Linear Grammars

Grammars with
at most one variable at the right side
of a production

Examples:

$$S \to aSb$$
$$S \to \lambda$$

$$S \to Ab$$
$$A \to aAb$$
$$A \to \lambda$$

A Non-Linear Grammar

Grammar G :

$$S \to SS$$

$$S \to \lambda$$

$$S \to aSb$$

$$S \to bSa$$

$$L(G) = \{w: \; n_a(w) = n_b(w)\}$$

Number of a in string w

Another Linear Grammar

Grammar G :

$$S \to A$$

$$A \to aB \mid \lambda$$

$$B \to Ab$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Right-Linear Grammars

All productions have form: $A \to xB$

or

$A \to x$

string of terminals

Example: $S \to abS$

$S \to a$

Left-Linear Grammars

All productions have form: $A \rightarrow Bx$

or

$A \rightarrow x$

string of terminals

Example:

$$S \rightarrow Aab$$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

Regular Grammars

Regular Grammars

A regular grammar is any
right-linear or left-linear grammar

Examples:

$$G_1$$

$$S \rightarrow abS$$

$$S \rightarrow a$$

$$G_2$$

$$S \rightarrow Aab$$

$$A \rightarrow Aab \mid B$$

$$B \rightarrow a$$

Observation

Regular grammars generate regular languages

Examples:

$$G_1$$

$$S \to abS$$

$$S \to a$$

$$L(G_1) = (ab)^*a$$

$$G_2$$

$$S \to Aab$$

$$A \to Aab \mid B$$

$$B \to a$$

$$L(G_2) = aab(ab)^*$$

Regular Grammars Generate Regular Languages

Theorem

$$\left\{\begin{array}{l}\text{Languages}\\\text{Generated by}\\\text{Regular Grammars}\end{array}\right\} = \left\{\begin{array}{l}\text{Regular}\\\text{Languages}\end{array}\right\}$$

Theorem - Part 1

$$\left\{\begin{array}{l}\text{Languages}\\ \text{Generated by}\\ \text{Regular Grammars}\end{array}\right\} \subseteq \left\{\begin{array}{l}\text{Regular}\\ \text{Languages}\end{array}\right\}$$

Any regular grammar generates a regular language

Theorem - Part 2

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \supseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

Any regular language is generated by a regular grammar

Proof - Part 1

$$\left\{ \begin{array}{l} \text{Languages} \\ \text{Generated by} \\ \text{Regular Grammars} \end{array} \right\} \subseteq \left\{ \begin{array}{l} \text{Regular} \\ \text{Languages} \end{array} \right\}$$

The language $L(G)$ generated by any regular grammar G is regular

The case of Right-Linear Grammars

Let G be a right-linear grammar

We will prove: $L(G)$ is regular

Proof idea: We will construct NFA M with $L(M) = L(G)$

Grammar G is right-linear

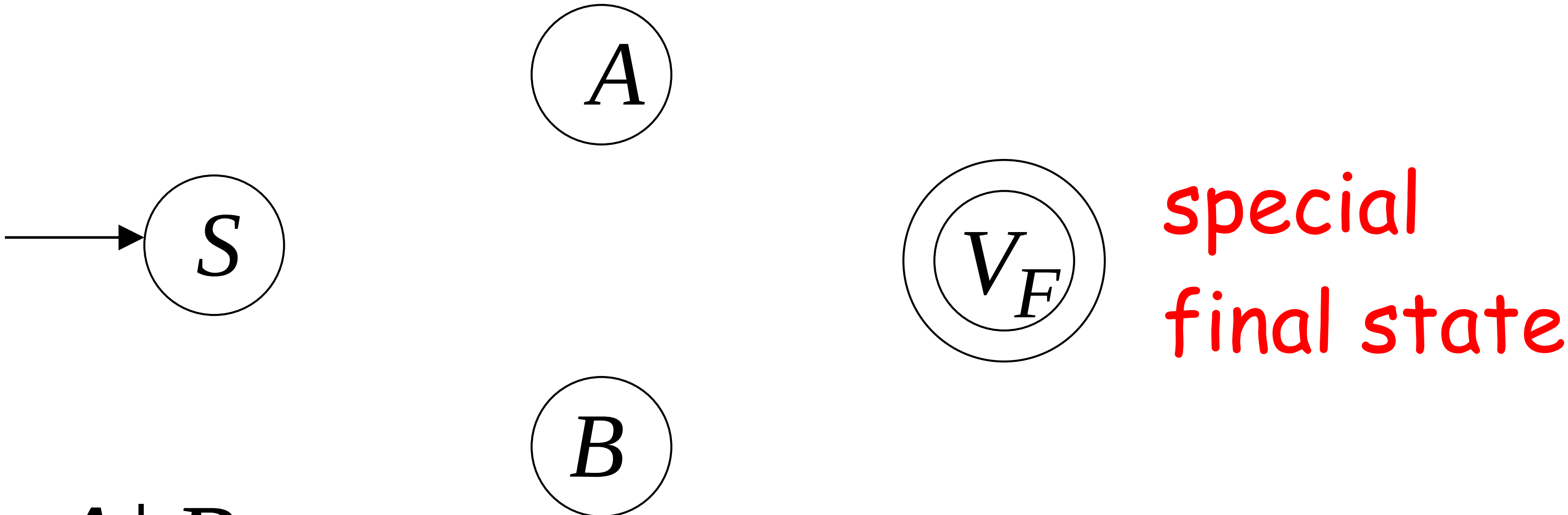
Example:

$$S \rightarrow aA \mid B$$

$$A \rightarrow aa\, B$$

$$B \rightarrow b\, B \mid a$$

Construct NFA M such that every state is a grammar variable:

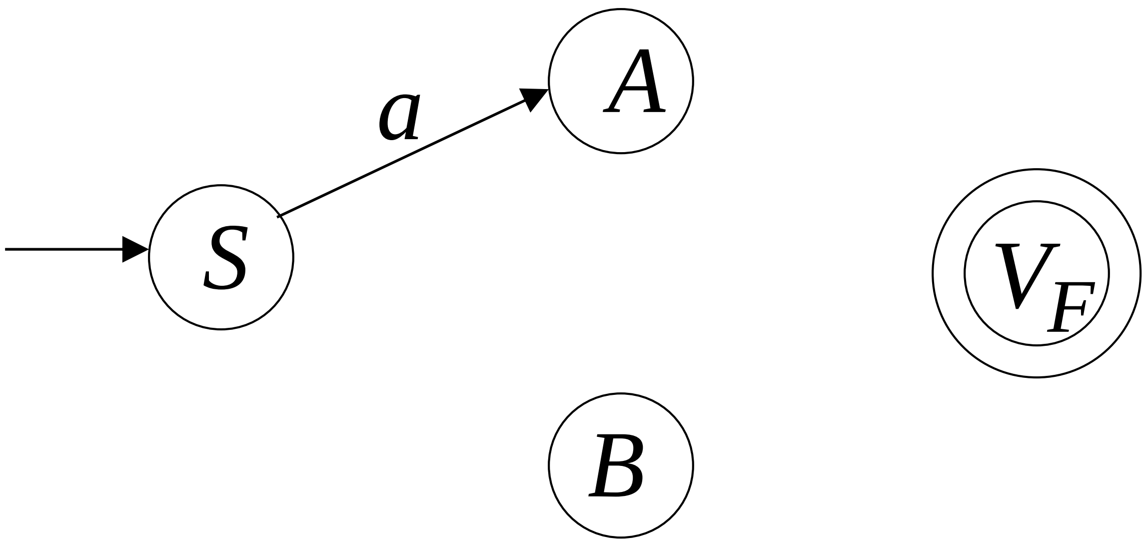


$$S \to aA \mid B$$

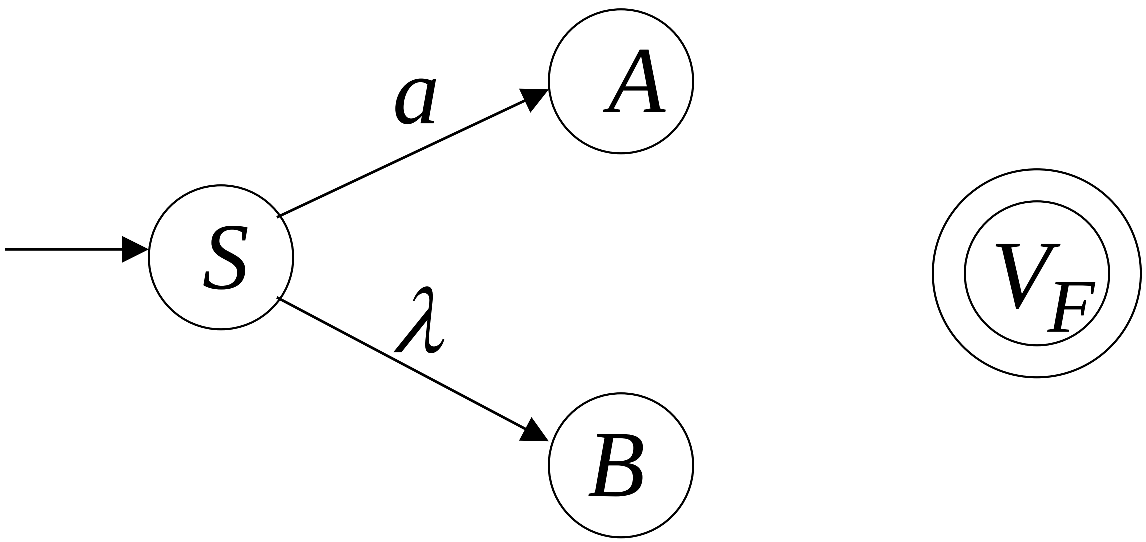
$$A \to aa\,B$$

$$B \to b\,B \mid a$$

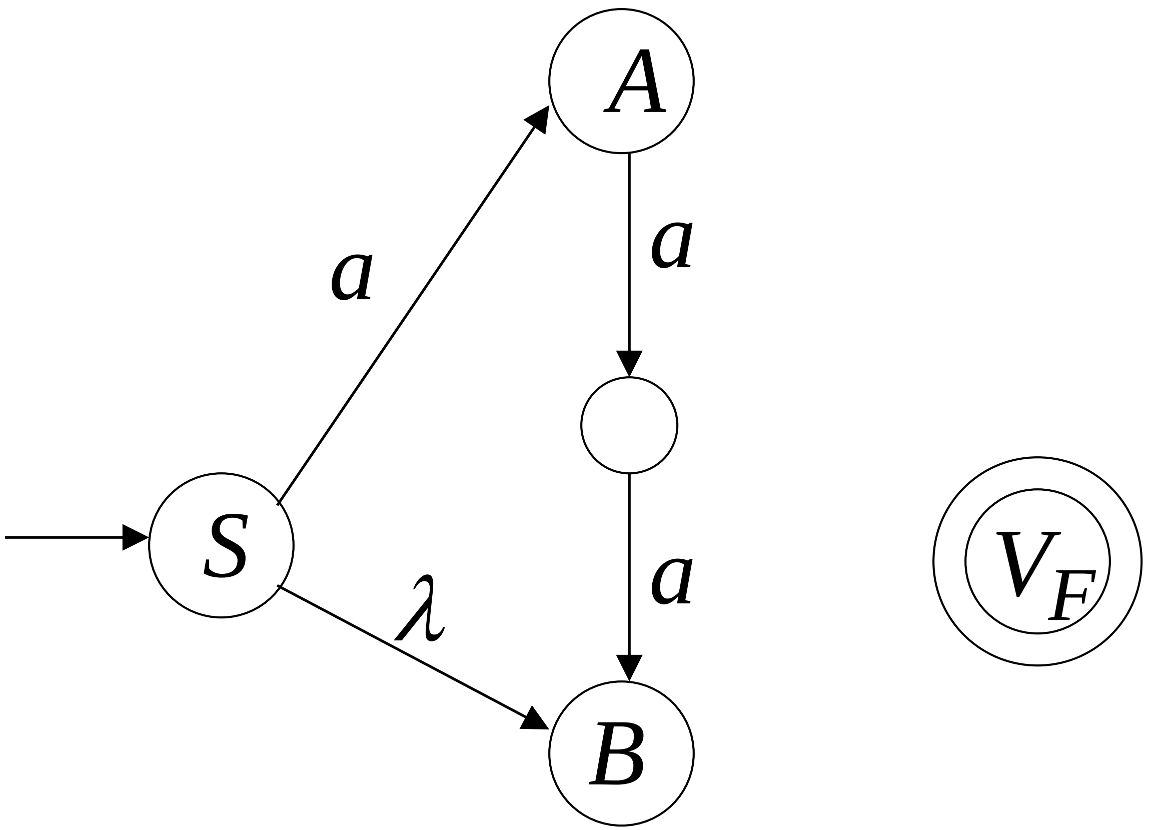
Add edges for each production:



