Q (7). [3-1], (PDF-Q(1)) Problems — 3.1

For each random variable defined here, describe the set of possible values for the variable, and state whether the variable is discrete.

a) X = the number of unbroken eggs in a randomly chosen standard egg carton.

b) Y = the number of students on a class list for a particular course who are absent on the first day of classes.

c) U = the number of times a duffer has to swing at a golf ball before hitting it.

d) X = the length of a randomly selected rattlesnake

e) Z = the amount of royalties earned from the sale of a first edition of 10,000 textbooks

f) Y = the PH of a randomly chosen soil sample

g) X = the tension (psi) at which a randomly selected tennis racket has been strung

h) X = the total number of coin tosses required for three individuals to obtain a match (HHH or TTT)

Ans : Discrete data are restricted to defined separated values, for example integers or counts.

Continuous data are not restricted to define separate values but can occupy any value over a continuous range.

(a) A standard egg carton contains 12 eggs.

∴ X = number of unbroken eggs in a randomly chosen standard egg carton

$= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Since it is possible to list all separate values, X is discrete.

(b) Since we don't know how many students are on the list, the number of students is some nonnegative integer.

∴ Y = number of students on the class list that are absent on the 1st day of classes

$= \{0, 1, 2, 3, \dashrightarrow N\}$, N = no. of students in the list

Since the possible values are all integers, X is discrete.

(c) Since we don't know how many swings are required until the first hit, the number of swings is some +ve ~~nonnegative~~ integer.

U = number of swings until hit

$= \{1, 2, 3 ----\}$

Since the possible values are all integers, U is discrete.

(d) The length of a rattle snake can be any +ve real number, as zero length is not possible.

X = the length of a randomly selected rattle snake

$= \{x : 0 < l(x) < \infty\}$, where $l(x)$ = length of the snake

which is a continuous random variable.

(e) Z = the amount of royalties earned from the sale of a 1st edition of 10,000 text books

$= \{0, c, 2c, 3c, \ldots, 10{,}000c\}$, c = royalty per book

Z is a discrete r. v.

(f) Since 0 is the smallest possible pH and 14 is the largest possible pH, possible values of Y are

$Y = \{y : 0 \leq y \leq 14\} = [0, 14]$

$\therefore$ Y is not discrete.

(g) X = the tension (psi) at which a randomly selected tennis racket has been strung

$= \{x : m \leq x \leq M\}$, where m = minimum possible tension

M = maximum possible tension

X is continuous r. v.

(h) The number of possible tries are 1, 2, 3 ---; and each try involve 3 racket spins.

$\therefore$ The possible values of X are

$X = \{3x : x = 1, 2, 3 \ldots\}$

$= \{3, 6, 9, 12 \ldots\}$

X is discrete r. v. as all possible values are integers.

Q.(6) [3.1]

Starting at a fixed time, each car entering an intersection is observed to see whether it turns left (L), right (R), or goes straight ahead (A). The experiment terminates as soon as a car is observed to turn left. Let x = the number of cars observed. What are possible x values? List five outcomes and their associated x values.

Ans: Given L = turn left, R = turn right, A = straight ahead.

Since the experiment terminates as soon as a car is observed to turn left (L), so the possible outcomes are

$$S = \{L, RL, AL, RAL, ARL, RRL, AAL, \dots\}$$

X counts no. of cars observed, so

$$X = \{1, 2, 2, 3, 3, 3, 3, \dots\}$$
$$= \{1, 2, 3, \dots\}$$

Q. (10) [3.1]

The number of pumps in use at both a six pump station and a four pump station will be determined. Give the possible values for each of the following random variables:

a) T = the total number of pums in use

b) X = the difference between the numbers in use at stations 1 and 2

c) U = the maximum number of pumps in use at either station.

d) Z = the number of stations having exactly two pumps in use

Ans:- (a) $T = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

0, 1, 2, 3, 4, 5, 6
0, 1, 2, 3, 4

(b) $X = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

where, X = difference between the numbers in use at Stations 1 and 2.

station 1 → 6 pumps-station → no. of pumps in use → 0, 1, 2, 3, 4, 5, 6
station 2 → 4 " " → " " " → 0, 1, 2, 3, 4

(c) $U = \{0, 1, 2, 3, 4, 5, 6\}$

(d) $Z = \{0, 1, 2\}$

Problems: 3.2

Q.(12) [3.2]

Airlines sometimes overbook flights. suppose that for a plane with 50 seats, 55 passengers have tickets. Define the random variable Y as the number of ticketed passengers who actually show up for the flight. The probability mass function of Y appears in the accompanying table.

y :	45	46	47	48	49	50	51	52	53	54	55
P(y) :	0.05	0.10	0.12	0.14	0.25	0.17	0.06	0.05	0.03	0.02	0.01

a) what is the probability that the flight will accommodate all ticketed passengers who show up?

b) what is the probability that not all ticketed passengers who show up can be accomodated?

c) ~~what is the probability~~ If you are the first person on the standby list (which means you will be the first one to get on the plane if there are any seats available after all ticketed passengers have been accommodated), what is the probability that you will be able to take the flight? what is the probability that if you are the third person on the standby list?

