Solution manual and Scheme of evaluation

MID SEMESTER EXAMINATION, SPRING 2023-2024

Subject: Discrete Mathematics

Code: MA21002

B. Tech.
Fourth Semester (_AB & Back)
Spring 2023-2024 (SAS)

Full Marks: 20 **Time: 90 minutes**

Answer any FOUR QUESTIONS including question No. 1 which is compulsory.
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All parts of a question should be answered at one place only.

Q.1 **Answer the following Questions**

a) Let p: It is below freezing, q: It is snowing.
Express the English sentence "That it is below freezing is necessary and sufficient for it to be snowing" as a proposition using p, q, and logical connectives.

Ans: $p \leftrightarrow q$ ………………..(1 mark)

b) Find the converse and contrapositive of the conditional statement "I come to class whenever there is going to be a quiz."

Ans:

Converse: If I come to class, then there is going to be a quiz. ………………..(0.5 mark)

Contrapositive: If I do not come to class, then there is no quiz. …………….(0.5 mark)

c) What is the negation of the statement "All Indians eat vegetables"

Ans: Some Indians do not eat vegetables ………………..(1 mark)

d) How many reflexive relations are there if the relation is defined on a set with 5 elements.

Ans: $2^{5^2-5} = 2^{20}$. ………………..(1 mark)

e) Find the power set of the set $A = \{\varphi, \{\varphi\}\}$

Ans: $\{\phi, \{\varphi\}, \{\{\varphi\}\}, \{\varphi, \{\varphi\}\}\}$. ………………..(1 mark)

Q. 2

a) Show that $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$ are logically equivalent.

Ans: Use truth table.

For each correct row 0.25 mark

For conclusion 0.5 mark

b) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology by developing a series of logical equivalences.

Ans: $(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ by conditional disjunction equivalence ….(0.5 mark)

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ by the De Morgan law ….(0.5 mark)

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ by the associative and commutative laws for disjunction ….(0.5 mark)

$\equiv T \vee T$ by Negation laws and the commutative law for disjunction….(0.5 mark)

$\equiv T$ by the domination law….(0.5 mark)

Q. 3

a) Using mathematical induction prove that for every positive integer n,

$$1.2 + 2.3 + 3.4 + \ldots\ldots\ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Ans: Let

$p(n)$: $1.2 + 2.3 + 3.4 + \ldots\ldots\ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$. ….(0.5 mark)

To prove that $\forall n \geq 1\ P(n)$.

Basis step: $P(1)$ is true, because $1.2 = \frac{1(1+1)(1+2)}{3}$. ….(0.5 mark)

Inductive step: For the inductive hypothesis we assume that $P(k)$ holds for an arbitrary positive integer $k > 1$. That is, we assume that

$$1.2 + 2.3 + 3.4 + \ldots\ldots\ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}. \quad \text{….(0.5 mark)}$$

Under this assumption, it must be shown that $P(k+1)$ is true, that is

$$1.2 + 2.3 + 3.4 + \ldots\ldots\ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

is also true.

Now

$$\begin{aligned}
&1.2 + 2.3 + 3.4 + \cdots \ldots \ldots + (k+1)(k+2) \\
&= \{1.2 + 2.3 + 3.4 + \cdots \ldots \ldots k(k+1)\} + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= (k+1)(k+2)\left\{\frac{k}{3} + 1\right\} \\
&= \frac{(k+1)(k+2)(k+3)}{3}.
\end{aligned}$$

This shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step. ….(0.5 mark)

We have completed the basis step and the inductive step, so by mathematical induction

$$\begin{array}{l}
\forall k > 1\ \big(P(k) \to P(k+1)\big) \\
\underline{P(1) \qquad\qquad} \\
\therefore \forall n \geq 1\ P(n)
\end{array}$$

….(0.5 mark)

b) Find M_{R^3}, where $R = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$ is a relation on $A = \{1,2,3\}$.

Ans: $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ ….(0.5 mark)

$$M_{R^2} = M_{RoR} = M_R \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ ….(1 mark)

$$M_{R^3} = M_{R^2oR} = M_{R^2} \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ ….(1 mark)

Q. 4

a) How many positive integers are not exceeding 1500 is divisible by 7, 13, or 21.