$$S \to aA$$

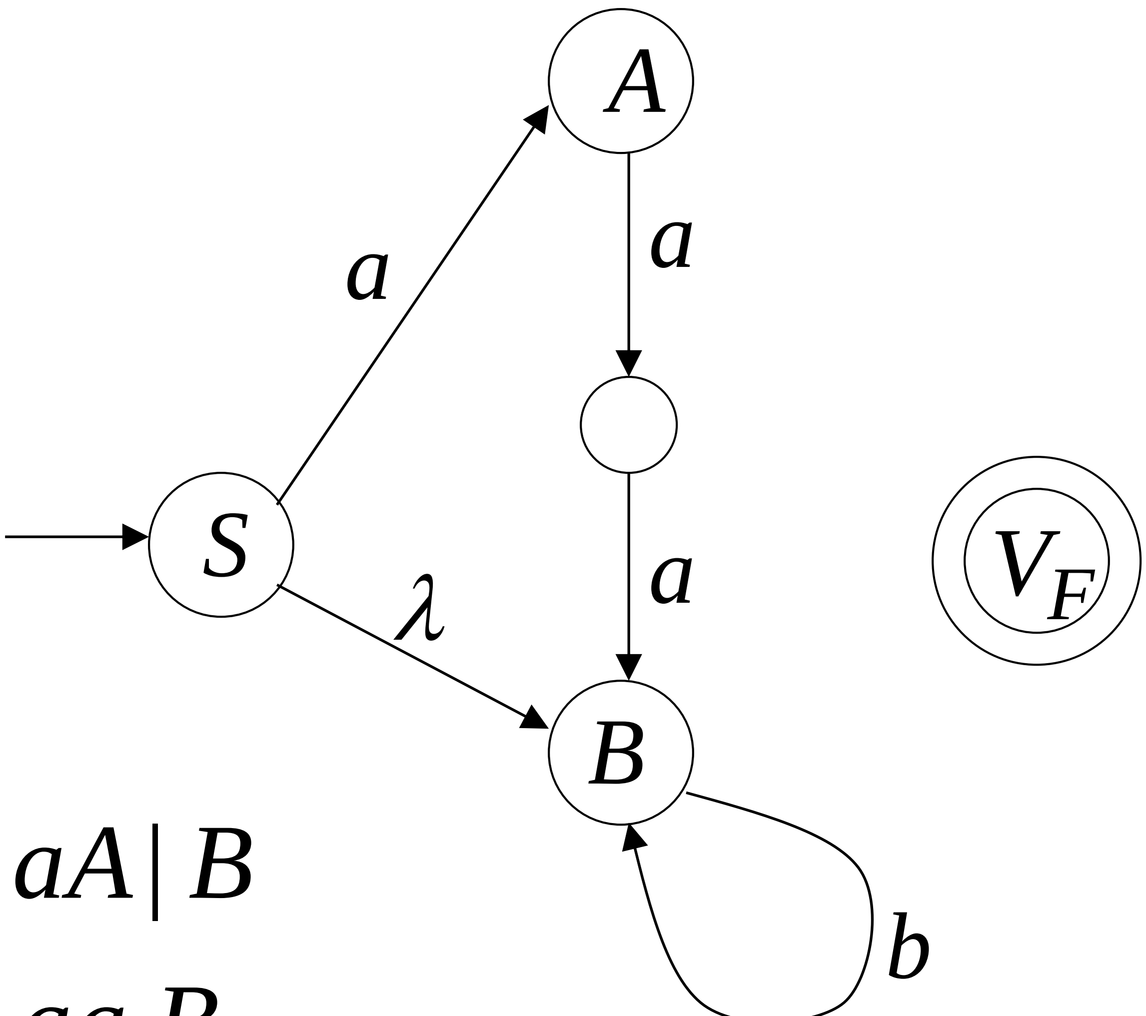


$$S \to aA \mid B$$



$$S \to aA \mid B$$

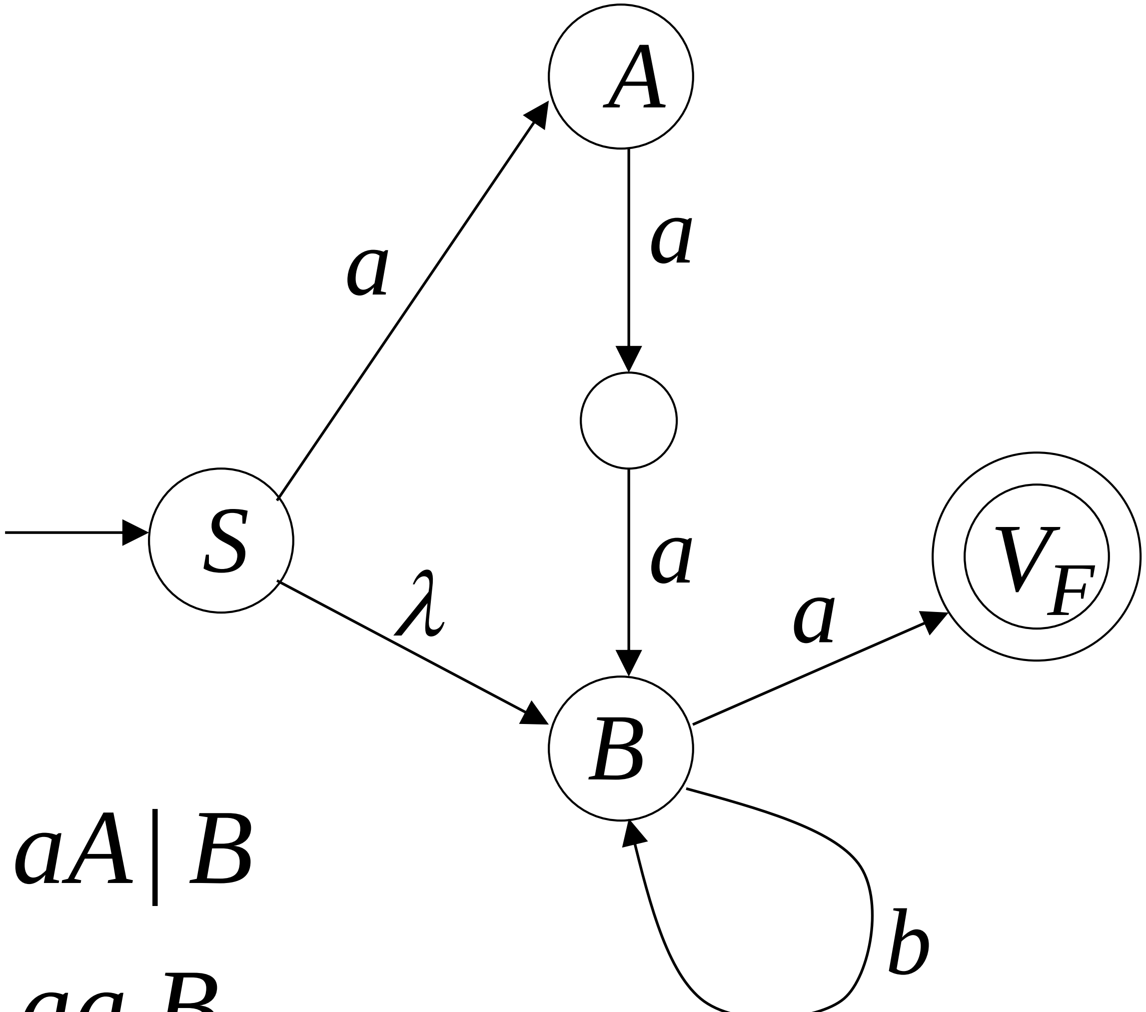
$$A \to aa\ B$$



$$S \to aA \mid B$$

$$A \to aa\,B$$

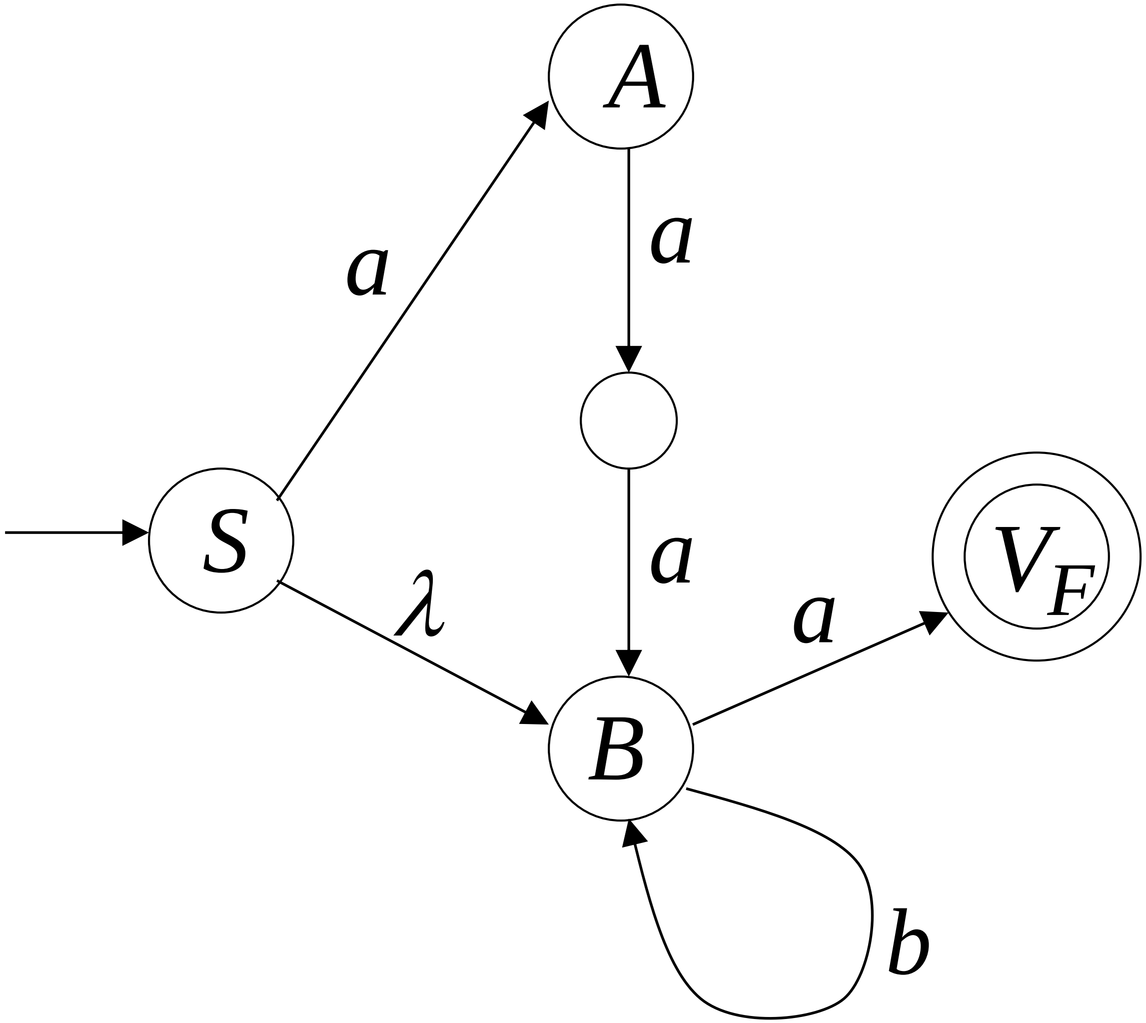
$$B \to bB$$



$$S \to aA \mid B$$

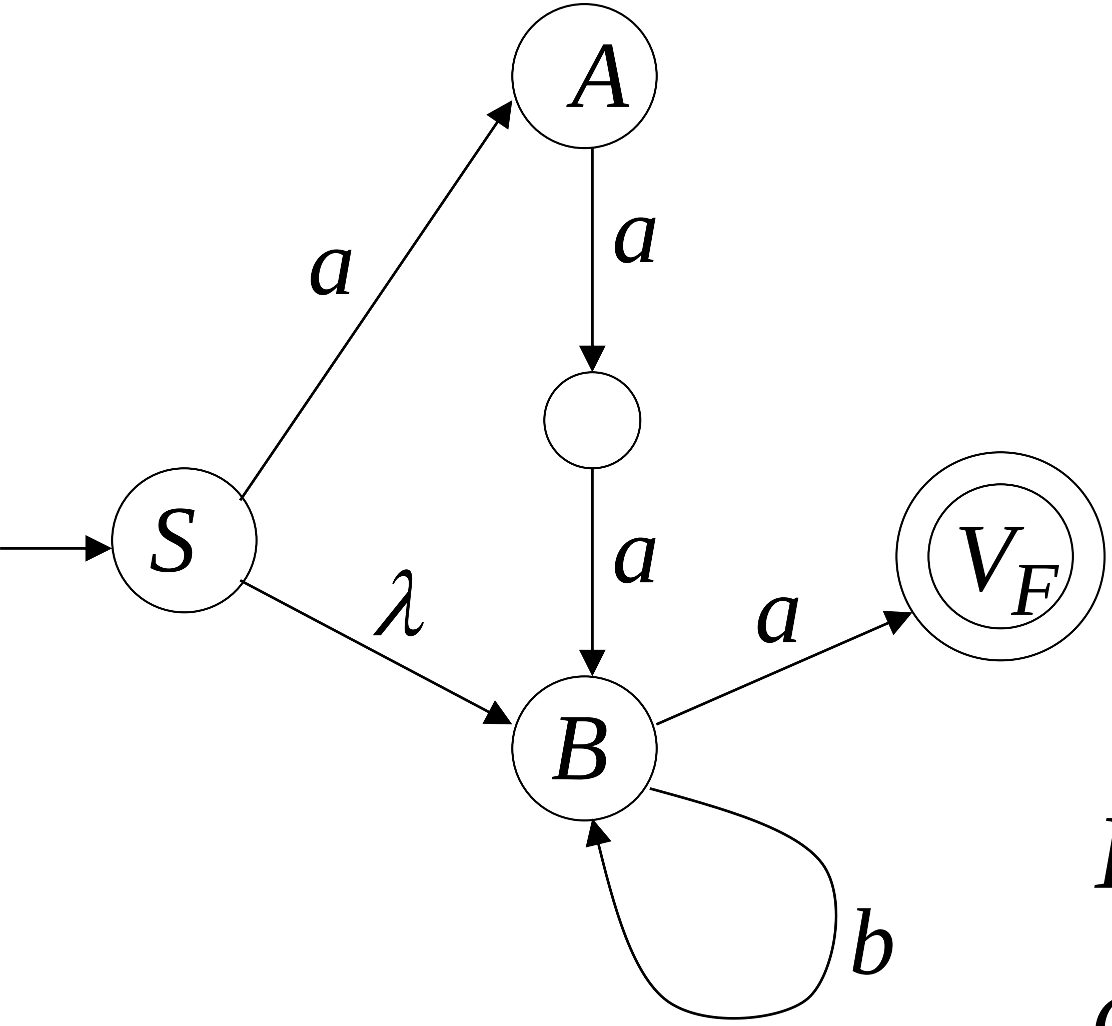
