Turing's Thesis

Computability

- Mathematician David Hilbert listed 23 problems in 1900.
  - These problems are challenges for mathematicians in 20th century.
- His 10th problem is to devise "a process according to which it can be determined by a finite number of operations," that tests whether a polynomial has an integral root.
  - In other words, Hilbert wants to find an algorithm to test whether a polynomial has an integral root.
- If such an algorithm exists, we just need to invent it.
- What if there is no such algorithm?
  - How can we argue Hilbert's 10th problem has no solution?
- We need a precise definition of algorithms!

Computability

In the 1930s, several independent attempts were made to formalize the notion of computability:

In 1933, Kurt Gödel formalized the definition of the class of general recursive functions: the smallest class of functions (with arbitrarily many arguments) that is closed under composition, recursion, and minimization, and includes zero, successor, and all projections.

In 1936, Alonzo Church created a method for defining functions called the λ-calculus: A function on the natural numbers is called λ-computable if the corresponding function on the Church numerals can be represented by a term of the λ-calculus.

Also in 1936, before learning of Church's work, Alan Turing created a theoretical model for machines, now called Turing machines, that could carry out calculations from inputs by manipulating symbols on a tape. A function on the natural numbers is called Turing computable if some Turing machine computes the corresponding function on encoded natural numbers.

Turing's thesis (1930):

Any computation carried out
by mechanical means
can be performed by a Turing Machine

Algorithm:

An algorithm for a problem is a Turing Machine which solves the problem

The algorithm describes the steps of the mechanical means

This is easily translated to computation steps of a Turing machine

When we say: There exists an algorithm

We mean: There exists a Turing Machine that executes the algorithm

Church-Turing Thesis

The Church-Turing thesis states that the above three formally-defined classes of computable functions coincide with the *informal* notion of an effectively calculable function.

Intuitive notion of algorithms	equals	*Turing machine algorithms*

FIGURE
The Church–Turing Thesis

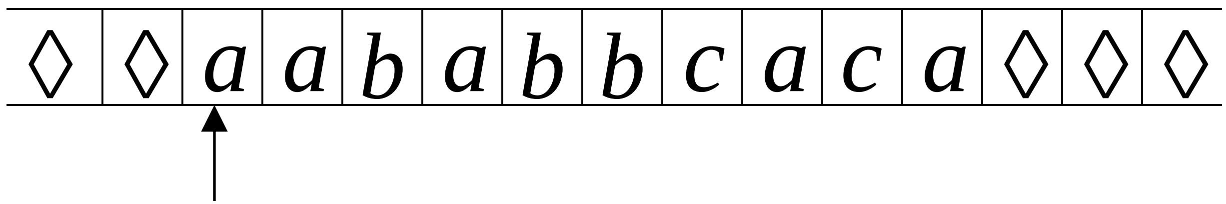
Computability

- In 1970, Yuri Matijasevič showed that Hilbert's 10th problem is not solvable.
  - That is, there is no algorithm for testing whether a polynomial has an integral root.
- Define $D = \{p : p \text{ is a polynomial with an integral root}\}$.
- Consider the following TM:
  $M =$ "The input is a polynomial p over variables $x_1, x_2, \ldots, x_k$
  1. Evaluate p on an enumeration of k-tuple of integers.
  2. If p ever evaluates to 0, accept."
- M recognizes D but does not decide D.

Variations of the Turing Machine

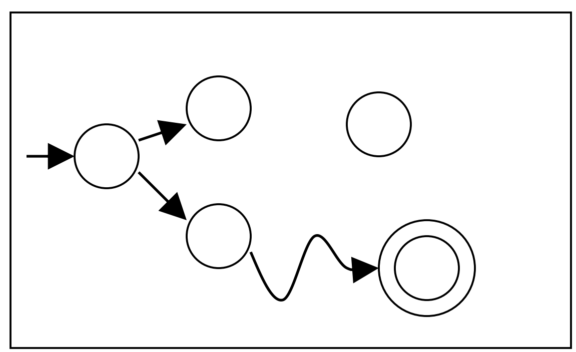
The Standard Model

Infinite Tape



Read-Write Head (Left or Right)

Control Unit



Deterministic

Variations of the Standard Model

Turing machines with:

- Stay-Option
- Semi-Infinite Tape
- Off-Line
- Multitape
- Multidimensional
- Nondeterministic

Different Turing Machine **Classes**

Same Power of two machine classes:

both classes accept the
same set of languages

We will prove:

each new class has the same power
with Standard Turing Machine

(accept Turing-Recognizable Languages)

Same Power of two classes means:

for any machine M_1 of first class

there is a machine M_2 of second class

such that: $L(M_1) = L(M_2)$

and vice-versa

Simulation: A technique to prove same power.

Simulate the machine of one class with a machine of the other class

First Class
Original Machine

M_1

Second Class
Simulation Machine

M_2

M_1

simulates M_1

Configurations in the Original Machine M_1 have corresponding configurations in the Simulation Machine M_2

$$\begin{array}{rccccccccc} & & & & \overset{M_1}{} & & & & \\ \text{Original Machine:} & d_0 & \succ & d_1 & \succ & \cdots & \succ & d_n \\ & \updownarrow & & \updownarrow & & & & \updownarrow \\ \text{Simulation Machine:} & d'_0 & \overset{*}{\succ} & d'_1 & \overset{*}{\succ} & \cdots & \overset{*}{\succ} & d'_n \\ & & & & \underset{M_2}{} & & & \end{array}$$

Accepting Configuration

Original Machine: d_f

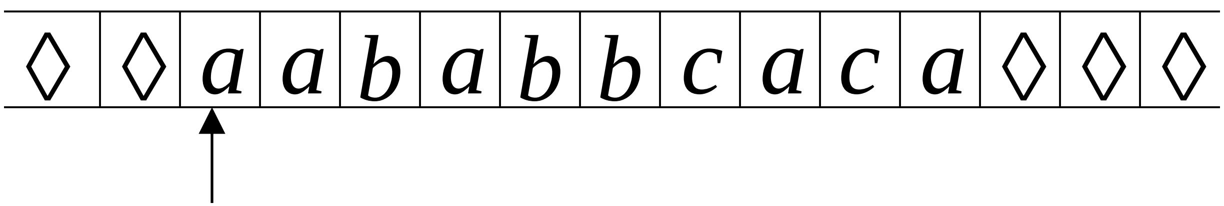
Simulation Machine: d'_f

the Simulation Machine
and the Original Machine
accept the same strings

$$L(M_1) = L(M_2)$$

Turing Machines with Stay-Option

The head can stay in the same position

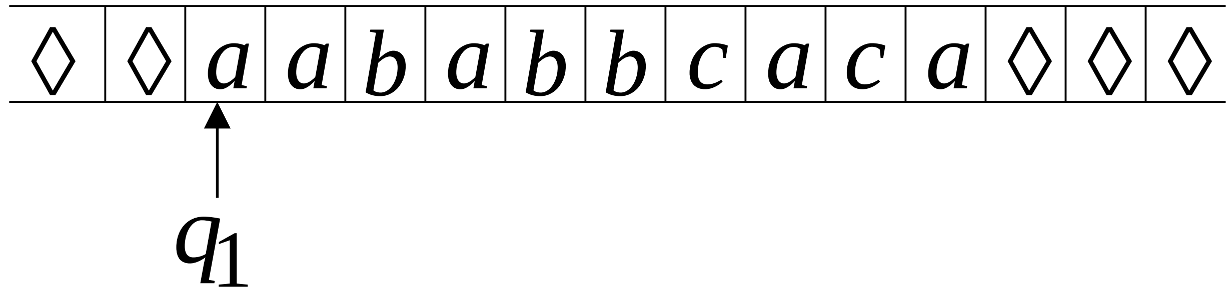


Left, Right, Stay

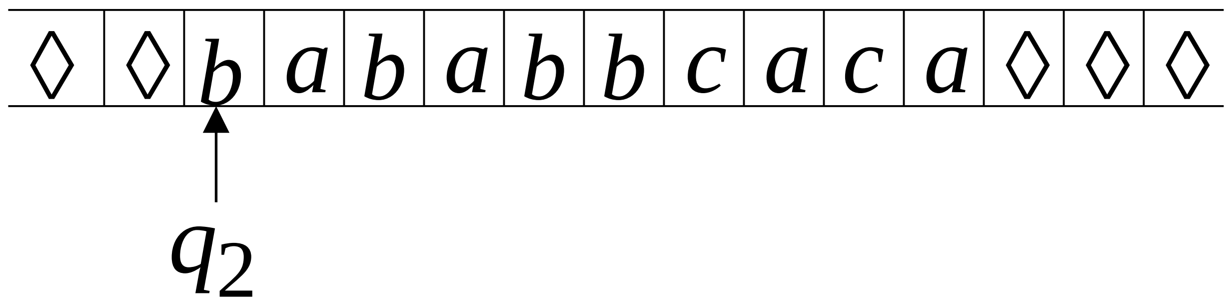
L,R,S: possible head moves

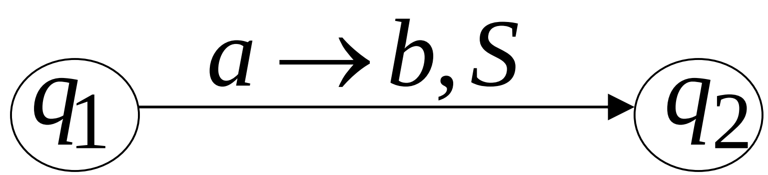
Example:

Time 1



Time 2





Theorem: Stay-Option machines have the same power with Standard Turing machines

Proof: 1. Stay-Option Machines simulate Standard Turing machines

2. Standard Turing machines simulate Stay-Option machines

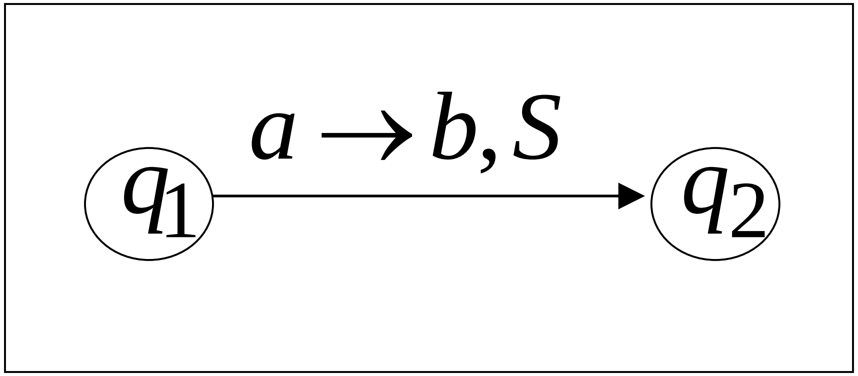
1. Stay-Option Machines simulate Standard Turing machines

