Coupled Oscillation

In earlier sections we discussed about different types of oscillations (free and forced) of individual isolated oscillators. The oscillation of these isolated oscillators is modeled with one degree of freedom. However, such isolated oscillators rarely exist in reality. Nearly all oscillating systems have more than one component which are interconnected. *The vibration of such a system depends on the components and the manner in which these components are coupled to one another. If any part of the coupled system is set into vibrations, its vibration energy is transmitted to other parts because of their coupling.* Then we require more number of degrees of freedom to describe oscillation of such an interconnected multi-component system. Such a system vibrates in different modes, and the frequencies of different modes of vibration of the system depend on the properties of the individual parts as well as the manner and strength of their coupling with other parts.

Examples:

Oscillation of:

1. Two pendulums connected by a light spring
2. The prongs of a tuning fork
3. Two oscillatory electrical circuits coupled inductively through mutual inductance.
4. Atoms and ions in molecules or in solids
5. Continuous systems such as vibrating strings and membranes
6. Acoustic and electric cavity oscillators

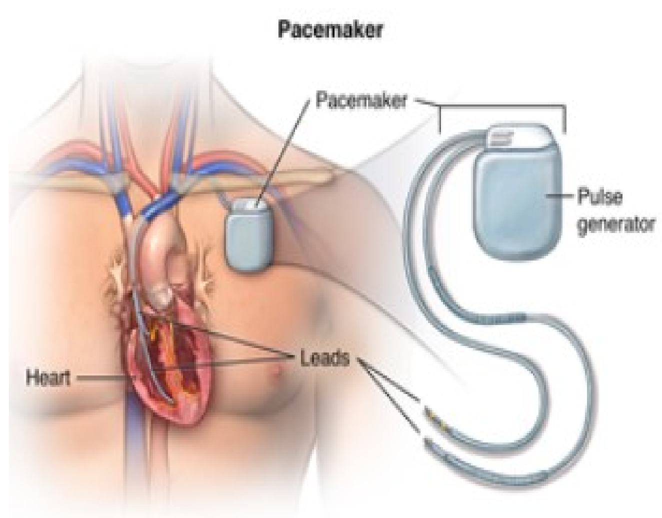
Three Practical applications in engineering:

1. In microwave communication, in order to steer a beam in a desired direction it is necessary to feed the antenna elements with a constant phase shift distribution. Traditionally, phase shifters and power distribution networks were used to feed phased antenna arrays from the source. These feeding network and phase shifters (or phasers) are simply bulky and too lossy at high

frequencies, thus making communication at high frequencies impossible. Communication at high frequencies is possible now because of phased array antenna architecture using coupled oscillators. Coupled oscillator chains are used for feeding antenna arrays and allowing for beam steering or pattern nulling. This not only reduces the size and the cost (by eliminating system of phase shifters and power distribution networks), but also possibility of direct delivery of power to each antenna makes the system compact and highly efficient even if a small amount of power is used for coupling and synchronization purposes. These Coupled oscillator arrays have intrinsic synchronization properties that make them suitable for a wide range of microwave and radiofrequency applications.

2. A tuning fork is an acoustic resonator in the form of a two-pronged fork with the prongs formed from a U-shaped bar of elastic metal. When the prong is stroke against a surface, coupled vibration is set up in it. Vibrations of different modes set up in tuning fork initially, but little of the energy of vibration goes into the overtone modes. These overtone modes die out faster, leaving a pure sine wave at the fundamental frequency. Finally, the prongs and base oscillates with the constant fundamental frequency. Hence then it is used to tune other instruments with this pure tone.

