2.4

Problems 2.4

Q.(45) The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying joint probability table gives the proportions of individuals in the various ethnic group-blood group combinations.

Blood Group

Ethnic Group	O	A	B	AB
1	0.082	0.106	0.008	0.004
2	0.135	0.141	0.018	0.006
3	0.215	0.200	0.065	0.020

Suppose that an individual is randomly selected from the population, and define events A={type A selected}, B={type B selected}, C={ethnic group 3 selected}.

a) Calculate P(A), P(C), and P(A ∩ C)

b) Calculate both P(A/C) and P(C/A), and explain in context what each of these probabilities represents.

c) If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

Ans (a) According to the given question

$P(A) = 0.106 + 0.141 + 0.200 = 0.447$

$P(C) = 0.215 + 0.200 + 0.065 + 0.020 = 0.500$

$P(A \cap C) = 0.2$ [∵ A ∩ C is the single ethnic group (group 3) and blood group (type A), i.e. row 3 and column A]

b) $P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{0.2}{0.5} = 0.4$,

$P(C/A) = \frac{P(C \cap A)}{P(A)} = \frac{0.2}{0.447} = 0.447$

$P(A/C)$ represents that if we know that an individual comes from ethnic group 3 (event C), the probability that the individual has blood type A (event A) is 0.4. Similarly, $P(C/A)$ represents that if we know that an individual has blood type A (event A), the probability that the individual comes from ethnic group 3 (event C) is 0.447.

c) Let D = "ethnic group 1 is selected". Then we need to find the probability $P(D/B')$

$$P(D/B') = \frac{P(D \cap B')}{P(B')}$$

Now $P(B') = 1 - P(B) = 1 - (0.008 + 0.018 + 0.065)$
$= 0.909$

$P(D \cap B')$ = probability of being from ethnic group 1 (row 1) and not blood type B (columns O, A and AB)

$= 0.082 + 0.106 + 0.004 = 0.192$

$\therefore P(D/B') = \frac{0.192}{0.909} = 0.211$

2.4 Q.(47)

Consider randomly selecting a student at a certain university, and let A denote the event that the selected individual has a Visa credit card and B be the analogous event for a Master card. Suppose that $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cap B) = 0.25$. Calculate and interpret each of the following probabilities.

a). $P(B/A)$ b). $P(B'/A)$ c). $P(A/B)$ d). $P(A'/B)$

e) Given that the selected individual has at least one card, what is the probability that he or she has a Visa card?

Ans:

$A = \{\text{visa card}\}$, $B = \{\text{Mastercard}\}$

$P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B) = 0.25$

(a) $P(B/A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.25}{0.5} = 0.50 = 50\%$

50% students with visa cards also have a master card.

(b) $P(B'/A) = \dfrac{P(A \cap B')}{P(A)}$ $[\because P(A \cap B') = P(A) - P(A \cap B)]$

$= \dfrac{P(A) - P(A \cap B)}{P(A)}$

$= \dfrac{0.5 - 0.25}{0.5} = 1 - \dfrac{0.25}{0.5} = 1 - 0.5 = 0.5 = 50\%$

50% students with visa cards don't have a master card.

(c) $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.25}{0.4} = \dfrac{25}{40} = \dfrac{5}{8} = 0.625 = 62.5\%$

62.5% students with a MasterCard also have a Visa card.

(d) $P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B) - P(A \cap B)}{P(B)}$

$= \frac{0.4 - 0.25}{0.4} = 1 - \frac{0.25}{0.4} = 1 - 0.625$

$= 0.375 = 37.5\%$

37.5% Students with a Mastercard don't have a Visa card.

(e) $A \cup B$ represents the event that the selected individual has at least one card.

Then we have to find $P(A/A \cup B)$

Now, $P(A/A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$

$= \frac{P(A)}{P(A \cup B)}$ $\left[\because A \subseteq A \cup B \quad \therefore A \cap (A \cup B) = A. \right.$

$= \frac{0.5}{P(A) + P(B) - P(A \cap B)}$

$= \frac{0.5}{0.5 + 0.4 - 0.25} = \frac{0.5}{0.65}$

$= \frac{50}{65} = \frac{10}{13} \simeq 0.7692 = 76.92\%$

Q.(50) [2.4]

A department store sells sport shirts in three sizes (small, medium, and large), three patterns (plaid, print, and stripe), and two sleeve lengths (long and short). The accompanying tables give the proportions of shirts sold in the various category combinations.

