1 Unit Step Function (Heaviside Function)

• The unit step function or Heaviside function is a fucntion $u(t-a)$ is 0 for $t < a$ and then rises instantaneously to 1 for $t > a$.

• The unit step function starting at zero time will be defined by $u(t)$ and that starting at time a is $u(t-a)$.

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a. \end{cases} \tag{1}$$

The Laplace transform of $u(t-a)$ is

$$L\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt$$

$$= \int_0^a e^{-st}u(t-a)dt + \int_a^\infty e^{-st}u(t-a)dt$$

$$= \int_a^\infty e^{-st}dt$$

$$= \left[-\frac{1}{s}e^{-st}\right]_a^\infty$$

$$= \frac{1}{s}e^{-sa}.$$

$$\Rightarrow L\{u(t-a)\} = \frac{1}{s}e^{-sa}.$$

When $a = 0$ i.e., the function instantaneously takes the value unity at zero time, then,

$$L\{u(t)\} = \frac{1}{s}.$$

The unit step function is a typical ”engineering function” made to measure engineering applications, which often involve functions that are either ”off” or ”on”. Multiplying functions $f(t)$ with $u(t-a)$, we can produce all sorts of effects.

Let, $f(t) = 0$ for all negative t, then $f(t-a)u(t-a)$ with $a > 0$ is $f(t)$ shifted towards the right by the amount a.

2 Second Shifting Theorem (t-Shifting)

The s-shifting theorem concerned transforms $F(s) = L\{f(t)\}$ and $F(s-a) = L\{e^{at}f(t)\}$. The second shifting theorem will concern functions $f(t)$ and $f(t-a)$.

Theorem 1: If $f(t)$ has the transform $F(s)$, then the "shifted function"

$$\widetilde{f(t)} = f(t-a)u(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t > a. \end{cases} \tag{2}$$

has the transform $e^{-as}F(s)$. That is, if $L\{f(t)\} = F(s)$, then

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s). \tag{3}$$

Or,

$$f(t-a)u(t-a) = L^{-1}\left\{e^{-as}F(s)\right\}. \tag{4}$$

Proof:
To prove eq.(3). By using the definition of Laplace transformation, from eq.(3) we have,

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau}f(\tau)d\tau = \int_0^\infty e^{-s(\tau+a)}f(\tau)d\tau$$

let, $\tau + a = t \Rightarrow d\tau = dt$. Thus,

$$e^{-as}F(s) = \int_a^\infty e^{-st}f(t-a)dt.$$

Now using eq.(2), we have,

$$e^{-as}F(s) = \int_0^\infty e^{-st}f(t-a)u(t-a)dt = \int_0^\infty e^{-st}\widetilde{f(t)}dt.$$

Hence proved.

Example 1: Write the following function using unit step function and find its transform.

$$f(t) = \begin{cases} 2, & 0 < t < 1 \\ \frac{1}{2}t^2, & 1 < t < \frac{\pi}{2} \\ \cos t, & t > \frac{\pi}{2}. \end{cases}$$

Solution:

Step-1: In terms of unit step function, given $f(t)$ becomes,

$$f(t) = 2(u(t-0) - u(t-1)) + \frac{1}{2}t^2\left(u(t-1) - u\left(t - \frac{\pi}{2}\right)\right) + \cos t\left(u\left(t - \frac{\pi}{2}\right)\right)$$

$$= 2(1 - u(t-1)) + \frac{1}{2}t^2\left(u(t-1)\right) - \frac{1}{2}t^2 u\left(t - \frac{\pi}{2}\right) + \cos t\left(u\left(t - \frac{\pi}{2}\right)\right)$$

Step-2: Now, to find the transformation, we need to apply theorem 1. Hence, we must write each term of $f(t)$ in the form of $f(t-a)u(t-a)$. Thus,

$$L\{2(1 - u(t-1))\} = \frac{2(1-e^{-s})}{s}.$$

$$L\left\{\frac{1}{2}t^2\left(u(t-1)\right)\right\} = L\left\{\left(\frac{1}{2}(t-1)^2 + (t-1) + \frac{1}{2}\right)u(t-1)\right\} = \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s}.$$

$$L\left\{\frac{1}{2}t^2 u\left(t - \frac{\pi}{2}\right)\right\} = L\left\{\left(\frac{1}{2}\left(t - \frac{\pi}{2}\right)^2 + \frac{\pi}{2}\left(t - \frac{\pi}{2}\right) + \frac{\pi^2}{8}\right)u\left(t - \frac{\pi}{2}\right)\right\} = \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{\left(\frac{-\pi s}{2}\right)}.$$

$$L\left\{(\cos t)u\left(t - \frac{\pi}{2}\right)\right\} = L\left\{\cos\left(t - \frac{\pi}{2} + \frac{\pi}{2}\right)u\left(t - \frac{\pi}{2}\right)\right\} = L\left\{\left(-\sin\left(t - \frac{\pi}{2}\right)\right)u\left(t - \frac{\pi}{2}\right)\right\} = -\left(\frac{1}{s^2+1}\right)e^{\left(\frac{-\pi s}{2}\right)}.$$

