## 1.4 Exact ODES. Integrating Factors

**Exact ODE:**

**Definition:** A first order ordinary differential equation of the form

$M(x, y)\,dx + N(x, y)\,dy = 0$ ……………………..(1)

is called exact if there exists a function $u(x, y)$ such that

$du(x,y) = M(x, y)\,dx + N(x, y)\,dy$ ……………..(2)

Where $du$ is called the differential of $u(x,y)$

Since,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \text{……………………..(3)}$$

where u = $u(x, y)$

Then from equations (1) and (2)

$du = 0$

Integrating, $u = c$ (constant)

i.e. $u(x, y) = c$, which is the general solution of equation (1).

**Example -1:-**

$xdy + ydx = 0$

The LHS of the given equation is d (xy)

So, $d(xy) = 0$

By integration, $xy = c$ (constant)

which is the general solution of the given differential equation

Also, the given equation is an exact ODE.

**Necessary and Sufficient condition for exactness:**

Let us suppose equation (1) is exact, then

$du(x, y) = M(x, y)\,dx + N(x, y)\,dy$ (by definition)

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = M(x,y)dx + N(x,y)dy$$

Comparing both sides of the equation,

$$\Rightarrow \frac{\partial u}{\partial x} = M(x,y)\, and\, \frac{\partial u}{\partial y} = N(x,y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Differentiating *M(x,y)* w.r.to *y* and *N(x, y)* w. r. to *x* we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}\, and\, \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

Since $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

which is not only the necessary but also the sufficient condition for equation (1) to be exact.

**Hence, the necessary and sufficient conditions for the differential equation (1) to be exact is**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

## Method of finding *u(x,y)*:–

If the ODE (1) is exact then from equation (4),

$$\frac{\partial u}{\partial x} = M,\; M = M(x,y)$$

Integrating both sides w.r. to *x*, we have

$$u = \int M dx + K(y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Differentiating (6) w.r. to *y* and then comparing with $\frac{\partial u}{\partial y} = N$, we will get $\frac{dK}{dy}$ and then *K(y)* after integration.

Similarly, we can also use

$$\frac{\partial u}{\partial y} = N, N = N(x,y)$$

Integrating both sides w.r.to y,

$$u = \int N dy + l(x) \qquad \text{.........................(7)}$$

Where *l(x)* is the integration constant as *x* is constant.

Differentiating equation (7) w.r.to *x* and the comparing with $\frac{\partial u}{\partial x} = M$,

we will get $\frac{dl}{dx}$ and then *l(x)* after integration.

## Reduction to Exact Form : Integrating Factors

### Integrating Factor (I.F.)

When the left-hand side of the equation

*M (x, y) dx + N(x, y) dy = 0*

is not an exact or a total differential, then it is necessary to choose a function *μ (x, y)* such that after multiplying by it, the left-hand side of the above equation becomes an exact differential *du* and is given by

*du = μ (x,y) M (x,y) dx + μ(x,y) N(x,y) dy*

Such a function *μ (x,y)* is called an I.F.

**Definition:**

An integrating factor is a function when multiplied by it, the left hand side of equation (1) becomes an exact differential.

**Example – 2 :**

$-ydx + xdy = 0$ ……… (1)

Here, $M = -y$, and $N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -1 \, and \, \frac{\partial N}{\partial x} = 1$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So, the given equation is not exact.

In order to make the equation exact, let us multiply the given equation (1) by $1/x^2$

$$\Rightarrow \frac{-ydx + xdy}{x^2} = 0.\frac{1}{x^2}$$

$$\Rightarrow \frac{-y}{x^2}dx + \frac{1}{x}dy = 0$$

$$\Rightarrow d\left(\frac{y}{x}\right) = 0 \Rightarrow \frac{y}{x} = C \text{ (constant)}$$

which is the general solution of the given problem.

Here, $\mu(x,y) = 1/x^2$ is the I.F and it is obtained by inspection.

