3.3 Expected values

Let X be a discrete r.v where

$$X = \{x_1, x_2, x_3, ---\}$$

and $p_1, p_2, p_3 ---$ be the probability values of $x_1, x_2, x_3 ---$ respectively satisfies the rule

$$p_i = P[X = x_i] \;, i = 1,2,3, --$$

then expectation of X or mean of X is defined by

$$\mu = \mu_X = E(X) = \sum_{i=1}^{\infty} x_i p_i = \sum_i x_i p_i$$

* $\mu = \mu_X = \text{mean}(X)$, $E(X) =$ expected value of X

If $f(x)$ is the pmf of X, then

$$\mu_X = E(X) = \sum_{x \in X} x f(x)$$

where $f(x) = P[X = x]$.

Ex Find $E(X)$ from the given data

x	0	1	2	3
p	1/8	3/8	3/8	1/8

$$E(X) = \sum_{i=1}^{4} x_i p_i \quad , x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 4$$

$$= 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = 1.5$$

Note: ① E is linear, i.e. $E(X+Y) = E(X) + E(Y)$ where X, Y has same pmf $f(x)$.

② $E(X) =$ constant for any r.v. X

③ $-\infty < E(X) < \infty$

④ If the r.v. X has pmf $f(x)$ and $Y = g(X)$ is a function of r.v. X then $\mu_Y = E(Y) = E(g(X)) = \sum_{x \in X} g(x) f(x)$,

Ex Find the mean of $Y = 2X$ where the r.v. X has following data

x	0	1	2	3
p	$1/8$	$3/8$	$3/8$	$1/8$

Soln

Given $x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 3$

$P_1 = 1/8,\ P_2 = 3/8,\ P_3 = 3/8,\ P_4 = \frac{1}{8}$

Since $Y = 2X$, so $y = 2x$

$\therefore y_1 = 2x_1 = 0,\ y_2 = 2x_2 = 2,\ y_3 = 2x_3 = 4$

$y_4 = 2x_4 = 6$

$\text{mean}(Y) = E(Y) = \sum_{i=1}^{4} y_i p_i$

$= 0 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + 4 \cdot \frac{3}{8} + 6 \cdot \frac{1}{8} = \frac{24}{8} = 3$

Q. Prove that for any r.v. $X = k$ (constant)

$E(1) = 1$ & $E(k) = k = kE(1)$

Proof

$$E(1) = \sum_{x \in X} 1 \cdot f(x) \quad \text{where } f(x) = \text{pmf of } X$$

$$= \sum_{x \in X} f(x) = 1 \quad \text{by unit property of } f.$$

$$E(k) = \sum_{x \in X} k \cdot f(x) = k \sum_{x \in X} f(x) = k = kE(1)$$

Q. If $Y = aX + b$ where a & b are constants where the r.v. X has the pmf $f(x)$, then find $E(Y)$ (or μ_Y).

Sol^n $\quad Y = aX + b$

$$\mu_Y = E(Y) = \sum_{x \in X} y f(x) = \sum_{x \in X} (ax + b) f(x)$$

$$= a \sum_{x \in X} x f(x) + b \sum_{x \in X} f(x)$$

$$= a E(X) + b$$

Alternatively

Since E is linear,

$$E(Y) = E(aX + b) = aE(X) + E(b)$$

$$= aE(X) + b$$

$$(\because E(\text{constant}) = \text{constant})$$

* Example of a pmf $f(x)$ without finite mean-

Q. If X is any r.v. with given pmf $f(x)$,

$$f(x) = \frac{K}{x^2}, \quad x = 1, 2, 3, \ldots$$

then find (i) K (ii) $E(X)$ or μ_X

Sol^n (i) By unit property of the pmf $f(x)$, we have

$$\sum_{x \in X} f(x) = 1$$

$$\Rightarrow \sum_{x=1}^{\infty} \frac{K}{x^2} = 1$$

$$\Rightarrow K\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right) = 1$$

$$\Rightarrow K\left(\frac{\pi^2}{6}\right) = 1 \Rightarrow K = \frac{6}{\pi^2}$$

$$\therefore f(x) = \frac{6}{\pi^2 x^2}, \quad x = 1, 2, 3, \ldots$$

(ii) $\mu_X = E(X) = \sum_{x \in X} x f(x)$

$$= \sum_{x=1}^{\infty} x \cdot \frac{6}{\pi^2 x^2}$$

$$= \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x} \to \infty \quad \left(\because \sum_{x=1}^{\infty} \frac{1}{x} \to \infty\right)$$

i.e. Expected values of X is undefined though the prob. func^n $f(x)$ is defined on X.