Ans: Let A be the set of positive integers not exceeding 1500 that are divisible by 7, B be the set of positive integers not exceeding 1500 that are divisible by 13 and C be the set of positive integers not exceeding 1500 that are divisible by 21. Then

$$|A| = \left\lfloor \frac{1500}{7} \right\rfloor = 214, |B| = \left\lfloor \frac{1500}{13} \right\rfloor = 115, |C| = \left\lfloor \frac{1500}{21} \right\rfloor = 71,$$ ….(0.5 mark)

Now, $A \cap B$ is the set of positive integers not exceeding 1500 that are divisible by 7 and 13. That is $A \cap B$ is the set of positive integers not exceeding 1500 that are divisible by LCM(7,13)=91. So

$|A \cap B| = \left\lfloor \frac{1500}{91} \right\rfloor = 16.$ $|A \cap C| = \left\lfloor \frac{1500}{21} \right\rfloor = 71$, $|B \cap C| = \left\lfloor \frac{1500}{273} \right\rfloor = 5$, $|A \cap B \cap C| = \left\lfloor \frac{1500}{273} \right\rfloor = 5.$ ….(0.5 mark)

Thus,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ ….(1 mark)

$= 214 + 115 + 71 - 16 - 71 - 5 + 5$

$= 313$ ….(0.5 mark)

Therefore, there are 313 positive integers not exceeding 1500 are divisible by 7, 13, or 21.

b) Show that the argument form is valid using rules of inference with premises $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, \neg s$ and conclusion $q \rightarrow r$.

Ans: As the conclusion is $q \rightarrow r$, it is enough to show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, \neg s, q$ and conclusion r is valid. That is

P_1 $(p \wedge t) \rightarrow (r \vee s)$
P_2 $q \rightarrow (u \wedge t)$
P_3 $u \rightarrow p$
P_4 $\neg s$
P_5 q

$\therefore$ r ….(0.5 mark)

For each step 0.25 mark

Step 1

$P_2 \quad q \to (u \land t)$

$P_5 \quad q$

$\therefore\ u \land t$ by Modus Ponens

Step 2

$u \land t$ result in step 1

$\therefore u$ by simplification

Step 3

$u \to p \quad P_3$

u result in step 1

$\therefore p$ by Modus Ponens

Step 4

$u \land t$ result in step 1

$\therefore t$ by simplification

Step 5

p result in step 2

t result in step 4

$\therefore p \land t$ by conjunction

Step 6

$(p \land t) \to (r \lor s) \quad P_1$

$p \land t$ result in step 5

$\therefore\ r \lor s$ by Modus Ponens

Step 7

$r \lor s$ result in step 6

$\neg s \quad P_4$

$\therefore r$ by Disjunctive syllogism

Thus, the argument form is valid. ….(0.25 mark)

Q. 5 Find reflexive closure and symmetric closure of the relation $R = \{(p,q),(q,p),(q,r),(r,s),(s,p)\}$ on the set $A = \{p,q,r,s\}$ Find the transitive closure of R using Warshall's algorithm.

Ans:

Reflexive closure of $R = R \cup \Delta = \{(p,p),(p,q),(q,p),(q,q),(q,r),(r,r),(r,s),(s,p),(s,s)\}$

….(1 mark)

Reflexive closure of $R = R \cup R^{-1} = \{(p,q),(p,s),(q,p),(q,r),(r,q),(r,s),(s,p),(s,r)\}$

….(1 mark)

The matrix of R is ….(0.25 mark)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Let $W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ ….(0.25 mark)

For each of the following step 0.5 mark

Step 1 : Construct W_1. First transfer all 1's of W_0 to W_1. In column 1 of W_0: Nonzero entry at position 2, 4. In row 1 of W_0: Nonzero entry at positions 2. Thus, at the position $(2,2)$ and $(4,2)$ of W_1 make the entries 1. Therefore

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Step 2 : Construct W_2. First transfer all 1's of W_1 to W_2. In column 2 of W_1: Nonzero entry at position 1, 2, 4. In row 2 of W_1: Nonzero entry at positions 1, 2, 3. Thus, at the position $(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(4,1),(4,2)$ and $(4,3)$ of W_2 make the entries 1. Therefore

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Step 3.

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Step 4

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

From W_4, we can conclude that the transitive closure of R is: ….(0.5 mark)

$$R^* = \{(p,p),(p,q),(p,r),(p,s),(q,p),(q,q),(q,r),(q,s),(r,p),(r,q),$$
$$(r,r),(r,s),(s,p),(s,q),(s,r),(s,s)\}.$$

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