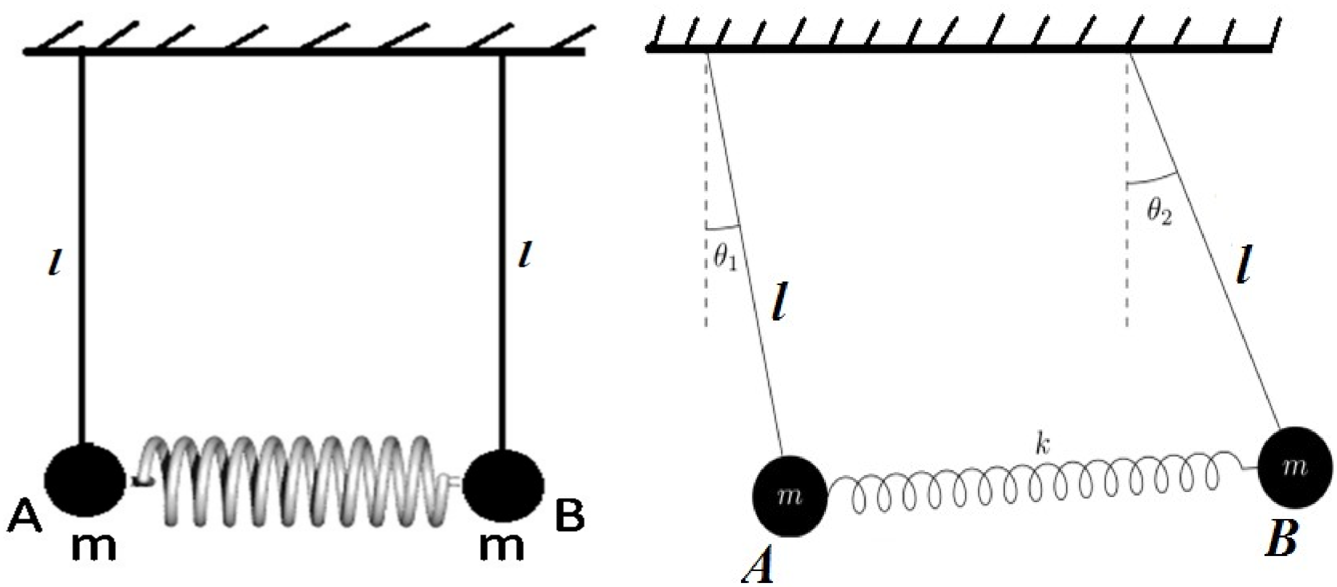
3. Another important application of coupled oscillation can be found in pacemaker, a man-made device. A pacemaker is a small electronic device that is usually placed in the chest (just below the collarbone) to help regulate slow or irregular heartbeat. A pacemaker is composed of (i) a pulse generator, (ii) one or more leads, and (iii) an electrode on each lead. The pulse generator is a small metal case that contains electronic circuitry with a small computer and a battery that regulates the impulses sent to the heart. The lead (or leads) is an insulated wire that is connected to the pulse generator on one end, with the other end placed inside one of the heart's chambers. The lead is almost always placed so that it runs through a large vein in the chest leading directly to the heart.



The electrode on the end of a lead touches the heart wall. The lead delivers the electrical impulses to the heart. It also senses the heart's electrical activity and relays this information back to the pulse generator. Pacemaker leads may be positioned in the atrium (upper chamber) or ventricle (lower chamber) or both, depending on the medical condition. If the heart's rate is slower than the programmed limit, an electrical impulse is sent through the lead to the electrode and causes the heart to beat at a faster rate. When the heart beats at a rate faster than the programmed limit, the pacemaker monitors the heart rate and does not permit it to pace.

