Lecture-1

5.1 Jointly Distributed Random Variables

A probability function f depending on more than one random variables is jointly distributed probability function.

If $X_1, X_2 \cdots X_n$ are n random variables then $X = (X_1, X_2 \cdots X_n)$ is jointly distributed random variable and the probability function $f = f(x_1, x_2 \cdots x_n)$, $x_1 \in X_1, x_2 \in X_2, \cdots x_n \in X_n$.

Two jointly distributed random variable

If X & Y are two random variables then $Z = (X, Y)$ is jointly distributed random variable. If X & Y are both discrete, then (X, Y) is jointly discrete r.v. If X & Y are both continuous, then (X, Y) is jointly continuous r.v. Otherwise jointly mixed random variable.

(a) Jointly discrete random Variable (JDRV)

For jointly discrete r.v. (X, Y), the jointly discrete mass function $p(x, y)$ is defined by

$$p(x, y) = P(X = x, Y = y).$$

Satisfying the condition

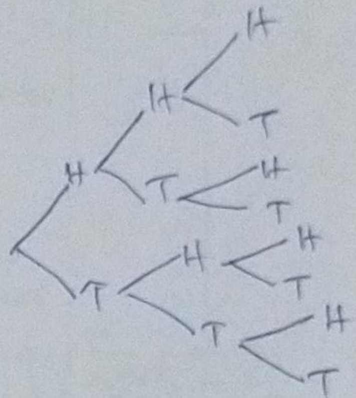
$$p(x, y) \geq 0 \text{ for each pair } (x, y)$$

and $\sum_y \sum_x p(x, y) = 1$ for all $(x, y) \in X \times Y$.

Example-1

If a fair coin tossed three times. The random variable counts number of head from 1st two trials and r.v. Y counts no. of heads from last two trials, then image space of X, Y and jointly distributed mass value $p(x,y)$ are obtained as follows

sample point	x	y	(x, y)	p(x,y)
HHH	2	2	(2,2) →	1/8
HHT	2	1	(2,1) →	1/8
HTH	1	1	(1,1) →	2/8
HTT	1	0	(1,0) →	1/8
THH	1	2	(1,2) →	1/8
THT	1	1	(1,1)	
TTH	0	1	(0,1) →	1/8
TTT	0	0	(0,0) →	1/8



Trials exp.

$X = \{0, 1, 2\}$

$Y = \{0, 1, 2\}$

$(X, Y) = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$

Jointy prob. mass value

X \ Y	0	1	2
0	$p(0,0) = 1/8$	$p(0,1) = 1/8$	$p(0,2) = 0$
1	$p(1,0) = 1/8$	$p(1,1) = 2/8$	$p(1,2) = 1/8$
2	$p(2,0) = 0$	$p(2,1) = 1/8$	$p(2,2) = 1/8$

Unit property of $p(x,y)$

$$\sum_{x=0}^{2}\sum_{x=0}^{2} p(x,y) = \frac{1}{8} + \frac{1}{8} + 0 + \frac{1}{8} + \frac{2}{8} + \frac{1}{8} + 0 + \frac{1}{8} + \frac{1}{8}$$

$$= \frac{8}{8} = 1$$

Jointly probability distribution function for discrete r.v. (X, Y) is defined by

$$F(x,y) = \sum_{y^* \le y} \sum_{x^* \le x} p(x^*, y^*) = P[X \le x, Y \le y]$$

evaluates the sum of prob. values with $X \le x$ and $Y \le y$

Example-2

From ex-1, we have

X \ Y	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

$F(0,0) = p(0,0) = \frac{1}{8}$

$F(0,1) = P(X \le 0, Y \le 1) = p(0,0) + p(0,1) = \frac{2}{8}$

$F(0,2) = P(X \le 0, Y \le 2) = p(0,0) + p(0,1) + p(0,2) = \frac{1}{8} + \frac{1}{8} + 0 = \frac{2}{8}$

$F(1,0) = P(X \le 1, Y \le 0) = p(0,0) + p(1,0) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$

$F(1,1) = P(X \le 1, Y \le 1)$

$= p(0,0) + p(0,1) + p(1,0) + p(1,1)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{2}{8} = \frac{5}{8}$

and so on.

Q. From Example-2, find $F(1,1)$ using complementarity theorem.

Sol^n $F(1,1) = P(X \le 1, Y \le 1)$

$= 1 - P(X > 1 \text{ or } Y > 1)$

$= 1 - [p(1,2) + p(2,1) + p(2,2)]$

$= 1 - \frac{3}{8} = 5/8$

Marginal probability mass function

Let (X, Y) be a discrete r.v. with given jointly probability mass function $p(x,y)$.

