.

$f_X(1) = P(X = 1)$ = Prob. of getting exactly one head in four tosses of a coin= 4/16 =1/4

$f_X(2) = P(X = 2)$ = Prob. of getting exactly two heads in four tosses of a coin= 6/16 =3/8

$f_X(3) = P(X = 3)$ = Prob. of getting exactly three heads in four tosses of a coin= 4/16 =1/4

$f_X(4) = P(X = 4)$ = Prob. of getting exactly four heads in four tosses of a coin= 1/16

X	0	1	2	3	4
$f(X)$	1/16	1/4	3/8	1/4	1/16

Example 7.

A random variable X has the following pmf:

X	-2	-1	0	1	2	3
$f(X)$	0.1	k	0.2	$2k$	0.3	k

Find the value of k.

Solution:

We know that, $\sum f(X) = 1$

$$\Rightarrow 0.1 + k + 0.2 + 2k + 0.3 + k = 1$$
$$\Rightarrow 4k + 0.6 = 1$$
$$\Rightarrow 4k = 1 - 0.6 = 0.4$$
$$\Rightarrow k = 0.4/4 = 0.1$$

Example 8.

Determine the value of c so that the following function can serve as a probability distribution of the discrete random variable X:

$$f(x) = c(x^2 + 4) \text{ for } x = 0,1,2,3.$$

Solution:

X	0	1	2	3
$f(X)$	$4c$	$5c$	$8c$	$13c$

We know that, $\sum f(X) = 1$

$$\Rightarrow 4c + 5c + 8c + 13c = 1$$
$$\Rightarrow 30c = 1$$
$$\Rightarrow c = 1/30$$

Cumulative Distribution Function (cdf):

The cumulative distribution function $F(x)$ of a discrete random variable with probability mass function $f(x)$ is defined for every number x by

$$F(x) = P(X \leq x) = \sum_{y:y \leq x} f(y)$$

where x = observed value/ values from range space.

Example 9.

In Example 6, find the cdf.

Solution:

$$F(0) = P(X \le 0) = \frac{1}{16}$$

$$F(1) = P(X \le 1) = f(0) + f(1) = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$

$$F(2) = P(X \le 2) = f(0) + f(1) + f(2) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} = \frac{11}{16}$$

$$F(3) = P(X \le 3) = f(0) + f(1) + f(2) + f(3) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} + \frac{1}{4} = \frac{15}{16}$$

$$F(4) = P(X \le 4) = f(0) + f(1) + f(2) + f(3) + f(4) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} + \frac{1}{4} + \frac{1}{16} = 1$$

$$F(x) = \begin{cases} 0, & if\ x < 0 \\ \frac{1}{16}, & if\ 0 \le x < 1 \\ \frac{5}{16}, & if\ 1 \le x < 2 \\ \frac{11}{16}, & if\ 2 \le x < 3 \\ \frac{15}{16}, & if\ 3 \le x < 4 \\ 1, & if\ x \ge 4 \end{cases}$$

Properties of cdf:

1. $F_X(\infty) = P(-\infty < x \le \infty) = 1$
2. $F(-\infty) = 0$
3. $0 \le F_X \le 1$
4. cdf is a non-decreasing function, that is, either it increases or remain constant.
5. $P(X > x) = 1 - P(X \le x)$

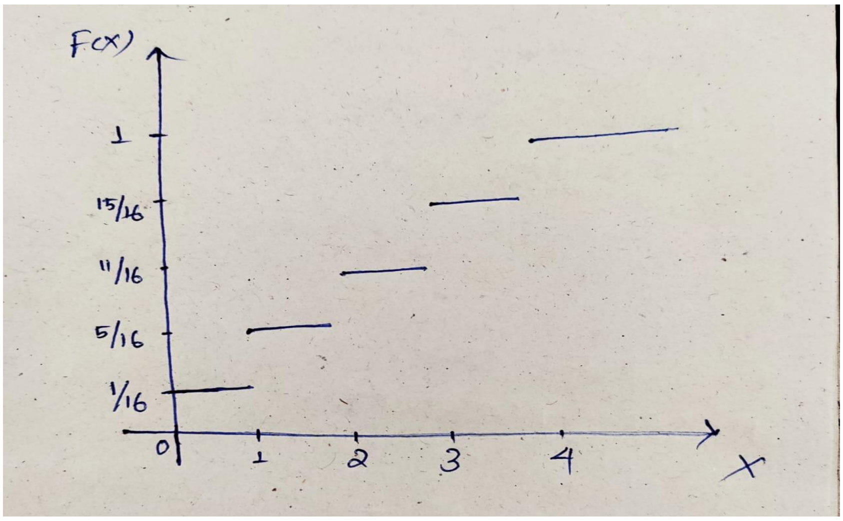
Interval properties of cdf (a<b):

1. $P(a < X \le b) = F(b) - F(a)$
2. $P(a < X \le b) = \sum_{a < x_j \le b} f(x_j)$
3. $\sum_j f(x_j) = 1$
4. $P(a < X < b) = P(a < X < b) + P(X = b) - P(X = b)$
   $= P(a < X \le b) - P(X = b)$
   $= F(b) - F(a) - P(X = b)$
5. $P(a \le X < b) = P(a < X < b) + P(X = a)$

$$= F(b) - F(a) - P(X = b) + P(X = a)$$

6. $P(a \le X \le b) = F(b) - F(a-1)$

- Plotting of the cdf of the previous example (Example 9)



Example 10.

