Chapter: 2.7

Nonhomogeneous ODEs

Consider the second order nonhomogeneous linear ODE of the following form.

$$y'' + p(x)y' + q(x)y = r(x) \qquad \textbf{(1)}$$

Where, $r(x) \neq 0$

General solution of the ODE (1):

The general solution of (1) on an open interval I, is defined as $\boldsymbol{y(x) = y_h(x) + y_p(x)}$ **(2)**

Where,

$y_h(x) = c_1y_1(x) + c_2y_2(x)$ is the general solution of the associated homogeneous ODE $y'' + p(x)y' + q(x)y = 0$ on I, known as the **complementary function** of (1) and $y_p(x)$ is any solution of (1) on I free from arbitrary constants, known as a **particular solution** of (1).

Method of Undetermined Coefficients:

In order to find the general solution of the nonhomogeneous ODE (1),

1. Find $y_h(x)$ on solving the homogeneous ODE $y'' + p(x)y' + q(x)y = 0$.
2. Find any solution $y_p(x)$ of the nonhomogeneous ODE (1).

Method of undermined coefficients is used to determine any solution $y_p(x)$ of (1). This method is appropriate to find $y_p(x)$ for linear ODEs with constant coefficients of the following form,

$$y'' + ay' + by = r(x) \qquad \textbf{(3)}$$

Where, a, b are constants and $r(x)$ is a function that exists in one of the following form,

1. $r(x)$ is,
   i. x^n, where $n \in \mathbb{N}$ or $n = 0$
   ii. e^{ax} where a is a non-zero constant.
   iii. $\sin(ax + b)$, where $b \neq 0$ and c are constants.
   iv. $\cos(ax + b)$, where $b \neq 0$ and c are constants.

2. $r(x)$ is defined as the sum or finite product of two or more functions of the types in (1).

Since, the derivatives of the function $r(x)$ in the above described form remain in the same or similar form, so $y_p(x)$ is choosen in form of $r(x)$ but with unknown coefficients whose values are to be determined which signifies the name of the "*Method of undetermined coefficients*".

As, $y_p(x)$ is a solution of the nonhomogeneous ODE (3),

$$y_p'' + ay_p' + by_p = r(x) \quad \textbf{(4)}$$

On comparing the coefficients in both sides of (4), the undetermined/unknown coefficients of $y_p(x)$ can be determined.

Consider the following table to choose $y_p(x)$ using Method of undetermined coefficients.

Table-1

<table>
<tr><th>$\boldsymbol{r(x)}$</th><th>$\boldsymbol{y_p(x)}$</th></tr>
<tr><td>$ke^{\gamma x}$</td><td>$Ce^{\gamma x}$</td></tr>
<tr><td>$kx^n\ (n = 0,1,\dots)$</td><td>$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$</td></tr>
<tr><td>$k\ cos\ \omega x$</td><td rowspan="2">$K\ cos\ \omega x + M\ sin\ \omega x$</td></tr>
<tr><td>$k\ sin\ \omega x$</td></tr>
<tr><td>$ke^{\alpha x}\ cos\ \omega x$</td><td rowspan="2">$e^{\alpha x}(K\ cos\ \omega x + M\ sin\ \omega x)$</td></tr>
<tr><td>$ke^{\alpha x}\ sin\ \omega x$</td></tr>
</table>

Rules to choose $y_p(x)$:

Basic Rule: If $r(x)$ in **(3)** is one of the function in the first column of *Table-1*, choose $y_p(x)$ from the corresponding row of the second column.

Modification Rule: If a term in the choice of $y_p(x)$ is a solution of the corresponding homogeneous ODE $y'' + ay' + by = 0$ then multiply this term by x or x^2 respectively when this solution corresponds to single or double root of the characteristic equation of the homogeneous ODE.

Sum Rule: If $r(x)$ is the sum of functions in the first column of *Table-1*, choose $y_p(x)$ as the sum of the functions in the corresponding rows of the second column.

Limitation of the method: If $p(x)$, $q(x)$ in the nonhomogeneous ODE (1) are not constants or $r(x)$ does not exist in one of the above specified form then this method is unsuitable to find $y_p(x)$.

