Interpolation and Approximation

This chapter is dedicated to find an approximation to a given function by a class of simpler functions, mainly polynomials. The main uses of polynomial interpolation are

- Reconstructing the function when it is not given explicitly and only the values of $f(x)$ and/or its certain order derivatives at a set of points, known as nodes/tabular points/arguments are given.
- Replace the function $f(x)$ by an interpolating polynomial $P(x)$ so that many common operations such as determination of roots, differentiation, and integration etc which are intended for the function $f(x)$ may be performed using $P(x)$.

Definition: A polynomial $P(x)$ is called *interpolating polynomial* if the values of $P(x)$ and/or its certain order derivatives coincide with those of $f(x)$ and/or its same order derivatives at one or more tabular points.

The reason behind choosing the polynomials ahead of any other functions is that polynomials approximate continuous functions with any desired accuracy. That is, for any continuous function $f(x)$ on an interval $a \leq x \leq b$ and error bound $\beta > 0$ there is a polynomial $p_n(x)$ (of sufficiently large degree) such that $|f(x) - p_n(x)| < \beta$ for all $x \in [a, b]$. This is the famous **Weierstrass approximation theorem.**

In this section we will be focusing on the following methods of interpolation

1. Lagrange's interpolation
2. Newton's divided difference interpolation
3. Newton's forward and backward difference interpolation

1. Lagrange's Interpolation

Assume that $f(x)$ is continuous on $[a, b]$ and further assume that we have $n + 1$ distinct points $a \leq x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n \leq b$. Let the values of the function $f(x)$ at these points are known and are denoted by $f_0 = f(x_0), f_1 = f(x_1), \dots, f_n = f(x_n)$. We aim to find a polynomial $P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ satisfying $P_n(x_i) = f(x_i), i = 0,1, \dots, n$.

Linear Interpolation

Let $n = 1$. Then the nodes are x_0 and x_1. The Lagrange linear interpolating polynomial is given by

$$P_1(x) = a_0 + a_1 x,$$

where the coefficients a_0 and a_1 can be evaluated solving the equations

$$f_0 = a_0 + a_1 x_0$$

$$f_1 = a_0 + a_1 x_1.$$

The Lagrange linear interpolating polynomial is given by

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1,$$

where $l_0(x) = \frac{x-x_1}{x_0-x_1}$ and $l_1(x) = \frac{x-x_0}{x_1-x_0}$ are called Lagrange's fundamental polynomials. The properties of the Lagrange fundamental polynomials are as follows:

(i) $l_0(x) + l_1(x) = 1$.
(ii) $l_0(x_0) = 1, l_0(x_1) = 0; l_1(x_0) = 0, l_1(x_1) = 1$.
(iii) The degrees of $l_0(x)$ and $l_1(x)$ are one.

The error in linear interpolation is given by

$$E_1(x, f) = \frac{1}{2}(x - x_0)(x - x_1) f''(\xi),\ x_0 \leq \xi \leq x_1.$$

The bound for the truncation error in linear interpolation is given by

$$|E_1(x, f)| \leq \frac{(x_1 - x_0)^2}{8} \max_{x_0 \leq x \leq x_1} |f''(x)|.$$

Example 1: Given that $f(2) = 4, f(2.5) = 5.5$. Find the linear interpolating polynomial using Lagrange's interpolation and hence find an approximate value of $f(2.2)$.

Answer: Given that $x_0 = 2, x_1 = 2.5, f_0 = 4, f_1 = 5.5$. The Lagrange fundamental polynomials are

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 2.5}{2 - 2.5} = 2(2.5 - x) = 5 - 2x,$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 2}{2.5 - 2} = 2(x - 2) = 2x - 4.$$

The linear Lagrange interpolating polynomial is given by

$$P_1(x) = l_0(x) f_0 + l_1(x) f_1 = 4(5 - 2x) + 5.5(2x - 4) = 3x - 2.$$

An approximate value of $f(2.2) \approx P_1(2.2) = 3 \times 2.2 - 2 = 4.6$.