Airlines sometimes overbook flights. Suppose that for a plane with 50 seats, 55 passengers have tickets. Define the random variable X as the number of ticketed passengers who show up for the flight. The probability mass function of X appears in the accompanying table.

x	45	45	47	48	49	50	51	52	53	54	55
$f(x)$	0.05	0.1	0.12	0.14	0.25	0.17	0.06	0.05	0.03	0.02	0.01

a) What is the probability that the flight will accommodate all ticketed passengers who show up?
b) What is the probability that the flight will accommodate all ticketed passengers who show up?
c) If you are the first person on the standby list (which means you will be the first one to get on the plane if there are any seats available after all ticketed passengers have been accommodated), what is the probability that you will be able to take the flight? and what is this probability if you are the third person on the standby list?

Solution:

a) As there are 50 seats the probability that the flight will accommodate all ticketed passengers who show up is

$$P(X \le 50) = f(45) + f(46) + f(47) + f(48) + f(49) + f(50)$$

$$= 0.05 + 0.1 + 0.12 + 0.14 + 0.25 + 0.17 = 0.83$$

b) The probability that not all ticketed passengers who show up can be accommodated is
$$P(X > 50) = 1 - P(X \le 50) = 1 - 0.83 = 0.17$$

c) Suppose you are the first person on the stand by list since there are 50 seats available at most 49 can show up for you to get the seat.
So, $P(X \le 49) = f(45) + f(46) + f(47) + f(48) + f(49) = 0.66$
Again, suppose you are the third person on the stand by list. At most 47 can show up for you to get a seat.
So, $P(X \le 47) = f(45) + f(46) + f(47) = 0.27$

Example 11.

A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose the pmf of X is as given in the accompanying table.

x	0	1	2	3	4	5	6
$f(x)$	0.1	0.15	0.2	0.25	0.2	0.06	0.04

Calculate the probability of each of the following events.

a) At most three lines are in use.
b) Fewer than three lines are in use.
c) At least three lines are in use.
d) Between two and five lines, inclusive, are in use.
e) Between two and four lines, inclusive, are not in use.
f) At least four lines are not in use.

Solution:

a) P {at most three lines are in use} = $P(X \le 3) = f(0) + f(1) + f(2) + f(3) = 0.7$.
b) P {fewer than three lines are in use} = $P(X < 3) = f(0) + f(1) + f(2) = 0.45$.
c) P {at least three lines are in use} = $P(X \ge 3) = 1 - P(X < 3) = 1 - 0.45 = 0.55$.
d) P {between two and five lines, inclusive, are in use} = $P(2 \le X \le 5)$
$$= f(2) + f(3) + f(4) + f(5) = 0.71.$$
e) P {between two and four lines, inclusive, are not in use}
If X is the number of lines in use then $6-X$ is the number of lines not in use. So here we must find $P(2 \le 6 - X \le 4) = P(-4 \le X \le -2) = P(2 \le X \le 4)$
$$= f(2) + f(3) + f(4) = 0.65.$$
f) P {at least four lines are not in use} = $P(6 - X \ge 4) = P(-X \ge -2)$
$$= P(X \le 2) = f(0) + f(1) + f(2) = 0.45.$$

Example 12.

Show that the cdf $F(x)$ is a non-decreasing function that is $x_1 < x_2$ implies $F(x_1) < F(x_2)$. Under what condition will $F(x_1) = F(x_2)$.

Solution:

$$F(x_2) = P(X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$$

$$= F(x_1) + P(x_1 < X \le x_2)$$

$$\Rightarrow F(x_2) \ge F(x_1)$$

If $P(x_1 < X \le x_2) = 0$ then $F(x_1) = F(x_2)$.

Example 13.

A consumer organization that evaluates new automobiles customarily reports the number of major defects in each car examined. Let X denote the number of major defects in a randomly selected car of a certain type. The cdf of X is as follows:

$$F(x) = \begin{cases} 0 & x < 0 \\ .06 & 0 \le x < 1 \\ .19 & 1 \le x < 2 \\ .39 & 2 \le x < 3 \\ .67 & 3 \le x < 4 \\ .92 & 4 \le x < 5 \\ .97 & 5 \le x < 6 \\ 1 & 6 \le x \end{cases}$$

Calculate the following probabilities directly from the cdf:

a) $P(X = 2)$
b) $P(X > 3)$
c) $P(2 \le X \le 5)$
d) $P(2 < X < 5)$

Solution:

a) $P(X = 2) = P(2 \le X \le 2) = F(2) - F(1) = 0.39 - 0.19 = 0.2$.
b) $P(X > 3) = 1 - P(X \le 3) = 1 - F(3) = 1 - 0.67 = 0.33$.
c) $P(2 \le X \le 5) = F(5) - F(1) = 0.97 - 0.19 = 0.78$.
d) $P(2 < X < 5) = F(5) - F(2) - P(X = 5) = F(5) - F(2) - P(5 \le X \le 5)$

$$= F(5) - F(2) - (F(5) - F(4))$$
$$= F(5) - F(2) - F(5) + F(4)$$
$$= F(4) - F(2) = 0.92 - 0.39 = 0.53.$$

Expected values of a random variable (Mean):

Let X be a discrete random variable with pmf $f_X(x)$. The expected value or mean value of X, denoted by $E(X)$ or μ_x or μ is given by

$$E(X) = \sum_j x_j f_X(x_j).$$

Example 14.

