SPECIAL FUNCTIONS
GAMMA FUNCTION AND BETA FUNCTION

Gamma and Beta functions are special functions which are expressed as integrals. Many integrals which cannot be expressed in terms of elementary functions can be evaluated in terms of Gamma and Beta functions.

GAMMA FUNCTION

Gamma function is denoted by $\Gamma(n)$ and defined by the improper integral which depends on the parameter n.

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad n > 0$$

Gamma function is also known as ***Euler's integral of second kind***.

Standard Results

1. For positive real number n,

$$\Gamma(n+1) = n\Gamma(n)$$

Proof: We have

$$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \quad n > 0$$

$$\Rightarrow \Gamma(n+1) = \int_0^\infty e^{-t} t^{(n+1)-1} dt = \int_0^\infty e^{-t} t^n dt = \left(-e^{-t} t^n\right)_{t=0}^{t=\infty} - \int_0^\infty \left[\frac{d}{dt} t^n \int e^{-t} dt\right] dt$$

$$= n \int_0^\infty e^{-t} t^{n-1} dt = n\Gamma(n)$$

Hence $\Gamma(n+1) = n\Gamma(n)$ .This is known as the functional relation or reduction or recurrence formula for Gamma function.

2. For positive integer n,

$$\Gamma(n+1) = n!$$

Proof: We have $\Gamma(n+1) = n\Gamma(n)$

$$\Rightarrow \Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2) = \cdots = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n!$$

This proves that $\Gamma(n+1) = n!$

For this reason, Gamma function is regarded as the generalization of elementary factorial function.

In particular,

$$\Gamma(1) = \int_0^\infty e^{-t} dt = (-e^{-t})_{t=0}^{t=\infty} = 1$$

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1!$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(1) = 2\cdot 1 = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3\Gamma(3) = 3\Gamma(2+1) = 3\cdot 2\Gamma(2) = 3\cdot 2\cdot 1 = 3!$$

3. For negative real numbers but not negative integers: $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

4. ***Gamma function is undefined*** **for** $n = 0$ **and negative integers**.

That is $\Gamma(0), \Gamma(-1), \Gamma(-2), \Gamma(-3), \cdots$ are undefined.

5. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Q. Prove that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Proof: Let $x^2 = t$

$\Rightarrow 2xdx = dt \qquad \Rightarrow dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

For $x = 0,\ t = 0 \ \ and \ \ x = \infty,\ t = \infty$

$$\Rightarrow \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{2}\int_0^\infty e^{-t} t^{\frac{1}{2}-1} dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2} = \frac{\sqrt{\pi}}{2}$$

Q. Calculate $\Gamma\left(-\frac{1}{2}\right),\ \Gamma\left(-\frac{3}{2}\right)$

Solution: We have

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

i. $$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

ii. $$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = \frac{4\sqrt{\pi}}{3}$$

Q. Evaluate $\int_0^\infty x^4 e^{-x^4} dx$

Solution: Put $x^4 = t \quad \Rightarrow 4x^3 dx = dt$

$\Rightarrow dx = \frac{dt}{4x^3} = \frac{1}{4} t^{-3/4} dt$

$$\Rightarrow \int_0^\infty x^4 e^{-x^4} dx = \frac{1}{4}\int_0^\infty t e^{-t} t^{-3/4} dt = \frac{1}{4}\int_0^\infty e^{-t} t^{1/4} dt = \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{5}{4}-1} dt = \frac{1}{4}\Gamma\left(\frac{5}{4}\right)$$

BETA FUNCTION

Beta function $\beta(p,\, q)$ is defined as

$$\beta(p,\, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad (p>0,\, q>0).$$

This function is also known as ***Euler's integral of first kind***.

Standard Results

1. **Symmetry**

$$\beta(p,\ q) = \beta(q,\ p)$$

Proof: We have

$$\beta(p,\, q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad (p>0,\, q>0).$$

Let $1-x = y \qquad \Rightarrow x = 1-y$

$dx = -dy, \qquad x=0,\ y=1$ and $x=1,\ y=0$

$$\beta(p,\, q) = \int_1^0 (1-y)^{p-1}y^{q-1}(-dy) = \int_0^1 y^{q-1}(1-y)^{p-1}dy = \beta(q,\ p).$$

