RUNGE-KUTTA METHODS

Let us consider the following IVP :

$$y'(x) = f(x, y), \ y(x_0) = y_0 \ .$$

Integrating the differential equation $y' = f(x, y)$ in the interval $[x_n, x_{n+1}]$, we get

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx}\, dx = \int_{x_n}^{x_{n+1}} f(x, y)\, dx \ .$$

Note that y' and hence $f(x, y)$ is the slope of the solution curve. Further, the integrand on the right hand side is the slope of the solution curve which changes continuously in $[x_n, x_{n+1}]$. By approximating the continuously varying slope in $[x_n, x_{n+1}]$ by a fixed slope, we obtain the Euler, Heun's and modified Euler methods. The basic idea of Runge-Kutta methods is to approximate the integral by a weighted average of slopes and approximate slopes at a number of points in $[x_n, x_{n+1}]$.

The general Runge-Kutta method can be written as

$$y_{n+1} = y_n + \sum_{i=1}^{v} w_i K_i, \tag{1}$$

where

$$K_i = hf\left(x_n + c_i h,\, y_n + \sum_{m=1}^{i-1} a_{im} K_m\right)$$

with $c_1 = 0$.

For $v = 1$, $w_1 = 1$, the equation (1) becomes the Euler method with order $p = 1$ and this is the lowest order Runge-Kutta method.

We now list a few Runge-Kutta methods.

Runge-Kutta method of second order

Euler-Cauchy method (Heun method)

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{2}(K_1 + K_2), \\ K_1 &= hf(x_n, y_n), \\ K_2 &= hf(x_n + h, y_n + K_1). \end{aligned}$$

Modified Euler-Cauchy method :

$$\begin{aligned} y_{n+1} &= y_n + K_2, \\ K_1 &= hf(x_n, y_n), \\ K_2 &= hf\left(x_n + \frac{h}{2}, y_n + \frac{K_1}{2}\right). \end{aligned}$$

Runge-Kutta method of fourth order

Classical method :

$$\begin{aligned} y_{n+1} &= y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\ K_1 &= hf(x_n, y_n), \\ K_2 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_1\right) \\ K_3 &= hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}K_2\right) \\ K_4 &= hf(x_n + h, y_n + K_3). \end{aligned}$$

Example Find the approximate value of $y(1.4)$ for the initial value problem

$$y' = x^2 + y^2, \; y(1) = 2$$

with $h = 0.2$, using the Heun's method.

Solution We have $f(x, y) = x^2 + y^2$, $h = 0.2$.

Heun's method is given by

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + h, y_n + k_1).$$

We have the following results.

$n = 0$: $x_0 = 1, \quad y_0 = 2.$

$$k_1 = hf(x_0, y_0) = 0.2\, f(1, 2) = 1,$$

$$k_2 = hf(x_0 + h, y_0 + k_1) = 0.2\, f(1.2, 3) = 2.088,$$

$$y(1.2) \approx y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 2 + \frac{1}{2}(1 + 2.088) = 3.544.$$

$n = 1$: $x_1 = 1.2, \quad y_1 = 3.544.$

$$k_1 = hf(x_1, y_1) = 0.2\, f(1.2, 3.544) = 2.8000,$$

$$k_2 = hf(x_1 + h, y_1 + k_1) = 0.2\, f(1.4, 6.344) = 8.4413,$$

$$y(1.4) \approx y_2 + \frac{1}{2}(k_1 + k_2) = 3.544 + \frac{1}{2}(2.8000 + 8.4413) = 9.1647.$$

Example *Given $y' = x^3 + y$, $y(0) = 2$, compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ using the Runge-Kutta method of fourth order.*

