Chapter-8
Matrix Eigenvalue Problems
8.1 Eigenvalues and Eigenvectors

Eigen values:

Let $A = [a_{jk}]$ be an $n \times n$ matrix and λ be a scalar. Then $D(\lambda) = \det(A - \lambda\, I) = |A - \lambda I| = 0$ is called characteristic equation of A.

A root of the characteristic equation, $|A - \lambda I| = 0$ is called an eigenvalue of characteristic value of A.

Here, $|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$.

Note: An $n \times n$ matrix A has at least one eigenvalue and at most n numerically different eigenvalues.

Examples:

Find the eigenvalues of the matrix$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1 - \lambda & 3 \\ 4 & 5 - \lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)(5 - \lambda) - 12 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 7 = 0$

$\Rightarrow (\lambda - 7)(\lambda + 1) = 0$

$\Rightarrow \lambda = -1, 7$

Hence the eigenvalues of A are $-1, 7$.

Example:

Find the eigen values of the matrix$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$.

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1 - \lambda & -1 & 0 \\ 1 & 2 - \lambda & -1 \\ 3 & 2 & -2 - \lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)\{-(2 - \lambda)(2 + \lambda) + 2\} + 1(-2 - \lambda + 3) = 0$

$\Rightarrow (1 - \lambda)(\lambda^2 - 2) + (1 - \lambda) = 0$

$\Rightarrow (1 - \lambda)(\lambda^2 - 1) = 0$

$\Rightarrow \lambda = 1, 1, -1$

Hence the eigenvalues of A are$1, 1 - 1$.

Eigen Vectors:

Let A be an $n \times n$ matrix. A non-null vector X which is a solution of the vector equation $AX = \lambda X$, is called an eigenvector or characteristic vector of A. Here λ is a scalar.

Example:

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -5 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$
$\Rightarrow \lambda = -1, -6$
Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to$\lambda = -1$. Then
$AX = \lambda X$
$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\Rightarrow -4x_1 + 2x_2 = 0,\ \ 2x_1 - x_2 = 0$
$\Rightarrow 2x_1 - x_2 = 0$
Let $x_1 = k,$ then $x_2 = 2k$ where, $k \neq 0$.
So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
Hence the eigenvector corresponding to $\lambda = -1$ is$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Similarly, let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigenvector corresponding to$\lambda = -6$. Then
$AX = \lambda X$
$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -6 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\Rightarrow x_1 + 2x_2 = 0,\ \ 2x_1 + 4x_2 = 0$
$\Rightarrow x_1 + 2x_2 = 0$
Let $x_1 = k,$ then $x_2 = -k/2$ where, $k \neq 0$.
So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k/2 \end{bmatrix} = k/2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
Hence the eigenvector corresponding to $\lambda = -6$ is$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Example:

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution: The characteristic equation of A is $|A - \lambda I| = 0$
$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$
$\Rightarrow (1-\lambda)\{(2-\lambda)(-1-\lambda) - 1\} - 1(\lambda + 1 - 0) - 2(-1-0) = 0$
$\Rightarrow (1-\lambda)\{(2-\lambda)(-1-\lambda) - 1\} + (1-\lambda)=0$
$\Rightarrow (1-\lambda)(\lambda^2 - \lambda - 2) = 0$
$\Rightarrow (1-\lambda)(\lambda - 2)(\lambda + 1) = 0$
$\Rightarrow \lambda = 1, 2, -1$
Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to$\lambda = 1$. Then
$AX = \lambda X$
$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$
$\Rightarrow 0x_1 + x_2 - 2x_3 = 0,$ …………………(1)
$-x_1 + x_2 + x_3 = 0,$ ………………….(2)
$0x_1 + x_2 - 2x_3 = 0$ ………………….(3)

Solving Eqs.(1) and (2) by cross multiplication method, we get

$$\frac{x_1}{1+2} = \frac{x_2}{2-0} = \frac{x_3}{0+1}$$

$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = k$ (say)

Then $x_1 = 3k,\ x_2 = 2k, x_3 = k.$

You can easily verify that this solution is also satisfied Eq. (3).

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = 2$. Then

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow -x_1 + x_2 - 2x_3 = 0,$ …………………(1)

$-x_1 + 0x_2 + x_3 = 0,$ ………………….(2)

$0x_1 + x_2 - 3x_3 = 0$ ………………….(3)

Solving Eqs.(1) and (2) by cross multiplication method, we get

$$\frac{x_1}{1-0} = \frac{x_2}{2+1} = \frac{x_3}{0+1}$$

$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1} = k$ (say)

Then $x_1 = k,\ x_2 = 3k, x_3 = k.$

You can easily verify that this solution is also satisfied Eq. (3).

