2.9 Modeling: Electric Circuits

The interconnection of various electric elements in a prescribed manner comprises as an electric circuit in order to perform a desired function. The electric elements include controlled and uncontrolled source of energy, resistors, capacitors, inductors, etc. Analysis of electric circuits refers to computations required to determine the unknown quantities such as voltage, current and power associated with one or more elements in the circuit. To contribute to the solution of engineering problems one must acquire the basic knowledge of electric circuit analysis and laws. Many other systems, like mechanical, hydraulic, thermal, magnetic and power system are easy to analyze and model by a circuit. To learn how to analyze the models of these systems, first one needs to learn briefly some of the basic circuit elements, the laws and some fundamental electric circuits.

Basic Definitions

Electromotive Force:

The Electromotive Force (EMF) is defined as, the amount of work done in the energy transformation (or conversion) and the amount of electricity that passes through the electrical source or the generator. The Electromotive Force (EMF) is measured in Volts.

Voltage Drop:

It refers to the decrease of electric potential along the path of a current flowing in an electrical circuit. In other words, the voltage drop is the arithmetical difference between a higher voltage and a lower voltage. It is measured by Voltmeter (current measuring device).

Voltage drop is encountered when current flows in an element (resistance or load) from the higher-potential terminal toward the lower potential terminal. Voltage rise is encountered when current flows in an element (voltage source) from lower potential terminal (or negative terminal of voltage source) toward the higher potential terminal (or positive terminal of voltage source).

Basic Elements of Electric Circuit:

The three basic elements used in electric circuits are the resistor, capacitor, and inductor. They each play an important role in how an electric circuit behaves. They also have their own standard symbols and units of measurement.

Resistor:

A passive electrical component with two terminals that are used for either limiting or regulating the flow of electric current in electrical circuits. The main purpose of resistor is to reduce the current flow and to lower the voltage in any particular portion of the circuit.

Note:

1. The resistance R of a resistor is measured in Ohms (Ω).

2. By Ohm's law, the voltage across resistor is
   $$E_R = RI$$
   Where R is the resistance and I is the current.
3. The symbol for resistor is a zigzag line as shown below:

Resistor Symbol

Capacitor

A capacitor represents the amount of capacitance in a circuit. The capacitance is the ability of a component to store an electrical charge. You can think of it as the "capacity" to store a charge.

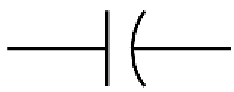
Note:

1. The Capacitance C is measured in Farads (F).
2. The voltage drop across capacitor is given by

$$E_c = \frac{Q}{C} = \frac{1}{C}\int I\,dt$$

Where Q is the charge in coulombs and $I = \frac{dQ}{dt}$ is the current.

3. The symbol for Capacitor is two parallel lines. Sometimes one of the lines is curved as shown below.



Capacitor Symbol

Inductor

An inductor represents the amount of inductance in a circuit. The inductance is the ability of a component to generate electromotive force due to a change in the flow of current. A simple inductor is made by looping a wire into a coil. Inductors are used in electronic circuits to reduce or oppose the change in electric current.

Note:

1. Inductance L is measured in Henrys (H).
2. The voltage drop across inductor is

$$E_L = L\frac{dI}{dt}$$

3. The symbol for inductor is a series of coils as shown below.

Inductor Symbol

Kirchhoff's Laws:

Kirchhoff's Current Law (KCL): KCL states that at any node (junction) in a circuit the algebraic sum of currents entering and leaving a node at any instant of time must be equal to zero.

Kirchhoff's Voltage Law (KVL): It states that in a closed circuit, the algebraic sum of all source voltages (total EMF) must be equal to the algebraic sum of all the voltage drops.

RLC-Circuit:

An RLC-circuit consists of a resistor, an inductor, a capacitor and an EMF.

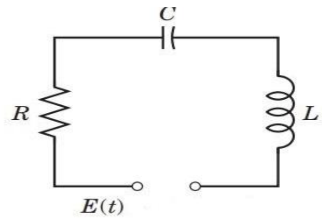


Fig. RLC-circuit

Modeling of RLC-Circuit:

The voltage drop across the Inductor is: $E_L = L\frac{dI}{dt}$

The voltage drop across the resistor is: $E_R = RI$

The voltage drop across the Capacitor is: $E_c = \frac{Q}{C} = \frac{1}{C}\int I\,dt$.

According to Kirchhoff's Voltage Law, the sum of the voltage drop across the three elements inductor, resistor and capacitor is equal to the total Electro Motive Force (EMF).

The ODE for RLC-circuit is: $L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$

Or, $L\frac{dI}{dt} + RI + \frac{1}{C}\int I\,dt = E(t)$ (1)

Equation (1) is an integro-differential equation.

Since, $I = \frac{dQ}{dt}, \frac{dI}{dt} = \frac{d^2Q}{dt^2}$ and $\int I\,dt = Q$.

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t) \quad (2)$$

Differentiating (1) w. r. to t:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}. \quad (3)$$

Solving the ODE (3) for the Current in an *RLC*-Circuit

A general solution of (1) is the sum $I = I_h + I_p$ where I_h is a general solution of the homogeneous ODE corresponding to (1) and I_p is a particular solution of (1).

