

Multiplication of Signed Integers

Booth's Algorithm

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Booth's Algorithm

It treats both +ve and -ve multiplier in the uniform way.

$$M \times 0011110[30] = M \times (2^5 - 2^1) [30 = 32 - 2 = 2^5 - 2^1]$$

We know that, multiplying something by 2^i is equivalent to shifting the number **left** by **i times**.

We know that, multiplying something by -2^i is equivalent to shifting the **2's complement** of the number **left** by **i times**.

Booth's Algorithm

In general, in the Booth scheme,

-1 times the shifted multiplicand is selected when moving from 0 to 1 in the multiplier.

+1 times the shifted multiplicand is selected when moving from 1 to 0, as the multiplier is scanned from right to left.

1 0 0 1 1 1 0 0 1 0 1 0 1 0 0

-1 0 +1 0 0 -1 0 +1 -1 +1 -1 +1 -1 0 0

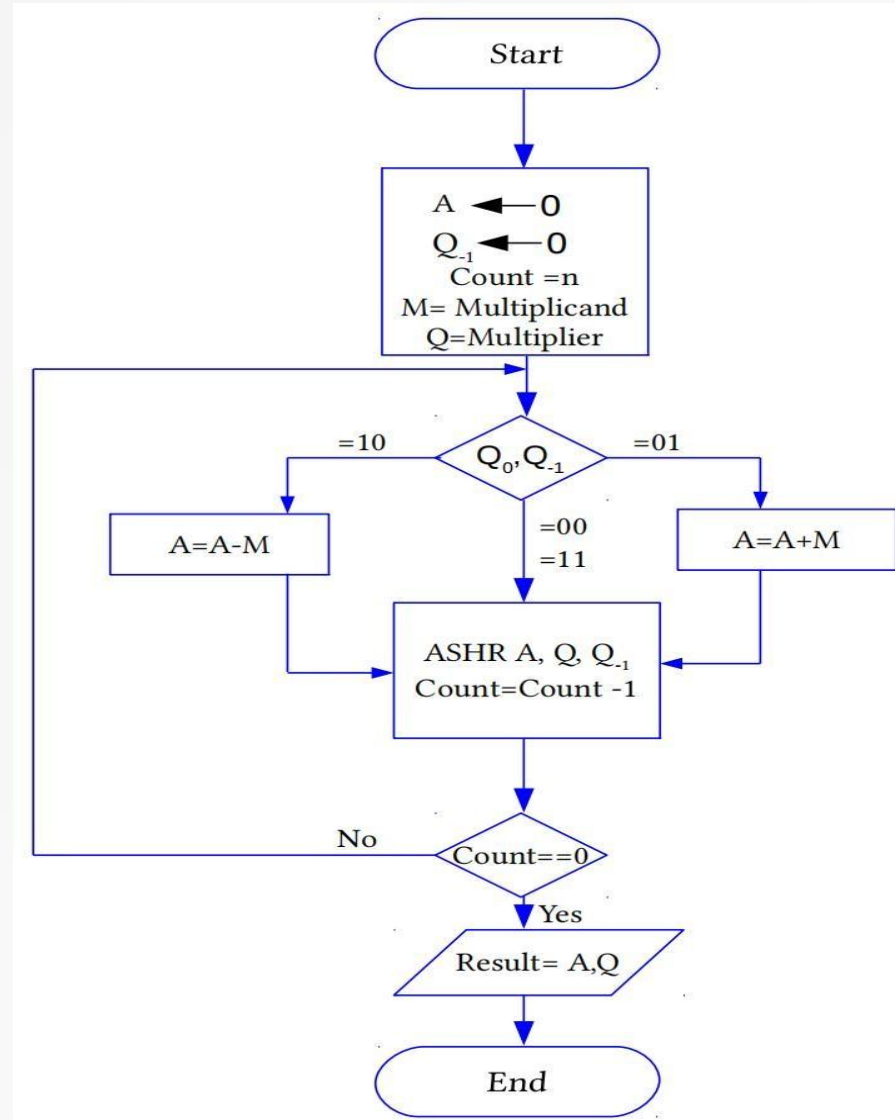
Booth's Algorithm

Multiplier		Version of multiplicand selected by i^{th} bit
Bit i	Bit $i-1$	
0	0	0 X M
0	1	+1 X M
1	0	-1 X M
1	1	0 X M

Booth multiplier recoding table.

