1) $f(x) = \begin{cases} 0.075x + 0.2, & 3 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$

a) $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{3} f(x)\,dx + \int_{3}^{5} f(x)\,dx + \int_{5}^{\infty} f(x)\,dx$$

$$= \int_{3}^{5} (0.075x + 0.2)\,dx$$

$$= 0.075 \times \left[\frac{x^2}{2}\right]_3^5 + 0.2\,[x]_3^5$$

$$= \cancel{0.075}\ 1$$

b) $$P(X \le 4) = \int_{-\infty}^{4} f(x)\,dx$$

$$= \int_{3}^{4} f(x)\,dx$$

$$= \int_{3}^{4} (0.075x + 0.2)\,dx$$

$$= 0.4625$$

c) $$P(3.5 \le X \le 4.5) = \int_{3.5}^{4.5} f(x)\,dx$$

$$= \int_{3.5}^{4.5} (0.075x + 0.2)\,dx$$

$$= 0.5$$

d) $$P(X > 4.5) = \int_{4.5}^{5} (0.075x + 0.2)\,dx = 0.278$$

Ex: 4.1

2) $A = -5,\ B = 5$

$$f(x) = \begin{cases} \frac{1}{10}, & -5 \le x \le 5 \\ 0, & \text{otherwise} \end{cases} \qquad \frac{1}{B-A},\ A \le x \le B$$

a) $$P(x<0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-5}^{0} \frac{1}{10}\,dx = \frac{1}{10} \times [x]_{-5}^{0}$$

$$= \frac{0+5}{10} = \frac{1}{2} = 0.5$$

b) $$P(-2.5 < X < 2.5) = \int_{-2.5}^{2.5} \frac{1}{10}\,dx = \frac{1}{10} \times [x]_{-2.5}^{2.5} = \frac{2.5+2.5}{10} = \frac{5}{10} = \frac{1}{2} = 0.5$$

c) $$P(-2 \le X \le 3) = \int_{-2}^{3} \frac{1}{10}\,dx = \frac{1}{10} \times [x]_{-2}^{3} = \frac{5}{10} = \frac{1}{2} = 0.5$$

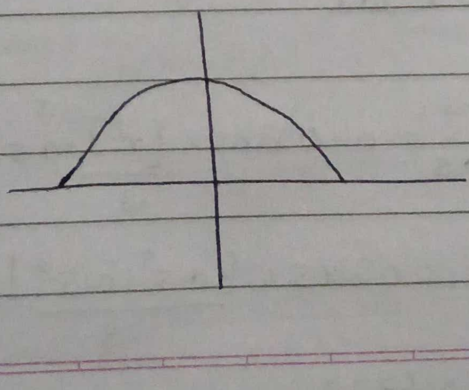
d) $P(k < X < k+4)$ where $-5 < k < k+4 < 5$

So, k and $k+4$ are within -5 to 5

$$\therefore P(k < X < k+4) = \int_{k}^{k+4} \frac{1}{10}\,dx = \frac{1}{10} \times [x]_{k}^{k+4} = \frac{k+4-k}{10} = 0.4$$

3) $$f(x) = \begin{cases} 0.09375(4-x^2), & -2 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

a)



3) b) $P(X>0)$

$1-P(X\le 0)$

$= 1 - \int_{-\infty}^{0} f(x)\,dx$

$= 1 - \int_{-2}^{0} 0.09375(4-x^2)\,dx$

$0^3 - (-2)^3$

8

$\cancel{= 1 - \left(0.375\times[x]_{-2}^{0} - \frac{[x^3]_{-2}^{0}}{3}\right)}$

$\cancel{= 1 - \left(0.375\times 2 - \frac{8}{3}\right)}$

(As this is a definite integral, you can do it directly by calculator)

