Chapter - 6,25

(mean, variance, S.D.)

Estimation of parameters

i) Point estimation of parameters :-

A point estimate of a parameter is a no. (a point on the real line) which is computed from a sample and serves as the exact value of an unknown parameter of the population.

Maximum likelihood method is used to compute the point estimation of parameters

Maximum likelihood method

Let, x be a discrete (or continuous) rv with pmf (or pdf) f(x) that depends on the parameter 'θ'. Then the parameter 'θ' can be estimated using the following steps.

1. Consider a corresponding sample of independent values $(x_1, x_2, \ldots, x_n)$ for which the pmfs (or the pdf) are $f(x_1), f(x_2), \ldots f(x_n)$

2. Consider the likelihood function as
$$L = f(x_1) f(x_2) \ldots f(x_n) = \prod_{i=1}^{n} f(x_i) \quad -(i)$$

3. Find the logarithm of both sides of eqn. (i)
i.e., $\ln L = \ln\left\{\prod_{i=1}^{n} f(x_i)\right\}$

4. Solve $\frac{\partial L}{\partial \theta} = 0$ to get the value of θ which is the estimation of the parameter 'θ'.

→ partial derivation

Note :-

If f(x) involves 'r' parameters $\theta_1, \theta_2, \ldots, \theta_r$, then by solving
$$\frac{\partial L}{\partial \theta_1} = 0,\ \frac{\partial L}{\partial \theta_2} = 0, \ldots \ldots \frac{\partial L}{\partial \theta_r} = 0$$

Ans:-

1) we know the pdf of the normal distribution is :

$$f(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \end{cases}$$

Let, $x_1, x_2, \ldots\ldots, x_n$ be the sample of n independent values for which the pmfs are $f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2}, i = 1, 2, \ldots, n$

$$\text{Let, } L = \prod_{i=1}^{n} \left(\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right)$$

$$= \frac{1}{\sigma^n (\sqrt{2\pi})^n} \times \prod_{i=1}^{n} \left(e^{-\frac{1}{2}\left(\frac{x_i^2 - 2x_i\mu + \mu^2}{\sigma^2}\right)} \right)$$

$$L = \frac{1}{\sigma^n (\sqrt{2\pi})^n} \times e^{-\frac{\left\{\sum_{i=1}^{n} x_i^2 - 2\mu \sum_{i=1}^{n} x_i + n\mu^2\right\}}{2\sigma^2}}$$

Take logarithm on both sides

$$\frac{\partial L}{\partial \mu} = 0$$

$$\Rightarrow \frac{\partial}{\partial \mu}\left(\frac{\mu \sum_{i=1}^{n} x_i}{\sigma_0^2}\right) - \frac{\partial}{\partial \mu}\left(\frac{n\mu^2}{2\sigma_0^2}\right) = 0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\sigma_0^2} - \frac{n\mu}{2\sigma_0^2} \times 2\mu = 0$$

$$\cancel{\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\sigma^2} - n\mu^3 = 0}$$

$$\cancel{\Rightarrow \mu^3 = \frac{\sum_{i=1}^{n} x_i}{n\sigma^2}}$$

$$\cancel{\Rightarrow \mu^3 = \frac{\bar{X}}{\sigma^2}}$$

$\cancel{\Rightarrow \sum_{i=1}^{n} x_i =}$ [illegible]

$$\Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\Rightarrow \mu = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow \mu = \bar{X}$, $\bar{x}$ is the sample mean.

So, mle of μ in the normal distribution is $\hat{\mu} = \bar{x}$.

Q) 3) Estimate the parameter 'p' in the binomial distribution.

