Oscillations

Notes for

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Contents

1 Brief Introduction to Oscillation

Oscillations are everywhere. Our heartbeat, as an important life process, is an example of oscillation that we all perceive every moment. Oscillating electric and magnetic fields (called electromagnetic waves) carry light to our eyes. Oscillating air carries sound to our ears. Through electromagnetic waves and sound waves (acoustic vibrations), we receive the major part of information about the world surrounding us. Without oscillation, neither can we see each other nor can we hear. Other examples of oscillatory motion include pendulum clock, swing, tuning fork, vibration of strings in musical instruments, alternating current, the motion of atoms in molecules and solids etc.

When we talk about the word 'oscillation', immediately a picture of to and fro motion comes to our mind. Yes, oscillation means to and fro motion of an object about a mean position. Some oscillations are periodic but not all periodic motions are oscillatory. Some examples of the periodic motion that are not oscillatory are motion of planets around the sun, motion of moon around the earth etc.

According to the physical nature of the phenomena involved, oscillations in various systems can be divided into mechanical oscillations and electromagnetic ones. Mechanical oscillations are characterized by alternating conversions of the kinetic energy into one (or several) kinds of potential energy and back. In electromagnetic oscillations, alternating conversions occur between the electric field energy (which is analogous to the potential energy in mechanical systems) and the magnetic field energy (the analogue of the kinetic energy). Sometimes oscillations have a combined mechanical and electromagnetic nature, e.g., oscillations in plasma. **However, we will confine our discussion to mechanical oscillations only.**

In order for mechanical oscillation to occur, a system must possess two quantities: elasticity and inertia. When the system is displaced from its equilibrium position, the elasticity provides a restoring force such that the system tries to return to equilibrium. The inertia property causes the system to overshoot equilibrium. This constant interplay between the elastic and inertia properties is what allows oscillatory motion to occur. The natural frequency of the oscillation is related to the elastic and inertia properties. The simplest example of an oscillating system is a spring mass system (Fig. 1), where the mass is connected to a rigid support by a spring. The spring constant k provides the elastic restoring force, and the inertia of the mass m provides the overshoot.

2 Simple Harmonic Oscillation

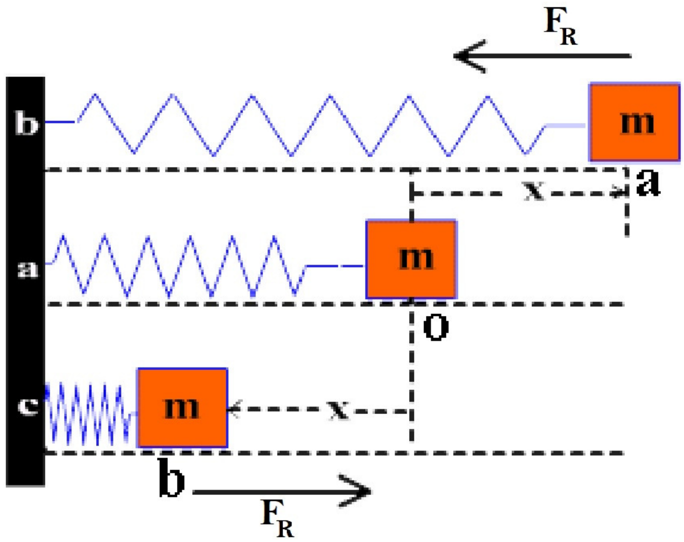

Figure 1: *The oscillation in a horizontal spring-Mass system placed in a friction-less surface. The equilibrium position is represented by the position 'O' (in the middle). When the spring is displaced by an amount "x" towards the right or left by extension or compression of the spring, the restoring force always tries to drive the mass towards the equilibrium point "O". This sets up an oscillation due to the interplay of inertia and restoring force.*

