Fourier Series:

Fourier series are infinite series that represent periodic functions in terms of cosines and sines.

Basic tools

Periodic function:

If $f(x+p) = f(x)$, p is positive real no.

then $f(x)$ is called periodic function with period p.

Ex. $\tan x$, $\sin x$, $\cos x$, $[x]-x$

* Let $f(x)$ is a given function of period 2π. Then the fourier series representation of the function is given by,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Here, a_ns and b_ns are called fourier coefficients

* Basic integral formulas:

i.] $\int_{-\pi}^{\pi} \sin mx\,dx = 0$

ii.] $\int_{-\pi}^{\pi} \cos mx\,dx = 0$

iii] $\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \pi\,\delta_{mn}$

iv.] $\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \pi\,\delta_{mn}$

v] $\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0$

~~2.~~ fourier series in ~~General~~ $[-L, L]$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where,
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

* Convergence of a fourier series:

Dirichlet conditions:

1. $f(x)$ must be absolutely integrable over a period
2. $f(x)$ must have a finite no of discontinuities in any given iterval
3. $f(x)$ must have a finite number of extrema over the interval.

If a function $f(x)$ having period 2π satisfies Dirichlet conditions in the interval $[-\pi, \pi]$, then it can be expaned in a fourier series which converges to the function at continuous points and mean of the positive and negative limits at points of discontinuity.

i.e. if x_0 be a point of discontinuity, ~~then, fo~~ of $f(x)$, then ~~f(x) converges to~~ fourier series of $f(x)$ converges to

$$\frac{f(x_0+) + f(x_0-)}{2}.$$

Problem set 11.1

3. Given, $f(x+p) = f(x)$

$g(x+p) = g(x)$

$h(x+p) = af(x+p) + bg(x+p) = af(x) + bg(x)$

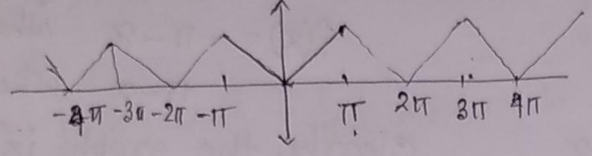
$= h(x)$

$\therefore$ $h(x)$ is also periodic with period p.

12. $f(x) = |x|, \quad -\pi < x < \pi, \quad$ with $f(x+2\pi) = f(x)$

i.e. $f(x) = -x \qquad -\pi < x < 0$

$\qquad = x \qquad 0 \le x < \pi$



Let the Fourier series expansion of $f(x)$ be

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where,
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right] = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - \int\frac{\sin nx}{n}\,dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

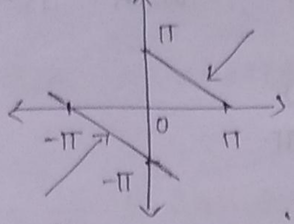
$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin nx\,dx = 0$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left[(-1)^n - 1\right]\cos nx$$

$$= \frac{\pi}{2} + \frac{-4}{\pi}\cos x + \frac{-4}{9\pi}\cos 3x + \frac{-4}{25\pi}\cos 5x + \cdots$$

21.



$\frac{x}{\pi} + \frac{y}{\pi} = 1$

or, $x + y = \pi$

$\Rightarrow y = \pi - x$

$\frac{x}{-\pi} + \frac{y}{-\pi} = 1$

or $x + y = -\pi$

or $y = -\pi - x$

$\therefore$ Equation of the graph

$f(x) = -\pi - x$ when $-\pi < x < 0$

$= \pi - x$ when $0 < x < \pi$

Clearly, the graph is defined in $(-\pi, \pi)$. To make it periodic with period 2π, we need to assert that $f(x+2\pi) = f(x)$.

The function $f(x)$ is discontinuous at $x = 0$.

