Forced Harmonic Oscillation

Notes for

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Forced Oscillation

Have you ever been on a swing, or helped someone swing by periodically tapping on the moving swing? Have you noticed how a regular tap after each cycle keeps the motion on for a long time without much effect on the amplitude? Also, the moment the periodic tapping is stopped, the swing gradually slows down and eventually stops. We have already seen that the presence of resistive forces reduce the amplitude of oscillation with time as energy is dissipated. In fact, the only way of maintaining the amplitude of a damped oscillator is to continuously feed energy into the system in such a manner so as to compensate the losses. A steady (i.e., constant amplitude) oscillation of this type is called *driven damped harmonic oscillation*. Consider the mass-spring system discussed in damped harmonic oscillator to which we now apply a periodic force externally. We will soon see that when a *periodic external force* is applied to an oscillating object, with time the natural oscillation frequency and amplitude of the oscillating system dies out. Eventually, the system starts to oscillate with the frequency of the applied force or the *driving force*.

Forced Harmonic Oscillation: an analytical treatment

Let us consider an oscillator (a simple mass spring system with some damping) of mass 'm' undergoing oscillation. In the absence of any damping factor, the equation of motion of the oscillator as we know is given by,

$$ma = -kx \Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x \qquad \text{(1)}$$

where, 'a' is the acceleration, 'x' is the instantaneous displacement at any given instant, 'k' is the force constant, and '$\omega = \sqrt{\frac{k}{m}}$' is the natural angular frequency of oscillation.

Let us now consider the same oscillator undergoing oscillatory motion in the presence of an external periodic force ($F=F_0 \sin pt$). The equation of motion would now be changed to incorporate the cumulative effects of periodic external applied force, damping and inertia of motion. Let us look at the various forces acting on the oscillating system.

I. Restoring Force: The oscillating mass would experience a restoring force proportional to the displacement (x) of the system from its equilibrium position at any given instant, such that

$$F_R = -k\,x \qquad \text{(2)}$$

II. <u>Damping Force</u>: Assuming a velocity dependent damping force with a damping constant 'b' is acting on the system, the damping force can be given as,

$F_D = -bv\ (b > 0)$..(3)

III. <u>Driving force</u>: Driving force is the external periodic force acting on the system i.e.

$F = F_0 \sin pt$,………... (4)

where *'p'* is the frequency of external force acting on the oscillator.

IV. <u>Force of Inertia</u>: The resultant force acting on the system allowing it to execute the motion.

$$F_I = ma = m\frac{d^2x}{dt^2} \qquad \text{……………………………………………………………(5)}$$

The resultant equation of motion can then be obtained as follows.

$$F_I = F_R + F_D + F_0 \sin pt \qquad \text{……………………………………………………………(6)}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_0 \sin pt \qquad \text{……………………………………………(7)}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x + \frac{F_0}{m}\sin pt \qquad \text{……………………………………………(8)}$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\sin pt \qquad \text{……………………………………………(9)}$$

where, ω, is the *natural frequency* of oscillation in the absence of damping, and r<u> = *b*/2m </u>is the <u>*damping coefficient*</u> (note ***b*** is the *damping constant*).

Thus, the equation of motion for forced oscillator or an oscillator driven by a periodic external force is,

$$\frac{d^2x}{dt^2} + 2r\frac{dx}{dt} + \omega^2 x = \frac{F_0}{m}\sin pt \qquad \text{……………………………………………(10)}$$

<u>Solution to equation of motion of forced oscillator:</u>

Equation (10) is an in-homogeneous, linear second order differential equation with constant coefficients ω^2 and *r*. You will soon learn (in mathematics) that the general solution to this equation would have two parts to it *viz. complementary solution* (x_C(t))and *particular integral* (x_P(t)).

i.e. $x(t) = x_P(t) + x_C(t)$

Complementary solution

The *complementary solution* is obtained by solving the homogeneous part of the equation or using,

$$\frac{d^2x}{dt^2}+2r\frac{dx}{dt}+\omega^2x=0 \quad \text{.......(11)}$$

which is same as that of a *damped harmonic oscillator*. For an under damped oscillator ($r<\omega$), the solution is given by

$$x_c(t)=ae^{-rt}\sin(\omega_1 t+\varphi) \quad \text{.......(12)}$$

where, the constant a corresponds to the <u>*amplitude of oscillation* in the absence of external driving force</u>, and 'ω_1' is the *frequency of damped oscillation.* Here, x_C decays exponentially with time and dies out ($x_C(t)\rightarrow(0)$ at sufficiently long time ($t >>1/r$). Thus, x_C represents a transient solution.

