Chapter-8
Matrix Eigenvalue Problems
8.1 Eigenvalues and Eigenvectors

Eigen values:

Let $A = [a_{jk}]$ be an $n \times n$ matrix and λ be a scalar. Then $D(\lambda) = \det(A - \lambda\, I) = \left|A - \lambda I\right| = 0$ is called characteristic equation of A .

A root of the characteristic equation, $\left|A - \lambda I\right| = 0$ is called an eigenvalue of characteristic value of A.

Here, $\left|A - \lambda I\right| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$.

Note: An $n \times n$ matrix A has at least one eigenvalue and at most n numerically different eigenvalues.

Examples:

Find the eigenvalues of the matrix$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$.

Solution: The characteristic equation of A is $\left|A - \lambda I\right| = 0$

$\Rightarrow \begin{vmatrix} 1 - \lambda & 3 \\ 4 & 5 - \lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)\left(5 - \lambda\right) - 12 = 0$

$\Rightarrow \lambda^2 - 6\lambda - 7 = 0$

$\Rightarrow (\lambda - 7)(\lambda + 1) = 0$

$\Rightarrow \lambda = -1, 7$

Hence the eigenvalues of A are -1, 7.

Example:

Find the eigen values of the matrix$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 3 & 2 & -2 \end{bmatrix}$.

Solution: The characteristic equation of A is $\left|A - \lambda I\right| = 0$

$\Rightarrow \begin{vmatrix} 1 - \lambda & -1 & 0 \\ 1 & 2 - \lambda & -1 \\ 3 & 2 & -2 - \lambda \end{vmatrix} = 0$

$\Rightarrow (1 - \lambda)\{-(2 - \lambda)(2 + \lambda) + 2\} + 1(-2 - \lambda + 3) = 0$

$\Rightarrow (1 - \lambda)\left(\lambda^2 - 2\right) + (1 - \lambda) = 0$

$\Rightarrow (1 - \lambda)\left(\lambda^2 - 1\right) = 0$

$\Rightarrow \lambda = 1, 1, -1$

Hence the eigenvalues of A are1, $1 - 1$.

Eigen Vectors:

Let A be an $n \times n$ matrix. A non-null vector X which is a solution of the vector equation $A\,X = \lambda X$, is called an eigenvector or characteristic vector of A . Here λ is a scalar.

Example:

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Solution: The characteristic equation of A is $\left| A - \lambda I \right| = 0$

$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow \lambda = -1, -6$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector corresponding to$\lambda = -1$. Then

$AX = \lambda X$

$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow -4x_1 + 2x_2 = 0, \quad 2x_1 - x_2 = 0$

$\Rightarrow 2x_1 - x_2 = 0$

Let $x_1 = k$, then $x_2 = 2k$ where, $k \neq 0$.

So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 2k \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = -1$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Similarly, let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigenvector corresponding to$\lambda = -6$. Then

$AX = \lambda X$

$\Rightarrow \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -6 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\Rightarrow x_1 + 2x_2 = 0, \quad 2x_1 + 4x_2 = 0$

$\Rightarrow x_1 + 2x_2 = 0$

Let $x_1 = k$, then $x_2 = -k/2$ where, $k \neq 0$.

So, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ -k/2 \end{bmatrix} = k/2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = -6$ is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Example:

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Solution: The characteristic equation of A is $\left| A - \lambda I \right| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)\{(2-\lambda)(-1-\lambda) - 1\} - 1(\lambda + 1 - 0) - 2(-1-0) = 0$

$\Rightarrow (1-\lambda)\{(2-\lambda)(-1-\lambda)-1\}+(1-\lambda)=0$
$\Rightarrow (1-\lambda)\left(\lambda^2-\lambda-2\right)=0$
$\Rightarrow (1-\lambda)(\lambda-2)(\lambda+1)=0$
$\Rightarrow \lambda = 1, 2, -1$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector corresponding to$\lambda = 1$. Then

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow 0x_1 + x_2 - 2x_3 = 0,$ …………………(1)
$-x_1 + x_2 + x_3 = 0,$ ………………….(2)
$0x_1 + x_2 - 2x_3 = 0$ ………………….(3)

Solving Eqs.(1) and (2) by cross multiplication method, we get

$$\frac{x_1}{1+2} = \frac{x_2}{2-0} = \frac{x_3}{0+1}$$
$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1} = k \text{ (say)}$$

Then $x_1 = 3k,\ x_2 = 2k,\ x_3 = k.$

You can easily verify that this solution is also satisfied Eq. (3).

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k \\ 2k \\ k \end{bmatrix} = k \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to$\lambda = 2$. Then

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\Rightarrow -x_1 + x_2 - 2x_3 = 0,$ …………………(1)
$-x_1 + 0x_2 + x_3 = 0,$ ………………….(2)
$0x_1 + x_2 - 3x_3 = 0$ ………………….(3)

Solving Eqs.(1) and (2) by cross multiplication method, we get

$$\frac{x_1}{1-0} = \frac{x_2}{2+1} = \frac{x_3}{0+1}$$
$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{1} = k \text{ (say)}$$

Then $x_1 = k,\ x_2 = 3k,\ x_3 = k.$

You can easily verify that this solution is also satisfied Eq. (3).

So, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ 3k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

Hence the eigenvector corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$.

Similarly, you can obtain the eigen value corresponding to $\lambda = -1$.

The eigenvalue corresponding to $\lambda = -1$ will be $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$