1) The marginal probability mass function for random variable X is defined by
$$p_X(x) = \sum_y p(x,y)$$ for each possible value x.

2) The marginal probability mass function for random variable Y is defined by
$$p_Y(y) = \sum_x p(x,y)$$ for each possible value y.

Derivation

By unit property of $p(x,y)$, we have
$$\sum_x \sum_y p(x,y) = 1$$
which can be written as
$$\sum_x p_X(x) = 1$$
where $p_X(x) = \sum_y p(x,y)$ is marginal pmf of X

Similarly
$$\sum_y \sum_x p(x,y) = 1$$
can be written as
$$\sum_y p_Y(y) = 1$$ where
$$p_Y(y) = \sum_x p(x,y)$$
is marginal pmf of Y.

Example-3

From jointly probability mass value of Ex-2, we have the marginal pmf of X & Y are

$x \backslash y$	0	1	2	$P_X(x)$
0	1/8	1/8	0	$P_X(0) = \frac{1}{8} + \frac{1}{8} + 0 = \frac{2}{8}$
1	1/8	2/8	1/8	$P_X(1) = \frac{1}{8} + \frac{2}{8} + \frac{1}{8} = \frac{4}{8}$
2	0	1/8	1/8	$P_X(2) = 0 + \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$
$P_Y(y)$	$P_Y(0) = 1/8 + 1/8 + 0 = 2/8$	$P_Y(1) = 1/8 + 2/8 + 1/8 = 4/8$	$P_Y(2) = 0 + \frac{1}{8} + 1/8 = 2/8$	$\sum_x P_X(x) = 1$, $\sum_y P_Y(y) = 1$

Note: 1) For each x, marginal pmf for r.v. X is sum of row elements containing x.

(2) For each y, marginal pmf for r.v. Y is sum of column elements containing y.

Independent discrete r.v.s

The r.v. X & Y are independent if for every pair (x, y),

$$p(x, y) = P_X(x)\, P_Y(y)$$

otherwise dependent.

Ex-4 In Ex-3, r.v. X & Y are independent or not.

Verification. Given $p(0,0) = \frac{1}{8}$

We have $P_X(0) = \frac{2}{8}$, $P_Y(0) = \frac{2}{8}$. So $p(0,0) \neq P_X(0)\, P_Y(0)$

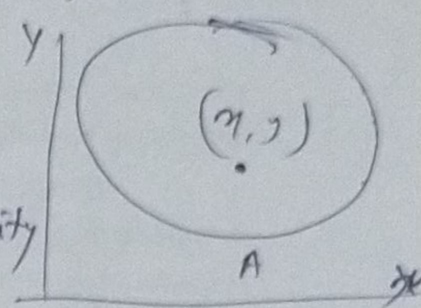
Hence X & Y are not independent.

Two jointly continuous random variables

For jointly continuous random variable (X, Y), the jointly probability density function (jpdf) is $f(x,y)$ defined by

$$f(x,y) \geq 0 \text{ for all } (x,y) \in A$$

and $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1.$$



For the region A, the probability value

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

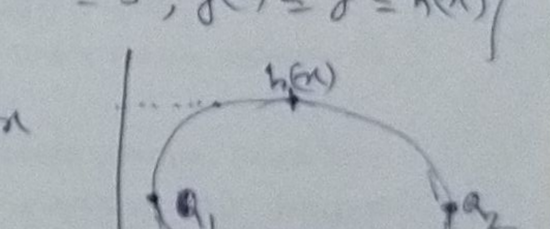
(a) If $A = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$ is the rectangular region, then

$$\therefore P[(X,Y) \in A] = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

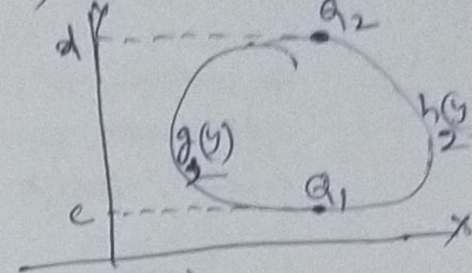
(b) If $A = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, g(x) \leq y \leq h(x)\}$

$$P[(X,Y) \in A] = \int_a^b \int_{g_1(x)}^{h(x)} f(x,y)\,dy\,dx$$



(c) If $A = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d,\ g(y) \le x \le h(y)\}$

then $$P[(X, Y) \in A] = \int_c^d \int_{g_2(y)}^{h_2(y)} f(x, y)\, dx\, dy$$



In general, for any region A

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \int_{g_1(x)}^{h_1(x)} f(x, y)\, dy\, dx = \int_c^d \int_{g_2(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

