3.2 Probability distributions for discrete random variable

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete (finite) r.v. with probabilities $p_1, p_2, \ldots p_n$ defined by

$$p_1 = P[X = x_1]$$
$$p_2 = P[X = x_2]$$
$$p_n = P[X = x_n]$$

i.e., $p_i = P[X = x_i]$, $i = 1, 2, \ldots n$

probability mass function (p.m.f)

The probability function $p(x)$ or $f(x)$ for any discrete r.v. X is defined by $p(x)$ or $f(x) = P[X = x]$, $x \in X$ and is known as probability mass function (pmf)

Properties

(a) $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 0$ for $x \notin X$.

(b) $\sum_{x \in X} f(x) = 1$ (unit property)

(c) For $a < b \in X$, $\sum_{x \leq a} f(x) \leq \sum_{x \leq b} f(x)$

Probability distribution function (PDF)

or

Cummulative distribution function (CDF)

For a finite discrete r.v. $X = \{x_1, x_2 - x_n\}$
with probabilities $p_j = P[X = x_j]$
$= p(x_j)$ or $f(x_j)$
$j = 1, 2, ---,$

the probability distribution function value at $x = x_j$ is def^n by

$$F(x_j) = p_1 + p_2 + --- + p_{j-1} + p_j$$
$$= F(x_{j-1}) + p_j \;, \; j \geq 2$$

& $F(x_1) = p_1$

Hence

x_0	x_1	x_2	x_3	---	x_n
p_0	p_1	p_2	p_3		p_n

$$F(x_1) = p_1$$
$$F(x_2) = p_1 + p_2 = F(x_1) + p_2$$
$$F(x_3) = p_1 + p_2 + p_3 = F(x_2) + p_3$$
$$\vdots$$
$$F(x_{n-1}) = p_1 + p_2 + --- + p_{n-2} + p_{n-1}$$
$$= F(x_{n-2}) + p_{n-1}$$
$$F(x_n) = p_1 + p_2 + --- + p_n = 1$$

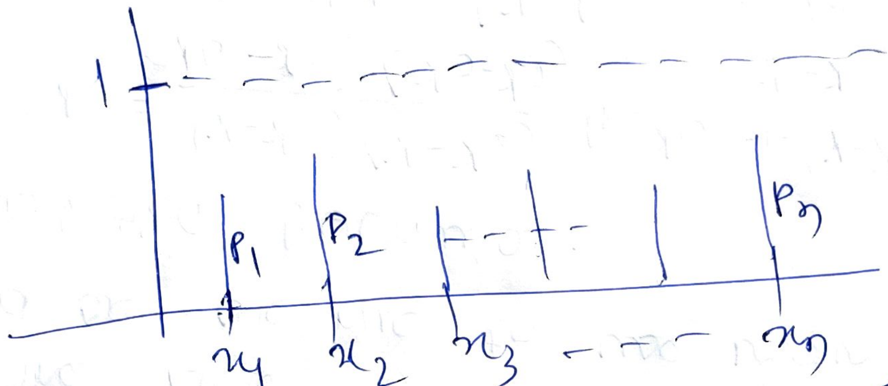
* $F(x) = 0$ for $x < x_1$ & $F(x) = 1$ for $x \geq x_n$
& $F(x) \in [0, 1]$ for $x \in X$

* $p_1 = F(x_1)$, $p_2 = F(x_2) - F(x_1)$, $p_3 = F(x_3) - F(x_2)$
--- $p_{n-1} = F(x_{n-1}) - F(x_{n-2})$, $p_n = F(x_n) - F(x_{n-1})$

Plotting of pmf & CDF

pmf for $X = \{x_1, x_2, \dots, x_n\}$, the pmf table is

x	x_1	x_2	---	x_n
$P(x)$	P_1	P_2		P_n

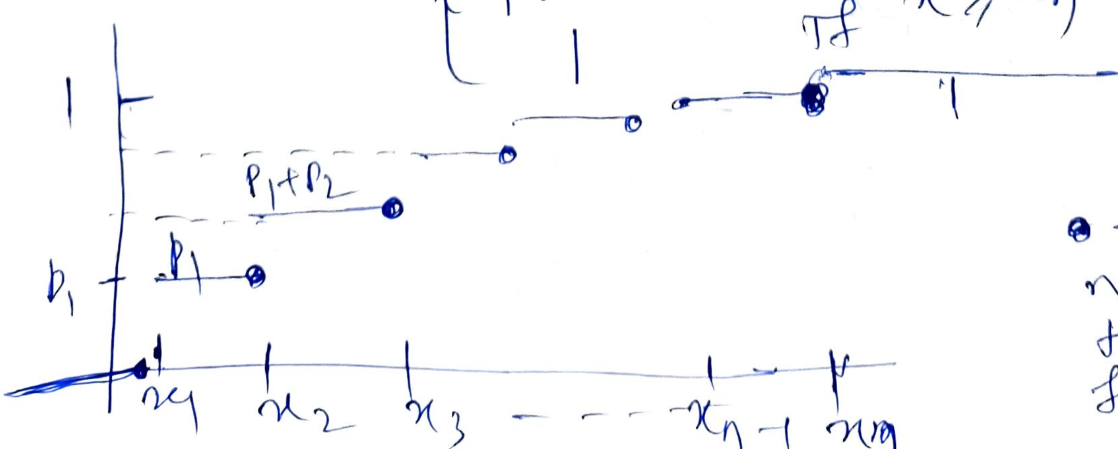


pmf of $X = P(x) = f(x) = P[X = x]$, $x \in X$

PDF or CDF of X is def^n by $F(x) = P[X \leq x]$

For $X = \{x_1, x_2, \dots, x_n\}$,

$$f(x) = \begin{cases} 0 & \text{if } x < x_1 \\ P_1 & \text{if } x_1 \leq x < x_2 \\ P_1 + P_2 & \text{if } x_2 \leq x < x_3 \\ P_1 + P_2 + P_3 & \text{if } x_3 \leq x < x_4 \\ \vdots & \\ P_1 + P_2 + \dots + P_{n-1} & \text{if } x_{n-1} \leq x < x_n \\ 1 & \text{if } x \geq x_n \end{cases}$$



