9/7/19

Date ___/___/___

* Phase velocity of de-Broglie waves:-

A/C to the de-Broglie hypothesis a free material Particle of energy 'E' and momentum 'P' can be represented by a wave travelling in the +ve 'x' direction as

$$\boxed{y = \underbrace{A}_{\text{Amplitude}} \sin(\underbrace{\omega}_{\text{Angular freq.} = (2\pi\nu)} t - Kx)} \quad \text{—— (5)}$$

$A \rightarrow$ Amplitude

$\omega \rightarrow$ Angular freq. $= (2\pi\nu)$

$K \rightarrow$ Propagation Constant $\left(\frac{2\pi}{\lambda}\right)$

Phase of the wave $\rightarrow \omega t - Kx$

→ Phase velocity is the velocity of a Particle with Constant phase

$$\omega t - Kx = \text{Const.}$$

$$\Rightarrow \frac{d(\omega t - Kx)}{dt} = 0$$

$$\Rightarrow \boxed{\frac{dx}{dt} = \frac{\omega}{K} = u}$$

where 'u' is the Phase vel. of the wave.

Page No.

9/7/19.

Date ___/___/___

$$u = \frac{\omega}{K} = \frac{\hbar\omega}{\hbar K} \quad ; \quad \left(\hbar = \frac{h}{2\pi}\right)$$

$$\hbar\omega = \frac{h}{2\pi} \times 2\pi\nu = h\nu = E$$

$$\hbar K = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \frac{h}{\lambda} = p$$

$$\boxed{u = \frac{E}{p}} \quad \text{--- (7)}$$

Phase vel.

Case : I

For Photons:-

$E = pc$ — from (a)

$\frac{E}{p} = c$

$$u = \frac{E}{p}$$

$$\boxed{u = \frac{E}{p} = c}$$

The Phase velocity of the de-Broglie waves — associated with a photon is equal to the vel. of light in Vacuum.

Page No.

Date ___/___/___

Case II For Non relativistic Particle:-

$$u = \frac{E}{P}$$

$$E = \frac{P^2}{2m}$$ (Energy of non-relativistic particle)

$$\boxed{u = \frac{E}{P} = \frac{P}{2m} = \frac{mv}{2m} = \frac{v}{2}}$$

The Phase velocity of the de-broglie waves associated with a non relativistic Particle is half the particle velocity.

Case III For Relativistic Particle:-

$$u = \frac{E}{P}$$

$E = mc^2$ (energy of relativistic Particle)

$$= \frac{m_0}{\sqrt{1 - v^2/c^2}} c^2 ;$$

(as $m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ for relativistic Particle)

$$P = mv$$

$$= \frac{m_0}{\sqrt{1 - v^2/c^2}} \times v.$$

$$\boxed{u = \frac{c^2}{v} >> c}$$

The Phase velocity of the de-broglie waves associated with relativistic Particles is greater than the velocity of light in vacuum this is in apparant contradiction with the special theory of relativity.

* Thus, the vel. of a Particle (both relativistic and non relativistic) is different from the Phase velocity of the de-broglie waves associated with it. This is sometimes reffered to as the velocity Paradox of matter waves.

* In our attempt to represent the Particles by waves of the type given by equation (5), we have encountered the velocity Paradox. The resulting Phase velocity does not agree with the Particle velocity infact a Particle is a localised object whereas a wave is an infinitely extended entity having no distinguishing marks to represent the location of the Particle.

Localisation can be achieved by modulating the wave to form a wave group or wave Packet which moves with a group velocity 'V_g'

Wave groups & group velocity:-

Date ___/___/___

A wave group or Packet can be constructed by superimposing a large number of waves so chosen that they superimpose constructively in a localised region of space and superimpose destructively else where. If the localisation is constant with time then it can represent a particle.

