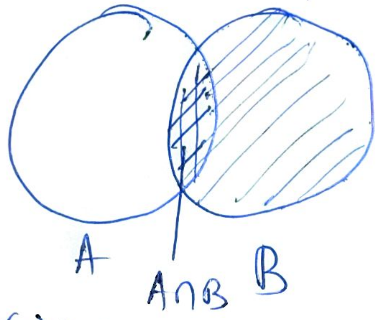
2-4 Conditional Probability

Let there are two events A & B. When the event A depends on the event B, then finding the probability of A with given that B occurs is the conditional probability of A with given B and is denoted by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) > 0$.



Note $P(A|B) + P(A'|B) = 1$ for $P(B) > 0$

Ex In a board exam,

A = student secured 90% in aggregate

B = student got 95% or more in Math.

$P(A|B)$ evaluates the prob. of the student secured 90% in aggregate with given that the student has got 95% ~~or~~ more in Math.

$P(B|A)$ evaluates the prob. of the student got 95% or more in Math with given that he has secured 90% in aggregate.

Ex: A box contains 10 screws, three of which are defective. Two screws are drawn at random, find the probability that none of the two screws is defective. with cases (a) replacement (b) without replacement

Sol^n let $|D| = 3$, $|N| = 7$, $|S| = 10$

A = 1st drawn screw is nondefective

B = 2nd drawn screw is nondefective

(a) With replacement:

For both events A & B, the box has 10 screws, so A & B are independent events & $P(A) = P(B) = \frac{7}{10}$

$\therefore$ Prob. that none of the two screw is deffective

= prob. that both are non-defective

$= P(A \cap B) = P(A)\,P(B) = \frac{7}{10} \cdot \frac{7}{10} = 0.49$

(b) Without replacement

$P(A) = \frac{7}{10}$, $P(B|A) = \frac{6}{9} = \frac{2}{3}$

$P(A \cap B) = P(B|A)\,P(A)$

$= \frac{2}{3} \cdot \frac{7}{10} = 0.47$

Ex Suppose that all individuals buying a certain digital camera, 60% include an optional memory card, 40% include an extra battery and 30% include both a card and battery in their purchase, then find

(a) the prob. that the individual is purchasing the camera with memory card which also has the extra battery.

(b) prob that the individual is purchasing the camera with ~~memory card~~ extra battery which also has a memory card.

Sol^n Consider $A = \{\text{memory card purchased}\}$

$B = \{\text{extra battery purchased}\}$

$A \cap B = \{\text{both memory card and battery purchased}\}$

so $P(A) = 0.6$, $P(B) = 0.4$, $P(A \cap B) = 0.3$

(a) $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$$

which means all those purchasing an extra battery, 75% purchased an optional memory card.

(b) $$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.3}{0.6} = 0.5$$

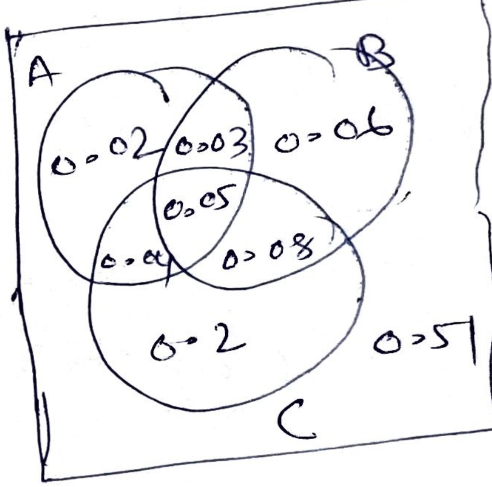
i.e., all those purchasing a memory card, 50% purchased an extra battery

Ex A news megazine publishes three columns entitled Art (A), Books (B), and Cinema (C). Reading habits of a randomly selected reader with respect to these columns are

read regularity	A	B	C	$A\cap B$	$A\cap C$	$B\cap C$	$A\cap B\cap C$
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05

then

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.08}{0.23} = 0.348$$



$$P(A|B\cup C) = \frac{P(A\cap(B\cup C))}{P(B\cup C)}$$

$$= \frac{P((A\cap B)\cup(A\cap C))}{P(B\cup C)}$$

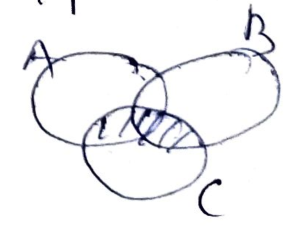
$$= \frac{0.03+0.05+0.04}{0.23+0.37-0.13}$$

$$= \frac{0.12}{0.47} = 0.255$$

$$P(A|\text{reads at least one}) = P(A|A\cup B\cup C)$$

$$= \frac{P(A\cap(A\cup B\cup C))}{P(A\cup B\cup C)} = \frac{P(A)}{0.49} = \frac{0.14}{0.49} = 0.286$$

$$P(A\cup B|C) = \frac{P((A\cup B)\cap C)}{P(C)} = \frac{0.04+0.05+0.08}{0.37}$$



$$= \frac{0.17}{0.37} = 0.459$$

Multiplication rule

For any two events A & B, we have

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow \boxed{P(A\cap B) = P(A|B)\,P(B) \text{ provided } P(B) > 0}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \Rightarrow \boxed{P(A\cap B) = P(B|A)\,P(A) \text{ provided } P(A) > 0}$$

