Chapter-2
Linear Differential Equations of Second Order
2.1Homogeneous Linear Differential Equations of second order

Linear ODEs of Second Order

A second order ODE is called linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

Where, $p(x)$, $q(x)$, $r(x)$ are any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the standard form (1) with y'' as the first term.

Homogeneous ODEs of Second Order

A second order ODE is called homogeneous if it can be expressed in the form $F(y'', y', y) = 0$. So a homogeneous ODE involves only the derivatives of y and terms involving y, and they set to 0.

Homogeneous Linear ODEs of Second Order

If $r(x) = 0$ in equation (1), then equation (1) reduces to

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

and is called homogeneous. If $r(x) \neq 0$, then (1) is called nonhomogeneous.

Solution of Second Order Linear ODEs

A solution of second order linear ODE on some open Interval I is a function $y = h(x)$, that has derivatives $y' = h'(x)$ and $y'' = h''(x)$ and satisfies the differential equation for all x in the interval I.

Superposition or Linearity Principle

If $y_1(x)$ and $y_2(x)$ are two solutions of a second order homogeneous linear ODE, then a function of the form $y = c_1 y_1(x) + c_2 y_2(x)$,the linear combination of y_1 and y_2 (c_1 and c_2 are arbitrary constants) is also the solution of the given differential equation.

Fundamental Theorem for the Homogeneous Linear ODE

For a homogeneous linear ODE (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Proof: Let y_1 and y_2 be solutions of ODE (2) and $y = c_1 y_1 + c_2 y_2$ is the linear combination of y_1 and y_2 Now $y' = c_1 y_1' + c_2 y_2'$ and $y'' = c_1 y_1'' + c_2 y_2''$. Substituting y, y' and y'' into the ODE (2) we get

$$y'' + py' + qy = c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2)$$

$$= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0 .$$

Since y_1 and y_2 are two solutions of ODE (2), $y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$. This shows that $y = c_1 y_1 + c_2 y_2$ is a solution of the ODE (2).

Linear independence

Two functions $y_1(x)$ and $y_2(x)$ defined on interval I are said to be linearly independent if $y_1(x)$ and $y_2(x)$ are not proportional, that is $\frac{y_1}{y_2} \neq k$ or $\frac{y_2}{y_1} \neq l$ where k and l are constants.

Or, $\alpha_1 y_1 + \alpha_2 y_2 = 0$ everywhere on I implies $\alpha_1 = 0$ and $\alpha_2 = 0$.

Linear dependence

Two functions $y_1(x)$ and $y_2(x)$ defined on interval I are said to be linearly dependent if $y_1(x)$ and $y_2(x)$ are proportional, that is $y_1 = ky_2$ or $y_2 = ly_1$ where k and l are constants.

Or, $\alpha_1 y_1 + \alpha_2 y_2 = 0$ everywhere on I implies $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$.

Example: Are the functions $\ln x$, $\ln(x^3)$, linearly independent on the interval $x > 0$

Solution: $\frac{\ln x}{\ln(x^3)} = \frac{\ln x}{3\ln x} = \frac{1}{3}$ so the functions are linearly dependent.

Basis

A basis of solutions of second order linear homogeneous ODE (2) on an open interval I is a pair of linearly independent solutions of (2) on I.

General solution and Particular solution

A general solution of an ODE (2) on an open interval I is a solution $y = c_1 y_1 + c_2 y_2$ in which y_1 and y_2 are solutions of (2) on I that are not proportional, and c_1 and c_2 are arbitrary constants.

A **particular solution** of (2) on I is obtained if we assign specific values to c_1 and c_2 in the general solution of (2) using the given initial conditions.

Example 1: Verify that the given functions are linearly independent and form a basis of solutions of the given ODE. Solve the IVP.

$y'' + 2y' + y = 0, \quad y(0) = 2, \; y'(0) = -1, \quad e^{-x}, xe^{-x}$.

Solution: Let $y_1 = e^{-x}, y_2 = xe^{-x}$.

Now $y_1' = -e^{-x}$ and $y_1'' = e^{-x}$. Similarly $y_2' = e^{-x} - xe^{-x}$ and $y_2'' = xe^{-x} - 2e^{-x}$

Substituting y_1' and y_1'' in the given ODE, we obtain $y_1'' + 2y_1' + y_1 = 0$, which implies $y_1 = e^{-x}$ is a solution of the given ODE.

Similarly by Substituting y_2' and y_2'' in the given ODE, we obtain $y_2'' + 2y_2' + y_2 = 0$, which implies $y_2 = xe^{-x}$ is a solution of the given ODE.

Since $\frac{y_1}{y_2} = \frac{1}{x}$, y_1 and y_2 are linearly independent.

The general solution of the given ODE is $y = c_1 e^{-x} + c_2 xe^{-x}$

Now $y' = -c_1 e^{-x} + c_2(e^{-x} - xe^{-x})$

$y(0)=2 \Rightarrow c_1 = 2$ and $y'(0)=-1 \Rightarrow c_2 = 1$

So the particular solution is $y = 2e^{-x} + xe^{-x}$.

Reduction of order

1. Equations of the form $F(x, y', y'') = 0$ can be reduced to first order by substituting $y' = z$ and $y'' = \dfrac{dz}{dx}$.
2. Equations of the form $F(y, y', y'') = 0$ can be reduced to first order by substituting $y' = z$ and $y'' = \left(\dfrac{dz}{dy}\right)z$.
3. Equations of the form $F(y', y'') = 0$ can be reduced to first order by substituting $y' = z$ and $y'' = \dfrac{dz}{dx}$.