Ans:

(a) The probability that the flight will accommodate all ticketed passengers who show up means that $Y \le 50$ [because there are 50 seats]

$$P(Y \le 50) = P(Y=45) + P(Y=46) + P(Y=47) + P(Y=48) + P(Y=49) + P(Y=50)$$
$$= p(45) + p(46) + p(47) + p(48) + p(49) + p(50)$$
$$= 0.05 + 0.1 + 0.12 + 0.14 + 0.25 + 0.17$$
$$= 0.83$$

(b) P(not all ticketed passengers who show up can be accommodated)

$$= P(Y > 50)$$
$$= 1 - P(Y \le 50)$$
$$= 1 - 0.83$$
$$= 0.17$$

or $P(Y > 50)$

$$= p(51) + p(52) + p(53) + p(54) + p(55)$$
$$= 0.17$$

(c) Assume that you are the 1st person on the standby. Since there are 50 available seats, at most 49 can show up for you to get the seat.

∴ we have to find $P(Y \le 49)$.

$$P(Y \le 49) = p(45) + p(46) + p(47) + p(48) + p(49)$$
$$= 0.05 + 0.1 + 0.12 + 0.14 + 0.25 = 0.66$$

Assume that you are the 3rd person on the standby. At most 47 can show up so you can get a seat.

$$\therefore P(Y \le 47) = p(45) + p(46) + p(47) = 0.05 + 0.1 + 0.12$$
$$= 0.27$$

Q(13) [3.2]

A mail-order computer business has six telephone lines. Let X denote the number of lines in use at a specified time. Suppose that the pmf of x is as given in the accompanying table.

x:	0	1	2	3	4	5	6
$P(x)$:	0.10	0.15	0.20	0.25	0.20	0.06	0.04

Calculate the probability of each of the following events.

a). {at most three lines are in use}

b) { fewer than three lines are in use}

c) {at least three lines are in use}

d) { between two and five lines, inclusive, are in use}

e) { between two and four lines, inclusive, are not in use}

f) { at least four lines are not in use}

Ans:

a) P(at most three lines are in use)

$= P(X \leq 3)$

$= P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= P(0) + P(1) + P(2) + P(3)$

$= 0.10 + 0.15 + 0.20 + 0.25$

$= 0.7$

(b) P{fewer than three lines are in use}

$= P(X<3) = P(X=0) + P(X=1) + P(X=2)$

$= p(0) + p(1) + p(2) = 0.1 + 0.15 + 0.2 = 0.45$

(c) P{at least three lines are in use}

$= P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= p(3) + p(4) + p(5) + p(6) = 0.25 + 0.2 + 0.06 + 0.04 = 0.55$

(d) P{between two and five lines, inclusive, are in use}

$= P(2 \leq X \leq 5)$

$= P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= p(2) + p(3) + p(4) + p(5)$

$= 0.2 + 0.25 + 0.2 + 0.06 = 0.71$

(e) The number of lines not in use is $6-X$ [$\because$ X counts how many lines are in use]

$\therefore$ P(between 2 and 4 lines, inclusive, are not in use)

$= P(2 \leq 6-X \leq 4) = P(-4 \leq -X \leq -2)$

$= P(2 \leq X \leq 4)$

$= p(2) + p(3) + p(4) = 0.2 + 0.25 + 0.2 = 0.65$

(f) P(at least 4 lines are not in use)

$= P(6-X \geq 4) = P(-X \geq -2)$

$= P(X \leq 2)$

$= p(0) + p(1) + p(2)$

$= 0.1 + 0.15 + 0.2$

$= 0.45$

Q. (14). [3.2]

A Contractor is required by a county planning department to submit one, two, three, four or five forms (depending on the nature of the project) in applying for a building permit. Let Y = the number of forms required of the next applicant. The Probability that y forms are required is known to be proportion to y that is $p(y) = ky$ $(y = 1, 2, \ldots, 5)$

a) what is the value of K?

b) what is the probability that at most three forms are required?

c) what is the probability that between two and four forms (inclusive) are required?

d) Could $p(y) = \frac{y^2}{50}$ for $y = 1, 2, \ldots, 5$ be the pmf of Y?