Trivial: any standard Turing machine
is also a Stay-Option machine

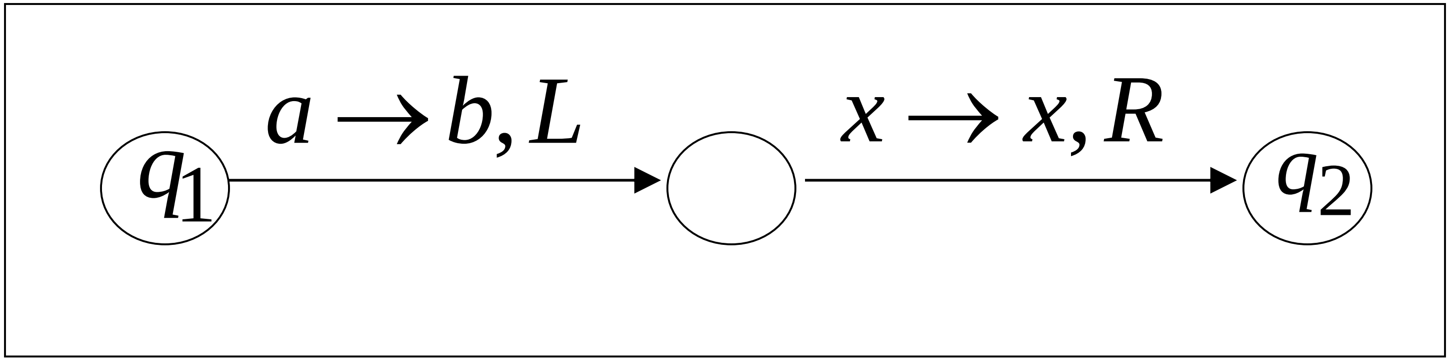
2. Standard Turing machines simulate Stay-Option machines

We need to simulate the stay head option with two head moves, one left and one right

Stay-Option Machine



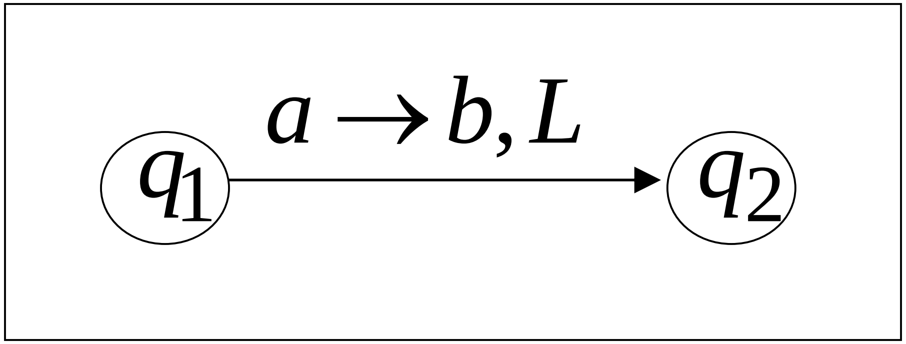
Simulation in Standard Machine



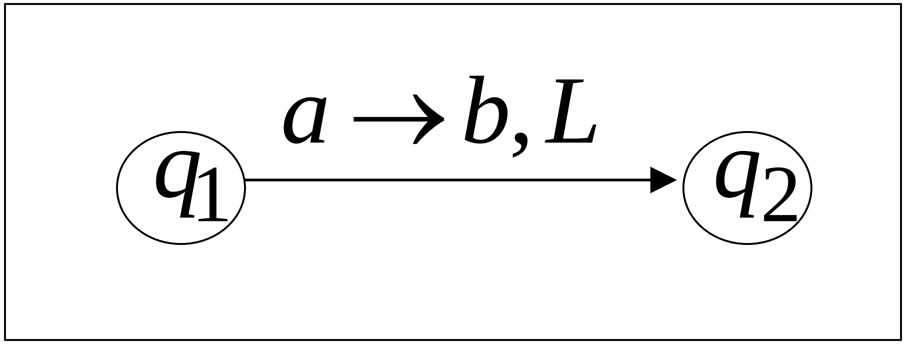
For every possible tape symbol x

For other transitions nothing changes

Stay-Option Machine



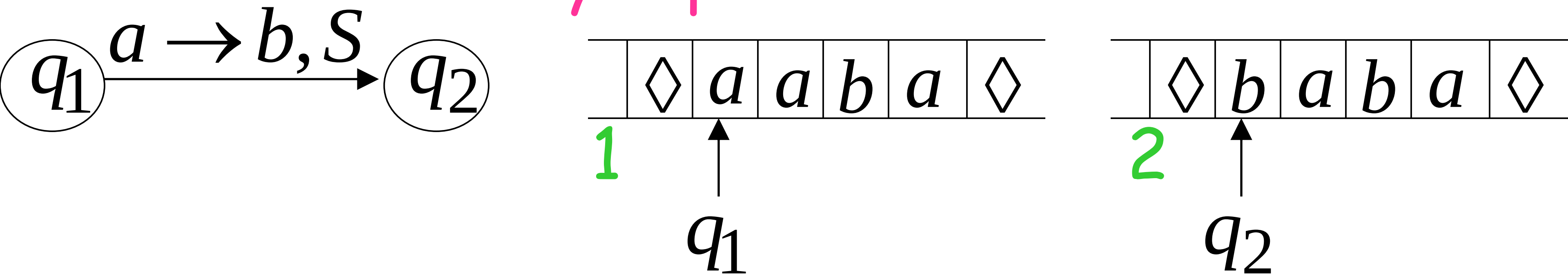
Simulation in Standard Machine



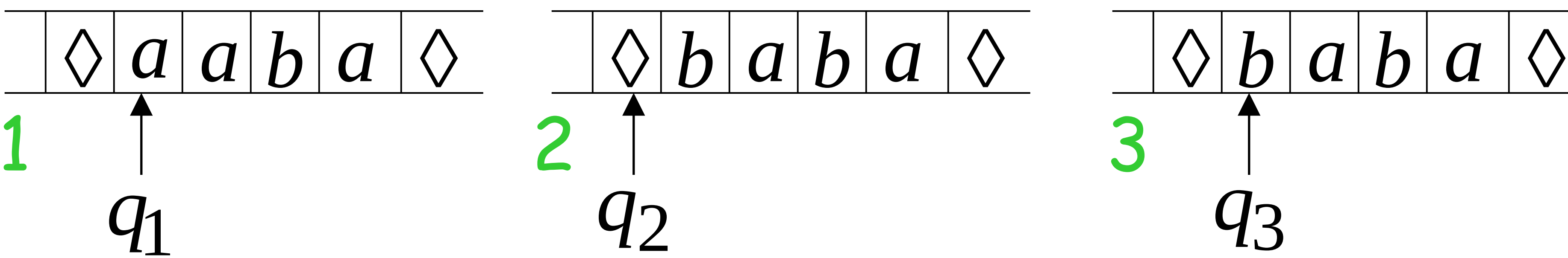
Similar for Right moves

example of simulation

Stay-Option Machine:



Simulation in Standard Machine:

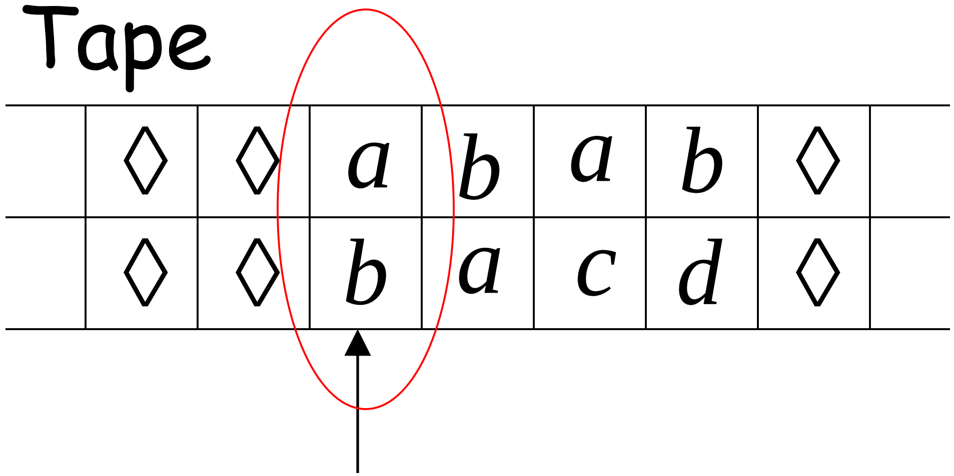


END OF PROOF

Multiple Track Tape

A useful trick to perform more complicated simulations

One Tape

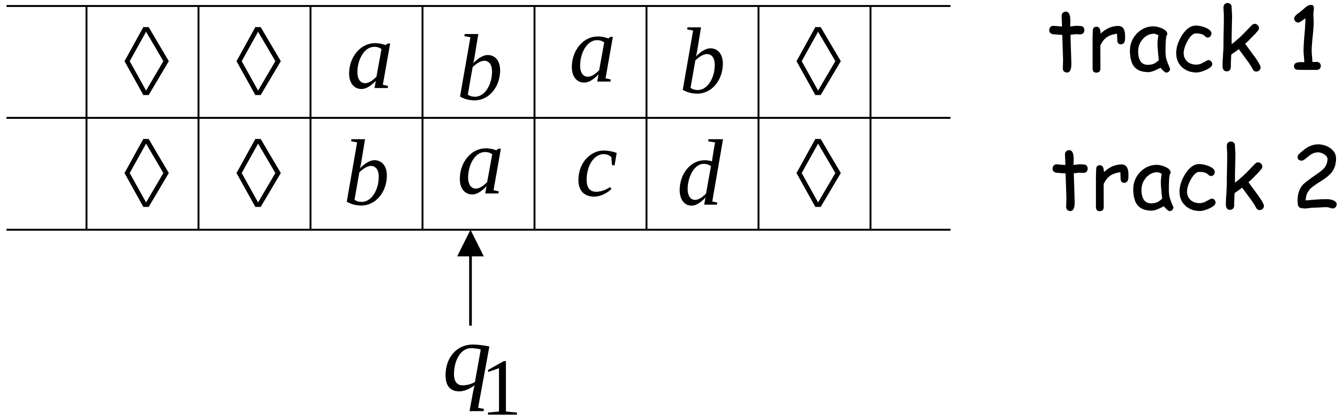


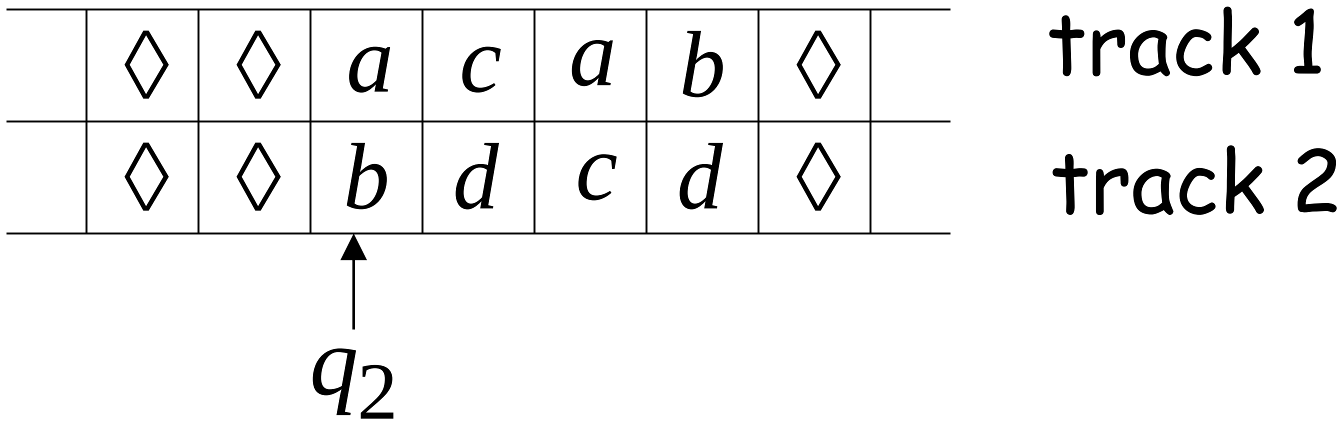
track 1

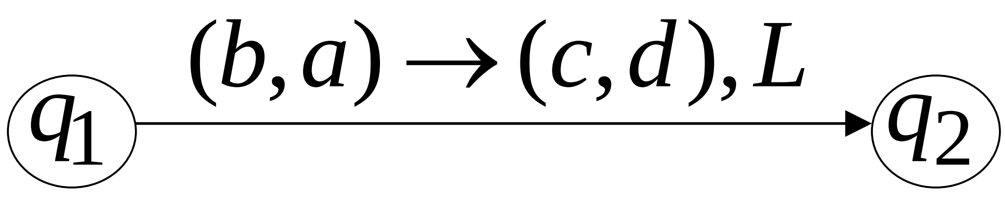
track 2

One head

One symbol (a,b)

