Approximations and Errors

Approximation of numbers by truncation and rounding-off, types of errors

Numerical methods are techniques by which mathematical problems are formulated so that they can be solved with arithmetic operations. Although there are many kinds of numerical methods, they have one common characteristic: they invariably involve large numbers of tedious arithmetic calculations. It is little wonder that with the development of fast, efficient digital computers, the role of numerical methods in engineering problem solving has increased dramatically in recent years.

A computer has a finite word length and so only a fixed number of digits are stored and used during computation. This would mean that even in storing an exact decimal number in its converted form in the computer memory, an error is introduced.

The quantity, $(True\ value\ -\ Approximate\ value)$ is called the ***error***.

(A) Reason of errors

The error in numerical computation can enter in two different ways:

(1) Inherent error: The inherent error is that quantity which is already present in the statement of the problem before its solution.

The inherent error arises either due to the simplified assumptions in the mathematical formulation of the problem or due to the physical measurements of the parameters of the problem.

(2) Round-off error: Error that arises due to the finite precision of an infinite decimal representation during arithmetic computation is known as round-off error.

Round-off error is occurred in two ways:

(i) Chopping (Truncation)

(ii) Rounding

In *chopping,* after n-significant digits we simply discard the *(* $n+1$*)-th* and later digits.

In *rounding*, to round-off a number to n-significant digits, we discard all the digits to the right of the *n-th* digit if the *(* $n+1$*)-th* digit is less than 5. If the *(* $n+1$*)-th* digit is more than *5*, we increase the *n-th* digit by 1. If the *(* $n+1$*)-th* digit is equal to *5*, then we increase the *n-th* digit by 1 if it is odd and leave the *n-th* digit unchanged if it is even.

Example. *Find round-off error in rounding and chopping correct up to 4-significant figures for the numbers (a) 1.7320508 (b) 0.014159.*

Solution. (a) Let $x_T = 1.7320508$

The approximate value after chopping is $x_A = 1.732$

Therefore, round-off error in chopping is $= x_T - x_A = 1.7320508 - 1.732 = 0.0000508$

Again, the approximate value after rounding is $x_A = 1.732$

Therefore, round-off error in rounding is $= x_T - x_A = 1.7320508 - 1.732 = 0.0000508$

(b) Let $x_T = 0.014159$.

The approximate value after chopping is $x_A = 0.01415$

Therefore, round-off error in chopping is $= x_T - x_A = 0.014159 - 0.01415 = 0.000009$

Again, the approximate value after rounding is $x_A = 0.01416$

Therefore, round-off error in rounding is $= x_T - x_A = 0.014159 - 0.01416 = - 0.000001$

(B) Measurement of errors

Whenever approximating a number, we mainly measure errors in three different ways

(1) Absolute error: It is the absolute value of the difference between true value and approximate value. Let us denote true value of a decimal number by x_T and approximate value of a decimal number by x_A. Then the absolute error is defined as $E_A = |x_T - x_A|$.

(2) Relative error: The relative error of a number is known as the absolute error divided by its true value, i.e., $E_R = \frac{|x_T - x_A|}{|x_T|}$.

(3) Percentage error: The percentage error is defined as the relative error multiplied by 100, i.e, $E_P = \frac{|x_T - x_A|}{|x_T|} \times 100$.

Example. *Find the absolute error, relative error and percentage error in approximating the number 10.42857 correct up to 3 decimal places.*

Solution. Let $x_T = 10.42857$. Therefore, $x_A = 10.429$

So, absolute error is $E_A = |x_T - x_A| = |-0.00043| = 0.00043$

Relative error is $E_R = \frac{|x_T - x_A|}{|x_T|} = \frac{0.00043}{10.42857} = 0.0000412$

Percentage error is $E_P = \frac{|x_T - x_A|}{|x_T|} \times 100 = 0.00412$