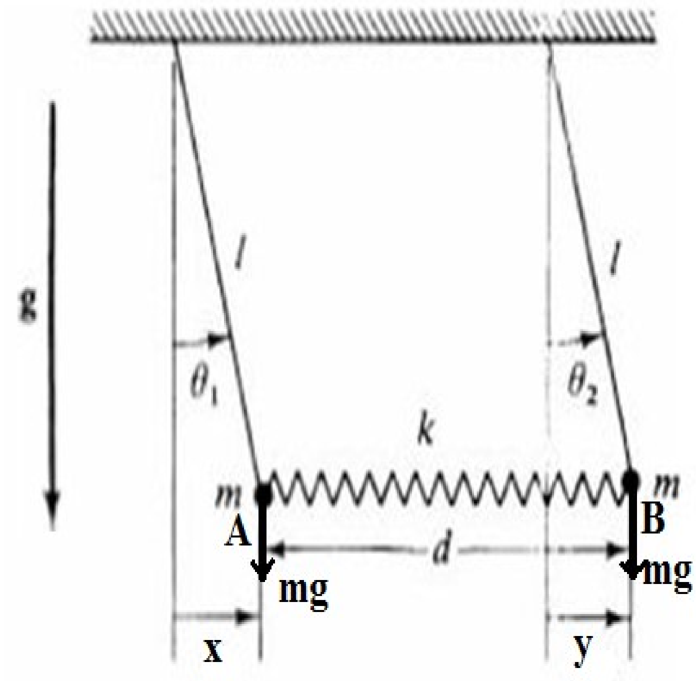
<u>Coupled system of two Pendulums</u>

Figure shows two identical pendulums ‘A’ and ‘B’ each having a bob of mass ‘m’, suspended by rigid, weightless rod of length from a rigid support. The two pendulum bobs are connected by a light spring of spring constant ‘k’. The normal length of the spring is equal to the distance between the bobs, when they are in their equilibrium positions. In this condition, the spring does not exert any force on the pendulum bobs. However, if the system is so disturbed that the bobs undergo unequal displacements, the springs gets either stretched or compressed, depending upon the relative displacement of the bobs. Then



deformed spring exerts force on the bobs. The pendulum bobs are set into oscillations in plane of pendulum.

Let at a given instant, the displacement of the bobs are ‘x’ and ‘y’ respectively (in the same direction). Then the restoring force, due to spring, on ‘A’ and ‘B’ are:



$$F_A = -k(x-y)$$

And

$$F_B = -k(y-x)$$

respectively.

But along with these spring forces, gravitational force is also acting on each mass. The component of gravitational forces (also playing the role of restoring force along with spring force) on ‘A’ and ‘B’ are:

$-mgs \quad {}_1 \approx -mg\frac{x}{l}$ (-ve sign because, it acts against increasing *x*) on A.

And $-mgsin\theta_2 \approx -mg\frac{y}{l}$ on B respectively.

The differential equation of motion of pendulum ‘A’ and ‘B’ are then:

$$m\frac{d^2x}{dt^2} = -mg\frac{x}{l} - k(x-y)$$

And

$$m\frac{d^2y}{dt^2} = -mg\frac{y}{l} - k(y-x)$$

respectively.

Mark that these two differential equations involve both coordinates *‘x’* and *‘y’*. Hence these are coupled equations. One can notice that, the second term on the R.H.S of eqns (1) and (2) arise due to coupling of two pendulums by deformed spring. In absence of coupling (no deformation of spring), eqns (1) and (2) reduce to the equation of motion of two independent simple pendulums.

Equations (1) and (2) also can be written as:

$$\frac{d^2x}{dt^2} + \frac{g}{l}x + \frac{k}{m}(x-y) = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + {\omega_2}^2 x + \frac{k}{m}(x-y) = 0 ------(1)$$

And $\frac{d^2y}{dt^2} + \frac{g}{l}y + \frac{k}{m}(y - x) = 0$

$$\Rightarrow \frac{d^2y}{dt^2} + {\omega_2}^2 y + \frac{k}{m}(y - x) = 0 - - - - - - - - (2)$$

Where we have substituted $\quad {\omega_2}^2 = \frac{g}{l} - - - (3)$

The pair of equations (1) and (2) involve variables both *'x'* and *'y'* and hence are called as coupled equations. They can be converted to a pair of decoupled equations in terms of two new variables as follows:

Adding (1) and (2):

$$\frac{d^2(x + y)}{dt^2} + {\omega_2}^2(x + y) = 0 - - - - - - -(4)$$

And subtracting (1) and (2):

$$\frac{d^2(x - y)}{dt^2} + \left\{{\omega_2}^2 + \frac{2k}{m}\right\}(x - y) = 0 - - - - - - -(5)$$