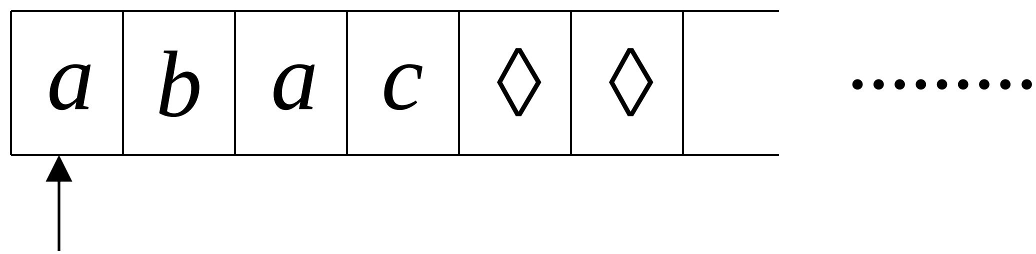

◊ ◊ a b a b ◊
◊ ◊ b a c d ◊
track 1
track 2
q_1


◊ ◊ a c a b ◊
◊ ◊ b d c d ◊
track 1
track 2
q_2


q_1
$(b,a) \rightarrow (c,d), L$
q_2

Semi-Infinite Tape

The head extends infinitely only to the right



- Initial position is the leftmost cell

- When the head moves left from the border, it returns to the same position

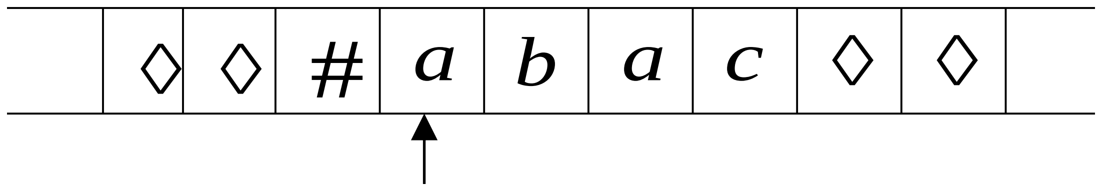
Theorem: Semi-Infinite machines have the same power with Standard Turing machines

Proof:

1. Standard Turing machines simulate Semi-Infinite machines

2. Semi-Infinite Machines simulate Standard Turing machines

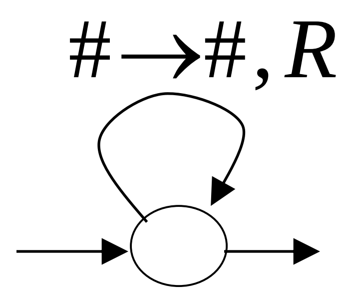
1. Standard Turing machines simulate Semi-Infinite machines:



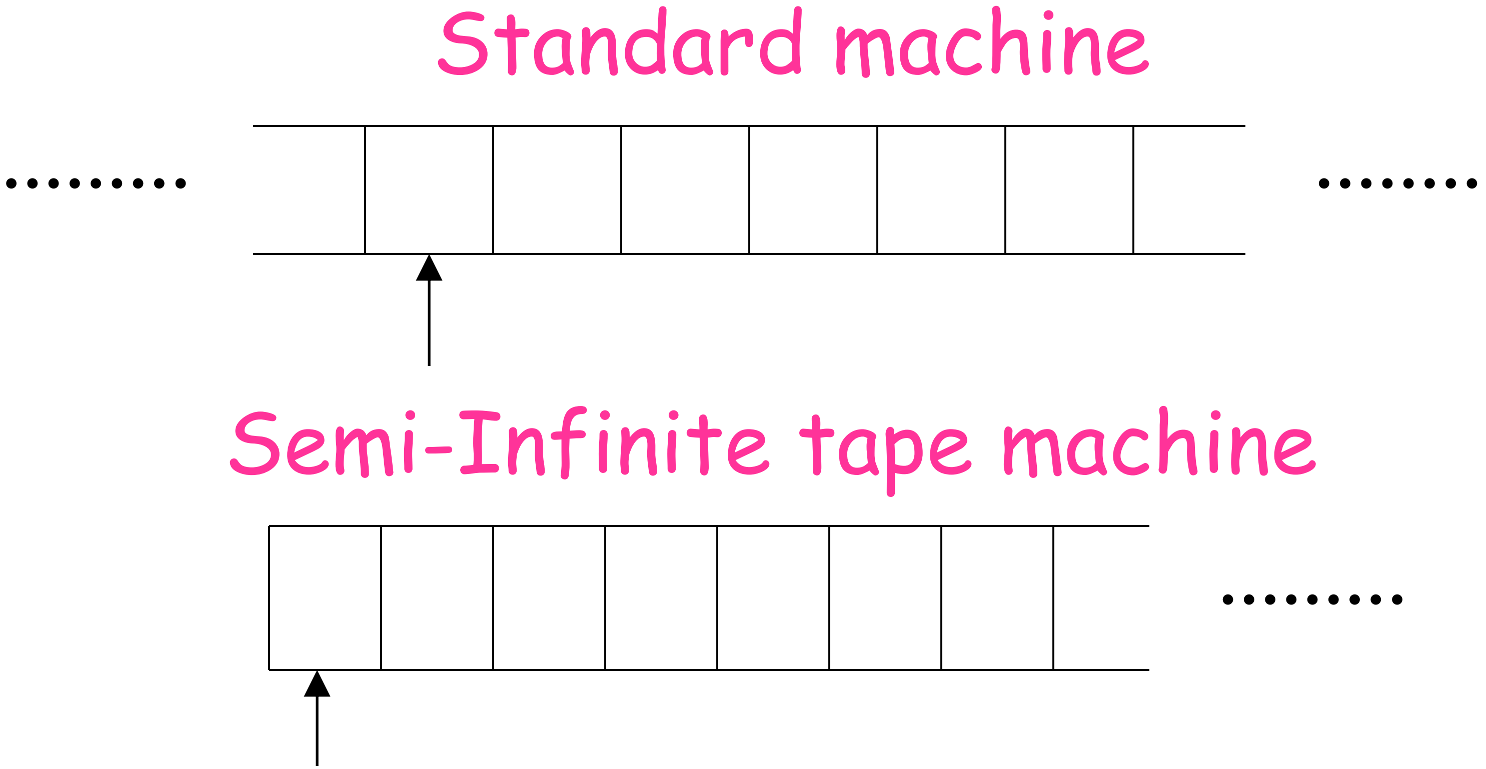
Standard Turing Machine

a. insert special symbol #
at left of input string

b. Add a self-loop
to every state
(except states with no
outgoing transitions)

