2.5 Independence of Events:

Two events A and B are called independent events if $P(A\cap B) = P(A)\,P(B)$ and are dependent otherwise.

Assuming $P(A)\neq 0$, $P(B)\neq 0$ then $P(A/B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\,P(B)}{P(B)} = P(A)$

$P(B/A) = \frac{P(A\cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$

This means that the probability of A does not depend on the occurrence or non occurrence of B and conversely.

* If A and B are independent then a) A' and B' are independent

b) A' and B are independent.

c) A and B' are independent.

proof: (a) $\because$ A and B are independent, $P(A\cap B) = P(A)P(B)$.

Now $P(A'\cap B') = P[(A\cup B)']$

$= 1 - P(A\cup B) = 1 - \{P(A) + P(B) - P(A\cap B)\}$

$= 1 - \{P(A) + P(B) - P(A)P(B)\}$

$= 1 - P(A) - P(B)[1 - P(A)]$

$= (1-P(A))[1-P(B)] = P(A')\,P(B')$

$\therefore$ A' and B' are independent. (proved)

* Three events A, B and C are independent/mutually independent iff

$P(A\cap B) = P(A)P(B)$

$P(B\cap C) = P(B)\,P(C)$

$P(C\cap A) = P(C)\,P(A)$

$P(A\cap B\cap C) = P(A)\,P(B)P(C)$

* Similarly, n events $A_1, A_2, \ldots, A_n$ are independent if
$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\,P(A_2) \cdots P(A_n)$$
as well as for every K different events $A_{j_1}, A_{j_2}, \ldots, A_{j_K}$
$$P(A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_K}) = P(A_{j_1})\,P(A_{j_2}) \cdots P(A_{j_K})$$
where $K = 2, 3, \ldots, n-1$

Problems (2.5)

Q.(71) [2.5]

An oil exploration company currently has two active projects, one in Asia and other in Europe. Let A be the event that the Asian project is successful. B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.

a) If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.

b) What is the probability that at least one of the two projects will be successful?

c) Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?

Ans: A = Asian project is successful.
B = European project is successful.
$P(A) = 0.4$, $P(B) = 0.7$

Given that A, B are independent

$$\Rightarrow \begin{cases} A', B & \text{''} \quad \text{''} \\ A, B' & \text{''} \quad \text{''} \\ A', B' & \text{''} \quad \text{''} \end{cases}$$

Also if A and B are independent then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

(a) $P(B'/A') = P(B')$ [$\because$ A' and B' are independent]

$= 1 - P(B)$

$= 1 - 0.7$

$= 0.3$

(b) probability of at least one of the two projects will be successful $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because P(A \cap B) = P(A)P(B) \text{ for independent events}]$$

$$= 0.4 + 0.7 - 0.4 \times 0.7$$

$$= 0.82$$

(c)

P (only Asian project is successful / At least one of the two project is successful)

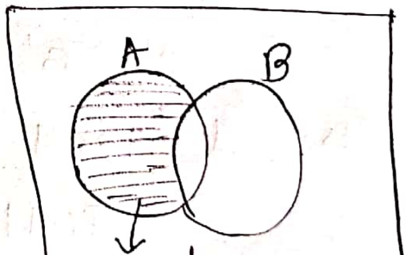
$$= P(A \cap B' \mid A \cup B)$$

$$= \frac{P[(A \cap B') \cap (A \cup B)]}{P(A \cup B)}$$

$$= \frac{P(A \cap B')}{P(A \cup B)} \quad [\because A \cap B' \subseteq A \cup B]$$

$$= \frac{P(A) \cdot P(B')}{P(A \cup B)} \quad [\because A \text{ and } B' \text{ are independent}]$$

$$= \frac{0.4(1 - 0.7)}{0.82} = 0.146$$



Q.(72) [2.5]

A computer consulting firm presently has bids out on three projects. Let $A_i = \{\text{Awarded project } i\}$, for $i=1,2,3$ and suppose that $P(A_1) = 0.22$, $P(A_2) = 0.25$, $P(A_3) = 0.28$, $P(A_1 \cap A_2) = 0.11$, $P(A_1 \cap A_3) = 0.05$, $P(A_2 \cap A_3) = 0.07$, $P(A_1 \cap A_2 \cap A_3) = 0.01$.

Is any A_i independent of any other A_j?

Answer using the multiplication property for independent events.

Ans;

$P(A_1 \cap A_2) = 0.11$

$P(A_1) P(A_2) = 0.22 \times 0.25 = 0.055$

$\therefore P(A_1 \cap A_2) \neq P(A_1) P(A_2)$

$\Rightarrow A_1$ and A_2 are not independent.