In simple harmonic oscillation, the body executes to and fro motion about its mean position. But in addition to that it has an important characteristic that is the restoring force is directly proportional to the displacement of the body but in the opposite direction. Our basic model of simple harmonic oscillator is a mass m moving back and forth along a line on a smooth (i.e. friction less) horizontal surface, connected to a horizontal spring, having spring constant k, the other end of the string being attached to a wall (see Figure 1). The spring exerts a restoring force equal to $-kx$ on the mass when it is a distance x from the equilibrium point. By "equilibrium point", we mean the point corresponding to the spring resting at its natural length, and therefore exerting no force on the mass. The restoring force (F_R) exerted by the spring on the mass can be written as,

$$\begin{aligned} F_R &\propto -x \\ \Rightarrow F_R &= -kx \end{aligned}$$

By applying Newton's second law $F_I = ma$ (i.e., the inertial force) to the mass, one can obtain the equation of motion for the system:

$$\begin{aligned} & F_I = F_R \\ \Rightarrow \quad & m\frac{d^2x}{dt^2} = -kx \\ \Rightarrow \quad & \frac{d^2x}{dt^2} + \omega^2 x = 0 \end{aligned}$$

where, $\omega = \sqrt{\frac{k}{m}}$ is the natural oscillation frequency.

One can solve the above differential equation. Solving this differential equation gives the position of the mass relative to the rest position as a function of time: However, we can verify by substitution that the solution to the above equation is given as follows

$$x(t) = A\sin(\omega t + \phi)$$

where A is the amplitude of the oscillation, ϕ is the phase constant of the oscillation, and $\omega t + \phi$ is called phase. Both A and ϕ are constants to be determined by the initial condition (i.e., initial displacement and velocity) at time $t = 0$ when one begins observing the oscillatory motion.

We discuss the various aspect of the solution below as part of the simple harmonic motion

3 Characteristics of Simple Harmonic Oscillation

The displacement of the pendulum w.r.t. time is represented with a sinusoidal function.

1. **Displacement (x):** At any instant of time, the distance of the particle from the mean position measured along the direction of motion is called as the displacement. The displacement (x) at any instant of time t is given by

$$x(t) = A\sin(\omega t + \phi) \tag{1}$$

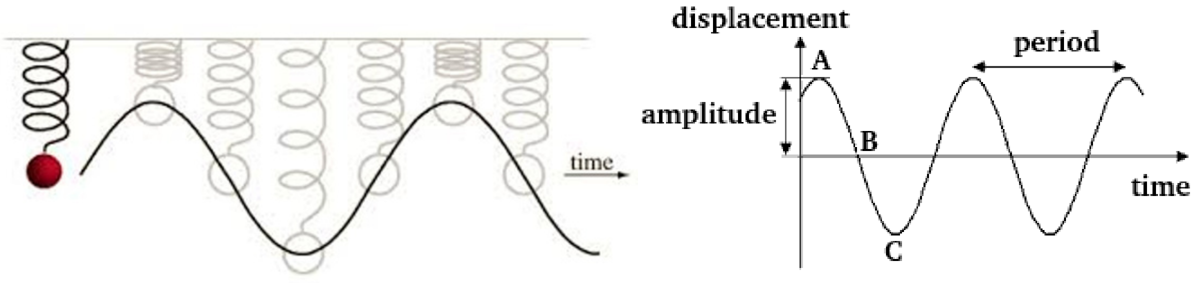

Figure 2: *(left) The oscillation of a vertical spring-mass system shows that the displacement can be represented as the sinusoidal function of the time. (right) The plot of displacement of the mass w.r.t. time.*

2. **Amplitude (A):** It is the maximum displacement of the particle on each side of the mean position.

3. **Time period (T):** It is the time taken by the particle to complete one oscillation.

4. **Frequency (f):** It is the number of complete oscillations taken by the particle in one unit of time. It can be written as the inverse of time period, i.e., ($f = \frac{1}{T}$).

5. **Velocity (v):** The velocity of particle at any instant of time t is given by,

$$\begin{aligned} v &= \frac{dx}{dt} \\ \Rightarrow v &= \frac{d}{dt}(A\sin(\omega t + \phi)) \\ \Rightarrow v &= \omega\sqrt{A^2 - x^2} \end{aligned} \tag{2}$$

At mean position ($x = 0$): $v_{max} = A\omega$, i.e., the velocity is maximum. Since the particle has maximum velocity at the mean position, therefore when the restoring force brings the particle towards the mean position, instead of stopping there, it moves the other side of the mean position with high velocity.
At extreme position ($x = A$): $v_{min} = 0$, i.e., the velocity is zero here, therefore the particle stops there and comes back towards its mean position.