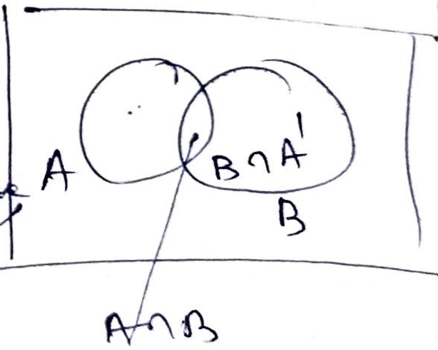
Hence

$$\boxed{P(A\cap B) = \begin{cases} P(A|B)\,P(B) & \text{if } P(B) > 0 \\ P(B|A)\,P(A) & \text{if } P(A) > 0 \end{cases}}$$

which is called multiplication rule (MR) for $A \cap B$.

Note For any event $B \subset S$,

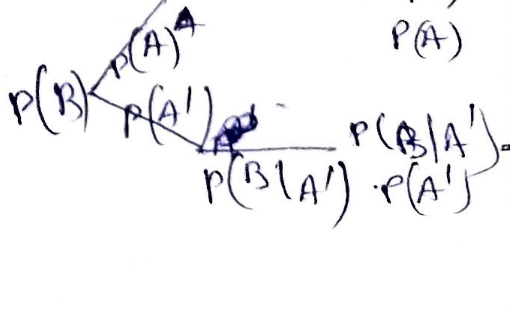
$$B = B \cap S = B \cap (A \cup A')$$
$$= (B \cap A) \cup (B \cap A')$$



since A & A' mutually exclusive & exhaustive

$\Rightarrow \cancel{P(B) = P(B\cap A) + P(B\cap A')}$

$$\Rightarrow P(B) = P(B\cap A) + P(B\cap A')$$
$$= P(B|A)P(A) + P(B|A')P(A')$$

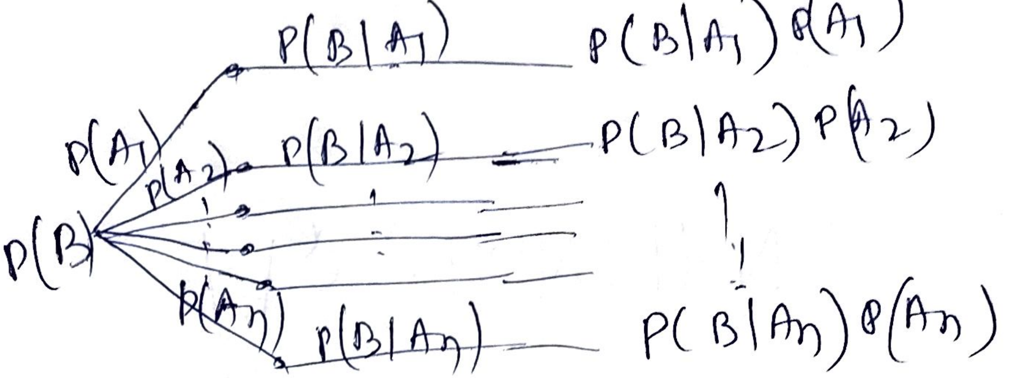


Law of total probability

Let $A_1, A_2, \ldots A_n$ be n mutually exclusive and exhaustive events, then for any event B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$



Proof Since $A_1, A_2 \ldots A_n$ are mutually exclusive & exhaustive, we have

$$A_1 \cup A_2 \cup \cdots \cup A_n = S, \quad A_i \cap A_j = \phi \quad \forall i \neq j$$

and for any event B,

$$B = B \cap S = B \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$$

$$= (B \cap A_1) \cup (B \cap A_2) \cup \cdots (B \cap A_n)$$

where $B \cap A_i$'s are mutually exclusive & exhaustive events for B.

Hence $P(B) = P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_n)$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$

Bayes' Theorem

Let $A_1, A_2, \ldots, A_n$ be a collection of n mutually exclusive & exhaustive events with prior probabilities $P(A_i) > 0$, $i = 1, 2, \ldots n$. Then for any event B for which $P(B) > 0$, the posterior probability of A_j given that B has occurred is

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)},$$

$j = 1, 2 \ldots n$

Proof Since A_i's are mutually exclusive & exhaustive, for any event B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

(by total prob. thm)

Now for any $j = 1, 2 \ldots n$

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B \cap A_j)}{P(B)}$$

$$= \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)} \quad \text{proved}$$