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

Similarly, you can obtain the eigen value corresponding to $\lambda = -1$.

The eigenvalue corresponding to $\lambda = -1$ will be $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Eigen Basis (Basis of eigenvectors):

Let A be an $n \times n$ matrix. Then an eigen basis is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

If A has n distinct eigenvalues, then A has a basis of eigenvectors $X_1,\ X_2, \dots, X_n$ for $\mathbb{R}^n$.

Eigen space:

The eigenvectors corresponding to one and the same eigenvalue λ of A together with null vector $\mathbf{0}$, form a vector space, called the Eigen space of A corresponding to that λ. It is usually denoted by E_λ.

Algebraic Multiplicity and Geometric Multiplicity:

The order of an eigenvalue λ is called the algebraic multiplicity of λ and it is denoted by M_λ.

The number of linearly independent eigenvectors corresponding to an eigenvalue λ is called the geometric multiplicity of λ and it is denoted by m_λ. Thus, the geometric multiplicity is the dimension of the Eigen space corresponding to this λ.
An eigenvalue λ is regular if $M_\lambda = m_\lambda$.
Example: Find the algebraic multiplicity and geometric multiplicity of the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$.
Solution: The characteristic equation of A is $|A - \lambda I| = 0$
$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 0 & 3-\lambda \end{vmatrix} = 0$
$\Rightarrow (\lambda - 3)^2 = 0$
$\Rightarrow \lambda = 3, 3$
Hence the algebraic multiplicity of $\lambda = 3$ is $M_3 = 2$.
Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = 3$. Then
$AX = \lambda X$
$\Rightarrow \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
$\Rightarrow 2x_2 = 0,\ \ 3x_2 = 3x_2$
$\Rightarrow x_2 = 0$
So, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
The eigen vector corresponding to $\lambda = 3$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
Hence the geometric multiplicity of $\lambda = 3$ is $m_3 = 1$.
Results:
1: If A is a $n \times n$ triangular matrix -upper triangular, lower triangular or diagonal, the eigenvalues of A are the diagonal entries of A.
2: $\lambda = 0$ is an eigenvalue of A if A is a singular (noninvertible) matrix.
3: A and A^T have the same eigenvalues.
4: $\det A$ is the product of the eigenvalues of A.
5: If λ be an eigenvalue of A, then

a) $k\lambda$ is an eigenvalue of kA, $k =$scalar.
b) λ^2 is an eigenvalue of A^2

8.3. Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric, Skew-Symmetric, and Orthogonal Matrices.

A real square matrix $A = [a_{jk}]$ is called

1. **symmetric** if transposition leaves it unchanged,
$$A^T = A, \qquad \text{thus} \qquad a_{kj} = a_{jk},$$
2. **skew-symmetric** if transposition gives the negative of A,
$$A^T = -A, \qquad \text{thus} \qquad a_{kj} = -a_{jk},$$
3. **orthogonal** if transposition gives the inverse of A,

$$A^T = A^{-1}.$$

Example: The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal, respectively.

Note:

1. The determinant of an orthogonal matrix has the value +1 or −1.
2. Every square matrix can be expressed as the addition of symmetric and skew-symmetric matrix.

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

Eigenvalues of Symmetric and Skew-Symmetric and Orthogonal Matrices:

(a) The eigenvalues of a symmetric matrix are real.

(b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.

(c) The eigenvalues of an orthogonal matrix **A** are real or complex conjugates in pairs and have absolute value 1.

Example:

The matrix $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ is symmetric and has eigenvalues 2 and 8.

The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric and has eigenvalues -i and i.

Example: The orthogonal matrix

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

has eigenvalues -1, $(5+i\sqrt{11})/6$ and $(5-i\sqrt{11})/6$, which have absolute value 1.

8.5. Complex Matrices and Different Forms.

Complex Conjugate Matrix

$\bar{A} = [\bar{a}_{jk}]$ is obtained from $A = [a_{jk}]$ by replacing each entry $a_{jk} = \alpha + i\beta$ (α, β real) with its complex conjugate $\bar{a}_{jk} = \alpha - i\beta$. Also, $\bar{A}^T = [\bar{a}_{kj}]$ is the transpose of $\bar{A}$, hence the conjugate transpose of A.

Hermitian, Skew-Hermitian, and Unitary Matrices.

A square matrix $A = [a_{kj}]$ is called

1. **Hermitian** if $\bar{A}^T = A$, that is, $\bar{a}_{kj} = a_{jk}$
2. **skew-Hermitian** if $\bar{A}^T = -A$, that is, $\bar{a}_{kj} = -a_{jk}$
3. **unitary** if $\bar{A}^T = A^{-1}$.

