1.3 Separable Ordinary Differential Equations, Modelling

Generally, it is difficult to solve first-order ordinary differential equations $y' = f(x, y)$ in the sense that no formulae exist for obtaining its solution in all cases. However, there are certain standard types of first-order differential equations of the first degree for which routine methods of solution are available. In this chapter, we shall discuss a few of these types.

<u>Separable Differential Equation:</u>
Differential equations of the form

$$g(y)dy = f(x)dx \tag{1}$$

are called equations with separated variables, the solutions of which are obtained by direct integration.
Thus, its solution is given by

$$\int g(y)dy = \int f(x)dx + c. \tag{2}$$

If f and g are continuous functions, the integrals in Eq. (2) exist, and by evaluating them we obtain a general solution of Eq. (1). This method of solving ODEs is called the **method of separating variables**, and Eq. (1) is called a **separable equation**.

Example: 1 **Solve** $y' = 1 + y^2$.

Solution: The given equation can be written as

$$\frac{dy}{1+y^2} = dx.$$

Integrating both the sides, we have

$$\tan^{-1} y = x + c$$

or

$$y = \tan(x + c).$$

Example: 2 Solve $\frac{dy}{dx} = e^{2x-y} + x^3 e^{-y}$.

Solution: The given equation can be written as

$$\frac{dy}{dx} = (e^{2x} + x^3)e^{-y}$$

Separating the variables, we have

$$e^y dy = (e^{2x} + x^3)dx$$

Integrating both the sides

$$e^y = \frac{e^{2x}}{2} + \frac{x^4}{4} + c.$$

HOMOGENEOUS DIFFERENTIAL EQUATIONS

A function $f(x, y)$ of two variables is said to be **homogeneous of degree n** if

$$f(tx, ty) = t^n f(x, y) \tag{3}$$

for every $t>0$ such that (tx, ty) is in the domain of f.

Example: Let $f(x,y) = 2x^4 - x^2y^2 + 5xy^3$

Now

$$\begin{aligned} f(tx,ty) &= 2(tx)^4 - (tx)^2(ty)^2 + 5(tx)(ty)^3 \\ &= t^4\left(2x^4 - x^2y^2 + 5xy^3\right) \\ &= t^4 f(x,y) \end{aligned}$$

Therefore, f is homogeneous of degree 4.

Example:

$$f(x, y) = \sin\left(\frac{x}{y}\right)$$

$$f(tx, ty) = \sin\left(\frac{tx}{ty}\right) = \sin\left(\frac{x}{y}\right) = t^0 f(x, y)$$

Thus $f(x, y) = \sin\left(\frac{x}{y}\right)$ is a homogeneous function of degree 0.

Homogeneous Differential Equation

A homogeneous differential equation is a differential equation which can be written in the form:

$$P(x,y)dx + Q(x,y)dy = 0 \tag{4}$$

where P and Q are **homogeneous functions** of the same degree.

Example: The differential equation $(x^2 + xy)dx + y^2dy = 0$ is a homogeneous differential equation because $(x^2 + xy)$ and y^2 are homogeneous functions of degree 2.

Note:

1. A first order differential equation $y' = f(x, y) = g\left(\frac{y}{x}\right)$ is known as homogeneous equation in which $f(x, y) = g\left(\frac{y}{x}\right)$ is a homogeneous function of degree 0.
2. Homogeneous differential equation does not involve *constant terms*.

Reduction to Separable form:

Type-I: If the given homogeneous differential equation of 1st order can be written as $y' = \frac{f(x, y)}{g(x, y)}$ then the solution of this type equation can be determined as follows:

Procedure:

I. Put $y = ux$

II. Calculate $\frac{dy}{dx} = u + x\frac{du}{dx}$

III. Substitute y and y' in the given ODE. The given ODE reduces to a separable differential equation.

IV. Separate the variables and integrate both sides to get the general solution.

V. Replace $u = \frac{y}{x}$ and find the general solution of the given ODE.

Example: Solve the ODE $2xyy' = y^2 - x^2$.