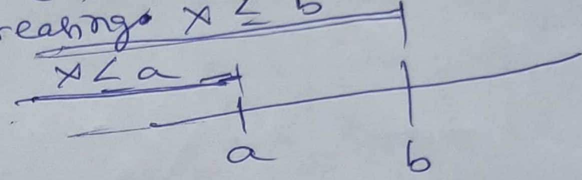
• → denotes not including the x point for prob. value.

Properties of PDF

For the discrete r.v. X, the PDF of X has the following properties

① $0 \le F(x) \le 1 \quad \forall x \in X$

② $F(a) \le F(b)$ for $a < b$ in X, i.e., F is monotonically increasing.



proof ②

We have

$$F(x) = P[X \le x]$$

$$F(b) = P[X \le b] = P[X \le a] + p \quad \text{for some } p \ge 0$$

$$\Rightarrow \boxed{p = F(b) - F(a)} \qquad = F(a) + p$$

since $p \ge 0 \Rightarrow F(b) - F(a) \ge 0$

$$\Rightarrow F(a) \le F(b)$$

③ for $a < x \le b$,

$$P[a < X \le b] = F(b) - F(a)$$

Proof We have

$$[a < X \le b] = [X \le b] - [X \le a]$$

so
$$P[a < X \le b] = P[(X \le b) - (X \le a)]$$
$$= P[X \le b] - P[X \le a]$$
$$= F(b) - F(a)$$

proved

event $\because [X \le a] \subset [X \le b]$
and $P(B - A) = P(B) - P(A)$ If $A \subseteq B$

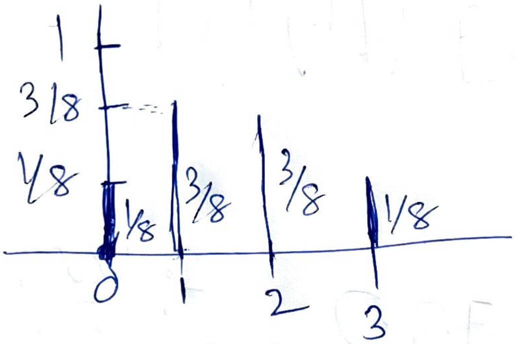
Ex A fair coined tossed three times and r.v. X counts no. of heads (H) in each outcome then find pmf & CDF and plot them.

Sol^n The sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

$$X = \{0, 1, 2, 3\}$$

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

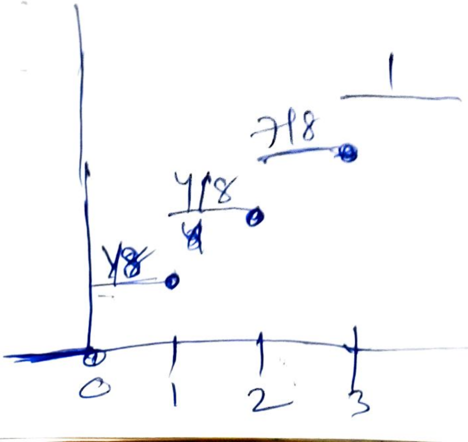


pmf P(x)

PDF or CDF

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{8} & \text{if } 0 \le x < 1 \\ \frac{1}{8} + \frac{3}{8} & \text{if } 1 \le x < 2 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

$$= \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{8} & \text{if } 0 \le x < 1 \\ 4/8 & \text{if } 1 \le x < 2 \\ 7/8 & \text{if } 2 \le x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$



A parameter of a probability distribution

For any parameter $0 < \alpha < 1$, the pmf is defined by

$$f(x) = P(x, \alpha) = \begin{cases} 1-\alpha & \text{If } x=0 \\ \alpha & \text{If } x=1 \\ 0 & \text{otherwise} \end{cases}$$

where $X = \{0, 1\}$

Ex (Birth problem)

In a hospital, it is observed that probability of the birth of a boy (B) is $P(B) = p$ and prob. of the birth of a girl (G) is $P(G) = q$ satisfying $p + q = 1 \Rightarrow q = 1-p$.

~~prob~~ Problem is to find the probability of the new born child is a boy. The r.v. ~~Sample~~ X counts no. of birth observed.

We have

Sample space $= \{B, GB, GGB, GGGB, \ldots\}$

& $X = \{1, 2, 3, \ldots\}$

$P(1) = P(B) = p$

$P(2) = P(GB) = P(G)P(B) = qp$

$P(3) = P(GGB) = P(G)P(G)P(B) = q^2 p$

In general:

$$\boxed{P(x) = q^{x-1} p = (1-p)^{x-1} p, \quad x = 1, 2, 3, \ldots \text{ and } 0 \text{ otherwise}}$$

Geometric distribution

$p \to$ parameter