## Numerical Solution to ODE

We know that an ODE of the first order is of the form $F(x, y, y') = 0$ and can often be written in the explicit form $y' = f(x, y)$. An **initial value problem** for this equation is of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

where $x_0$ and $y_0$ are given and we assume that the problem has a unique solution on some open interval $a < x < b$ containing $x_0$. We shall discuss methods of computing approximate numeric values of the solution of the above initial value problem at the equidistant points on the $x$-axis $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \ldots$ where the **step size** $h$ is a fixed positive real number. These methods are suggested by the Taylor series expansion of an infinitely differentiable function.

**Taylor series Method.**

Suppose $f$ is an infinitely differentiable function (i.e., $f$ can be differentiated infinitely often). Then, the **Taylor series** expansion of $f$ about $x_0$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Let $f$ be infinitely differentiable function on the interval $(x_0 - r, x_0 + r)$ and $x \in (x_0 - r, x_0 + r)$. Define

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Then

$$R_n(x) = \frac{f^{(n+1)}(w)}{(n+1)!} (x - x_0)^{n+1}$$

for some $w$ between $x_0$ and $x$. Moreover, if

$$\lim_{n \to \infty} R_n(x) = 0$$

then the Taylor series for $f$ expanded about $x_0$ converges to $f(x)$, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

for all $x \in (x_0 - r, x_0 + r)$.

To solve the **initial value problem** of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

where $x_0$ and $y_0$ are given, we assume that the problem has a unique solution on some open interval $a < x < b$ containing $x_0$ and $y$ is an infinitely differentiable function on $(a, b)$. Let $x = x_0 + h$, then by Taylor series expansion of $y$, we have

$$y(x_0 + h) = y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!} h^k .$$

That is,

$$y(x_0 + h) = y(x_0) + h\, y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots.$$

To compute approximate numeric values of the solution of the above initial value problem at the equidistant points on the $x$-axis $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \ldots,$ we use the above form of Taylor series expansion. Thus we have the approximate values are

$$y(x_1) = y(x_0 + h) = y(x_0) + h\, y'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots,$$

$$y(x_2) = y(x_1 + h) = y(x_1) + h\, y'(x_1) + \frac{h^2}{2} y''(x_1) + \frac{h^3}{3!} y'''(x_1) + \cdots,$$

$$y(x_3) = y(x_2 + h) = y(x_2) + h\, y'(x_2) + \frac{h^2}{2} y''(x_2) + \frac{h^3}{3!} y'''(x_2) + \cdots,$$

and so on. In general

$$y(x_{k+1}) = y(x_k + h) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k) + \cdots.$$

**Example:** Solve the initial value problem

$$y' = xy, \quad y(0) = 1.$$

in the interval $[0,1]$ with $h = 0.2$ using Taylor series method upto 3rd order derivative.

Solution: We have to solve the initial value problem

$$y' = xy, \quad y(0) = 1.$$

in the interval $[0,1]$ with $h = 0.2$ . That is we have to find the values of $y(x)$ at

$$x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0 .$$

It is given that $x_0 = 0$, $y(x_0) = 1$, $h = 0.2$. Let denote $y(x_k) = y_k$. We have to use Taylor series method up to 3$^{rd}$ order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\, y^{'}(x_k) + \frac{h^2}{2} y^{''}(x_k) + \frac{h^3}{3!} y^{'''}(x_k).$$

That is,

$$y_{k+1} = y_k + h\, y^{'}_k + \frac{h^2}{2} y^{''}_k + \frac{h^3}{3!} y^{'''}_k.$$

Given that, $y^{'} = xy$. So, $y^{''} = (1 + x^2)y$ and $y^{'''} = (3x + x^3)y$. Now using $h = 0.2$ and the given initial condition $y(0) = 1$, that is $x_0 = 0$, $y_0 = 1$, we find,

$$y^{'}_0 = x_0 y_0 = 0, \quad y^{''}_0 = (1 + x_0^2)y_0 = 1, \quad y^{'''}_0 = (x_0 + x_0^3)y_0 = 0.$$