$$A \to aa\,B$$

$$B \to bB \mid a$$



$$S \Rightarrow aA \Rightarrow aaaB \Rightarrow aaabB \Rightarrow aaaba$$

NFA M



Grammar

G

$$S \to aA \mid B$$

$$A \to aa\,B$$

$$B \to bB \mid a$$

$$L(M) = L(G) =$$

$$aaab^*a + b^*a$$

In General

A right-linear grammar G

has variables: $V_0, V_1, V_2, \ldots$

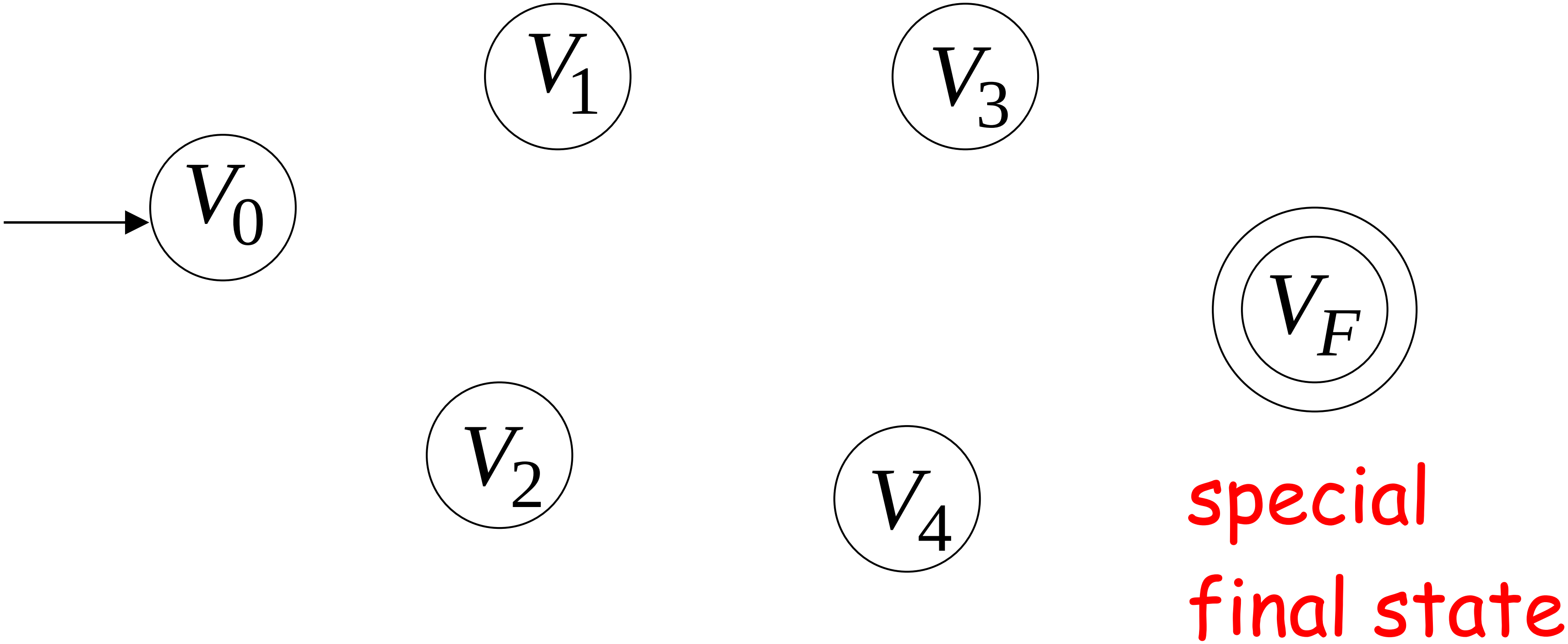
and productions: $V_i \to a_1 a_2 \cdots a_m V_j$

or

$V_i \to a_1 a_2 \cdots a_m$

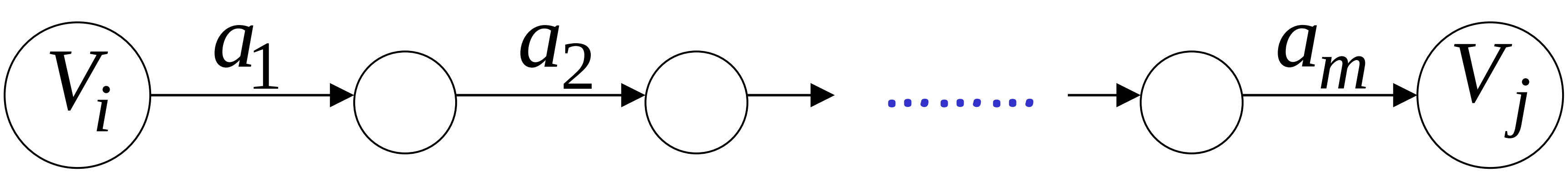
We construct the NFA M such that:

each variable V_i corresponds to a node:



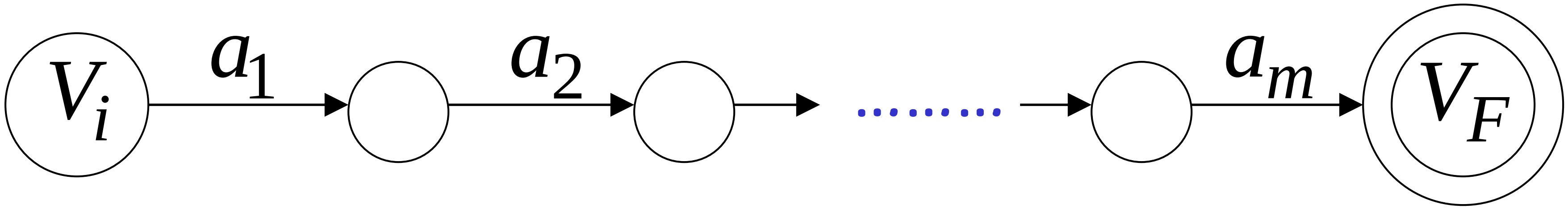
For each production: $V_i \to a_1 a_2 \cdots a_m V_j$