Let us introduce two new variables given by:

$Q_1 = x + y \qquad$ and $\qquad Q_2 = x - y$

$$\Rightarrow \frac{d^2(x + y)}{dt^2} = \frac{d^2Q_1}{dt^2}$$

And $\quad \frac{d^2(x-y)}{dt^2} = \frac{d^2Q_2}{dt^2}$

Then eqns. (4) and (5) transforms:

$$\frac{d^2Q_1}{dt^2} + {\omega_2}^2 Q_1 = 0 - - - - - - - - (6)$$

And $\quad \frac{d^2Q_2}{dt^2} + {\omega_3}^2 Q_2 = 0 - - - - - - -(7)$

Where $\quad {\omega_3}^2 = {\omega_2}^2 + \frac{2k}{m} = \frac{g}{l} + \frac{2k}{m} - - - - - - (8)$

The equations of motion (6) and (7) expressed in terms of new coordinates and are decoupled and each equation describes oscillation of a simple harmonic oscillator.

Normal Coordinates:

The coordinates Q_1 and Q_2 are called as the normal coordinates of the coupled System (Normal coordinates are the coordinates with which the coupled system is described after decoupling the differential equation of motions).

The normal coordinates are linear combinations of original variables ('x' and 'y'here). The oscillations described in terms of the normal coordinates are independent and are called as *"normal modes of oscillation"*.

Normal Mode Frequencies:

After describing the coupled oscillation in terms of normal coordinates and the corresponding angular frequencies of oscillation to each normal coordinates and are:

$$\omega_2 = \sqrt{\frac{g}{l}}$$

And

$$\omega_3 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$$

The corresponding frequencies:

$$f_2 = \frac{\omega_2}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

And

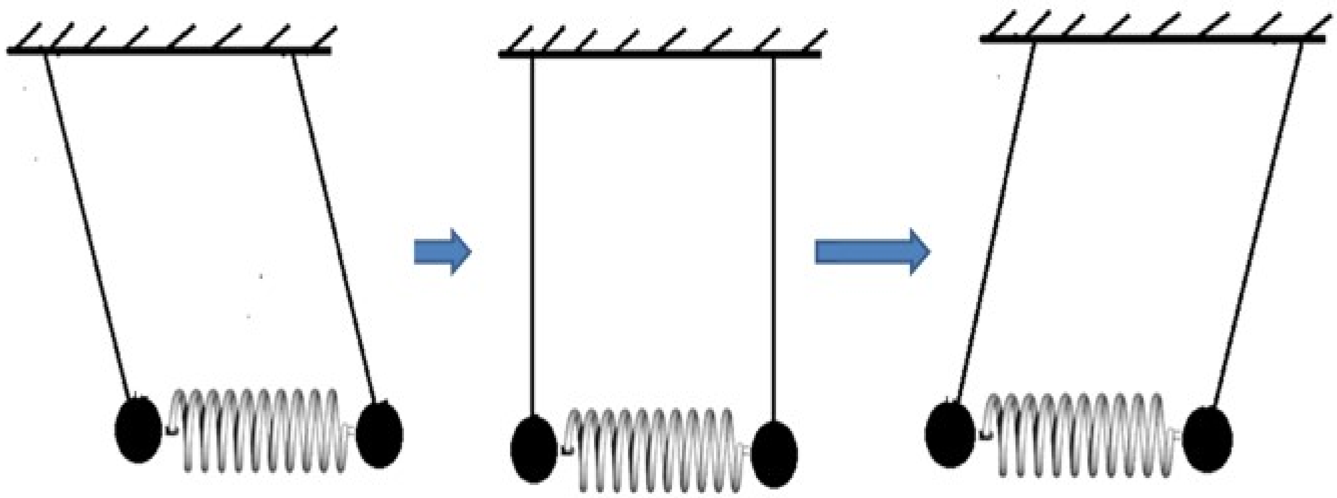
$$f_3 = \frac{\omega_3}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l} + \frac{2k}{m}}$$

f_2 are f_3 called the normal mode frequencies of the coupled oscillator. With suitable choice of the initial conditions, it is always possible to describe the oscillation of the coupled oscillator in terms of only one normal coordinate. The system oscillates with corresponding normal mode frequency. The oscillation of the coupled system in terms of each normal coordinate is called as *"Normal mode Of Oscillation"*.