Thus,

$$L\{f(t)\} = \frac{2}{s} - \frac{2}{s}e^{-s} + \left(\frac{1}{s^3} + \frac{1}{s^2} + \frac{1}{2s}\right)e^{-s} - \left(\frac{1}{s^3} + \frac{\pi}{2s^2} + \frac{\pi^2}{8s}\right)e^{\left(\frac{-\pi s}{2}\right)} - \left(\frac{1}{s^2+1}\right)e^{\left(\frac{-\pi s}{2}\right)}.$$

Example 2: Write the following function using unit step function and find its transform.

$$f(t) = e^t (0 < t < \frac{\pi}{2}).$$

Solution:

$$\text{Given, } f(t) = e^t, (0 < t < \frac{\pi}{2}).$$

In terms of unit step function, we have,

$$f(t) = e^t \left[u(t-0) - u\left(t - \frac{\pi}{2}\right) \right] = e^t \left[1 - u\left(t - \frac{\pi}{2}\right) \right].$$

To apply the second shifting theorem, we have to write $f(t)$ in terms of $f(t-a)u(t-a)$. Thus we have,

$$f(t-a)u(t-a) = e^t - exp\left[\frac{\pi}{2} + \left(t - \frac{\pi}{2}\right)\right] u\left(t - \frac{\pi}{2}\right) = e^t - e^{\pi/2} e^{t-\pi/2} u\left(t - \frac{\pi}{2}\right).$$

Now applying the second shifting theorem we have the required transformation as,

$$L\{f(t)\} = \frac{1}{s-1} - \frac{e^{\pi/2} e^{-(\pi/2)s}}{s-1} = \frac{1}{s-1} \left[1 - exp\left(\frac{\pi}{2} - \frac{\pi}{2}s\right) \right].$$

Example 3: Find $f(t)$ if $L(f)$ equals $6(1-e^{-\pi s})/(s^2+9)$.

Solution:
We have,

$$\frac{6}{s^2+9} = 2\left(\frac{3}{s^2+3^2}\right).$$

Hence, we have the inverse function of $\frac{6}{s^2+9}$ is $2\sin 3t$. Also,

$$\frac{-6e^{-\pi s}}{s^2+9} = -2\left(\frac{3e^{-\pi s}}{s^2+3^2}\right).$$

Hence, by shifting theorem, we have that $\left(\frac{3e^{-\pi s}}{s^2+3^2}\right)$ has the inverse $\sin 3(t-\pi)u(t-\pi)$.
Since,

$$\sin 3(t-\pi) = -\sin 3t \text{ (periodicity)}$$

we have,

$$\sin 3(t-\pi)u(t-\pi) = -\sin(3t)u(t-\pi).$$

Thus, we have,

$$f(t) = 2\sin 3t - [-\sin(3t)u(t-\pi)] = 2\sin 3t + 2\sin(3t)u(t-\pi) = 2[1+u(t-\pi)]\sin 3t.$$

Thus, we have,

$$f(t) = \begin{cases} 2\sin 3t, & 0 < t < \pi \\ 4\sin 3t, & t > \pi. \end{cases}$$

3 Short impulses and Dirac's Delta function

A ship being hit by a single high wave, a tennis ball being hit by a racket and many other similar examples appear in everyday life.

The are the phenomena of an impulsive nature where actions of forces-mechanical, electrical etc.-are applied over short intervals of time.

We can model such type of phenomena as,

$$f_k(t-a) = \begin{cases} \frac{1}{k}, & a \leqslant t \leqslant a+k \\ 0, & otherwise. \end{cases} \tag{5}$$

This function represents, a force of magnitude $\frac{1}{k}$ acting from $t = a$ to $t = a+k$, where k is positive and small.

In mechanics, the integral of a force acting over a time interval $a \leqslant t \leqslant a+k$ is called the "impulse" of the force.

