1 Convolution (6.5)

In general the Laplace transform of a product of two functions is different from the product of the transforms of the factors i.e.

$$\mathcal{L}(f(t)g(t)) \neq \mathcal{L}(f(t))\mathcal{L}(g(t)).$$

Verify!!! $\mathcal{L}(e^{2t}\sin t) \neq \mathcal{L}(e^{2t})\mathcal{L}(\sin t)$.
However, the convolution theorem states that $\mathcal{L}(f(t) * g(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t))$ where $f(t) * g(t)$ denotes the convolution of two functions $f(t)$ and $g(t)$ which is defined by the integral

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Theorem 1 (Convolution Theorem) *Let us consider two functions $f(t)$ and $g(t)$ satisfy the assumption in the existence theorem and denote that $F(s) = \mathcal{L}(f(t))$ and $G(s) = \mathcal{L}(g(t))$, then $\mathcal{L}^{-1}(F(s)G(s)) = f(t) * g(t)$.*

Example 1 *Find the convolution of t and e^t.*

Solution: The convolution of t and e^t is given by

$$\begin{aligned} t * e^t &= \int_0^t \tau \cdot e^{(t-\tau)} d\tau \\ &= e^t \int_0^t \tau \cdot e^{-\tau} d\tau \\ &= e^t[-\tau e^{-\tau} - e^{-\tau}]_0^t \\ &= e^t[-te^{-t} - e^{-t} + 1] \\ &= e^t - t - 1 \end{aligned}$$

Example 2 *Find Laplace inverse of the function $\frac{2\pi s}{(s^2+\pi^2)^2}$.*

Solution: Let us take $\frac{2\pi s}{(s^2+\pi^2)^2} = \frac{2\pi}{s^2+\pi^2} \cdot \frac{s}{s^2+\pi^2} = F(s) \cdot G(s)$, where $F(s) = \frac{2\pi}{s^2+\pi^2}$ and $G(s) = \frac{s}{s^2+\pi^2}$.
Now, by Convolution Theorem,

$$\mathcal{L}^{-1}\left(\frac{2\pi s}{(s^2+\pi^2)^2}\right) = \mathcal{L}^{-1}(F(s) \cdot G(s)) = f(t) * g(t),$$

where, $f(t) = \mathcal{L}^{-1}(F(s))$ and $g(t) = \mathcal{L}^{-1}(G(s))$.
We calculate $f(t)$ and $g(t)$ as follows:

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(\frac{2\pi}{s^2+\pi^2}\right) = 2\sin(\pi t)$$

and

$$g(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\pi^2}\right) = \cos(\pi t)$$

Then,

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{2\pi s}{(s^2+\pi^2)^2}\right) &= \mathcal{L}^{-1}(F(s)\cdot G(s)) \\
&= f(t) * g(t) \\
&= \int_0^t f(\tau)g(t-\tau)d\tau \\
&= \int_0^t 2\sin(\pi\tau)\cos\pi(t-\tau)d\tau \\
&= \int_0^t \sin(\pi t) - \sin\pi(2\tau - t)d\tau \\
&= \int_0^t \sin(\pi t)d\tau - \int_0^t \sin\pi(2\tau-t)d\tau \\
&= \sin(\pi t)[\tau]_0^t + \frac{1}{2\pi}[\cos\pi(2\tau-t)]_0^t \\
&= t\sin(\pi t) + \frac{1}{2\pi}[\cos(\pi t) - \cos(-\pi t)] \\
&= t\sin(\pi t)
\end{aligned}$$

Apart from finding Laplace inverse, Convolution theorem also helps in solving certain **integral equations (Volterra integral equations)** where the integral is of the form of a convolution.

Example 3 *Solve the integral equation* $y(t) + 4\int_0^t y(\tau)(t-\tau)d\tau = 2t$ *by using Laplace transform.*

Solution: Let us denote $\mathcal{L}(y(t)) = Y(s)$*. Now, observe that* $\int_0^t y(\tau)(t-\tau)d\tau = y(t) * t$*.*
So, the given equation can be written as

$$y(t) + 4y(t) * t = 2t$$

Taking Laplace transform on both sides,

$$\begin{aligned}
&\mathcal{L}(y(t)) + 4\mathcal{L}(y(t) * t) = 2\mathcal{L}(t) \\
\Rightarrow &Y(s) + 4Y(s)\cdot\frac{1}{s} = \frac{2}{s} \\
\Rightarrow &(1+\frac{4}{s})Y(s) = \frac{2}{s} \\
\Rightarrow &Y(s) = \frac{2}{s+4} \\
\Rightarrow &y(t) = \mathcal{L}^{-1}\left(\frac{2}{s+4}\right) = 2e^{-4t}
\end{aligned}$$