Let the Fourier series representation of $f(x)$ is,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Then, $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{0}(-\pi - x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= -\frac{1}{2\pi}\left[+\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= 0$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-\pi - x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\,dx$$

$$= -\int_{-\pi}^{0}\cos nx\,dx - \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= \int_{0}^{-\pi}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{-\pi} x\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$= -\int_{0}^{\pi}\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$= 0$ for $n = 1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{0} (-\pi - x) \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x) \sin nx\,dx$$

$$= -\int_{-\pi}^{0} \sin nx\,dx - \frac{1}{\pi}\int_{-\pi}^{0} x \sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\,dx$$

$$= \int_{0}^{-\pi} \sin nx\,dx + \frac{1}{\pi}\int_{0}^{-\pi} x \sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\,dx$$

$$= \int_{0}^{\pi} \sin nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x \sin nx\,dx + \int_{0}^{\pi} \sin nx\,dx - \frac{1}{\pi}\int_{0}^{\pi} x \sin x\,dx$$

$$= 2\int_{0}^{\pi} \sin nx\,dx - \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\,dx$$

$$= 2\left[-\frac{\cos nx}{n}\right]_0^{\pi} - \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= 2\left[-\frac{\cos n\pi}{n} + \frac{1}{n}\right] - \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]$$

$$= -\frac{2\cos n\pi}{n} + \frac{2}{n} + \frac{2\cos n\pi}{n}$$

$$= \frac{2}{n}, \quad \text{for } n = 1, 2, 3, \ldots$$

$$\therefore\ f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

$$= 2\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots\right] \longrightarrow (1)$$

23. In the previous prob. the function $f(x)$ is discontinuous at $x = 0$.

Now, At $x = 0$, RHS of (1) is 0 and LHS will be $f(0)$.

But $f(0)$ is not defined.

So, to get the value 0 in LHS, we need to calculate

$$\frac{f(0+) + f(0-)}{2} = \frac{\lim_{x\to 0^+} f(x) + \lim_{x\to 0^-} f(x)}{2} = \frac{\lim_{x\to 0^+}(+\pi - x) + \lim_{x\to 0^-}(-\pi - x)}{2}$$

$$= \frac{\pi - \pi}{2} = 0$$

$\therefore$ Always at the point of discontinuity, the series converges to average of left limit and right limit at that point.

* Fourier series in $[-L, L]$ i.e. of period $2L$

The Fourier series of a function $f(x)$ defined in $[-L, L]$ with the period $2L$ is represented as,

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

* Fourier series of Even and Odd functions:

If $f(x)$ is an even function, then its Fourier series reduces to a Fourier cosine series as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where,
$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

If $f(x)$ is an odd function, then its Fourier series reduces to a Fourier sine series as

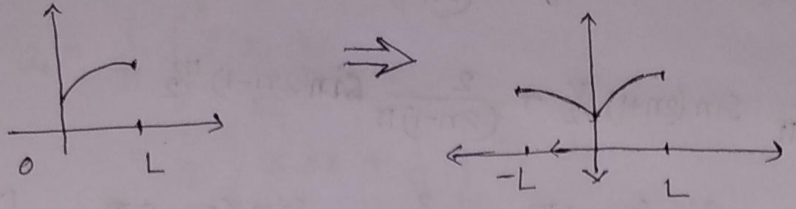
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where,
$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

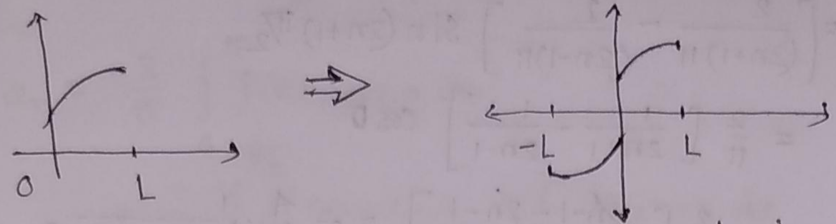
* Half-Range Expansions:

A half-range Fourier series is a Fourier series which is generally defined on an interval $[0, L]$ instead of taking more common interval $[-L, L]$.

