Numerical Solutions of Nonlinear Equations

A number ξ is a solution of $f(x) = 0$ if $f(\xi) = 0$. Such a solution ξ is a root or a zero of $f(x) = 0$.

Geometrically, a root of the equation $f(x) = 0$ is the value of x at which the graph of $y = f(x)$ intersects x- axis.

1. Bisection Method

This method is consists in locating the root of the equation $f(x) = 0$ between a and b. If $f(x)$ is continuous in between a and b, and $f(a)$ and $f(b)$ are of opposite signs then there is a root in between a and b. Then the first approximation of the root is $x_1 = \frac{1}{2}(a + b)$. If $f(x_1) = 0$, then x_1 is a root of $f(x) = 0$. Otherwise, the root lies between a and x_1 or x_1 and b according to $f(a)f(x_1)$ is negative or positive. Then we bisect the interval as before and continue the process until the root is found to be desired accuracy.

Example 1. *Find a root of the equation $x^3 - 4x - 9 = 0$, using the bisection method correct to two decimal places.*

Let $f(x) = x^3 - 4x - 9$.

Then $f(2) = -9 < 0$ and $f(3) = 6 > 0$. Therefore, root lies between 2 and 3.

$\therefore$ first approximation to the root is $x_1 = \frac{2+3}{2} = 2.5$.

Now $f(x_1) = f(2.5) = -3.375 < 0$. Thus the second approximation of the root is $x_2 = \frac{1}{2}(x_1 + 3) = 2.75$.

Then $f(x_2) = f(2.75) = 0.7969 > 0$. $\therefore$ the root lies between x_1 and x_2. Thus the third approximation of the root is $x_3 = \frac{1}{2}(x_1 + x_2) = 2.625$.

Then $f(x_3) = f(2.625) = -1.4121 < 0$. $\therefore$ the root lies between x_2 and x_3. Thus the fourth approximation to the root is $x_4 = \frac{1}{2}(x_2 + x_3) = 2.6875$.

Repeating this process, the successive approximations are $x_5 = 2.71875$, $x_6 = 2.70313$, $x_7 = 2.71094$, $x_8 = 2.70703$, $x_9 = 2.70508$, $x_{10} = 2.70605$, $x_{11} = 2.70654$.

Hence the root is 2.70.

2. Fixed point iteration method

This iteration method is based on the principle of finding a sequence $\{x_n\}$ each element of which successively approximates a real root α of the equation $f(x) = 0$ in $[a, b]$.

We re-write $f(x) = 0$ as:

$$x = \phi(x). \tag{1}$$

Thus, a root α of the given equation satisfies $\alpha = \phi(\alpha)$. Therefore, the point α remains fixed under the mapping ϕ so a root of the equation is a fixed point of ϕ.

The function $\phi(x)$ is called iteration function. Here we also assume that $\phi(x)$ is continuously differentiable in $[a, b]$.

We first find a location or crude approximation $[a_0, b_0]$ of a real root α say, of $f(x) = 0$ and let $x = x_0$ $(a_0 \leq x_0 \leq b_0)$ be the initial approximation of α. Thus α satisfies the equation

$$\alpha = \phi(\alpha). \tag{2}$$

Putting $x = x_0$ in (1), we get first approximation x_1 of α as

$$x_1 = \phi(x_0)$$

and then the successive approximations are calculated as

$$x_2 = \phi(x_1),\ x_2 = \phi(x_2), \cdots,\ x_n = \phi(x_{n-1}),\ x_{n+1} = \phi(x_n). \tag{3}$$

The above iteration is generated by the formula $x_{n+1} = \phi(x_n)$ and is called iteration formula, where x_n is the nth approximation of the root α of $f(x) = 0$.

The sequence $\{x_n\}$ of iterations or the successive better approximations may or may not converge to a limit. If $\{x_n\}$ converges, then it converges to α and the number of iterations required depends upon the desired degree of accuracy of the root α.

Remark 1. *(Concergence) The method of fixed point iteration is conditionally convergent. The condition of convergence is* $|\phi'(x)| < 1$. *Before starting the computation, one should confirm that* $\phi(x)$ *must be such that* $-1 < \phi'(x) < 1$.