If the system is initially disturbed in such a way that only one normal mode is excited, the system continues to oscillate in that mode and no other mode is excited (since the different modes are decoupled). However, if the system is initially disturbed in an arbitrary manner, the general oscillation will be a linear combination of different normal of oscillation.

Q_1 mode:

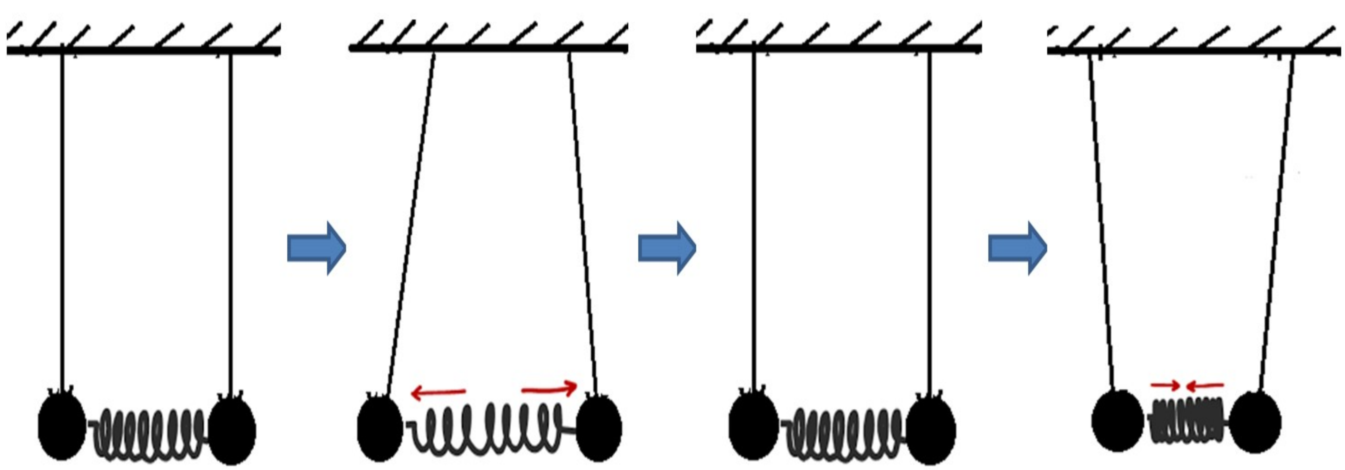
If the initial conditions are chosen such that $x = y$, i.e. both pendulum bobs are displaced by the same amount in the same direction, then $Q_2 = x - y = 0$. Thus only the Q_1 mode is excited and the equation of motion is described only one equation of motion (6). Since both the bobs have same displacement in the same direction, the spring is always in the normal state and both the bobs oscillate with same amplitude, frequency and phase. This is called as ***in phase mode of oscillation***.



In "in phase mode of oscillation", the spring does not play any role and the two pendulum oscillate as if there is no coupling. The frequency of oscillation in this case is same as that of any one independent oscillator.

Q_2 mode:

If the initial conditions are such that $x = -y$, i.e. the bobs are displaced by same amount in opposite directions, then $x+y = 0$ and hence, $Q_1 = x + y = 0$. So here only the Q_2 mode is excited and the



oscillation is described by equation of motion (7). Then the angular frequency of oscillation becomes ω_2 (greater than the individual bob frequency). In this case, bobs always vibrate in opposite phase. This is called as *"out of phase mode"* of oscillation. As $\omega_3 > \omega_2$, so the *frequency of out of phase mode is greater than frequency of in phase mode.*