Numerical Solution to ODE

We know that an ODE of the first order is of the form $F(x, y, y') = 0$ and can often be written in the explicit form $y' = f(x, y)$. An **initial value problem** for this equation is of the form

$$y' = f(x, y), \ \ y(x_0) = y_0$$

where x_0 and y_0 are given and we assume that the problem has a unique solution on some open interval $a < x < b$ containing x_0. We shall discuss methods of computing approximate numeric values of the solution of the above initial value problem at the equidistant points on the x-axis $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \dots$ where the **step size** h is a fixed positive real number. These methods are suggested by the Taylor series expansion of an infinitely differentiable function.

Taylor series Method.

Suppose f is an infinitely differentiable function (i.e., f can be differentiated infinitely often). Then, the **Taylor series** expansion of f about $x_0$is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Let f be infinitely differentiable function on the interval $(x_0 - r, x_0 + r)$ and $x \in (x_0 - r, x_0 + r)$. Define

$$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Then

$$R_n(x) = \frac{f^{(n+1)}(w)}{(n+1)!}(x - x_0)^{n+1}$$

for some w between x_0 and x. Moreover, if

$$\lim_{n \to \infty} R_n(x) = 0$$

then the Taylor series for f expanded about x_0 converges to $f(x)$, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

for all $x \in (x_0 - r, x_0 + r)$.

To solve the **initial value problem** of the form

$$y' = f(x, y), \ \ y(x_0) = y_0$$

where x_0 and y_0 are given, we assume that the problem has a unique solution on some open interval $a < x < b$ containing x_0 and y is an infinitely differentiable function on (a, b). Let $x = x_0 + h$, then by Taylor series expansion of y, we have

$$y(x_0 + h) = y(x) = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}(x - x_0)^k = \sum_{k=0}^{\infty} \frac{y^{(k)}(x_0)}{k!}h^k.$$

That is,

$$y(x_0 + h) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots.$$

To compute approximate numeric values of the solution of the above initial value problem at the equidistant points on the x-axis $x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, \dots,$ we use the above form of Taylor series expansion. Thus we have the approximate values are

$$y(x_1) = y(x_0 + h) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots,$$

$$y(x_2) = y(x_1 + h) = y(x_1) + h\,y'(x_1) + \frac{h^2}{2}y''(x_1) + \frac{h^3}{3!}y'''(x_1) + \cdots,$$

$$y(x_3) = y(x_2 + h) = y(x_2) + h\,y'(x_2) + \frac{h^2}{2}y''(x_2) + \frac{h^3}{3!}y'''(x_2) + \cdots,$$

and so on. In general

$$y(x_{k+1}) = y(x_k + h) = y(x_k) + h\,y'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{3!}y'''(x_k) + \cdots.$$

Example: Solve the initial value problem

$$y' = xy,\ y(0) = 1.$$

in the interval $[0,1]$ with $h = 0.2$ using Taylor series method upto 3rd order derivative.

Solution: We have to solve the initial value problem

$$y' = xy,\ y(0) = 1.$$

in the interval $[0,1]$ with $h = 0.2$. That is we have to find the values of $y(x)$ at

$$x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8, x_5 = 1.0\,.$$

It is given that $x_0 = 0,\ \ y(x_0) = 1,\ \ h = 0.2$. Let denote $y(x_k) = y_k$. We have to use Taylor series method up to 3rd order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\,y'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{3!}y'''(x_k).$$

That is,

$$y_{k+1} = y_k + h\,y_k' + \frac{h^2}{2}y_k'' + \frac{h^3}{3!}y_k'''.$$

Given that, $y' = xy$. So, $y'' = (1 + x^2)y$ and $y''' = (3x + x^3)y$. Now using $h = 0.2$ and the given initial condition $y(0) = 1$, that is $x_0 = 0, y_0 = 1$, we find,

$$y_0' = x_0 y_0 = 0,\ \ y_0'' = (1 + x_0^2)y_0 = 1,\ \ y_0''' = (x_0 + x_0^3)y_0 = 0.$$