Quadratic Interpolation

Here $n = 2$. We need to find an interpolating polynomial of the form

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

where a_0, a_1 and a_2 are arbitrary constants which satisfies the condition $P_2(x_0) = f_0, P_2(x_1) = f_1$ and $P_2(x_2) = f_2$. That is, we need to solve the following system of equations:

$$a_0 + a_1 x_0 + a_2 x_0^2 = f_0$$

$$a_0 + a_1 x_1 + a_2 x_2^2 = f_1$$

$$a_0 + a_1 x_2 + a_2 x_2^2 = f_2.$$

The Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = l_0(x) f_0 + l_1(x) f_1 + l_2(x) f_2$$

where $l_0(x)$, $l_1(x)$ and $l_2(x)$ are Lagrange's fundamental polynomial and are defined by

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}, \quad l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \; and \; l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The truncation error in Lagrange's quadratic polynomial is given by

$$E_2(x, f) = \frac{1}{3!}(x - x_0)(x - x_1)(x - x_2) f'''(\xi), \quad x_0 \le \xi \le x_2.$$

The bound for the quadratic Lagrange's interpolating polynomial is

$$|E_2(x, f)| \le \frac{M_3}{6} \max_{x_0 \le x \le x_2} |(x - x_0)(x - x_1)(x - x_3)|, \; M_3 = \max_{x_0 \le x \le x_2} \left|f'''(x)\right|.$$

Example:

General Formula

The general Lagrange's interpolating polynomial for $n + 1$ nodes $x_0, x_1, \ldots, x_n$ is given by

$$P_n(x) = \sum_{k=0}^{n} l_k(x) f_k,$$

where the n-th degree Lagrange's fundamental polynomial is given by

$$l_k(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)(x_k - x_1)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \; for \; k = 0,1,2, \ldots, n.$$

The truncation error in Lagrange's interpolation is

$$E_n(x,f) = \frac{w(x)}{(n+1)!} f^{(n+1)}(\xi), \;\; x_0 \le \xi \le x_n,$$

where $w(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$.

Example 2. Given that $f(0) = 1, f(1) = 3$ and $f(3) = 55$. Find the unique polynomial of degree 2 or less, which fits the given data. Find the truncation error.

Answer: By hypothesis, $x_0 = 0, x_1 = 1, x_2 = 3$; $f_0 = 1, f_1 = 3, f_2 = 55$. The Lagrange fundamental polynomials are

$$l_0(x) = \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{1}{3}(x-1)(x-3)$$

$$l_1(x) = \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{1}{2}x(x-3), l_2(x) = \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{1}{6}x(x-1).$$

The Lagrange quadratic interpolating polynomial is

$$P_2(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 = \frac{1}{3}(x-1)(x-3) - \frac{3}{2}x(x-3) + \frac{55}{6}x(x-1)$$
$$= 8x^2 - 6x + 1$$

The truncation error is

$$E_2(x) = \frac{1}{6}x(x-1)(x-3)f'''(\xi), \;\; 0 \le \xi \le 3.$$

<u>Newton's Divided Difference Interpolation</u>

Let $f(x)$ be a function defined on the interval $[a,b]$. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ be a partition of $[a,b]$. The divided difference of $f(x)$ at x_0, written as $f[x_0]$, is the value of the function at x_0. That is,

$$f[x_0] = f(x_0) = f_0.$$

The first order divided difference at the nodes x_0 and x_1 is defined as

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}.$$

The second order divided difference at the nodes x_0, x_1, x_2 is defined by

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$
$$= \frac{f_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{f_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{f_2}{(x_2 - x_0)(x_2 - x_1)}$$

The n-th order divided difference is defined by

$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f_i}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}.$$

Divided difference table:

Nodes	Functional Values	1st order Div. Diff.	2nd order Div. Diff.
x_0	$f[x_0]$		
x_1	$f[x_1]$	$f[x_0, x_1]$	
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$

The Newton divided difference interpolating polynomial $P_n(x)$ interpolating at $n+1$ points $x_0, x_1, \ldots, x_n$ is given by

$$P_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots$$
$$+ (x - x_0)\cdots(x - x_{n-1})f[x_0, x_1, \cdots, x_n].$$

Example 4.10 Construct the divided difference table for the data

x	0.5	1.5	3.0	5.0	6.5	8.0
$f(x)$	1.625	5.875	31.0	131.0	282.125	521.0

Hence find the interpolating polynomial and an approximation to the value $f(7)$.

