1 Differential operator (2.3)

In operational calculus, an **operator** is a transformation that transforms a function into another function. Hence, differential calculus involves an operator, the **differential operator** D, which transforms a (differentiable) function into its derivative. In operator notation we write:

$$Dy\,(x) = y'\,(x) = \frac{dy}{dx}$$

.

Double D allows to obtain the second derivative of the function $y\,(x)$:

$$D^2y\,(x) = D\,(Dy\,(x)) = Dy'\,(x) = y''\,(x)\,.$$

Similarly, the n^{th} power of D leads to the n^{th} derivative:

$$D^n y\,(x) = y^{(n)}\,(x)$$

. Here we assume that the function $y\,(x)$ is n times differentiable.

So, a homogeneous linear ODE of second order of the form $y'' + ay' + by = 0$ can be represented as

$$\begin{aligned} &D^2y + aDy + by = 0 \\ \Rightarrow &(D^2 + aD + bI)y = 0, \qquad \text{where } I \text{ is the identity operator defined as } Iy = y \\ \Rightarrow &P(D)y = 0, \qquad \text{where } P(D) = D^2 + aD + bI \end{aligned}$$

We call $P(D)$ a **second-order differential operator**.

Example 1 *a.* $(D^2 - 3D)(e^{2x}) = D^2(e^{2x}) - 3D(e^{2x}) = 4e^{2x} - 6e^{2x} = -2e^{2x}$.

b. $(D + 11I)(x \sin x) = D(x \sin x) + 11I(x \sin x) = x \cos x + \sin x + 11x \sin x = x(\cos x + 11 \sin x) + \sin x$.

c. $(D - I)(2D + I) \sinh 3x = (2D^2 - D - I) \sinh 3x = 2D^2 \sinh 3x - D \sinh 3x - I \sinh 3x = 18 \sinh 3x - 3 \cosh 3x - \sinh 3x = 17 \sinh 3x - 3 \cosh 3x$.

Example 2 *Solve* $(4D^2 - I)y = 0$ *by using factorization of* $P(D) = 4D^2 - I$.

Solution: The given equation can be written as

$$(2D - I)(2D + I)y = 0$$

Now, we have two different equations

$$\begin{aligned} (2D + I)y = 0 \\ \textit{and } (2D - I)y = 0 \end{aligned}$$

The first equation is nothing but a first order linear ODE. The solution is $y = e^{-0.5x}$ *(verify!!!). Similarly, for the second equation the solution is* $y = e^{0.5x}$ *(verify!!!)*

Since the solutions are linearly independent, the general solution is

$$y = c_1e^{-0.5x} + c_2e^{0.5x}.$$

Note1: *If the factors of* $P(D)$ *are repeated then the above method is not that much useful.*

Note2: *If it is only the matter to solve the problem, then one can use the techniques of Section 2.2.*

2 Euler–Cauchy Equations (2.5)

Euler–Cauchy equations are ODEs of the form

$$x^2y'' + axy' + by = 0 \tag{1}$$

with given constants a and b and unknown function $y(x)$. To find the general solution of it, we naturally choose a solution of the form

$$y = x^m. \tag{2}$$

Substituting (2) in equation (1) we get the **auxiliary equation**

$$m(m-1) + am + b = 0.$$

or,

$$m^2 + (a-1)m + b = 0. \tag{3}$$

Let m_1 and m_2 be the roots of (3). There are three different cases on which the general solution of (1) depends and which are as follows

Case I: (m_1 and m_2 are real and distinct) General solution is

$$y = c_1x^{m_1} + c_2x^{m_2}$$

Case II: (m_1 and m_2 are real and equal) General solution is

$$y = (c_1 + c_2 \ln x)x^{m_1}$$

Case III: (m_1 and m_2 are complex) Since complex roots occur in conjugate pair, we consider $m_1 = p + \iota q$ and $m_2 = p - \iota q$, where p and q are real numbers. Then the general solution is

$$y = x^p(c_1 \cos(q \ln x) + c_2 \sin(q \ln x))$$

Example 3 *Solve* $(x^2D^2 - 4xD + 6I)y = 0$

Solution: Let $y = x^m$ *be a trial solution. Then the auxiliary equation is*

$$\begin{aligned}
&m(m-1) - 4m + 6 = 0\\
\Rightarrow &m^2 - 5m + 6 = 0\\
\Rightarrow &(m-2)(m-3) = 0\\
\Rightarrow &m = 2,\ 3
\end{aligned}$$

