3.3 : Expected values :

Mean, variance, standard deviation

Expected value or mean value of X :

Let X be the discrete r.v. with set of possible values D and pmf p(x). The expected value or mean value of X, denoted by E(X) or μ_X or just μ, is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

Let X be a discrete r.v whose distribution is given by

$$\begin{array}{llllll} X : & x_0 & x_1 & x_2 \; ---- & x_n \; ----- & \\ p(x): & p_0 & p_1 & p_2 \; ---- & p_n \; ------ & , \quad p_j = f(x_j) \\ & & & & & j = 0, 1, 2, --- \end{array}$$

Then $\mu_X = E(X) = \sum_j x_j p_j = x_0 p_0 + x_1 p_1 + x_2 p_2 + -----$

Expected value of a function:

If the r.v. X has a set of possible values D and pmf p(x), then the expected value of any function h(X), denoted by E[h(X)] or $\mu_{h(X)}$, is computed by

$$E[h(X)] = \sum_{x \in D} h(x) \cdot p(x)$$

Note: The expectation of a random variable X, E(X) gives the average value of X to be expected in many trials.

Rules of Expected value:

i) $E(a) = a$

ii) $E(aX) = a\,E(X)$

iii) $E(aX+b) = a\,E(X)+b$ i.e. $\mu_{aX+b} = a\mu_X + b$

iv) $E(X \pm Y) = E(X) \pm E(Y)$

The variance and standard deviation of x:

Let X have pmf $p(x)$ and mean μ. Then the variance of X, denoted by $V(X)$, or $var(X)$ or σ_X^2 or just σ^2, is

$$V(X) = \sum_{x \in D} (x-\mu)^2 \cdot p(x) = E[(X-\mu)^2], \quad \mu = E(X).$$

The standard deviation (S.D) of X is

$$\sigma_X = +\sqrt{var(X)}.$$

i.e. $\sigma_X = \sigma = \sqrt{\sigma_X^2}$ or $\sigma^2 = var(X)$

Significance of the variance of a distribution:

The variance describes how widely the probability masses are spread out about the mean. $Var(X)$ gives the mean value of the squares of the deviations of the value of X from the mean μ.

If $var(X)$ is small then it is highly probable that values of X will very close to mean μ and if $V(X)$ is large, then it is highly probable that values of X will deviate much from the mean μ.

$var(X) = 0$ only when $X-\mu = 0$ i.e $X = \mu$, so in that case the whole mass is concentrated at the mean.

Shortcut formula for σ^2

$$V(X) = \sigma^2 = E(X^2) - \{E(X)\}^2 = E(X^2) - \mu^2$$

proof:

$$V(X) = E\{(X-\mu)^2\}$$
$$= E(X^2 - 2\mu X + \mu^2)$$
$$= E(X^2) - E(2\mu X) + E(\mu^2) \qquad [\because E(aX) = aE(X)]$$
$$= E(X^2) - 2\mu E(X) + \mu^2$$
$$= E(X^2) - 2\mu.\mu + \mu^2$$
$$= E(X^2) - \mu^2 \quad \text{(proved)}, \quad \mu = E(X).$$

Rules of variance:

$$\sigma^2_{h(X)} = V\{h(X)\} = \sum_x \{h(x) - E[h(X)]\}^2 . p(x)$$

Proposition:

If $h(X) = aX + b$, then $V[h(X)] = a^2 V(X)$

i.e $V(aX+b) = a^2 V(X)$.

proof:

$$V[h(X)] = V(aX+b)$$
$$= \sum_x \{h(x) - E[h(X)]\}^2 . p(x)$$

$E[h(X)]$
$= E(aX+b)$
$= aE(X) + b$
$= a\mu + b$

$$= \sum_x \{ax + b - (a\mu + b)\}^2 . p(x)$$
$$= \sum_x (ax - a\mu)^2 . p(x) = a^2 \sum_x (x-\mu)^2 . p(x)$$
$$= a^2 V(X) \quad \text{(proved)}$$

Note: $V(aX+b) = \sigma^2_{aX+b} = a^2 \sigma^2_X$,

$\sigma_{aX+b} = |a| . \sigma_X$

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Problems – 3.3

Q.29 [3.3]

The pmf of the amount of memory X (GB) in a purchased flash drive was given as in Example 3.13 as

x :	1	2	4	8	16
$p(x)$:	0.05	0.10	0.35	0.40	0.10

Compute the following:

a) $E(X)$

b) $V(X)$ directly from the definition

c) The standard deviation of X

d) $V(X)$ using the shortcut formula

Ans: (a) $E(X) = \mu_X = \sum_x x \cdot p(x)$

$$= 1\times 0.05 + 2\times 0.10 + 4\times 0.35 + 8\times 0.40 + 16\times 0.10$$

$$= 6.45$$

(b) $$V(X) = E\{(X-\mu_X)^2\}$$

$$= \sum (x-\mu_X)^2 \cdot p(x)$$

$$= (1-6.45)^2\times 0.5 + (2-6.45)^2\times 0.10 + (4-6.45)^2\times 0.35 + (8-6.45)^2\times 0.40 + (16-6.45)^2\times 0.10$$

$$= 15.6475$$

(c) The standard deviation of X is

$$\sigma_X = \sqrt{V(X)} = \sqrt{15.6475} = 3.956.$$

(d) $V(X) = E(X^2) - \{E(X)\}^2 = E(X^2) - \mu^2$

Now $E(X^2) = \sum x^2 \cdot p(x) = 1^2\times 0.05 + 2^2\times 0.10 + 4^2\times 0.35 + 8^2\times 0.40 + 16^2\times 0.10$

$$= 57.25$$

$\therefore V(X) = E(X^2) - \mu^2 = 57.25 - (6.45)^2 = 15.6475$

Q. (30) [3.3]

An individual who has automobile insurance from a certain company is randomly selected. Let Y be the number of moving violations for which the individual was cited during the last 3 years. The pmf of Y is

y :	0	1	2	3
$p(y)$:	0.60	0.25	0.10	0.05

a) Compute $E(Y)$.

b) Suppose that an individual Y violations incurs a surcharge of \$ $100Y^2$. Calculate the expected amount of the surcharge.