Thus,

$$y_1 = y_0 + h\,y_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{3!}y_0''' = 1.02.$$

Similarly we calculate y_2, y_3, y_4, y_5 and are given in the below table.

k	x_k	y_k	y_k'	y_k''	y_k'''	y_{k+1}
0	0	1	0	1	0	1.0200
1	0.2	1.0200	0.2040	1.0608	0.6202	1.0828
2	0.4	1.0828	0.4331	1.2561	1.3687	1.1964
3	0.6	1.1964	0.7179	1.6271	2.4120	1.3757
4	0.8	1.3757	1.1006	2.2562	4.0062	1.6463
5	1.0	1.6463	---	---	---	---

From the above table we get the solution of the initial value problem as

$$y(0.2) = 1.0200, y(0.4) = 1.0828, y(0.6) = 1.1964, y(0.8) = 1.3757, y(1.0) = 1.6463\,.$$

Note: The exact solution is $y(x) = e^{\frac{x^2}{2}}$, and so the exact value of $y(1)$ is 1.6487.

Example: Solve the initial value problem

$$y' = x + y,\ y(0) = 0.$$

in the interval $[0,1]$ with $h = 0.25$ using Taylor series method upto 3rd order derivative.

Solution: We have to solve the initial value problem

$$y' = x + y,\ y(0) = 0.$$

in the interval $[0,1]$ with $h = 0.25$. That is we have to find the values of $y(x)$ at

$$x_1 = 0.25, x_2 = 0.50, x_3 = 0.75, x_4 = 1.0\,.$$

It is given that $x_0 = 0,\ \ y(x_0) = 0,\ \ h = 0.25$. Let denote $y(x_k) = y_k$. We have to use Taylor series method up to 3rd order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\,y'(x_k) + \frac{h^2}{2}y''(x_k) + \frac{h^3}{3!}y'''(x_k).$$

That is,

$$y_{k+1} = y_k + h\,y_k' + \frac{h^2}{2}y_k'' + \frac{h^3}{3!}y_k'''.$$

Given that, $y' = x + y$. So, $y'' = 1 + x + y$ and $y''' = 1 + x + y$. Now using $h = 0.25$ and the given initial condition $y(0) = 0$, that is $x_0 = 0, y_0 = 0$, we find,

$$y_0' = x_0 + y_0 = 0,\ \ y_0'' = 1 + x_0 + y_0 = 1,\ \ y_0''' = 1 + x_0 + y_0 = 1$$

$y_0 = x_0 + y_0 = 0, \; y_0 = 1 + x_0 + y_0 = 1, \; y_0 = 1 + x_0 + y_0 = 1.$

Thus,

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' = 0.0339.$$

Similarly, we calculate y_2, y_3, y_4 and are given in the below table.

k	x_k	y_k	y_k'	y_k''	y_k'''	y_{k+1}
0	0	0	0	1	1	0.0339
1	0.25	0.0339	0.2839	1.2839	1.2839	0.1483
2	0.50	0.1483	0.6483	1.6483	1.6483	0.3662
3	0.75	0.3662	1.1162	2.1162	2.1162	0.7168
4	1.0	0.7168	---	---	---	---

From the above table we get the solution of the initial value problem as

$$y(0.25) = 0.0339, y(0.50) = 0.1483, y(0.75) = 0.3662, y(1.0) = 0.7168.$$

Note: The exact solution is $y(x) = e^x - x - 1$, and so the exact value of $y(1)$ is 0.7183.

Example: Solve the initial value problem

$$y' = (x+y)^2, \; y(0) = 0; \quad h = 0.1$$

using Taylor series method up to 3rd order derivative. Do 10 steps.