Ans: 1- Consider a sample of 'n' independent values $x_1, x_2, \dots, x_n$ for which the pmfs are given by

$$f(x_i) = {}^{n}C_{x_i}\, p^{x_i} q^{n-x_i} \quad , i=1,2,\dots\dots,n$$

Let, $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} {}^{n}C_{x_i}\, p^{x_i}(1-p)^{n-x_i}$

$$= \prod_{i=1}^{n}\left\{\frac{\lfloor n}{\lfloor x_i \lfloor n-x_i} p^{x_i}(1-p)^{n-x_i}\right\}$$

$$= \left(\prod_{i=1}^{n}\frac{\lfloor n}{\lfloor x_i \lfloor n-x_i}\right)(p)^{\sum_{i=1}^{n} x_i}(1-p)^{\sum_{i=1}^{n}(n-x_i)}$$

$$= \left(\prod_{i=1}^{n}\frac{\lfloor n}{\lfloor x_i \lfloor n-x_i}\right) p^{\sum_{i=1}^{n} x_i}(1-p)^{n^2-\sum_{i=1}^{n} x_i}$$

$\sum_{i=1}^{n} 1 = n$

$$\Rightarrow \ln L = \ln\left\{\left(\prod_{i=1}^{n}\frac{\lfloor n}{\lfloor x_i \lfloor n-x_i}\right) p^{\sum_{i=1}^{n} x_i}(1-p)^{n^2-\sum_{i=1}^{n} x_i}\right\}$$

$p^{x_1} \times p^{x_2} \times p^{x_3}$

$$= \ln\left(\prod_{i=1}^{n}\frac{\lfloor n}{\lfloor x_i \lfloor n-x_i}\right) + \left(\sum_{i=1}^{n} x_i\right)\ln p + \left(n^2-\sum_{i=1}^{n} x_i\right)\ln(1-p)$$

Now, $\frac{\partial(\ln L)}{\partial P} = 0$

$\frac{d(1-p)}{dP} = 0-1$

$$\Rightarrow 0 + \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\left(n^2-\sum_{i=1}^{n} x_i\right)}{1-p} = 0$$

$$\Rightarrow (1-p)\sum_{i=1}^{n} x_i - p\left(n^2-\sum_{i=1}^{n} x_i\right) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - pn^2 + p\sum_{i=1}^{n} x_i = 0$$

Continuing classwork

3) $\sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - pn^2 + p\sum_{i=1}^{n} x_i = 0$

$\Rightarrow pn^2 = \sum_{i=1}^{n} x_i$

$\Rightarrow p = \frac{\sum_{i=1}^{n} x_i}{n^2}$ $\quad \because \bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\downarrow$
Sample mean

$\Rightarrow p = \frac{\bar{X}}{n}$

$\therefore$ Maximum likelihood estimate (MLE) of 'p' in the Binomial distribution is $\hat{p} = \frac{1}{n^2}\sum_{i=1}^{n} x_i = \frac{\bar{X}}{n}$

Q) 9) Find the maximum likelihood estimate of μ in the poisson's distribution.

Ans: We know the pmf of the poisson's distribution is

$$f(x) = \frac{e^{-\mu} \times \mu^{x}}{\lfloor x}, \quad x = 0, 1, 2, \ldots, \infty$$

x is the no of trials. because n value is very very large

Let, $x_1, x_2, \ldots, x_n$ be the sample of 'n' independent values for which the pmfs are

$$f(x_i) = \frac{e^{-\mu} . \mu^{x_i}}{\lfloor x_i}, \quad x_i = 0, 1, 2, \ldots\ldots \quad i = 1, 2, \ldots\ldots, n$$

Let, $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-\mu} . \mu^{x_i}}{\lfloor x_i}, \quad x_i = 0, 1, 2, \ldots\ldots \quad i = 1, 2, \ldots\ldots, n$

$$\Rightarrow L = \left(\prod_{i=1}^{n} \frac{1}{\lfloor x_i}\right)\left(\prod_{i=1}^{n} e^{-\mu}\right) . \left(\prod_{i=1}^{n} \mu^{x_i}\right)$$

$$\Rightarrow L = \left(\prod_{i=1}^{n} \frac{1}{\lfloor x_i}\right) e^{-n\mu} . \mu^{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \ln L = \ln\left\{\prod_{i=1}^{n} \frac{1}{\lfloor x_i}\right\} + \ln e^{-n\mu} + \ln \mu^{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \ln L = \ln\left\{\prod_{i=1}^{n} \frac{1}{\lfloor x_i}\right\} - n\mu + \sum_{i=1}^{n} x_i \ln \mu$$