2 Differentiation and integration of Laplace transforms (6.6)

If a function $f(t)$ satisfies the assumption in the existence theorem and $\mathcal{L}(f(t)) = F(s)$,

$$\mathcal{L}(tf(t)) = -F'(s) \quad \text{or} \quad \mathcal{L}^{-1}(F'(s)) = -tf(t)$$

Also,

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s})\ d\bar{s} \quad \text{or} \quad \mathcal{L}^{-1}\left(\int_s^\infty F(\bar{s})\ d\bar{s}\right) = \frac{f(t)}{t}$$

Example 4 *Find the Laplace transform of $t^2 sin\omega t$.*

Solution: To find $\mathcal{L}(t^2 sin\omega t)$, we have to evaluate $\mathcal{L}(tsin\omega t) first$. Now,

$$\mathcal{L}(tsin\omega t) = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Then,

$$\mathcal{L}(t^2 sin\omega t) = \mathcal{L}(t \cdot tsin\omega t) = -\frac{d}{ds}\left(\frac{2\omega s}{(s^2+\omega^2)^2}\right) = -\frac{2\omega(s^2+\omega^2)^2 - 8\omega s^2(s^2+\omega^2)}{(s^2+\omega^2)^4} = \frac{2\omega(3s^2-\omega^2)}{(s^2+\omega^2)^3}$$

Example 5 *Find the Laplace inverse of $\ln\frac{s^2+1}{(s-1)^2}$.*

Solution: Let $F(s) = \ln\frac{s^2+1}{(s-1)^2}$. Then,

$$\begin{aligned}
&F(s) = \ln(s^2+1) - 2\ln(s-1) \\
&\Rightarrow F'(s) = \frac{2s}{s^2+1} - \frac{2}{s-1} \\
&\textit{taking Laplace inverse on both sides} \\
&\Rightarrow \mathcal{L}^{-1}(F'(s)) = \mathcal{L}^{-1}\left(\frac{2s}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{2}{s-1}\right) \\
&\Rightarrow -tf(t) = 2\cos t - 2e^t \\
&\Rightarrow f(t) = -\frac{2}{t}(\cos t - e^t).
\end{aligned}$$

Therefore, $\mathcal{L}^{-1}\left(\ln\frac{s^2+1}{(s-1)^2}\right) = -\frac{2}{t}(\cos t - e^t)$.

Example 6 *Find the Laplace inverse of $\frac{2s+6}{(s^2+6s+10)^2}$.*

Solution: Let $F(s) = \frac{2s+6}{(s^2+6s+10)^2}$. Then,

$$\begin{aligned}
&F(\bar{s}) = \frac{2\bar{s}+6}{(\bar{s}^2+6\bar{s}+10)^2} \\
&\Rightarrow \int_s^\infty F(\bar{s})\ d\bar{s} = \int_s^\infty \frac{2\bar{s}+6}{(\bar{s}^2+6\bar{s}+10)^2}\ d\bar{s} \\
&\Rightarrow \int_s^\infty F(\bar{s})\ d\bar{s} = \int_s^\infty \frac{d(\bar{s}^2+6\bar{s}+10)}{(\bar{s}^2+6\bar{s}+10)^2}\ d\bar{s} \\
&\Rightarrow \int_s^\infty F(\bar{s})\ d\bar{s} = -\left[\frac{1}{(\bar{s}^2+6\bar{s}+10)}\right]_s^\infty \\
&\Rightarrow \int_s^\infty F(\bar{s})\ d\bar{s} = \frac{1}{(s^2+6s+10)} \\
&\textit{taking Laplace inverse both sides} \\
&\Rightarrow \mathcal{L}^{-1}\left(\int_s^\infty F(\bar{s})\ d\bar{s}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s^2+6s+10)}\right) \\
&\Rightarrow \frac{f(t)}{t} = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^2+1}\right) \\
&\Rightarrow \frac{f(t)}{t} = e^{-3t}\sin t \quad \textit{(by applying s-shifting property)} \\
&\Rightarrow f(t) = e^{-3t}t\sin t
\end{aligned}$$

Therefore, $\mathcal{L}^{-1}\left(\frac{2s+6}{(s^2+6s+10)^2}\right) = e^{-3t}t\sin t$.