Bernoulli trial and Bernoulli pmf

A trial or experiment with two possible outcomes is Bernoulli trial. The r.v. associated with Bernoulli trial is Bernoulli r.v. and its pmf is Bernoulli pmf. If success and failure are two possible outcomes, the the image of r.v. X can be taken as

$$X = \begin{cases} 0 & \text{if } \cancel{X} = \text{Failure (F)} \\ 1 & \text{if } \cancel{S} = \text{Success (S)} \end{cases}$$

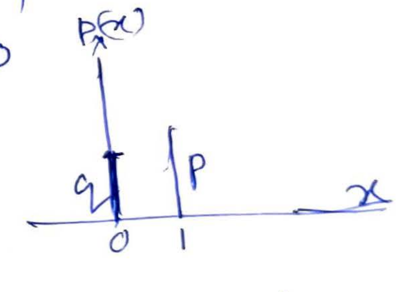
i.e. $X = \{0, 1\}$

The pmf of the Bernoulli r.v. X can be def^n by

$$p_X(x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = 0 \end{cases}$$

or

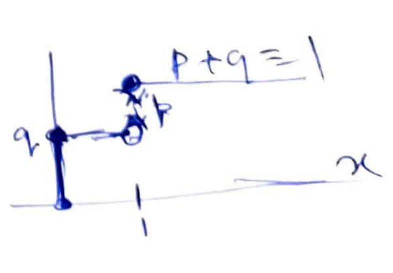
$$p_X(0) = P[X=0] = q$$
$$p_X(1) = P[X=1] = p$$



where p, q satisfies $p + q = 1 \Rightarrow q = 1 - p$

The probability distribution function (PDF) or cdf is def^n by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ q & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



Binomial experiment

An experiment is Bernoulli experiment with finite no. of trials that satisfies the following properties:

ⓐ The experiment consists of a sequence of ~~A~~ smaller experiments (called trials) where n is fixed in advance of experiment.

ⓑ ~~Each~~ Each trial can result in one of the two possible outcomes (discontinuous trials) which generally denoted by success(S) & failure(F).

ⓒ The trials are independent, i.e. outcome on any particular trial does not influence the outcome on any other trial.

ⓓ The probability of success(S) is $P(S)$ is constant from trial to trial and denoted by $p = P(S)$.

Binomial random variable

The r.v. X associated with the binomial experiment ~~with~~ with n trials is def^n as follows: X counts no. of successes (k) out of n trials, i.e.

X = the no. of successes (S) out of n trials.

Binomial pmf

Consider a sequence of n independent Bernoulli trials with probability of success p on each trial and probability of failure q on that same trial satisfying $p+q=1$.

The r.v. X counts no. of successes out of n trials. Let no. of success be k, i.e. $X=k$ then k no. of successes can be obtained out of n trials in $\binom{n}{k}$ ways.

probability of k successes $= p^k$ and
probability of $n-k$ failures $= q^{n-k}$

The probability of getting k no. of successes out of n no. of trials is

$$p_k = P[X=k] = \binom{n}{k} p^q q^{n-k}, \quad q=1-p, \quad k=0,1,2 \ldots n$$

$$= b(k;n,p)$$

We have

$$1=(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} b(k;n,p)$$

where $b(k;n,p) = \binom{n}{k} p^k q^{n-k}$ is the term of the binomial expansion of $(p+q)^n$.

Binomial pmf

Consider a sequence of n independent Bernoulli trials with probability of success p on each trial and probability of failure q on that same trial satisfying $p+q=1$.

The r.v. X counts no. of successes out of n trials. Let no. of success be k, i.e. $X=k$

then k no. of successes can be obtained out of n trials in $\binom{n}{c_k}$ ways.

Probability of k successes $= p^k$ and probability of $n-k$ failures $= q^{n-k}$

The probability of getting k no. of successes out of n no. of trials is

$$p_k = P[X=k] = \left({}^{n}c_k\right) p^q q^{n-k}, \quad q=1-p, \; k=0,1,2 \ldots n$$

$$= b(k;n,p)$$

We have

$$1=(p+q)^n = \sum_{k=0}^{n} \left({}^{n}c_k\right) p^k q^{n-k}$$

$$= \sum_{k=0}^{n} b(k;n,p)$$

where $b(k;n,p) = \left({}^{n}c_k\right) p^k q^{n-k}$ is the term of the binomial expansion of $(p+q)^n$.

The binomial pmf of the Binomial distribution is defined by

$$f(x) = P[X=x] = b(x;n,p)$$
$$= \begin{cases} \binom{n}{c_x} p^x q^{n-x}, & x=0,1,2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

where $q = 1-p$. $\boxed{p < 0.5}$

* n, p are parameters.

The binomial distribution or the cdf associated with binomially distributed r.v. X is denoted by $X \sim B(n;p)$ where

$$F(x) = B(x;n,p) = P[X \le x] = \sum_{y=0}^{x} \binom{n}{c_y} p^y q^{n-y}$$
$$= \sum_{y=0}^{x} b(y;n,p)$$

or

$$F(x) = P[X \le x] = \sum_{y \le x} f(y) = \sum_{y=0}^{x} b(y;n,p)$$
$$= \sum_{y=0}^{x} \binom{n}{c_y} p^y q^{n-y}$$
$$= B(x;n,p)$$

Q. $b(x;n,1-p) = b(n-x,n,p)$. $x = 0,1,2 \ldots n$. prove it.

proof

$$b(x;n,1-p) = \binom{n}{c_x} (1-p)^x p^{n-x}$$
$$= \binom{n}{c_{n-x}} p^{n-x} (1-p)^x$$