we add transitions and intermediate nodes

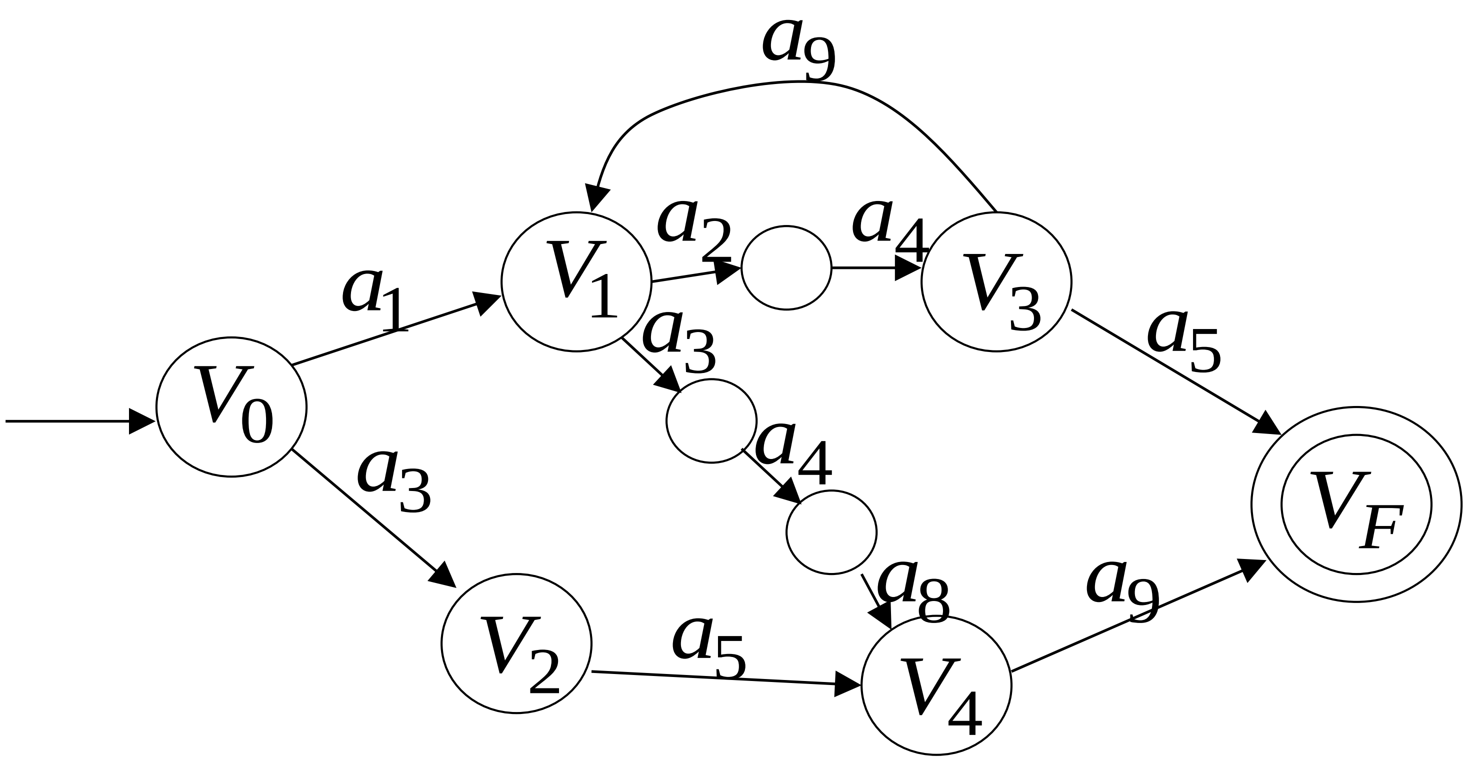


For each production: $V_i \to a_1 a_2 \cdots a_m$

we add transitions and intermediate nodes



Resulting NFA M looks like this:



It holds that: $L(G) = L(M)$

The case of Left-Linear Grammars

Let G be a left-linear grammar

We will prove: $L(G)$ is regular

Proof idea:

We will construct a right-linear grammar G' with $L(G) = L(G')^R$

Example

$$C \to Bc$$

$$B \to Ab$$

$$A \to a$$

The derivation of abc is: $C \to Bc \to Abc \to abc$

Equivalently,

$$C \to cB$$

$$B \to bA$$

$$A \to a$$

$C \to cB \to cbA \to cba$ and then take the reverse.

It is easy to see that: $L(G) = L(G')^R$

Since G' is right-linear, we have:

$L(G')$	$\Rightarrow$	$L(G')^R$	$\Rightarrow$	$L(G)$
Regular Language		Regular Language		Regular Language

Since G is left-linear grammar the productions look like:

$$A \to Ba_1a_2\cdots a_k$$

$$A \to a_1a_2\cdots a_k$$

Construct right-linear grammar G'

Left linear G

$$A \to Ba_1a_2\cdots a_k$$

$$A \to Bv$$

Right linear G'

$$A \to a_k\cdots a_2a_1B$$

$$A \to v^R B$$

Construct right-linear grammar G'

Left linear G

$$A \to a_1 a_2 \cdots a_k$$

$$A \to v$$

$$\Downarrow$$

Right linear G'

$$A \to a_k \cdots a_2 a_1$$

$$A \to v^R$$

Proof - Part 2

$$\left\{\begin{array}{l}\text{Languages}\\\text{Generated by}\\\text{Regular Grammars}\end{array}\right\} \supseteq \left\{\begin{array}{l}\text{Regular}\\\text{Languages}\end{array}\right\}$$

Any regular language L is generated by some regular grammar G

Any regular language L is generated by some regular grammar G

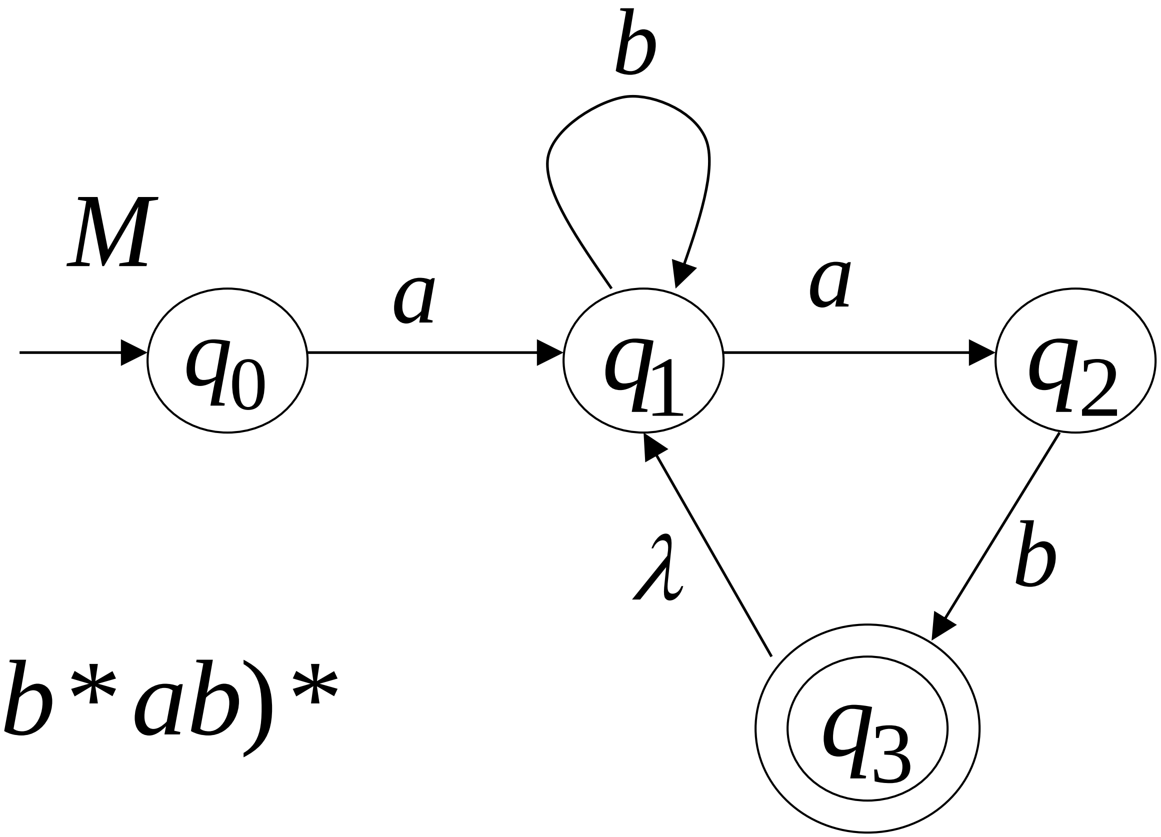
Proof idea:

Let M be the NFA with $L = L(M)$.