Here, we attempt to construct a wave packet by the combination of two waves of same amplitude and slightly different frequencies.

$$y_1 = A \sin(\omega t - Kx) \quad \text{——} \; (8)$$

$$y_2 = A \sin(\omega' t - K' x) \quad \text{——} \; (9)$$

By the principle of superposition:-

$$y = y_1 + y_2 = A\,[\underbrace{\sin(\omega t - Kx)}_{\sin B} + \underbrace{\sin(\omega' t - K' x)}_{\sin A}]$$

$$= 2A \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$= 2A \sin\left[\left(\frac{\omega' + \omega}{2}\right)t - \frac{K' + K}{2}x\right] \cdot \left[\cos \frac{\omega' - \omega}{2} t - \frac{K' - K}{2} x\right]$$

Let $\omega' - \omega = \Delta\omega$ & $K' - K = \Delta K$

$\because \omega' \simeq \omega$ & $K' \simeq K$ i.e. the waves vary slightly in their frequencies

$\rightarrow \omega' = \Delta\omega + \omega$ & $K' = \Delta K + K$.

11/7/19

Date __/__/__

Substituting in the above eqⁿ:-

$$y = 2A \sin\left[\frac{2\omega + \Delta\omega}{2} t - \frac{2K + \Delta K}{2} x\right] \cdot \cos\left[\frac{\Delta\omega}{2} t - \frac{\Delta K}{2} x\right]$$

$$\Rightarrow \boxed{y = 2A \underbrace{\sin[\omega t - Kx]}_{\text{Contribution of a single wave}} \times \underbrace{\cos\left[\frac{\Delta\omega}{2} t - \frac{\Delta K}{2} x\right]}_{\text{Contribution of the wave group.}}} \quad (10)$$

$\because \Delta\omega << 2\omega$ & $\Delta K << 2K$, it is neglected in the sine term.

→ The Resultant wave in eqⁿ (10) has two Parts

i) A sine wave of frequency ω and Propagation Constant 'K', the Phase velocity given by this Part is:

$$u = \frac{\omega}{K}$$

ii) Another wave of frequency $\frac{\Delta\omega}{2}$ and Propagation Constant $\frac{\Delta K}{2}$ keeps rise to the group vel. 'V_g'

$$V_g = \lim_{\Delta K \to 0} \frac{\Delta\omega}{\Delta K} \Rightarrow \boxed{V_g = \lim_{\Delta K \to 0} \frac{\Delta\omega}{\Delta K}} = \frac{d\omega}{dK}$$

Date ___/___/___

* V_g is called as the group vel. and it is the velocity of the wave group or wave packet

$$\boxed{V_g = \frac{d\omega}{dk} = \frac{d(\hbar\omega)}{d(\hbar k)} = \frac{dE}{dp}}$$ —— (11)

$\left\{ \hbar = \frac{h}{2\pi} \right.$ refer previous topics.

Case. I For Photons:-

$$V_g = \frac{dE}{dp} = c$$

as;
$$E = pc$$
$$dE = c\,dp$$
$$\frac{dE}{dp} = c$$

The group velocity of the de-broglie waves associated with the photons is equal to the speed of light in vacuum.

Case: II For Non-relativistic Particle :-

$$V_g = \frac{dE}{dp} = \frac{p}{m} = \frac{mv}{m} = v$$

$$E = \frac{p^2}{2m}$$
$$dE = \frac{p\,dp}{m}$$
$$\frac{dE}{dp} = \frac{p}{m}$$

11/7/19

Date __/__/__

Thus, the group vel. of the de-broglie waves associated with a non-relativistic Particle is equal to the Particle velocity.

Case: III ; For relativistic Particle :-

$$V_g = \frac{dE}{dP} = \frac{c^2 P}{E}$$

$$E = [P^2C^2 + m_0^2 C^4]^{1/2}$$

$$dE = \frac{1}{2}[P^2c^2 + m_0^2 c^4]^{-1/2} \cdot [2PdPc^2]$$

$$\frac{dE}{dP} = \frac{\cancel{2}c^2 P}{\cancel{2}[P^2c^2 + m_0^2 c^4]^{1/2}}$$

$$\frac{dE}{dP} = \frac{c^2 P}{E}$$

$$V_g = c^2 \frac{\frac{m_0 v}{\sqrt{1-v^2/c^2}}}{\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}}$$

$$V_g = \frac{dE}{dP} = v$$

The group velocity of the de-broglie waves associated with a relativistic Particle is equal to its Particle vel.