2. **Beta function in terms of trigonometric function**

$$\beta(p,\, q) = 2\int_0^{\pi/2} \sin^{2p-1}x\cos^{2q-1}x\,dx$$

3. **Beta function expressed as an improper integral**

$$\beta(p,\, q) = \int_0^\infty \frac{y^{p-1}}{(1+y)^{p+q}}dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$$

4. **Relation between Beta and Gamma functions**

$$\beta(p,\, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

Q. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: Let $p = \frac{1}{2},\ \ q = \frac{1}{2}$

We have $\beta(p,\, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$$\Rightarrow \beta\left(\frac{1}{2},\, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2}{\Gamma(1)} = \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 \qquad (1)$$

Again, $\beta\left(\frac{1}{2},\, \frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx = \int_0^1 x^{-1/2}(1-x)^{-1/2}dx$

Let $x=\sin^2\theta \quad \Rightarrow 1-x=1-\sin^2\theta=\cos^2\theta$

$dx=2\sin\theta\cos\theta\, d\theta$

$x=0,\ \ \theta=0\ \ and\ \ x=1,\ \ \theta=\pi/2$

$$\Rightarrow \beta\left(\frac{1}{2},\ \frac{1}{2}\right)=\int_0^{\pi/2}(\sin^2\theta)^{-1/2}(\cos^2\theta)^{-1/2}2\sin\theta\cos\theta\ d\theta=2\int_0^{\pi/2}d\theta=2\times\frac{\pi}{2}=\pi$$

From (1), we get

$$\Rightarrow \beta\left(\frac{1}{2},\ \frac{1}{2}\right)=\left\{\Gamma\left(\frac{1}{2}\right)\right\}^2=\pi \qquad\qquad \Rightarrow \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\ .$$

Q. Evaluate $\int_0^1 x^{10}(1-x)^{15}dx$

Solution: We have $\beta(p,\ q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx,\ \ (p>0,\ q>0).$

The given integral can be written as

$$\int_0^1 x^{10}(1-x)^{15}dx=\int_0^1 x^{11-1}(1-x)^{16-1}dx=\beta(11,16)$$

$$=\frac{\Gamma(11)\Gamma(16)}{\Gamma(11+16)}=\frac{\Gamma(10+1)\Gamma(15+1)}{\Gamma(26+1)}=\frac{10!\times 15!}{26!}$$

Q. Calculate $\beta\left(\frac{5}{2},\ \frac{3}{2}\right)$

Solution: We have

$$\beta(p,\ q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$\Rightarrow \beta\left(\frac{5}{2},\ \frac{3}{2}\right)=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}=\frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(4)}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)^2}{\Gamma(3+1)}=\frac{3\left\{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right\}^2}{2\times 3!}=\frac{1}{4}\times\frac{1}{4}\pi=\frac{\pi}{16}$$

Assignment

1. Evaluate $\int_0^1 t^{-1/2}(1-t)^{5/2}dt$
2. Evaluate $\int_0^\infty e^{-x}\sqrt{x}\, dx$
3. Evaluate $\int_0^\infty \frac{e^{-4t}}{\sqrt{t}}dt.$
4. Calculate the value of $\boldsymbol{\beta}\left(-\frac{3}{2},\ \frac{1}{2}\right)$

5.1 Power Series

An infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$$

is a ***Power series*** in powers of $(x-x_0)$ or ***Power series*** about the point $x = x_0$.

Here the constants $a_0,\ a_1,\ a_2,\cdots$ are the ***coefficients*** of the power series, the constant x_0 is called as ***centre*** of the power series and x is a variable. The constant m is known as summation index.

In Particular, a power series about the point $x_0 = 0$ is of the form

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

Note: Power series does not include series of negative powers or series involving fractional powers of x.

Radius of Convergence

A power series will converge for some values of the variables x and may diverge for others. All power series in powers of $(x-x_0)$ will converge at $x = x_0$. If x_0 is not only the convergent point then there exists a real number with $0 \le R \le \infty$ such that the infinite series $\sum_{m=0}^{\infty} a_m (x-x_0)^m$ ***converges*** whenever $|x-x_0| < R$ and ***diverges*** whenever $|x-x_0| > R, \quad R \ge 0$.

This non-negative real number R is known as ***Radius of convergence*** and the interval $(x_0 - R,\ \ x_0 + R)$ is known as ***convergence interval***.

Determination of Radius of Convergence

Let the power series be $\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots$.

There are two formulae to determine the radius of convergence of a power series.