Variance of X and standard deviation of X

For any r.v. X with pmf $f(x)$ and finite mean μ, the variance of X is denoted by $V(X)$ or $Var(X)$ and symbolically σ_X^2 or σ^2 where

$$\sigma_X^2 = V(X) = \sum_{x \in X} (x-\mu)^2 f(x) \quad \text{where } \mu = \sum_{x \in X} x f(x)$$
$$= E(X-\mu)^2$$

* If $X = \{x_1, x_2, \ldots x_n\}$ & $p_i = P[X = x_i]$, $i = 1, 2, \ldots n$,

$$\sigma^2 = V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i \quad \text{where } \mu_X = \sum_{i=1}^{n} x_i p_i$$

If $n = \infty$, then

$$\sigma^2 = V(X) = \sum_{i=1}^{\infty} (x_i - \mu)^2 p_i \quad \text{where } \mu = \sum_{i=1}^{\infty} x_i p_i$$

Ex Find variance of the r.v. X from the given data

x	0	1	2	3
p	1/8	3/8	3/8	1/8

Given $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 3$

$p_1 = 1/8$, $p_2 = 3/8$, $p_3 = 3/8$, $p_4 = 1/8$

We have mean $= \mu = \sum_{i=1}^{4} x_i p_i = 1.5$

So variance of X $= V(X) = \sum_{i=1}^{4} (x_i - \mu)^2 p_i$

$= (0-1.5)^2 \cdot \frac{1}{8} + (1-1.5)^2 \cdot \frac{3}{8} + (2-1.5)^2 \cdot \frac{3}{8} + (3-1.5)^2 \cdot \frac{1}{8} = 0.75$

Q. For any r. v. X, prove that

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2$$

Proof

Let $f(x)$ be the pmf of X, then

$$\sigma^2 = V(X) = \sum_{x \in X} (x-\mu)^2 f(x)$$

$$= E(X-\mu)^2, \quad \mu = E(X) = \text{constant}$$

$$= E[X^2 - 2X\mu + \mu^2]$$

$$= E(X^2) - 2\mu E(X) + \mu^2$$

$$= E(X^2) - 2E(X) \cdot E(X) + (E(X))^2$$

$$= E(X^2) - (E(X))^2$$

proved

$$\boxed{\begin{array}{l} \because \mu = \text{const}, \\ E(\mu^2) = \mu^2 \\ \mu = E(X) \end{array}}$$

Q. Let X be a discrete r.v. with pmf $f(x)$,

$$V(aX+b) = a^2 V(X) \text{ ; } a, b \text{ are constants}$$

– prove it.

Proof

$$V(aX+b) = E[(aX+b)^2] - [E(aX+b)]^2$$

$$= E[a^2X^2 + 2abX + b^2] - [aE(X) + b]^2$$

$$= a^2 E(X^2) + 2ab\, E(X) + b^2$$

$$- (a^2(E(X))^2 + 2ab\, E(X) + b^2)$$

$$= a^2 [E(X^2) - (E(X))^2]$$

$$= a^2 V(X) \quad \text{proved}$$

$E(K) = K$

$K = \text{const}$

Standard deviation of X

Standard deviation of X is defined by $\sigma = S(D) = +\sqrt{\text{variance}} = +\sqrt{V(X)}$, $\sigma > 0$

Note If $Y = aX + b$, a, b are constants with variance of X is $V(X)$ or σ_X^2, then variance of Y is

$$\sigma_Y^2 = V(Y) = a^2 V(X) = a^2 \sigma_X^2 .$$

implying $\boxed{\sigma_Y = |a| \sigma_X}$

$\Rightarrow \boxed{SD(Y) = |a| \, SD(X)}$

K^{th} moment and K^{th} central moment

1) For any r.v. X, K^{th} moment of X is defined by $E(X^K)$, $K = 1, 2, \ldots$

for $K = 1$; $E(X) = \mu$ = mean(X)

for $K = 2$, $E(X^2)$ = 2nd moment of X

2) K^{th} central moment of X is $E(X-\mu)^K$, $K = 1, 2, \ldots$

for $K = 1$, 1st central moment is

$E(X-\mu) = E(X) - E(\mu) = E(X) - \mu = 0$

for $K = 2$, 2nd central moment is $E(X-\mu)^2 = V(X)$

$\Rightarrow V(X) = \text{2nd moment} - (\text{1st moment})^2$

Markov Inequality

If X is any random variable with mean $\mu = E(X)$ where X is nonnegative and μ is finite, then for any $t > 0$,

