The Negative Binomial Distribution:

The negative binomial r.v. and distribution are based on an experiment satisfying the following conditions:

1) The experiment consists of a sequence of independent trials.
2) Each trial can result in either a success (S) or a failure (F).
3) The probability of success is constant from trial to trial, so $P(\text{success on trial } i) = p$, $i = 1, 2, 3, \ldots$
4) The experiment continues until a total r successes have been observed, where r is a specified positive integer.

The random variable of interest is X = the number of failures that precede the r-th success. X is called negative binomial random variable because, in contrast to the binomial r.v., the number of successes is fixed and the number trials is random.

We denote the p.m.f of X by $nb(x; r, p)$.
Possible values of X are $0, 1, 2, \ldots$
For example $nb(7; 3, p) = P(X=7)$ means the probability that exactly 7 failures occur before the 3rd success.

The p.m.f of negative binomial random variable:

The p.m.f of the negative binomial r.v. X with parameters r = number of successes and $p = P(S)$ is

$$nb(x; r, p) = \binom{x+r-1}{r-1} p^r (1-p)^x, \quad x = 0, 1, 2, \ldots \longrightarrow (1)$$

In the special case when $r = 1$, the p.m.f (1) is

$$nb(x; 1, p) = (1-p)^x p \longrightarrow (2), \quad x = 0, 1, 2, \ldots$$

This p.m.f (2) is called geometric distribution.

Proposition:

If X is a negative binomial r.v. with pmf $nb(x; r, p)$,

then $E(X) = \frac{r(1-p)}{p}$, $V(X) = \frac{r(1-p)}{p^2}$

Example:

A pediatrician ~~who~~ wishes to recruit 5 couples, each of whom is expecting their first child, to participate in a new natural childbirth regimen. Let $p = P$(a randomly selected couples agrees to participate). If $p = 0.2$, (a) what is the probability that 15 couples must be asked before 5 are found who agree to participate?
ie, what is the probability that 10 F's occur before the 5 S?

(b) what is the probability that at most 10 F's are observed
ie atmost 15 couples are asked?

Ans:

S = {agrees to participate}, F = {Not agrees to participate}

(a) Here $p = 0.2$, $r = 5$, $x = 10$

$\therefore$ P(10 F occur before the 5 S)

$= nb(10; 5, 0.2) = \binom{10+5-1}{5-1}(0.2)^5(1-0.2)^{10}$

$= \binom{14}{4}(0.2)^5(0.8)^{10} = 0.034$

(b) P(at most 10 F are observed)

$= P(X \leq 10) = \sum_{x=0}^{10} nb(x; 5, 0.2)$

$= \sum_{x=0}^{10} \binom{x+5-1}{5-1}(0.2)^5(1-0.2)^x$

$= \sum_{x=0}^{10} (0.2)^5(0.8)^x \binom{x+4}{4} = 0.164$

Q(7.5) [3.5]

Suppose that P(male birth) = 0.5. A couple wishes to have exactly two female children in their family. They will have children until this condition is fulfilled.

a) What is the probability that the family has x male children?

b) What is the probability that the family has four children?

c) What is the probability that the family has at most four children?

d) How many male children would you expect this family to have? How many children would you expect this family to have?

Ans: The negative binomial distribution is the probability distribution of a variable X that measures the number failures needed to obtain the r-th success. Here X has a negative binomial distribution where,

r = Number of successes = 2 (No. of female children)

p = P(female birth) = 1 − P(male birth)

= 1 − 0.5 = 0.5

The p.m.f of negative binomial distribution is

$$nb(x; r, p) = \binom{x+r-1}{r-1} p^r (1-p)^x, \quad x = 0, 1, 2, \ldots$$

(a) Here x = no. of failures before the r-th successes

= no. of male children before the 2nd female children

$\therefore$ P (family has x male children)

$= nb(x; 2, 0.5)$

$= \binom{x+2-1}{2-1} (0.5)^2 (1-0.5)^x$

$= \binom{x+1}{1} 0.25 (1-0.5)^x = 0.25(x+1)(0.5)^x$

(b) If the family has four children then out of these four children 2 will be female and 2 will be male.

Thus we need to find $P(X=2)$

$\therefore P(X=2) = nb(2; 2, 0.5)$

$= \binom{2+2-1}{2-1} (0.5)^2 (1-0.5)^2$

$= 3C_1 \times 0.25 \times (0.5)^2 = 3 \times 0.25 \times 0.25 = 0.1875$

(c) The family has at most 4 children of which 2 are female. That's means the family has at most 2 males. Thus we need to find $P(X \le 2)$

$\boxed{x = 0, 1, 2}$

now $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= nb(0; 2, 0.5) + nb(1; 2, 0.5) + nb(2; 2, 0.5)$

$= 0.25 + 0.25 + 0.1875 = 0.6875 = 68.75\%$

(d) The mean of the negative binomial distribution is

$$\mu = E(X) = \frac{r(1-p)}{p} = \frac{2(1-0.5)}{0.5} = 2$$

Thus we expect the family to have 2 male children.

Hence since the family needs to have 2 femal children, we expect the family to have $2+2=4$ children in total.