Chapter-7: Matrices, Vectors & Linear Systems

Linear System of Equations. Gauss-Elimination

Linear System of Equations

A ***linear system*** of m number of equations in n number of variables $x_1,\ x_2,\ x_3,\cdots x_n$ is a set of equations of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

The matrix form of linear system of equations is

$$AX = B$$

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The matrix A is known as ***Coefficient matrix***.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The column matrix of unknowns is $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ and the column matrix of constants is $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}$.

Classification of Linear System of Equations

I. According to the elements of column matrix $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}$ there are two types of linear system of equations.

   i. Homogeneous Linear system of equations

   If all the elements of column matrix B are zero then the linear system of equations is known as ***Homogeneous***. That is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

   ii. Nonhomogeneous Linear System of Equations

   If at least one of the elements of the column matrix B is not zero then the linear system of equations is ***Nonhomogeneous***.

II. According to the involvement of number of equations and number of variables there are three types of linear system of equations.

   i. Over-determined Linear System

If $\text{No. of equations} > \text{No. of unknowns}$ then linear system of equations is ***Over-determined***.

ii. Determined Linear System

If $\text{No. of equations} = \text{No. of unknowns}$ then linear system of equations is ***Determined***.

iii. Under-determined Linear System

If $\text{No. of equations} < \text{No. of unknowns}$ then linear system of equations is ***Under-determined***.

III. According to the solution of linear system of equations there are two types of linear system of equations.

i. Consistent Linear System

If the linear system of equations has at least one solution then the linear system of equations is ***Consistent***.

ii. Inconsistent Linear system

If there is no solution at all for a linear system of equations then the linear system of equations is ***Inconsistent***.

Elementary Row Operations

i. Interchange of two rows
ii. Multiplication of a row by nonzero constant.
iii. Addition of a constant multiple of one row to another row.

Row-equivalent linear system

A linear system S_2 is ***row-equivalent*** to a linear system S_1 if S_2 can be obtained from S_1 by using elementary row operations.

Note: Row-equivalent linear systems have same set of solutions.

Gauss-Elimination Method

The Gauss-Elimination process is a standard method for solving linear system of equations. It is a systematic elimination process which reduces the augmented matrix to upper triangular form and then by Back substitution method we will get the values of the unknowns.

Basic Definitions

Augmented Matrix

The ***Augmented matrix*** is of the form $\tilde{A} = [A|B]$ i.e.

$$\tilde{A} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Pivot element

The 1st row 1st column element i.e. $a_{11} \neq 0$ of the augmented matrix which is used to eliminate other elements in the elimination process is known as ***Pivot element***.

Note: Pivot element never be zero.

Pivot Row

The row containing pivot element is called ***Pivot row*** and it is used in the elimination process.

Partial Pivoting

The process of interchanging of rows when the pivot element is zero is called ***partial pivoting***.

Solution of Linear System of Equations by Gauss-Elimination Method

Let the Linear system of equations be

$$a_1x_1 + a_2x_2 + a_3x_3 = b_1$$
$$a_4x_1 + a_5x_2 + a_6x_3 = b_2$$
$$a_7x_1 + a_8x_2 + a_9x_3 = b_3$$

In matrix form the given linear system can be written as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

This can be solved by Gauss-elimination method as follows:

Procedure

I. Write the augmented matrix.

$$\tilde{A} = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ a_4 & a_5 & a_6 & b_2 \\ a_7 & a_8 & a_9 & b_3 \end{array}\right]$$

II. Take $a_{11} = a_1 \neq 0$ as pivot element and 1st row as pivot row. If $a_{11} = 0$ then make it nonzero by using partial pivoting.

III. Use pivot element and pivot row in the elimination process.

IV. The elements of pivot row remains unchanged.

V. Eliminate all the 1st column elements which are below the pivot element from other rows by using the row operations

$$R_i \to R_i - \frac{a_{i1}}{a_{11}} R_1, \quad i > 1$$

In particular, $R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$ and $R_3 \to R_3 - \frac{a_{31}}{a_{11}} R_1$.

VI. Now augmented matrix becomes $\tilde{A} = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ 0 & c_1 & c_2 & d_1 \\ 0 & c_3 & c_4 & d_2 \end{array}\right]$

VII. First row elements remains untouched.

VIII. Take $a_{22} = c_1 \neq 0$ as pivot element and 2nd row as pivot row. If $a_{22} = 0$ then make it nonzero by using partial pivoting.

IX. Eliminate all the 2nd column elements which are below the pivot element from other rows by using the row operations

$$R_i \to R_i - \frac{a_{i2}}{a_{22}} R_2, \quad i > 2$$

The 2nd row elements remain unchanged.