Thus, the impulse of f_k iin eq.(5) is,

$$I_k = \int_0^\infty f_k(t-a)dt = \int_a^{a+k} \frac{1}{k}dt = 1. \tag{6}$$

To find out what will happen if k becomes smaller and smaller, we take the limit of f_k as $k \to 0 (k > 0)$ denoted by,

$$\delta(t-a) = \lim_{k\to 0} f_k(t-a).$$

Where, $\delta(t-a)$ is called ”Dirac delta function” or the ”unit impulse function”.

If we take the impulse I_k of f_k is 1, then from eq.(5) and eq.(6) taking the limit as $k \to 0$, we have,

$$\delta(t-a) = \begin{cases} \infty, & t = a \\ 0, & otherwise. \end{cases} and \int_0^\infty \delta(t-a)dt = 1. \tag{7}$$

In particular, for a continuous function $g(t)$ we use the property [often called shifting property of $\delta(t-a)$] as,

$$\int_0^\infty g(t)\delta(t-a)dt = g(a). \tag{8}$$

Now to find the Laplace transform of $\delta(t-a)$, we can write,

$$f_k(t-a) = \frac{1}{k}[u(t-a) - u(t-(a+k))]$$

and now taking the transform,

$$L\{f_k(t-a)\} = \frac{1}{ks}[e^{-as} - e^{-(a+k)s}] = e^{-as}\left(\frac{1-e^{-ks}}{ks}\right).$$

Now taking the limit as $k \to 0$ on both the sides, we have,

$$L\{\delta(t-a)\} = e^{-as}. \tag{9}$$

Example 1: Find the solution of the IVP,

$$y'' + 4y = \delta(t-\pi),\ y(0) = 8,\ y'(0) = 0.$$

Solution:
Given IVP is

$$y'' + 4y = \delta(t-\pi),\ y(0) = 8,\ y'(0) = 0$$

which models an undamped motion that starts with initial displacement 8 and initial velocity 0 and receives a hammerblow at a later instant at $t = \pi$.

Applying the Laplace transformation on the IVP, we obtain,

$$s^2Y - 8s + 4Y = e^{-\pi s},$$
$$\text{thus } (s^2+4)Y = e^{-\pi s} + 8s.$$
$$\Rightarrow Y = \frac{8s}{s^2+2^2} + \frac{e^{-\pi s}}{s^2+2^2}.$$

$$\Rightarrow y = L^{-1}(Y) = 8L^{-1}\left(\frac{s}{s^2+2^2}\right) + L^{-1}\left(\frac{e^{-\pi s}}{s^2+2^2}\right).$$

Now

$$L^{-1}\left(\frac{e^{-\pi s}}{s^2+2^2}\right) = \sin(2(t-\pi)).\frac{1}{2}u(t-\pi) = \sin(2t-2\pi).\frac{1}{2}u(t-\pi) = \sin 2t.\frac{1}{2}u(t-\pi).$$

Thus,

$$y = 8\cos 2t + (\sin 2t)\frac{1}{2}u(t-\pi). \text{ (Ans)}$$

Example 2: Find the solution of the IVP,

$$y'' + y = \delta(t-\pi) - \delta(t-2\pi),\ y(0) = 8,\ y'(0) = 1.$$

Solution:
Given undamped force motion with two impulses at $t = \pi$ and $t = 2\pi$ as the driving force is,

$$y'' + y = \delta(t-\pi) - \delta(t-2\pi),\ y(0) = 8,\ y'(0) = 1.$$

. Taking the Laplace transform on the left hand side we have,

$$(s^2Y - s(0) - 1) + Y = e^{-\pi s} - e^{-2\pi s}.$$

$$\Rightarrow (s^2+1)Y = e^{-\pi s} - e^{-2\pi s} + 1.$$

Hence,

$$Y = \frac{1}{s^2+1}(e^{-\pi s} - e^{-2\pi s} + 1).$$

Using linearity and applying the inverse Laplace transform to each term we get,

$$L^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) = \sin(t-\pi).u(t-\pi) = -(\sin t)u(t-\pi)$$

$$L^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) = \sin(t-2\pi).u(t-2\pi) = -(\sin t)u(t-2\pi)$$

$$L^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

Thus,

$$y = -(\sin t)u(t-\pi) - (\sin t)u(t-2\pi) + \sin t.$$

Thus, we have,

$$y = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \\ -\sin t, & t > 2\pi. \end{cases}$$