Relation b/w group velocity and Phase velocity :-

$$V_g = \frac{d\omega}{dK} = \frac{d(2\pi\nu)}{d\left(\frac{2\pi}{\lambda}\right)} = \frac{d\nu}{d\left(\frac{1}{\lambda}\right)} \quad \text{—— (9)}$$

$$u = \frac{\omega}{K} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda$$

$$\Rightarrow \nu = \frac{u}{\lambda} \quad \text{——} (b)$$

Eqn (b) $\longrightarrow$ eqn (a)

$$v_g = \frac{d(u/\lambda)}{d(1/\lambda)} = \frac{d\left(u \cdot \frac{1}{\lambda}\right)}{d(1/\lambda)} = u + \frac{1}{\lambda}\frac{du}{d(1/\lambda)}$$

dividing $d\lambda$ both in num. & deno.

$$v_g = u + \frac{1}{\lambda}\frac{\left(\frac{du}{d\lambda}\right)}{\frac{d(1/\lambda)}{(d\lambda)}}$$

$$\Rightarrow v_g = u + \frac{1}{\cancel{\lambda}}\frac{\frac{du}{d\lambda} \times \lambda^{\cancel{2}}}{-1}$$

$$\Rightarrow \boxed{v_g = u - \lambda\left(\frac{du}{d\lambda}\right)} \quad (12)$$

→ For a dispersive medium, $\frac{du}{d\lambda}$ is non zero.

$$\Rightarrow \boxed{v_g < u.}$$

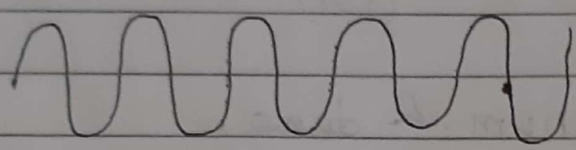
11/7/19.