Example 2. *Find the root of the equation* $3x - \cos x - 1 = 0$, *by the iteration method, correct to four significant figures.*

Let $f(x) = 3x - \cos x - 1$.
$\therefore\ f(0) = -2 < 0,\ f(1) = 1.45 > 0$. Thus, one root of $f(x) = 0$ lies between 0 and 1.
We re-write the equation as

$$x = \frac{\cos x + 1}{3} = \phi(x),\ \therefore \phi'(x) = -\frac{\sin x}{3}.$$

$\therefore |\phi'(x)| < 1$ as $|\sin x| \leq 1$.
Here, we take $x_0 = 0$ and iteration equation as $x_{n+1} = \frac{\cos x_n + 1}{3}$.

n	x_n	$\phi(x_n)$
0	0	0.6
1	0.6	0.61
2	0.61	0.606
3	0.606	0.6073
4	0.6073	0.660706
5	0.60706	0.60711
6	0.60711	0.60710

Thus, 0.6071 is a root of the equation, correct up to four significant figures.

3. Newton-Raphson Method

This is also an iterative method and is used to find isolated roots of an equation $f(x) = 0$. The object of this method is to correct the approximate root x_0 (say) successively to its exact value α. Initially, a crude approximation small interval $[a_0, b_0]$ is found out in which only one root α (say) of $f(x) = 0$ lies.

Let $x = x_0$ ($a_0 \leq x_0 \leq b_0$) is an approximation of the root α of the equation $f(x) = 0$. Let h be a small correction on x_0, then $x_1 = x_0 + h$ is the correct root.

Therefore, $f(x_1) = 0 \implies f(x_0 + h) = 0$.

By Taylor series expansion, we get,

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0.$$

As h is small, neglecting the second and higher power of h, we get, $h = -\frac{f(x_0)}{f'(x_0)}$. Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}. \tag{4}$$

Further, if h_1 be the correction on x_1, then $x_2 = x_1 + h_1$ is the correct root.

Therefore, $f(x_2) = 0 \implies f(x_1 + h_1) = 0$.

By Taylor series expansion, we get,

$$f(x_1) + h_1 f'(x_1) + \frac{h_1^2}{2!}f''(x_1) + \cdots = 0.$$

Neglecting the second and higher power of h_1, we get, $h_1 = -\frac{f(x_1)}{f'(x_1)}$. Therefore,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}. \tag{5}$$

Proceeding in this way, we get the $(n+1)$th corrected root as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{6}$$

The formula (6) generates a sequence of successive corrections on an approximate root α to get the correct root α of $f(x) = 0$, provided the sequence is convergent. The formula (6) is known as the iteration formula for Newton-Raphson Method. The number of iterations required depends upon the desired degree of accuracy of the root.

3.1. **Convergence of Newton-Raphson Method.** Comparing with the iteration method, we may assume the iteration as

$$\phi(x) = x - \frac{f(x)}{f'(x)}.$$

Thus, the above sequence will be convergent if and only if,

$$|\phi'(x)| = \left|1 - \frac{\{f'(x)\}^2 - f(x)f''(x)}{\{f'(x)\}^2}\right| < 1 \text{ i.e.,} \left|\frac{f(x)f''(x)}{\{f'(x)\}^2}\right| < 1.$$

Therefore, $|\{f'(x)\}^2| > |f(x)f''(x)|$.

Example 3. *Find the positive root of $x^4 - x - 10 = 0$ correct to four decimal places, using Newton-Raphson method.*

Let $f(x) = x^4 - x - 10$. Then $f'(x) = 4x^3 - 1$.
Also $f(1) = -10$ and $f(2) = 4$. $\therefore$ $f(x) = 0$ has a root lies in between 1 and 2.
Let us take the initial approximation $x_0 = 2$. Iteration formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1} = x_n - \frac{x_n^4 - x_n - 10}{4x_n^3 - 1} \; i.e., x_{n+1} = \frac{3x_n^4 + 10}{4x_n^3 - 1}.$$

n	x_n	$x_{n+1} = \frac{3x_n^4+10}{4x_n^3-1}$
0	2	1.87097
1	1.87097	1.85578
2	1.85578	1.85558
3	1.85558	1.85558

Thus 1.8556 is a root of the given equation correct to four decimal places.

3.2. **Rate of convergence.** An iterative method is said to be of order p or has the rate of convergence p, if p is the largest positive real number for which there exists a finite constant $C \neq 0$ such that $|\epsilon_{n+1}| \leq C|\epsilon_n|^p$ where ϵ_n is the error in the nth iterate.