Multiplication of Signed Integers Booth's Algorithm[Method 2]

Flowchart for Booth's Algorithm



Booth's Algorithm (-7 x 3) [+7=0111 -7=1001]

Register	Multiplier Register		Multiplicand Register	Operation	Remark
A	Q	Q ₋₁	M		
0000	0011	0	1001	Initially	
0111	0011	0		A=A-M=A+ 2s(M)	1st Cycle
0011	1001	1		Shift A, Q, Q ₋₁ right	
0011	1001	1		No add/sub	2nd Cycle
0001	1100	1		Shift A, Q, Q ₋₁ right	
1010	1100	1		A=A+M	3rd Cycle
1101	0110	0		Shift A, Q, Q ₋₁ right	
1101	0110	0		No add/sub	4th Cycle
1110	1011	0		Shift A, Q, Q ₋₁ right	

Booth's Algorithm (14 x -7) [+14=01110 -14=10010]

Register	Multiplier Register		Multiplicand Register	Operation	Remark
A	Q	Q ₋₁	M		
00000	11001	0	01110	Initially	Count=5
10010	11001	0		A=A-M=A+ 2s(M)	Count=4
11001	01100	1		AShift A, Q, Q ₋₁ right	
100111	01100	1		A=A+M	Count=3
00011	10110	0		AShift A, Q, Q ₋₁ right	
00011	10110	0		No add/sub	Count=2
00001	11011	0		AShift A, Q, Q ₋₁ right	
10011	11011	0		A=A-M=A+ 2s(M)	Count=1
11001	11101	1		AShift A, Q, Q ₋₁ right	

Booth's Algorithm (14 x -7) [+14=01110 -14=10010]

Register	Multiplier Register		Multiplicand Register	Operation	Remark
A	Q	Q ₋₁	M		
11001	11101	1		No add/sub	Count=0
11100	11110	1		AShift A, Q, Q ₋₁ right	

Booth's Algorithm (-15 x -5) [+15=01111 -15=10001]

Register	Multiplier Register		Multiplicand Register	Operation	Remark
A	Q	Q ₋₁	M	+5= 00101(-5= 11011)	
00000	11011	0	10001	Initially	Count=5
01111	11011	0		A=A-M=A+ 2s(M)	Count=4
00111	11101	1		AShift A, Q, Q ₋₁ right	
00111	11101	1		No add/sub	Count=3
00011	11110	1		AShift A, Q, Q ₋₁ right	
10100	11110	1		A=A+ M	Count=2
11010	01111	0		AShift A, Q, Q ₋₁ right	
1 01001	01111	0		A=A-M=A+ 2s(M)	Count=1
00100	10111	1		AShift A, Q, Q ₋₁ right	

Booth's Algorithm (-15×-5) [$+15=01111$ $-15=10001$]

Register	Multiplier Register		Multiplicand Register	Operation	Remark
A	Q	Q_{-1}	M		
00100	10111	1		No add/sub	Count=0
00010	01011	1		AShift A, Q, Q_{-1} right	

Analysis of Booth's Algorithm

1 0 0 1 1 1 0 0 1 0 1 0 1 0 0

-1 0 +1 0 0 -1 0 +1 -1 +1 -1 +1 -1 0 0

The general case.

1 0 1 0 1 0 1 0 1 0 1 0 1 0 1

-1 +1 -1 +1 -1 +1 -1 +1 -1 +1 -1 +1 -1 +1 -1

The **Worst** case.