Solution: We have to solve the initial value problem

$$y' = (x+y)^2, \; y(0) = 0.$$

with $h = 0.25$. That is we have to find the values of $y(x)$ at

$$x_1 = 0.1, x_2 = 0.1, x_3 = 0.3, \ldots, x_9 = 0.9, x_{10} = 1.0.$$

It is given that $x_0 = 0, \; y(x_0) = 0, \; h = 0.1$. Let denote $y(x_k) = y_k$. We have to use Taylor series method up to 3rd order derivative. That is we have to use

$$y(x_{k+1}) = y(x_k) + h\, y'(x_k) + \frac{h^2}{2} y''(x_k) + \frac{h^3}{3!} y'''(x_k).$$

That is,

$$y_{k+1} = y_k + h\, y_k' + \frac{h^2}{2} y_k'' + \frac{h^3}{3!} y_k'''.$$

Given that, $y' = (x+y)^2$.

So,

$$y'' = 2(x+y)(1+y')$$

And

$$y''' = 2(1 + 4y' + 3y'^2)$$

Using $h = 0.1$ and the given initial condition $y(0) = 0$, that is $x_0 = 0, y_0 = 0$, we do the following 10 steps.

Step 1.

$$y_0' = (x_0 + y_0)^2 = 0, \; y_0'' = 2(x_0 + y_0)(1 + y_0') = 0, \; y_0''' = 2(1 + 4y_0' + 3y_0'^2) = 2.$$

Thus,

$$y_1 = y_0 + h\, y_0' + \frac{h^2}{2} y_0'' + \frac{h^3}{3!} y_0''' = .$$

Similarly, we do the rest steps and are given in the below table.

k	x_k	y_k	y_k'	y_k''	y_k'''	y_{k+1}
1	0	0.0000	0.0000	0.0000	2.0000	0.0003
2	0.1	0.0003	0.0101	0.2007	2.0811	0.0027
3	0.2	0.0027	0.0411	0.4061	2.3388	0.0092
4	0.3	0.0092	0.0956	0.6241	2.8198	0.0224
5	0.4	0.0224	0.1784	0.8716	3.6181	0.0452
6	0.5	0.0452	0.2972	1.1867	4.9077	0.0816
7	0.6	0.0816	0.4646	1.6576	7.0124	0.1376
8	0.7	0.1376	0.7015	[illegible]	10.5649	0.2220
9	0.8	0.2220	1.0444	4.2736	16.9005	0.3506
10	0.9	0.3506	1.5640	8.6194	29.1887	0.5550

11	1	0.5550	---	---	---	---

Note: The exact solution is $y(x) = \tan x - x$, and so the exact value of $y(1)$ is 0.05574.

Euler's method

From use Taylor series method, we have,

$$y(x_{k+1}) = y(x_k) + h\,y^{'}(x_k) + \frac{h^2}{2}y^{''}(x_k) + \frac{h^3}{3!}y^{'''}(x_k) + \cdots.$$

For small h the higher powers h^2, h^3, ..., are very small and if we drop all of them to get the crude approximation

$$y(x_{k+1}) = y(x_k) + h\,y^{'}(x_k)$$

i.e.,

$$y_{k+1} = y_k + h\,f(x_k, y_k).$$

Geometrically, this is an approximation of the curve of by a polygon whose first side is tangent to this curve at x_k.

Example: Apply the Euler method to the initial value problem

$$y^{'} = (y-x)^2,\ y(0) = 0;$$

choosing $h = 0.1$ and computing ten steps.

Solution: Given that

$$y^{'} = (y-x)^2,\ y(0) = 0;\ h = 0.1.$$

We have to do ten steps. That is, we have to calculate $y_1, y_2, \ldots, y_{10}$. Given that $x_0 = 0$, $y_0 = 0$, $h = 0.1$. So $x_1 = 0.1, x_2 = 0.2, \ldots, x_{10} = 1.0$. Here $f(x,y) = (y-x)^2$. By Euler method

$$y_{k+1} = y_k + h\,f(x_k, y_k).$$

Thus

$$y_1 = y_0 + hf(x_0, y_0) = 0 + 0.1 \cdot (0-0)^2 = 0$$

Similarly, we calculate $y_2, \ldots, y_{10}$ and are given in the below table.

k	x_k	y_k
1	0.1	0.0000
2	0.2	0.0010
3	0.3	0.0050
4	0.4	0.0137
5	0.5	0.0286
6	0.6	0.0508
7	0.7	0.0810
8	0.8	0.1193
9	0.9	0.1656
10	1.0	0.21[illegible]6

Modified Euler's method.