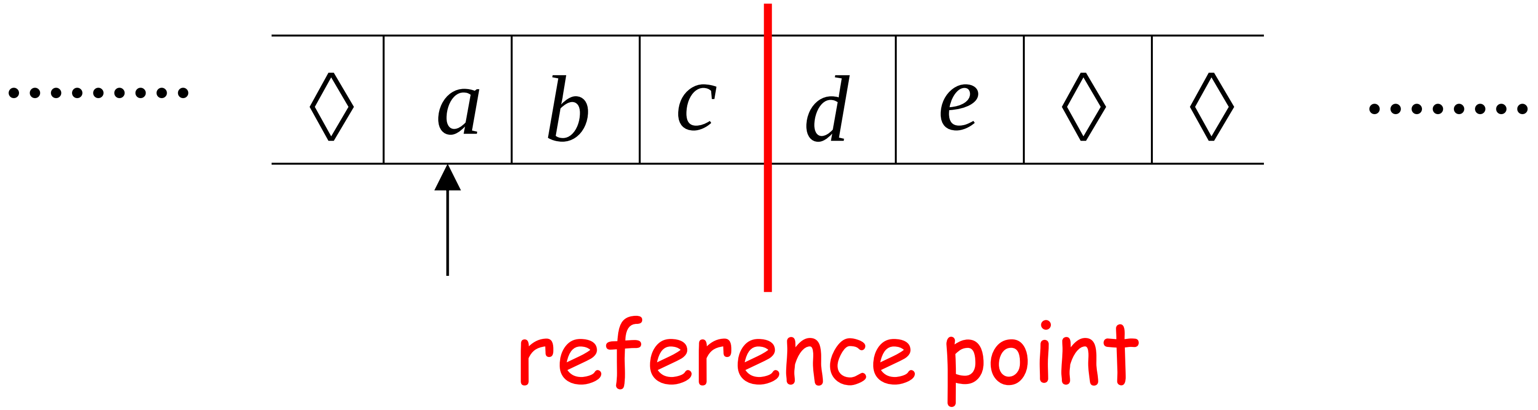


2. Semi-Infinite tape machines simulate Standard Turing machines:

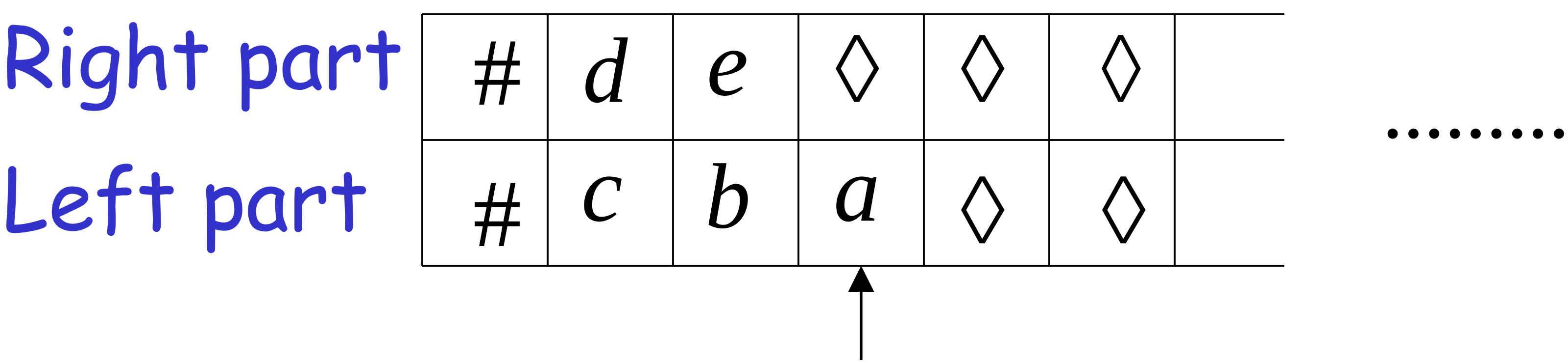


Squeeze infinity of both directions in one direction

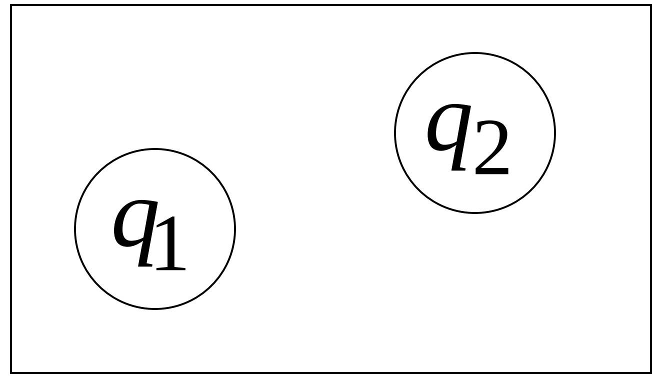
Standard machine



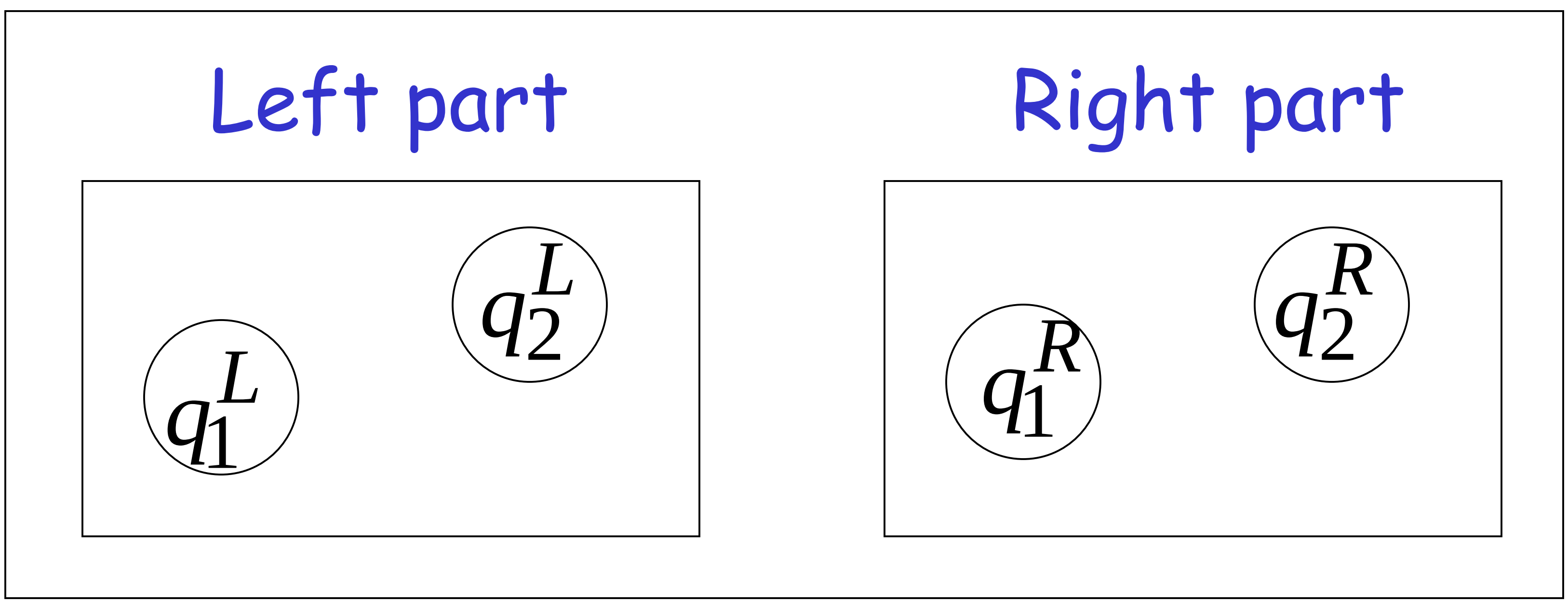
Semi-Infinite tape machine with two tracks



Standard machine



Semi-Infinite tape machine



Standard machine

$$q_1 \xrightarrow{a \rightarrow g, R} q_2$$

Semi-Infinite tape machine

Right part

$$q_1^R \xrightarrow{(a, x) \rightarrow (g, x), R} q_2^R$$

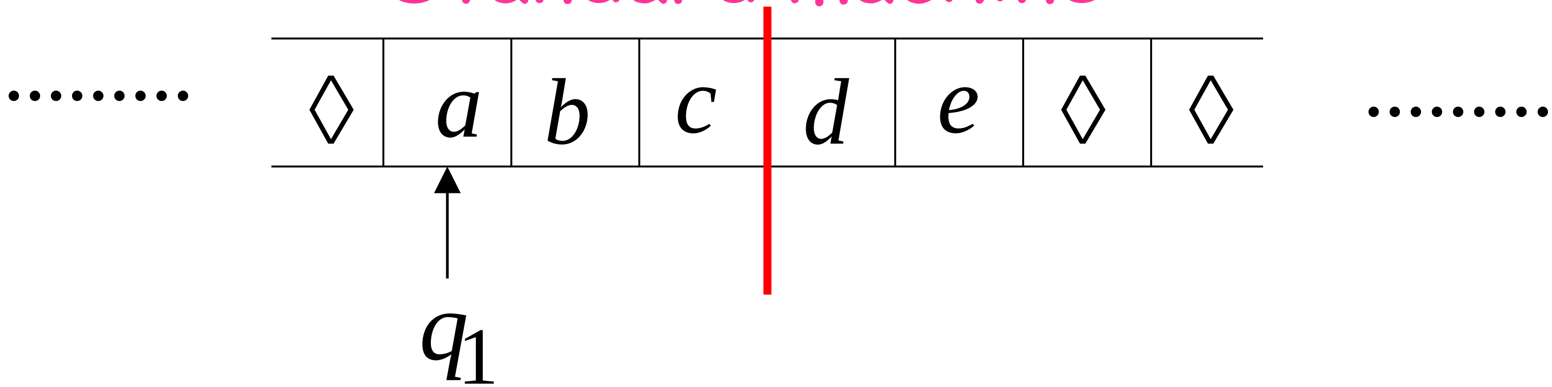
Left part

$$q_1^L \xrightarrow{(x, a) \rightarrow (x, g), L} q_2^L$$

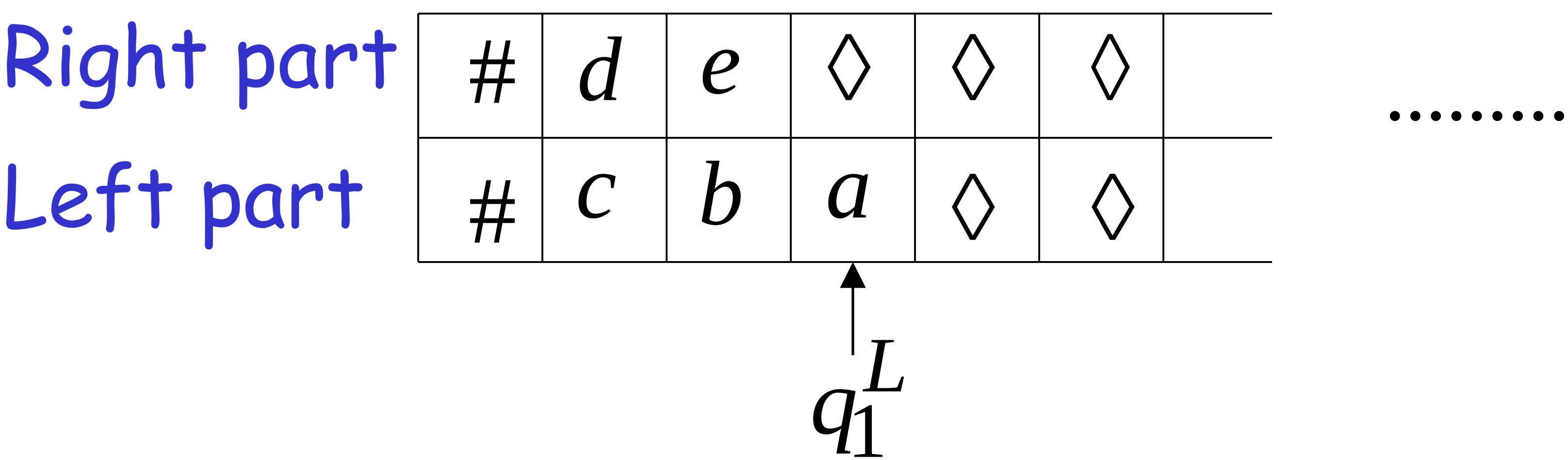
For all tape symbols x

Time 1

Standard machine

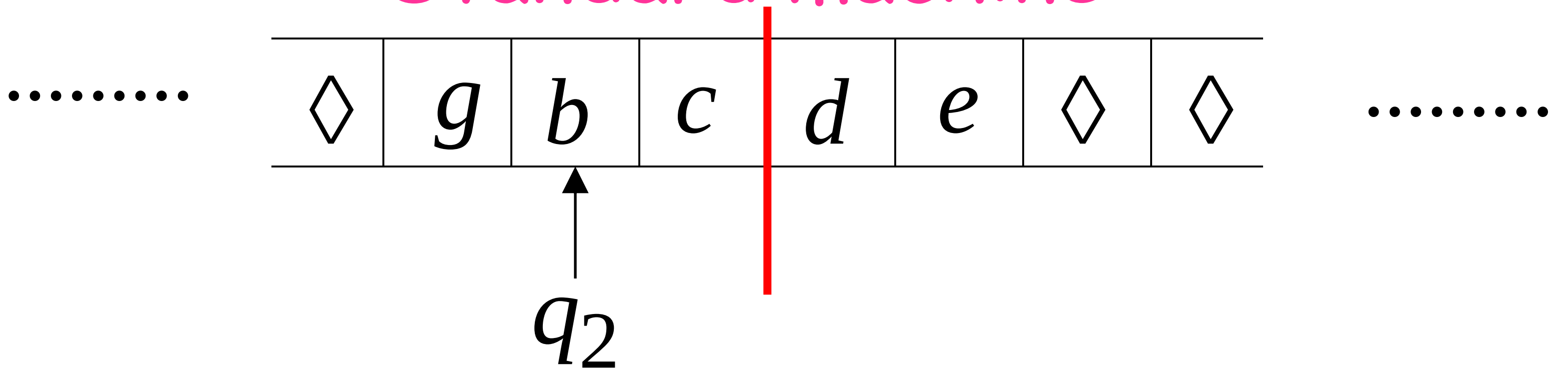


Semi-Infinite tape machine

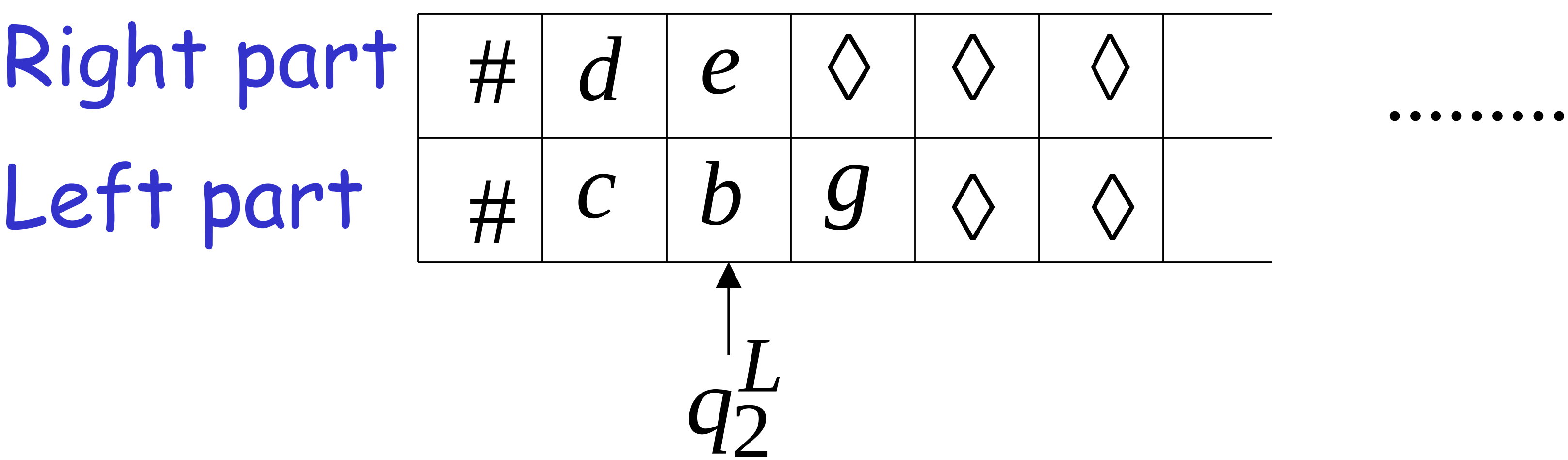


Time 2

Standard machine



Semi-Infinite tape machine



At the border:

Semi-Infinite tape machine

Right part

$$q_1^R \xrightarrow{(\#,\#)\rightarrow(\#,\#),R} q_1^L$$

Left part

$$q_1^L \xrightarrow{(\#,\#)\rightarrow(\#,\#),R} q_1^R$$

Semi-Infinite tape machine

Time 1

Right part	#	d	e	◊	◊	◊	
Left part	#	c	b	g	◊	◊	

....

$\uparrow$ q_1^L

Time 2

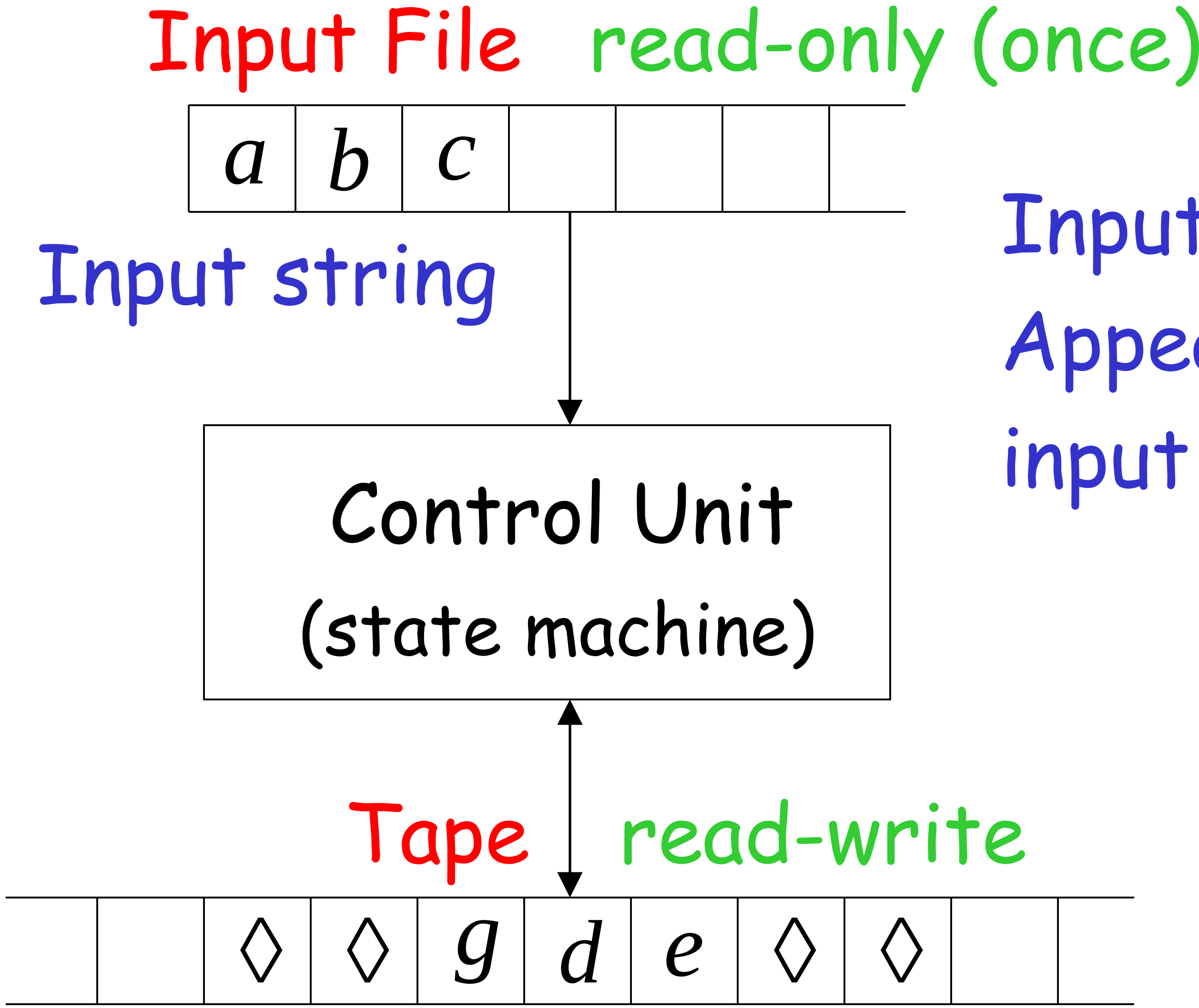
Right part	#	d	e	◊	◊	◊	
Left part	#	c	b	g	◊	◊	

....

$\uparrow$ q_1^R

END OF PROOF

The Off-Line Machine



Input string
Appears on
input file only

Theorem: Off-Line machines have the same power with Standard Turing machines

Proof:

1. Off-Line machines simulate Standard Turing machines

2. Standard Turing machines simulate Off-Line machines

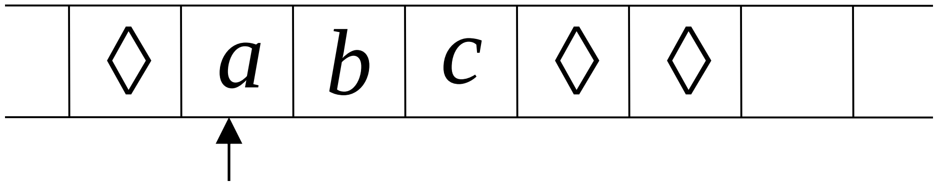
1. Off-line machines simulate Standard Turing Machines

Off-line machine:

1. Copy input file to tape

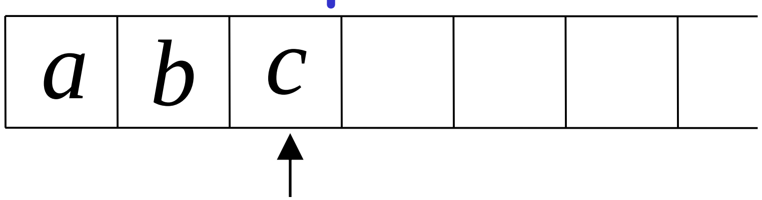
2. Continue computation as in Standard Turing machine

Standard machine

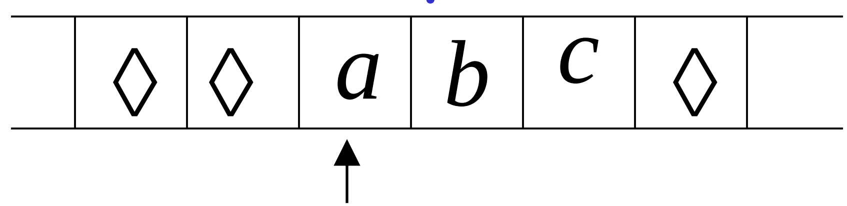


Off-line machine

Input File

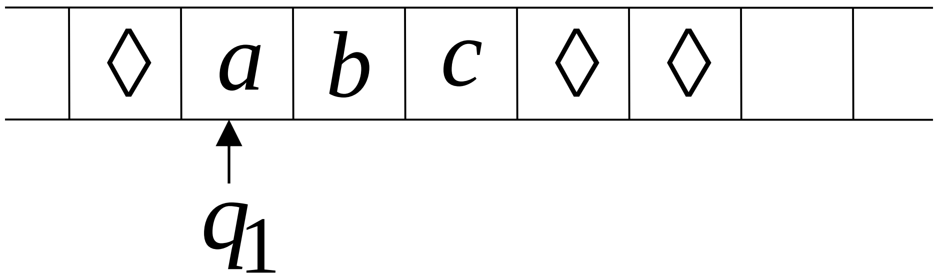


Tape



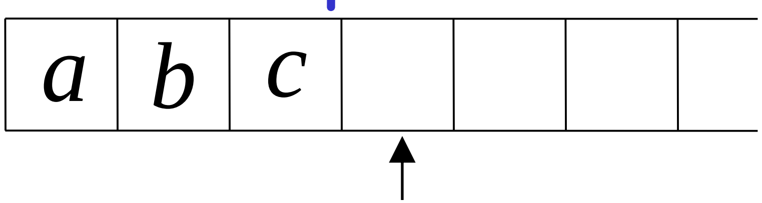
1. Copy input file to tape

Standard machine

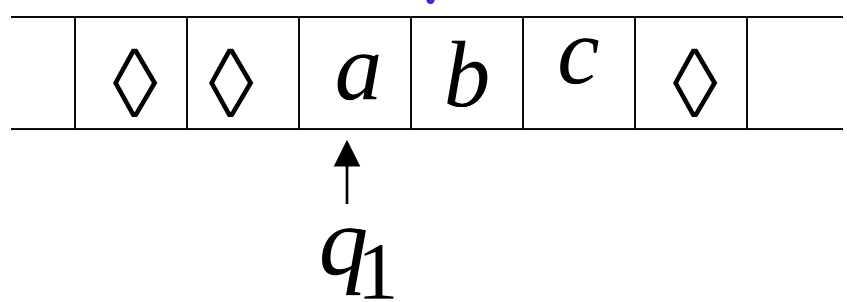


Off-line machine

Input File



Tape



2. Do computations as in Turing machine

2. Standard Turing machines simulate Off-Line machines:

Use a Standard machine with
a four-track tape to keep track of
the Off-line input file and tape contents

Off-line Machine

Input File

a	b	c	d			

Tape

	◊	◊	e	f	g	◊

Standard Machine -- Four track tape

	#	a	b	c	d			Input File
	#	0	0	1	0			head position
		e	f	g				Tape
		0	1	0				head position

Reference point (uses special symbol #)