If x_{n+1} be the $(n+1)$th approximation of a root α of $f(x) = 0$ and ϵ_{n+1} be the corresponding error, we have

$$\alpha - x_{n+1} = \epsilon_{n+1} \tag{7}$$

$$\text{and } x_{n+1} = x_n + h_n$$

$$\implies \alpha - x_{n+1} = \alpha - x_n - h_n \implies \epsilon_{n+1} = \epsilon_n - h_n$$

$$\epsilon_n = \epsilon_{n+1} + h_n \tag{8}$$

where

$$h_n = \frac{f(x_n)}{f'(x_n)} \text{ or } f(x_n) + h_n f'(x_n) = 0. \tag{9}$$

Again we have

$$\alpha = x_n + \epsilon_n$$

$$\text{or, } f(\alpha) = 0 = f(x_n + \epsilon_n) = f(x_n) + \epsilon_n f'(x_n) + \frac{\epsilon_n^2}{2!} f''(\xi_n); (x_n < \xi_n < \alpha) \tag{10}$$

Substracting (9) from (10), we get,

$$(\epsilon_n - \xi_n)f'(x_n) + \frac{\epsilon_n^2}{2!}f''(\xi_n) = 0, \text{ or } \epsilon_{n+1}f'(x_n) + \frac{\epsilon_n^2}{2!}f''(\xi_n) = 0$$

$$\left|\frac{\epsilon_{n+1}}{\epsilon_n^2}\right| = \frac{1}{2}\left|\frac{f''(\xi_n)}{f'(\xi_n)}\right|.$$

$$\lim_{n\to\infty}\left|\frac{\epsilon_{n+1}}{\epsilon_n^2}\right| = \frac{1}{2}\left|\frac{f''(\xi_n)}{f'(\xi_n)}\right|$$

If the iteration converges i.e., $x_n,\ \xi_n \to \alpha$, as $n \to \infty$

$$\lim_{n\to\infty}\left|\frac{\alpha - x_{n+1}}{(\alpha - x_n)^2}\right| = \frac{1}{2}\left|\frac{f''(\xi_n)}{f'(\xi_n)}\right|.$$

Thus, it is clear that Nweton-Raphson iteration method is a second order iterative process.

4. Regula-Falsi method

This is the method of finding the real root of the equation $f(x) = 0$ and closely resembles the bisection method. Here we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Then, $f(x) = 0$ has a root lies between x_0 and x_1.

Equation of the chord joining $A(x_0, f(x_0))$ and $A(x_1, f(x_1))$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0).$$

The method consists in replacing the curve AB by the chord AB and taking the point of intersection of the chord with the x axis as an approximation to the root. So, the abscissa of the point where the chord cuts the x axis $(y = 0)$ is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}f(x_0) \tag{11}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_1)$ are of opposite signs, then the root lies between x_0 and x_2. So, replacing x_1 by x_2 in (11), we obtain the next approximation x_3. (The root could as well lie between x_1 and x_2 and we would obtain x_3 accordingly). This procedure is repeated till the root is found to desired accuracy.

Example 4. *Find the positive root of $x^4 - 32 = 0$ correct to two decimal places, using Regula-Falsi method.*

Let $f(x) = x^4 - 32$. Then, $f(2) = -16$ and $f(3) = 49$. $\therefore$ $f(x) = 0$ has a root lies between 2 and 3.

$\therefore$ taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -16$, $f(x_1) = 49$ in Regula-Falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}f(x_0) = 2 + \frac{16}{65} = 2.2462.$$

Now, $f(x_2) = f(2.2462) = -6.5438$. So, the root lies between 2.2462 and 3.

Taking $x_0 = 2.2462$, $x_1 = 3$, $f(x_0) = -6.5438$, $f(x_1) = 49$ in Regula-Falsi method, we get

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}f(x_0) = 2.2462 - \frac{3 - 2.2462}{49 + 6.5438}(-6.5438) = 2.335.$$

Repeating this process, the successive approximations are $x_4 = 2.3645$, $x_5 = 2.3770$, $x_6 = 2.3779$.

$\therefore$ 2.38 is a root of the given equation correct to two decimal places.

5. Secant Method

Iteration formula for Newton-Raphson method is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_0 is the initial approximation of the root.