Note:

1. Let A be a unitary matrix. Then its determinant has absolute value one, that is, $|\det A| = 1$.
2. Every complex square matrices can be expressed as the addition of Hermitian and Skew-Hermitian matrices.

$$A = \frac{A + (\bar{A})^T}{2} + \frac{A - (\bar{A})^T}{2}$$

Eigenvalues of Hermitian and Skew-Hermitian and Unitary Matrices.

(a) The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.

(b) The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.

(c) The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value 1.

Example:

The matrix $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is Hermitian and has eigenvalues 9 and 2.

The matrix $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is skew-Hermitian and has eigenvalues $4i$ and $-2i$.

The matrix $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$ is unitary and has eigenvalues $(\sqrt{3}+i)/2$ and $(-\sqrt{3}+i)/2$, which have absolute value 1.

Different Forms

1. **Quadratic Form**

   Quadratic form is an expression of the form

   $Q = x^T Ax$

   Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

   $$\Rightarrow Q = x^T Ax = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + a_{21}x_1x_2 + a_{22}x_2^2 + \cdots + a_{n1}x_1x_n + \cdots + a_{nn}x_n^2$$

   $$\Rightarrow Q = x^T Ax = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + \cdots + (a_{1n} + a_{n1})x_1x_n$$

   In particular

   Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

   $$\Rightarrow Q = x^T Ax = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{23} + a_{32})x_2x_3$$

Determination of Symmetric matrix from Quadratic form

From the given Quadratic form $Q = x^T Ax$ we can obtain symmetric matrix as follows:

I. Place coefficients of x_i^2 diagonally.

II. Divide the coefficients of $x_ix_j, i \neq j$ equally and place equal values at a_{ij} and a_{ji} places respectively.

Example: Find the symmetric matrix from the following quadratic form

$$Q = 2x_1^2 + 4x_2^2 + 3x_3^2 + 10x_1x_2 + 12x_2x_3 + 8x_3x_1$$

Solution: In case of symmetric matrix $a_{ij} = a_{ji}$

$$\Rightarrow Q = 2x_1^2 + 4x_2^2 + 3x_3^2 + (5 + 5)x_1x_2 + (6 + 6)x_2x_3 + (4 + 4)x_3x_1$$

The symmetric matrix is

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 5 & 4 & 6 \\ 4 & 6 & 3 \end{bmatrix}$$

2. **Hermitian Form**

   Hermitian form H is given by

   $$H = \bar{x}^T Ax$$

   Where A is a Hermitian matrix and x is a column matrix.

Note: Hermitian form is always real.

Example: Is the following matrix Hermitian or Skew-Hermitian matrix? Find the Hermitian form or Skew-Hermitian form.

$$A = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}, x = \begin{bmatrix} -2i \\ 1+i \end{bmatrix}$$

Solution: $\bar{A} = \begin{bmatrix} 4 & 3+2i \\ 3-2i & -4 \end{bmatrix}, \bar{A}^T = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} = A$

Since $\bar{A}^T = A$, so A is Hermitian.

Hermitian form is $H = \bar{x}^T A x$

Here $x = \begin{bmatrix} -2i \\ 1+i \end{bmatrix} \quad \Rightarrow \bar{x} = \begin{bmatrix} 2i \\ 1-i \end{bmatrix}$

$$\Rightarrow H = \bar{x}^T A x = [2i \quad 1-i] \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} \begin{bmatrix} -2i \\ 1+i \end{bmatrix} = [7i+5 \quad 10i] \begin{bmatrix} -2i \\ 1+i \end{bmatrix} = 4$$

3. **Skew-Hermitian Form**

   Skew-Hermitian form S is given by

$$H = \bar{x}^T A x$$

   Where A is a Skew-Hermitian matrix and x is a column matrix.

Note: Skew-Hermitian form is always 0 or purely imaginary.

Example: Is the following matrix Hermitian or Skew-Hermitian matrix? Find the Hermitian form or Skew-Hermitian form.

$$A = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}, x = \begin{bmatrix} i \\ 4 \end{bmatrix}$$

Solution: $\overline{A} = \begin{bmatrix} -i & -2-3i \\ 2-3i & 0 \end{bmatrix}, \overline{A}^T = \begin{bmatrix} -i & 2-3i \\ -2-3i & 0 \end{bmatrix} = -\begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} = -A$

Since $\overline{A}^T = -A$, so A is Skew-Hermitian.

