4.0 Continuous random variables

The random variable X defined by

$$X = \{x \mid -\infty < x < \infty\}$$
$$= \{x \mid -\infty < X(s) < -\infty,\ X(s) = x,\ s \in S\}$$

is continuous random variable.

In particular, If $X = [a, b]$, $a, b \in \mathbb{R}$ is continuous r.v.

4.1 Probability density function (pdf)
and
4.2 Probability distribution function (PDF)

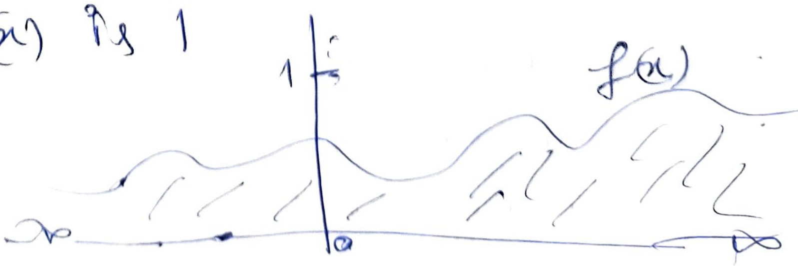
Let X be a continuous random variable.
The probability density function (pdf) of the r.v. X is defined by $f : X \to [0, 1]$,
i.e. $0 \le f(x) \le 1$ for all $x \in X$.

The density function satisfies the rule

① $0 \le f(x) \le 1$

② $\int_{-\infty}^{\infty} f(x)\, dx = 1$ (unit property),

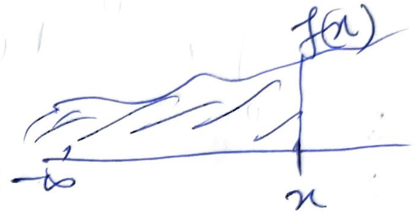
i.e. the area under the entire graph of $f(x)$ is 1



Probability Distribution function (PDF)

Let X be a continuous r.v. with pdf $f(x)$. The probability distribution function $F(x)$ is defined by

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(y)\,dy$$



i.e. $f(x) = F'(x)$ almost everywhere,

i.e. the set of all points $x \in X$ for which $f(x) \neq F'(x)$ is of measure zero.

Properties of PDF

(1) $0 \le F(x) \le 1$ for all $x \in X \subseteq \mathbb{R}$

(2) F is monotonically increasing, i.e.,

$F(a) \geq F(b)$ for $a > b$ in X

(c) $F(\infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1 = \lim_{x\to\infty} \int_{-\infty}^{x} f(y)\,dy$

(d) $F(-\infty) = \lim_{x\to-\infty} \int_{-\infty}^{x} f(y)\,dy = 0$

(e) for $a < b$ in X, we have

$$P[a < X \le b] = F(b) - F(a) = \int_a^b f(x)\,dx$$

Note (a) for $a < b$ in X

$$P[a < X < b] = P[a < X \le b] = P[a \le X < b] = P[a \le X \le b] = F(b) - F(a) = \int_a^b f(x)\,dx$$

(b) $P[X = c] = \lim_{y\to 0} \int_{c-y}^{c+y} f(x)\,dx = 0$

Uniform distribution

A continuous r.v. X with density function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

is uniform distribution.

The distribution function is

$$F(x) = \int_{-\infty}^{x} f(y)\, dy , \quad x \leq b$$

$$= \int_{a}^{x} \frac{1}{b-a} dy , \quad x \leq b$$

$$F(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x \geq b \end{cases} = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b \end{cases}$$

Expectation or mean of X

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{a}^{b} x \cdot \frac{1}{b-a} dx$$

$$= \frac{a+b}{2}.$$

Variance of X

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left(\frac{a+b}{2}\right)^2$$

$$= \int_{a}^{b} x^2 \cdot \frac{1}{b-a} dx - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

1.2 Cummulative Distribution Function and Expected values

Let X be any continuous r.v. with pdf $f(x)$, $-\infty < x < \infty$, then the cdf or PDF of X is defined by

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(y)\,dy, \quad x \in (-\infty, \infty)$$

* The median $\bar{x}$ of X is the 50th percentile, satisfies $0.5 = F(\bar{x})$

Expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx, \quad E(h(X)) = \int_{-\infty}^{\infty} h(x) f(x)\,dx$$

Thm Let X is any cont. r.v. with pdf $f_X(x)$, $x \in I$ and $Y = g(X)$ is any function of X which is monotonically increasing, then the pdf of Y is

$$f_Y(y) = f_X(g^{-1}(y))(g^{-1}(y))', \quad g^{-1}(y) \in I$$

for all $y \in Y$.

Proof Given $f_X(x)$, $x \in I$ is the pdf of X, with distribution function $F_X(x)$, then

$$F_X'(x) = f_X(x), \quad x \in I$$

Let $f_Y(y)$ be the pdf of Y with distribution function $F_Y(y)$, then $F_Y'(y) = f_Y(y)$

We have

$$F_Y(y) = P[Y \le y] = P[g(X) \le y]$$
$$= P[X \le g^{-1}(y)], \quad g^{-1}(y) \in I$$
$$= F_X(g^{-1}(y))$$

$$\Rightarrow f_Y(y) = F_Y'(y) = F_X'(g^{-1}(y))(g^{-1}(y))' = f_X(g^{-1}(y))(g^{-1}(y))', \quad g^{-1}(y) \in I$$

Percentile of a cont. distribution

Let p be a number between 0 & 1. The $100p$th percentile of the distribution of a cont. r.v. X denoted by $\eta = \eta(p)$ is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(x)\,dx$$

Note:- for any fixed $p \in [0,1]$, our aim to find the value of $\eta = \eta(p)$ form the above eqn.

Ex The distribution of the amount of gravel (in tons) sold by a particular construction supply company in a given week is a cont. r.v. with pdf

$$f(x) = \begin{cases} 1.5(1-x^2), & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the 50th percentile of the distribution.

Soln Given $f(x) = \begin{cases} 1.5(1-x^2), & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

is the pdf of X.

So the distribution function is

$$F(x) = \int_0^x 1.5(1-x^2)\,dx = 1.5\left(x - \frac{x^3}{3}\right)$$

For 50th percentile, we have $p = 0.5$ that satisfies

$$P = F(\eta) = 1.5\left(\eta - \frac{\eta^3}{3}\right)$$

$$\Rightarrow 0.5 = \frac{1.5}{3}(3\eta - \eta^3)$$

$$\Rightarrow \eta^3 - 3\eta + 1 = 0 \Rightarrow \eta = 0.3473, -1.8794, 1.5324$$

Hence 50th percentile is $\eta = 0.3473$