Solution Manual & Scheme of Evaluation
Autumn Mid Semester Examination-2022
Subject-Differential Equations & Linear Algebra
Subject Code: MA 11001
Scheme-I

Question No.	Answers	Step Marks
Q.1(a)	Find the general solution of the differential equation $$xy' = x + y.$$ **Solution:** Given ODE: $$y' = 1 + \frac{y}{x} \qquad (1)$$ Let $\frac{y}{x} = u,$ $\Rightarrow y' = u + x\frac{du}{dx}$ Equation (1) becomes: $$u + x\frac{du}{dx} = 1 + u \qquad (2)$$ Solving Equation (2), we obtain: $$y(x) = x(\ln cx)$$	**0.5** **0.5**
(b)	Find the orthogonal trajectory for $x = ce^{\frac{y}{4}}$. **Solution:** The given equation can be written as: $$f(x,y) = xe^{-\frac{y}{4}} = c$$ $\Rightarrow f_x = e^{\frac{-y}{4}}$ and $f_y = -\frac{x}{4}e^{\frac{-y}{4}}$ The ODE for O. T. is: $y' = \frac{f_y}{f_x} = -\frac{x}{4}$ The orthogonal trajectory is: $y(x) = -\frac{x^2}{8} + c^*$	**0.5** **0.5**
(c)	What will be the exact differential equation whose solution is $\sin x - \cos x - x^2 + y = c$, where c is an arbitrary constant? **Solution:** Let $u(x,y) = \sin x - \cos x - x^2 + y$ Now, $\frac{\partial u}{\partial x} = \cos x + \sin x - 2x$ and $\frac{\partial u}{\partial y} = 1$ The exact ODE: $$(\cos x + \sin x - 2x)dx + dy = 0$$	**0.5** **0.5**
(d)	Apply the operator $(D^2 - 2D + I)$ on $\sinh(x+2)$. **Solution:** $(D^2 - 2D + I)(\sinh(x+2))$ $= D^2(\sinh(x+2)) - 2D(\sinh(x+2)) + \sinh(x+2)$ $= D(\cosh(x+2)) - 2\cosh(x+2) + \sinh(x+2)$ $= \sinh(x+2) - 2\cosh(x+2) + \sinh(x+2)$ $= 2[\sinh(x+2) - \cosh(x+2)]$.	**0.5** **0.5**
(e)	Find an integrating factor for the ODE $\frac{dx}{dy} = 2(x-1)\tanh 2y$. **Solution:** Given ODE can be written as $$\frac{dx}{dy} - (2\tanh 2y)x = -2\tanh 2y\,.$$ Here, $P(y) = -2\tanh 2y$ Integrating factor is: $e^{\int P(y)\,dy} = e^{-2\int \tanh 2y\,dy} = e^{-\ln(\cosh 2y)} = \operatorname{sech} 2y$	**0.5** **0.5**

Q.2(a)	Find the particular solution of $$xy' = y + 4x^5 cos^2(y/x)\ , \quad y(2) = 0$$ **Solution:** Given ODE can be written as $$y' = \frac{y}{x} + 4x^4 cos^2\left(\frac{y}{x}\right)$$ (1) Let $\frac{y}{x} = u, \quad \Rightarrow y' = u + x\frac{du}{dx}$ Equation (1), becomes: $u + x\frac{du}{dx} = u + 4x^4 cos^2 u$ (2) From Equation (2): $\frac{du}{dx} = 4x^3 cos^2 u$ (3) Separating variable and integrating both sides: $$\int sec^2 u\, du\ = \int 4x^3 dx$$ $\Rightarrow \tan u = x^4 + c$ $\tan\left(\frac{y}{x}\right) = x^4 + c$ (4) Using the initial condition $y(2) = 0$ in Equation (4), $$c = -16$$ The particular solution is: $\tan\left(\frac{y}{x}\right) = x^4 - 16.$	**1** **1** **1**
(b)	Reduce to first order and then solve $$y'' + (y')^3 \sin y = 0$$ **Solution:** Let $y' = u, \ \ \Rightarrow y'' = u\left(\frac{du}{dy}\right)$ Now, Given ODE becomes: $$u\frac{du}{dy} + (u)^3 \sin y = 0$$ If $u = 0$, then solution is: $y = c$. If $u \neq 0$, then $$\frac{du}{dy} + u^2 \sin y\ = 0$$ Separating the variables and on integration we get: $$-\frac{1}{u} = \cos y + A$$ $\Rightarrow \frac{dx}{dy} = -\cos y - A$ Required Solution: $x = -\sin y - Ay + B$	**1** **1**
Q.3(a)	Find the integrating factor of the following differential equation which will make it exact. Hence solve the equation. $$(3xy - 2y^2)dx + (x^2 - 2xy)dy = 0.$$ **Solution:** Here, $M(x, y) = 3xy - 2y^2$ and $N(x, y) = x^2 - 2xy$ **Exactness:**	**1**

