2.2 **Homogeneous linear ODEs with constant coefficients**

A differential equation of the form

$$y'' + ay' + by = 0 \qquad (1)$$

where a and b are constants is known as second order homogeneous linear ODE with constant coefficients.

General Solution

Let $y = e^{\lambda x}$ be a solution of equation (1).

Now $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$.

Substituting y, y' and y'' in equation (1) we get

$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$

$\Rightarrow (\lambda^2 + a\lambda + b)e^{\lambda x} = 0$

$\Rightarrow (\lambda^2 + a\lambda + b) = 0 \qquad (2)$

Equation (2) is known as ***auxiliary equation or characteristics equation*** of the ODE (1).

Now $\lambda = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$

Case I: If $a^2 - 4b > 0$ we have ***two real and distinct roots*** λ_1 and λ_2.

Let $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two solutions of the ODE (1) which constitutes a basis of solution of the given ODE (1) on any interval.

The corresponding general solution is given by $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example: Find the general solution of the ODE $y'' + 8y' + 15y = 0$.

Solution: The auxiliary equation of the given ODE is

$$\lambda^2 + 8\lambda + 15 = 0$$
$$\Rightarrow (\lambda + 3)(\lambda + 5) = 0$$
$$\Rightarrow \lambda = -3, -5$$

Here the roots are real and distinct so the general solution is $y = c_1 e^{-3x} + c_2 e^{-5x}$.

Case II: If $a^2 - 4b = 0$ we have ***two real and repeated roots*** $\lambda_1 = \lambda_2 = \frac{-a}{2}$.

Since the roots are real and repeated, $y_1 = e^{\frac{-ax}{2}}$ is one solution.

The another solution y_2 can be obtained by using the formula $y_2 = y_1 \int \frac{1}{y_1^2} e^{-\int p(x)dx} dx$

$$\Rightarrow y_2 = e^{\frac{-ax}{2}} \int \frac{1}{e^{-ax}} e^{-\int a dx} dx = xe^{-\frac{ax}{2}}$$

The corresponding general solution is given by $y = (c_1 + c_2 x)e^{\frac{-ax}{2}} = (c_1 + c_2 x)e^{\lambda x}$.

Example: Find the general solution of the ODE.

Solution: The auxiliary equation of the given ODE is

$$\lambda^2 + 2\pi\lambda + \pi^2 = 0$$

$$\Rightarrow (\lambda+\pi)^2 = 0$$
$$\Rightarrow \lambda = -\pi, -\pi$$

Here the roots are real and repeated so the general solution is $y = (c_1 + c_2 x)e^{-\pi x}$.

Case III: If $a^2 - 4b < 0$ we have ***complex conjugate roots*** $\lambda = \alpha \pm i\beta$.

Let $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, then the general solution is

$$y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$$
$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$$
$$= e^{\alpha x}[c_1(\cos\beta x + i\sin\beta x) + c_2(\cos\beta x - i\sin\beta x)]$$
$$= e^{\alpha x}[(c_1+c_2)\cos\beta x + i(c_1-c_2)\sin\beta x]$$
$$= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$$

where $A = c_1 + c_2,\ B = i(c_1 - c_2)$.

Example: Solve the IVP $y'' + 9y = 0,\ y(0) = 0.2,\ y'(0) = -1.5$.

Solution: The auxiliary equation of the given ODE is

$$\lambda^2 + 9 = 0$$
$$\Rightarrow \lambda = \pm 3i$$

Here the roots are complex conjugate so the general solution is $y = A\cos 3x + B\sin 3x$.

Now, $y' = -3A\sin 3x + 3B\cos 3x$

$y(0) = 0.2 \Rightarrow A = \frac{1}{5}$ and $y'(0) = -1.5 \Rightarrow B = -\frac{1}{2}$

So, $y = \frac{1}{5}\cos 3x - \frac{1}{2}\sin 3x$.

Example: Find an ODE $y'' + ay' + by = 0$ for the given basis $e^{2.6x},\ e^{-4.3x}$.

Solution: If $e^{2.6x},\ e^{-4.3x}$ form a basis of solutions of an ODE, then $y = c_1 e^{2.6x} + c_2\ e^{-4.3x}$ is the general solution of that ODE.

From the general solution of the ODE we can obtain $\lambda = 2.6, -4.3$ which are real and distinct. So the corresponding auxiliary equation is

$$(\lambda - 2.6)(\lambda + 4.3) = 0$$
$$\Rightarrow \lambda^2 + 1.7\lambda - 11.18 = 0$$

So the corresponding ODE is $y'' + 1.7y' - 11.18y = 0$.

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