In half range Fourier series, a function $f(x)$ which is defined in $[0, L]$ is extended to even periodic function or odd periodic function in the interval $[-L, L]$.

$f(x) \to$ 

Even extension

$f(x) \to$ 

Odd extension

So, basically half-range ~~fw~~ expansions of a functions are ~~basically~~ Fourier sine series (for odd extension) and Fourier cosine series (for even extension).

Problem set 11.2

14. $f(x) = \cos \pi x \qquad -\frac{1}{2} < x < \frac{1}{2}, \quad p = 1$

The given function is an even function with period 1. $\therefore L = \frac{1}{2}$ and the Fourier series of it will be a Fourier ~~cone~~ cosine series.

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\frac{1}{2}} = a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

where

$$a_0 = \frac{1}{\frac{1}{2}} \int_0^{\frac{1}{2}} f(x)\,dx = 2 \int_0^{\frac{1}{2}} \cos \pi x\,dx = \frac{2}{\pi}\left[\sin \pi x\right]_0^{\frac{1}{2}}$$
$$= \frac{2}{\pi}\left[\sin \frac{\pi}{2}\right] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\frac{1}{2}} \int_0^{\frac{1}{2}} f(x) \cos 2n\pi x\,dx = 4 \int_0^{\frac{1}{2}} \cos \pi x \cos 2n\pi x\,dx$$
$$= 2 \int_0^{\frac{1}{2}} \left[\cos (2n+1)\pi x + \cos (2n-1)\pi x\right] dx$$

$$= 2\int_0^{1/2} \cos(2n+1)\pi x\,dx + 2\int_0^{1/2} \cos(2n-1)\pi x\,dx$$

$$= \frac{2}{(2n+1)\pi}\Big[\sin(2n+1)\pi x\Big]_0^{1/2} + \frac{2}{(2n-1)\pi}\Big[\sin(2n-1)\pi x\Big]_0^{1/2}$$

$$= \frac{2}{(2n+1)\pi}\sin(2n+1)\pi/2 + \frac{2}{(2n-1)\pi}\sin(2n-1)\pi/2$$

$$= \frac{2}{(2n+1)\pi}\sin(2n+1)\pi/2 - \frac{2}{(2n-1)\pi}\sin(2n+1)\pi/2 \qquad [\because \sin(\pi+x) = -\sin x]$$

$$= \left[\frac{2}{(2n+1)\pi} - \frac{2}{(2n-1)\pi}\right]\sin(2n+1)\pi/2$$

$$= \frac{2}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right] \cancel{\cos 0}\ \cos n\pi$$

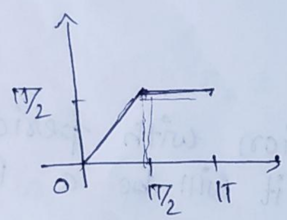
$$= \frac{2}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{(2n+1)(2n-1)}\right] = -\frac{4}{\pi}\frac{(-1)^n}{(2n+1)(2n-1)} = \frac{4}{\pi}\frac{(-1)^{n+1}}{(2n+1)(2n-1)}$$

$\therefore$ The fourier series representation is

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} + \frac{4}{\pi}\frac{(-1)^{n+1}}{(2n+1)(2n-1)}\cos 2n\pi x$$

$$= \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{1}{3\cdot 1}\cos 2\pi x - \frac{1}{5\cdot 3}\cos 4\pi x + \frac{1}{7\cdot 5}\cos 6\pi x - \dots\right]$$

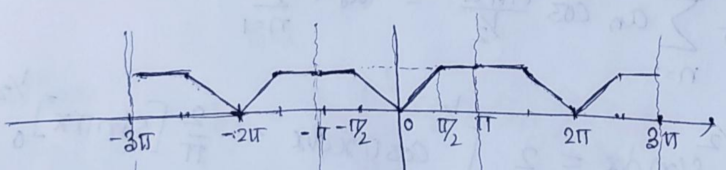
26.



