$$\therefore E(r^2) = \int_{-\infty}^{\infty} r^2 \cdot f(r)\, dr$$

$$= \int_9^{11} r^2 \cdot \frac{3}{4}\left(1-(10-r)^2\right) dr$$

$$= \frac{3}{4}\int_9^{11} r^2 \left(1-100-r^2+20r\right) dr$$

$$= \frac{501}{5}.$$

$$\therefore E(A) = \pi E(r^2) = \frac{501\times\pi}{5}$$

Q12 The cdf for X

$$F(x) = \begin{cases} 0 & \text{for } x < -2 \\ \frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right) & \text{for } -2 \le x < 2 \\ 1 & \text{for } 2 \le x. \end{cases}$$

a. Compute $P(X < 0)$

$$= F(0) = \frac{1}{2} + \frac{3}{32}\left(4\times 0 - \frac{0^3}{3}\right)$$

$$= \frac{1}{2}$$

b. $P(-1 < X < 1) = F(1) - F(-1)$

$$= \left(\frac{1}{2} + \frac{3}{32}\left(4 - \frac{1}{3}\right)\right) - \left(\frac{1}{2} + \frac{3}{32}\left(-4 - \frac{-1}{3}\right)\right)$$

$$= \left(\frac{1}{2} + \frac{11}{32}\right) - \left(\frac{1}{2} - \frac{11}{32}\right) = \frac{11}{16}.$$

c. $P(0.5 < X) = 1 - F(0.5)$

$$= 1 - \left(\frac{1}{2} + \frac{3}{32}\left(4(0.5) - \frac{(0.5)^3}{3}\right)\right)$$

$$= 1 - (0.5 + 0.1836) = 0.3614$$

d. $$f(x) = \frac{d}{dx}\left[F(x)\right] = \frac{d}{dx}\left[\frac{1}{2} + \frac{3}{32}\left(4x - \frac{x^3}{3}\right)\right]$$

$$= \frac{3}{32}\left(4 - x^2\right).$$

$$f(x) = \begin{cases} 0 & x < -2 \\ \frac{3}{32}(4-x^2) & -2 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

Q13

X = the headway betⁿ two randomly selected consecutive cars.

The distribution of time headway has the form

$$f(x) = \begin{cases} \frac{K}{x^4} & \text{for } x > 1 \\ 0 & \text{for } x \leq 1. \end{cases}$$

a) Determine the value of 'K'.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_1^{\infty} \frac{K}{x^4}\,dx = K \left. \frac{-1}{3x^3} \right|_1^{\infty}$$

$$= \frac{K}{3} = 1$$

$$\Rightarrow K = 3. \qquad \Rightarrow f(x) = \begin{cases} \frac{3}{x^4} & \text{for } x > 1 \\ 0 & \text{for } x \leq 1 \end{cases}$$

b) Find cdf.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_1^{x} \frac{3}{x^4}\,dx$$

$$= \left. \frac{-1}{x^3} \right|_1^{x} = 1 - \frac{1}{x^3}.$$

$$\therefore F(x) = \begin{cases} 1 - \frac{1}{x^3} & \text{for } x > 1 \\ 0 & \text{for } x \leq 1. \end{cases}$$

c) Use the cdf to determine the probability that headway exceeds 2 sec and also the probability that headway is betⁿ 2 and 3 sec.

$$P(X > 2) = 1 - P(X \leq 2) = 1 - F(2)$$

$$= 1 - \left(1 - \frac{1}{2^3}\right) = \frac{1}{8}$$

$$P(2 < X < 3) = F(3) - F(2)$$

$$= \left(1 - \frac{1}{3^3}\right) - \left(1 - \frac{1}{2^3}\right) = \frac{1}{8} - \frac{1}{27} = \frac{19}{216}.$$

