2.10 Solution by Variation of Parameters

The second order nonhomogeneous linear differential equation is given by

$$y'' + p(x)y' + q(x)y = r(x)$$

Here the functions $p(x)$, $q(x)$ and $r(x)$ are any given continuous functions of x or may be constants.

General solution

The ***general solution*** of nonhomogeneous linear differential equation is given by

$$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$$

Method of Variation of Parameters

This method is used to determine the Particular Integral of nonhomogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = r(x)$$

In this method the constants in the corresponding homogeneous equation (Complementary Function) are considered to be functions of the independent variable.

Particular Integral by Variation of Parameters method

Let's consider a second order linear nonhomogeneous differential equation be

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

Let the Complementary Function be

$$y_h(x) = c_1 y_1 + c_2 y_2 \tag{2}$$

In order to apply this method, let's consider c_1 and c_2 in (2) as functions of x.
Therefore,

$$y_p(x) = c_1(x)y_1 + c_2(x)y_2 \tag{3}$$

Now we have to determine $c_1(x)$ and $c_2(x)$ so that (3) will be a solution of Equation (1).

If y_p happens to be a solution of Equation (1), then it must satisfy Equation (1).

$$\Rightarrow y_p' = [c_1(x)y_1' + c_2(x)y_2'] + [c_1'(x)y_1 + c_2'(x)y_2] \tag{4}$$

To keep y_p' in the same form as (3), we have to assume

$$c_1'(x)y_1 + c_2'(x)y_2 = 0 \tag{5}$$

Now expression (4) reduces to

$$\Rightarrow y_p' = c_1(x)y_1' + c_2(x)y_2'$$

$$\Rightarrow y_p'' = c_1(x)y_1'' + c_1'(x)y_1' + c_2(x)y_2'' + c_2'(x)y_2'$$

Substituting y_p, y_p' and y_p'' in Equation (1), we get

$$c_1(x)y_1'' + c_1'(x)y_1' + c_2(x)y_2'' + c_2'(x)y_2' + p(x)[c_1(x)y_1' + c_2(x)y_2'] + q(x)[c_1(x)y_1 + c_2(x)y_2] = r(x)$$

$$\Rightarrow c_1(x)[y_1'' + p(x)y_1' + q(x)y_1] + c_2(x)[y_2'' + p(x)y_2 + q(x)y_2] + c_1'(x)y_1' + c_2'(x)y_2' = r(x)$$

Since y_1 and y_2 are solutions of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = r(x)$$

So, $y_1'' + p(x)y_1' + q(x)y_1 = 0$ and $y_2'' + p(x)y_2 + q(x)y_2 = 0$

$$\Rightarrow c_1'(x)y_1' + c_2'(x)y_2' = r(x) \tag{6}$$

Solving Equations (5) and (6) we obtain

$$c_1(x) = -\int \frac{y_2 r(x)}{W(y_1,\ y_2)}dx \quad \text{and} \quad c_2(x) = \int \frac{y_1 r(x)}{W(y_1,\ y_2)}dx$$

Substituting $c_1(x)$ and $c_2(x)$ in (3) we get

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\ y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\ y_2)} dx$$

Hence by ***Variation of Parameters method*** the Particular Integral is calculated by the formula

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\ y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\ y_2)} dx$$

Solution of a 2nd order Nonhomogeneous ODE by Method of Variation of Parameters

The general solution of a second order linear Nonhomogeneous ODE $y'' + p(x)y' + q(x)y = r(x)$ can be obtained by Variation of Parameters method as follows:

I. Find the Complementary function by solving the corresponding homogeneous ODE
$y'' + p(x)y' + q(x)y = 0$
C.F. $= y_h(x) = c_1 y_1 + c_2 y_2$

II. Write the basis of solutions $\{y_1,\ y_2\}$.

III. Find the Wronskian $W(y_1,\ y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

IV. Write the given ODE in the standard form of linear nonhomogeneous ODE
$y'' + p(x)y' + q(x)y = r(x)$ and find $r(x)$.

V. Find the Particular Integral using the formula
$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\ y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\ y_2)} dx$$

VI. Find the general solution
$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$

Problem Set-2.10

Q.1-Q.13: Solve the following differential equations by variation of parameters method.