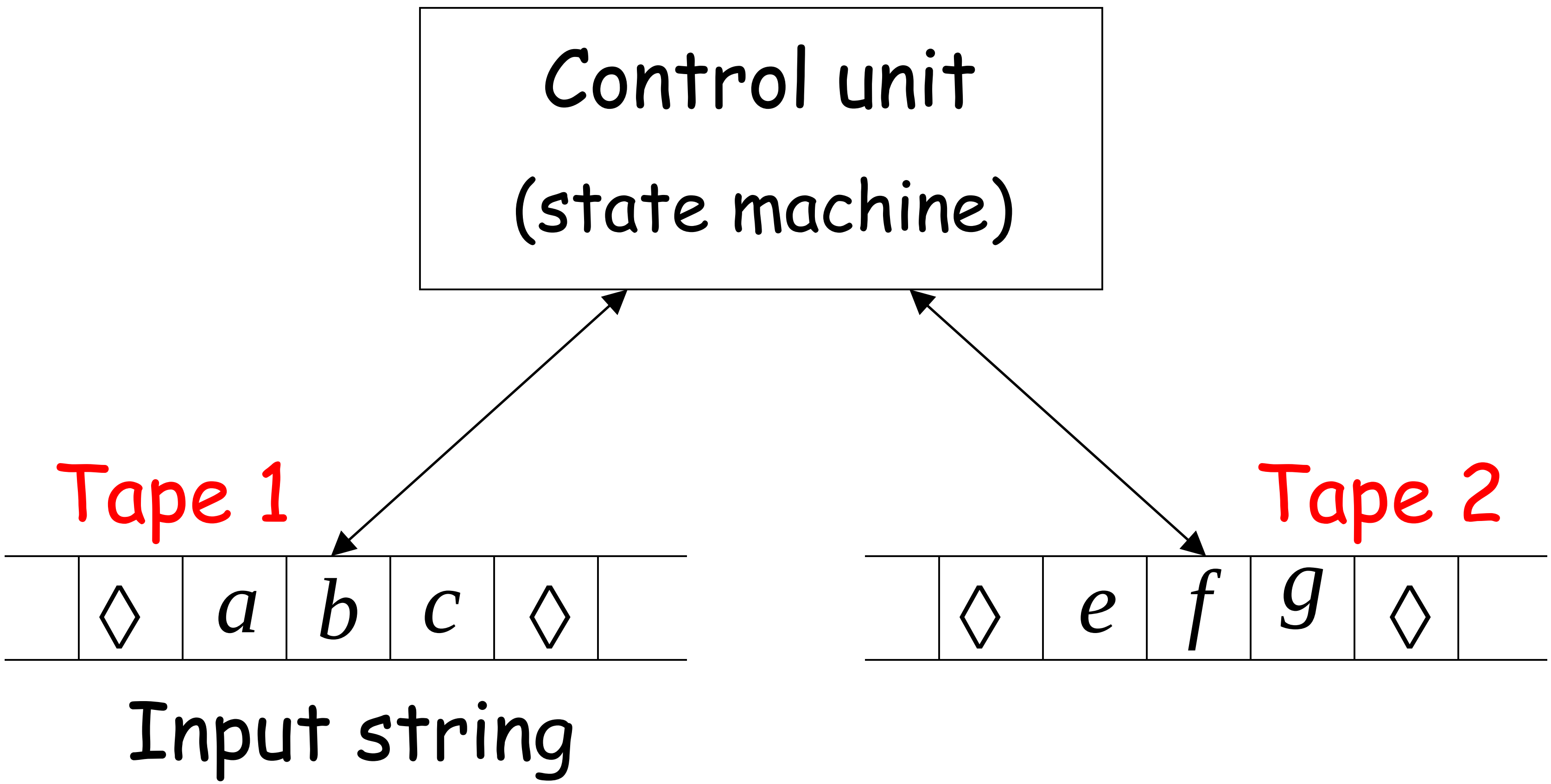
#	a	b	c	d			Input File
#	0	0	1	0			head position
#	e	f	g				Tape
#	0	1	0				head position

Repeat for each state transition:

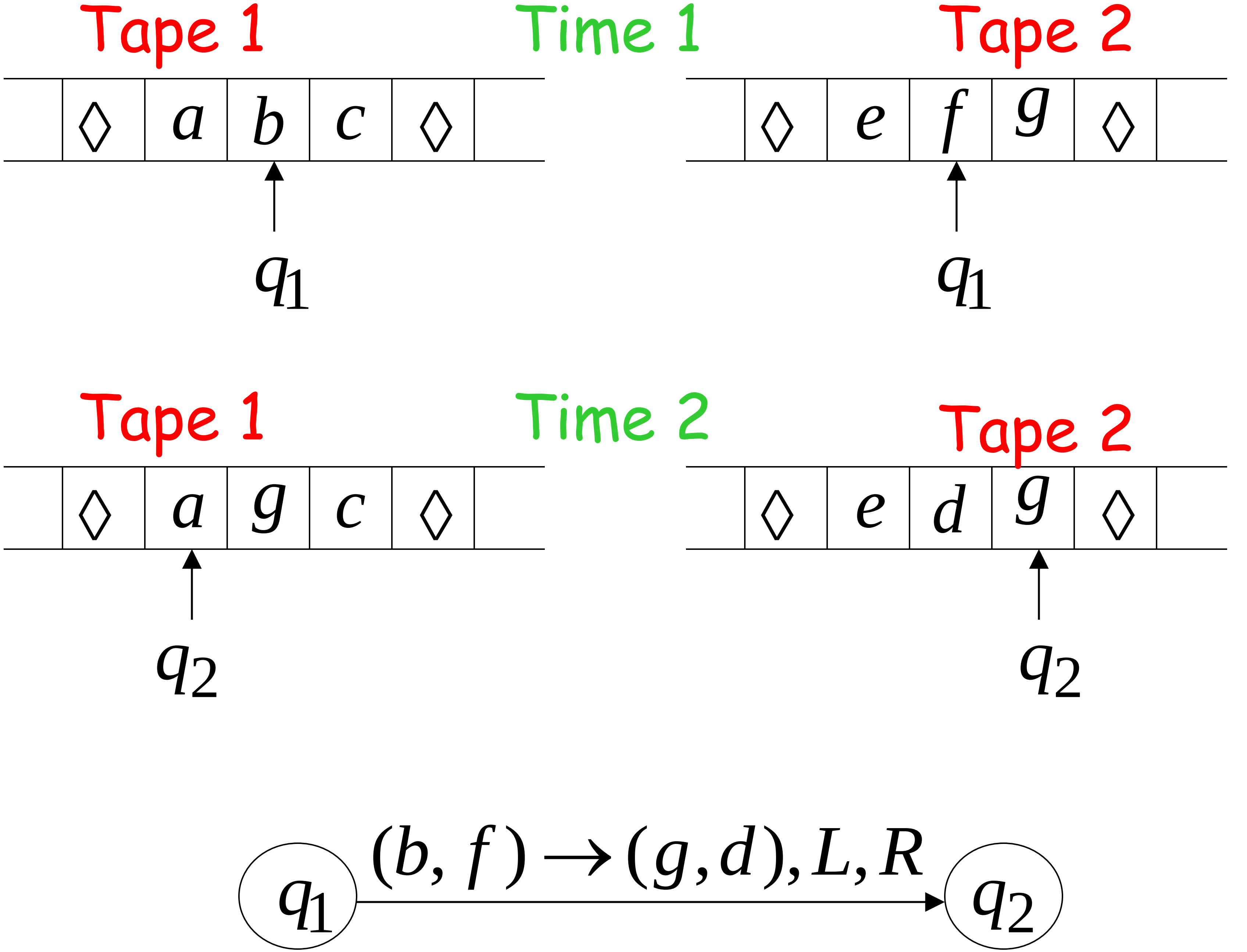
1. Return to reference point
2. Find current input file symbol
3. Find current tape symbol
4. Make transition

END OF PROOF

Multi-tape Turing Machines



Input string appears on Tape 1



Tape 1
Time 1
Tape 2
◊ a b c ◊
◊ e f g ◊
q_1
q_1
Tape 1
Time 2
Tape 2
◊ a g c ◊
◊ e d g ◊
q_2
q_2
$(b, f) \rightarrow (g, d), L, R$
q_1
q_2

Theorem: Multi-tape machines have the same power with Standard Turing machines

Proof: 1. Multi-tape machines simulate Standard Turing machines

2. Standard Turing machines simulate Multi-tape machines

1. Multi-tape machines simulate Standard Turing Machines:

Trivial: Use just one tape

2. Standard Turing machines simulate Multi-tape machines:

Standard machine:

- Uses a multi-track tape to simulate the multiple tapes

- A tape of the Multi-tape machine corresponds to a pair of tracks

Multi-tape Machine

Tape 1

	◊	a	b	c	◊	

Tape 2

	◊	e	f	g	h	◊

Standard machine with four track tape

		a	b	c				Tape 1
		0	1	0				head position
		e	f	g	h			Tape 2
		0	0	1	0			head position

Reference point

	#	a	b	c				Tape 1
	#	0	1	0				head position
	#	e	f	g	h			Tape 2
	#	0	0	1	0			head position

Repeat for each state transition:

1. Return to reference point
2. Find current symbol in Tape 1
3. Find current symbol in Tape 2
4. Make transition

END OF PROOF

Same power doesn't imply same speed:

$$L = \{a^n b^n\}$$

Standard Turing machine: $O(n^2)$ time

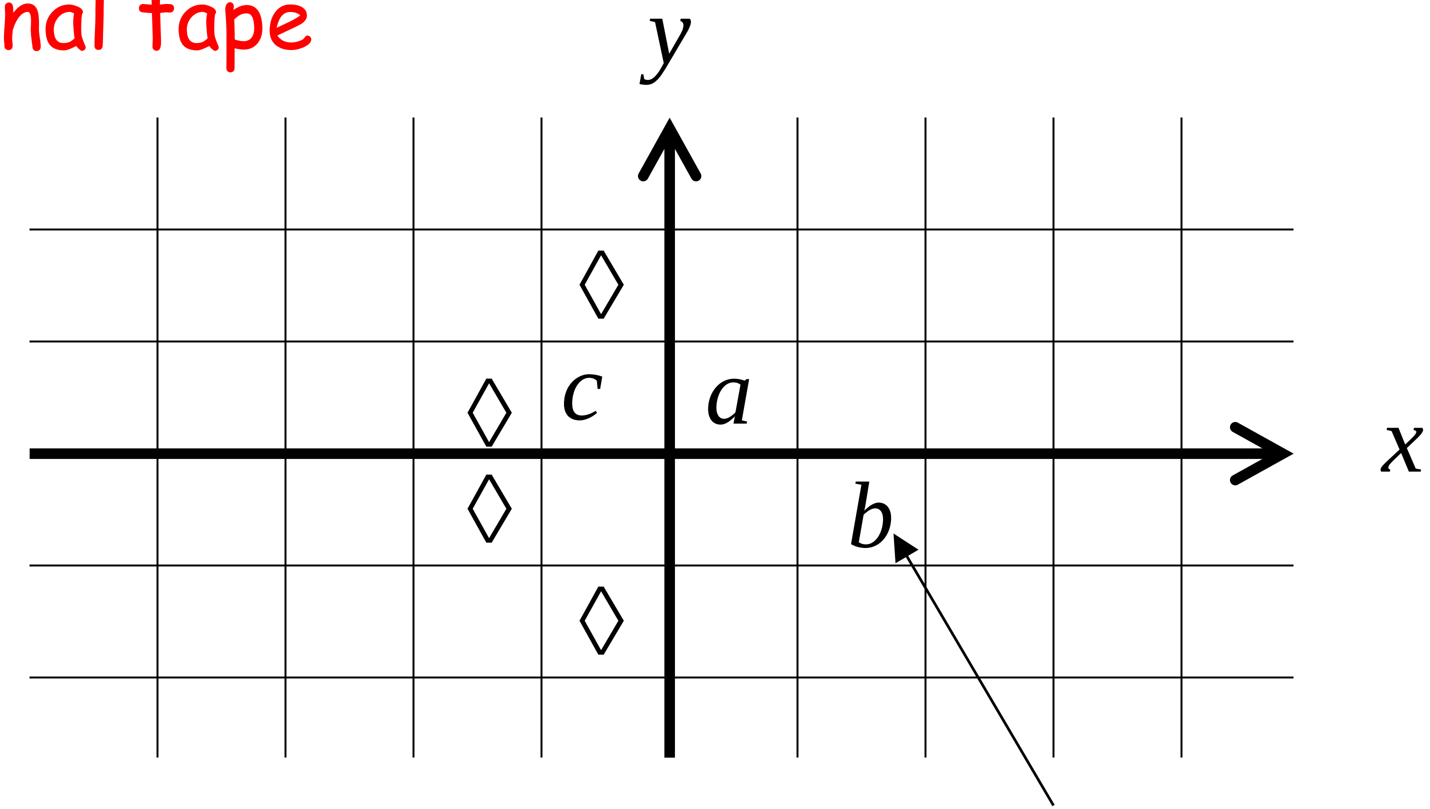
Go back and forth $O(n^2)$ times
to match the a's with the b's