1. $R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right|$, Here a_m is the coefficient of $(x-x_0)^m$ and a_{m+1} is the coefficient of $(x-x_0)^{m+1}$
2. $R = \dfrac{1}{\lim_{m\to\infty} |a_m|^{1/m}}$

Example: Determine the radius of convergence.

1. $\sum_{m=0}^{\infty} x^m$

   In the given series $a_m = 1$ and $a_{m+1} = 1$

   $$\Rightarrow R = \lim_{m\to\infty} \left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty} 1 = 1$$

   Radius of convergence is $R = 1$, hence the series converges for $|x| < 1$.

Therefore, the convergence interval is $(-1, 1)$.

2. $\sum_{m=0}^{\infty} \frac{x^m}{m!}$

In the given series $a_m = \frac{1}{m!}$ and $a_{m+1} = \frac{1}{(m+1)!}$

$$\Rightarrow R = \lim_{m\to\infty}\left|\frac{a_m}{a_{m+1}}\right| = \lim_{m\to\infty}\frac{(m+1)!}{m!} = \lim_{m\to\infty}(m+1) = \infty$$

Radius of convergence is $R = \infty$, hence the series converges for $|x| < \infty$.

Therefore, the convergence interval is $(-\infty, \infty)$.

3. $\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2m!}$

In the given series $a_{2m} = \frac{(-1)^m}{2m!}$ and $a_{2(m+1)} = \frac{(-1)^{m+1}}{2(m+1)!} = \frac{(-1)^{m+1}}{(2m+2)!}$

$$\Rightarrow R^2 = \lim_{m\to\infty}\left|\frac{a_{2m}}{a_{2m+2}}\right| = \lim_{m\to\infty}\frac{(2m+2)!}{2m!} = \lim_{m\to\infty}(2m+1)(2m+2)$$

$$= \lim_{m\to\infty} m^2(2+1/m)(2+2/m) = \infty$$

$\Rightarrow R = \infty$

Radius of convergence is $R = \infty$, hence the series converges for $|x| < \infty$.

Therefore, the convergence interval is $(-\infty, \infty)$.

4. $\sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$

In the given series $a_{2m+1} = \frac{(-1)^m}{(2m+1)!}$ and $a_{2(m+1)+1} = \frac{(-1)^{m+1}}{[2(m+1)+1]!} = \frac{(-1)^{m+1}}{(2m+3)!}$

$$\Rightarrow R^2 = \lim_{m\to\infty}\left|\frac{a_{2m+1}}{a_{2m+3}}\right| = \lim_{m\to\infty}\frac{(2m+3)!}{(2m+1)!} = \lim_{m\to\infty}(2m+2)(2m+3)$$

$$= \lim_{m\to\infty} m^2(2+2/m)(2+3/m) = \infty$$

$\Rightarrow R = \infty$

Radius of convergence is $R = \infty$, hence the series converges for $|x| < \infty$.

Therefore, the convergence interval is $(-\infty, \infty)$.

5. $\sum_{m=0}^{\infty} (-1)^m \frac{x^{3m}}{8^m}$

In the given series $a_{3m} = \frac{(-1)^m}{8^m}$ and $a_{3(m+1)} = \frac{(-1)^{m+1}}{8^{m+1}}$

$$\Rightarrow R^3 = \lim_{m\to\infty}\left|\frac{a_{3m}}{a_{3m+3}}\right| = \lim_{m\to\infty}\frac{8^{m+1}}{8^m} = \lim_{m\to\infty} 8 = 8$$

$$\Rightarrow R^3 = 8 \qquad \Rightarrow R = 2$$

Radius of convergence is $R = 2$, hence the series converges for $|x| < 2$.

Therefore, the convergence interval is $(-2, 2)$.