6. **Acceleration (a):** Acceleration is the rate of change of velocity, therefore,

$$\begin{aligned} a &= \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A\sin(\omega t + \phi)) \\ \Rightarrow a &= -A\omega^2 \sin\omega t = -\omega^2 x \end{aligned} \tag{3}$$

The negative sign implies that acceleration acts in the opposite direction to the displacement.

7. **Phase (ϕ):** This is the physical quantity which differentiate between two particles at the same time of oscillation.

   We saw above that $x(t) = A\sin(\omega t + \phi)$, where $\omega^2 = \frac{k}{m}$. The sine function goes through one complete cycle when its argument increases by 2π, so we require that,

$$\begin{aligned}
&(\omega(t+T)+\phi) - (\omega t + \phi) = 2\pi \\
\Rightarrow &\omega T = 2\pi \\
\Rightarrow &\omega = \frac{2\pi}{T} = 2\pi f = \sqrt{\frac{k}{m}}
\end{aligned} \tag{4}$$

   This parameter is determined by the system: the particular mass and spring used. For a linear system, the frequency is independent of amplitude. This is a hallmark of simple harmonic motion. So far, in our discussion, we considered an idealized spring mass system on a friction-less surface. As a result, the oscillations once started will continue forever with the same amplitude and such oscillations are called free or undamped oscillation. But in reality, there will always be resistive forces such as friction, air resistance, viscous drag etc at play. These dissipative forces will oppose the free oscillation and thus reduce the amplitude of oscillation with time. Such an oscillator is called damped harmonic oscillator and is of practical utility. We next discuss such an oscillator.

4 Damped Harmonic Oscillation

So far, what we have discussed is the situation where the amplitude of the oscillation remains constant with time. But actually in practice, this doesn't happen. There always present a resistive force, which opposes the free oscillation. Hence, the amplitude always decreases with time. Such oscillation with decrease in amplitude is called as *damped harmonic oscillation*. For example, in simple pendulum, when the bob sets into oscillation, the amplitude decreases and slowly the oscillation comes to rest.

Consider a body of mass m, which is subjected to damped oscillation. If x be the instantaneous displacement at time t, then various forces acting on the body are,

1. Restoring force (F_R): This force acts opposite to the displacement and tries to bring the body back to its mean position.

$$F_R \propto -x \Rightarrow F_R = -kx \tag{5}$$

2. Damping force (F_D): Most oscillating physical systems dissipate their energy over time. We will consider the special cases where the force is a function of velocity $F_D = -bv - cv^2$, both b & c are constants. The damping force is in the opposite direction of the velocity. For motion at low velocities, When velocity is small enough, or c is is small enough, only the first term is important. We will consider a damping force that is proportional to velocity i.e.

$$F_D \propto -v \Rightarrow F_D = -bv \Rightarrow F_D = -b\frac{dx}{dt} \tag{6}$$

3. Force of inertia (F_I): Each moving particle has inertia force which is proportional to its acceleration.

$$F_I \propto \frac{d^2x}{dt^2} \Rightarrow F_I = m\frac{d^2x}{dt^2} \tag{7}$$

The force of inertia balances the restoring and the damping forces, i.e.,

$$\begin{aligned}
&F_I = F_R + F_D \\
&\Rightarrow m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \\
&\Rightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \\
&\Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0 \\
&\Rightarrow \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = 0
\end{aligned} \tag{8}$$

The above equation is the second order differential equation for the damped oscillation. Here, b is damping constant, $2r = \frac{b}{m} \Rightarrow r = \frac{b}{2m}$ is the damping coefficient. $\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the undamped oscillator.