Solution We have $x_0 = 0,\ y_0 = 2,\ f(x, y) = x^3 + y,\ h = 0.2.$

For $n = 0$, we have $x_0 = 0,\ y_0 = 2.$

$$k_1 = hf(x_0, y_0) = 0.2\,f(0, 2) = (0.2)(2) = 0.4,$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = 0.2\,f(0.1, 2.2)$$

$$= (0.2)(2.201) = 0.4402,$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = 0.2\,f(0.1, 2.2201)$$

$$= (0.2)(2.2211) = 0.44422,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2\,f(0.2, 2.44422)$$

$$= (0.2)(2.45222) = 0.490444,$$

$$y(0.2) \approx y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.0 + \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444]$$

$$= 2.443214.$$

For $n = 1$, we have $x_1 = 0.2,\ y_1 = 2.443214.$

$$k_1 = h\,f(x_1, y_1) = 0.2\,f(0.2, 2.443214) = (0.2)(2.451214) = 0.490243,$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = 0.2\,f(0.3, 2.443214 + 0.245122)$$

$$= (0.2)(2.715336) = 0.543067,$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2\right) = 0.2\,f(0.3, 2.443214 + 0.271534)$$

$$= (0.2)(2.741748) = 0.548350,$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2\,f(0.4, 2.443214 + 0.548350)$$

$$= (0.2)(3.055564) = 0.611113,$$

$$y(0.4) \approx y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.443214 + \frac{1}{6}[0.490243 + 2(0.543067) + 2(0.548350) + 0.611113]$$

$$= 2.990579.$$

For $n = 2$, we have $x_2 = 0.4,\ y_2 = 2.990579$.

$$k_1 = hf(x_2, y_2) = 0.2\,f(0.4, 2.990579) = (0.2)(3.054579) = 0.610916,$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{1}{2}k_1\right) = 0.2\,f(0.5, 2.990579 + 0.305458)$$

$$= (0.2)(3.421037) = 0.684207,$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{1}{2}k_2\right) = 0.2f(0.5, 2.990579 + 0.342104)$$

$$= (0.2)(3.457683) = 0.691537,$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = 0.2\,f(0.6, 2.990579 + 0.691537)$$

$$= (0.2)(3.898116) = 0.779623.$$

$$y(0.6) \approx y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.990579 + \frac{1}{6}[0.610916 + 2(0.684207) + 2(0.691537) + 0.779623]$$

$$= 3.680917.$$

Example *Solve the initial value problem*

$$y' = -2xy^2,\ y(0) = 1$$

with $h = 0.2$ *on the interval* $[0, 0.4]$ *using the fourth order classical Runge-Kutta method.*

Solution

We have $x_0 = 0$, $y_0 = 1$, $h=0.2$ and $f(x, y) = -2xy^2$.

For $n = 0$, we have $x_0 = 0$, $y_0 = 1$.

$$k_1 = hf(x_0, y_0) = -2(0.2)(0)(1)^2 = 0,$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) = -2(0.2)(0.1)(1)^2 = -0.04,$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) = -2\,(0.2)(0.1)(0.98)^2 = -0.038416,$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = -2(0.2)(0.2)(0.961584)^2 = -0.0739715,$$

$$y(0.2) \approx y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.0 + \frac{1}{6}[0.0 - 0.08 - 0.076832 - 0.0739715] = 0.9615328.$$

For $n = 1$, we have $x_1 = 0.2$, $y_1 = 0.9615328$.

$$k_1 = hf(x_1, y_1) = -2(0.2)(0.2)(0.9615328)^2 = -\ 0.0739636,$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_1\right) = -2(0.2)(0.3)\ (0.924551)^2 = -\ 0.1025753,$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{1}{2}k_2\right) = -2(0.2)(0.3)(0.9102451)^2 = -\ 0.0994255,$$

$$k_4 = hf\ (x_1 + h, y_1 + k_3) = -2(0.2)(0.4)(0.86210734)^2 = -\ 0.1189166,$$

$$y(0.4) \approx y_2 = y_1 + \frac{1}{6}\ (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.9615328 + \frac{1}{6}[-0.0739636\ -0.2051506\ -0.1988510\ -\ 0.1189166]$$

$$= 0.8620525$$