1 1 1 1 1 0 0 0 0 0 1 1 1 1 1

0 0 0 0 -1 0 0 0 0 +1 0 0 0 0 -1

The **Best** case.

Bit Pair Recoding of Multipliers

1 1 1 0 1 0 **0**

0 **0** **-1** **+1** **-1** **0**

0 **-1** **-2**

Bit Pair recoding halves the maximum number of summands .

Bit-Pair Recoding of Multipliers

Multiplier bit pair		Multiplier bit on the right $i-1$	Version of multiplicand selected by i^{th} bit
Bit $i+1$	Bit i		
0	0	0	$0 \times M$
0	0	1	$+1 \times M$
0	1	0	$+1 \times M$
0	1	1	$+2 \times M$
1	0	0	$-2 \times M$
1	0	1	$-1 \times M$
1	1	0	$-1 \times M$
1	1	1	$0 \times M$

Bit-Pair Recoding of Multipliers

$$+11 \times -9$$

$$+11 = 01011, -11 = 10101$$

$$+9 = 01001$$

$$-9 = 10111$$

Bit pair recorded multiplier will be

$$1101110$$

$$-1 \quad +2 \quad -1$$

$$\begin{array}{r}
 01011 \\
 -1 \quad +2 \quad -1 \\
 \hline
 1111110101 \\
 00010110 \\
 110101 \\
 \hline
 11110011101
 \end{array}$$

$$\begin{aligned}
 &\text{Ans. 2's comp}(1110011101) \\
 &= 0001100011 \\
 &= -99
 \end{aligned}$$

X (-2)= Take the 2's comp n then shift left by 1 position

X (-1)= Take the 2's comp of the operand

X (2)= Shift the operand by 1 bit position to the left

Bit-Pair Recoding of Multipliers

$$+18 \times -11$$

$$+18 = 010010, -18 = 101110$$

$$+11 = 001011$$

$$-11 = 110101$$

Bit pair recorded multiplier will be

$$1 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0$$

$$-1 \quad +1 \quad +1$$

								0	1	0	0		1	0
								-1		+1			+1	
<hr/>														
0	0	0	0	0	0	0	0	1	0	0	1	0		
0	0	0	0	0	1	0	0	1	0					
1	1	1	0	1	1	1	0							
<hr/>														
1	1	1	1	0	0	1	1	1	0	1	0			

$$\begin{aligned} \text{Ans. 2's comp}(1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 0) \\ = 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0 \ 1 \ 1 \ 0 \\ = 198 \end{aligned}$$

Division of Positive Integers

Restoring Algorithm

Restoring Division

Initialization:

$A=0$

$Q = \text{Dividend}$

$M = \text{Divisor}$

$n = \text{No of bits in the operand}$

Note: Both Q and M are positive integers represented using equal no of bits.
In A one extra bit is considered to keep track of the sign of the result of subtraction operation.

Step 1: Shift A and Q left one binary position.

Step 2: Subtract M from A , and place the answer back in A

If the sign of A is 1, set q_0 to 0 and add M back to A (restore A); otherwise, set q_0 to 1

Repeat these steps n times.

Result: Quotient: Q

Remainder : A

Restoring Division Example (15 / 5)

		A	Q
Initially	0	0 0 0 0	1 1 1 1
M	0	0 1 0 1	-M= 1 1 0 1 1
ShiftL	0	0 0 0 1	<input type="checkbox"/>
Subtract	1	1 0 1 1	<input type="checkbox"/>
	1	1 1 0 0	<input type="checkbox"/>
Restore	0	0 1 0 1	
	1 0	0 0 0 1	
ShiftL		0 0 0 1 1	<input type="checkbox"/> <input type="checkbox"/>
Subtract		1 1 0 1 1	
		1 1 1 1 0	<input type="checkbox"/> <input type="checkbox"/>
Restore	0	0 1 0 1	
	1 0	0 0 1 1	
ShiftL	0	0 1 1 1	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>

Restoring Division Example (15 / 5)