Differentiation of a Power series

Let $f(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m$ be a power series which converges for $|x - x_0| < R,\ R > 0$. Then the series obtained by differentiating term by term also converges for those x and represents the derivatives of the function $f(x)$ for those x.

$$\Rightarrow \frac{d}{dx} f(x) = \frac{d}{dx}\sum_{m=0}^{\infty} a_m (x - x_0)^m$$

$$= \frac{d}{dx}\left(a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots\right)$$

$$= a_1 + 2a_2(x - x_0)^2 + 3a_3(x - x_0)^2 \cdots$$

$$= \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1}$$

Similarly, $\frac{d^2}{dx^2} f(x) = \frac{d^2}{dx^2}\sum_{m=0}^{\infty} a_m (x - x_0)^m = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$

Example-1: Solve the following ODE by power series method.

$$y' + y = 0$$

Solution: Given ODE is

$$y' + y = 0$$

Let $y(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$be the series solution of the given ODE.

$$\Rightarrow y'(x) = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

Substituting y and y' in (2) we get

$$\sum_{m=1}^{\infty} m a_m x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots) + (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots) = 0$$

Now, equating the different powers of x to 0, we get

$a_1 = -a_0, \qquad a_2 = -\frac{a_1}{2} = \frac{a_0}{2!}, \qquad a_3 = -\frac{a_2}{3} = -\frac{a_0}{3!}$ and so on.

Putting the values of coefficients in the series solution, we obtain the required solution.

$$y(x) = a_0 - a_0 x + \frac{a_0}{2!}x^2 - \frac{a_0}{3!}x^3 + \cdots - \cdots = a_0\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) = a_0 e^{-x}.$$

5.2 Legendre's Equation. Legendre's Polynomials

Legendre's Equation

The second order linear differential equation of the form

$$(1-x^2)y''-2xy'+n(n+1)y=0, \quad n\geq 0, \quad |x|<1$$

is known as ***Legendre's equation***.

The solution of Legendre's equation is known as Legendre's function or Legendre's polynomial.

Solution of Legendre's Equation

The Legendre's equation is

$$(1-x^2)y''-2xy'+n(n+1)y=0$$

The point $x_0=0$ is an ordinary point for ODE (1).

Let $y(x)=\sum_{m=0}^{\infty}a_m x^m = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$ be the power series solution of ODE.

$\Rightarrow y'=\sum_{m=1}^{\infty}m a_m x^{m-1}$ and $y''=\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}$

Putting $y,\ y'$ and y'' in equation (1) we get

$$(1-x^2)\sum_{m=2}^{\infty}m(m-1)a_m x^{m-2}-2x\sum_{m=1}^{\infty}m a_m x^{m-1}+n(n+1)\sum_{m=0}^{\infty}a_m x^m=0$$

Equating the coefficient of different powers of x to 0 we fin

$a_2=-\frac{n(n+1)}{2!}a_0$ $\qquad$ $a_3=-\frac{(n-1)(n+2)}{3!}a_1$

$a_4=\frac{(n-2)n(n+1)(n+3)}{4!}a_0$ $\qquad$ $a_5=\frac{(n-3)(n-1)(n+2)(n+4)}{5!}a_1$

In this way we get $a_6,\ a_7,\cdots$

Putting the values of the coefficients in the series solution we get

$$y(x)=\sum_{m=0}^{\infty}a_m x^m = a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\cdots$$

$$\Rightarrow y(x)=a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\cdots+\cdots\right]$$

$$a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\cdots+\cdots\right]$$

$$\Rightarrow y(x)=a_0y_1(x)+a_1y_2(x)$$

Where,

$$y_1(x)=\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\cdots+\cdots\right]$$

$$y_2(x)=\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5-\cdots+\cdots\right]$$

Legendre's Polynomial
Case-I
If n is a positive even integer then $y_1(x)$ reduces to a polynomial of degree n while $y_2(x)$ remains an infinite series.
Case-II
If n is a positive odd integer then $y_2(x)$ reduces to a polynomial of degree n while $y_1(x)$ remains an infinite series.
In either of these cases, the series which reduces to a polynomial multiplied by some constant a_0 or a_1 is known as Legendre's Polynomial and it is denoted by $P_n(x)$.
The ***Legendre's Polynomial*** is

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m}$$

Where, $M = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even} \\ \dfrac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$

Values of Legendre's Polynomials

1. $n = 0: \quad M = 0, \ m = 0$

$$P_0(x) = \sum_{m=0}^{0} (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m} = 0! = 1$$

$\Rightarrow P_0(x) = 1$

2. $n = 1: \quad M = 0, \ m = 0$

$$P_1(x) = \sum_{m=0}^{0} (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m} = \frac{2!}{2 \times 1! \times 1!} x^{1-0} = x$$