Particular integral (The steady state solution)

At sufficiently long time i.e. $t >>1/r$, the complementary solution (x_C(t)) vanishes, and the general solution reduces to a *steady-state* solution given by particular integral (x_P(t)).While the complementary solution (x_C) is the transient solution, the particular integral (x_P) yields the steady state solution to the equation of motion of the forced/driven oscillator

The particular integral (x_P) may be obtained by using a trial function, $x=x_p(t)=A\sin(pt-\theta)$ as the solution to the equation of motion in steady state. Here A is the *amplitude* of oscillation in the presence of the applied force and θ is the *phase difference* between the driving force and displacement of the oscillator.

When we use $x=A\sin(pt-\theta)$, we must substitute,

$$\frac{dx}{dt}=pA\cos(pt-\theta),\&\frac{d^2x}{dt^2}=-p^2A\sin(pt-\theta)$$

in equation (10)which then yields,

$$-p^2A\sin(pt-\theta)+2prA\cos(pt-\theta)+\omega^2A\sin(pt-\theta)=\frac{F_0}{m}\sin pt=f_0\sin pt,\text{where} f_0=\frac{F_0}{m}$$

$$-p^2 A\sin(pt-\theta)+2prA\cos(pt-\theta)+\omega^2 A\sin(pt-\theta)=f_0\sin[(pt-\theta)+\theta]$$
$$\Rightarrow A(\omega^2-p^2)\sin(pt-\theta)+2rpA\cos(pt-\theta)=f_0\sin(pt-\theta)\cos\theta+f_0\cos(pt-\theta)\sin\theta \qquad (13)$$

For the above relation to hold good for all values of *t*, the coefficients of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ on both sides of the equation must be equal i.e., by comparison of the coefficient of $\sin(pt-\theta)$ and $\cos(pt-\theta)$ on both sides, we obtain

$$A(\omega^2-p^2)=f_0\cos\theta \qquad (14)$$

$$2rpA=f_0\sin\theta \qquad (15)$$

Squaring and then adding (14) & (15) one can obtain the amplitude, *A* of the forced vibration as follows

$$\because f_0^{\,2}=\{f_0\cos\theta\}^2+\{f_0\sin\theta\}^2$$
$$\Rightarrow f_0^{\,2}=A^2\{(\omega^2-p^2)^2+4p^2r^2\}$$
$$\Rightarrow A=\frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}}$$
$$\Rightarrow A=\frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}} \qquad (16)$$

The *amplitude of forced/driven oscillations 'A'* thus depends on the *natural angular frequency* of oscillation 'ω', the *damping coefficient*, 'r' and the *angular frequency of the external force*, 'p'.

The *phase difference* (θ) between the driving force and the driven system may be obtained as tangent of the angle θ, by dividing equation (15) by (14)

$$\tan\theta=\left(\frac{2rp}{\omega^2-p^2}\right)\Rightarrow\theta=\tan^{-1}\left(\frac{2rp}{\omega^2-p^2}\right) \qquad (17)$$

Thus, for a given oscillator, θ depends on the frequency of the external force. The *particular integral or* the steady state solution *is* therefore given as,

$$\Rightarrow x_p(t)=\frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}}\sin(pt-\theta) \qquad (18)$$

Therefore, the complete solution to the equation of motion of a forced oscillator driven by an external periodic force $F_0\sin pt$, may be obtained as

$$x(t) = x_C(t) + x_p(t) = ae^{-rt}\sin(\omega_1 t + \varphi) + \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}}\sin(pt-\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots(19)$$

Note that both parts of the solution contribute at the beginning, however, the first part of the solution quickly dies out *depending on the degree of damping*. Ultimately, *once the transient part of the solution vanishes* the oscillator attains a steady state and oscillates with the frequency of the external force.

So, the steady state solution of the forced oscillator is given by,

$$x(t) = x_p(t) = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}}\sin(pt-\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots(20)$$

<u>Variation of Amplitude of Forced Vibration & Amplitude Resonance</u>

As evident from equation (16), *amplitude* of forced vibration depends on the difference between the *driving frequency, p* and the natural frequency, ω. Thus, not only the driving frequency but also how far the driving frequency is from the natural frequency (i.e. the undamped oscillator) is crucial in determining fate of a driven oscillator and so is the damping coefficient. While, frequency of undamped oscillator (ω) is constant, the driving frequency, *p can be tuned* to match the oscillatory frequency. Such situation where both the frequencies nearly match each other; amplitude and energy of the oscillator system increases dramatically. This phenomenon is called *resonance*. We will soon see that if the damping constant *b* is small, the amplitude *A* gets *very large* when the frequency of the driver approaches the natural frequency of the oscillator and it can sometimes lead to catastrophes such as collapse of the Tacoma Narrows Bridge. This bridge was destroyed as the wind (driving force) was at the same as the natural frequency. The bridge vibrated and shook itself apart. It is for this same reason a marching troop is asked to break their rhythm if they have to cross a narrow bridge. Of course, it has its positive aspects, from getting a swing going to tuning a radio. We now discuss in details how the amplitude changes as we drive the oscillator with frequency near and far from the frequency of the undamped oscillator.