Q.2) $y'' + 9y = \cos ec3x$

Solution: Complementary Function: $y_h(x)$

The corresponding homogeneous ODE is

$$y'' + 9y = 0$$

The Auxiliary equation is $\lambda^2 + 9 = 0 \quad \Rightarrow \lambda = \pm 3i$

Hence the Complementary Function is $y_h(x) = A\cos 3x + B\sin 3x$

Basis of solutions is $\{\cos 3x,\ \sin 3x\} = \{y_1,\ y_2\}$

Wronskian:

$$W(\cos 3x,\ \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3(\cos^2 3x + \sin^2 3x) = 3$$

The given ODE is in the standard form, so $r(x) = \cos ec3x$

Particular Integral: $y_p(x)$

According to method of variation of parameters the P. I. can be obtained by

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\ y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\ y_2)} dx$$

$$\Rightarrow y_p(x) = -\cos 3x \int \frac{\sin 3x \times \cos ec3x}{3} dx + \sin 3x \int \frac{\cos 3x \times \cos ec3x}{3} dx$$

$$\Rightarrow y_p(x) = -\frac{\cos 3x}{3}\int dx + \frac{\sin 3x}{3}\int \frac{\cos 3x}{\sin 3x}dx$$

$$\Rightarrow y_p(x) = -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9}\ln|\sin 3x|$$

General Solution:

The general solution of the 2nd order linear nonhomogeneous ODE is

$$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$$

Hence the required general solution is

$$y(x) = A\cos 3x + B\sin 3x - \frac{x\cos 3x}{3} + \frac{\sin 3x}{9}\ln|\sin 3x|$$

Q.10) $(D^2 + 2D + 2I)y = 4e^{-x}\sec^3 x$

Solution: The given differential equation is

$$y'' + 2y' + 2y = 4e^{-x}\sec^3 x$$

Complementary Function: $y_h(x)$

The corresponding homogeneous differential equation is

$$y'' + 2y' + 2y = 0$$

The Auxiliary equation is

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\Rightarrow (\lambda + 1)^2 = -1 \qquad \Rightarrow \lambda = -1 \pm i$$

The Complementary Function is:

$$y_h(x) = e^{-x}(A\cos x + B\sin x)$$

Basis of Solution is: $\{e^{-x}\cos x,\ e^{-x}\sin x\} = \{y_1,\ y_2\}$

Wronskian:

$$W(e^{-x}\cos x,\ e^{-x}\sin x) = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\cos x - e^{-x}\sin x & -e^{-x}\sin x + e^{-x}\cos x \end{vmatrix} = e^{-2x}$$

Particular Integral: $y_p(x)$

In the given differential equation $r(x) = 4e^{-x}\sec^3 x$

By Variation of parameter method, the Particular is obtained by

$$y_p(x) = -y_1\int \frac{y_2 r(x)}{W(y_1,\ y_2)}dx + y_2\int \frac{y_1 r(x)}{W(y_1,\ y_2)}dx$$

$$\Rightarrow y_p(x) = -e^{-x}\cos x\int \frac{e^{-x}\sin x \times 4e^{-x}\sec^3 x}{e^{-2x}}dx + e^{-x}\sin x\int \frac{e^{-x}\cos x \times 4e^{-x}\sec^3 x}{e^{-2x}}dx$$

$$\Rightarrow y_p(x) = -4e^{-x}\cos x\int \frac{\sin x}{\cos^3 x}dx + 4e^{-x}\sin x\int \sec^2 x dx = -2e^{-x}\sec x\cos 2x$$