Examples

Example-1: (Basic Rule)

Solve the initial value problem, $y'' + y = 0.001x^2,\ y(0) = 0, y'(0) = 1.5$

Solution:

Step-1: General solution of the homogeneous ODE $y'' + y = 0$ is $y_h = A\cos x + B\cos x$

Step-2: As $r(x) = 0.001x^2$, let $y_p = K_2x^2 + K_1x + K_0$. Substituting y_p and y_p'' in the given ODE,

$y_p'' + y_p = 2K_2 + K_2x^2 + K_1x + K_0 = 0.001x^2$. Equating the coefficients of x^2, x, x^0 on both the sides, $K_2 = 0.001, K_1 = 0, K_0 = -0.002$. So, $y_p = 0.001x^2 - 0.002$.

The general solution of the given ODE is $y = y_h + y_p = A\cos x + B\cos x + 0.001x^2 - 0.002$

Step-3: Using the initial conditions $y(0) = 0, y'(0) = 1.5$, we get $A = 0.002, B = 1.5$.

So the solution of the IVP is $y = 0.002\cos x + 1.5\cos x + 0.001x^2 - 0.002$

Example-2: (Modification Rule)

Solve the initial value problem, $y'' + 3y' + 2.25y = -10e^{-1.5x},\ y(0) = 1, y'(0) = 0$

Solution:

Step-1: General solution of the homogeneous ODE $y'' + 3y' + 2.25y = 0$ is $y_h = (c_1 + c_2x)e^{-1.5x}$ since the characteristic equation of the homogeneous ODE is $\lambda^2 + 3\lambda + 2.25 = (\lambda + 1.5)^2 = 0$.

Step-2: As $r(x) = -10e^{-1.5x}$, , let $y_p = Ce^{-1.5x}$ but from y_h, it is clearly a solution of the homogeneous ODE having double root of the characteristic equation. So, using Modification rule y_p is multiplied by x^2 and it is modified as, $y_p = Cx^2e^{-1.5x}$.

$$y_p' = C(2x - 1.5x^2)e^{-1.5x}, \qquad y_p'' = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}$$

Substituting y_p, y_p' and y_p'' in the given ODE and comparing the coefficients of x^2, x, x^0 and $e^{-1.5x}$ on both the sides, $= -5$. So, $y_p = -5x^2e^{-1.5x}$.

The general solution of the given ODE is $y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x}$

Step-3: Using the initial conditions $y(0) = 1, y'(0) = 0$, we get $c_1 = 1.5,\ c_2 = 1.5$.

So the solution of the IVP is $y = (1 + 1.5x - 5x^2)e^{-1.5x}$

Example-3: (Sum Rule)

Solve the initial value problem,

$y'' + 2y' + 0.75y = 2\,cosx - 0.25\,sinx + 0.09x,\ y(0) = 2.78, y'(0) = -0.43$

Solution:

Step-1: General solution of the homogeneous ODE $y'' + 2y' + 0.75y = 0$ is $y_h = c_1e^{-x/2} + c_2e^{-3x/2}$ since the characteristic equation of the homogeneous ODE is $\lambda^2 + 2\lambda + 0.75 = \left(\lambda + \frac{1}{2}\right)\left(\lambda + \frac{3}{2}\right) = 0$.

Step-2: As $r(x) = (2cosx - 0.25sinx) + 0.09x$, , Choose $y_p = y_{p_1} + y_{p_2}$ where $y_{p_1} = K\,cosx + M\,sinx$ and $y_{p_2} = K_1x + K_0$.

$$y_p' = -Ksinx + Mcosx + K_1, y_p'' = -Kcosx - Msinx$$

Substituting y_p, y_p' and y_p'' in the given ODE and comparing the coefficients of $x,\ x^0,\ cosx,\ sinx$ on both the sides, $K = 0,\ M = 1,\ K_1 = 0.12,\ K_0 = -0.32$. So, $y_p = sinx + 0.12x - 0.32$

The general solution of the given ODE is $y = y_h + y_p = c_1 e^{-x/2} + c_2 e^{-3x/2} + sinx + 0.12x - 0.32$

Step-3: Using the initial conditions $y(0) = 2.78,\ y'(0) = -0.43$, we get $c_1 = 3.1,\ c_2 = 0$.

So the solution of the IVP is $y = 3.1e^{-x/2} + sinx + 0.12x - 0.32$

Problems

Solve the following problems.

1. $y'' + 5y' + 6y = 2e^{-x}$
2. $y'' + 3y' + 2y = 12x^2$
3. $(3D^2 + 27I)y = 3\cos x + \cos 3x$
4. $y'' + 4y = -12\sin 2x\,,\ y(0) = 1.8,\ y'(0) = 5$
5. $8y'' - 6y' + y = 6\cosh x\,, y(0) = 0.2,\ y'(0) = 0.05$

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