Construct from M a regular grammar G such that $L(M) = L(G)$

Since L is regular
there is an NFA M such that $L = L(M)$

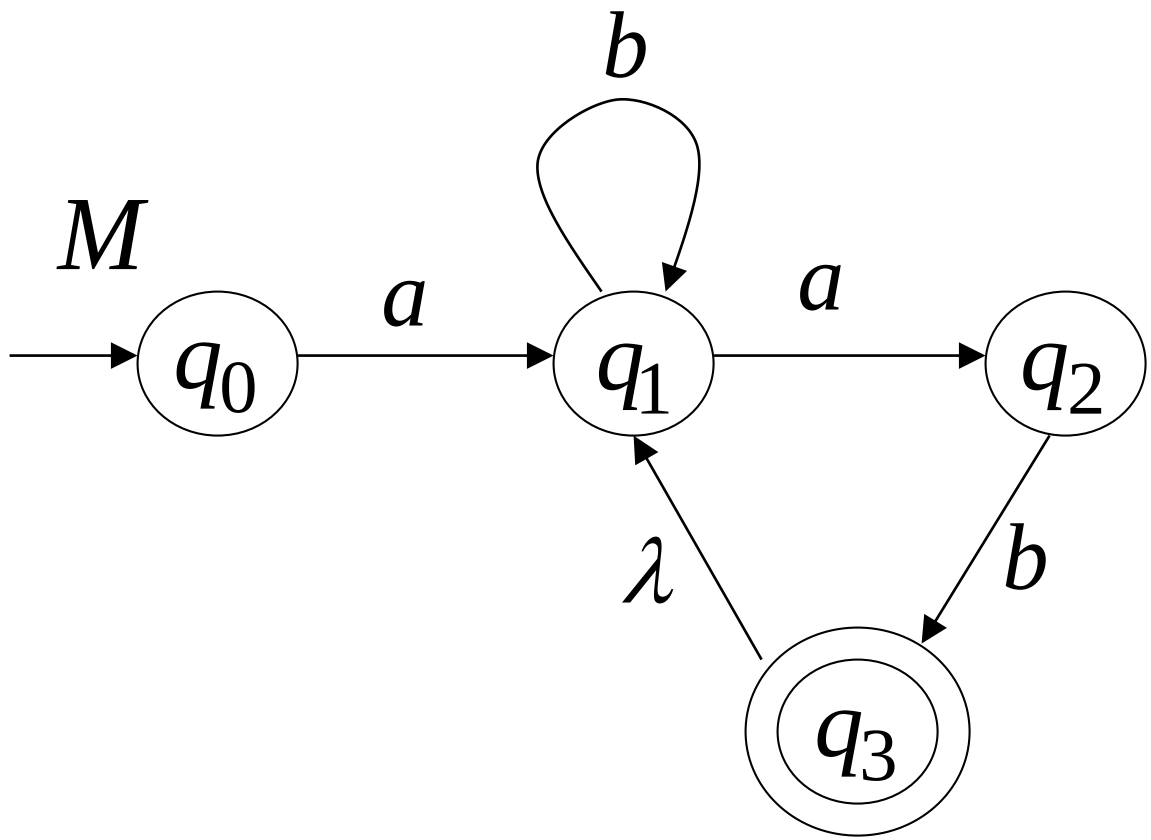
Example:



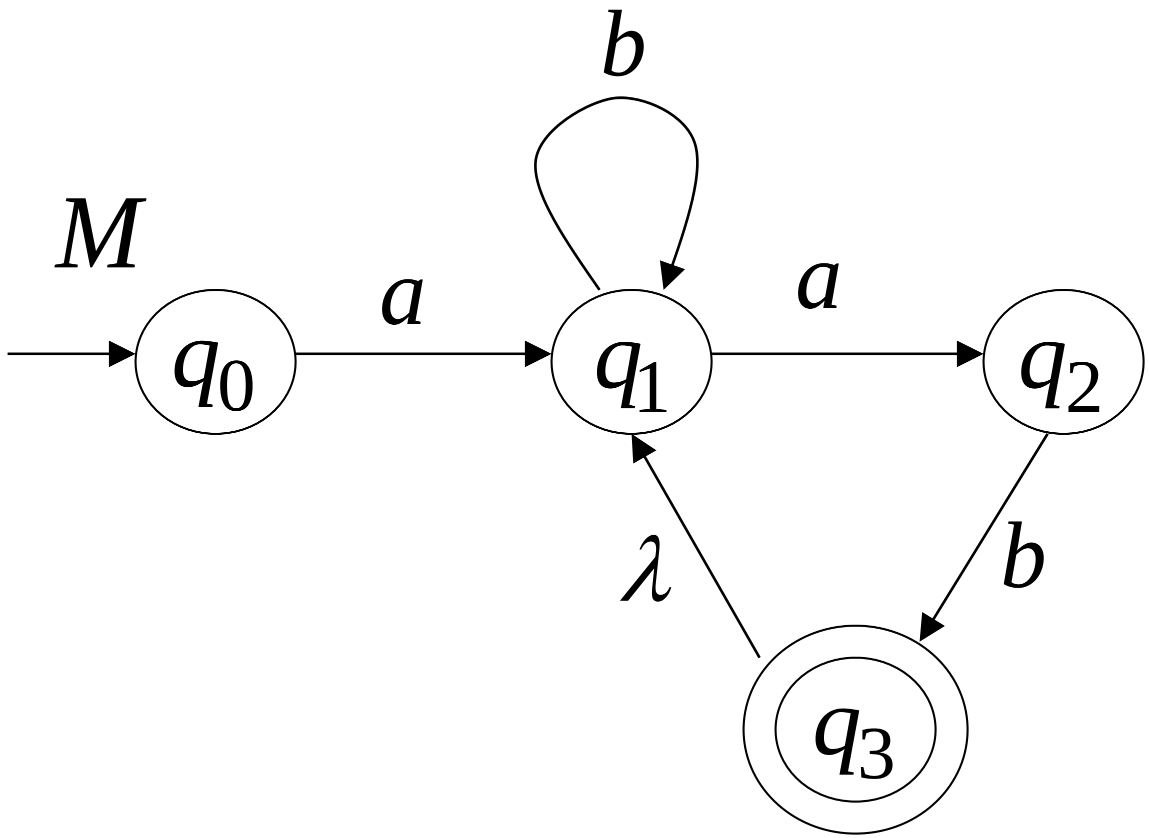
$$L = ab^*ab(b^*ab)^*$$

$$L = L(M)$$

Convert M to a right-linear grammar



$$q_0 \rightarrow aq_1$$



$$q_0 \to aq_1$$

$$q_1 \to bq_1$$

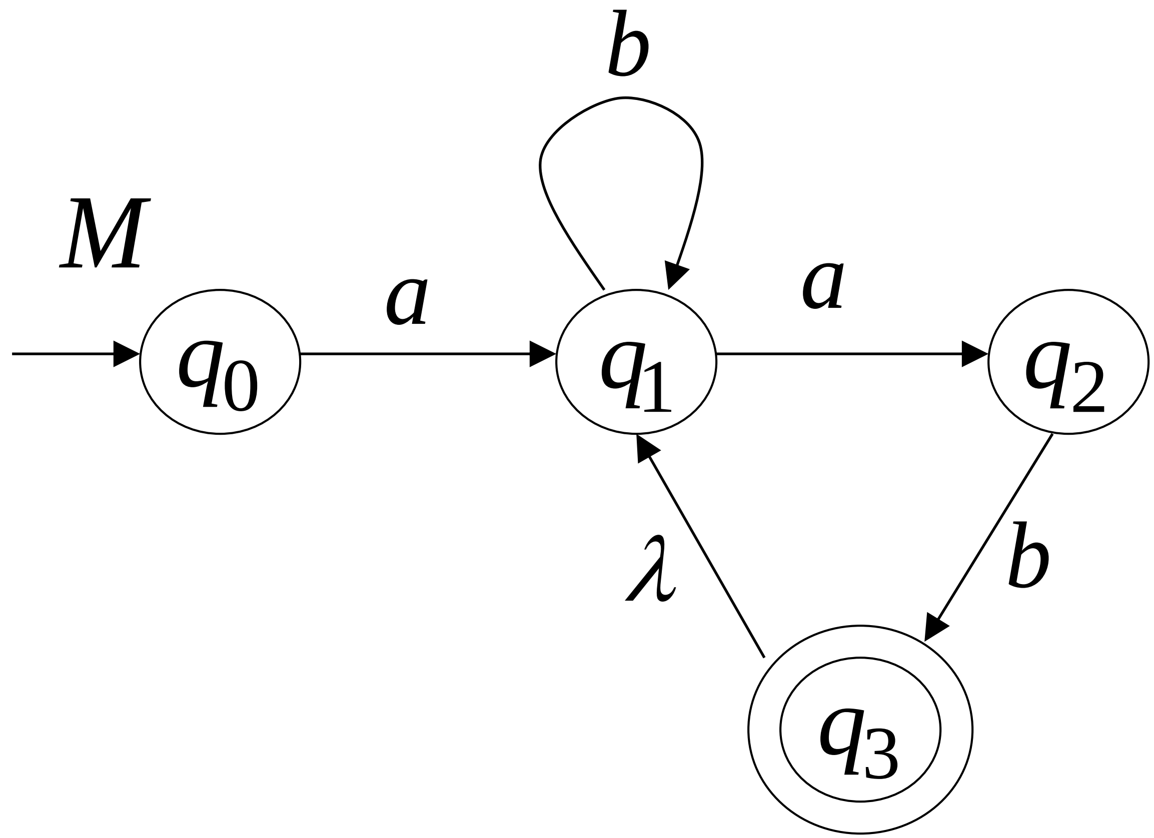
$$q_1 \to aq_2$$

$$q_0 \rightarrow aq_1$$

$$q_1 \rightarrow bq_1$$

$$q_1 \rightarrow aq_2$$

$$q_2 \rightarrow bq_3$$



$$L(G) = L(M) = L$$

G

$$q_0 \rightarrow a q_1$$

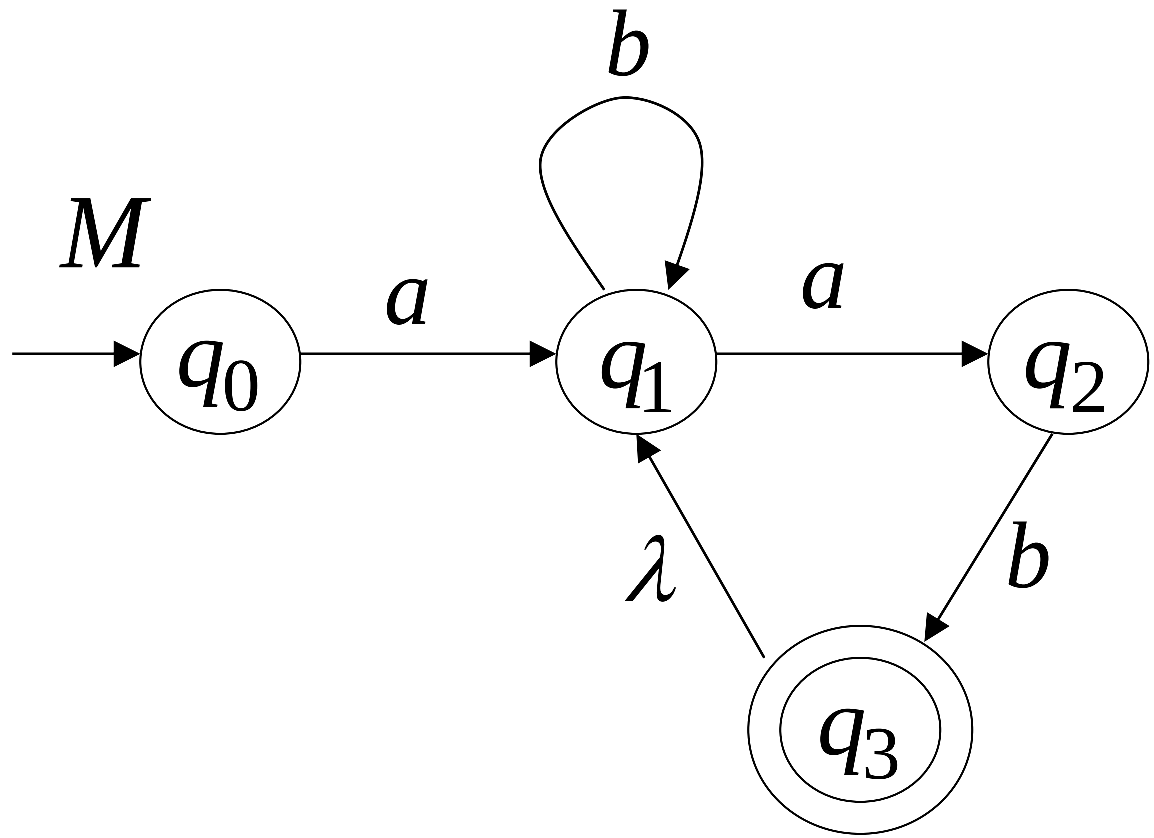
$$q_1 \rightarrow b q_1$$

$$q_1 \rightarrow a q_2$$

$$q_2 \rightarrow b q_3$$

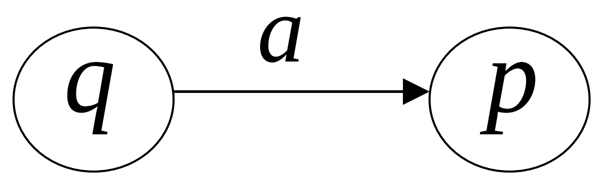
$$q_3 \rightarrow q_1$$

$$q_3 \rightarrow \lambda$$



In General

For any transition:

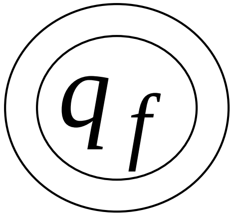


Add production:

$$q \to ap$$

variable   terminal   variable

For any final state:



Add production:

$$q_f \to \lambda$$

Since G is right-linear grammar

G is also a regular grammar

with $L(G) = L(M) = L$