Example: Reduce the ODE $y'' + (1 + y^{-1})y'^2 = 0$ to first order and solve.

Solution: Substituting $y' = z$ and $y'' = \left(\dfrac{dz}{dy}\right)z$ in the given ODE the equation becomes

$$z\frac{dz}{dy} + (1 + y^{-1})z^2 = 0$$

$$\Rightarrow z\frac{dz}{dy} = -1(1 + y^{-1})z^2$$

$$\Rightarrow \frac{dz}{z} = -(1 + y^{-1})dy$$

Integrating both sides we get

$$\int \frac{dz}{z} = -\int \left(1 + \frac{1}{y}\right)dy$$

$$\Rightarrow \ln z = -y - \ln y + c_1$$

$$\Rightarrow \ln z + \ln y = c_1 - y$$

$$\Rightarrow \ln(zy) = c_1 - y$$

$$\Rightarrow yz = e^{c_1 - y} = c_2 e^{-y} \quad (c_2 = e^{c_1})$$

$$\Rightarrow y\frac{dy}{dx} = c_2 e^{-y}$$

$$\Rightarrow \int ye^{y}dy = \int c_2 dx$$

$$\Rightarrow ye^{y} - e^{y} = c_2 x + c_3$$

$$\Rightarrow (y-1) = (c_2 x + c_3)e^{-y}$$

$\Rightarrow y = 1 + (c_2 x + c_3)e^{-y}$.

How to find a basis if one of the solutions is known

Let $y'' + p(x)y' + q(x)y = 0$ be a second order linear homogeneous ODE and y_1 be a known solution of the given ODE. Another solution y_2 can be obtained by using the formula $y_2 = y_1 \int \frac{1}{y_1^{\,2}} e^{-\int p(x)dx} dx$,

where y_1 and y_2 are linearly independent.

Proof: Let y_1 be a given solution of the ODE $y'' + p(x)y' + q(x)y = 0$.

Let $y_2 = uy_1$ be another solution of the given differential equation, where u is an unknown function of variable x .

Differentiating y_2 w.r.t. x once and twice, we get $y_2' = u'y_1 + uy_1'$ and $y_2'' = u''y_1 + 2u'y_1' + uy_1''$.

Since y_2 is a solution of the given ODE, y_2 and its derivatives satisfy the given equation $y'' + p(x)y' + q(x)y = 0$.

Hence $y_2'' + py_2' + qy_2 = 0$

$\Rightarrow u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1 = 0$

$\Rightarrow u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$

$\Rightarrow u''y_1 + u'(2y_1' + py_1) = 0$ [$y_1'' + py_1' + qy_1 = 0$ as y_1 is a solution of the given ODE]

$$\Rightarrow u'' + u'\left(\frac{2y_1'}{y_1} + p\right) = 0$$

By taking $u' = v,\ u'' = v'$ we obtain

$$v' + \left(\frac{2y_1'}{y_1} + p\right)v = 0$$

$$\Rightarrow \frac{dv}{v} = -\left(\frac{2y_1'}{y_1} + p\right)dx$$

Integrating both sides, we get

$$\int \frac{dv}{v} = -\int\left(\frac{2y_1'}{y_1} + p\right)dx$$

$$\Rightarrow \ln v = -2\ln y_1 - \int p\,dx$$

$$\Rightarrow v = e^{-2\ln y_1 - \int p\,dx}$$

$$\Rightarrow v = \frac{1}{y_1^{\,2}} e^{-\int p\,dx}$$

$$\Rightarrow u' = \frac{1}{y_1^{\,2}} e^{-\int p\,dx}$$

$$\Rightarrow du = \frac{1}{y_1^{\,2}} e^{-\int p dx} dx$$

Taking integration on both sides, we get

$$\Rightarrow \int du = \int \frac{1}{y_1^{\,2}} e^{-\int p dx} dx$$

$$\Rightarrow u = \int \frac{1}{y_1^{\,2}} e^{-\int p dx} dx$$

As we have considered $y_2 = uy_1$, we can obtain y_2 using the formula

$$y_2 = y_1 \int \frac{1}{y_1^{\,2}} e^{-\int p dx} dx \,.$$

Example: Reduce the ODE $xy'' + 2y' + xy = 0$ to first order and solve, where $y_1 = \dfrac{(\cos x)}{x}$.

Solution: The standard form of the given ODE is $y'' + \dfrac{2}{x} y' + y = 0$.

Here $p(x) = \dfrac{2}{x}$ and $e^{-\int p(x) dx} = e^{-\int \frac{2}{x} dx} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$.

$$y_2 = y_1 \int \frac{1}{y_1^{\,2}} e^{-\int p(x) dx} dx = \frac{\cos x}{x} \int \frac{x^2}{\cos^2 x} x^{-2} dx$$

$$= \frac{\cos x}{x} \int \sec^2 x \, dx = \frac{\cos x}{x} \tan x = \frac{\sin x}{x}$$

The general solution is $y = c_1 y_1 + c_2 y_2 = c_1 \dfrac{\cos x}{x} + c_2 \dfrac{\sin x}{x}$.

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