$(\because n-(n-x) = x)$

$$= b(n-x;n,p) \quad \text{proved}$$

(b) We have for $X \sim B(n;p)$,

$$B(x;n,p) = P[X \le x] = \sum_{y=0}^{x} b(y;n,p)$$

We have, for $X \sim B(n;1-p)$,

$$B(x;n,1-p) = \sum_{y=0}^{x} b(y;n,1-p)$$

$$= \sum_{y=0}^{x} b(n-y;n,p)$$

Now $P[X > x] = 1 - P[X \le x]$

$$= 1 - B(x;n,1-p)$$

$$= 1 - \sum_{y=0}^{x} b(y;n,1-p)$$

$$= 1 - \sum_{y=0}^{x} b(n-y;n,p)$$

$$= \sum_{y=x+1}^{n} b(n-y;n,p)$$

$$= b(n-x-1;n,p) + b(n-x-2;n,p) + \cdots + b(1;n,p) + b(0;n,p)$$

$$= b(0;n,p) + b(1;n,p) + \cdots + b(n-x-2;n,p) + b(n-x-1;n,p)$$

$$= \sum_{y=0}^{n-x-1} b(y;n,p)$$

$$= B(n-x-1;n,p)$$

Hence

$$B(x;n,1-p) = P[X \le x] = 1 - P[X > x]$$
$$= 1 - B(n-x-1;n,p) \text{ (proved)}$$

Ex-1 (a) $b(2; 10; 0.4) = \binom{10}{2}(0.4)^2(0.6)^8 = 0.1209$

$b(8; 10; 0.6) = \binom{10}{8}(0.6)^8(0.4)^2 = 0.1209$

$\therefore b(2; 10; 0.4) = b(8; 10, 0.6)$

(b) $B(2; 10, 0.4) = \sum_{x=0}^{2} b(x; 10, 0.4) = 0.1673$

$B(10-2-1; 10, 1-0.4) = B(7; 10, 0.6)$

$= \sum_{x=0}^{7} b(x; 10, 0.6) = 0.8327$

$1 - B(10-2-1; 10, 1-0.4) = B(2; 10, 0.4)$

$= 0.1673$

Ex-2 A fair coin tossed continuously. What is probability of getting 2 heads out of 10 tosses. What is prob. of getting at most 2 heads.

Sol From the given Bernoulli trial, we have

$p = P(H) = \frac{1}{2}$ & $q = P(T) = \frac{1}{2}$

Given no. of trials $= n = 10$

R.V. X counts no. of heads $x = 2$.

Prob. of getting 2 heads out of 10 trials

$= f(2) = P[X=2] = b(2; 10, 0.5)$

$= \binom{10}{2}(0.5)^2(0.5)^8 = 0.0439$

Prob. of getting at most 2 heads $= f(2)$

$= B(2; 10, 0.5) = \sum_{x=0}^{2} b(x; 10, 0.5) = 0.0547$

Ex-3 A fair die is rolled continuously.

(a) What prob. of getting 2 sixes from 10 trials.

(b) What is prob. of getting at most 2 sixes from 10 trials.

Soln

Prob. getting a six $= \frac{1}{6} = p$

prob. of not getting a six $= q = 1 - \frac{1}{6} = \frac{5}{6}$

Given no. of trials $= n = 10$

Let R.V $X =$ no. of sixes

(a) Prob. of getting 2 sixes out of 10 trials $= f(2) = b(2; 10; \frac{1}{6})$

$$= \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8 = 0.2907$$

(b) Prob. of getting at most 2 sixes out of 10 trials

$$= F(2) = B(2; 10, 1/6)$$

$$= \sum_{x=0}^{2}\binom{10}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{10-x}$$

$$= 0.7752$$

Note (1) A distribution is symmetric if

$f(x) = f(n-x)$, $x = 0, 1, 2 \ldots n$

→ Binomial dist. is symmetric if $p = \frac{1}{2}$

since $f(x) = \binom{n}{x}(1/2)^n = \binom{n}{n-x}(1/2)^n$ for any n.

Mean, variance and standard deviation of binomial distribution

The pmf of binomial distribution $X \sim B(n, p)$ is $f(x) = \binom{n}{x} p^x q^{n-x}$, $q = 1-p$

$x = 0, 1, 2, \dots n$

Mean $(X) = \mu_X$

$$\mu_X = E(X) = \sum_{x=0}^{n} x f(x) = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$\because n-x = (n-1)-(x-1)$

$$= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)![(n-1)-(x-1)]!} p^{x-1} q^{n-x}$$

$$= np(p+q)^{n-1} = np \quad (\because p+q=1)$$

Variance $(V(X))$

$$V(X) = E(X^2) - (E(X))^2$$

$$= E(X^2) - n^2 p^2$$

where $E(X^2) = E(X(X-1) + X)$

$$= E(X(X-1)) + E(X)$$

$$= \sum_{x=0}^{n} x(x-1)\, f(x) + np$$

$$= \sum_{x=0}^{n} x(x-1) \left({}^{n}C_{x}\right) p^{x} q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^{x} q^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} p^{x} q^{n-x} + np$$

$$= n(n-1)p^{2} \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\,(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^{2}(p+q)^{n-2} + np$$

$$= n(n-1)p^{2} + np$$

Hence

$$V(X) = E(X^2) - (E(X))^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$\Rightarrow \sigma^2 = npq$$

$$S.D(X) = \sigma = \sqrt{npq}$$