**How to Find an Integrating Factor: -**

**Rule-1:**

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x)$$

If, a function of $x$ alone,

Then $I.F. = e^{\int f(x)dx}$

**Rule – 2 :**

If, $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{M} = F(y)$

a function of $y$ alone then $I.F. = e^{\int -F(y)dy}$

## Problem Set 1.4

Test for exactness. If exact, solve If not, use an integrating factor (I.F) to solve.

**Q-1** $2xy\,dx + x^2\,dy = 0$

**Solution:-**

$2xy\,dx + x^2\,dy = 0$…………….. (1)

$$\Rightarrow \mathrm{d}\left(x^2 y\right) = 0$$

Integrating, we have

$x^2y = C$ is the general solution.

**Or**

The given equation is of the form *M (x, y) dx + N (x, y) dy = 0*

So, $M = 2xy,\ N = x^2$

$$\Rightarrow \frac{\partial M}{\partial y} = 2x \text{ and } \frac{\partial N}{\partial x} = 2x$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given equation is exact.

By using formula (4), $\frac{\partial u}{\partial y} = M$

$$\Rightarrow u = \int M dx + k(y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\Rightarrow u = \int 2xy dx + k(y)$$

$$\Rightarrow u = x^2 y + k(y)$$

Differentiating both sides w.r.to *y*,

$$\frac{\partial u}{\partial y} = x^2 + \frac{dk}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

But

$$\frac{\partial u}{\partial y} = N = x^2$$

Substituting in equation (*)

$$x^2 = x^2 + \frac{dk}{dy}$$

$$\Rightarrow \frac{dk}{dy} = 0$$

$$\Rightarrow k = c\ (constant)$$

So, from equation (2)

$u = x^2 y = c$ is the general solution

i.e. $y = c/x^2$

**Q – 2:** $x^3dx + y^3dy = 0$

**Solution: -**

Comparing with the equation *M (x, y) dx + N (x, y) dy = 0*

We have

$M = x^3$ *and* $N = y^3$

$$\Rightarrow \frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = 0$$

Since, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So, the given equation is exact

Since,

$$u = \int M dx + K(y)$$

i.e $u = \int x^3 dx + K(y)$

$$u = \frac{x^4}{4} + K(y)$$

Differentiating both sides w.r.to *y*,

$$\frac{\partial u}{\partial y} = \frac{dk}{dy} \quad \left[\because \frac{\partial K}{\partial y} = \frac{dK}{dy}\right]$$

But $\frac{\partial u}{\partial y} = N = y^3$

So $\frac{dK}{dy} = y^3$

Integrating w. r.to y,

$$\int \frac{dK}{dy} dy = \int y^3 dy$$

$$\Rightarrow K = \frac{y^4}{4} + C$$

$$u = \frac{x^4}{4} + \frac{y^4}{4} = C$$

is the general solution.

**Q-5** $(x^2+y^2)dx - 2xydy = 0$ ….(1)

**Solution:**

Here, $M = x^2 + y^2$, $N = -2xy$

$$\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = -2y$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So, the equation is not exact.

Now $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y+2y}{-2xy} = \frac{4y}{-2xy} = -\frac{2}{x} = f(x)$$

So, $$I.F = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\ln x}$$

$$= e^{\ln x^{-2}}$$

$$= 1/x^2$$

Now multiplying $1/x^2$ in equation (1),

$$\frac{1}{x^2}\left(x^2 + y^2\right)dx - \frac{2xy}{x^2}dy = 0$$

$$\Rightarrow \left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$$ …(2)

Which is of the form $M\,dx + Ndy = 0$

Now equation (2) is exact.

So ,

$$u = \int\left(1 + \frac{y^2}{x^2}\right)dx + K(y)$$

$$= x + \left(\frac{-y^2}{x}\right) + K(y)$$

$$= x - \frac{y^2}{x} + K(y)$$

Now differentiating both sides w. r. to *y*

$$\frac{\partial u}{\partial y} = \frac{-2y}{x} + \frac{\partial K}{\partial y} = N$$

$$\Rightarrow \frac{\partial K}{\partial y} = 0 \Rightarrow K = C$$

Hence, the general solution is

$$u(x, y) = x - \frac{y^2}{x} = C$$

Where $C$ is a constant.

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