		A	Q
	0	0 1 1 1	1 0 0 <input type="text"/>
Subtract	1	1 0 1 1	1 0 0 1
	1 0	0 0 1 0	
ShiftL	0	0 1 0 1	0 0 1 <input type="text"/>
Subtract	1	1 0 1 1	
	1 0	0 0 0 0	0 0 1 1

Result:

Quotient: 0011

Remainder: 0000

Non restoring Division

In case of restoring division, after unsuccessful division, M is added to A then shifted to left and M is subtracted from it.

$$A+M$$

$$2(A+M)$$

$$2(A+M) - M$$

$$2A+2M - M = 2A + M$$

i.e., Shift A to the left and ADD M directly to it.

Non restoring Division

Initialization:

A=0

Q= Dividend

M= Divisor

n= No of bits in the operand

Note: Both Q and M are positive integers represented using equal no of bits.
In A one extra bit is considered to keep track of the sign of the result of subtraction operation.

Repeat these step <1> n times:

Step 1: If the sign of A is 0, shift A and Q left one bit position and subtract M from A; otherwise, shift A and Q left and add M to A.

Now, if the sign of A is 0, set q_0 to 1; otherwise, set q_0 to 0.

Step 2: If the sign of A is 1, add M to A

Non restoring Division Example(15 / 5)

		A	Q
Initially	0	0 0 0 0	1 1 1 1
M	0	0 1 0 1	-M= 1 1 0 1 1
ShiftL	0	0 0 0 1	1 1 1 <input type="checkbox"/>
Subtract	1	1 0 1 1	1 1 1 <input type="checkbox"/>
	1	1 1 0 0	1 1 1 <input type="checkbox"/>
ShiftL	1	1 0 0 1	1 1 <input type="checkbox"/> <input type="checkbox"/>
Add	0	0 1 0 1	
	1	1 1 1 0	1 1 <input type="checkbox"/> <input type="checkbox"/>
ShiftL	1	1 1 0 1	1 <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
Add	0	0 1 0 1	
	1 0	0 0 1 0	1 <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
ShiftL	0	0 1 0 1	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>
Subtract	1	1 0 1 1	
	1 0	0 0 0 0	<input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/> <input type="checkbox"/>

Quotient= 0011=3

Remainder= 0000=0

Floating Point Numbers

IEEE Number Representation

Why Floating Point Numbers?

The maximum value that can be represented using 32 bits is 4,294,967,295, is an integer equal to $2^{32} - 1$.

For scientific calculations, sometimes, we need very big number like 6.0247×10^{23} or a very small number like 9.1×10^{-31}

But for representing such numbers 32 bits are not sufficient enough, we need larger number of bits in our conventional methods.

Solution:

IEEE 754 Standard: Single Precision(32 bits)

Double Precision(64 bits)

Introduction to Basic Terms

For a number 6.0247×10^{23}

It consists of 3 components:

- a. Mantissa: [Significant Digits] 6.0247**
- b. base: 10**
- c. Exponent : 23**
- d. Scale factor : 10^{23}**

IEEE Representation

IEEE Floating Point notation is the standard representation in use. There are two representations:

- **Single precision.**
- **Double precision.**



Fig. IEEE Single Precision



Fig. IEEE Double Precision

Biased Exponent

Using 8 bits we can have exponent value in the range - 2^7 to $+(2^7-1)$

Biased exponent says that we don't want our exponent as a signed number, we want it only as a **positive number**.

So to get only +ve values, we are using biased exponent, where we will **add bias** to the actual exponent and the result will be stored as the exponent of the number.

Say the exponent is -5, it will be stored as **-5 + bias**

Bias is represented as 2^{k-1} , if exponent is represented using **k** no of bits. [In general]

Biased Exponent

In the IEEE Single precision, using 8 bits exponent can be from 0 to 255(using 8bits)

But 0 and 255 is used for representing special numbers.

So, we are going to store exponent in the range 1 to 254.