Find the expected value of X, where X represents the outcome when a die is tossed.

Solution:

The probability distribution of X is:

x	1	2	3	4	5	6
$f(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Thus, the expected value of X is

$$E(X)=\sum_j x_j f(x_j)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}=\frac{7}{2}=3.5.$$

Properties of Expectation:

1. $E(a)=a$, for any constant a.
   Proof: $\sum_j af(x_j)=a\sum_j f(x_j)=a.1=a$.
2. For any constant b, $E(X+b)=E(X)+b$.
3. For any constant a and b, $E(aX+b)=aE(X)+b$.

Variance of a random variable:

Let X have pmf $f_X(x)$ and expected value μ. Then the variance of X, denoted by $V(X)$ or σ_x^2 or σ^2 is given by

$$V(X)=E[(x-\mu)^2]=\sum_x (x-\mu)^2 f(x).$$

Standard deviation of a random variable:

The standard deviation (SD) of a random variable X is the positive square root of variance and it is denoted by σ.

That is, $\sigma=+\sqrt{\sigma^2}$.

- Let X be a random variable, then variance of X is denoted by

$$\begin{aligned}V(X)&=E[(X-\mu)^2]\\&=E[(X^2-2\mu X+\mu^2)]\\&=E(X^2)-2\mu E(X)+\mu^2E(1)\\&=E(X^2)-2\mu\mu+\mu^2=E(X^2)-2\mu^2+\mu^2\\&=E(X^2)-[E(X)]^2\end{aligned}$$

$$\Rightarrow V(X)=E(X^2)-[E(X)]^2$$

Properties of variance:

1. $V(a)=E(a^2)-[E(a)]^2=a^2-(a)^2=0$.

2. $V(aX+b) = E((aX+b)^2) - [E(aX+b)]^2$

$$= E(a^2X^2) + E(b^2) + 2E(abX) - [aE(X)+b]^2$$
$$= a^2E(X^2) + b^2 + 2abE(X) - a^2[E(X)]^2 - 2abE(x) - b^2$$
$$= a^2E(X^2) - a^2[E(X)]^2$$
$$= a^2[E(X^2) - [E(X)]^2] = a^2V(X)$$

$\Rightarrow V(aX+b) = a^2V(X)$.

Example 15.

If three coins are tossed, find the mean, variance, and standard deviation of number of heads.

Solution:

Let X be a random variable representing the number of heads obtained in a random throw of three coins, then the probability distribution of X is

x	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

Mean number of heads $= \mu = E[X] = \sum_x xf(x) = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8} = \frac{3}{2} = 1.5$.

$$E[X^2] = \sum_x x^2 f(x) = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 4\times\frac{3}{8} + 9\times\frac{1}{8} = \frac{24}{3} = 3.$$

Variance $= \sigma^2 = E[X^2] - [E[X]]^2 = 3 - (1.5)^2 = 3 - 2.25 = 0.75$.

Standard deviation (SD) $= \sigma = +\sqrt{\sigma^2} = +\sqrt{0.75} = 0.86$.

Moment generating function for discrete random variable:

- *Moment about origin*: The k^{th} moment about origin of a random variable X is denoted by $E[X^k]$ and defined as $E[X^k] = \sum_j x_j^k f_X(x_j)$.
- *Moment about mean*: The k^{th} moment about mean of a random variable X is denoted by $E[(X-\mu)^k]$ and defined as $E[(X-\mu)^k] = \sum_j (x_j-\mu)^k f_X(x_j)$.

The moment generating function (MGF) of the random variable X is denoted by $M_X(t)$ and defined as

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} f_X(x).$$

If X is discrete random variable with pmf $f_X(x)$, we call $M_X(t)$ the moment generating function because all of the moments of X can be obtained by successively differentiating $M_X(t)$ and then evaluating the result at $t = 0$. For example,

$$M'_X(t) = \frac{d}{dt}E(e^{tX}) = E\left(\frac{d}{dt}(e^{tX})\right) = E(Xe^{tX}).$$

For $t = 0$, $M'_X(0) = E(X)$.

Similarly, $$M''_X(t) = \frac{d}{dt}(M'_X(t)) = \frac{d}{dt}(Xe^{tX}) = E(X^2e^{tX}).$$

For $t = 0$, $M''_X(0) = E(X^2)$.

In general, the n^{th} derivative of $M_X(t)$ is given by

$$M_X^{(n)}(t) = E(X^n e^{tX}).$$

and for $t = 0$, $M_X^{(n)}(0) = E(X^n)$.