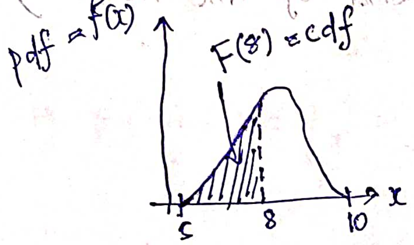
Cumulative Distribution Functions

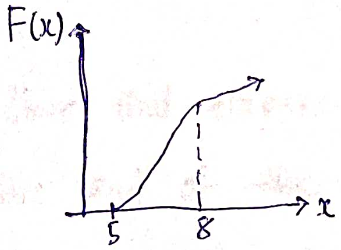
The cumulative distribution function $F(x)$ for a continious r.v X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy.$$

→ For each x, $F(x)$ is the area under the density-curve to the left of x.



→ $F(x)$ increases smoothly as x increases.



properties of CDF

(1) From Def^n: $F(x) = P(X \leq x)$

(2) $0 \leq F(x) \leq 1$

(3) $F(x)$ is non-decreasing (If $a \leq b$ then $F(a) \leq F(b)$)

(4) $\lim_{x \to +\infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$

(5) $P(a \leq X \leq b) = F(b) - F(a) = \int_a^b f(x)\,dx$

(But, in discrete $P(a \leq X \leq b) = F(b) - F(a-1)$)

(6) $F'(x) = f(x)$.

(7) $P(X > a) = 1 - F(a)$.

(8) $P(a \leq X \leq b) = P(a < X \leq b)$

$= P(a \leq X < b)$

$= P(a < X < b) = F(b) - F(a)$.

percentile.

Let 'p' be a number between 0 and 1. The percentile of the distribution of a continuous rv X, denoted by "$\eta(p)$", is defined by

$$\boxed{p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\,dy.}$$

Selected questions. 11, 12, 13, 15, 20, 21.

Q.11 Let X denote the amount of time a book on two-hour reserve is actually checked out, and suppose the cdf is

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$

Use the cdf to obtain the following.

(a) $P(X \leq 1) = F(1) = \frac{1^2}{4} = 0.25$

(b) $P(0.5 \leq X \leq 1) = F(1) - F(0.5) = \frac{1^2}{4} - \frac{0.5^2}{4}$

$= \frac{1 - 0.25}{4} = \frac{0.75}{4} = 0.1875$.

(c) $P(X > 1.5) = 1 - F(1.5) = 1 - \frac{1.5^2}{4} = 0.4375$

(d) $F'(x)$ to obtain the density function $f(x)$

$\therefore f(x) = \frac{d}{dx}\left(\frac{x^2}{4}\right) = \frac{x}{2}$ for $0 \leq x < 2$.

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & 0 \leq x < 2 \\ 0, & \text{otherwise.} \end{cases}$$

e) $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$= \int_0^2 x \cdot \frac{x}{2}\, dx$

$= \frac{1}{2} \left. \frac{x^3}{3} \right|_0^2 = \frac{4}{3}.$

f) $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$

$$\boxed{= \int_0^2 x^2 \cdot \frac{x}{2}\, dx = \frac{1}{2} \left. \frac{x^4}{4} \right|_0^2 = 2.}$$

$\therefore V(X) = E(X^2) - (E(X))^2$

$= 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9} = 0.2222$ //

(g) If $h(X) = X^2$, then

$E[h(X)] = E(X^2) = 2.$ //

21 An ecologist wishes to mark off a circular sampling region having radius 10 m. However, the radius of the resulting region is actually a random variable R with pdf

$$f(r) = \begin{cases} \frac{3}{4}\left(1 - (10-r)^2\right) & 9 \le r \le 11 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected area of the resulting circular region?

Sol^n. We know $A = \pi r^2$

Expected area is $E(A) = E(\pi r^2) = \pi E(r^2)$.

$$\therefore\ E(r^2) = \int_{-\infty}^{\infty} r^2 \cdot f(r)\, dr$$

$$= \int_{9}^{11} r^2 \cdot \frac{3}{4}\left(1-(10-r)^2\right) dr$$

$$= \frac{3}{4}\int_{9}^{11} r^2 \left(1-100-r^2+20r\right) dr$$

$$= \frac{501}{5}.$$

$$\therefore\ E(A) = \pi E(r^2) = \frac{501 \times \pi}{5}.$$