If if exponent is represented using k(8) no of bits, we are taking bias as $2^{k-1}-1$,
($2^7-1=127$)

Original Exponent	Biased (Excess-127) Exponent
-126	1
-125	2
....	
.....	
127	254

Implicit Normalized Number

Say a number is 1001.01

Then to perform implicit normalization we need to bring the decimal point to the left of 1st one in the number.

$$1001.01 = 1.00101 \times 2^3$$

Steps to convert a given decimal number into IEEE Format

Step 1: Convert the number into Binary.

Step 2: Normalize (Implicit) the number.

if decimal point is taken to the left by i positions, then multiply the number by 2^i
if decimal point is taken to the right by i positions, then multiply the number by 2^{-i}

Step 3: If the number is +ve $S=0$, Else $S=1$

Step 4: Add 127 to the exponent, then write the binary of it for E'

Step 5: Mantissa is whatever is to the right of the decimal point in the normalized number. If it is not of 23 bits, then append zeros to the right of the actual mantissa to make it of length 23 bits.

Example : Convert -13.25 into IEEE Single Precision Format

Step1: Convert the number into Binary.

$$13.25 = 1101.01$$

Step 2: Normalize (Implicit) the number.

$$1101.01 = 1.10101 \times 2^3$$

Step 3: If the number is +ve S=0, Else S=1
Here, S=1

Step 4: Add 127 to the exponent , then write the binary of it for E'

$$E' = 127 + 3 = 130 = 10000010$$

Step 5: Mantissa is whatever is to the right of the decimal point in the normalized number. If it is not of 23 bits, then append zeros to the right of the actual mantissa to make it of length 23 bits.

$$M = 1010\ 1000\ 0000\ 0000\ 0000\ 000$$

Solution: S=1

$$E' = 1000\ 0010$$

$$M = 1010\ 1000\ 0000\ 0000\ 0000\ 000$$

Example : Convert -13.25 into IEEE Double Precision Format

Step1: Convert the number into Binary.

$$13.25 = 1101.01$$

Step 2: Normalize (Implicit) the number.

$$1101.01 = 1.10101 \times 2^3$$

Step 3: If the number is +ve $S=0$, Else $S=1$
Here, $S=1$

Step 4: Add **1023 to the exponent** , then write the binary of it for E'

$$E' = 1023 + 3 = 1026 = 100\ 0000\ 0010$$

Step 5: Mantissa is whatever is to the right of the decimal point in the normalized number. If it is not of 52 bits, then **append zeros to the right** of the actual mantissa to make it of length 23 bits.

$$M = 1010\ 1000\ 0000\ 0000\ 0000\ 000\ldots\ldots\ldots 0$$

Solution: $S=1$

$$E' = 100\ 0000\ 0010$$

$$M = 1010\ 1000\ 0000\ 0000\ 0000\ 000\ldots\ldots\ldots 0$$

Example : Convert -17 into IEEE Format

Step1: Convert the number into Binary.

$$17 = 10001$$

Step 2: Normalize (Implicit) the number.

$$10001.0 = 1.0001 \times 2^4$$

Step 3: If the number is +ve $S=0$, Else $S=1$
Here, $S=1$

Step 4: Add **127 to the exponent** , then write the binary of it for E'

$$E' = 127 + 4 = 131 = 10000011$$

Step 5: Mantissa is whatever is to the right of the decimal point in the normalized number. If it is not of 23 bits, then **append zeros to the right** of the actual mantissa to make it of length 23 bits.

$$M = 0001\ 0000\ 0000\ 0000\ 0000\ 000$$

Solution: $S=1$

$$E' = 1000\ 0011$$

$$M = 0001\ 0000\ 0000\ 0000\ 0000\ 000$$

Example : Convert -0.35 into IEEE Format

Step1: Convert the number into Binary.

$$0.35 = 0.01011$$

Step 2: Normalize (Implicit) the number.