Let us try to solve this second order differential equation. Let the solution is of the form,

$$x = Ae^{\alpha t} \tag{9}$$

Here, A and α are the arbitrary constants. From the above equation, we get,

$$\frac{dx}{dt} = A\alpha e^{\alpha t}, \quad \text{and} \quad \frac{d^2x}{dt^2} = A\alpha^2 e^{\alpha t}$$

Therefore, the differential equation (6) can now be written as,

$$A\alpha^2 e^{\alpha t} + 2rA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0$$

$$\Rightarrow A e^{\alpha t}\left(\alpha^2 + 2r\alpha + \omega^2\right) = 0$$

$$\Rightarrow \alpha^2 + 2r\alpha + \omega^2 = 0 \quad \text{as} \quad A e^{\alpha t} = x \neq 0$$

$$\Rightarrow \alpha = -r \pm \sqrt{(r^2 - \omega^2)} \tag{10}$$

Hence, the solution can be written as,

$$x(t) = A_1 e^{\left(-r+\sqrt{(r^2-\omega^2)}\right)t} + A_2 e^{\left(-r-\sqrt{(r^2-\omega^2)}\right)t} \tag{11}$$

In the above equation, the relative term under the square-root, i.e., r^2 and ω^2 can give rise to three different situations. These three cases are discussed here.

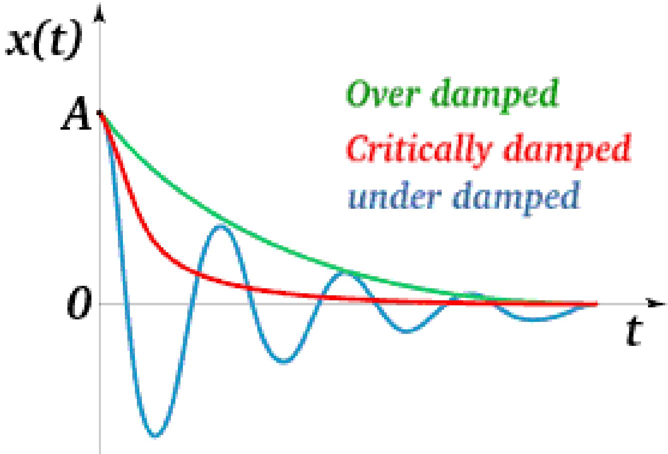

Figure 3: *The displacement vs time curve for three different types of damping.*

4.1 Case I: $r > \omega$ (over-damped oscillation):

In this situation, the damping coefficient is more than the angular frequency of oscillation. Hence it is called as the over-damped situation. When $r > \omega \Rightarrow r^2 > \omega^2 \Rightarrow \sqrt{r^2 > \omega^2}$ is real and less than r. That is, both $\left(-r + \sqrt{(r^2 - \omega^2)}\right)$ and $\left(-r - \sqrt{(r^2 - \omega^2)}\right)$ are negative. Hence, both the terms of equation (11) represent the situation where, the displacement decay exponentially to zero with time t (the green colour curve in Fig. 3). Thus the motion is no more oscillatory as mass does not oscillate. It gradually approaches the equilibrium position at $x = 0$.

Though both the terms decay exponentially, but the second term, i.e., $\left(-r - \sqrt{(r^2 - \omega^2)}\right)$ decays to zero faster than the first term, i.e., $\left(-r + \sqrt{(r^2 - \omega^2)}\right)$. Therefore, the displacement is primarily governed by the first term.

Overdamping is typically used in door dampers as the system goes to equilibrium (without overshooting it) in a relatively longer time. A shock absorber is basically a damped spring oscillator, the damping is from a piston moving in a cylinder filled with oil. If the oil is really thick, or the piston too tight, the shock absorber will be too stiff - it won't absorb the shock, and you will! This is the case of overdamping. So we need to tune the damping so that the car responds smoothly to a bump in the road, but doesn't continue to bounce after the bump. this is achieved in using critical damping as discussed below.

4.2 Case II: $r = \omega$ (critically damped oscillation):

In this case, the damping coefficient is approximately equal to the angular frequency of oscillation. If we consider $r = \omega \Rightarrow r^2 = \omega^2 \Rightarrow r^2 - \omega^2 = 0$, then the solution of the equation (11) will have only one constant term, hence will not represent the general solution of equation (11). Therefore, we shall consider $\sqrt{r^2 - \omega^2} = h \to 0$, i.e., h is very small quantity. Using this term, the solution is of the form,

$$
\begin{aligned}
x &= A_1 e^{(-r+h)t} + A_2 e^{(-r-h)t} \\
\Rightarrow x &= e^{-rt}\left[A_1 e^{ht} + A_2 e^{-ht}\right] \\
&\quad \text{using the exponent series, } e^{ht} = \sum_{n=0}^{\infty} \frac{(ht)^n}{n!} \\
\Rightarrow x &= e^{-rt}\left[(A_1 + A_2) + ht(A_1 - A_2)\right] \\
&\quad \text{since } h \text{ is very small so its higher orders are neglected} \\
\Rightarrow x &= e^{-rt}(p + qt) \qquad (12)
\end{aligned}
$$