Euler's method is generally much too inaccurate and methods of higher order and precision are obtained by taking more terms in Taylor series into account. But this involves an important practical problem of calculation of successive derivatives. The **general strategy** now is to avoid the computation of these derivatives and to replace it by computing functions for one or several suitably chosen auxiliary values of (x, y). The **Modified Euler's method** for solving initial value problem of the form

$$y^{'} = f(x,y),\ \ y(x_0) = y_0$$

where x_0 and y_0 are given is a predictor–corrector method as described below.

Suppose y_k has been calculated and to calculate y_{k+1} we do two steps

Step 1. Find the ***predictor***

$$y_{k+1}^{(0)} = y_k + h\,f(x_k, y_k).$$

Step 2. Find the **corrector** by iterations

$$y_{k+1}^{(i+1)} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i)})\right] \text{ for } i = 0,1,2, \ldots.$$

In this step we repeat the iteration till two consecutive $y_{k+1}^{(i)}$ are equal up to certain decimal places and treat it as y_{k+1}.

Example: Solve the initial value problem

$$y^{'} = xy^2,\ y(0) = 1; h = 0.25$$

in the interval $[0,1]$ using Modified Euler's method correct to four decimal places.

Solution: According to the question $f(x,y) = xy^2, x_0 = 0, y_0 = 1, h = 0.25$ .We have to solve in the interval $[0,1]$, that is we have to find the values of $y(x)$ at $x_1 = 0.25, x_2 = 0.5, x_3 = 0.75,$ and $x_4 = 1.0$. By Modified Euler's method we have

$$y_{k+1}^{(0)} = y_k + h\, f(x_k, y_k),$$

and

$$y_{k+1}^{(i+1)} = y_k + \frac{h}{2}\left[f(x_k, y_k) + f(x_{k+1}, y_{k+1}^{(i)})\right] \text{ for } i = 0,1,2, \ldots.$$

First to calculate $y_1 = y(x_1)$, we have

$$y_1^{(0)} = y_0 + hf(x_0, y_0) = 1$$

Now by iteration we calculate $y_1^{(i+1)}$; for $i = 0,1,2, \ldots$, till it repeats. That is

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right] = 1.0313.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1.0332.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right] = 1.0334.$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right] = 1.0334.$$

Thus, $y_1 = 1.0334$.

In similar way we calculate y_2 as bellow:

$y_2^{(0)} = 1.1001, y_2^{(1)} = 1.1424, y_2^{(2)} = 1.1483, y_2^{(3)} = 1.1492, y_2^{(4)} = 1.1493, y_2^{(5)} = 1.1493.$

Thus, $y_2 = 1.1493$. Again

$y_3^{(0)} = 1.3144$, $y_3^{(1)} = 1.3938$, $y_3^{(2)} = 1.4140$, $y_3^{(3)} = 1.4193$, $y_3^{(4)} = 1.4207$, $y_3^{(5)} = 1.4212$, $y_3^{(6)} = 1.4212$.

Thus $y_3 = 1.4212$. Finally,

$y_4^{(0)} = 1.7999,\ y_4^{(1)} = 2.0155,\ \ldots, y_4^{(15)} = 2.2349,\ y_4^{(16)} = 2.2349.$

Thus, $y_4 = 2.2349$.

Therefore, $y_1 = 1.0334, y_2 = 1.1493, y_3 = 1.4212, y_4 = 2.2349$.

Note: The exact solution of the above initial value problem is $y(x) = \frac{2}{2-x^2}$ and $y_1 = 1.0323$, $y_2 = 1.1429, y_3 = 1.3913$ and $y_4 = 2$.

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