Ans: Given that $p(y) = ky$, $y = 1, 2, 3, 4, 5$.

(a) For the pmf $p(y)$, we know that

$$\sum_y p(y) = 1$$

$$\Rightarrow \sum_{y=1}^{5} p(y) = 1$$

$$\Rightarrow p(1) + p(2) + p(3) + p(4) + p(5) = 1$$

$$\Rightarrow K + 2K + 3K + 4K + 5K = 1$$

$$\Rightarrow 15K = 1$$

$$\Rightarrow \boxed{K = \frac{1}{15}}$$

(b) P(at most 3 forms are required)

$= P(Y \le 3)$

$= p(1) + p(2) + p(3)$

$= k + 2k + 3k = 6k = 6 \times \frac{1}{15} = \frac{2}{5} = 0.4 = 40\%$ $\left[\because k = \frac{1}{15}\right]$

(c) P(between 2 and 4 forms (inclusive) are required)

$= P(2 \le Y \le 4)$

$= p(2) + p(3) + p(4)$

$= 2k + 3k + 4k = 9k = \frac{9}{15} = \frac{3}{5} = 0.6 = 60\%.$

(d) $p(y) = \frac{y^2}{50}$ $(y = 1, 2, --, 5)$ will be the pmf of Y

if $\sum_y p(y) = 1$

Now $\sum_{y=1}^{5} p(y) = \sum_{y=1}^{5} \frac{y^2}{50}$

$= \frac{1^2}{50} + \frac{2^2}{50} + \frac{3^2}{50} + \frac{4^2}{50} + \frac{5^2}{50}$

$= \frac{1}{50}(1 + 4 + 9 + 16 + 25)$

$= \frac{55}{50} = \frac{11}{10} = 1.1 \neq 1$

$\therefore$ The given probability distribution is not a valid probability distribution.

$\therefore$ $p(y) = \frac{y^2}{50}$ $(y = 1, 2, --, 5)$ can not be pmf of Y.

Q(17) [3.2]

A new battery's voltage may be acceptable (A) or unacceptable (U). A certain flashlight requires two batteries, so batteries will be independently selected and tested until two acceptable ones have been found. Suppose that 90% of all batteries have acceptable voltages. Let Y denote the number of batteries that must be tested.

a) what is $p(2)$, that is $P(Y=2)$?

b) What is $p(3)$?

c) To have $Y=5$, what must be true of fifth battery selected? List the four outcomes for which $Y=5$ and then determine $p(5)$

d) Use the pattern in your answers for parts (a)-(c) to obtain a general formula for $p(y)$.

Ans:- $P(A) = 90\% = 0.9$

$P(U) = 10\% = 0.1$, $Y =$ no. of batteries must be tested.

(a) Let $F_2 = \{$ first two batteries are acceptable $\}$

$\therefore p(2) = P(Y=2) = P(F_2) = \cancel{P(A \cap A)} = P(A \cap A)$

$= P(A) \cdot P(A) = 0.9 \times 0.9 = 0.81.$

[$\because$ A, B are independent iff $P(A \cap B) = P(A) P(B)$]

(b) $P(3) = P(Y=3) = P\{(U \cap A \cap A) \cup (A \cap U \cap A)\}$

$= P(U \cap A \cap A) + P(A \cap U \cap A)$

$= P(U) P(A) P(A) + P(A) P(U) P(A)$

$= 0.1 \times 0.9 \times 0.9 + 0.9 \times 0.1 \times 0.9 = 0.162$

[$\because$ only 2 possible outcomes are UAA, AUA]

(c) To have $Y=5$, fifth battery must give acceptable voltage.

$\therefore$ Four outcomes for which $Y=5$ are

AUUUA, UAUUA, UUAUA, UUUAA

$$\therefore p(5) = P(Y=5)$$

$$= P[(AUUUA) \cup (UAUUA) \cup (UUAUA) \cup (UUUAA)]$$

$$= 4 \times P(A)\,P(U)\,P(U)\,P(U)\,P(A)$$

$$= 4 \times 0.9 \times 0.1 \times 0.1 \times 0.1 \times 0.9 = 0.00324$$

(d) from (a), $p(2) = 0.9^2 \qquad = 1 \times 0.81$

(b), $p(3) = 2 \times 0.1 \times 0.9^2 \quad = 2 \times 0.81 \times 0.1$

(c) $p(5) = 4 \times 0.9^2 \times 0.1^3 = 4 \times 0.81 \times 0.1^3$

$\vdots$

$$p(y) = (y-1) \times 0.81 \times (0.1)^{y-2}$$

$$= 0.81\,(y-1)\,(0.1)^{y-2}$$

Q(23) [3.2]

