1.5 Linear Ordinary Differential Equation. Bernoulli Equation

First order Linear Ordinary Differential Equation:

A differential equation is said to be ***linear*** if

i) The dependent variable and all its derivatives are of degree one.

ii) No product terms of dependent variable and/or any of its derivatives are present.

iii) No transcendental functions of dependent variable and/or its derivatives occur.

Otherwise differential equation is non-linear.

The standard form of ***a first order linear ordinary differential equation*** is

$$y' + p(x)y = r(x), \tag{1}$$

where $p(x)$ and $r(x)$ are any given functions of x only or they may be constant.

The function $r(x)$ on the right may be a force, and the solution $y(x)$ a displacement in a motion or an electrical current or some other physical quantity. In engineering, $r(x)$ is frequently called the input, and $y(x)$ is called the output or the response to the input.

Note: Every linear differential equations are of degree one but every first degree equations are not linear.

Example:

The ODEs $y' + y = e^x$, $y' = y$ etc. are the example of linear ODE.

The differential equations $(y')^2 + y = e^x$, $y' + y = \sin y$ etc. are non-linear ODEs.

Homogeneous Linear ODE: If each term of a linear ODE involves the dependent variable or its derivative, then the ODE is known as ***Homogeneous Linear ODE***. Otherwise ODE is non-homogeneous. The standard form of ***a Homogeneous Linear ODE*** is given by

$$y' + p(x)y = 0 \tag{2}$$

Solution of Homogeneous Linear ODE:

By separating variables and integrating we then obtain

$$\frac{dy}{dx} = -p(x)y$$

$$\Rightarrow \frac{dy}{y} = -p(x)dx$$

$$\Rightarrow \int \frac{dy}{y} = -\int p(x)dx$$

$$\Rightarrow \ln|y| = -\int p(x)dx + c^*$$

Taking exponents on both sides, we obtain the general solution of the homogeneous ODE

$$y(x) = ce^{-\int p(x)dx} \qquad \left(c = \pm e^{c^*}\right) \tag{3}$$

If we choose $c = 0$ then we obtain trivial solution $y(x) = 0$ for all$x \in (a, b)$.

Solution of Nonhomogeneous linear ODE:

We now solve linear differential equation (1) $y' + p(x)y = r(x)$, in the case that $r(x)$ is not everywhere zero in the given interval. Then the ODE (1) is called nonhomogeneous linear ODE.

We have to solve ODE (1) by the method that provided in previous section. That means, first we have to check the given ODE (1) is exact or not. ODE (1) can be expressed as

$$\big(p(x)y - r(x)\big)dx + dy = 0 .$$

Here, $M = \big(p(x)y - r(x)\big)$ and $N = 1$. $\dfrac{\partial M}{\partial y} = p(x) \neq \dfrac{\partial N}{\partial x} = 0$, so it is not exact.

Again, $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = p(x)$ and integrating factor is $e^{\int p(x)dx}$.

Multiplying both side of (1) by $e^{\int p(x)dx}$ we obtain,

$$e^{\int p(x)dx}y'(x) + e^{\int p(x)dx}p(x)y(x) = r(x)e^{\int p(x)dx} \tag{4}$$

Since $\frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = e^{\int p(x)dx}y'(x) + y(x)p(x)e^{\int p(x)dx}$

Equation (4) becomes

$$\frac{d}{dx}\left(y(x)e^{\int p(x)dx}\right) = r(x)e^{\int p(x)dx}$$

This implies

$$y(x)e^{\int p dx} = \int r(x)e^{\int p(x)dx} + c$$

That is $$y(x)(I.F) = \int r(x)(I.F)dx + c$$

This implies

$$y(x) = e^{-\int p(x)dx}\left(\int r(x)e^{\int p(x)dx} + c\right)$$ is the general solution of (1).

Example 1: Solve the initial value problem.

$$y' + y\tan x = \sin 2x, \qquad y(0) = 1$$

Solution: Here $p(x) = \tan x$, $r(x) = \sin 2x = 2\sin x\cos x$.

Integrating factor is $e^{\int p(x)dx} = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$

$$y(x)(I.F) = \int r(x)(I.F)\, dx + c$$

$$y(x)\sec x = \int 2\sin x\cos x\sec x\, dx + c$$

The general solution of our equation is

$y(x) = \cos x\,(-2\cos x + c) = -2\cos^2 x + c.\cos x.$ (5)

Putting initial condition in (5),$1 = -2 + c$, thus $c = 3$.