General Solution

The general solution is

$$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$$

Required general solution is

$$y(x) = e^{-x}(A\cos x + B\sin x) - 2e^{-x}\sec x\cos 2x = e^{-x}(A\cos x + B\sin x - 2\sec x\cos 2x)$$

Q.12) $(D^2 - I)y = \dfrac{1}{\cosh x}$

Solution: The given differential equation is

$$y'' - y = \frac{1}{\cosh x}$$

Complementary Function: $y_h(x)$

The corresponding homogeneous ODE is

$$y'' - y = 0$$

Auxiliary equation is $\lambda^2 - 1 = 0$ $\quad\Rightarrow \lambda = \pm 1$

Hence the complementary function is

$$y_h(x) = c_1 e^x + c_2 e^{-x}$$

Since e^x and e^{-x} are solutions of $y'' - y = 0$, so $\cosh x$ and $\sinh x$ are also the solutions.

Hence the Complementary Function is

$$y_h(x) = A\cosh x + B\sinh x$$

Basis of solutions is: $\{\cosh x, \sinh x\} = \{y_1, y_2\}$

Wronskian:

$$W(\cosh x,\ \sinh x) = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x = 1$$

Particular Integral: $y_p(x)$

By Variation of Parameters method

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\ y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\ y_2)} dx$$

$$\Rightarrow y_p(x) = -\cosh x \int \frac{\sinh x}{\cosh x} dx + \sinh x \int \frac{\cosh x}{\cosh x} dx = -\cosh x \ln|\cosh x| + x\sinh x$$

General Solution:

The general solution is

$$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$$

Hence the required general solution is

$$y(x) = A\cosh x + B\sinh x - \cosh x \ln|\cosh x| + x\sinh x$$

Q.13) $(x^2 D^2 + xD - 9I)y = 48x^5$

Solution: The given ODE is

$$x^2 y'' + xy' - 9y = 48x^5$$

Complementary Function: $y_h(x)$

The corresponding homogeneous ODE is

$$x^2 y'' + xy' - 9y = 0$$

Here $a = 1$ and $b = -9$

The Auxiliary equation is $m^2 + (a-1)m + b = 0$

$\Rightarrow m^2 - 9 = 0$ $\quad\Rightarrow m = \pm 3$

The Complementary Function is

$$y_h(x) = c_1 x^3 + c_2 x^{-3}$$

Basis of solutions is: $\{x^3,\ x^{-3}\} = \{y_1,\ y_2\}$

Wronskian:

$$W(x^3,\, x^{-3}) = \begin{vmatrix} x^3 & x^{-3} \\ 3x^2 & -3x^{-4} \end{vmatrix} = -3x^{-1} - 3x^{-1} = -6x^{-1}$$

Particular Integral: $y_p(x)$

The given differential equation can be written as

$$y'' + \frac{1}{x} y' - \frac{9}{x^2} y = 48x^3$$

Hence $r(x) = 48x^3$

By Variation of Parameters method

$$y_p(x) = -y_1 \int \frac{y_2 r(x)}{W(y_1,\, y_2)} dx + y_2 \int \frac{y_1 r(x)}{W(y_1,\, y_2)} dx$$

$$\Rightarrow y_p(x) = -x^3 \int \frac{x^{-3} \times 48x^3}{-6x^{-1}} dx + x^{-3} \int \frac{x^3 \times 48x^3}{-6x^{-1}} dx$$

$$\Rightarrow y_p(x) = 8x^3 \int x dx - 8x^{-3} \int x^7 dx = 4x^5 - x^5 = 3x^5$$

General Solution:

The general solution is

$$y(x) = \text{C.F.} + \text{P.I.} = y_h(x) + y_p(x)$$

The required general solution is

$$y(x) = c_1 x^3 + c_2 x^{-3} + 3x^5$$

Course Faculty
Dr. Madhusmita Sahoo

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