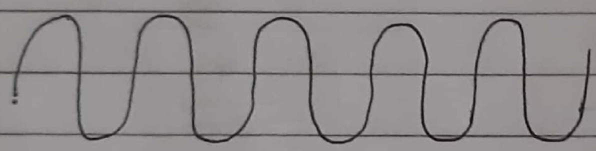
Date ___/___/___

→ For a non-dispersive medium, $\frac{du}{d\lambda} = 0$

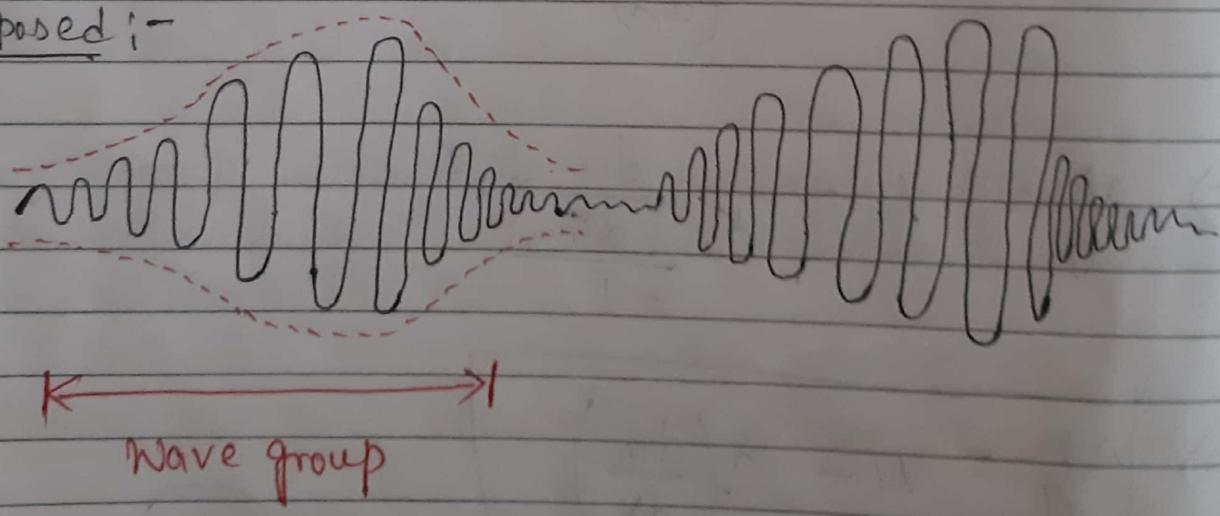
$\Rightarrow \boxed{V_g = u}$

Superimposition of wave.

(a) 

(b) 

Superimposed :-



* Properties of de-Broglie waves:-

i) Lighter is the particle greater is the wavelength associated with it.

ii) Smaller is the velocity of the particle greater is the wavelength associated with it.

iii) The de-Broglie waves are produced irrespective of the fact that the particle is charged or uncharged. (if the particle is moving with some speed). Whereas, the electromagnetic waves are produced from accelarated charged particles.

P-T-O

Date ___/___/___ Saathi

iv) The velocity of the matter waves depends on the velocity of the material Particle. i.e. it is not a Constant, whereas; The velocity of the electromagnetic waves is the constant.

v) The wave nature of matter introduced the concept of uncertainity in the location of a particle within the wave packet.

* Heisenberg's Uncertainity Principle :-

A/c Heisenberg's uncertainity Principle, it is impossible to determine simultaneously the exact Position & momentum (velocity) of a small moving Particle like electron with unlimited accuracy.

$$\Delta x \cdot \Delta p \geq \hbar$$

If we write h or $\frac{\hbar}{2}$ instead of $\hbar$ that is also correct but $\hbar$ is most appropiate

Other Pairs :-

$$\Delta E \cdot \Delta t \geq \hbar$$

$$\Delta \theta \cdot \Delta L\theta \geq \hbar$$

ΔE = uncertainity in Energy.

Δt = " " time

$\Delta \theta$ = " " Angular displacement

$\Delta L\theta$ = " " " momentum.

* <u>Application of Heisenberg's uncertainity Principle:-</u>

→ <u>Non-Existence of electrons inside the nucleus:-</u>

Radioactive nucli emmits 'β' Particles (which are electrons), in the 'β' decay Process, we may presume that the electrons are existing inside the nucleus before emmission, but the nucleus has only Protons & neutrons as its Constituents, the electron is not a Part of it.

A qualitative explanation for the non-existence of electrons inside the nucleus is provided by the Heisenberg's uncertainity Principle.

⇝ Diameter of a typical nucleus is of the order of 10^{-14} m. If the electron is assumed to be present inside the nucleus then the uncertainity in the location of the electron inside the nucleus is :-

$$\Delta x_{max} \simeq 10^{-14} m$$

By using the Heisenberg's uncertainity Principle.

$$\Delta x_{max} \cdot \Delta P_{min} \simeq \hbar$$

$$\Rightarrow \Delta P_{min} \simeq \frac{\hbar}{\Delta x_{max}} = \frac{6.628 \times 10^{-34} \, J \cdot S}{2 \times 3.14 \times 10^{-14} \, m} \simeq 10^{-21} \, kg \cdot m/s$$

Page No.

The min. value of momentum 'p' of the electron is of the order of its uncertainity ΔP_{min}.

$$P \approx \Delta P_{min} \approx 10^{-21} \, kg \cdot m/s$$

⇝ <u>Energy of the emitted electron</u> :-

$$E = [p^2c^2 + m_0^2 c^4]^{1/2}$$

$$= \left[(10^{-21})^2 \times (3\times10^8)^2 + (9.1\times10^{-31})^2 \times (3\times10^8)^4\right]^{1/2}$$

$$\approx 10 \, MeV$$

But the energy of the electrons ejected in the 'β' decay process is much less than the above estimated value this shows that the electrons cannot be a part of the nucleus.