Skew-Hermitian form is $S = \overline{x}^T A x$

Here $x = \begin{bmatrix} i \\ 4 \end{bmatrix} \quad \Rightarrow \overline{x} = \begin{bmatrix} -i \\ 4 \end{bmatrix}$

$$\Rightarrow S = \overline{x}^T A x = [-i \quad 4] \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} \begin{bmatrix} i \\ 4 \end{bmatrix} = [9+12i \quad 2i+3] \begin{bmatrix} i \\ 4 \end{bmatrix} = 17i$$

8.4 Eigen basis. Diagonalization

Similarity of Matrices

A square matrix $\hat{A}$ of order n is said to be similar to a $n \times n$ square matrix A if there exists a non-singular matrix P such that

$$\hat{A} = P^{-1} A P$$

This transformation of A to $\hat{A}$ is known as ***Similarity transformation***.

Note:

1. Eigenvalues of $A =$ Eigenvalues of $\hat{A}$
2. If X is an eigenvector of A corresponding to the eigenvalue λ i.e. $AX = \lambda X,\ X \neq 0$ then $Y = P^{-1} X$ is an eigenvector of $\hat{A}$ corresponding to same eigenvalue λ, means $\hat{A}(P^{-1}X) = \lambda(P^{-1}X), \quad P^{-1}X = Y \neq 0$

Basis of Eigenvectors

If A is a square matrix of order n and it has n distinct eigenvalues then the corresponding n eigenvectors are L. I. and they will form a basis which is known as ***basis of eigenvectors or Eigen basis.***

Note:

1. A symmetric matrix has an orthonormal basis of eigenvectors.
2. A Hermitian, Skew-Hermitian or unitary matrix has a basis of eigenvectors which is a unitary system or unitary Eigen basis.
3. A basis of eigenvectors is possible if algebraic multiplicity of $\lambda =$ geometric multiplicity.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Eigenvalues

The characteristic equation is $|A - \lambda I| = 0.$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow \lambda^2 = 0 \qquad \Rightarrow \lambda = 0,\ 0$

Hence for $\lambda = 0$, A. M. =2

Eigenvectors

The homogeneous system is $(A - \lambda I)X = 0, \quad X \neq 0.$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \Rightarrow 0x_1 + x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

Here $x_2 = 0$ but x_1 is arbitrary.

$\Rightarrow X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, This is the only one eigenvector.

Hence G. M. =1

Therefore A cannot provide a basis of eigenvectors.

Diagonalization

Let A be any square matrix and X be the Modal matrix whose column vectors are eigenvectors of A. Then the ***Diagonalization*** of the matrix A is denoted by D and defined by

$$D = X^{-1}AX$$

Note:

1. Only those matrices can be diagonalized if there exists a basis of eigenvectors.
2. A square matrix A of order n with n L. I. eigenvectors is similar to a diagonal matrix D whose diagonal elements are eigenvalues of A.
3. Powers of a matrix A

   We have $D = X^{-1}AX$

   $\Rightarrow D^2 = DD = (X^{-1}AX)(X^{-1}AX) = X^{-1}A^2X$

   Similarly $D^3 = X^{-1}A^3X$

   In general, $D^n = X^{-1}A^nX$

   To obtain A^n we have to pre-multiply by X and post multiply by X^{-1}

   $\Rightarrow XD^nX^{-1} = XX^{-1}A^nXX^{-1} = A^n$

   Hence $A^n = XDX^{-1}$.

Procedure of Diagonalization

Any square matrix which has a basis of eigenvectors can be diagonalized as follows:

I. Find the eigenvalues of the given matrix A

II. Find eigenvectors.

III. Form a Modal matrix X by taking eigenvectors as its columns.

IV. Calculate X^{-1}.

V. The diagonal matrix is $D = X^{-1}AX$ (Main diagonal elements are eigenvalues of A).

Example: Diagonalize the following matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Solution: Eigenvalues

The characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$.

Eigenvector for $\lambda_1 = 0$

The homogeneous system is $(A - \lambda_1 I)X_1 = 0, \quad X_1 \neq 0$.

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \Rightarrow 2x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

Both the equations are identical.

$\Rightarrow x_2 = -2x_1$, Here x_1 is arbitrary.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenvector for $\lambda_1 = 0$ is

$$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenvector for $\lambda_2 = 3$

The homogeneous system is $(A - \lambda_2 I)X_2 = 0, \quad X_2 \neq 0$.

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \Rightarrow -x_1 + x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned}$$

Both the equations are identical.

$\Rightarrow x_2 = x_1$, Here x_1 is arbitrary.

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvector for $\lambda_2 = 3$ is

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The **modal matrix** is $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow |X| = 1 + 2 = 3$

$\Rightarrow X^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$

$\Rightarrow D = X^{-1}AX = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

This is the required diagonal matrix whose diagonal elements are eigenvalues of the matrix A.

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