$$0.01011 = 1.011 \times 2^{-2}$$

Step 3: If the number is +ve $S=0$, Else $S=1$
Here, $S=1$

Step 4: Add 127 to the exponent , then write the binary of it for E'

$$E' = 127 + (-2) = 125 = 0111\ 1101$$

Step 5: Mantissa is whatever is to the right of the decimal point in the normalized number. If it is not of 23 bits, then append zeros to the right of the actual mantissa to make it of length 23 bits.

$$M = 0110\ 0000\ 0000\ 0000\ 0000\ 000$$

Solution: $S=1$

$$E' = 0111\ 1101$$

$$M = 0110\ 0000\ 0000\ 0000\ 0000\ 000$$

Value Represented

Value represented $= (-1)^s \times 1.M \times 2^{E'-127}$

What is the value represented by the following 32 bits in IEEE -754 REPRESENTATION?

$S=0$

$E'=10000011$

$M=11000\ldots\ldots\ldots 00$

Value represented $= (-1)^0 \times 1.11 \times 2^{131-127}$

$[10000011=131]$

$= +1.11 \times 2^4$

$= (11100)_2$

$= +(28)_{10}$

Value Represented

Value represented $= (-1)^s \times 1.M \times 2^{E'-127}$

What is the value represented by the following 32 bits in IEEE -754 REPRESENTATION?

$S=0$

$E'=10000000$

$M=11000\ldots\ldots\ldots00$

Value represented $= (-1)^0 \times 1.11 \times 2^{128-127}$

[10000000=128]

$= +1.11 \times 2^1$

$= (11.1)_2$

$= +(3.5)_{10}$

Value Represented

Value represented $= (-1)^s \times 1.M \times 2^{E'-127}$

What can be the maximum value represented by the 32 bits in IEEE -754 REPRESENTATION?

$S=0$

$E'=11111110$

$M=11111\dots\dots\dots 11$

$$\begin{aligned}\text{Value represented} &= (-1)^0 \times 01.11\dots\dots\dots 1 \times 2^{254-127} \\ &= 1111\dots\dots\dots 1 \times 2^{-23} \times 2^{127} \\ &= (2^{24}-1) \times 2^{104} \\ &= 2^{128}\end{aligned}$$

Value Represented

$$\text{Value represented} = (-1)^s \times 1.M \times 2^{E'-127}$$

What can be the minimum value represented by the 32 bits in IEEE -754 REPRESENTATION?

$$S=1$$

$$E'=11111110$$

$$M=11111\dots\dots\dots 11$$

$$\begin{aligned}\text{Value represented} &= (-1)^1 \times 0.11\dots\dots\dots 1 \times 2^{254-127} \\ &= -1111\dots\dots\dots 1 \times 2^{-23} \times 2^{127} \\ &= -(2^{24}-1) \times 2^{104} \\ &= -2^{128}\end{aligned}$$

Value Represented

Value represented $= (-1)^s \times 1.M \times 2^{E'-127}$

What can be the minimum positive value represented by the 32 bits in IEEE - 754 REPRESENTATION?

$S=0$

$E'=00000001$

$M=0000\dots0000$

Value represented $= (-1)^0 \times 1.00\dots\dots\dots 0 \times 2^{1-127}$
 $= 1 \times 2^{-126}$
 $= 2^{-126}$

Value Represented

$$\text{Value represented} = (-1)^s \times 1.M \times 2^{E'-127}$$

What can be the maximum negative value represented by the 32 bits in IEEE-754 REPRESENTATION?

$$S=1$$

$$E'=00000001$$

$$M=0000\dots0000$$

$$\begin{aligned}\text{Value represented} &= (-1)^1 \times 1.00\dots\dots\dots 0 \times 2^{1-127} \\ &= -1 \times 2^{-126} \\ &= -2^{-126}\end{aligned}$$

Denormalized Number

Denormalized Number: is a very very small number which cannot be normalized(implicit) [For single precision]

Let say a number be, 0.0000000.....11 [After decimal 131 places are there]

After Implicit normalization =>

$$1.1 \times 2^{-130}$$

$$M=1$$

$$E'=-130+127=-3$$

In conventional method this number cannot be stored as the exponent is -ve.
But , in IEEE Standard, this will be treated as a special number.