Answer: The divided difference table is given by

x	$f(x)$	first order d.d.	second order d.d.	third order d.d.	fourth order d.d.
0.5	1.625				
		4.25			
1.5	5.875		5.0		
		16.75		1.0	
3.0	31.000		9.5		0
		50.00		1.0	
5.0	131.000		14.5		0
		100.75		1.0	
6.5	282.125		19.5		
		159.25			
8.0	521.000				

We write the divided difference interpolating polynomial as

$$
\begin{aligned}
f(x) &= f[x_0] + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] \\
&\quad + (x - x_0)(x - x_1)(x - x_2)\, f[x_0, x_1, x_2, x_3] \\
&= 1.625 + (x - 0.5)(4.25) + 5(x - 0.5)(x - 1.5) \\
&\quad + (x - 0.5)(x - 1.5)(x - 3.0)
\end{aligned}
$$

$$
\begin{aligned}
&= (1.625 - 2.125 + 3.75 - 2.25) + x(4.25 - 10.0 + 6.75) \\
&\quad + x^2(5 - 5) + x^3 \\
&= x^3 + x + 1.
\end{aligned}
$$

Hence, $f(7.0) = 351.$

Finite Difference Operators

Let the nodes $x_0, x_1, \ldots, x_n$ be equally spaced. That is, $x_i = x_0 + ih,\ i = 0,1, \ldots, n$. We now define the following operators:

The Shift Operator: $E\big(f(x_i)\big) = f(x_i + h)$.

The forward difference Operator: $\Delta f(x_i) = f(x_i + h) - f(x_i)$.

The backward difference operator: $\nabla f(x_i) = f(x_i) - f(x_i - h)$.

The central difference operator: $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$.

The average operator: $\mu f(x_i) = \frac{1}{2}\left(f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right)$.

It can be easily verified that

$$\Delta f_i = \nabla f_{i+1} = \delta f_{i+1/2}, \quad \Delta = E - 1,$$

$$\nabla = 1 - E^{-1}, \ \delta = E^{1/2} - E^{-1/2} \text{ and } \mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right).$$

Repeated application of the difference operators lead to the following higher order differences.

$$E^n f(x_i) = f(x_i + nh).$$

$$\Delta^n f(x_i) = \Delta^{n-1} f_{i+1} - \Delta^{n-1} f_i$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n-k}.$$

$$\nabla^n f(x_i) = \nabla^{n-1} f_i - \nabla^{n-1} f_{i-1}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k}.$$

$$\delta^n f(x_i) = \delta^{n-1} f_{i+1/2} - \delta^{n-1} f_{i-1/2}$$

$$= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n/2-k}.$$

where $\quad f_i = f(x_i)$.

We may also write

$$\Delta^n f(x_i) = (E - 1)^n f(x_i), \ \nabla^n f(x_i) = (1 - E^{-1})^n f(x_i)$$

and expand $(E - 1)^n$, $(1 - E^{-1})^n$, symbolically, to obtain the same result.

The forward and backward difference tables can be computed as follows:

Table 4.3 Forward Difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	f_0			
		Δf_0		
x_1	f_1		$\Delta^2 f_0$	
		Δf_1		$\Delta^3 f_0$
x_2	f_2		$\Delta^2 f_1$	
		Δf_2		
x_3	f_3			

Note that the differences $\Delta^k f_0$ lie on a straight line sloping downward to the right.

Table 4.4 Backward Difference Table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	f_0			
		∇f_1		
x_1	f_1		$\nabla^2 f_2$	
		∇f_2		$\nabla^3 f_3$
x_2	f_2		$\nabla^2 f_3$	
		∇f_3		
x_3	f_3			

Note that the differences $\nabla^k f_3$ lie on a straight line sloping upward to the right.