$\cancel{= 1 - (-1.917)}$

$\cancel{= 2.917}$

$= 1 - \left(0.375\times[x]_{-2}^{0} - 0.09375\,\frac{[x^3]_{-2}^{0}}{3}\right)$

$= 1 - \left(0.375\times 2 - \frac{0.09375\times 8}{3}\right)$

$= 1 - 0.5$

$= 0.5$

3) c) $P(-1<X<1) = \int_{-1}^{1} 0.09375(4-x^2)\,dx$

$= 0.375\times[x]_{-1}^{1} - 0.09375\times\frac{[x^3]_{-1}^{1}}{3}$

$= 0.375\times 2 - 0.09375\times\frac{[1+1]}{3}$

$= 0.6875$

3) d) $P(X<-0.5 \text{ or } X>0.5)$

$= 1 - P(-0.5\le X\le 0.5)$

$= 1 - \int_{-0.5}^{0.5} 0.09375(4-x^2)\,dx$

$= 1 - \left(0.375\times[x]_{-0.5}^{0.5} - 0.09375\times\frac{[x^3]_{-0.5}^{0.5}}{3}\right)$

$= 1 - \left(0.375\times 1 - 0.09375\times\frac{[0.5^3+0.5^3]}{3}\right)$

$= 1 - 0.3672 = 0.6328$

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4) $f(x;\theta) = \begin{cases} \frac{x}{\theta^2} \times e^{-x^2/(2\theta^2)}, & x>0 \\ 0, & \text{otherwise} \end{cases}$

a) $\int_{-\infty}^{\infty} f(x)\,dx$ should be equal to 1 for it to be a legitimate pdf.

$\int_0^{\infty} \frac{x}{\theta^2} \times e^{-x^2/(2\theta^2)}\,dx$

Let, $\frac{x^2}{2\theta^2} = t$

$\Rightarrow \frac{2x}{2\theta^2}\,dx = dt$

$\Rightarrow \frac{x\,dx}{\theta^2} = dt$

x	0	∞
t	0	∞

$= \int_0^{\infty} e^{-t}\,dt$

$= -\left[e^{-t}\right]_0^{\infty}$

$= -\left[0-1\right]$

$= 1$

Hence, proved.

b) $\theta = 100$

$P(X \le 200) = \int_0^{200} \frac{x}{100^2} \times e^{-x^2/2\times 100^2}\,dx$

x	0	200
t	0	2

$\frac{200^2}{2\times 100^2} = t$

$= \frac{\cancel{2}\cancel{0}\cancel{0}\times 2\cancel{0}\cancel{0}}{\cancel{2}\times 1\cancel{0}\cancel{0}\times 1\cancel{0}\cancel{0}} = t$

$= \int_0^{2} e^{-t}\,dt$

$= -\left[e^{-t}\right]_0^{2}$

$= -\left[e^{-2}-1\right]$

$= 1-e^{-2}$

$= 0.8645$

$\frac{100^2}{2\times 100^2}$

$P(X < 200) = 0.865$

$P(X \geq 200) = 1 - P(X < 200) = 1 - 0.865 = 0.135$

c) $P(100 < X < 200) = \int_{100}^{200} \frac{x}{100^2} \times e^{-x^2/2\times 100^2}\,dx$

x	100	200
t	0.5	2

$= \int_{0.5}^{2} e^{-t}\,dt = -\left[e^{-t}\right]_{0.5}^{2} = -\left[e^{-2}-e^{-0.5}\right]$

$= 0.4712$

d) $P(X \le x) = \int_{0}^{x} \frac{x}{\sigma^2} \cdot e^{-x^2/(2\sigma^2)} dx$

$= \int_{0}^{x} e^{-t} dt$

$= -\left[e^{-x^2/2\sigma^2}\right]_0^{x}$

$= -\left[e^{-x^2/2\sigma^2} - 1\right]$

$= 1 - e^{-x^2/2\sigma^2}, \; x > 0$

~~P(x ≤ x~~ $P(X \le x) = 0 \; ; \; x \le 0$

$$P(X \le x) = \begin{cases} 1 - e^{-x^2/2\sigma^2}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

5) $$f(x) = \begin{cases} kx^2, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow k\int_{0}^{2} x^2 dx = 1$

$\Rightarrow k \times \left[\frac{x^3}{3}\right]_0^2 = 1$

$\Rightarrow k = \frac{3}{8}$

b) $\frac{3}{8}\int_{0}^{1} x^2 dx = \frac{3}{8} \times \left[\frac{x^3}{3}\right]_0^1 = \frac{3}{8} \times \frac{1}{3} = \frac{1}{8}$

c) $P(1 < X < 1.5) = \int_{1}^{1.5} \frac{3}{8} x^2 dx = \frac{3}{8} \times \left[\frac{x^3}{3}\right]_1^{1.5} = \frac{3}{8} \times \frac{(1.5^3 - 1)}{3}$