In Secant method the derivative $f'(x)$ is replaced by the difference quotient

$$f'(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}.$$

$\therefore$ iteration formula for Secant method is

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

where x_0 and x_1 are initial approximation of the root.

Example 5. *Find a root of the equation $x^3 - 5x + 1 = 0$ correct to three decimal places.*

Let $f(x) = x^3 - 5x + 1$. Then $f(0) = 1$ and $f(1) = -3$.

$\therefore$ $f(x) = 0$ has a root in between 0 and 1. Let us take $x_0 = 0$ and $x_1 = 1$. Then $f(x_0) = 1$ and $f(x_1) = -3$. By Secant method

$$x_2 = x_1 - f(x_1)\frac{x_1 - x_0}{f(x_1) - f(x_0)} = 1 - (-3)\frac{(1-0)}{(-3-1)} = 0.25, ; f(x_2) = -0.234375.$$

Similarly,

$$x_3 = x_2 - f(x_2)\frac{x_2 - x_1}{f(x_2) - f(x_1)} = 0.186441, ; f(x_3) = -0.074276.$$

Repeating this process, the successive approximations are $x_4 = 0.201736$, $x_5 = 0.201640$.

$\therefore$ 0.202 is a root of the given equation correct to three decimal places.

5.1. **Rate of convergence.** If x_{n+1} be the $(n+1)$th approximation of a root α of $f(x) = 0$ and ϵ_{n+1} be the corresponding error, we have

$$\epsilon_{n+1} = \alpha - x_{n+1} = \alpha - x_n + f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$\implies \epsilon_{n+1} = \epsilon_n + f(\alpha - \epsilon_n)\frac{\alpha - \epsilon_n - \alpha + \epsilon_{n-1}}{f(\alpha - \epsilon_n) - f(\alpha - \epsilon_{n-1})}$$

$$\implies \epsilon_{n+1} = \epsilon_n + \frac{(\epsilon_{n-1} - \epsilon_n)\left[f(\alpha) - \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2} f''(\alpha) + \cdots\right]}{\left[f(\alpha) - \epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2} f''(\alpha) + \cdots\right] - \left[f(\alpha) - \epsilon_{n-1} f'(\alpha) + \frac{\epsilon_{n-1}^2}{2} f''(\alpha) + \cdots\right]}$$

$$\implies \epsilon_{n+1} = \epsilon_n + \frac{(\epsilon_{n-1} - \epsilon_n)\left[-\epsilon_n f'(\alpha) + \frac{\epsilon_n^2}{2} f''(\alpha) + \cdots\right]}{(\epsilon_{n-1} - \epsilon_n) f'(\alpha) - \frac{1}{2}(\epsilon_{n-1}^2 - \epsilon_n^2) f''(\alpha) + \cdots}$$

$$\implies \epsilon_{n+1} = \epsilon_n + \left[-\epsilon_n + \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 - \frac{(\epsilon_n + \epsilon_{n-1})}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$\implies \epsilon_{n+1} = +\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\epsilon_n\epsilon_{n-1} + O(\epsilon_n^2\epsilon_{n-1} + \epsilon_n\epsilon_{n-1}^2)$$

Thus we have,

$$\epsilon_{n+1} = C\epsilon_n\epsilon_{n-1} \tag{12}$$

where $C = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$ and higher powers of ϵ_n are neglected.

The relation (12) is known as error equation. Keeping in view the definition of convergence, we seek a relation of the form

$$\epsilon_{n+1} = A\epsilon_n^p \tag{13}$$

where A and p are to be determined.

From (13), we have $\epsilon_n = A\epsilon_{n-1}^p$ or $\epsilon_{n-1} = A^{-\frac{1}{p}}\epsilon_n^{\frac{1}{p}}$.

Substituting ϵ_{n+1} and ϵ_{n-1} in (12), we have

$$\epsilon_n^p = CA^{-(1+\frac{1}{p})}\epsilon_n^{1+\frac{1}{p}}. \tag{14}$$

Comparing the power of ϵ_n on both sides, we get

$$p = 1 + \frac{1}{p} \implies p = \frac{1}{2}(1 \pm \sqrt{5}).$$

Neglecting the minus sign, we find the rate of convergence of the Secant method is $p = \frac{1}{2}(1+\sqrt{5}) \approx 1.618$.

Also we have, $A = C^{\frac{p}{p+1}}$. Hence, the Secant method has linear rate of convergence 1.62.