3.6 The Poisson Probability Distribution:

A discrete random variable x is said to have a poisson distribution with parameter μ $(\mu > 0)$ if the p.m.f of x is

$$P(X=x) = P(x;\mu) = \frac{e^{-\mu}\mu^{x}}{x!}, \quad x = 0, 1, 2, 3 \cdots\cdots$$

Clearly, $p(x,\mu) \geq 0$, as $\mu > 0$ and

$$\sum_{x} p(x;\mu) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^{x}}{x!} = e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^{x}}{x!} = e^{-\mu}.e^{\mu} = e^{0} = 1$$

The poisson distribution as a Limit:

Poisson distribution is a limiting case of binomial distribution when n is very large and p is very small so that $\mu = np$ is of finite magnitude.

Theorem:

Suppose that in the binomial pmf $b(x;n,p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value $\mu > 0$. Then $b(x;n,p) \longrightarrow p(x;\mu)$.

Note:

According to this theorem in any binomial experiment in which n is large and p is small, $b(x;n,p) \approx p(x;\mu)$ where $\mu = np$. This approximation can safely be applied if $n > 50$ and $np < 5$.

Theorem:

If x has a poisson distribution with parameter μ, then

$$E(X) = V(X) = \mu.$$

Proof:

The p.m.f of poisson distribution is

$$p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0, 1, 2, \ldots$$

$\therefore$ mean, $E(X) = \sum_x x\,p(x;\mu) = \sum_{x=0}^{\infty} x \frac{e^{-\mu}\mu^x}{x!}$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\mu}\mu^x}{x!} = e^{-\mu}\sum_{x=1}^{\infty} \frac{\mu^x}{(x-1)!}$$

$$= e^{-\mu}\sum_{r=0}^{\infty} \frac{\mu^{r+1}}{r!} \qquad \text{let } x-1 = r$$

$$= e^{-\mu}\cdot\mu\sum_{r=0}^{\infty}\frac{\mu^r}{r!}$$

$$= e^{-\mu}\cdot\mu\cdot e^{\mu}$$

$$= \mu$$

$$\Rightarrow \boxed{E(X) = \mu}$$

$\because e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$

$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

To find $V(X)$

We know that, $V(X) = E\{X(X-1)\} - E(X)\{E(X)-1\}$

Now, $E\{X(X-1)\} = \sum_x x(x-1)\,p(x;\mu) = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\mu}\mu^x}{x!}$

$$= e^{-\mu}\sum_{x=2}^{\infty}\frac{x(x-1)\mu^x}{x(x-1)(x-2)!}$$

$$= e^{-\mu}\sum_{x=2}^{\infty}\frac{\mu^x}{(x-2)!}$$

let $x-2 = r$

$$\Rightarrow E\{x(x-1)\} = e^{-\mu} \sum_{r=0}^{\infty} \frac{\mu^{r+2}}{r!} = e^{-\mu} \mu^2 \sum_{r=0}^{\infty} \frac{\mu^r}{r!}$$

$$= e^{-\mu} \mu^2 . e^{\mu} = \mu^2$$

$$\therefore V(x) = E\{x(x-1)\} - \mu(\mu-1)$$

$$= \mu^2 - \mu^2 + \mu = \mu$$

$\Rightarrow \boxed{V(x) = \mu}$ $\therefore E(x) = V(x) = \mu$ (proved)

Example:
If a publisher of nontechnical books takes great pains to ensure that its books are free of typographical errors, so that the probability of any given page containing at least one such error is 0.005 and errors are independent from page to page; what is the probability that one of its 400 page novels will contain exactly one page with errors? At most three pages with errors?

Ans. $S = \{$a page containing ~~on~~ at least one error$\}$
$F = \{$an error free page$\}$
X = the number of pages containing at least one error
Here, $n = 400$, $p = 0.05$ $\Rightarrow np = 400 \times 0.05 = 2$
Here n is very large and $p = 0.05$ is small but $\mu = np = 2$
$\therefore$ Using the poisson approximation we have
P(exactly one page with errors)
$= P(x=1) = b(1; 400, 0.05) \simeq p(1; 2) = \frac{e^{-2}(2)^1}{1!} = 0.270671$
Also the binomial value is $b(1; 400, 0.05) = 0.270669$.
So the approximation is very good.

