4.4 Exponential Distribution

The random variable 'X' with exponential distribution has a pdf with parameter $\lambda > 0$, given by

$$f(x) = f(x, \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Q. Find mean & variance of exponential distribution.

Sol^n mean(X)

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \cdot \int_{0}^{\infty} x e^{-\lambda x} dx$$

$$= \lambda \cdot L(x)$$

$$= \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

Laplace of x^n

$$= \int_0^{\infty} x^n e^{-\lambda x} dx$$

$$= \frac{n!}{\lambda^{n+1}}$$

Variance of X

$$\sigma^2 = V(X) = E(X^2) - (E(X))^2$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx - \frac{1}{\lambda^2}$$

$$= \lambda \cdot L(x^2) - \frac{1}{\lambda^2}$$

$$= \lambda \cdot \frac{2}{\lambda^3} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\sigma = \frac{1}{\lambda}$$

Thus mean & st. deviation of exponential dist. are same & $\mu = \sigma = \frac{1}{\lambda}, \lambda > 0$

Prob. dist. fun^n

The r.v. X with exponential dist. has pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

The distribution function is

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(y)\,dy$$

$$= \int_{0}^{x} \lambda e^{-\lambda y}\,dy$$

$$= \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & \text{as } x \to 0 \\ 1, & \text{as } x \to \infty \end{cases}$$

Q. Find the density function of $Y = \beta X^{1/\alpha}$, where $\alpha > 0$, $\beta > 0$ are the parameters of Y if X is exponentially distributed r.v. with parameter $\lambda = 1$.

Sol^n X has the pmf of exponential distribution,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Given $Y = \beta X^{1/\alpha} \Rightarrow y = \beta x^{1/\alpha}$

$$\Rightarrow x^{1/\alpha} = \frac{y}{\beta} \Rightarrow x = \left(\frac{y}{\beta}\right)^{\alpha}, \quad \alpha > 0$$

$$= g(y),$$

Now pdf of y is

$$f_Y(y) = f_X(g^{-1}(y))\,(g^{-1}(y))' \quad \text{where } x = g^{-1}(y) = \left(\frac{y}{\beta}\right)^{\alpha}$$

$$= f_X\left(\frac{y}{\beta}\right)^{\alpha} \alpha \frac{y^{\alpha-1}}{\beta^{\alpha}}$$

$$= e^{-(y/\beta)^{\alpha}} \alpha \frac{y^{\alpha-1}}{\beta^{\alpha}}$$

$$= \frac{\alpha}{\beta^{\alpha}} y^{\alpha-1} e^{-(y/\beta)^{\alpha}}, \quad y>0, \ \alpha \geq 1, \ \beta > 0$$

Thus the pdf

$$f_X(x) = \begin{cases} \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(x/\beta)^{\alpha}}, & x>0 \\ 0 & \text{otherwise} \end{cases}$$

is called Weibull distribution.

Q. Find the density function of $Y = e^{X}$ if X is normal r.v. with mean μ and variance σ^2, i.e. $X \sim N(\mu, \sigma^2)$

Sol^n X is normal r.v. with mean μ and variance σ^2, so the normal curve is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

As $Y = e^{X}$, we have $y = e^{x} \Rightarrow x = \ln y$, $y > 0$
$= g^{-1}(y)$ (say)

pdf of Y

$$f_Y(y) = f_X(g^{-1}(y))\,(g^{-1}(y))'$$

$$= f_X(\ln y)(\ln y)'$$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln y - \mu}{\sigma}\right)^2} \cdot \frac{1}{y}, \quad y > 0$$

Hence the r.v. X with pdf

$$f_X(x) = \begin{cases} \frac{1}{\sigma\sqrt{2\pi}\, x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is called log-normal distribution

Q. verify the function

$$f_X(x, \alpha) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is the generalized form of pdf of exponential distribution with parameter $\alpha = 1$

Proof The pdf of exponential dist. with parameter 1 is

$$f_X(x) = \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

For $\alpha = 1$, we have

$$f_X(x, 1) = \begin{cases} \frac{x^0}{\Gamma(1)} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases} = f_X(x)$$

Q: Find the density function of $Y = \beta X$, $\beta > 0$ If X has the pdf with parameter $\alpha > 0$ given by

$$f_X(x) = f_X(x, \alpha) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Sol^n

Given $$f_X(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Given

$$Y = \beta X \Rightarrow y = \beta x, \quad \beta > 0$$

Now $$\Rightarrow x = \frac{y}{\beta} = g^{-1}(y)$$

$$f_Y(y) = f_X(g^{-1}(y)) \, (g^{-1}(y))'$$

$$= f_X\left(\frac{y}{\beta}\right) \cdot \left(\frac{y}{\beta}\right)'$$

$$= \frac{1}{\Gamma(\alpha)} \left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y/\beta} \cdot \frac{1}{\beta}$$

$$= \frac{1}{\Gamma(\alpha) \beta^{\alpha}} y^{\alpha-1} e^{-y/\beta}, \quad y > 0$$

The r.v. X with pdf

$$f_X(x, \alpha, \beta) = \begin{cases} \frac{1}{\Gamma(\alpha) \beta^{\alpha}} x^{\alpha-1} e^{-x/\beta}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is called Gamma distribution

Note ($\nu \in \mathbb{Z}_+$) for $\alpha = \nu/2$, $\beta = 2$, the pdf of gamma dist reduces to $$f_X(x, \nu) = \begin{cases} \frac{1}{2^{\nu} \Gamma(\nu/2)} x^{\frac{\nu}{2}-1} e^{-x/2}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

is the pdf of χ^2-dist.

Q. Find the density function of $Y = \frac{X}{\beta}$ If X ~~[illegible]~~ has the pdf ~~with~~ ~~pdf~~ parameter $\alpha > 0$, i.e.

$$f_X(x) = f_X(x, \alpha) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}, & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)} \qquad y = \frac{x}{\beta}$$

$$\frac{\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)} \cdot \left(\frac{x}{\beta}\right)'$$

$$= \frac{\left(\frac{x}{\beta}\right)^{\alpha-1}}{\Gamma(\alpha)} e^{-x/\beta} \left(\frac{1}{\beta}\right)$$

$$= \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$