Thus,

$$y_1 = y_0 + h\, y^{'}_0 + \frac{h^2}{2} y^{''}_0 + \frac{h^3}{3!} y^{'''}_0 = 1.02.$$

Similarly we calculate $y_2, y_3, y_4, y_5$ and are given in the below table.

| $k$ | $x_k$ | $y_k$ | $y^{'}_k$ | $y^{''}_k$ | $y^{'''}_k$ | $y_{k+1}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 1 | 0 | 1.0200 |
| 1 | 0.2 | 1.0200 | 0.2040 | 1.0608 | 0.6202 | 1.0828 |
| 2 | 0.4 | 1.0828 | 0.4331 | 1.2561 | 1.3687 | 1.1964 |
| 3 | 0.6 | 1.1964 | 0.7179 | 1.6271 | 2.4120 | 1.3757 |
| 4 | 0.8 | 1.3757 | 1.1006 | 2.2562 | 4.0062 | 1.6463 |
| 5 | 1.0 | 1.6463 | --- | --- | --- | --- |

From the above table we get the solution of the initial value problem as

$$y(0.2) = 1.0200, y(0.4) = 1.0828, y(0.6) = 1.1964, y(0.8) = 1.3757, y(1.0) = 1.6463.$$

**Note:** The exact solution is $y(x) = e^{\frac{x^2}{2}}$, and so the exact value of $y(1)$ is 1.6487.

**Example:** Solve the initial value problem

$$y^{'} = x + y, \; y(0) = 0.$$

in the interval $[0,1]$ with $h = 0.25$ using Taylor series method upto 3$^{rd}$ order derivative.

Solution:  We have to solve the initial value problem

$$y' = x + y, \ y(0) = 0.$$

in the interval $[0,1]$ with $h = 0.25$ . That is we have to find the values of $y(x)$ at

$$x_1 = 0.25, x_2 = 0.50, x_3 = 0.75, x_4 = 1.0 .$$

It is given that $x_0 = 0$, $y(x_0) = 0$, $h = 0.25$ . Let denote $y(x_k) = y_k$ . We have to use

Taylor series method up to $3^{rd}$ order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k).$$

That is,

$$y_{k+1} = y_k + h\, y'_k + \frac{h^2}{2} y''_k + \frac{h^3}{3!} y'''_k.$$

Given that,  $y' = x + y$ . So,  $y'' = 1 + x + y$  and  $y''' = 1 + x + y$ .  Now using

$h = 0.25$ and the given initial condition  $y(0) = 0$, that is $x_0 = 0, y_0 = 0$, we find,

$$y'_0 = x_0 + y_0 = 0, \ y''_0 = 1 + x_0 + y_0 = 1, \ y'''_0 = 1 + x_0 + y_0 = 1.$$

Thus,

$$y_1 = y_0 + h\, y'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{3!} y'''_0 = 0.0339.$$

Similarly, we calculate $y_2, y_3, y_4$ and are given in the below table.

| $k$ | $x_k$ | $y_k$ | $y'_k$ | $y''_k$ | $y'''_k$ | $y_{k+1}$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 | 0.0339 |
| 1 | 0.25 | 0.0339 | 0.2839 | 1.2839 | 1.2839 | 0.1483 |
| 2 | 0.50 | 0.1483 | 0.6483 | 1.6483 | 1.6483 | 0.3662 |
| 3 | 0.75 | 0.3662 | 1.1162 | 2.1162 | 2.1162 | 0.7168 |
| 4 | 1.0 | 0.7168 | --- | --- | --- | --- |

From the above table we get the solution of the initial value problem as

$$y(0.25) = 0.0339, y(0.50) = 0.1483, y(0.75) = 0.3662, y(1.0) = 0.7168.$$

**Note:** The exact solution is $y(x) = e^x - x - 1$, and so the exact value of $y(1)$ is 0.7183.

**Example:** Solve the initial value problem

$$y' = (x + y)^2, \ y(0) = 0; \quad h = 0.1$$

using Taylor series method up to 3$^{rd}$ order derivative. Do 10 steps.