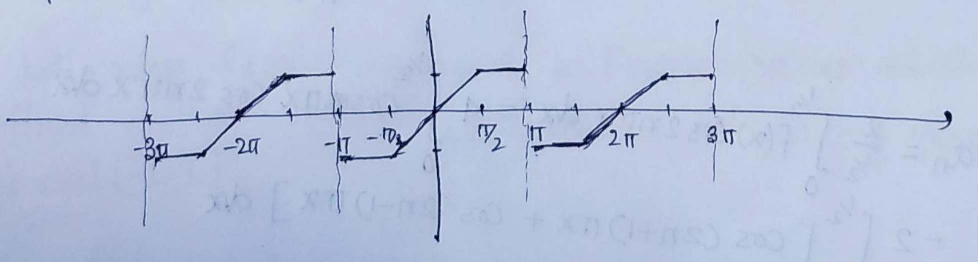
Eq^n of the graph

$$f(x) = x \qquad 0 \le x < \pi/2$$
$$= \pi/2 \qquad \pi/2 \le x \le \pi$$

Even periodic extension:



Odd periodic extension:



(a) To find Fourier cosine series we consider

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

Where,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} x\,dx + \frac{1}{\pi}\int_{\pi/2}^{\pi} \pi/2\,dx$$

$$= \frac{1}{\pi}\times\frac{\pi^2}{8} + \frac{1}{\pi}\cdot\frac{\pi}{2}\cdot\frac{\pi}{2} = \pi/8 + \pi/4 = \frac{3\pi}{8}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\cos nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi/2} + \left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{\cos n\pi/2}{n^2} - \frac{1}{n^2}\right] \cancel{+} - \frac{\sin n\pi/2}{n}$$

$$= \frac{1}{n}\cancel{\sin}\frac{n\pi}{2} + \frac{2}{\pi n^2}\cos\frac{n\pi}{2} - \frac{1}{n}\cancel{\sin}\frac{n\pi}{2} - \frac{2}{\pi n^2}$$

$$= \frac{2}{\pi n^2}\cos\frac{n\pi}{2} - \frac{2}{\pi n^2}$$

$\therefore$

$$f(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos nx$$

$$= \frac{3\pi}{8} + \frac{2}{\pi}\left[\cancel{-\frac{1}{2^2}\cos 2x} \cancel{+} \cancel{\frac{1}{4^2}\cos 4x} - \cancel{\frac{1}{6^2}\cos 6x} \cancel{+} \cdots\right]$$

$$= \frac{3\pi}{8} + \frac{2}{\pi}\left[-\cos x - \frac{1}{2}\cos 2x - \frac{1}{3^2}\cos 3x - \frac{2}{5^2}\cos 5x - \cdots\right]$$

(b) For Fourier sine series we consider,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\,dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}\frac{\pi}{2}\sin nx\,dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi/2} + \left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin\frac{n\pi}{2}}{n^2}\right] + \frac{\cos\frac{n\pi}{2}}{n} - \frac{\cos n\pi}{n}$$

$$= -\frac{1}{n}\cancel{\cos\frac{n\pi}{2}} + \frac{2}{\pi n^2}\sin\frac{n\pi}{2} + \frac{1}{n}\cancel{\cos\frac{n\pi}{2}} - \frac{1}{n}\cos n\pi$$

$$= \frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{(-1)^n}{n}$$

$\therefore$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2}{\pi n^2}\sin\frac{n\pi}{2} - \frac{(-1)^n}{n}\right\}\sin nx$$

$$= \left(\frac{2}{\pi}+1\right)\sin x - \frac{1}{2}\sin 2x + \left(-\frac{2}{9\pi}+\frac{1}{3}\right)\sin 3x - \frac{1}{4}\sin 4x + \cdots\cdots$$