## 1.6 Orthogonal Trajectories

**Definition of Orthogonal Trajectory**

A curve which cuts every member of a given family of curves at a right angle is called an ***Orthogonal Trajectory*** of the given family of curves.

**Examples**

1. In the electrical field, the paths along which the current flows are orthogonal trajectories of the equipotential curves.
2. In fluid mechanics, the stream lines and the equipotential lines are orthogonal trajectories of one another.
3. In thermodynamics, the lines of heat flow are perpendicular to isothermal curves.

**Method for finding Orthogonal Trajectories**

Let

$$f(x, y, c) = 0 \qquad \text{Or, } f(x, y) = c \tag{1}$$

represent the equation of a given family of curves with single parameter $c$.

The orthogonal trajectory of the given family of curves (1) can be obtained as follows:

I. Find a first order differential equation from Equation (1). Let it be
$$\frac{dy}{dx} = \varphi(x,\ y) \tag{2}$$
II. Let $m_1$ be the slope of the curve of the given family and $m_2$ be the slope of an orthogonal trajectory. Since the curves cut at right angles, we have $m_1 m_2 = -1$.
III. Thus, $m_1 = \frac{dy}{dx} = \varphi(x, y)$ and $m_2 = \frac{-1}{m_1} = \frac{-1}{\varphi(x,y)}$
IV. Now the differential equation for the orthogonal trajectories is
$$\frac{dy}{dx} = m_2 = \frac{-1}{\varphi(x,y)} \qquad \text{or} \qquad \frac{dx}{dy} = -\varphi(x, y) \tag{3}$$
V. Solve Equation (3) and find the equation for the orthogonal trajectories.

**Example-1:** Find the orthogonal trajectories of the rectangular hyperbolas $x^2 - y^2 = c$, where $c$ is a parameter.

**Solution:** Equation of the given curves is

$$x^2 - y^2 = c \tag{4}$$

Differentiating Equation (4) w. r. to $x$, we get the differential equation as

$$\frac{dy}{dx} = \frac{x}{y} \tag{5}$$

The differential equation for orthogonal trajectories is

$$\frac{dx}{dy} = -\frac{x}{y} \tag{6}$$

Solving Equation (6), we find

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\ln x + \ln y = \ln k_1$$

The equation for orthogonal trajectories is $xy = k$.

**Alternative Method for finding Orthogonal Trajectories**

Let the equation of the given family of curves be

$$f(x, y) = c$$

$$\Rightarrow df = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$$

The differential equation for orthogonal trajectories is

$$\frac{dy}{dx} = \frac{f_y}{f_x}$$

Solving it we find the equation for orthogonal trajectories.

**Example-2:** Find the orthogonal trajectories for $y = ce^{x^2}$.

**Solution:** Given equation is $ye^{-x^2} = c$

$$\Rightarrow f(x, y) = ye^{-x^2}$$

$$\Rightarrow \frac{\partial f}{\partial x} = -2xye^{-x^2}, \ \frac{\partial f}{\partial y} = e^{-x^2}$$

The differential equation for orthogonal trajectory is

$$\frac{dy}{dx} = -\frac{1}{2xy}$$

Solving it, we get

$$\int 2y\, dy = -\int \frac{dx}{x}$$

$$\Rightarrow y^2 = -\ln x + \ln k_1 = \ln\left(\frac{k_1}{x}\right)$$

The equation of orthogonal trajectories is $xe^{y^2} = k$.

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