A general solution of the homogeneous equation corresponding to (1) is

$$I_h = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}$$

where, λ_1 and λ_1 are roots of the characteristic equation

$$L\lambda^2 + R\lambda + \frac{1}{c} = 0$$

Let $\lambda_1 = -\alpha + i\beta$ and $\lambda_2 = -\alpha - i\beta$, where $\alpha = \frac{R}{2L}$ and $\beta = \frac{1}{2L}\sqrt{R^2 - \frac{4L}{c}}$.

In an actual circuit, R is never zero. From this it follows that I_h approaches zero, theoretically as $t \to \infty$ but practically after a relatively short time. And hence the current I tends to I_p after some time. The current I_p it is called the steady-state current and the current $I = I_h + I_p$ is known as the transient current.

The transient current I_p can be determined by the method of undetermined coefficients.

Note:

1. In the IVP, the Equation $L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$ is used.

2. The ***transient current*** is $I(t) = I_h + I_p$

As $t \to \infty$, $I_h \to 0$.

Therefore, the transient current $I(t)$ approaches ***steady state current*** I_p.

Example: Find the transient current in RLC-circuit, where $R = 20$ Ohms, $L = 5$ Henrys,

$C = 10^{-2}$ Farad, $E = 85 \sin 4t$ Volts.

Solution: The ODE for RLC-circuit is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dE}{dt}$$

$\Rightarrow 5I'' + 20I' + 100I = 340 \cos 4t$

$\Rightarrow I'' + 4I' + 20I = 68 \cos 4t$

C. F.: $I_h(t)$

$I'' + 4I' + 20I = 0$

Characteristic equation: $\lambda^2 + 4\lambda + 20 = 0$

$\Rightarrow \lambda = -2 \pm 4i$

$\Rightarrow I_h = e^{-2t}(A \cos 4t + B \sin 4t)$

Basis: $\{e^{-2t} \cos 4t, e^{-2t} \sin 4t\}$

P. I.: $I_p(t)$

The particular integral is: $I_p = C \cos 4t + D \sin 4t$

$\Rightarrow I_p' = -4C \sin 4t + 4D \cos 4t$ and $I_p'' = -16C \cos 4t - 4D \sin 4t$

Now, $(4C + 16D) \cos 4t + (4D - 16C) \sin 4t = 68 \cos 4t$

$\Rightarrow 4C + 16D = 68$ and $4D - 16C = 0$

Solving the above equations, we find: $C = 1$ and $D = 4$.

$\Rightarrow$ P. I. is: $I_p = \cos 4t + 4 \sin 4t$

The transient current is: $I(t) = e^{-2t}[A \cos 4t + B \sin 4t] + \cos 4t + 4 \sin 4t.$

Modeling of LC-Circuit:

The voltage drop across the Inductor is: $E_L = L\frac{dI}{dt}$

The voltage drop across the Capacitor is: $E_c = \frac{Q}{C} = \frac{1}{C}\int I\, dt.$

According to Kirchhoff's Voltage Law, the sum of the voltage drop across the three elements inductor, resistor and capacitor is equal to the total Electro Motive Force (EMF).

The ODE for LC-circuit is: $L\frac{dI}{dt} + \frac{Q}{C} = E(t)$

$$\text{Or, } L\frac{dI}{dt} + \frac{1}{C}\int I\, dt = E(t) \qquad (4)$$

Since, $I = \frac{dQ}{dt}, \frac{dI}{dt} = \frac{d^2Q}{dt^2}$ and $\int I\, dt = Q$.

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} = E(t) \qquad (5)$$

Differentiating (4) w. r. to t:

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = \frac{dE}{dt}. \qquad (6)$$

Example: Find the current when $L = 1\,H$, $C = 0.25\,F$, $E = 30 \sin t\, V$, assuming zero initial current and charge.

Solution: The ODE for LC-circuit is

$$L\frac{d^2I}{dt^2} + \frac{1}{C}I = \frac{dE}{dt}$$

$\Rightarrow \frac{d^2I}{dt^2} + \frac{1}{0.25}I = 30\frac{d}{dt}(\sin t)$

$$\Rightarrow I'' + 4I = 30 \cos t \qquad (7)$$

C. F. is: $I_h = A \cos 2t + B \sin 2t$

Basis of solutions is: $\{\cos 2t, \sin 2t\}$

The choice for P. I. is: $I_p = C \cos t + D \sin t$

$\Rightarrow I_p{}' = -C \sin t + D \cos t, I_p{}'' = -C \cos t - D \sin t$

From Eq. (6) we have: $3C \cos t + 3D \sin t = 30 \cos t$

$\Rightarrow C = 10$ and $D = 0$.

The particular integral is: $I_p = 10 \cos t$

The general solution is: $I(t) = A \cos 2t + B \sin 2t + 10 \cos t$ (8)

Initial conditions are: $I(0) = 0,\ Q(0) = 0$

Using $I(0) = 0$ in (7), $A = -10$

The solution (7) becomes

$I(t) = -10 \cos 2t + B \sin 2t + 10 \cos t$ (9)

We have, $L \frac{dI}{dt} + \frac{Q}{C} = E(t)$

$\Rightarrow I'(0) = \frac{1}{L}\left(E(0) - \frac{Q(0)}{C}\right)$

Now, $I'(0) = 0$

From (8): $I'(t) = 20 \sin 2t + 2B \cos 2t - 10 \sin t$

Using $I'(0) = 0$: $B = 0$.

The Particular solution is: $I(t) = 10(\cos t - \cos 2t)$.

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