$\Rightarrow P_1(x) = x$

3. $n = 2: \quad M = 2/2 = 1, \ m = 0 \text{ to } 1$

$$P_2(x) = \sum_{m=0}^{1} (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m} = \frac{4!}{2^2 \times 2! \times 2!} x^{2-0} - \frac{(4-2)!}{2^2 \times 1! \times 1! \times 0!} x^0$$

$$= \frac{3}{2} x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

$\Rightarrow P_2(x) = \frac{1}{2}(3x^2 - 1)$

4. $n = 3: \quad M = (3-1)/2 = 1, \ m = 0 \text{ to } 1$

$$P_3(x) = \sum_{m=0}^{1} (-1)^m \frac{(2n-2m)!}{2^n\, m!\, (n-m)!\, (n-2m)!} x^{n-2m} = \frac{1}{2}(5x^3 - 3x)$$

$\Rightarrow P_3(x) = \frac{1}{2}(5x^3 - 3x)$

5. $n=4$: $M=4/2=2,\ m=0$ to 2

$$P_4(x)=\sum_{m=0}^{2}(-1)^m\frac{(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}x^{n-2m}=\frac{1}{8}(35x^4-30x^2+3)$$

$$\Rightarrow P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$$

6. $n=5$: $M=(5-1)/2=2,\ m=0$ to 2

$$P_5(x)=\sum_{m=0}^{2}(-1)^m\frac{(2n-2m)!}{2^n\, m!\,(n-m)!\,(n-2m)!}x^{n-2m}=\frac{1}{8}(63x^5-70x^3+15x)$$

$$\Rightarrow P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$$

Rodrigues's Formula for Legendre's Polynomial

The ***Rodrigues's formula for Legendre's polynomial*** is

$$P_n(x)=\frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2-1)^n$$

Values of Legendre's polynomial by using Rodrigues's formula

The Rodrigues's formula is

$$P_n(x)=\frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$n=0$:

$P_0(x)=1$

$n=1$:

$$P_1(x)=\frac{1}{2^1 1!}\frac{d}{dx}(x^2-1)=\frac{1}{2}(2x)=x$$

$n=2$:

$$P_2(x)=\frac{1}{2^2\, 2!}\frac{d^2}{dx^2}(x^2-1)^2=\frac{1}{8}\frac{d}{dx}[2(x^2-1)2x]=\frac{1}{2}\frac{d}{dx}(x^3-x)=\frac{1}{2}(3x^2-1)$$

$n=3$:

$$P_3(x)=\frac{1}{2^3\, 3!}\frac{d^3}{dx^3}(x^2-1)^3=\frac{1}{48}\frac{d^2}{dx^2}[3(x^2-1)^2 2x]=\frac{1}{8}\frac{d^2}{dx^2}[x(x^2-1)^2]$$

$$=\frac{1}{8}\frac{d}{dx}[(x^2-1)^2+4x^2(x^2-1)]$$

$$=\frac{1}{8}\frac{d}{dx}[5x^4-6x^2+1]=\frac{1}{2}(5x^3-3x)$$

Similarly, $P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$ and $P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$

Generating Function for Legendre's polynomial

The function $G(u, x) = \sum_{n=0}^{\infty} P_n(x)u^n = (1-2xu+u^2)^{-1/2}$ is known as ***generating function*** for Legendre's polynomial.

Q. Prove the followings using generating function for Legendre's polynomial.

1. $P_n(1) = 1$
2. $P_n(-1) = (-1)^n$

Proof: The Generating function for Legendre's polynomial is

$$(1-2xu+u^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)u^n \tag{1}$$

1. We have to prove $P_n(1) = 1$

Putting $x = 1$ in (1) we get

$$(1-2u+u^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1)u^n$$

$$\Rightarrow [(1-u)^2]^{-1/2} = (1-u)^{-1} = \sum_{n=0}^{\infty} P_n(1)u^n$$

We have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

$$\Rightarrow \frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots = \sum_{n=0}^{\infty} P_n(1)u^n$$

Comparing the coefficients of u^n on both sides we get

$$P_n(1) = 1$$

2. We have to prove $P_n(-1) = (-1)^n$

Putting $x = -1$ in (1) we get

$$(1+2u+u^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(-1)u^n$$

$$\Rightarrow [(1+u)^2]^{-1/2} = (1+u)^{-1} = \sum_{n=0}^{\infty} P_n(-1)u^n$$

We have

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} x^n = 1 - x + x^2 - x^3 + \cdots - \cdots$$

$$\Rightarrow \frac{1}{1+u} = 1 - u + u^2 - \cdots + \cdots + (-1)^n u^n + \cdots = \sum_{n=0}^{\infty} P_n(-1)u^n$$