Here $p = (A_1 + A_2)$ and $q = h(A_1 - A_2)$, are the two constants. Initially, for small value of t, the $(p + qt)$ term is dominated. But with further increase in t, e^{-rt} will be dominated. Therefore, the nature of this curve is also exponentially decaying (as shown in red curve in Fig 3). Such type of motion is called critically damped. So the system returns to equilibrium in the shortest possible time without undergoing any oscillation or overshooting the equilibrium. As discussed earlier, critical damping is used in car suspension systems for absorbing shocks.

4.3 Case III: $r < \omega$ (under-damped oscillation):

When $r < \omega \Rightarrow r^2 < \omega^2$, then $\sqrt{r^2 - \omega^2}$ is imaginary. Let us say, $\sqrt{r^2 - \omega^2} = i\omega_1$, where, $\omega_1 = \sqrt{\omega^2 - r^2}$. Hence, the solution given by equation (11) can be written as,

$$
\begin{aligned}
x &= A_1 e^{(-r+i\omega_1)t} + A_2 e^{(-r-i\omega_1)t} \\
&= e^{-rt}\left[(A_1 + A_2)\cos\omega_1 t + i(A_1 - A_2)\sin\omega_1 t\right] \\
&= a e^{-rt}\sin(\omega_1 t + \phi) \\
&\qquad\qquad \text{here, } (A_1 + A_2) = a\sin\phi \text{ and } i(A_1 - A_2) = a\cos\phi\text{, are the constants} \\
x &= a e^{-rt}\sin\left(\sqrt{\omega^2 - r^2}\, t + \phi\right) \qquad (13)
\end{aligned}
$$

Equation (13) represents the solution for the under-damped oscillation which is sinusoidal. It is clear from the above equation that the amplitude term, i.e., ae^{-rt} is exponential decaying (as shown in Fig. 3 blue color curve). And the time period of this oscillation is given by,

$$
T = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{\omega^2 - r^2}} \qquad (14)
$$

Equation (14) shows that oscillation frequency gradually decreases and is slightly lower than the undamped oscillator.

When the damping coefficient, $r = 0$, i.e., in the absence of damping, the displacement and time period will give the solutions for simple harmonic oscillation. The example of under-damped oscillation is simple pendulum.

5 Energy Decay

The energy of oscillation is proportional to the square of the amplitude. For under-damped oscillation amplitude decay is proportional to e^{-rt}, therefore, energy decay is proportional to e^{-2rt}. Hence energy decay is given by,

$$
\begin{aligned}
&E \propto e^{-2rt} \\
&E = E_0\, e^{-2rt} \qquad (15)
\end{aligned}
$$

Where, E_0 is the initial energy when the oscillation starts at time $t = 0$.

6 Logarithmic Decrement (δ)

It is defined as the natural log of the ratio of two successive amplitude and is given by,

$$
\begin{aligned}
\delta &= \log \frac{A(t)}{A(t+T)} \\
\Rightarrow \delta &= \log \frac{ae^{-rt}}{ae^{-r(t+T)}} \\
\Rightarrow \delta &= \log(e^{rT}) = rT = r\frac{2\pi}{\omega_1} = \frac{2\pi r}{\sqrt{\omega^2 - r^2}} \qquad (16)
\end{aligned}
$$

7 Relaxation Time (τ)

It is the time taken for the energy to decay to $\frac{1}{e}$ times of its original value. That is, in equation (15), at $t = \tau$, energy will be $E = \dfrac{E_0}{e}$, i.e.,

$$
\begin{aligned}
E &= E_0\, e^{-2rt} \\
\Rightarrow \frac{E_0}{e} &= E_0\, e^{-2r\tau} \\
\Rightarrow \tau &= \frac{1}{2r} \qquad (17) \\
\Rightarrow \tau &= \frac{m}{b} \qquad (18)
\end{aligned}
$$

Equation (17) and (18) represent the relaxation time in terms of damping coefficient and damping constant, respectively.