2-tape machine: $O(n)$ time

1. Copy b^n to tape 2 ($O(n)$ steps)
2. Compare a^n on tape 1 and b^n tape 2 ($O(n)$ steps)

Multidimensional Turing Machines

2-dimensional tape



MOVES: L,R,U,D

U: up D: down

HEAD

Position: +2, -1

Theorem: Multidimensional machines have the same power with Standard Turing machines

Proof: 1. Multidimensional machines simulate Standard Turing machines

2. Standard Turing machines simulate Multi-Dimensional machines

1. Multidimensional machines simulate Standard Turing machines

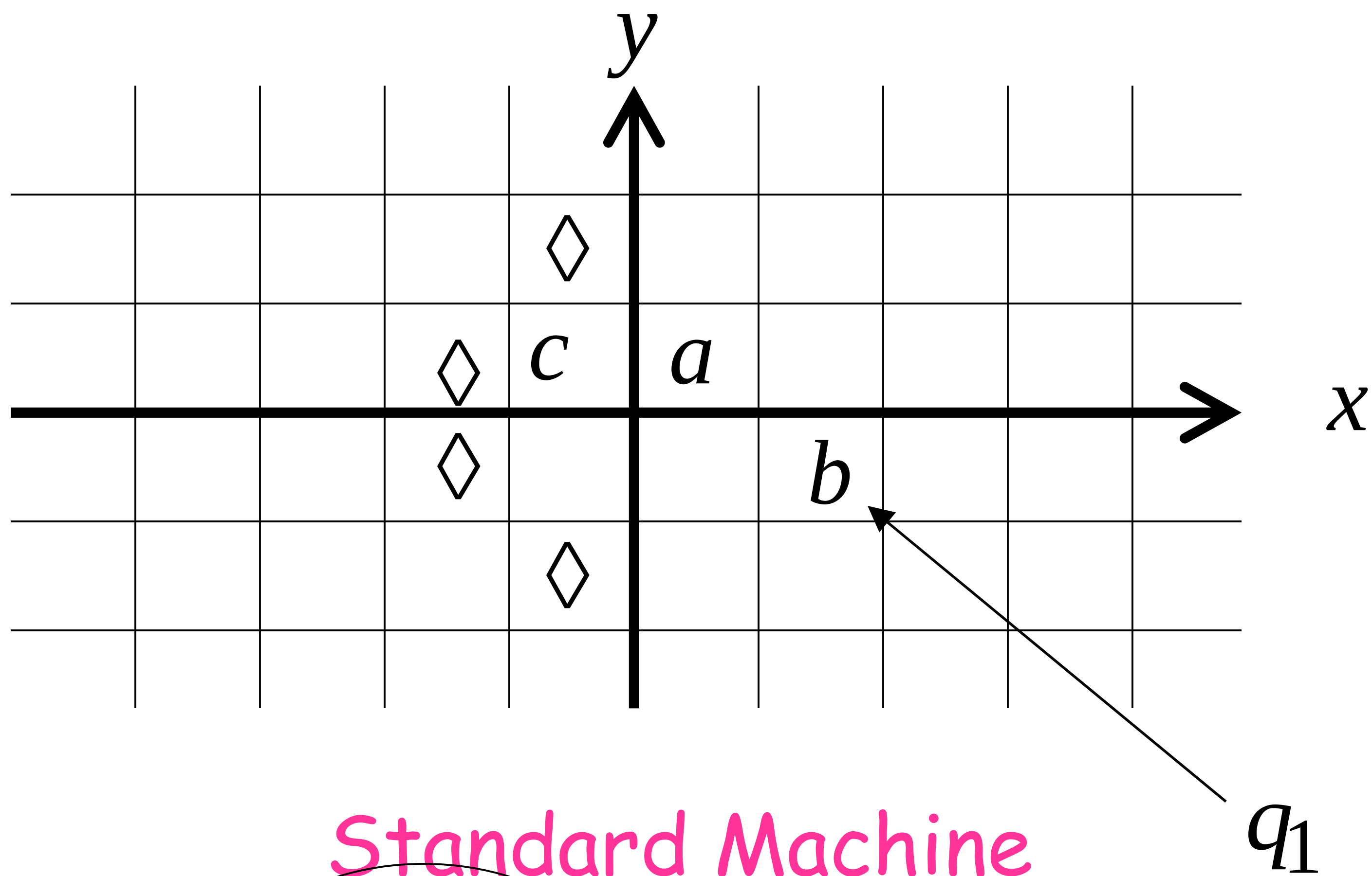
Trivial: Use one dimension

2. Standard Turing machines simulate Multidimensional machines

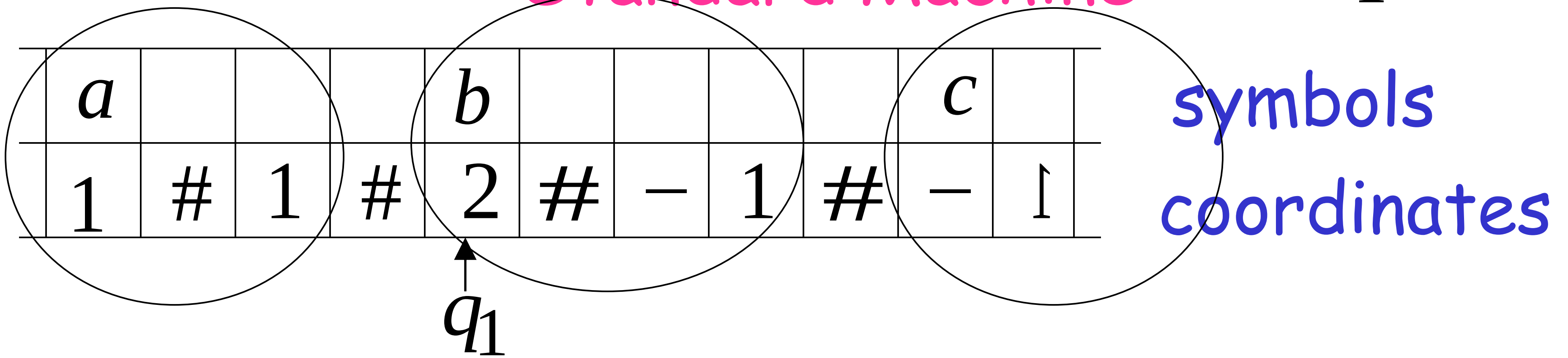
Standard machine:

- Use a two track tape
- Store symbols in track 1
- Store coordinates in track 2

2-dimensional machine



Standard Machine



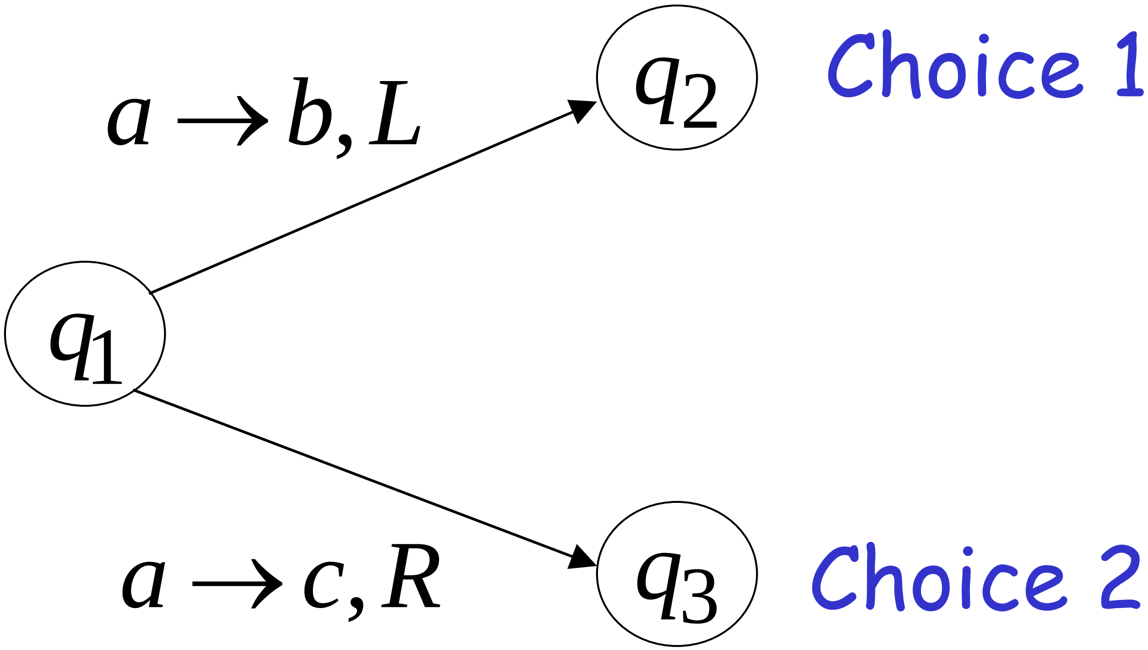
Standard machine:

Repeat for each transition followed in the 2-dimensional machine:

1. Update current symbol
2. Compute coordinates of next position
3. Go to new position

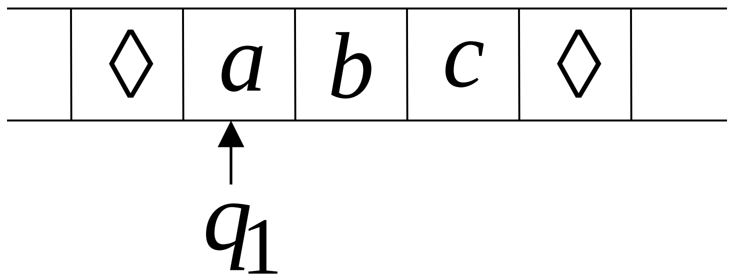
END OF PROOF

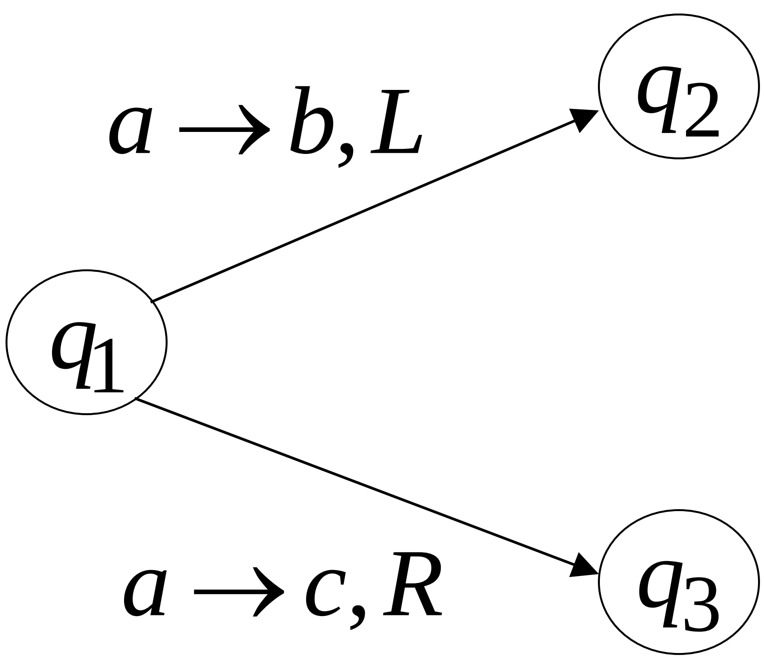
Nondeterministic Turing Machines



Allows Non Deterministic Choices

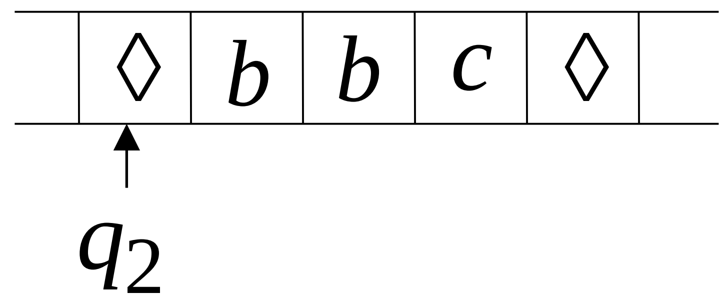
Time 0



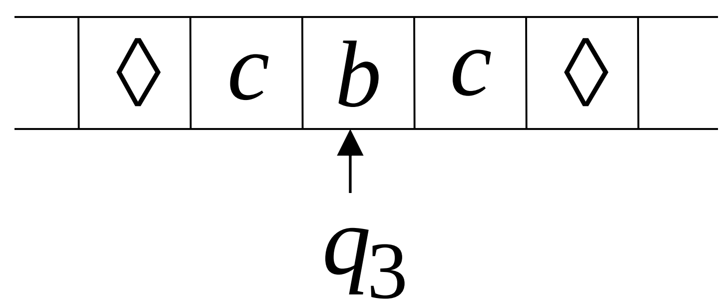


Time 1

Choice 1



Choice 2



Input string w is accepted if there is a computation:

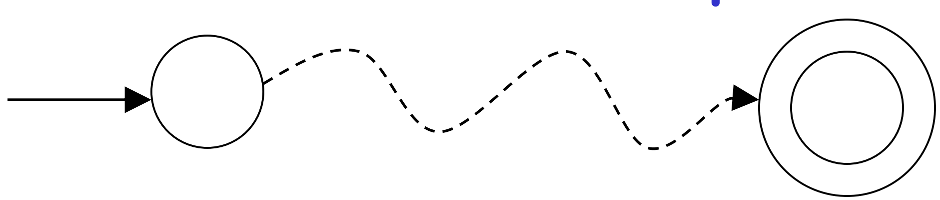
$$q_0 w \overset{*}{\succ} x\, q_f\, y$$

Initial configuration

Final Configuration

Any accept state

There is a computation:



Theorem: Nondeterministic machines have the same power with Standard Turing machines

Proof:

1. Nondeterministic machines simulate Standard Turing machines

2. Standard Turing machines simulate Nondeterministic machines

1. Nondeterministic Machines simulate Standard (deterministic) Turing Machines

Trivial: every deterministic machine is also nondeterministic

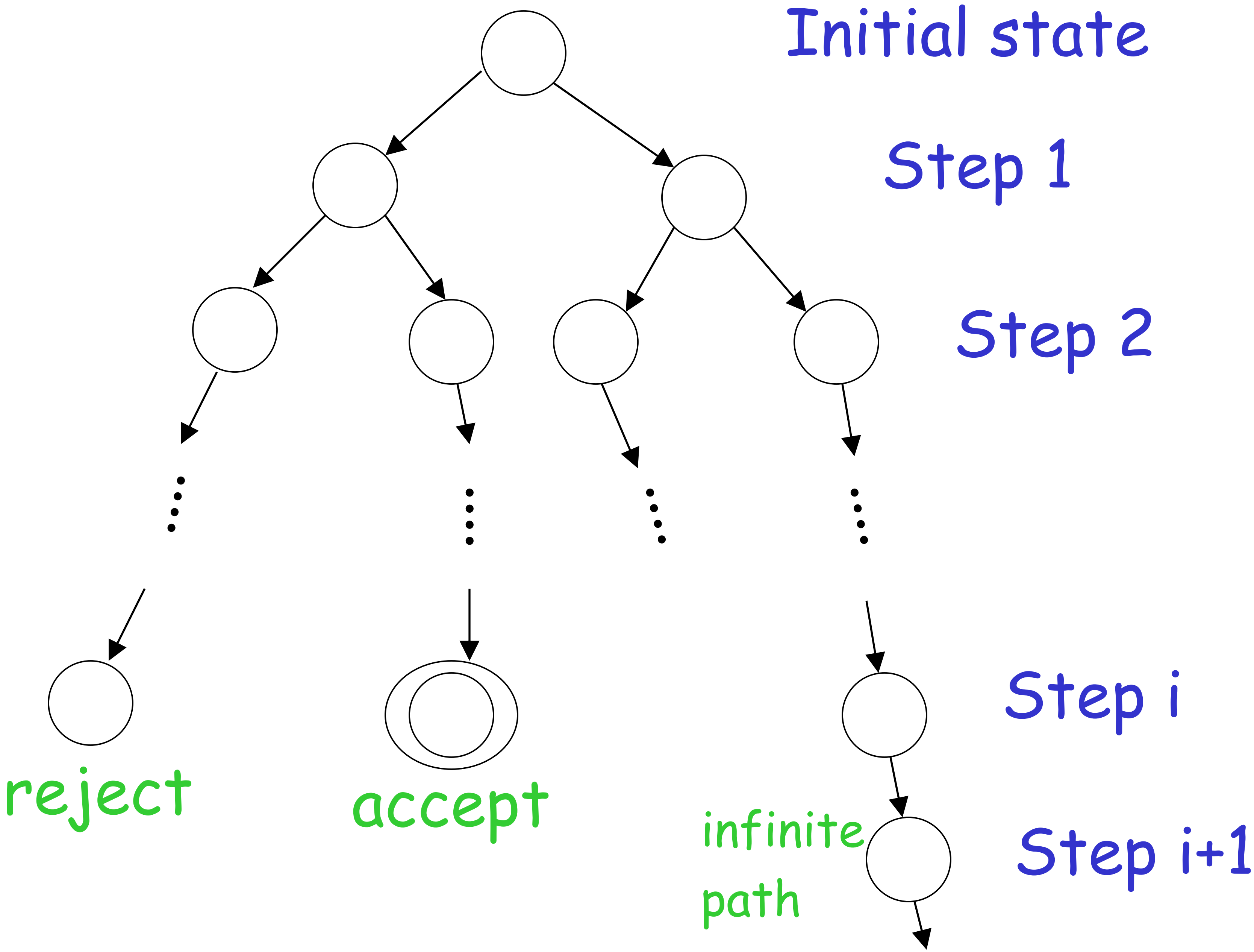
2. Standard (deterministic) Turing machines simulate Nondeterministic machines:

Deterministic machine:

- Uses a 2-dimensional tape
  (which is equivalent to 1-dimensional tape)

- Stores all possible computations
  of the non-deterministic machine
  on the 2-dimensional tape

All possible computation paths



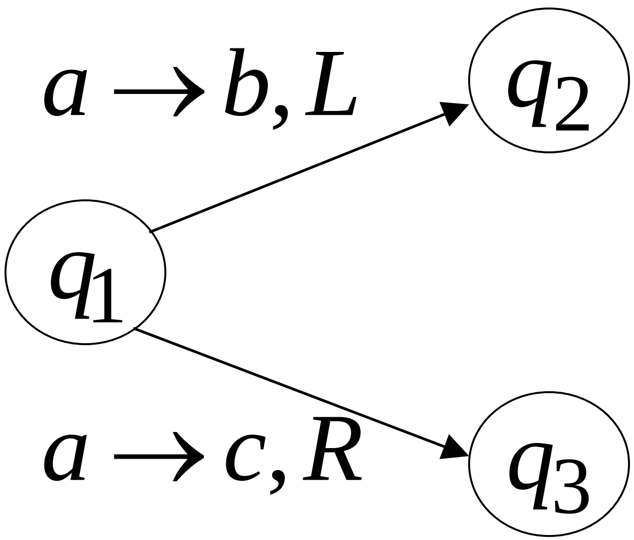
The Deterministic Turing machine simulates all possible computation paths:

- simultaneously

- step-by-step

- in a breadth-first search fashion

NonDeterministic machine



Time 0

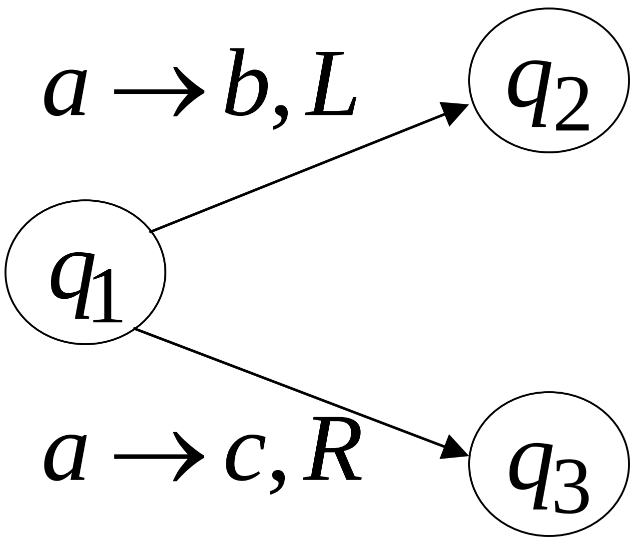
	◊	a	b	c	◊	
		↑ q_1				

Deterministic machine

	#	#	#	#	#	#	
	#	a	b	c	#		
	#	q_1			#		
	#	#	#	#	#		

current configuration

NonDeterministic machine



Time 1

Choice 1

	$\Diamond$	b	b	c	$\Diamond$	

$\uparrow$
q_2

Choice 2

	$\Diamond$	c	b	c	$\Diamond$	

$\uparrow$
q_3

Deterministic machine

	#	#	#	#	#	#	
#		b	b	c	#		
#	q_2				#		
#		c	b	c	#		
#			q_3		#		

Computation 1

Computation 2

Deterministic Turing machine

Repeat

For each configuration in current step
of non-deterministic machine,
if there are two or more choices:

1. Replicate configuration
2. Change the state in the replicas

Until either the input string is accepted
or rejected in all configurations

END OF PROOF

Remark:

The simulation takes in the worst case <u>exponential time</u> compared to the shortest accepting path length of the nondeterministic machine