The Normal Distribution.

A continious rv X is said to have a normal distribution with parameters μ and σ, where $-\infty < \mu < \infty$ and $\sigma > 0$, if the pdf of X is

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \quad \text{for } -\infty < x < \infty.$$

$X \sim N(\mu, \sigma^2)$

X is normally distributed with parameters μ and σ^2.

Note: The curve of normal distribution is bell shaped and it's symmetric about it's mean value. The curve of normal distribution is always fixed.

Q. 1) Verify proper pdf.

2) $E(X)$ and 3) $V(X)$.

(1) $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = t \Rightarrow \frac{dx}{\sigma} = dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{t^2}{2}} dt$$

Let $\frac{t^2}{2} = u \Rightarrow t^2 = 2u \Rightarrow t = \sqrt{2u}$

$$\Rightarrow dt = \frac{du}{t} = \frac{du}{\sqrt{2u}}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\infty} e^{-u} \times \frac{du}{\sqrt{2} \times \sqrt{u}}$$

$$= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-u} u^{-1/2} du$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(\tfrac{1}{2}) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1.$$

(ii) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \, dx$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = t$

$\Rightarrow x = \mu + \sigma t$

$\Rightarrow dx = \sigma \, dt$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma t) e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu e^{-\frac{t^2}{2}} dt + \underbrace{\cancel{\int_{-\infty}^{\infty} \sigma t e^{-\frac{t^2}{2}} dt}}_{\text{Odd func.}} \right]$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty \mu e^{-t^2/2} dt$$

$$= \sqrt{\frac{2}{\pi}} \mu \int_0^\infty e^{-t^2/2} dt$$

$$= \sqrt{\frac{2}{\pi}} \mu \sqrt{\frac{\pi}{2}} = \mu.$$

$$\boxed{\therefore E(X) = \mu}$$

(iii) $V(X) = E(X^2) - (E(X))^2$

$$\therefore E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = t \Rightarrow x = \mu + \sigma t$

$dx = \sigma dt$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu+\sigma t)^2 e^{-t^2/2} dt$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} \mu^2 e^{-t^2/2} dt + \underbrace{\cancel{\int_{-\infty}^{\infty} 2\mu\sigma t\, e^{-t^2/2} dt}}_{\text{odd func}} + \int_{-\infty}^{\infty} \sigma^2 t^2 e^{-t^2/2} dt \right]$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_{0}^{\infty} \mu^2 e^{-t^2/2} dt + \int_{0}^{\infty} \sigma^2 t^2 e^{-t^2/2} dt \right]$$

Let $\frac{t^2}{2} = u$

$\Rightarrow dt = \frac{du}{\sqrt{2u}}$

$$= \frac{2}{\sqrt{2\pi}} \left[\sigma^2 \int_0^{\infty} \sqrt{2}\, u^{1/2} e^{-u} du + \mu^2 \sqrt{\frac{\pi}{2}} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\sigma^2 \times \sqrt{2} \times \frac{\sqrt{\pi}}{2\sqrt{2}} + \mu^2 \sqrt{\frac{\pi}{2}} \right)$$

$= \sigma^2 + \mu^2$

$\therefore V(x) = \sigma^2 + \mu^2 - \mu^2$

$$\boxed{V(x) = \sigma^2}$$

<u>Note</u>: cdf of normal distribution doesn't exist in closed form.