<u>Marginal probability density function</u>

(a) For random variable X, the marginal density function is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{for } -\infty < x < \infty$$

(b) for random variable Y, the marginal density function is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx \quad \text{for } -\infty < y < \infty$$

<u>Derivation</u> By unit property of $f(x, y)$, we have

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x, y)\, dx \right] dy = \int_{-\infty}^{\infty} f_Y(y)\, dy$$

where $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$ is mpdf of Y.

Similarly

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx$$

where $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ is the mpdf of X

Jointly probability distribution function (JPDF)

(a) The JPDF is defined by

$$F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$$

$$= P[X \le x, Y \le y] \text{ if } -\infty < x < \infty,\ -\infty < y < \infty$$

(b) for any region $A = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le h_1(x)\}$

$$= \{(x,y) \in \mathbb{R}^2 \mid c \le y \le d,\ g_2(y) \le x \le h_2(y)\}$$

(i) mpdf of X is

$$f_X(x) = \int_{g_1(x)}^{h_1(x)} f(x,y)\,dy$$

(ii) mpdf of Y is

$$f_Y(y) = \int_{g_2(y)}^{h_2(y)} f(x,y)\,dx$$

and

(iii) MPDF of X is $\int_a^x f_X(x)\,dx$

(iv) MPDF of Y is $\int_c^y f_Y(y)\,dy$

* X & Y are independent if $f(x,y) = f_X(x)\, f_Y(y)$

Q. For mixed continuous r.v. (X, Y) with jointly pdf

$$f(x,y) = \begin{cases} k(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) find the value of k.
(b) find joint PDF $F(x,y)$.
(c) find Marginal pdf for X.
(d) find marginal pdf for Y
(e) find $P[0 \le X \le \frac{1}{4},\ 0 \le Y \le \frac{1}{4}]$
(f) find $P[\frac{1}{4} \le X \le \frac{3}{4}]$
(g) find $P[\frac{1}{4} \le Y \le \frac{3}{4}]$
(h) X & Y are independent or not.

Soln (a) By unit property of jpdf, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^1\int_0^1 k(x+y^2)\,dx\,dy = 1$$

$$\Rightarrow \frac{5}{6}k = 1 \Rightarrow k = \frac{6}{5}$$

$$\therefore f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Joint PDF is

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx$$

(Partial Integration yeilds)

$$= \int_0^x \int_0^y \frac{6}{5}(x+y^2)\,dy\,dx$$

$$= \frac{6}{5}\left[\frac{x^2 y}{2} + \frac{x y^3}{3}\right]$$

(c) Marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy$$

$$= \frac{6}{5}\left(x+\frac{1}{3}\right), \quad 0 \le x \le 1,$$

and 0 otherwise

(d) Marginal pdf of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{5}(x+y^2)\,dx$$

$$= \frac{6}{5}\left[\frac{1}{2}+y^2\right), \quad 0 \le y \le 1$$

& 0 otherwise

(e)

$$P\left[0 \le X \le \frac{1}{4}, 0 \le Y \le \frac{1}{4}\right]$$

$$= \int_0^{1/4}\int_0^{1/4} f(x,y)\,dy\,dx$$

$$= \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dy\,dx$$

$$= \frac{7}{640} = 0.0109$$

(h) Since

$$f(x,y) = \frac{6}{5}(x+y^2) \ne \left[\frac{6}{5}\left(x+\frac{1}{3}\right)\right]\left[\frac{6}{5}\left(\frac{1}{2}+y^2\right)\right] = f_X(x)\,f_Y(y),$$

so X & Y are not independent

(f)

$$P\left[\frac{1}{4} \le X \le \frac{3}{4}\right] = \int_{1/4}^{3/4} f_X(x)\,dx$$

$$= \int_{1/4}^{3/4} \frac{6}{5}\left(x+\frac{1}{3}\right)dx = \frac{1}{2} = 0.5$$

(g)

$$P\left[\frac{1}{4} \le Y \le \frac{3}{4}\right] = \int_{1/4}^{3/4} f_Y(y)\,dy$$

$$= \int_{1/4}^{3/4} \frac{6}{5}\left(\frac{1}{2}+y^2\right)dy = \frac{37}{80} = 0.4625$$