Solution: Given ODE is

$$y' = \frac{y^2 - x^2}{2xy} \tag{5}$$

Let $y = ux, \quad \Rightarrow y' = u + x\frac{du}{dx}$

Substituting y and y' in the given ODE (5) we find

$$u + x\frac{du}{dx} = \frac{x^2(u^2-1)}{2ux^2} = \frac{u^2-1}{2u}$$

$$\Rightarrow x\frac{du}{dx} = \frac{u^2-1}{2u} - u = \frac{u^2-1-2u^2}{2u} = \frac{-(1+u^2)}{2u}$$

This is a separable differential equation.

Separating variables and integrating we get

$$\int \frac{2u}{1+u^2}du = -\int \frac{dx}{x}$$

$$\Rightarrow \ln\left|1+u^2\right| = -\ln|x| + \ln C$$

$$\Rightarrow 1 + u^2 = \frac{C}{x}$$

Putting $u = \frac{y}{x}$ we obtain

$$\Rightarrow 1 + \frac{y^2}{x^2} = \frac{x^2+y^2}{x^2} = \frac{C}{x}$$

$\Rightarrow x^2 + y^2 = Cx$, This is the required general solution of the given ODE.

Type-II: An equation of the form

$$y' = f(ax + by + c)$$

Can be reduced to a separable equation by substituting $ax + by + c = v$.

Example: Solve $y' = (y + 4x)^2$

Solution: Given ODE is

$$y' = (y + 4x)^2 \tag{6}$$

Let $y + 4x = v$

Differentiating both sides w.r.to x we get

$$y' + 4 = \frac{dv}{dx}, \quad \Rightarrow y' = \frac{dv}{dx} - 4$$

The given ODE (6) becomes

$$\frac{dv}{dx} - 4 = v^2, \qquad \Rightarrow \frac{dv}{dx} = v^2 + 4$$

This is a separable differential equation.

Separating variables and integrating both sides we get

$$\Rightarrow \int \frac{dv}{v^2+4} = \int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C_1$$

$$\Rightarrow \tan^{-1}\left(\frac{v}{2}\right) = 2x + 2C_1 = 2x + C$$

$$\Rightarrow v = 2\tan(2x + C)$$

Putting $y + 4x = v$ we find

$$\Rightarrow y + 4x = 2\tan(2x + C)$$

This is the required solution of the given ODE.

Example: Solve the ODE $y' = \cos(x + y + 1)$

Solution: Let $x + y + 1 = v$

Differentiating w.r.to x we get

$$y' = \frac{dv}{dx} - 1$$

The given ODE becomes

$\dfrac{dv}{dx} = 1 + \cos v = 2\cos^2(v/2)$, This is a separable differential equation.

Separating variables and integrating both sides we find

$$\frac{1}{2}\int \sec^2(v/2)dv = \int dx$$

$$\Rightarrow \tan(v/2) = x + C$$

$$\Rightarrow \tan\left(\frac{x+y+1}{2}\right) = x + C$$

This is the general solution of the given differential equation.

Type-III: Nonhomogeneous ODEs Reducible to Homogeneous Form

If the differential equation is of the form

$$y' = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \tag{7}$$

Suppose $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ or $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

Equation (7) can be solved as follows:

Let $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = k(\text{constant}), \quad \Rightarrow a_1 = a_2k$ and $b_1 = b_2k$

Equation (7) becomes

$$y' = \frac{k(a_2x + b_2y) + c_1}{a_2x + b_2y + c_2} \tag{8}$$

Putting $a_2x + b_2y = u$, the differential equation (8) reduces to a separable equation in terms of u and x.

