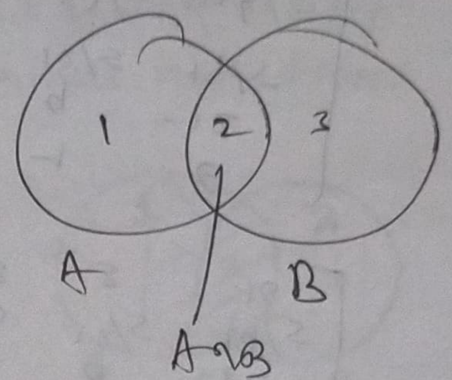
Q. Prove $P(A\cup B) = P(A) + P(B) - P(A\cup B)$ without any probability measuring concept.

Proof

We have



$A\cup B = A\cup(B-(A\cap B))$

where A, $B-(A\cap B)$ are mutually exclusive events. So

$3 \to B-(A\cap B)$

$2 \to A\cap B$

$1 \to A-(A\cap B)$

$P(A\cup B) = P(A) + P(B-(A\cap B))$

$= P(A) + P(B) - P(A\cap B)$,

since $A\cap B \subset B$

proved

$\boxed{\because \text{① } P(B-A) = P(B)-P(A) \text{ if } A\subseteq B \quad \text{② } A\cap B\subset B}$

Q. Prove that

$$P(A\cap B') = P(A) - P(A\cap B)$$

Proof $P(A\cap B') = P[A\cap(S-B)]$

$= P[(A\cap S)-(A\cap B)]$

$= P[A-(A\cap B)]$

$= P(A) - P(A\cap B)$ $(\because A\cap B\subset A)$

Q. Prove that $P(A\cap B'\cap C') = P(A) - P(A\cap B) - P(A\cap C) + P(A\cap B\cap C)$

Proof we have $P(B-A) = P(B) - P(A)$ if $A\subset B$

and $P(A\cap B') = P(A) - P(A\cap B)$.

Hence $P(A\cap B'\cap C') = P(A\cap(B\cup C)') = P(A) - P[A\cap(B\cup C)]$

$$= P(A) - P[(A \cap B) \cup (A \cap C)]$$
$$= P(A) - [P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

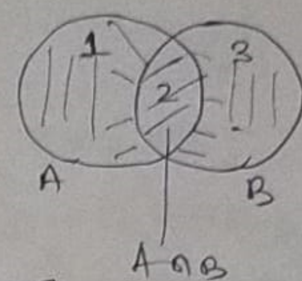
proved

Q Prove that

$$P(A \cup B) - P(A \cap B) = P[A - (A \cap B)] + P[B - (A \cap B)]$$

where '$-$' sign is set minus notation.

Proof



$A \cup B$ can be written as

$$A \cup B = [A - (A \cap B)] \cup (A \cap B) \cup [B - (A \cap B)]$$

where $[A - (A \cap B)]$, $A \cap B$ & $[B - (A \cap B)]$ are mutually exclusive events, so

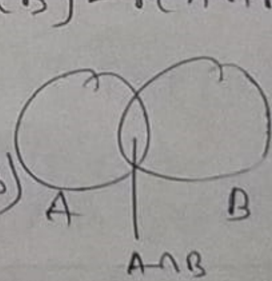
$$P(A \cup B) = P[A - (A \cap B)] + P(A \cap B) + P[B - (A \cap B)]$$

$$\Rightarrow P(A \cup B) - P(A \cap B) = P[A - (A \cap B)] + P[B - (A \cap B)]$$

Q. Prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof



We have

$$A = [A - (A \cap B)] \cup (A \cap B)$$

where $A \cap B$ & $A - (A \cap B)$ are mutually exclusive

so $P(A) = P(A - (A \cap B)) + P(A \cap B)$

Similarly

$$P(B) = P[B-(A\cap B)] + P(A\cap B)$$

Hence

$$P(A) + P(B) = P[A-(A\cap B)] + P[B-(A\cap B)] + 2P(A\cap B)$$

$$= P(A\cup B) - P(A\cap B) + 2P(A\cap B)$$

$$= P(A\cup B) + P(A\cap B)$$

$$\Rightarrow P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

proved

Q. Prove that

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$$

Proof

Considering $B\cup C = D$, we have

$$P(A\cup B\cup C) = P(A\cup D)$$

$$= P(A) + P(D) - P(A\cap D)$$

$$= P(A) + P(B\cup C) - P[A\cap (B\cup C)]$$

$$= P(A) + P(B) + P(C) - P(B\cap C) - P[(A\cap B)\cup(A\cap C)]$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - [P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]]$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