Similarly, P(at most 3 pages with errors)

$$= P(X \leq 3) \approx P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$$\approx p(0;2)+p(1;2)+p(2;2)+p(3;3)$$

where $p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}$, $x=0,1,2,\ldots$

$$\therefore P(X\leq 3) = \frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}+\frac{e^{-2}2^3}{3!}$$

$$= 0.135335+0.270671+0.270671+0.180447$$

$$= 0.8571$$

which again is quite close to the binomial value $P(X\leq 3) = B(3;400,0.05) = 0.8576$

The poisson process:

Let us consider a sequence of changes. If the random variable $X(t)$ denotes the number of changes during the interval $(0,t)$ then $X(t)$ assumes the values $0,1,2,3,\ldots$

It can be shown that

$$P(X(t)=x) = P_x(t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

where λ = no. of changes per unit time

$x=0,1,2,3\ldots$

So, the number of events during a time interval of length t is a Poisson r.v. with parameter $\mu=\lambda t$. The expected number of events during any such time interval is then λt, so the expected number during a unit interval of time is λ.

Q. (79) [3.6] Problems 3.6

Let X be the number of flaws on the surface of a randomly selected boiler of a certain type, have a Poisson distribution with parameter $\mu = 5$. Use Appendix Table A.2 to compute the following probabilities

a) $P(X \le 8)$ b) $P(X = 8)$ c) $P(9 \le X)$

d) $P(5 \le X \le 8)$ e) $P(5 < X < 8)$

Ans: X = no. of flaws

$\mu = 5$

For poisson distribution, the p.m.f is $p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}$

$x = 0, 1, 2, \ldots$

and the cdf is $F(x;\mu) = P(X \le x) = \sum_{y=0}^{x} \frac{e^{-\mu}\mu^y}{y!}$

so using Appendix A.2

$\therefore$ (a) $P(X \le 8) = F(8;5) = 0.932$

~~(b) $P(X = 8) = p(8;5) = \frac{e^{-5}5^8}{8!}$~~

(b) $P(X=8) = P(8 \le X \le 8)$

$= F(8;5) - F(7;5)$

$= 0.932 - 0.867$

$= 0.065$

[$\because$ $P(a \le X \le b)$ $= F(b) - F(a)$ a, b are integer]

(c) $P(9 \le X) = P(X \ge 9) = 1 - P(X < 9) = 1 - P(X \le 8)$

$= 1 - F(8;5) = 1 - 0.932 = 0.068$

(d) $P(5 \le X \le 8) = F(8;5) - F(4;5) = 0.932 - 0.440 = 0.492$

(e) $P(5 < X < 8) \ne$ ~~$P(6 \le X \le 7$~~

$= P(6 \le X \le 7) = F(7;5) - F(5;5) = 0.867 - 0.616 = 0.251$

Q. (80)

Let X be the number of material anomalies occuring in a particular region of an aircraft gas-turbine disk. The article " Methodology for probabilistic Life Prediction of Multiple-Anomaly Materials" propose a poisson distribution for X. Suppose that $\mu = 4$.

a) compute both $P(X \leq 4)$ and $P(X < 4)$

b) Compute $P(4 \leq X \leq 8)$

c) Compute $P(8 \leq X)$

d) what is the probability that the number of anomalies exceeds its mean value by no more than one standard deviation ?

Ans: X = no. of material anomalies.

$\mu = 4$ (given)

The p.m.f Poisson distribution is

$$P(X=x) = \frac{e^{-\mu}\mu^{x}}{x!}, \quad x = 0, 1, 2 \ldots$$

(a) $P(X \leq 4)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{e^{-4}4^{0}}{0!} + \frac{e^{-4}4^{1}}{1!} + \frac{e^{-4}4^{2}}{2!} + \frac{e^{-4}4^{3}}{3!} + \frac{e^{-4}4^{4}}{4!}$$

$= 0.0183 + 0.0733 + 0.1465 + 0.1954 + 0.1954 = 0.6288$

and $P(X<4)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 0.0183 + 0.0733 + 0.1465 + 0.1954$