Example: Solve $y' = \dfrac{x+y+4}{x+y-6}$

Solution: Given differential equation is

$$y' = \frac{x+y+4}{x+y-6} \tag{9}$$

In the equation (9), $\frac{a_1}{a_2} = \frac{b_1}{b_2} = 1$

Putting $x+y=u$ and $y' = \frac{du}{dx} - 1$ in equation (9) we find

$\frac{du}{dx} - 1 = \frac{u+4}{u-6}, \quad \Rightarrow \frac{du}{dx} = \frac{u+4}{u-6} + 1 = \frac{2(u-1)}{u-6}$, This is a separable equation in terms of u and x.

Separating variables and integrating both sides we find

$$\int \frac{u-6}{u-1} du = \int 2dx, \quad \Rightarrow \int \left(1 - \frac{5}{u-1}\right) du = \int 2dx$$

$$\Rightarrow u - 5\ln|u-1| = 2x + C$$

$$\Rightarrow x + y - 5\ln|x+y-1| = 2x + C$$

$$\Rightarrow y - x - 5\ln|x+y-1| = C.$$

Solve the following ODEs.

1. $(2x - 4y + 5)y' + x - 2y + 3 = 0$
2. $(2x + y + 1)dx + (4x + 2y - 1)dy = 0$
3. $(2x - 3y + 2)dx + 3(4x - 6y - 1)dy = 0$

Problems Set 1.3

Question: Find a general solution. Show the steps of derivation. Check your answer by substitution

Q1: $xy' = y + 2x^3 \sin^2\left(\frac{y}{x}\right)$

Answer: Let $u = \frac{y}{x} \quad \Rightarrow y = ux$

Differentiating we get $y' = u + xu'$

Now we can substitute y and y ' in the equation:

$$x(u + xu') = ux + 2x^3 \sin^2 u$$

$$\Rightarrow xu + x^2u' = ux + 2x^3 \sin^2 u$$

$$\Rightarrow x^2 u' = 2x^3 \sin^2 u$$

$$\Rightarrow \frac{u'}{\sin^2 u} = 2\frac{x^3}{x^2}$$

$$\Rightarrow \frac{u'}{\sin^2 u} = 2x$$

Integrating both the sides we get

$$\Rightarrow \int \frac{du}{\sin^2 u} = \int 2x\, dx$$

$$\Rightarrow -\cot u = 2\frac{x^2}{2} - c$$

$$\Rightarrow -\cot u = x^2 - c$$

$$\Rightarrow -\cot u = x^2 - c$$

$$\Rightarrow \cot u = c - x^2$$

$$\Rightarrow u = \cot^{-1}(c - x^2)$$

Using back substitution method $u = \dfrac{y}{x}$, we get

$$\frac{y}{x} = \cot^{-1}(c - x^2)$$

$$\Rightarrow y = x\cot^{-1}(c - x^2)$$ **Ans.**

Q2: $y' = (y + 4x)^2$

Answer: Let $v = y + 4x \quad \Rightarrow y = v - 4x$

Differentiating we get $y' = v' - 4$

Now we can substitute y and y ' in the given equation:

$$v' - 4 = (v - 4x + 4x)^2$$

$$\Rightarrow v' = v^2 + 4$$

$$\Rightarrow \frac{dv}{dx} = v^2 + 4$$

$$\Rightarrow \frac{dv}{v^2 + 4} = dx$$

Integrating both the sides we get

$$\Rightarrow \int \frac{dv}{v^2 + 4} = \int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}(\frac{v}{2}) = x + \frac{c}{2}$$

$$\Rightarrow \tan^{-1}(\frac{v}{2}) = 2x + c$$

$$\Rightarrow \frac{v}{2} = \tan(2x + c)$$

$$\Rightarrow v = 2\tan(2x + c)$$

Using back substitution method, we get $v = y + 4x$

$$\Rightarrow y + 4x = 2\tan(2x + c)$$

$$\Rightarrow y = 2\tan(2x + c) - 4x$$ **Ans.**

Q3. $xy' = y^2 + y$

Answer: Let $u = \frac{y}{x} \quad \Rightarrow y = ux$

Differentiating we get $y' = u + xu'$

Now we can substitute y and y ' in the equation:

$$x(u + xu') = (ux)^2 + ux$$

$$\Rightarrow xu + x^2u' = (ux)^2 + ux$$

$$\Rightarrow u' = u^2$$

$$\Rightarrow \frac{du}{dx} = u^2$$

$$\Rightarrow \frac{du}{u^2} = dx$$

Integrating both the sides we get

$$\Rightarrow \int \frac{du}{u^2} = \int dx - c$$

$$\Rightarrow -\frac{1}{u} = x - c$$

$$\Rightarrow \frac{1}{u} = c - x$$

$$\Rightarrow u = \frac{1}{c - x}$$

Using back substitution method $u = \frac{y}{x}$, we get

$$\frac{y}{x} = \frac{1}{c - x}$$

Ans.

$$\Rightarrow y = \frac{x}{c - x}$$

Q4. $y' = (x + y - 2)^2, \qquad y(0) = 2$

Answer: Let $v = x + y - 2 \quad \Rightarrow y = v - x + 2$

Differentiating we get $y' = v' - 1$

Now we can substitute y and y ' in the given equation:

$$v' - 1 = (v)^2$$

$$\Rightarrow v' = v^2 + 1$$

$$\Rightarrow \frac{dv}{dx} = v^2 + 1$$

$$\Rightarrow \frac{dv}{v^2 + 1} = dx$$

Integrating both the sides we get

$\Rightarrow \int \frac{dv}{v^2+1} = \int dx$

$\Rightarrow \tan^{-1}(v) = x + c$

$\Rightarrow v = \tan(x+c)$

Using back substitution method, we get $v = x + y - 2$

$\Rightarrow x + y - 2 = \tan(x+c)$

$\Rightarrow y = \tan(x+c) - x + 2$

All we need to do is to determine c from IVP

Given $y(0) = 2$ i.e. when $x = 0$ then $y = 2$

$2 = \tan(0+c) - 0 + 2$

$\Rightarrow \tan c = 0$

$\Rightarrow c = 0$

Hence the particular solution is: $y = \tan x - x + 2$ **Ans.**

Q5. $xy' = y + 3x^4 \cos^2\left(\frac{y}{x}\right), \qquad y(1) = 0$

Answer: Let $u = \frac{y}{x} \Rightarrow y = ux$

Differentiating we get $y' = u + xu'$

Now we can substitute y and y ' in the equation:

$x(u + xu') = ux + 3x^4 \cos^2 u$

$\Rightarrow xu + x^2u' = ux + 3x^4 \cos^2 u$

$\Rightarrow x^2u' = 3x^4 \cos^2 u$

$\Rightarrow \frac{u'}{\cos^2 u} = 3\frac{x^4}{x^2}$

$\Rightarrow \frac{u'}{\cos^2 u} = 3x^2$

Integrating both the sides we get

$\Rightarrow \int \frac{du}{\cos^2 u} = \int 3x^2\, dx$

$\Rightarrow \tan u = 3\frac{x^3}{3} + c$

$\Rightarrow \tan u = x^3 + c$

$\Rightarrow u = \tan^{-1}(x^3 + c)$

Using back substitution method $u = \frac{y}{x}$, we get

$\frac{y}{x} = \tan^{-1}(x^3 + c)$

$\Rightarrow y = x\tan^{-1}(x^3 + c).$

All we need to do is to determine c from IVP

Given $y(1)=0$ i.e. when $x=1$ then $y=0$

$0=1.\tan^{-1}(1+c)$
$\Rightarrow (c+1)=\tan 0$
$\Rightarrow c+1=0$
$\Rightarrow c=-1$

Hence the particular solution is: $y=x\tan^{-1}(x^3-1)$ **Ans.**

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