	$M_y = 3x - 4y$ and $N_x = 2x - 2y$ Given ODE is not exact. **Integrating Factor:** $$\frac{M_y - N_x}{N(x,y)} = \frac{x-2y}{x(x-2y)} = \frac{1}{x}$$ Integrating Factor: $$e^{\int \frac{dx}{x}} = x$$ Now multiplying the integration factor by given ODE: $$(3x^2y - 2xy^2)dx + (x^3 - 2x^2y)dy = 0$$ **General Solution:** Solving the exact ODE, we get: $$x^3y - x^2y^2 = c$$	**1** **1**
(b)	Find a general solution of $(D^2 + 6D + 13I)y = 0$. **Solution:** The Auxiliary Equation is: $\lambda^2 + 6\lambda + 13 = 0$ $\Rightarrow (\lambda + 3)^2 = -4$ $\Rightarrow \lambda = -3 \pm 2i$ The general solution is: $y(x) = e^{-3x}(A\cos 2x + B\sin 2x)$.	**0.5** **0.5** **1**
Q.4(a)	Find a differential equation for which the given $y(x)$ is a general solution and then determine the constants so that the given initial conditions are satisfied. $$y(x) = e^x(c_1\cos x + c_2\sin x), \quad y(0) = 1, \quad y'(0) = 3.$$ **Solution:** Basis of solutions: $\{e^x\cos x, e^x\sin x\}$ Hence, the roots of Auxiliary equation is: $\lambda = 1 \pm i$ The auxiliary equation is: $\lambda^2 - 2\lambda + 2 = 0$ The ODE is: $y'' - 2y' + 2y = 0$. Particular Solution: The given general solution is: $$y(x) = e^x(c_1\cos x + c_2\sin x)$$ Using the initial condition $y(0) = 1$ we find: $c_1 = 1$. $\Rightarrow y(x) = e^x(\cos x + c_2\sin x)$ Now, $y' = e^x(\cos x + c_2\sin x) + e^x(-\sin x + c_2\cos x)$ Using the initial condition $y'(0) = 3$, we get: $c_2 = 2$. The particular solution is: $y(x) = e^x(\cos x + 2\sin x)$.	**0.5** **1** **0.5** **0.5** **0.5**
(b)	Find a general solution of the differential equation $$y' + 2y = 4\cos 2x.$$ **Solution:** The Integrating factor is: e^{2x} The general solution is: $$ye^{2x} = 4\int e^{2x}\cos 2x \ dx + c$$ $\Rightarrow y(x) = \cos 2x + \sin 2x + ce^{-2x}$ is the required solution.	**0.5** **0.5** **1**

Q.5(a)	A thermometer, reading 10^0C is brought into a room whose temperature is 23^0C. Two minutes later the thermometer reading is 18^0C. Find the temperature in the thermometer after 2 hours. **Solution:** Let $T(t)$ be the temperature in the thermometer at time t. The temperature in the room: $T_s = 23$ By Newton's law of cooling $$\frac{dT}{dt} = k(T - T_s)$$ $\Rightarrow$ The ODE: $\frac{dT}{dt} = k(T - 23)$ (1) The initial conditions: $$T(0) = 10, \;\; T(2) = 18$$ Solving ODE (1): $T(t) = 23 + ce^{kt}$ Using the initial condition, $T(0) = 10$ $\Rightarrow c = -13$ $\Rightarrow T(t) = 23 - 13e^{kt}$ Again using $T(2) = 18$, $\Rightarrow k \cong -0.47775$ The temperature in the thermometer at time t is $$T(t) = 23 - 13e^{-0.47775t}$$ The temperature after 2 hours: $T(120) = 23 - 13e^{-0.47775\times 120} = 23^o$C.	**0.5** **0.5** **0.5** **1** **0.5**
(b)	Find a basis of solutions of the ODE by method of reduction of order**.** $$x^2y'' + 3xy' + y = 0, \quad y_1 = \frac{1}{x}$$ **Solution:** The given ODE can be written as: $$y'' + \frac{3}{x}y' + \frac{1}{x^2}y = 0$$ Here, $P(x) = \frac{3}{x}$ Given solution is: $y_1 = \frac{1}{x}$ $\Rightarrow$By method of reduction of order, the second solution is obtained by $$y_2 = y_1 \int \frac{1}{(y_1)^2} e^{-\int P dx} dx$$ $y_2 = \frac{1}{x} \int x^2 e^{-\int \frac{3}{x} dx} dx = x^{-1} \int \frac{1}{x} dx = x^{-1} \ln x$. Basis of solutions is: $\{x^{-1}, x^{-1}\ln x\}$	**0.5** **1** **0.5**