Case I: $p<<\omega$ (when the driving frequency is lower than natural frequency)

$A \approx \frac{f_0}{\omega^2}$……………………………………………………………………………………..(21)

Since $p << \omega$, here we have neglected the terms containing p^2.

This shows that the amplitude of vibration is independent of the frequency of driving force.

Case II: $p >>\omega$ (when the driving frequency is greater than natural frequency)

i.e. $\omega^2 \ll p^2$, considering the expression for driven amplitude, $A = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}}$,

$$\Rightarrow A = \frac{f_0}{\sqrt{p^4\left(\frac{\omega^2}{p^2}-1\right)^2+4p^2r^2}}$$

$$\Rightarrow A = \frac{f_0}{\sqrt{p^4+4p^2r^2}} = \frac{f_0}{\sqrt{p^4\left(1+\frac{4r^2}{p^2}\right)}} \quad \text{(neglecting } \frac{\omega^2}{p^2} \text{ and } \frac{r^2}{p^2}\text{)}$$

$\Rightarrow A \approx \frac{f_0}{p^2}$ ……………………………………………………………………………………(22)

Thus, the amplitude A goes on decreasing with the increase in the driving force frequency.

Case III: $p\approx\omega$ or Resonance (when the driving frequency is nearly same as natural frequency)

$A = \frac{f_0}{\sqrt{(\omega^2-p^2)^2+4p^2r^2}} = \frac{f_0}{\sqrt{4p^2r^2}} \approx \frac{f_0}{2pr} \approx \frac{F_0}{b\omega}$ (Since $f_0 = \frac{F_0}{m}, r = \frac{b}{2m}$ *and* $p \approx \omega$)……………(23)

Thus, the amplitude is governed by the damping constant and is inversely proportional to it. For small damping, the amplitude of vibration will be quite large. Thus, a weakly damped oscillator can be driven to large amplitude by the application of a relatively small amplitude external driving force that oscillates at a frequency close to the resonant frequency. If you vary the driver frequency (variable p) for a given oscillator (fixed ω), what is the value of p for which A is maximum? This is the condition for *amplitude resonance*.

Amplitude Resonance

Looking at eqn (16), the amplitude is maximum, when $\sqrt{(\omega^2 - p^2)^2 + 4p^2r^2}$ is minimum**,** i.e.

$$\frac{d}{dp}\left[\sqrt{(\omega^2 - p^2)^2 + 4p^2r^2}\right] = 0$$

$$\Rightarrow \frac{2(-2p)(\omega^2 - p^2) + 4(2p)r^2}{2\sqrt{(\omega^2 - p^2)^2 + 4p^2r^2}} = 0$$

$$\Rightarrow -p(\omega^2 - p^2) + 2pr^2 = 0$$

$$\Rightarrow (\omega^2 - p^2) + 2r^2 = 0$$

$$\Rightarrow p^2 = \omega^2 - 2r^2$$

$$\Rightarrow p = \sqrt{\omega^2 - 2r^2}$$

$$\Rightarrow \textit{At resonance},\ p = \omega_R = \sqrt{\omega^2 - 2r^2} \quad \text{......}(24)$$

Here *driving frequency*, $p = \omega_R$ i.e. the resonant frequency, and is not equal to ω but slightly lesser than it.

Thus, substituting equation (24) in (16) we can obtain the maximum amplitude at resonance as,

$$A_{max} = \frac{f_0}{\sqrt{(-2r^2)^2 + 4p^2r^2}} = \frac{f_0}{\sqrt{4r^2(r^2+p^2)}} = \frac{f_0}{2r\sqrt{(r^2+p^2)}} \quad \text{......}(25)$$

Thus, for low/weak damping it reduces to

$$\Rightarrow A_{\max} = \frac{f_0}{2rp} \quad \text{......}(26)$$

When, $\omega \approx$ p, the *amplitude* becomes maximum, and this condition is known as *resonance*. At resonance,

$$\Rightarrow A = \frac{F_0}{2mr\omega} = \frac{F_0}{b\omega} \quad \text{......}(27)$$

Thus, the smaller the *b*, the larger is the *resonance amplitude*. In principle for $b = 0$, A= ∞, though it is practically unattainable as in a real system there will always be dissipative forces leading to nonzero damping.