Ans;

(a) $E(Y) = \sum_y y \cdot p(y) = 0 \times 0.60 + 1 \times 0.25 + 2 \times 0.10 + 3 \times 0.05 = 0.06$

(b) Let $h(Y) = 100Y^2$, Then the expected amount of the surcharge,

$$E[h(Y)] = E(100Y^2)$$
$$= 100\, E(Y^2) \qquad [\because E\{a\,h(x)\} = aE\{h(x)\}]$$
$$= 100 \sum y^2 \cdot p(y)$$
$$= 100[0^2 \times 0.60 + 1^2 \times 0.25 + 2^2 \times 0.10 + 3^2 + 0.05]$$
$$= 110$$

Q. (31) [Q(41) in PDF]

prove that $V(ax+b) = a^2 \cdot \sigma_x^2$, where σ_x = standard deviation of x.

Ans : Try yourself.

Q. 38 [3.3]

Let X = the outcome when a fair die is rolled once. If before the die is rolled you offered either $(1/3.5)$ dollars or $h(x) = 1/x$ dollars, would you accept the guaranteed amount or would you gamble?

Ans:

$X = 1, 2, 3, 4, 5, 6$

$h(x) = \frac{1}{x}$, and the pmf is

x :	1	2	3	4	5	6
$p(x)$:	1/6	1/6	1/6	1/6	1/6	1/6

$\therefore E\{h(x)\} = E\left(\frac{1}{x}\right)$

$= \sum \frac{1}{x} \cdot p(x)$

$= \frac{1}{1}\cdot\frac{1}{6} + \frac{1}{2}\cdot\frac{1}{6} + \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{6} + \frac{1}{5}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6}$

$= 0.408333 > \frac{1}{3.5} = 0.28571$

$\therefore$ Gambling is better option.

Q. (45) [3.3]

If $a \le x \le b$, show that $a \le E(x) \le b$.

proof:- Given $a \le x \le b$

$\Rightarrow a\,p(x) \le x\,p(x) \le b\,p(x)$

$\Rightarrow \sum a\,p(x) \le \sum x\,p(x) \le \sum b\,p(x)$

$\Rightarrow a \sum p(x) \le \sum x\,p(x) \le b \sum p(x)$

$\Rightarrow a \le E(x) \le b$ $[\because \sum p(x) = 1]$

(proved)

$\because$ For pmf, $p(x)$

$\sum p(x) = 1$

$E(x) = \sum x\,p(x)$

Q. (39) [3.3]

A chemical supply company currently has in stock 100 lb of a certain chemical, which it sells to customers in 5-lb batches. Let X = the number of batches ordered by a randomly chosen customer, and suppose that X has pmf

x :	1	2	3	4
$p(x)$:	0.2	0.4	0.3	0.1

Compute E(X), and V(X). Then compute the expected number of pounds left after the next customer's order is shipped and the variance of the number of pounds left.

Ans:

$$E(X) = \mu = \sum x\, p(x)$$

$$= 1\times 0.2 + 2\times 0.4 + 3\times 0.3 + 4\times 0.1 = 2.3$$

$$V(X) = E(X^2) - \mu^2$$

$$= \sum x^2 \cdot p(x) - \mu^2$$

$$= (1^2\times 0.2 + 2^2\times 0.4 + 3^2\times 0.3 + 4^2\times 0.1) - (2.3)^2$$

$$= 6.1 - (2.3)^2 = 0.81$$

We are given that a company has 100 lb of a certain chemical in a stock, the customers orders in 5 lb batches, therefore we have to compute expected value and variance of the number of pounds left, which is $100 - 5X$

Now $E(100-5X) = 100 - 5E(X) = 100 - 5\times 2.3 = 88.5$

and $V(100-5X) = V(-5X+100)$ $\quad [\because V(aX+b) = a^2 V(X)]$

$= (-5)^2 V(X)$ $\quad E(aX+b) = aE(X)+b$

$= 25 \times 0.81$

$= 20.25$

Q.(42) [3.3]

Suppose $E(X) = 5$ and $E[X(X-1)] = 27.5$. what is

a) $E(X^2)$

b) $V(X)$

c) The general relationship among the quantities $E(X)$, $E[X(X-1)]$, and $V(X)$?

Ans: Given $E(X) = 5$, $E[X(X-1)] = 27.5$

(a) From $E[X(X-1)] = E(X^2 - X)$
$= E(X^2) - E(X)$

$\Rightarrow E(X^2) = E[X(X-1)] + E(X)$
$= 27.5 + 5 = 32.5$

(b) $V(X) = E(X^2) - \{E(X)\}^2$
$= 32.5 - 5^2 = 7.5$

(c) Using short-cut formula,

$V(X) = E(X^2) - \{E(X)\}^2$
$= E\{X(X-1) + X\} - \{E(X)\}^2$
$= E\{X(X-1)\} + E(X) - \{E(X)\}^2$
$= E\{X(X-1)\} + E(X)\{E(X) - 1\}$
$= E\{X(X-1)\} + \mu(\mu - 1)$, $\mu = E(X)$