Laplace Transform (6.1)

Laplace transform of a function $f(t)$, $\forall\, t \geq 0$ is denoted by $L\{f(t)\}$ and defined as the integral $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = F(s)$.

In reverse, Laplace inverse of $F\{s\}$ is $f\{t\}$,

i.e. $L^{-1}\{F(s)\} = f(t)$.

Laplace Transform of Some Functions:

$L\{1\} = \frac{1}{s}$, $L\{t\} = \frac{1}{s^2}$, $L\{t^n\} = \frac{n!}{s^{n+1}}, n = 0,1,2,\ldots$	$L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$
$L\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$	$L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$
$L\{\sin\omega t\} = \dfrac{\omega}{s^2 + \omega^2}$	$L\{e^{at}\sin\omega t\} = \dfrac{\omega}{(s-a)^2 + \omega^2}$
$L\{\cos\omega t\} = \dfrac{s}{s^2 + \omega^2}$	$L\{e^{at}\cos\omega t\} = \dfrac{s-a}{(s-a)^2 + \omega^2}$

Linearity Principle of Laplace Transform:

Laplace transform is a linear operation which satisfies the following property.

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Where, $f\{t\}$ and $g\{t\}$ are any two functions whose Laplace transform exist and a, b are arbitrary constants.

Ex $L\{2e^{3t} - 5cos2t\} = 2L\{e^{3t}\} - 5L\{cos2t\}$

First Shifting Theorem:

If $\boldsymbol{L\{f(t)\} = F(s)}$ where $s > k$ for some k, then $\boldsymbol{L\{e^{at}f(t)\} = F(s-a)}$ where $s - a > k$

In other words, $\boldsymbol{L^{-1}\{F(s-a)\} = e^{at}L^{-1}\{F(s)\} = e^{at}f(t)}$

Ex $L\{e^{-2t}cos2t\} = \frac{s+2}{(s+2)^2+4}$, $L^{-1}\{\frac{1}{(s-3)^2}\} = e^{3t}t$.

Existence and Uniqueness of Laplace Transforms:

Laplace transform of a function $f\{t\}\ \forall\, t \geq 0$ exists $\forall\, s > k$ if,

1. $f(t)$ is defined and piecewise continuous on every finite interval (i.e., the interval can be divided into a finite number of subintervals such that the function is continuous in each subinterval) on the axis $t \geq 0$

   Ex Greatest integer function $\lfloor t \rfloor$

2. $|f(t)| \leq Me^{kt}$, for some constants M and k (*Growth restriction condition*)

- Some examples of functions satisfying growth restriction.
  - $sinht = \frac{e^t - e^{-t}}{2} < e^t$

- $e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots$ so, $t^n < n!\, e^t$
- $cost = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots < e^t$

➢ *There are some functions which don't satisfy growth restriction condition and therefore their Laplace transform don't exist.*
Ex: e^{t^2}, $t^t, t!$ etc.

Laplace Transform (6.2)

Laplace Transform of Differentials:

$$L\{f\,'(t)\} = sL\{f(t)\} - f(0)$$
$$L\{f\,''(t)\} = s^2 L\{f(t)\} - sf(0) - f\,'(0)$$
$$L\{f\,'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf\,'(0) - f\,''(0)$$
$$\vdots$$
$$\vdots$$
$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f\,'^{(0)} - s^{n-3} f\,''^{(0)} - \cdots - f^{\,(n-1)}(0)$$

Where, $f, f\,', \ldots . f^{\,n-1}$ are continuous $\forall\, t \geq 0$ and satisfy growth restriction. Besides, $f^{\,(n)}$ is piecewise continuous on every finite interval on the axis $t \geq 0$.

Laplace Transform of Integrals :

Let $L\{f(t)\} = F(s)$.

Laplace transform of the integral is defined as follows,

$$L\{\int_0^t f(\rho)d\rho\} = \frac{1}{s}F(s)$$

and $$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t L^{-1}(F(s))d\rho = \int_0^t f(\rho)d\rho$$

Note: *Laplace Transform can be used to find the solution of a differential equation associated with some initial conditions i.e., initial value problems (IVP).*

<u>Ex</u>

Solve $y'' - y = t,\ y(0) = 1,\ y'(0) = 1$

Solution: Taking Laplace transform of both sides of the given differential equation,

$$s^2 Y - sy(0) - y'(0) - Y = \frac{1}{s^2} \text{ where, } Y = L\{y(t)\}$$
$$(s^2 - 1)Y = s + 1 + \frac{1}{s^2}$$
$$Y = \frac{1}{s-1} + \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right)$$
$$y(t) = L^{-1}\{Y\} = L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{s^2-1}\right\} - L^{-1}\left\{\frac{1}{s^2}\right\}$$
$$y(t) = e^t + sinht - t \quad \textit{\textbf{(ANS)}}$$

Shifted Data Problem:

Suppose the initial conditions of an IVP are defined at a non-zero value of $t = t_0$. In order to shift the initial conditions at $t = 0$, assume that $t = \bar{t} + t_0$. Such IVP are called shifted data problem.

Here, $y(t) = y(\bar{t} + t_0) = \bar{y}(\bar{t}),\ y'(t) = \bar{y}'(\bar{t}),\ y''(t) = \bar{y}''(\bar{t})$

Ex

$$Solve\ the\ IVP\quad y'' + y = 2t,\qquad y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\qquad y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$$

Solution: Let $t = \bar{t} + \frac{\pi}{4}$,

$y(t) = y\left(\bar{t} + \frac{\pi}{4}\right) = \bar{y}(\bar{t}),\ y'(t) = \bar{y}'(\bar{t}),\ y''(t) = \bar{y}''(\bar{t})$

Initial conditions, $\bar{y}(0) = \frac{\pi}{2},\ \bar{y}'^{(0)} = 2 - \sqrt{2}$

Given IVP is reformulated as, $\bar{y}'' + \bar{y} = 2(\bar{t} + \frac{\pi}{4}),\ \bar{y}(0) = \frac{\pi}{2},\ \bar{y}'(0) = 2 - \sqrt{2}$

Taking Laplace transform of both sides and on simplifying,

$$\bar{Y} = \frac{2}{(s^2+1)s^2} + \frac{\pi/2}{(s^2+1)s} + \frac{\pi s/2}{(s^2+1)} + \frac{2-\sqrt{2}}{(s^2+1)}$$

Where $\bar{Y} = L\{\bar{y}\}$

$$\bar{y} = L^{-1}\{\frac{2}{(s^2+1)s^2}\} + L^{-1}\{\frac{\frac{\pi}{2}}{(s^2+1)s}\} + L^{-1}\{\frac{\frac{\pi s}{2}}{(s^2+1)}\} + L^{-1}\{\frac{2-\sqrt{2}}{(s^2+1)}\}$$

$$\bar{y} = 2\bar{t} + \frac{\pi}{2} - \sqrt{2}sin\bar{t}$$

$$y(t) = 2t - sint + cost \quad \textit{\textbf{(ANS)}}$$