Now, $\frac{\partial(\ln L)}{\partial \mu} = 0$

$$\Rightarrow 0 - n + \frac{1}{\mu}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow n = \frac{1}{\mu}\sum_{i=1}^{n} x_i = 0 \quad \Rightarrow \mu = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

So, mle of 'μ' is

$$\boxed{\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}}$$

11) $f(x) = \begin{cases} \theta e^{-\theta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

Let, $x_1, x_2, \ldots\ldots, x_n$ be the sample of n independent values for which the pmf's are $f(x_i) = \theta e^{-\theta x_i}$, for $i = 1, 2, \ldots\ldots, n$, $x_i \geq 0$

$$\text{Let, } L = \prod_{i=1}^{n} \theta e^{-\theta x_i}, \quad x_i \geq 0, \; i = 1, 2, \ldots\ldots, n$$

$$L = \theta^n \times e^{-\theta \sum_{i=1}^{n} x_i}$$

- Taking logarithm on both sides of the above 'eqn.

$$\ln L = n \ln \theta - \theta \sum_{i=1}^{n} x_i$$

$$\frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \frac{n}{\theta} = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \theta = \frac{\sum_{i=1}^{n} x_i}{n}$$

$\Rightarrow \theta = \bar{X}$, $\bar{X}$ is the sample mean

So, the mle of θ is $\hat{\theta} = \bar{X}$

Q) Find the mle of λ in the exponential distribution.

we know the pdf of the exponential distribution is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Let, $x_1, x_2, \dots, x_n$ be the sample of n independent values for which the pdfs are

$$f(x_i) = \lambda e^{-\lambda x_i} \; ; \; x_i \geq 0 \text{ and } i = 1, 2, \dots, n$$

$$\text{Let, } L = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} \; ; \; x_i \geq 0 \; \& \; i = 1, 2, \dots, n$$

$$\Rightarrow L = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

$$\Rightarrow \ln L = \ln \left\{ \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} \right\}$$

$$\Rightarrow \ln L = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

$$\text{Now, } \frac{\partial (\ln L)}{\partial \lambda} = 0$$

$$\Rightarrow \frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \frac{n}{\lambda} = \sum_{i=1}^{n} x_i$$

$$\Rightarrow \lambda = \frac{n}{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \lambda = \frac{1}{\bar{x}}$$

So, mle of λ is

$$\boxed{\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}}$$

Q) Find the mle of p in the geometric distribution.

$$f(x) = p(1-p)^x$$ No. of trials $= x+1$

$$\underbrace{F\,F\,F \dots F}_{x\text{-failures}}$$

$$\underbrace{S\,S\,S \dots S}_{r\text{-successes}}$$

when $r=1$, $x+1$

pmf of the geometric distribution is $f(x) = p(1-p)^{x}$, where $x+1$ is the total no of trials.

Let, $x_1, x_2, \ldots\ldots, x_n$ be the sample of 'n' independent values for which the pmfs are $f(x_i) = p(1-p)^{x_i}$; ~~x_i+1 is~~ x_i+1 is the no of trials and $i = 1, 2, \ldots, n$

Let, $L = \prod_{i=1}^{n} (1-p)^{x_i} \cdot p$

$$\Rightarrow L = (p)^{n} \cdot (1-p)^{\sum_{i=1}^{n} x_i}$$

$$\Rightarrow \ln L = n \ln p + \sum_{i=1}^{n} x_i \ln(1-p)$$

Now, $\frac{\partial(\ln L)}{\partial p} = 0$

$$\Rightarrow \frac{n}{p} - \frac{\sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\Rightarrow (1-p)n - p\sum_{i=1}^{n} x_i = 0 \qquad \because \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

$$\Rightarrow n - np - np\bar{x} = 0$$

$$\Rightarrow np + np\bar{x} = n$$

$$\Rightarrow np(1+\bar{x}) = n$$

$$\Rightarrow p = \frac{1}{1+\bar{x}}$$

So, mle of p is

$$\boxed{\hat{p} = \frac{1}{1+\bar{x}}}$$

If we take, x trials, $\hat{p} = \frac{1}{\bar{x}}$

Here, we have taken $x+1$ trials.