Comparing the coefficients of u^n on both sides we get, $P_n(-1) = (-1)^n$

Bonnet's recursion

$$(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$$

Proof: The generating function for Legendre's polynomial is

$$(1-2xu+u^2)^{-1/2}=\sum_{n=0}^{\infty}P_n(x)u^n \qquad (2)$$

Differentiating (2) partially w. r. to u we find

$$-\frac{1}{2}(1-2xu+u^2)^{-3/2}(-2x+2u)=\sum_{n=0}^{\infty}nP_n(x)u^{n-1}$$

$$\Rightarrow (x-u)\frac{(1-2xu+u^2)^{-1/2}}{(1-2xu+u^2)}=\sum_{n=0}^{\infty}nP_n(x)u^{n-1}$$

$$\Rightarrow (x-u)\sum_{n=0}^{\infty}P_n(x)u^n=(1-2xu+u^2)\sum_{n=0}^{\infty}nP_n(x)u^{n-1}$$

$$\Rightarrow \sum_{n=0}^{\infty}xP_n(x)u^n-\sum_{n=0}^{\infty}P_n(x)u^{n+1}=\sum_{n=0}^{\infty}nP_n(x)u^{n-1}-2x\sum_{n=0}^{\infty}nP_n(x)u^n+\sum_{n=0}^{\infty}nP_n(x)u^{n+1}$$

Comparing the coefficients of u^n on both sides we get

$$xP_n(x)-P_{n-1}(x)=(n+1)P_{n+1}(x)-2nxP_n(x)+(n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x)=xP_n(x)-P_{n-1}(x)+2nxP_n(x)-nP_{n-1}(x)+P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$$

This is called ***Bonnet's recursion***.

Q. Express $P_2(x)$ in terms of $P_1(x)$ and $P_0(x)$.

Solution: The Bonnet's recursion is $(n+1)P_{n+1}(x)=(2n+1)xP_n(x)-nP_{n-1}(x)$.

For $n=1$:

$$(2)P_2(x)=(3)xP_1(x)-P_0(x)$$

$$\Rightarrow P_2(x)=\frac{3}{2}xP_1(x)-\frac{1}{2}P_0(x)$$

Orthogonality of Legendre's Polynomials

The Legendre's polynomials $P_m(x)$ and $P_n(x)$ are said to be orthogonal in the interval $-1\le x\le 1$ if

$$\int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0, & m\ne n\\ \dfrac{2}{2n+1}, & m=n\end{cases}$$

Q. Evaluate

i. $\displaystyle\int_{-1}^{1}P_7(x)P_2(x)dx$

ii. $\displaystyle\int_{-1}^{1}[P_4(x)]^2dx$

Solution: We have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

i. $$\int_{-1}^{1} P_7(x)P_2(x)dx = 0$$

ii. $$\int_{-1}^{1} [P_4(x)]^2 dx = \frac{2}{2\times 4+1} = \frac{2}{9}$$

5.4 Bessel's Equation. Bessel Functions

Bessel's Equation

The second order linear equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0$$

is known as ***Bessel's equation***. Where the parameter $\nu \geq 0$ is a real number.
The solution of Bessel's equation is called ***Bessel function***.

Bessel function of 1st kind of order $\boldsymbol{\vartheta}$

$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} (-1)^m \dfrac{x^{2m}}{2^{2m+\nu} m!\Gamma(m+\nu+1)}$ is the Bessel function of 1st kind of order ν.

Bessel function of 1st kind of order $\boldsymbol{n}$

$J_n(x) = x^n \sum_{m=0}^{\infty} (-1)^m \dfrac{x^{2m}}{2^{2m+n} m!(m+n)!}$ is the Bessel function of 1st kind of order n.

General Solution of Bessel's Equation

Case-I:* ν *is not an integer

The ***general solution of Bessel's equation, when*** ν ***is not an integer is***

$y(x) = c_1 y_1 + c_2 y_2 = c_1 J_\nu(x) + c_2 J_{-\nu}(x)$

Case-II:* $\nu = n$ *is an integer

The ***general solution of Bessel's equation, when*** $\nu = n$ ***is an integer is***

$y(x) = c_1 y_1 + c_2 y_2 = c_1 J_n(x) + c_2 Y_n(x)$

Here $Y_n(x)$ is the Bessel function of 2nd kind of order n.