$= 0.4335$

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(b) $P(4 \le X \le 8)$

$= P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$$= \sum_{x=4}^{8} \frac{e^{-\mu} \mu^{x}}{x!} = e^{-4} \sum_{x=4}^{8} \frac{4^{x}}{x!} = 0.5451$$

(c) $P(8 \le X) = P(X \ge 8) = 1 - P(X < 8)$

$= 1 - P(X \le 7) = 1 - \{P(X=0) + P(X=1) + \cdots + P(X=7)\}$

$$= 1 - \sum_{x=0}^{7} \frac{e^{-\mu} \mu^{x}}{x!} = 1 - e^{-4} \sum_{x=0}^{7} \frac{4^{x}}{x!}$$

$= 1 - 0.9489 = 0.0511$

(d)

mean, $\mu = 4$, $var(X) = 4$

$\therefore$ s.d, $\sigma = \sqrt{4} = 2$

The probability that x exceeds the mean value by no more than one standard deviation is

$P(X - \mu \le \sigma)$

$= P(X \le \mu + \sigma)$

$= P(X \le 4 + 2)$

$= P(X \le 6)$

$= P(X=0) + P(X=1) + \cdots + P(X=6)$

$= 0.8893.$

Q.(84)

Suppose that only 0.10% of all computers of a certain type experience CPU failure during the warranty period. Consider a sample of 10,000 computers.

a) What are the expected value and standard deviation of the number of computers in the sample that have the defect?

b) What is the (approximate) probability that more than 10 sampled computers have the defect?

c) What is the (approximate) probability that no sampled computers have the defect?

Ans:

(a) Here $n = 10{,}000$, $p = 0.10\% = 0.001$ (probability of success)

mean, $\mu = np = 10{,}000 \times 0.001 = 10$

$V(X) = \sigma^2 = npq$

n = large
p = small

S.D., $\therefore \sigma = \sqrt{npq} = \sqrt{1000 \times 0.001 \times 0.999} = 3.161$

(b) Let X = no. of computers in the sample that have the defect.

Then X has poisson distribution with parameter $\mu = 10$

Here n is very large and p is small but $\mu = np = 10$

The p.m.f of poisson distribution is

$$P(X=x) = \frac{e^{-\mu}\mu^x}{x!}, \quad x = 0,1,2,3 - - -$$

$$\therefore P(X>10) = 1 - P(X \le 10)$$

$$= 1 - \{P(X=0) + P(X=1) + \cdots + P(X=10)\}$$

$$= 1 - \sum_{x=0}^{10} \frac{e^{-10}\,10^x}{x!} = 1 - 0.5826 = 0.4174$$

(c) $P(X=0) = \dfrac{e^{-10}\, 10^0}{0!} \simeq 0.0000454 = 0.00454\%$

Q.(87)

The number of request for assistance received by a towing service is a Poisson process with rate $\alpha = 4$ per hour.

a) compute the probability that exactly 10 requests are received during a particular ~~hour~~ 2 hour period?

(b) ~~what is the probability that exactly 4 arrivals occur during a particular hour?~~

c) How many ~~people do you expect to arrive during a 45 min period?~~ calls would you expect during their break?

Ans. Number of events during a time interval of length t can be modeled using Poisson random variable with parameter $\mu = \alpha t$. This indicates that

$$P_x(t) = \frac{e^{-\alpha t}(\alpha t)^x}{x!}$$

which is called Poisson process with rate α.

(a) Given $\alpha = 4$ per hour, which means that for a 2 hour period we have $\mu = \alpha t = 4 \times 2 = 8$

Let $X(t)$ = no. of request recieved during the interval $(0, t)$

$\therefore P(X=10) = P_{10}(2) = \dfrac{e^{-8}\, 8^{10}}{10!} = 0.099$

(b) If the operators of the towing service take a 30-min break for lunch, what is the probability that they do not miss any calls for assistance?

Ans: Given a rate $\alpha = 4$ per hour

which means that for a 30 min (0.5 hour) period,

$$\mu = \alpha t = 4 \times 0.5 = 2$$

P (Not missing any calls for assistance)

= P(received no request during break)

$= P(X = 0)$

$$= P_0(0.5) = e^{-2} \frac{2^0}{0!} = 0.135$$

(c) $E(X) = \mu = 2$ (during break).