(d) Obtain the mean value and standard deviation of headway

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{1}^{\infty} x \cdot \frac{3}{x^4}\, dx = \int_{1}^{\infty} 3x^{-3}\, dx$$

$$= \frac{3}{-2}\, x^{-2} \Big|_1^{\infty} = \frac{3}{2} = 1.5$$

$$V(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)\, dx$$

or $\quad E(X^2) - (E(X))^2$

$$E(X^2) = \int_{1}^{\infty} x^2 \cdot \frac{3}{x^4}\, dx = \int_{1}^{\infty} \frac{3}{x^2}\, dx = 3\, \frac{x^{-1}}{-1}\Big|_1^{\infty}$$

$$= 3$$

$$\cancel{V(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{\ \ }{12}}$$

$$V(X) = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$$

$$\therefore\ \sigma = \sqrt{V(X)} = \frac{\sqrt{3}}{2}$$

(e) P(headway is within 1 standard deviation of the mean value)

$$= P(\mu - \sigma < x < \mu + \sigma)$$

$$= P\left(\frac{3}{2} - \frac{\sqrt{3}}{2} < x < \frac{3}{2} + \frac{\sqrt{3}}{2}\right)$$

$$= P\left(\frac{3-\sqrt{3}}{2} < x < \frac{3+\sqrt{3}}{2}\right) = F\left(\frac{3+\sqrt{3}}{2}\right) - F\left(\frac{3-\sqrt{3}}{2}\right)$$

$$= \left(1 - \frac{1}{((3+\sqrt{3})/2)^3}\right) - 0 = \frac{10\sqrt{3}}{9}$$

Q20 Consider the pdf for total waiting time Y for two buses

$$f(y) = \begin{cases} \frac{y}{25} & 0 \leq y < 5 \\ \frac{2}{5} - \frac{1}{25}y & 5 \leq y \leq 10 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find cdf of Y.

$$F(y) = \int_{-\infty}^{y} f(y)\,dy = \int_{0}^{y} f(y)\,dy$$

$$= \int_{0}^{5} \frac{y}{25}\,dy + \int_{5}^{y} \left(\frac{2}{5} - \frac{1}{25}y\right)dy$$

$$= \frac{y^2}{50}\Big|_0^5 + \frac{2y}{5} - \frac{y^2}{50}\Big|_5^y$$

$$= \frac{5^2}{50} + \frac{2y}{5} - \frac{y^2}{50} - \left(\frac{10}{5} - \frac{25}{50}\right)$$

$$= -\frac{y^2}{50} + \frac{2y}{5} - 1.$$

$$\therefore F(y) = \begin{cases} 0 & y < 0 \\ \frac{y^2}{50} & 0 \leq y < 5 \\ -\frac{y^2}{50} + \frac{2y}{5} - 1 & 5 \leq y \leq 10 \\ 1 & y > 10. \end{cases}$$

b) Compute $E(Y)$ and $V(Y)$

$$E(Y) = \int_0^5 y\cdot\frac{y}{25}\,dy + \int_5^{10} y\left(\frac{2}{5} - \frac{1}{25}y\right)dy = 5.$$

$$V(Y) = \int_{-\infty}^{\infty} (y-\mu)^2 f(y)\,dy = \int_0^5 (y-5)^2 \frac{y}{25}\,dy + \int_5^{10} (y-5)^2\left(\frac{2}{5} - \frac{y}{25}\right)dy$$

$$= \frac{25}{6}.$$

How do these compare with the expected waiting time and variance for a single bus when the time is uniformly distributed on [0,5] ?

Here, $A=0$, $B=5$

$$E(X) = \frac{A+B}{2} = \frac{5}{2}.$$

$$V(X) = \frac{B^2+A^2+AB}{3} - \frac{(B+A)^2}{4}$$

$$= \frac{5^2}{3} - \frac{5^2}{4} = \frac{100-75}{12} = \frac{25}{12}.$$

$E(Y) = 5$ and $V(Y) = \frac{25}{12}$

∴ The expected waiting time for a single bus is half the expected waiting time for two buses and the variance for a single bus is half the variance of two buses.