Table 4.5 Central Difference Table

x	$f(x)$	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
x_0	f_0				
		$\delta f_{1/2}$			
x_1	f_1		$\delta^2 f_1$		
		$\delta f_{3/2}$		$\delta^3 f_{3/2}$	
x_2	f_2		$\delta^2 f_2$		$\delta^4 f_2$
		$\delta f_{5/2}$		$\delta^3 f_{5/2}$	
x_3	f_3		$\delta^2 f_3$		
		$\delta f_{7/2}$			
x_4	f_4				

The differences $\delta^{2k} f_2$ lie on a horizontal line.

Table 4.6 Relationship between the Operators

	E	Δ	∇	δ
E	E	$\Delta + 1$	$(1-\nabla)^{-1}$	$1 + \frac{1}{2}\delta^2 + \delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
Δ	$E-1$	Δ	$(1-\nabla)^{-1} - 1$	$\frac{1}{2}\delta^2 + \delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
∇	$1-E^{-1}$	$1-(1+\Delta)^{-1}$	∇	$-\frac{1}{2}\delta^2 + \delta\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$
δ	$E^{1/2} - E^{-1/2}$	$\Delta(1+\Delta)^{-1/2}$	$\nabla(1-\nabla)^{-1/2}$	δ
μ	$\frac{1}{2}(E^{1/2} + E^{-1/2})$	$\left(1+\frac{1}{2}\Delta\right)(1+\Delta)^{1/2}$	$\left(1-\frac{1}{2}\nabla\right)(1-\nabla)^{-1/2}$	$\sqrt{\left(1+\frac{1}{4}\delta^2\right)}$

Interpolating polynomials using the forward difference operator

The Gregory-Newton forward difference interpolating polynomial is given by

$$P_n(x) = f_0 + \frac{x-x_0}{h}\,\Delta f_0 + \frac{(x-x_0)(x-x_1)}{2!\,h^2}\,\Delta^2 f_0 + \cdots + \frac{(x-x_0)\cdots(x-x_{n-1})}{n!\,h^n}\,\Delta^n f_0.$$

Putting $u = (x-x_0)/h$, the interpolating polynomial using forward difference operator becomes

$$P_n(x) = \sum_{i=0}^{n} \binom{u}{i}\,\Delta^i f_0,$$

With the error

$$E_n(x,f) = \frac{u(u-1)\cdots(u-n)}{(n+1)!}\,h^{n+1} f^{(n+1)}(\xi), \quad x_0 \le \xi \le x_n.$$

Interpolating polynomials using the backward difference operator

Let $u = \frac{x-x_n}{h}$. The Gregory-Newton backward difference interpolating polynomial is given by

$$P_n(x) = f_n + u\,\nabla f_n + \frac{u(u+1)}{2!}\,\nabla^2 f_n + \cdots + \frac{u(u+1)\cdots(u+n-1)}{n!}\,\nabla^n f_n.$$

The truncation error becomes

$$E_n(x,f) = \frac{u(u+1)\cdots(u+n)}{(n+1)!}\,h^{n+1} f^{(n+1)}(\xi), \quad x_0 \le \xi \le x_n.$$

Example 4: For the following data calculate the differences, obtain the forward and backward difference polynomials. Interpolate at $x = 0.25$ and $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

Solution: The forward difference table is obtained as

The difference table is obtained as

0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

The forward difference polynomial is given by

$$P(x) = 1.4 + (x - 0.1)\,\frac{0.16}{0.1} + \frac{(x-0.1)(x-0.2)}{2}\,\frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

The backward difference polynomial is obtained as

$$P(x) = 2.28 + (x - 0.5)\,\frac{0.28}{0.1} + \frac{(x-0.5)(x-0.4)}{2}\,\frac{0.04}{0.01}$$

$$= 2x^2 + x + 1.28.$$

Both the polynomials are identical and we obtain

$f(0.25) = 1.655$ and $f(0.35) = 1.875$.