Short-sleeved

Size	Pattern Pl	Pr	St
S	0.04	0.02	0.05
M	0.08	0.07	0.12
L	0.03	0.07	0.08

Long-sleeved

Size	Pattern Pl	Pr	St
S	0.03	0.02	0.03
M	0.10	0.05	0.07
L	0.04	0.02	0.08

a). What is the probability that the next shirt sold is a medium, long-sleeved, print shirt?

b). What is the probability that the next sold is a medium print shirt?

c). What is the probability that the next shirt sold is a short sleeved shirt? A long-sleeved shirt?

d). What is the probability that the size of the next shirt sold is medium? That the pattern of the next shirt sold is a print?

e). Given that the shirt just sold was a short-sleeved plaid what is the probability that its size was medium?

f). Given that the shirt just sold was a medium plaid, what is the probability that it was short-sleeved? Long-sleeved?

Ans: Let M = {sold shirt is a medium}
LS = {sold shirt is a long-sleeved shirt}
SS = {sold shirt is a short-sleeved shirt}
Pr = {sold shirt is a print shirt}
Pl = {sold shirt is a plaid shirt}

(a) $P(M, LS, Pr) = 0.05$

(b) $P(M, Pr) = P(M, Pr, SS) + P(M, Pr, LS)$
$= 0.07 + 0.05 = 0.12$

(c) $P(SS) =$ sum of probabilities of all 9 combinations
$= 0.04 + 0.02 + 0.05 + 0.08 + 0.07 + 0.12 + 0.03 + 0.07 + 0.08$
$= 0.56$

$P(LS) = 1 - P(SS)$ [using $P(A) + P(A') = 1$ $\Rightarrow P(A') = 1 - P(A)$]
$= 1 - 0.56 = 0.44$

(d) $P(M) = 0.08 + 0.07 + 0.12 + 0.10 + 0.05 + 0.07 = 0.49$
$P(Pr) = 0.02 + 0.07 + 0.07 + 0.02 + 0.05 + 0.02 = 0.25$

(e) $P(M / SS \cap Pl) = \dfrac{P(M \cap SS \cap Pl)}{P(SS \cap Pl)} = \dfrac{0.08}{0.04 + 0.08 + 0.03} = 0.533$

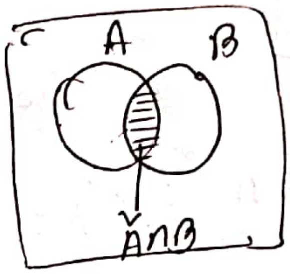
(f) $P(SS / M \cap Pl) = \dfrac{P(SS \cap M \cap Pl)}{P(M \cap Pl)} = \dfrac{0.08}{0.08 + 0.1} = 0.444$

$P(LS / M \cap Pl) = \dfrac{P(LS \cap M \cap Pl)}{P(M \cap Pl)} = \dfrac{0.10}{0.08 + 0.1} = \dfrac{0.10}{0.18}$
$= 0.556.$

Q.(56) [2.4]

For any events A and B with $P(B) > 0$, show that $P(A|B) + P(A'|B) = 1$

Proof,



We know that $P(A' \cap B) = P(B) - P(A \cap B) \longrightarrow (1)$

Now, $P(A|B) + P(A'|B)$

$$= \frac{P(A \cap B)}{P(B)} + \frac{P(A' \cap B)}{P(B)}$$

$$= \frac{P(A \cap B) + P(A' \cap B)}{P(B)}$$

$$= \frac{P(A \cap B) + P(B) - P(A \cap B)}{P(B)} \quad [\text{using } (1)]$$

$$= \frac{P(B)}{P(B)} = 1 \quad \text{(proved)}$$

2.4

Q. (58) show that for any three events A, B, and C with $P(C) > 0$, $P(A \cup B | C) = P(A|C) + P(B|C) + P(A \cap B/C)$.

Proof:

From the definition of the conditional probability,

$$P(A \cup B / C) = \frac{P[(A \cup B) \cap C]}{P(C)}$$

$$= \frac{P[(A \cap C) \cup (B \cap C)]}{P(C)}$$

$$= \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P[(A \cap B) \cap C)]}{P(C)}$$

$$= P(A/C) + P(B/C) - P(A \cap B/C)$$

(proved)

Q.(54) [2.4]

