Problem Set - 4.3

(28) Z is a standard normal random variable

(a) $P(0 \leq Z \leq 2.17) = \varphi(2.17) - \varphi(0)$

$$\boxed{\varphi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz}$$

$= 0.9850 - 0.5$

$= 0.4850$

(b) $P[0 \leq Z \leq 1] = \varphi(1) - \varphi(0)$

$= 0.8413 - 0.5 = 0.3413$

(c) $P[-2.5 \leq Z \leq 0] = \varphi(0) - \varphi(-2.5)$

$= \varphi(0) - (1 - \varphi(2.5))$

$= \varphi(0) - 1 + \varphi(2.5)$

$= 0.5 - 1 + 0.9938$

$= 0.4939$

(d) $P[-2.5 \leq Z \leq 2.5] = \varphi(2.5) - \varphi(-2.5)$

$= \varphi(2.5) - (1 - \varphi(2.5))$

$= 2\varphi(2.5) - 1$

$= 2 \times 0.9938 - 1$

$= 0.9876$

(f) $P[-1.75 \leq Z] = P[Z \geq -1.75]$

$= 1 - P[Z < -1.75]$

$= 1 - \varphi(-1.75)$

$= 1 - (1 - \varphi(1.75)) = \varphi(1.75)$

$= 0.9599$

(j) $P[|Z| \leq 2.50]$

$= P[-2.5 \leq Z \leq 2.5]$

$= 0.9876$

(20) Determine the value of c

(a) $\varphi(c) = 0.9836 \Rightarrow c = 2.135$

(b) $P[0 \le Z \le c] = 0.291$

$\Rightarrow \varphi(c) - \varphi(0) = 0.291$

$\Rightarrow \varphi(c) - 0.5 = 0.291$

$\Rightarrow \varphi(c) = 0.791 \Rightarrow c = 0.81$

(c) $P[c \le Z] = 0.121 \Rightarrow P[Z \geq c] = 0.121$

$\Rightarrow 1 - P[Z < c] = 0.121$

$\Rightarrow 1 - \varphi(c) = 0.121$

$\Rightarrow \varphi(c) = 1 - 0.121 = 0.879$

$\Rightarrow c = 1.17$

(d) $P[-c \le Z \le c] = 0.668$

$\Rightarrow \varphi(c) - \varphi(-c) = 0.668$

$\Rightarrow 2\varphi(c) - 1 = 0.668$

$\Rightarrow \varphi(c) = \frac{1.668}{2} = 0.8340$

$\Rightarrow c = 0.97$

Note

$P[|Z| \le c]$

$= P[-c \le Z \le c]$

$= 2\varphi(c) - 1$

(e) $P(c \le |Z|) = 0.016$

$\Rightarrow P(|Z| \geq c) = 0.016$

$\Rightarrow 1 - P[|Z| < c] = 0.016$

$\Rightarrow P[|Z| < c] = 1 - 0.016 = 0.984$

$\Rightarrow P[-c < Z < c] = 0.984$

$\Rightarrow 2\varphi(c) - 1 = 0.984$

$\Rightarrow \varphi(c) = \frac{1 + 0.984}{2} = 0.992$

$\Rightarrow c = 2.41$

(3.5) Suppose the diameter at breast height (in.) of trees of a certain type is normally distributed with $\mu = 8.8$ and $\sigma = 2.8$.

(a) What is the prob. that the diameter of a randomly selected tree will be at least 10 in.? will exceed 10 in.?

(b) What is the prob. that the diameter of a randomly selected tree with exceed 20 in.

(c) What is the prob. that the diameter of a randomly selected tree will be between 5 and 10 in?

(d) What is the value c such that the interval $(8.8-c, 8.8+c)$ included 98% of all diameter values?

(e) If 4 trees are independently selected, what is prob. that at least one has a diameter exceeding 10 in.?

Soln Given $X \sim N(\mu, \sigma^2)$, $\mu = 8.8$, $\sigma = 2.8$

(a) and X measures the diameter of the tree at breast height

$P\{X \text{ is at least } 10 \text{ in}\} = P[X \geq 10] = 1 - P[X < 10]$

$= 1 - F_X(10) = 1 - \varphi\left(\frac{10 - 8.8}{2.8}\right)$

$= 1 - \varphi(0.4286)$

$= 1 - 0.6659 = 0.3341$

(5) $P\{X \text{ is between } 5 \text{ and } 10 \text{ in}\}$

$= P[5 < X < 10] = F_X(10) - F_X(5)$

$= \varphi\left(\frac{10 - 8.8}{2.8}\right) - \varphi\left(\frac{5 - 8.8}{2.8}\right)$

$= \varphi(0.4286) - \varphi(-1.3571) = 0.5785.$

$\boxed{\begin{array}{l} \varphi(-1.3751) \\ = 1 - \varphi(1.3751) \\ = 1 - 0.9126 \end{array}}$