Since the roots are real and distinct, by Case I we have the general solution

$$y = c_1x^2 + c_2x^3,$$

where c_1, c_2 *are arbitrary constants.*

Example 4 *Solve* $(x^2D^2 - 3xD + 4I)y = 0$

Solution: Let $y = x^m$ *be a trial solution. Then the auxiliary equation is*

$$\begin{aligned}
&m(m-1) - 3m + 4 = 0\\
\Rightarrow &m^2 - 4m + 4 = 0\\
\Rightarrow &(m-2)^2 = 0\\
\Rightarrow &m = 2,\ 2
\end{aligned}$$

Since the roots are real and equal, by Case II we have the general solution

$$y = (c_1 + c_2 \ln x)x^2,$$

where c_1, c_2 *are arbitrary constants.*

3 Existence and Uniqueness of Solutions. Wronskian (2.6)

In this section we shall discuss the general theory of homogeneous linear ODEs

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

with continuous, but otherwise arbitrary, variable coefficients $p(x)$ and $q(x)$. This will concern the existence and form of a general solution of (4) as well as the uniqueness of the solution of initial value problems consisting of such an ODE with two initial conditions

$$y(x_0) = K_0, \qquad y'(x_0) = K_1 \tag{5}$$

with given x_0, K_0 and K_1.

Theorem 1 (Existence and Uniqueness Theorem for Initial Value Problems) *If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is in I, then the initial value problem consisting of* (4) *and* (5) *has a unique solution on the interval I.*

3.1 Linear Independence of Solutions

Let y_1 and y_2 be two solutions of (4) on an open interval I. We call y_1 and y_2 **linearly independent** on I if the equation

$$c_1y_1 + c_2y_2 = 0 \text{ on } I \text{ has the only solution } c_1 = 0 \text{ and } c_2 = 0$$

We call y_1, y_2 **linearly dependent** on I if this equation also holds for constants c_1, c_2 not both 0.

Also, in another way we can say if there exists some constant $k \neq 0$ for which $y_2 = ky_1$, then y_1, y_2 are linearly dependent.

Note: If solutions y_1, y_2 are linearly independent, then the set $\{y_1, y_2\}$ is called **basis of solutions** of (4).

Theorem 2 (Wronskian) *Let the ODE* (4) *have continuous coefficients $p(x)$ and $q(x)$ on an open interval I. Then two solutions y_1 and y_2 of* (4) *on I are linearly dependent on I if and only if their "Wronskian"*

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$$

is 0 at some x_0 in I. Furthermore, if $W(y_1, y_2) = 0$ at an $x = x_0$ in I, then $W(y_1, y_2) = 0$ on I; hence, if there is an x_1 in I at which $W(y_1, y_2)$ is not 0, then y_1, y_2 are linearly independent on I.

3.2 A general solution of (4)

Theorem 3 (Existence of a General Solution) *If $p(x)$ and $q(x)$ are continuous on an open interval I, then* (4) *has a general solution on I.*

Theorem 4 *If the ODE* (4) *has continuous coefficients $p(x)$ and $q(x)$ on some open interval I, then every solution of* (4) *on I is of the form*

$$y = c_1y_1 + c_2y_2$$

where $\{y_1, y_2\}$ is any basis of solutions of (4) *on I and c_1, c_2 are suitable constants.*

Example 5 *We see that $W(x, \frac{1}{x}) = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{1}{x} - \frac{1}{x} = -\frac{2}{x} \neq 0$. So, the set $\{x, \frac{1}{x}\}$ is linearly independent.*

Example 6 *We see that $W(2x, 3x) = \begin{vmatrix} 2x & 3x \\ 2 & 3 \end{vmatrix} = 0$. So, the set $\{2x, 3x\}$ is linearly dependent.*

Note: *One can check that $2x = \left(\dfrac{2}{3}\right) 3x$. It also implies that the set $\{2x, 3x\}$ is linearly dependent.*