Solution of our initial value problem is$y(x) = 3\cos x - 2\cos^2 x$.

Note:

Sometimes a first order ODE can be brought to the linear form provided we regard x as the dependent variable and y as the independent variable. In this case the equation will be of the form

$\dfrac{dx}{dy} + p(y)x = r(y)$ where p and r are functions of y only or constant.

Example 2: Solve $(x + 2y^3)dy - ydx = 0$

Solution: The given equation may be written as

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2$$

Here $$p(y) = -\frac{1}{y}, r(y) = 2y^2 .$$

So the integrating factor is

$$e^{\int p dy} = e^{\int (-\frac{1}{y}) dy} = e^{-\log y} = \frac{1}{y}$$

Thus we have the general solution as

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c$$

$$x\left(\frac{1}{y}\right) = y^2 + c$$

$$x = y\left(y^2 + c\right)$$

Equations Reducible to Linear Form:

I. An equation of the form

$$f'(y)\frac{dy}{dx} + p(x)f(y) = r(x)$$

where $p(x)$ and $r(x)$ are any given functions of x only or constants can also be reduced to the liner form.

If we put $f(y) = v$ then $f'(y)\frac{dy}{dx} = \frac{dv}{dx}$.

So the given equation reduces to

$$\frac{dv}{dx} + pv = r$$

which is a linear differential equation in v.

Example: Solve $2xyy' + (x-1)y^2 = x^2 e^x$

Solution: Let $y^2 = v$

Differentiating w.r.to x we obtain

$$2yy' = \frac{dv}{dx}$$

The given equation becomes

$$x\frac{dv}{dx} + (x-1)v = x^2 e^x$$

Again it can be written as

$$\frac{dv}{dx} + \left(1 - \frac{1}{x}\right)v = xe^x$$

This is a linear nonhomogeneous ODE.

I.F. is $$e^{\int\left(1-\frac{1}{x}\right)dx} = e^{x - \ln x} = \frac{e^x}{x}$$

The general solution is

$$v\left(\frac{e^x}{x}\right) = \int xe^x\left(\frac{e^x}{x}\right)dx + C = \int e^{2x}dx + C = \frac{e^{2x}}{2} + C$$

$$\Rightarrow v(x) = \frac{xe^x}{2} + Cxe^{-x}$$

Hence the general solution of the given ODE is $y^2 = \frac{xe^x}{2} + Cxe^{-x}$.

II. **Bernoulli Equation (Reduction to linear form):**

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the ***Bernoulli equation***

$$y' + p(x)y = g(x)y^a, \qquad a \in R \tag{6}$$

If $a = 0$ or $a = 1$, Equation (6) is linear. Otherwise it is nonlinear.

Note:

1. Putting $a = 0$ in equation (6) we have $y' + p(x)y = g(x)$ which is non-homogeneous linear ODE similar to (1).

2. Putting $a = 1$ in equation (6), we have $y' + \left(p(x) - g(x)\right)y = 0$ which is homogeneous linear ODE similar to (2).

For $a \neq 0$ and $a \neq 1$, the equation (6) is nonlinear. Then we set

$$u(x) = [y(x)]^{1-a}.$$

We differentiate this and substitute $y' = g(x)y^a - p(x)y$, obtaining

$$u' = (1-a)y^{-a}y' = (1-a)y^{-a}(gy^a - py).$$

Simplification gives

$$u' + (1-a)pu = (1-a)g. \tag{7}$$

Equation (7) is a linear ODE w.r.t. u.

Example: Solve the nonlinear ODE $y' + xy = xy^{-1}$.

Solution: Here $p(x) = x$, $g(x) = x$ and $a = -1$.

Using (7), we get the linear ODE as $u' + 2xu = 2x$, where $u = y^2$.

Now I.F$= e^{\int 2x dx} = e^{x^2}$.

Then,

$$u(x)(e^{x^2}) = \int 2x.e^{x^2} dx + c = e^{x^2} + c.$$

Or,

$$u(x) = 1 + c.e^{-x^2}.$$

So, solution of given ODE is $y(x) = \sqrt{1 + c.e^{-x^2}}$.

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