$P(A_1) \cdot P(A_3) = 0.22 \times 0.28 = 0.0616 \neq P(A_1 \cap A_3)$

$\Rightarrow A_1$ and A_3 are not independent.

$P(A_2) P(A_3) = 0.25 \times 0.28 = 0.07 = P(A_2 \cap A_3)$

$\Rightarrow A_2$ and A_3 are independent.

Q.(78) [2.5]

A boiler has five identical relief valves. The probability that any particular valve will open on demand is 0.95. Assuming ~~that~~ independent operation of the valves, calculate P(at least one valve opens) and P(at least one valve fails to open).

Ans; Let A = "valve opens"
A' = "valve does not open"

then $P(A) = 0.95$, $P(A') = 0.05$

Also let X be the number of valves which open
Y be the number of valves which don't open.

∴ The P(at least one valve opens) | $X = 0,1,2,3,4,5$; $Y = 0,1,2,3,4,5$

$$= P(X \geq 1)$$
$$= 1 - P(X < 1)$$
$$= 1 - P(X = 0) = 1 - P(Y = 5)$$
$$= 1 - (0.05)^5 \simeq 1$$

The probability at least one valve fails to open

$$= P(Y \geq 1)$$
$$= 1 - P(Y = 0)$$
$$= 1 - P(X = 5)$$
$$= 1 - (0.95)^5$$
$$= 0.2262$$

Q.(83) [2.5]

Components arriving at a distributor are checked for defects by two different inspectors (each component is checked by both inspectors). The first inspector detects 90% of all defectives that are present, and the second inspector likewise. At least one inspector does not detect a defect on 20% of all defective components. What is the probability that the following occur?

a) A defective component will be detected only by the first inspector? By exactly one of the two inspectors?

b) All three defective components in a batch escape detection by both inspectors (assuming inspections of different components are independent of one another)?

Ans:

P(atleast one does not detect a defect) = 0.20

P(both inspectors detect the defect) = 1 − P(atleast one does not)

= 1 − 0.20 = 0.80

(a) P(1st detects ∩ 2nd does not), Let A = 1st detects, B = 2nd detects

$= P(A \cap B')$

$= P(A) - P(A \cap B)$

$= 0.9 - 0.8 = 0.1$

[∵ $P(A \cap B) = 0.80$, $P(A) = 0.9$, $P(B) = 0.9$]

Similarly P(1st does not ∩ 2nd detects)

$= P(A' \cap B) = P(B) - P(A \cap B) = 0.9 - 0.8 = 0.1$

∴ P(exactly one does) $= P(A \cap B') + P(A' \cap B)$

$= 0.1 + 0.1 = 0.2$

(b) P (neither detects a defect)

$= P(A' \cap B') = P[(A \cup B)']$

$= 1 - P(A \cup B)$

$= 1 - \{P(A) + P(B) - P(A \cap B)\}$

$= 1 - (0.9 + 0.9 - 0.80)$

$= 1 - 1 = 0$

That is under this model there is 0% probability neither inspector detects a defect.

$\therefore P(\text{all 3 escape}) = (0)(0)(0) = 0.$

Q.(87) [2.5]

Consider randomly selecting a single individual and having that person test drive 3 different vehicles. Defines events A_1, A_2 and A_3 by

A_1 = likes vehicle #1, A_2 = likes vehicle #2,
A_3 = likes Vehicle #3

Suppose that $P(A_1) = 0.55$, $P(A_2) = 0.65$, $P(A_3) = 0.70$, $P(A_1 \cup A_2) = 0.80$, $P(A_2 \cap A_3) = 0.40$ and $P(A_1 \cup A_2 \cup A_3) = 0.88$.

a) what is the probability that the individual likes both vehicle #1 and vehicle #2?

b) Determine and interpret $P(A_2 | A_3)$.

c) Are A_2 and A_3 independent events? Answer in two different ways.

d) If you learn that the individual did not like vehicle #1, what is now the probability that he/she liked at least one of the other two vehicles?

Ans:

(a) $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$
$= 0.55 + 0.65 - 0.80 = 0.40$

(b) $P(A_2/A_3) = \dfrac{P(A_2 \cap A_3)}{P(A_3)} = \dfrac{0.40}{0.70} = 0.5714$

If a person likes vehicle #3, there is a 57.14% chance he/she will also like vehicle #2

(c) NO, from (b), $P(A_2 | A_3) = 0.5714 \neq P(A_2) (= 0.65)$
$\therefore$ A_2 and A_3 are not independent.