Solution:  We have to solve the initial value problem

$$y' = (x + y)^2, \ y(0) = 0.$$

with $h = 0.25$ . That is we have to find the values of $y(x)$ at

$$x_1 = 0.1, x_2 = 0.1, x_3 = 0.3, ..., x_9 = 0.9, x_{10} = 1.0.$$

It is given that $x_0 = 0, \ y(x_0) = 0, \ h = 0.1$ . Let denote $y(x_k) = y_k$. We have to use Taylor series method up to 3$^{rd}$ order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k).$$

That is,

$$y_{k+1} = y_k + h\, y'_k + \frac{h^2}{2} y''_k + \frac{h^3}{3!} y'''_k.$$

Given that, $y' = (x + y)^2.$

So,

$$y'' = 2(x + y)(1 + y')$$

And

$$y''' = 2(1 + 4y' + 3y'^2)$$

Using $h = 0.1$ and the given initial condition $y(0) = 0$, that is $x_0 = 0, y_0 = 0$, we do the following 10 steps.

Step 1.

$$y'_0 = (x_0 + y_0)^2 = 0, \ y''_0 = 2(x_0 + y_0)(1 + y'_0) = 0, \ y'''_0 = 2(1 + 4y'_0 + 3y_0'^2) = 2.$$

Thus,

$$y_1 = y_0 + h\, y'_0 + \frac{h^2}{2} y''_0 + \frac{h^3}{3!} y'''_0 = .$$

Similarly, we do the rest steps and are given in the below table.

| $k$ | $x_k$ | $y_k$ | $y'_k$ | $y''_k$ | $y'''_k$ | $y_{k+1}$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 0.0000 | 0.0000 | 0.0000 | 2.0000 | 0.0003 |
| 2 | 0.1 | 0.0003 | 0.0101 | 0.2007 | 2.0811 | 0.0027 |
| 3 | 0.2 | 0.0027 | 0.0411 | 0.4061 | 2.3388 | 0.0092 |
| 4 | 0.3 | 0.0092 | 0.0956 | 0.6241 | 2.8198 | 0.0224 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 0.4 | 0.0224 | 0.1784 | 0.8716 | 3.6181 | 0.0452 |
| 6 | 0.5 | 0.0452 | 0.2972 | 1.1867 | 4.9077 | 0.0816 |
| 7 | 0.6 | 0.0816 | 0.4646 | 1.6576 | 7.0124 | 0.1376 |
| 8 | 0.7 | 0.1376 | 0.7015 | 2.4995 | 10.5649 | 0.2220 |
| 9 | 0.8 | 0.2220 | 1.0444 | 4.2736 | 16.9005 | 0.3506 |
| 10 | 0.9 | 0.3506 | 1.5640 | 8.6194 | 29.1887 | 0.5550 |
| 11 | 1 | 0.5550 | --- | --- | --- | --- |

**Note:** The exact solution is $y(x) = \tan x - x$, and so the exact value of $y(1)$ is 0.05574.

## Euler's method

From use Taylor series method, we have,

$$y(x_{k+1}) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k) + \cdots.$$

For small $h$ the higher powers $h^2, h^3, \ldots$, are very small and if we drop all of them to get the crude approximation

$$y(x_{k+1}) = y(x_k) + h\, y'(x_k)$$

i.e.,

$$y_{k+1} = y_k + h\, f(x_k, y_k).$$

Geometrically, this is an approximation of the curve of by a polygon whose first side is tangent to this curve at $x_k$.

**Example:** Apply the Euler method to the initial value problem

$$y' = (y - x)^2, \; y(0) = 0;$$

choosing $h = 0.1$ and computing ten steps.