$$P[X \geq t] \leq \frac{\mu}{t}$$

<u>Proof</u> For a fixed $t > 0$, define the r.v. Y as
$$Y = \begin{cases} 0 & \text{if } X < t \\ t & \text{if } X \geq t. \end{cases}$$

then Y is a discrete r.v. with pmf

$$p_Y(0) = P[X < t]$$

and $p_Y(t) = P[X \geq t]$. where $Y = \{0, t\}$

Thus
$$\text{mean}(Y) = E(Y) = \sum_y y\, p_Y(y)$$
$$= 0 \cdot p_Y(0) + t\, p_Y(t)$$
$$= t\, P[X \geq t]$$

Since $X \geq Y$, we have

$$E(X) \geq E(Y)$$
$$\geq t\, P[X \geq t]$$
$$\Rightarrow \mu \geq t\, P[X \geq t]$$
$$\Rightarrow P[X \geq t] \leq \frac{\mu}{t} \quad \text{proved}$$

Note ① Taking $Y = \begin{cases} 0 & \text{if } X < t \\ t^k & \text{if } X \geq t, \end{cases}$

we get $P[X^k \geq t^k] \leq \frac{E(X^k)}{t^k}$, $k = 1, 2, \ldots$

and 0 otherwise

② Replacing X by $X - \mu$, we have

$$P[(X-\mu)^k \geq t^k] \leq \frac{E(X-\mu)^k}{t^k}, \quad k = 1, 2, \ldots$$

and 0 otherwise

Chebyshev Inequality

Let μ and σ^2 be the mean and variance of any r.v. X, then

$$P[|X-\mu| \geq t] \leq \frac{\sigma^2}{t^2}, \quad t > 0$$

Proof By Markov inequality, we have

$$P[(X-\mu)^2 \geq t^2] \leq \frac{E(X-\mu)^2}{t^2} \quad \boxed{\because \sigma^2 = E(X-\mu)^2}$$

$$= \sigma^2/t^2$$

Since $[(X-\mu)^2 \geq t^2] = [|X-\mu| \geq t]$,

so we have

$$P[|X-\mu| \geq t] \leq \frac{\sigma^2}{t^2}, \quad t > 0$$

proved

Note

Replacing t by $k\sigma$, $k \geq 1$, we have

$$P[|X-\mu| \geq k\sigma] \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

Q. Let μ & σ be mean & SD of any r.v. X then prove that $P[|X-\mu| \geq k\sigma] = \frac{1}{k^2}$ for any $k \geq 1$

Ex The r.v. Y has the probabilities given in the accompanying table

y	45	46	47	48	49	50	51	52	53	54	55
$p(y)$	0.05	0.10	0.12	0.14	0.25	0.17	0.06	0.05	0.03	0.02	0.01

Find $E(Y)$, $SD(Y)$ and $P(|Y-\mu| \geq t)$ for $t = 2\sigma, 3\sigma$

Sol^n Expectation of $Y = E(Y) = \sum_{y=45}^{55} y\,p(y)$

$= 48.84$

Variance $= V(Y) = E(Y^2) - (E(Y))^2 = \sum_{y=45}^{55} y^2 p(y) - (48.84)^2$

$= 2389.84 - (48.84)^2$

$\Rightarrow \sigma^2 = 4.4944$

$\Rightarrow \sigma = 2.12$

By Chebyshev inequality, we have

$$P[|Y-\mu| \geq t] \leq \frac{\sigma^2}{t^2}$$

(a) For $t = 2\sigma$, $P[|Y-\mu| \geq 2\sigma] \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4}$

$P[|Y-\mu| \geq 2\sigma] = P[Y-\mu \geq 2\sigma] + P[Y-\mu \leq -2\sigma]$

$= P[Y \geq \mu+2\sigma] + P[Y \leq \mu-2\sigma]$

$= P[Y \geq 53.08] + P[Y \leq 44.60]$

$= 1 - P[Y < 53.08] + P[Y \leq 44]$

$= 1 - P[Y \leq 53] + P[Y \leq 44]$

$= 1 - (F(53) - F(44))$

$= 1 - (0.97 - 0) = 0.03$

(b) Find $P[|Y-\mu| \geq 3\sigma]$ (Task)