Denormalized Number

IEEE- 754 standard says that, for a small number normalize it till -126 bits.

Let say a number be, 0.0000000.....011 [After decimal 130 places are there]

After Implicit normalization =>

$$0.0011 \times 2^{-126}$$

$$M=0011$$

$$E'=00000000$$

This is an example of **denormalized** number

Value formula for denormalized number : $(-1)^s \times 0.M \times 2^{-126(\text{bias}-1)}$

SPECIAL VALUES

S	E'	M	Number
0	00000000	000..... 0	+0
1	00000000	000..... 0	-0
0	11111111	000..... 0	$+\infty$
1	11111111	000..... 0	$-\infty$
0/1	11111111	$M \neq 0$	NaN
0/1	00000000	$M \neq 0$	Denormalized Number
0/1	E' \neq 00000000 E' \neq 11111111	M = XXX.....X	Implicit Normalized Number

Floating point arithmetic: ADD/SUB rule

Choose the number with the smaller exponent.

Shift its mantissa right until the exponents of both the numbers are equal.

Add or subtract the mantissas.

Determine the sign of the result.

Normalize the result if necessary and truncate/round to the number of mantissa bits.

Floating point arithmetic: MUL rule

Add the exponents.

Subtract the bias.

Multiply the mantissas and determine the sign of the result.

Normalize the result (if necessary).

Truncate/round the mantissa of the result.

Floating point arithmetic: DIV rule

Subtract the exponents

Add the bias.

Divide the mantissas and determine the sign of the result.

Normalize the result if necessary.

Truncate/round the mantissa of the result.



Floating Point Arithmetic: ADD/SUB rule

Number 1:

$S=0$

$E' = 1000\ 0100$

$M = 000111100\dots 0$

Number 2:

$S=0$

$E' = 1000\ 0011$

$M = 0100100\dots 00000$

Number 1:

1.0001111×2^5

Number 2:

1.01001×2^4

Choose the number with the smaller exponent. and shift its mantissa right until the exponents of both the numbers are equal. So number 2 will become

0.1010010×2^5 and now perform the addition

1.0001111×2^5

0.1010010×2^5

1.1100001×2^5

Value Represented

Value represented $= (-1)^s \times 0.M \times 2^{-126}$

What is the value represented by the following 32 bits in IEEE -754 REPRESENTATION?

$S=0$

$E'=00000000$

$M=11000\ldots\ldots\ldots00 \neq 0 \Rightarrow$ Denormalized number

Value represented $= (-1)^0 \times 0.11 \times 2^{-126}$

$= 11.0 \times 2^{-2} \times 2^{-126}$

$= 3 \times 2^{-128}$

GATE QUESTIONS

Consider three registers R1, R2, and R3 that store numbers in IEEE–754 single precision floating point format. Assume that R1 and R2 contain the values (in hexadecimal notation) 0x42200000 and 0xC1200000, respectively.

If $R3 = R1 / R2$, what is the value stored in R3 ?

(A) 0x40800000

(B) 0xC0800000

(C) 0 x 8 3 4 0 0 0 0 0

(D) 0xC8500000

R1= 0x42200000

R1= 0**100 0010 0010 0000****0000**

R1, S=0

E'=10000100 = 132-127=5

M= 1.010000000.....0

R1, $1.0100 \times 2^5 = 101000=40$

R2= 0xC1200000

R2= 1**100 0001 0010 0000****0000**

R2, S=1

E'=1000 0010 = 130-127=3

M= 1.010000000.....0

R2, $1.0100 \times 2^3 = 1010=-10$

So, $R1/R2 = 40/-10 = -4$

$-4 = 100.0 \times 2^0 = 1.00 \times 2^2$, S=1, E'=1000 0001, M=0000000.....0

1**100 0000 100**.....0= C0800000

THANK YOU