8.4 Eigen basis. Diagonalization

Similarity of Matrices

A square matrix $\hat{A}$ of order n is said to be similar to a $n \times n$ square matrix A if there exists a non-singular matrix P such that

$$\hat{A} = P^{-1}AP$$

This transformation of A to $\hat{A}$ is known as ***Similarity transformation***.

Note:

1. Eigenvalues of A = Eigenvalues of $\hat{A}$
2. If X is an eigenvector of A corresponding to the eigenvalue λ i.e. $AX = \lambda X,\ X \neq 0$ then $Y = P^{-1}X$ is an eigenvector of $\hat{A}$ corresponding to same eigenvalue λ, means $\hat{A}(P^{-1}X) = \lambda(P^{-1}X), \quad P^{-1}X = Y \neq 0$

Basis of Eigenvectors

If A is a square matrix of order n and it has n distinct eigenvalues then the corresponding n eigenvectors are L. I. and they will form a basis which is known as ***basis of eigenvectors or Eigen basis.***

Note:

1. A symmetric matrix has an orthonormal basis of eigenvectors.
2. A Hermitian, Skew-Hermitian or unitary matrix has a basis of eigenvectors which is a unitary system or unitary Eigen basis.
3. A basis of eigenvectors is possible if algebraic multiplicity of λ = geometric multiplicity.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Eigenvalues

The characteristic equation is $|A - \lambda I| = 0.$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = 0 \qquad \Rightarrow \lambda^2 = 0 \qquad \Rightarrow \lambda = 0,\ 0$$

Hence for $\lambda = 0$, A. M. =2

Eigenvectors

The homogeneous system is $(A - \lambda I)X = 0, \quad X \neq 0.$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{aligned} 0x_1 + x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

Here $x_2 = 0$ but x_1 is arbitrary.

$\Rightarrow X = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, This is the only one eigenvector.

Hence G. M. =1

Therefore A cannot provide a basis of eigenvectors.

Diagonalization

Let A be any square matrix and X be the Modal matrix whose column vectors are eigenvectors of A. Then the ***Diagonalization*** of the matrix A is denoted by D and defined by

$$D = X^{-1}AX$$

Note:

1. Only those matrices can be diagonalized if there exists a basis of eigenvectors.
2. A square matrix A of order n with n L. I. eigenvectors is similar to a diagonal matrix D whose diagonal elements are eigenvalues of A.
3. Powers of a matrix A

   We have $D = X^{-1}AX$

   $\Rightarrow D^2 = DD = (X^{-1}AX)(X^{-1}AX) = X^{-1}A^2X$

   Similarly $D^3 = X^{-1}A^3X$

   In general, $D^n = X^{-1}A^nX$

   To obtain A^n we have to pre-multiply by X and post multiply by X^{-1}

   $\Rightarrow XD^nX^{-1} = XX^{-1}A^nXX^{-1} = A^n$

   Hence $A^n = XDX^{-1}$.

Procedure of Diagonalization

Any square matrix which has a basis of eigenvectors can be diagonalized as follows:

I. Find the eigenvalues of the given matrix A

II. Find eigenvectors.

III. Form a Modal matrix X by taking eigenvectors as its columns.

IV. Calculate X^{-1}.

V. The diagonal matrix is $D = X^{-1}AX$ (Main diagonal elements are eigenvalues of A).

Example: Diagonalize the following matrix

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

Solution: Eigenvalues

The characteristic equation is $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0 \qquad \Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$.

Eigenvector for $\lambda_1 = 0$

The homogeneous system is $(A - \lambda_1 I)X_1 = 0, \quad X_1 \neq 0$.

$$\Rightarrow \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \Rightarrow 2x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$$

Both the equations are identical.

$\Rightarrow x_2 = -2x_1$, Here x_1 is arbitrary.

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenvector for $\lambda_1 = 0$ is

$$X_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Eigenvector for $\lambda_2 = 3$

The homogeneous system is $(A - \lambda_2 I)X_2 = 0, \quad X_2 \neq 0.$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} \Rightarrow -x_1 + x_2 &= 0 \\ 2x_1 - 2x_2 &= 0 \end{aligned}$$

Both the equations are identical.

$\Rightarrow x_2 = x_1$, Here x_1 is arbitrary.

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The eigenvector for $\lambda_2 = 3$ is

$$X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The **modal matrix** is $X = [X_1 \quad X_2] = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$\Rightarrow |X| = 1 + 2 = 3$

$$\Rightarrow X^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow D = X^{-1}AX = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

This is the required diagonal matrix whose diagonal elements are eigenvalues of the matrix A.

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