Let X and Y be continious RV's. A joint probability density function $f(x,y)$ for these two variables is a function satisfying $f(x,y) \geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1.$$

Then,

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_{a}^{b}\int_{c}^{d} f(x,y)\, dy\, dx$$

The marginal probability density functions of X and Y, denoted $f_X(x)$ and $f_Y(y)$, respectively, given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{for } -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx \quad \text{for } -\infty < y < \infty.$$

Two continious RV's X and Y are said to be independent if for every pair of x and y values

$$f(x,y) = f_X(x) \cdot f_Y(y)$$ when X and Y are continious.

If it is not satisfied for all (x,y), then X and Y are said to be dependent.

Eg. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 26 psi. Suppose the actual air pressure in each tire is a rv X for the right tire and Y for the left tire, with joint pdf

$$f(x,y) = \begin{cases} K(x^2+y^2) & 20 \leq x \leq 30,\ 20 \leq y \leq 30 \\ 0 & \text{otherwise.} \end{cases}$$

a) What is the value of "K"

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy$$

$$= K\left(\int_{20}^{30}\int_{20}^{30} (x^2)\,dx\,dy + \int_{20}^{30}\int_{20}^{30} y^2\,dx\,dy\right)$$

$$= K\int_{20}^{30} x^2\cdot\left(y\Big|_{20}^{30}\right)dx + \int_{20}^{30} y^2\left(x\Big|_{20}^{30}\right)dy$$

$$= K\left(10\cdot\frac{x^3}{3}\Big|_{20}^{30} + 10\cdot\frac{y^3}{3}\Big|_{20}^{30}\right)$$

$$= 20K\left(\frac{30^3}{3} - \frac{20^3}{3}\right) = \frac{380{,}000}{3}K = 1$$

$$\Rightarrow K = \frac{3}{380{,}000}$$

b) What is the probability that both tires are underfilled.

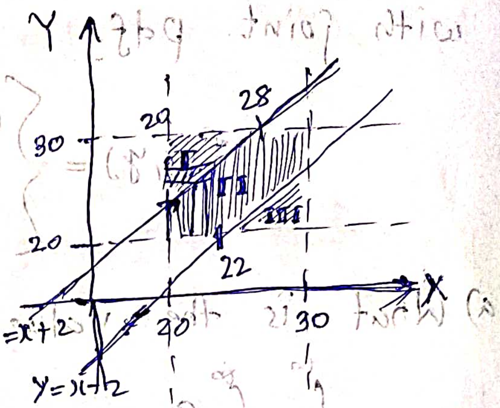
$$P(X < 26 \text{ and } Y < 26) = P(20 \leq X < 26 \text{ and } 20 \leq Y < 26)$$

$$= \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\,dx\,dy = \frac{3}{380{,}000}\left(\int_{20}^{26}\int_{20}^{26} x^2\,dx\,dy + \int_{20}^{26}\int_{20}^{26} y^2\,dx\,dy\right)$$

$$= 0.3024$$

(c.) What is the probability that the difference in air pressure bet^n two tires is at most 2 psi?

$$P\{|X-Y| \leq 2\} = P(-2 \leq X-Y \leq 2)$$



$$= P(X-2 \leq Y \leq X+2)$$

$$= \iint_{II} f(x,y)\,dx\,dy$$

$$= 1 - \left(\iint_{I} f(x,y)\,dx\,dy + \iint_{III} f(x,y)\,dx\,dy \right)$$

$$= 1 - \underbrace{\int_{20}^{28}\int_{x+2}^{30} K(x^2+y^2)\,dx\,dy}_{} - \int_{22}^{30}\int_{20}^{x-2} K(x^2+y^2)\,dx\,dy$$

$$= 1 - 0.3203 - 0.3203$$

$$= 0.3594.$$

$$\therefore K\int_{20}^{28}\int_{x+2}^{30} x^2\,dx\,dy + K\int_{20}^{28}\int_{x+2}^{30} y^2\,dx\,dy$$

$$= K\left(\int_{20}^{28} \left(y\Big|_{x+2}^{30}\right) x^2\,dx + \int_{20}^{28}\left(\frac{y^3}{3}\right)_{x+2}^{30} dx\right)$$

$$= K\left[\int_{20}^{28} x^2(30-x-2)\,dx + \frac{1}{3}\int_{20}^{28}\left(30^3-(x+2)^3\right)dx\right]$$

$$= K \times \left(16554.67 + \frac{1}{3}\cdot 72064\right) = 0.3203.$$

(d) Determine the (marginal) distribution of air pressure in the night time alone.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{20}^{30} K(x^2+y^2)\, dy$$

$$= K\left(x^2 \cdot y \Big|_{20}^{30} + \frac{y^3}{3}\Big|_{20}^{30} \right)$$

$$= K(10x^2 + 0.05)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{20}^{30} K(x^2+y^2)\, dx = K(10y^2 + 0.05)$$

(e) Are X and Y independent rv's.

To check $f(x,y) = f_X(x) \cdot f_Y(y)$

$$K(x^2+y^2) \neq K(10x^2+0.05) \cdot K(10y^2+0.05).$$

(Not Independent.)