In exercise 13 $A_i = \{\text{awarded project } i\}$, for $i=1,2,3$.
Given that $P(A_1) = 0.22$, $P(A_2) = 0.25$, $P(A_3) = 0.28$,
$P(A_1 \cap A_2) = 0.11$, $P(A_1 \cap A_3) = 0.05$, $P(A_2 \cap A_3) = 0.07$,
$P(A_1 \cap A_2 A_3) = 0.01$. Compute the following probabilities

a). $P(A_2 | A_1)$, b). $P(A_2 \cap A_3 | A_1)$

c). $P(A_2 \cup A_3 / A_1)$ d). $P(A_1 \cap A_2 \cap A_3 | A_1 \cup A_2 \cup A_3)$

Ans: (a)

$$P(A_2 | A_1) = \frac{P(A_2 \cap A_1)}{P(A_1)} = \frac{0.11}{0.22} = 0.5$$

(b)

$$P(A_2 \cap A_3 | A_1) = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1)}$$

$$= \frac{0.01}{0.22} = \frac{1}{22} = 0.0455$$

(c)

$$P(A_2 \cup A_3 | A_1) = \frac{P[A_1 \cap (A_2 \cup A_3)]}{P(A_1)}$$

$$= \frac{P[(A_1 \cap A_2) \cup (A_1 \cap A_3)]}{P(A_1)}$$

$$= \frac{-P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2) + P(A_1 \cap A_3)}{P(A_1)} = \frac{0.11 + 0.05 - 0.01}{0.22}$$

$$= \frac{15}{22} = 0.682$$

(d)

$$P(A_1 \cap A_2 \cap A_3 / A_1 \cup A_2 \cup A_3) = \frac{P(A_1 A_2 A_3 \cap (A_1 + A_2 + A_3))}{P(A_1 + A_2 + A_3)}$$

$$= \frac{P(A_1 A_2 A_3)}{P(A_1 + A_2 + A_3)} \quad [\because A_1 A_2 A_3 \subseteq A_1 + A_2 + A_3]$$

$$= \frac{P(A_1 A_2 A_3)}{P(A_1) + P(A_2) + P(A_3) - P(A_1 A_2) - P(A_1 A_3) - P(A_2 A_3) + P(A_1 A_2 A_3)}$$

$$= \frac{0.01}{0.22 + 0.25 + 0.28 - 0.11 - 0.05 - 0.07 + 0.01} = \frac{0.01}{0.53} = 0.0189$$

Q. (59) [2.4]

At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus gas 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

a). what is the probability that the next customer will request plus gas and fill the tank ($A_2 \cap B$)?

b) what is the probability that the next customer fills the tank?

c) If the next customer fill the tank, what is the probability that the ~~next customer~~ regular gas is requested? plus? premium?

Ans: Given that,

$P(A_1) = 0.4$, $P(A_2) = 0.35$ $P(A_3) = 0.25$;

$P(B|A_1) = 0.30$, $P(B|A_2) = 0.60$, $P(B|A_3) = 0.50$.

(a) $P(A_2 \cap B) = P(B|A_2)\,P(A_2)$ (By multiplication rule)

$= 0.60 \times 0.35$

$= 0.210$

(b) Given that

B = next customer fill the tank

We have to find $P(B)$.

using law of total probability,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)\,P(A_i) = \sum_{i=1}^{3} P(B/A_i)\,P(A_i)$$

$$= P(B|A_1)P(A_1) + P(B/A_2)P(A_2) + P(B/A_3)P(A_3)$$

$$= 0.30 \times 0.4 + 0.60 \times 0.35 + 0.50 \times 0.25$$

$$= 0.12 + 0.210 + 0.1250$$

$$= 0.455$$

(c) probability that the regular gas is requested given that the next customer fill the tank

$$= P(A_1/B)$$

$$= \frac{P(A_1 \cap B)}{P(B)} = \frac{P(B|A_1)\,P(A_1)}{P(B)} = \frac{0.30 \times 0.4}{0.455} = 0.264$$

Probability that the ~~regula~~ plus gas is requested, given ~~that~~ the next customer fill the tank

$$= P(A_2/B)$$

$$= \frac{P(A_2 \cap B)}{P(B)} = \frac{P(B/A_2)\,P(A_2)}{P(B)} = \frac{0.6 \times 0.35}{0.455} = 0.462$$

Probability that the premium gas is requested, given that the next customer fill the tank

$$P(A_3|B)$$

$$= \frac{P(A_3 \cap B)}{P(B)} = \frac{P(B/A_3)\,P(A_3)}{P(B)} = \frac{0.5 \times 0.25}{0.455} = 0.274$$