A consumer organization that evaluates new automobiles customarily reports the number of major defects in each car examined. Let X denote the number of major defects in a randomly selected car of a certain type. The cdf of X is as follows

$$F(x) = \begin{cases} 0, & x<0 \\ 0.06, & 0\le x<1 \\ 0.19, & 1\le x< 2 \\ 0.39, & 2\le x<3 \\ 0.67, & 3\le x<4 \\ 0.92, & 4\le x<5 \\ 0.97, & 5\le x<6 \\ 1, & 6\le x \end{cases}$$

Calculate the following probabilities directly from the cdf:

a) $P(2)$ that is $P(X=2)$

b) $P(X>3)$

c) $P(2\le X\le 5)$

d) $P(2<X<5)$

Ans: The cdf $F(x) = P(X\le x)$.

Again for $a, b \in \mathbb{R}$, with $a\le b$, the following holds

$$P(a\le X\le b) = F(b) - F(a-)$$

where $a-$ stands for largest possible value of X that is less than a.

If a, b are integers, then $P(a\le X\le b) = F(b) - F(a-1)$

(a) $p(2) = P(X=2) = P(2 \leq X \leq 2)$

$= F(2) - F(2-1)$

$= F(2) - F(1) = 0.39 - 0.19 = 0.2$

(b) $P(X > 3) = 1 - P(X \leq 3)$

$= 1 - F(3)$

$= 1 - 0.67 = 0.33$

(c) $P(2 \leq X \leq 5) = F(5) - F(2-1)$

$= F(5) - F(1)$

$= 0.97 - 0.19 = 0.78$

(d) $P(2 < X < 5) = P(3 \leq X \leq 4)$

$= F(4) - F(3-1)$

$= F(4) - F(2)$

$= 0.92 - 0.39$

$= 0.53$

Q. (24) [3.2]

An insurance company offers its policy holders a number of different premium payment options. For a randomly selected policy holder, let X = the number of months between successive payments. The cdf of X is as follows:

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.30, & 1 \le x < 3 \\ 0.40, & 3 \le x < 4 \\ 0.45, & 4 \le x < 6 \\ 0.60, & 6 \le x < 12 \\ 1, & 12 \le x \end{cases}$$

a) What is pmf of X?

b) Using just the cdf, compute $P(3 \le X \le 6)$ and $P(4 \le X)$.

Ans: The pmf is $p(x) = P(X = x)$

cdf is $F(x) = P(X \le x) = \sum_{y \le x} p(y)$

(a) cdf is given. We have to find pmf.

The values that random variable X takes are the "jump" values (the values where the function F jumps) and those are 1, 3, 4, 6, 12.

The probability is the size of particular jump.

$\therefore$ $p(1) = 0.30 - 0 = 0.30$

$p(3) = 0.40 - 0.30 = 0.10$

$p(4) = 0.45 - 0.40 = 0.05$

$p(6) = 0.60 - 0.45 = 0.15$

$p(12) = 1 - 0.60 = 0.40$

and $p(x) = 0$ for $x \notin \{1, 3, 4, 6, 12\}$.

$\therefore$ The pmf is $p(x) = \begin{cases} 0.3, & x=1 \\ 0.1, & x=3 \\ 0.05, & x=4 \\ 0.15, & x=6 \\ 0.4, & x=12 \\ 0, & x \neq \{1, 3, 4, 6, 12\} \end{cases}$

(b) $P(3 \le X \le 6) = F(6) - F(3-1)$

$= F(6) - F(2)$

$= 0.60 - 0.30 = 0.30$

$P(4 \le X) = P(X \geq 4)$

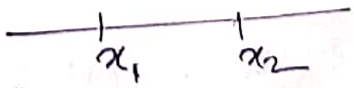
$= 1 - P(X < 4)$

$= 1 - 0.40 = 0.60$

Q.(28) [3.2]

Show that the cdf $F(x)$ is a non-decreasing function, that is $x_1 < x_2$ implies $F(x_1) \le F(x_2)$. Under what condition will $F(x_1) = F(x_2)$?

Ans: Given $x_1 < x_2$



now $F(x_2) = P(X \le x_2) = P\{(X \le x_1) \cup \{x_1 < X \le x_2\}\}$

$= P(X \le x_1) + P(x_1 < X \le x_2)$

$= F(x_1) + P(x_1 < X \le x_2)$

$\therefore F(x_2) \geq F(x_1)$ $[\because P(x_1 < X \le x_2) \geq 0]$

$\Rightarrow F(x_1) \le F(x_2)$ (proved)

If $P(x_1 < X \le x_2) = 0$ then $F(x_1) = F(x_2)$.