(b) $P\{X \text{ exceeds } 20 \text{ in}\} = P[X > 20]$

$= 1 - P[X \leq 20] = 1 - \varphi\left(\frac{20 - 8.8}{2.8}\right)$

$= 1 - \varphi(4) = 1 - 1 = 0$

(d) Given $P\{X \text{ is in between } 8.8 - c \text{ and } 8.8 + c\} = 0.98$

$\Rightarrow P[8.8 - c < X < 8.8 + c) = 0.98$

$\Rightarrow F_X(8.8 + c) - F_X(8.8 - c) = 0.98$

$\Rightarrow \varphi\left(\frac{8.8 + c - 8.8}{2.8}\right) - \varphi\left(\frac{8.8 - c - 8.8}{2.8}\right) = 0.98$

$\Rightarrow \varphi\left(\frac{c}{2.8}\right) - \varphi\left(-\frac{c}{2.8}\right) = 0.98$

$\Rightarrow 2\varphi\left(\frac{c}{2.8}\right) - 1 = 0.98$ $\left(\because \varphi(z) - \varphi(-z) = 2\varphi(z) - 1\right)$

$\Rightarrow \varphi\left(\frac{c}{2.8}\right) = \frac{1.98}{2} = 0.99$

$\Rightarrow \frac{c}{2.8} = 2.33 \Rightarrow c = 2.33 \times 2.8 = 6.5240$

(e) $P\{$at least one of 4 trees has diameter exceeding 10 in$\}$

$= 1 - P\{$none of 4 trees has diameter exceeding 10 in$\}$

$= 1 - (P[X < 10])^4$

$P[X < 10] = P\{$a tree has diameter not exceeding 10 in$\}$ ($\because$ trees are independent)

$= 1 - \left(\varphi\left(\frac{10 - 8.8}{2.8}\right)\right)^4$

$= 1 - (\varphi(0.4286))^4$

$= 1 - (0.6659)^4 = 0.8034$

(13) Determine Z_α for the following

(a) $\alpha = 0.0055$ (b) $\alpha = 0.09$

(c) $\alpha = 0.663$

Sol^n

$Z_\alpha = 100(1-\alpha)^{th}$ percentile of Z, satisfies

$p = \varphi(n)$, $p = 1-\alpha$

(a) for $\alpha = 0.0055$, $1-\alpha = 0.9945$

Thus $0.9945 = \varphi(n)$

$\Rightarrow n = 2.54$

(b) for $\alpha = 0.09$, we have

$1-\alpha = \varphi(n)$

$\Rightarrow 0.91 = \varphi(n)$

$\Rightarrow n = 1.34$

(c) for $\alpha = 0.663$, we have

$1-\alpha = \varphi(n)$

$\Rightarrow 0.337 = \varphi(n)$

$\Rightarrow n = -0.421$ $\left(\because \varphi(-0.421) = 0.3369\right)$

(53) Let X have a binomial distribution with parameters $n=25$ and p. Calculate each of the following probabilities using the normal approximation (with the continuity correction) for the cases $p=0.5$, $p=0.6$ and $p=0.8$ and compare to the exact probabilities calculated from the normal table.

(a) $P(15 \le X \le 20)$ (b) $P[X \le 15]$

(c) $P[20 \le X]$

Sol^n Case-1 $n=25,\ p=0.5 \Rightarrow \mu = np = 12.5 \geq 0$

The PDF is $\quad$ & $nq = 12.5 \geq 0$

$$f_X(x) = B(x; n, p) \sim \varphi\left(\frac{x+0.5-\mu}{\sigma}\right)$$

$$= \varphi\left(\frac{x+0.5-12.5}{2.5}\right)$$

$\mu = np = 12.5$
$\sigma^2 = npq = 6.25$
$\sigma = 2.5$

(a) $P[15 \le X \le 20] = f_X(20) - f_X(14)$

$$= B(20, n, p) - B(14, n, p)$$

$$\sim \varphi\left(\frac{20+0.5-12.5}{2.5}\right) - \varphi\left(\frac{14+0.5-12.5}{2.5}\right)$$

$$= \varphi(3.2) - \varphi(0.8)$$

$$= 0.9993 - 0.7881 = 0.2112$$

(b) $P[X \le 15] = P[0 \le X \le 15] = f_X(15) - f_X(-1)$

$$= \varphi\left(\frac{15+0.5-12.5}{2.5}\right) - 0 \quad (\because f_X(-1) = 0)$$

$$= \varphi(1.2) = 0.8849$$

(c) $P[20 \le X] = P[X \geq 20] = 1 - P[X < 20]$

$$= 1 - P[X \le 19] = 1 - f_X(19) = 1 - \varphi\left(\frac{19+0.5-12.5}{2.5}\right)$$

$$= 1 - \varphi(2.8) = 1 - 0.9974$$

$$= 0.0026$$