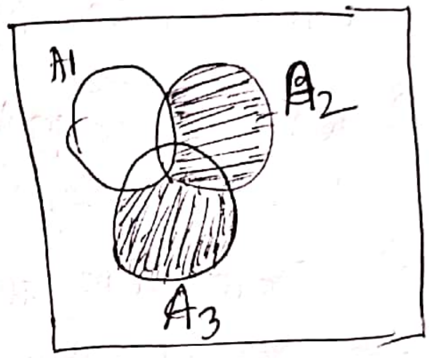
Alternatively, $P(A_2)\, P(A_3) = 0.65 \times 0.70 = 0.4550 \neq P(A_2 \cap A_3)$
$\Rightarrow$ A_2 and A_3 are not independent.

(d) we have to find $P(A_2 \cup A_3 / A_1')$

Now, $P(A_2 \cup A_3 / A_1')$
$$= \frac{P[A_1' \cap (A_2 \cup A_3)]}{P(A_1')}$$

Now, $P(A_1') = 1 - P(A_1) = 1 - 0.55 = 0.45$

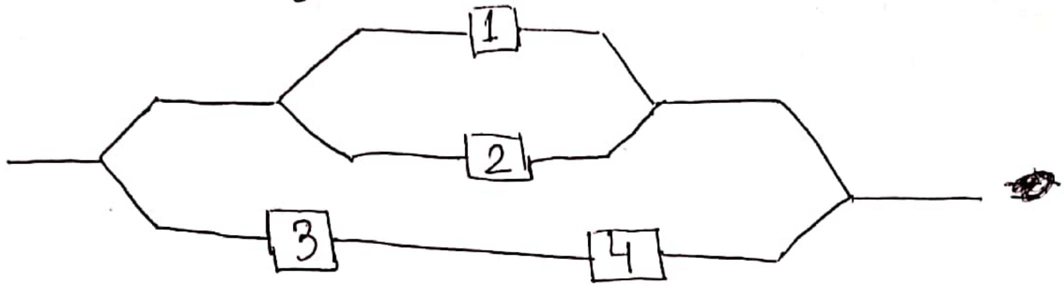
and $P[A_1' \cap (A_2 \cup A_3)]$
$= P(A_1 \cup A_2 \cup A_3) - P(A_1)$
$= 0.88 - 0.55 = 0.33$



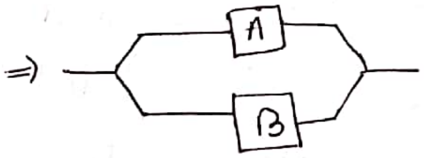
$\therefore P(A_2 \cup A_3 / A_1') = \frac{0.33}{0.45} = 0.7333$

Q. (80) [2.5]

consider the system of components connected as in the accompanying picture. Components 1 and 2 are connected in parallel, so that subsystems works iff either 1 or 2 works; since 3 and 4 are connected in series, that subsystem works iff both 3 and 4 work. If components work independently of one another and P(component works) = 0.9. calculate P(system works).



Ans: Let $A = 1 \cup 2$, $B = 3 \cap 4$ $\Rightarrow$



$\therefore$ P(System works)

$= P(A \cup B)$ [$\because$ A and B are parallel]

$= P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)P(B)$ [$\because$ both the systems A and B are independent of each other]

Now $P(A) = P(1 \cup 2)$

$= P(1) + P(2) - P(1 \cap 2)$

$= P(1) + P(2) - P(1)P(2)$ [$\because$ All components are independent]

$= 0.9 + 0.9 - 0.9 \times 0.9$ [$\because$ P(component works) = 0.9]

$= 0.99$

$P(B) = P(3 \cap 4) = P(3)P(4) = 0.9 \times 0.9 = 0.81$

$\therefore$ P(System works) $= 0.99 + 0.81 - 0.99 \times 0.81$

$= 0.9981$

Q.(81) [2.5]

Ans: Using the hints, let $P(A_i) = p$, and $x = p^2$. Following the solution provided in the example,

P(system lifetime exceeds t_0) $= p^2 + p^2 - p^4$

$= 2p^2 - p^4 = 2x - x^2$.

Now set this equal to 0.99, we have

$2x - x^2 = 0.99$

$\Rightarrow x^2 - 2x + 0.99 = 0 \Rightarrow x = 0.9$ or 1.1

$\Rightarrow p = 1.049$ or 0.9487.

$\Rightarrow p = 0.9487$ ($\because p \neq 1.049$)