$= 0.2969$

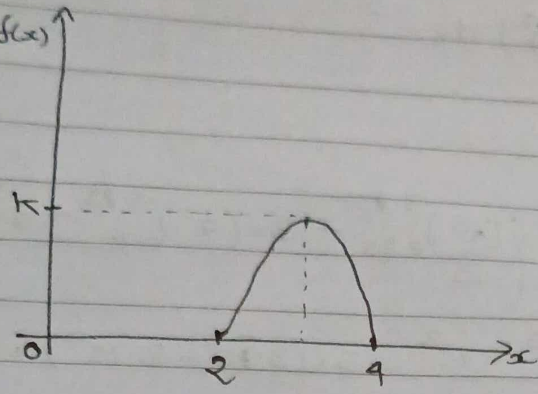
d) $P(X > 1.5) = 1 - P(X \le 1.5)$

$= 1 - \int_{0}^{1.5} \frac{3}{8} x^2 dx$

$= 1 - \frac{\cancel{3}}{8} \times \left[\frac{x^3}{\cancel{3}}\right]_0^{1.5} = 1 - \frac{1.5^3}{8} = 0.578$

6) $f(x) = \begin{cases} k[1-(x-3)^2], & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$

a)



b) $\int_{-\infty}^{\infty} f(x) = 1$

$\Rightarrow k\int_2^4 (1-(x-3)^2)dx = 1$

$\Rightarrow k\left[\int_2^4 dx - \int_2^4 (x-3)^2 dx\right] = 1$

$\Rightarrow k\left[[x]_2^4 - \int_2^4 (x^2-6x+9)dx\right] = 1$

$\Rightarrow k\left[2 - \left(\frac{x^3}{3}\right)_2^4 + 6\times\left[\frac{x^2}{2}\right]_2^4 - 9\times 2\right] = 1$

$\Rightarrow k \times 1.333 = 1$

$\Rightarrow k = \frac{1}{1.333}$

$\Rightarrow k = 0.75$

c) It is given that the actual prescribed weight is 3g.

$P(x>3) = \int_3^4 0.75(1-(x-3)^2)\,dx$

$= 0.75\times 1 - 0.75\times\int_3^4 (x^2-6x+9)\,dx$

$= 0.75 - 0.75\times\left\{\left[\frac{x^3}{3}\right]_3^4 - 6\left(\frac{x^2}{2}\right)_3^4 + 9\right\}$

$= 0.75 - 0.75\times\left[\frac{64-27}{3} - 3\times(16-9) + 9\right]$

$= 0.5$

d) $P(3-0.25 < x < 3+0.25)$

$= P(2.75 < x < 3.25)$

$= \int_{2.75}^{3.25} 0.75 \times (1-(x-3)^2)\, dx$

~~$= 0.75 \times [x$~~

$= 0.75 \times 1 - 0.75 \times \left\{ \left[\frac{x^3}{3}\right]_{2.75}^{3.25} - 3 \times (x^2)_{2.75}^{3.25} + 9 \right\}$

$= 0.75 - 0.75 \left\{ \frac{3.25^3 - 2.75^3}{3} - 3 \times (3.25^2 - 2.75^2) + 9 \right\}$

$= 0.3672$

e) ~~$P(3-0.5 < x < 3+0.5)$~~

~~$= P(2.5 < x <$~~

$P(x < 2.5) + P(x > 3.5)$

$= 1 - P(2.5 < x < 3.5)$

$= 1 - \int_{2.5}^{3.5} 0.75 \times (1-(x-3)^2)\, dx$

$= 0.3125$ (calculated directly from the calculator)

8) $f(y) = \begin{cases} \frac{y}{25}, & 0 \leq y < 5 \\ \frac{2}{5} - \frac{1}{25}y, & 5 \leq y \leq 10 \\ 0, & y < 0 \text{ or } y > 10 \end{cases}$

b) To prove $\int_{-\infty}^{\infty} f(y)\,dy = 1$

$\int_0^5 \frac{y}{25}\,dy + \int_5^{10} \left(\frac{2}{5} - \frac{y}{25}\right) dy$