proved

considering
$A = A_1$, $B = A_2$ & $C = A_3$

we have

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= \sum_{j=1}^{3} P(A_j) - \sum_{1 \le i < j \le 3} P(A_i \cap A_j) + (-1)^2 P(A_1 \cap A_2 \cap A_3)$$

Similarly

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum_{j=1}^{4} P(A_j) - \sum_{1 \le i < j \le 4} P(A_i \cap A_j) + \sum\sum_{1 \le i < j < k \le 4} P(A_i \cap A_j \cap A_k) + (-1)^3 P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

$$= P(A_1) + P(A_2) + P(A_3) + P(A_4)$$
$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_1 \cap A_4)$$
$$- P(A_2 \cap A_3) - P(A_2 \cap A_4) - P(A_3 \cap A_4)$$
$$+ P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_4)$$
$$+ P(A_1 \cap A_3 \cap A_4) + P(A_2 \cap A_3 \cap A_4)$$
$$- P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

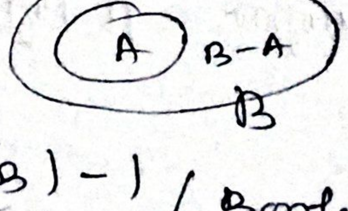
In general

$$P(A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= \sum_{j=1}^{n} P(A_j) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$
$$+ \sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) + \cdots$$
$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

Note

① If $A \subseteq B$, then

$$P(B - A) = P(B) - P(A)$$



② For any events A & B,

$$P(A \cap B) \ge P(A) + P(B) - 1$$

$(\because P(A \cup B) \le 1)$ (Bonferroni's inequality).

③ For any events, $A_1, A_2 \ldots A_n$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \ge \sum_{j=1}^{n} P(A_j) - (n-1)$$

④ For any events A & B

$$P(A \cup B) \le P(A) + P(B)$$ (Boole's Ineq)

Examples (Sample space & events)

① Experiment: Tossing two fair coins

Sol Sample space = {HH, HT, TH, TT}

At least one head event

= {HH, HT, TH}

At most one head event = {HT, TH, TT}

1st head event = {HH, HT}

② Experiment:

Drawing 4 screws from a lot of right-handed (R) and left-handed (L) screws

Sol^n Total no. of outcomes

$= 2^4 = 16$

Sample space is

{RRRR, RRRL, RRLR, RLRR, LRRR

RRLL, RLRL, LRRL, RLLR,

LRLR, LLRR, RLLL, LRLL, LLRL,

LLLR, LLLL}

③ Experiment: Rolling 2 dice

Event = sum of $x+y = 6$

Sol^n Total no. of outcomes $= 6^2 = 36$

Sample space $= \{(x, y) \mid 1 \le x \le 6, 1 \le y \le 6\}$

event $= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

(4) Experiment: Tossing a coin until the 1st head (H) appears

Sol^n

Sample space

$= \{H, TH, TTH, TTTH, ---\}$

(5) Experiment: Rolling a die until the 1st 'six' appears

Sol^n

Getting a six = S (success)

Not getting a six = F (failure)

Sample space

$= \{S, FS, FFS, FFFS, ---\}$

(6) Experiment: (Male card = M, female card = F)

Getting 2 female cards before getting 3 male cards.

Sol^n Sample space $= \{FF, FMF, MFF, FMMF, MFMF, MMFF\}$

(6) Three screws are drawn at random from a lot of 100 screws, 10 of which are defective. Find the event of getting at ~~least~~ most two ~~non~~ ~~heads~~ non-defective screw.

Sol^n Drawing deffective screw = D

" Non defective screw = N

Event (E) = At ~~least~~ most two ~~heads~~ nondefective screw

$= \{DNN, NDN, NND, DDN, DND, NDD, DDD\}$