Q. (63) [2.4] (sol), Q. (55) (print)

For customers purchasing a refrigerator at a certain appliance store, let A be the event that the refrigerator was manufactured in the U.S., B be the event that the refrigerator had an icemaker, and C be the event that the customer purchased an extended warranty. Relevant probabilities are

$P(A) = 0.75$, ~~P(B)~~ $P(B|A) = 0.9$, $P(B|A') = 0.8$

$P(C|A \cap B) = 0.8$, $P(C|A \cap B') = 0.6$, $P(C|A' \cap B) = 0.7$

$P(C|A' \cap B') = 0.3$

a) construct a tree diagram consisting of first, second, and third generation branches, and place an event label and appropriate probability next to each branch.

b) compute $P(A \cap B \cap C)$

c) compute $P(B \cap C)$

d) compute $P(C)$

e) compute $P(A | B \cap C)$, the probability of a U.S. purchase given that an icemaker and extended warranty are also purchased.

Ans (a).

The tree diagram which represents the experimental situation given in the exercise is given below.

1st generation branches → 1st layer of events and the adequate probabilities,

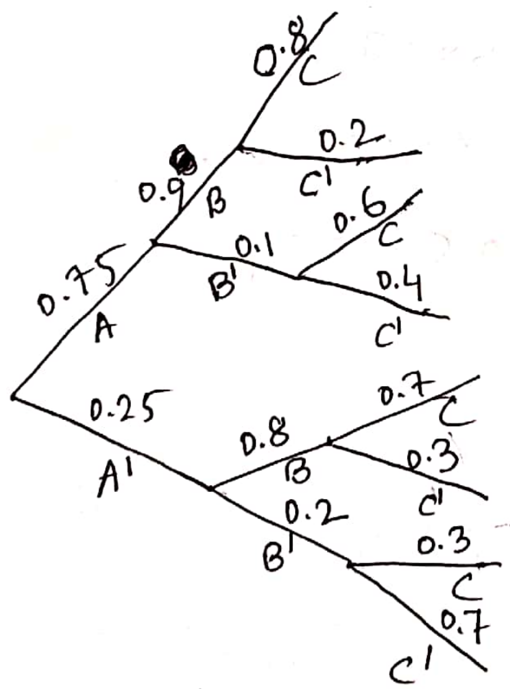
2nd " " → 2nd layer of events, with the adequate conditional probabilities given,

3rd " " → 3rd layer of events, with the adequate conditional probabilities given.

Given probabilities are

$P(A) = 0.75$, $P(B/A) = 0.9$, $P(B|A') = 0.8$, $P(C|A\cap B) = 0.8$

$P(C|A\cap B') = 0.6$, $P(C|A'\cap B) = 0.7$ $P(C|A'\cap B') = 0.3$

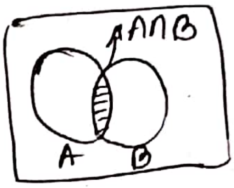


(b) $P(A \cap B \cap C) = P(C|A \cap B)\, P(A \cap B)$ [using multiplication Rule]

$= P(C|A \cap B)\, P(B|A)\, P(A)$ [" " "]

$= 0.8 \times 0.9 \times 0.75 = 0.54$

(c) For any three events A, B, C, we have



$P(A) = P(A \cap B) + P(A \cap B')$

$P(B \cap C) = P(B \cap C \cap A) + P(B \cap C \cap A')$

$= P(B \cap C \cap A) + P(C \cap A' \cap B)$

$= P(A \cap B \cap C) + P(C|A' \cap B)\, P(A' \cap B)$

$= P(A \cap B \cap C) + P(C|A' \cap B)\, P(B|A')\, P(A')$

$= 0.54 + 0.7 \times 0.8 \times 0.25 = 0.68$

(d) $P(C) = P(A \cap B \cap C) + P(A' \cap B \cap C) + P(A \cap B' \cap C) + P(A' \cap B' \cap C)$

$= 0.75 \times 0.9 \times 0.8 + 0.25 \times 0.8 \times 0.7$

$+ 0.75 \times 0.1 \times 0.6 + 0.25 \times 0.2 \times 0.3$

$= 0.54 + 0.14 + 0.045 + 0.015 = 0.74$

(e) $P(A|B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)}$

$= \dfrac{0.54}{0.68} = 0.7941$