$= \frac{1}{25} \times \left[\frac{y^2}{2}\right]_0^5 + \frac{2}{5} \times [y]_5^{10} - \frac{1}{25} \times [y^2]_5^{10} \times \frac{1}{2}$

$= \frac{1}{\cancel{25}} \times \frac{\cancel{25}}{2} + \frac{2}{\cancel{5}} \times \cancel{5} - \frac{1}{25} \times (100-25) \times \frac{1}{2} = 1$

Hence, proved.

c) $P(Y \le 3) = \int_0^3 \frac{y}{25}\,dy = \frac{1}{25} \times \left[\frac{y^2}{2}\right]_0^3 = \frac{9}{50} = 0.18$

d) $P(Y \le 8) = \int_0^5 \frac{y}{25}\,dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25}y\right) dy$

$= \frac{1}{25} \times \left[\frac{y^2}{2}\right]_0^5 + \frac{2}{5} \times [y]_5^8 - \frac{1}{25} \times \left[\frac{y^2}{2}\right]_5^8$

$= 0.5 + \frac{2}{5} \times 3 - \frac{(64-25)}{50}$

$= 0.92$

e) $P(3 < X < 8) = \int_3^5 \frac{y}{25}\,dy + \int_5^8 \left(\frac{2}{5} - \frac{1}{25}y\right) dy$

$= \frac{1}{25} \times \left[\frac{y^2}{2}\right]_3^5 + \frac{2}{5} \times 3 - \frac{(64-25)}{50}$

$= \frac{25-9}{50} + \frac{6}{5} - \frac{(64-25)}{50}$

$= 0.74$

f) $P(X < 2)$ or $P(X > 6)$

$1 - P(2 \le X \le 6)$

$= 1 - \left[\int_2^5 \frac{y}{25}\,dy + \int_5^6 \left(\frac{2}{5} - \frac{1}{25}y\right) dy\right]$

$= 1 - \left[\frac{1}{25} \times \left(\frac{y^2}{2}\right)_2^5 + \frac{2}{5} - \frac{1}{25} \times \left(\frac{y^2}{2}\right)_5^6\right]$

$= 1 - \left[\frac{25-4}{50} + \frac{2}{5} - \frac{36-25}{50}\right]$

$= 0.4$

10) $f(x; k, \theta) = \begin{cases} \frac{k \cdot \theta^k}{x^{k+1}}, & x \ge \theta \\ 0, & x < \theta \end{cases}$

b) To prove $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$x^{-(k+1)}$

$\frac{1}{x^k}$

$$\int_{\theta}^{\infty} \frac{k \cdot \theta^k}{x^{k+1}} dx$$

$$= k \cdot \theta^k \times \left[\frac{x^{-k-1+1}}{-k}\right]_{\theta}^{\infty}$$

$$= -\theta^k \times \left(x^{-k}\right)_{\theta}^{\infty}$$

$$= -\theta^k \times \left(0 - \frac{1}{\theta^k}\right)$$

$$= 1$$

Hence, proved

c) $P(X \leq b) = \int_{\theta}^{b} \frac{k\theta^k}{x^{k+1}} dx$

$$= -\cancel{k}\theta^k \times \left[\frac{x^{-k}}{\cancel{k}}\right]_{\theta}^{b}$$

$$= \cancel{-\theta^k \times \left(x^{-b} - x^{-\theta}\right)}$$

$$= \cancel{-\theta^k \times \left(\frac{1}{x^{\theta}}\right)}$$

$$= -\theta^k \times \left(\frac{1}{b^k} - \frac{1}{\theta^k}\right)$$

$$= -\frac{\theta^k}{b^k} + 1$$

d) $P(a \leq X \leq b) = \int_{a}^{b} \frac{k\theta^k}{x^{k+1}} dx$

$$= -\cancel{k}\theta^k \times \left[\frac{x^{-k}}{\cancel{k}}\right]_{a}^{b}$$

$$= -\theta^k \times \left[\frac{1}{b^k} - \frac{1}{a^k}\right]$$

$$= \frac{\theta^k}{a^k} - \frac{\theta^k}{b^k}$$