2.4 Conditional probability:

For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has already occurred is defined by
$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Similarly, $P(B/A) = \frac{P(A \cap B)}{P(A)}$ – $P(A) > 0$.

Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(A/B).P(B) = P(B/A).P(A)$$

* For 3 events show that $P(A \cap B \cap C) = P(A)\,P(B/A)\,P(C/A \cap B)$

Proof: R.H.S $= P(A)\,P(B/A)\,P(C/A \cap B)$

$= P(A)\,\frac{P(A \cap B)}{P(A)}\,\frac{P(C \cap A \cap B)}{P(A \cap B)}$

$= P(A \cap B \cap C) =$ L.H.S (proved)

* For n events $A_1, A_2, A_3, \ldots, A_n$

$P(A_1 \cap A_2 \cap \ldots A_n) = P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2)\,P(A_4/A_1 \cap A_2 A_3) \ldots$
$\ldots P(A_n/A_1 \cap A_2 \cap \ldots \cap A_{n-1})$

Proof: R.H.S $P(A_1)\,P(A_2/A_1)\,P(A_3/A_1 \cap A_2) \ldots P(A_n/A_1 \cap A_2 \cap \ldots A_{n-1})$

$= P(A_1)\,P\frac{(A_1 \cap A_2)}{P(A_1)}\,\frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} \ldots \frac{P(A_1 \cap A_2 \cap \ldots \cap A_n)}{P(A_1 \cap A_2 \ldots \cap A_{n-1})}$

$= P(A_1 \cap A_2 \cap \ldots \cap A_n)$

(proved)

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Example:

Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery and 30% include both a card and battery. Consider selecting a buyer and let $A = \{\text{memory card purchased}\}$ and $B = \{\text{Battery purchased}\}$. Then $P(A) = 0.60$, $P(B) = 0.40$, and $P(A \cap B) = 0.30$.

Then $P(A|B)$ = the probability that an optional card was also purchased, Given that the selected individual purchased an extra battery

$$= \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.40} = 0.75$$

Similary $P(\text{battery} / \text{memory card})$

$$= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.30}{0.60} = 0.50$$

Total probability and Bayes' theorem :-

Recall that events $A_1, A_2, \ldots, A_K$ are mutually exclusive if no two have any common outcomes i.e. if $A_i \cap A_j = \phi$, $i \neq j$

The events $A_1, A_2, \ldots, A_K$ are exhaustive if one A_i must occur, so that $A_1 \cup A_2 \cup \ldots \cup A_K = S$.

The Law of Total probability :

Let $A_1, A_2, \ldots, A_K$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B/A_1)P(A_1) + P(B/A_2)P(A_2) + \ldots + P(B/A_K)P(A_K)$$
$$= \sum_{i=1}^{K} P(B/A_i)P(A_i)$$

Proof: Since $A_1, A_2, \ldots, A_K$ are exhaustive, we have

$$\bigcup_{i=1}^{K} A_i = S$$

Now, $B = B \cap S \quad (\because B \subseteq S)$

$$= B \cap \bigcup_{i=1}^{K} A_i$$
$$= B \cap [A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_K]$$
$$= (B \cap A_1) \cup (B \cap A_2) \cup (B \cap A_3) \cup \ldots \cup (B \cap A_K)$$

$\Rightarrow P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_K)$ ($\because$ the events are mutually exclusive)

$$= P(A_1)P(B/A_1) + P(A_2)P(B/A_2) + \ldots + P(A_n)P(B/A_K)$$
$$= \sum_{i=1}^{K} P(A_i)P(B/A_i)$$

(proved)

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Bayes' theorem:

Let $A_1, A_2, \ldots, A_K$ be a collection of K mutually exclusive and ~~each~~ exhaustive events with prior probabilities $P(A_i)$ $(i = 1, 2, \ldots K)$. Then for any other event B for which $P(B) > 0$, the probability of A_j given that B has occurred is

$$P(A_j/B) = \frac{P(A_j \cap B)}{P(B)}$$

$$= \frac{P(B|A_j)\,P(A_j)}{\sum_{i=1}^{K} P(B|A_i)\,P(A_i)}$$

[Using multiplication rule, $j = 1, 2, \ldots, K$]

[Using total probability]

Example (Incidence of rare disease)

Only 1 in 1000 adults is afflicted with a rare disease for which a diagnostic test has been developed. The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, whereas an individual without the disease will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Ans: Let A_1 = individual has the disease

A_2 = individual does not have the disease

B = positive test result.

Then,

then, $P(A_1) = \frac{1}{1000} = 0.001$, $P(A_2) = \frac{999}{1000} = 0.999$

$P(B/A_1) = \frac{99}{100} = 0.99$, $P(B/A_2) = \frac{2}{100} = 0.02$.

$\therefore$ The probability that the individual has the disease, given that selected individual is tested positive is

$$P(A_1/B) = \frac{P(B/A_1)\,P(A_1)}{\sum_{i=1}^{2} P(B/A_i)\,P(A_i)} \quad \text{(By Baye's theorem)}$$

$$= \frac{P(B/A_1)\,P(A_1)}{P(B/A_1)\,P(A_1) + P(B/A_2)\,P(A_2)}$$

$$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.02 \times 0.999}$$

$$= \frac{0.00099}{0.02097}$$

$$= 0.047$$

Ans