Case-III: $\nu = 0$

The ***general solution of Bessel's equation, when*** $\nu = 0$ ***is***

$y(x) = c_1 y_1 + c_2 y_2 = c_1 J_0(x) + c_2 Y_0(x)$

Here $J_0(x)$ is Bessel function of 1st kind of order 0 and $Y_0(x)$ is Bessel function of 2nd kind of order 0.

Recurrence Relations of Bessel's Function

1. $J_{-n}(x) = (-1)^n J_n(x)$.

2. $J_{\nu-1}(x) + J_{\nu+1}(x) = \dfrac{2\nu}{x} J_\nu(x)$

3. $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_\nu(x)$

Q. Prove that $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$

Proof: We have

$$J_\nu(x)=x^\nu\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m+\nu}m!\Gamma(m+\nu+1)}$$

$$J_\nu(x)=x^\nu\left[\frac{1}{2^\nu\Gamma(\nu+1)}-\frac{x^2}{2^{2+\nu}\Gamma(\nu+2)}+\frac{x^4}{2^{4+\nu}2!\Gamma(\nu+3)}-\cdots+\cdots\right]$$

$$\Rightarrow J_\nu(x)=\frac{x^\nu}{2^\nu\Gamma(\nu+1)}\left[1-\frac{x^2}{2^2(\nu+1)}+\frac{x^4}{2^4 2!(\nu+2)(\nu+1)}-\cdots+\cdots\right]$$

For $\nu=\frac{1}{2}$

$$\Rightarrow J_{\frac{1}{2}}(x)=\frac{x^{1/2}}{2^{1/2}\Gamma\left(\frac{1}{2}+1\right)}\left[1-\frac{x^2}{2^2\left(\frac{1}{2}+1\right)}+\frac{x^4}{2^4 2!\left(\frac{1}{2}+2\right)\left(\frac{1}{2}+1\right)}-\cdots+\cdots\right]$$

$$\Rightarrow J_{\frac{1}{2}}(x)=\frac{x^{1/2}}{2^{1/2}\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left[1-\frac{x^2}{2^2\left(\frac{3}{2}\right)}+\frac{x^4}{2^4 2!\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}-\cdots+\cdots\right]$$

$$\Rightarrow J_{\frac{1}{2}}(x)=\sqrt{\frac{2x}{\pi}}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+\cdots\right]=\sqrt{\frac{2x}{\pi}}\times\frac{x}{x}\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots+\cdots\right]$$

$$\Rightarrow J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\left[1-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\cdots\right]=\sqrt{\frac{2x}{\pi}}\sin x$$

Hence $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$.

Q. Prove that $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Proof: We have

$$J_\nu(x)=x^\nu\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{2^{2m+\nu}m!\Gamma(m+\nu+1)}$$

$$J_\nu(x)=x^\nu\left[\frac{1}{2^\nu\Gamma(\nu+1)}-\frac{x^2}{2^{2+\nu}\Gamma(\nu+2)}+\frac{x^4}{2^{4+\nu}2!\Gamma(\nu+3)}-\cdots+\cdots\right]$$

$$\Rightarrow J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)}\left[1 - \frac{x^2}{2^2(\nu+1)} + \frac{x^4}{2^4 2!(\nu+2)(\nu+1)} - \cdots + \cdots\right]$$

For $\nu = -\frac{1}{2}$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\left(-\frac{1}{2}+1\right)}\left[1 - \frac{x^2}{2^2\left(-\frac{1}{2}+1\right)} + \frac{x^4}{2^4 2!\left(-\frac{1}{2}+2\right)\left(-\frac{1}{2}+1\right)} - \cdots + \cdots\right]$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2^2\left(\frac{1}{2}\right)} + \frac{x^4}{2^4 2!\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)} - \cdots + \cdots\right]$$

$$\Rightarrow J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \cdots\right] = \sqrt{\frac{2x}{\pi}}\cos x$$

Hence $J_{-(1/2)}(x) = \sqrt{\frac{2}{\pi x}}\cos x$.

Q. Find the value of $J_{-(1/2)}(\pi/2)$

Solution: We have

$$J_{-(1/2)}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

$$\Rightarrow J_{-(1/2)}(\pi/2) = 0$$

***********************************END*************************************