Solution: Given that

$$y' = (y - x)^2, \; y(0) = 0; \; h = 0.1.$$

We have to do ten steps. That is, we have to calculate $y_1, y_2, \ldots, y_{10}$. Given that $x_0 = 0$, $y_0 = 0$, $h = 0.1$. So $x_1 = 0.1$, $x_2 = 0.2$, ..., $x_{10} = 1.0$ . Here $f(x, y) = (y - x)^2$. By Euler method

$$y_{k+1} = y_k + h\, f(x_k, y_k).$$

Thus

$$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1 \cdot (0 - 0)^2 = 0$$

Similarly, we calculate $y_2, ..., y_{10}$ and are given in the below table.

| $k$ | $x_k$ | $y_k$ |
|---|---|---|
| 1 | 0.1 | 0.0000 |
| 2 | 0.2 | 0.0010 |
| 3 | 0.3 | 0.0050 |
| 4 | 0.4 | 0.0137 |
| 5 | 0.5 | 0.0286 |
| 6 | 0.6 | 0.0508 |
| 7 | 0.7 | 0.0810 |
| 8 | 0.8 | 0.1193 |
| 9 | 0.9 | 0.1656 |
| 10 | 1.0 | 0.2196 |

## Modified Euler's method.

Euler's method is generally much too inaccurate and methods of higher order and precision are obtained by taking more terms in Taylor series into account. But this involves an important practical problem of calculation of successive derivatives. The **general strategy** now is to avoid the computation of these derivatives and to replace it by computing functions for one or several suitably chosen auxiliary values of $(x, y)$. The **Modified Euler's method** for solving initial value problem of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

where $x_0$ and $y_0$ are given is a predictor–corrector method as described below.

Suppose $y_k$ has been calculated and to calculate $y_{k+1}$ we do two steps

Step 1. Find the **_predictor_**

$$y_{k+1}^{(0)} = y_k + h\, f(x_k, y_k).$$

Step 2. Find the **corrector** by iterations

$$y_{k+1}^{(i+1)} = y_k + \frac{h}{2}\left[ f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i)}) \right] \text{ for } i = 0,1,2, ....$$

In this step we repeat the iteration till two consecutive $y_{k+1}^{(i)}$ are equal up to certain decimal places and treat it as $y_{k+1}$.

**Example:** Solve the initial value problem

$$y' = xy^2, \ y(0) = 1; h = 0.25$$

in the interval $[0,1]$ using Modified Euler's method correct to four decimal places.

Solution: According to the question $f(x, y) = xy^2$, $x_0 = 0$, $y_0 = 1$, $h = 0.25$ .We have to solve in the interval $[0,1]$, that is we have to find the values of $y(x)$ at $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, and $x_4 = 1.0$. By Modified Euler's method we have

$$y_{k+1}^{(0)} = y_k + h f(x_k, y_k),$$

and

$$y_{k+1}^{(i+1)} = y_k + \frac{h}{2}\left[ f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i)}) \right] \text{ for } i = 0,1,2, \dots$$

First to calculate $y_1 = y(x_1)$, we have

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1$$

Now by iteration we calculate $y_1^{(i+1)}$; for $i = 0,1,2, \dots$, till it repeats. That is

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1.0313.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1.0332.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1.0334.$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] = 1.0334.$$

Thus, $y_1 = 1.0334$.


In similar way we calculate $y_2$ as bellow:

$y_2^{(0)} = 1.1001$, $y_2^{(1)} = 1.1424$, $y_2^{(2)} = 1.1483$, $y_2^{(3)} = 1.1492$, $y_2^{(4)} = 1.1493$, $y_2^{(5)} = 1.1493$.

Thus, $y_2 = 1.1493$. Again

$y_3^{(0)} = 1.3144$, $y_3^{(1)} = 1.3938$, $y_3^{(2)} = 1.4140$, $y_3^{(3)} = 1.4193$, $y_3^{(4)} = 1.4207$, $y_3^{(5)} = 1.4212$, $y_3^{(6)} = 1.4212$.

Thus $y_3 = 1.4212$. Finally,

$y_4^{(0)} = 1.7999$, $y_4^{(1)} = 2.0155$, $\dots$, $y_4^{(15)} = 2.2349$, $y_4^{(16)} = 2.2349$.

Thus, $y_4 = 2.2349$.

Therefore, $y_1 = 1.0334$, $y_2 = 1.1493$, $y_3 = 1.4212$, $y_4 = 2.2349$.

**Note:** The exact solution of the above initial value problem is $y(x) = \frac{2}{2-x^2}$ and $y_1 = 1.0323$, $y_2 = 1.1429$, $y_3 = 1.3913$ and $y_4 = 2$.

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