Now augmented matrix becomes $\tilde{A} = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ 0 & c_1 & c_2 & d_1 \\ 0 & 0 & c_5 & d_3 \end{array}\right]$

X. Stop the process when the coefficient matrix A reduces to upper triangular matrix.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & c_1 & c_2 \\ 0 & 0 & c_5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ d_1 \\ d_3 \end{bmatrix}$$

XI. Write the row equivalent linear system of equations

$$c_5x_3 = d_3$$

$$c_1x_2 + c_2x_3 = d_1$$

$$a_1x_1 + a_2x_2 + a_3x_3 = b_1$$

XII. Use back substitution to find the values of unknowns.

Echelon Form

The matrix which is obtained in the last step of Gauss-elimination is called Echelon form.

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & b_1 \\ 0 & c_1 & c_2 & d_1 \\ 0 & 0 & c_5 & d_3 \end{array}\right]$$

Example: Solve the linear system of equations by using Gauss-elimination method.

$$-x_1 + x_2 + 2x_3 = 2$$

$$3x_1 - x_2 + x_3 = 6$$

$$-x_1 + 3x_2 + 4x_3 = 4$$

Solution: The augmented matrix is

$$\tilde{A} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 3 & -1 & 1 & 6 \\ -1 & 3 & 4 & 4 \end{array}\right]$$

Since $a_{11} = -1 \neq 0$, so $a_{11} = -1$ is pivot element and 1st row is pivot row.

We have to eliminate $a_{21} = 3$ and $a_{31} = -1$ by the row operations

$R_2 \to R_2 + 3R_1,\ R_3 \to R_3 - R_1$

$$\tilde{A} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 2 & 2 & 2 \end{array}\right]$$

Now $a_{22} = 2 \neq 0$ is pivot element and we have to eliminate $a_{32} = 2$ by using the row operation

$R_3 \to R_3 - R_2$

$$\tilde{A} = \left[\begin{array}{ccc|c} -1 & 1 & 2 & 2 \\ 0 & 2 & 7 & 12 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

This is the last step of Gauss-elimination.

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & 7 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 12 \\ -10 \end{bmatrix}$$

It gives $-5x_3 = -10, \quad 2x_2 + 7x_3 = 12, \quad -x_1 + x_2 + 2x_3 = 2$

Solving these equations, we get $x_3 = 2,\ x_2 = -1,\ x_1 = 1$.

Rank of a matrix

The ***rank*** of a matrix is the number of L. I. row/column vectors of the matrix. In other words, the Rank of the matrix=Number of non-zero rows in the last step of Gauss elimination method.

The rank of a matrix A is denoted by $r(A)$.

Example: Find rank of the following matrix

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

Solution: By Gauss-elimination we get:

$$A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

$$\begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 7R_1 \end{matrix} \qquad A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix}$$

$$R_3 \to R_3 + (1/2)R_2 \qquad A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 non-zero rows.

Hence $r(A) = 2$.

Results

1. $r(A) = r(A^T)$
2. Rank of a Null matrix is 0.
3. For a rectangular matrix $A = [a_{jk}]_{m\times n},\ r(A) \le \min(m,\ n)$.
4. For any square matrix $A = [a_{jk}]_{n\times n}$, if $r(A) = n$ then $|A| \ne 0$ that means matrix is non-singular.
5. For any square matrix $A = [a_{jk}]_{n\times n}$, if $r(A) < n$ then $|A| = 0$ that means matrix is singular.
6. If the elements of rows or columns are in AP then rank of the matrix is always 2.
   Example: Let the matrix be

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$r(A)=2.$

7. If elements of rows or columns are in AP but the multiples are present then rank of the matrix is always 1.
   Example: The following matrices have rank 1.
$$\begin{bmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 3 & 3 & 3\end{bmatrix}, \begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{bmatrix}$$
8. If all the elements of a matrix are equal then rank of the matrix is 1.
   Example: The following matrix has rank 1.
$$\begin{bmatrix}k & k & k\\ k & k & k\\ k & k & k\end{bmatrix}, \ k \neq 0$$

Row-equivalent matrices

Two matrices A and B are said to be ***row-equivalent*** if A can be reduced to B by a sequence of elementary row operations.

Note: Row-equivalent matrices have same rank.

Three Cases of Gauss-Elimination method

Case-I: Unique Solution

If $r(A)=r(\tilde{A})=n$ (number of variables) then the system has unique solution.

Example: Solve the linear system of equations by Gauss-elimination method.
$$x+y-z=9$$
$$8y+6z=-6$$
$$-2x+4y-6z=40$$
Solution: The given linear system of equations is
$$x+y-z=9$$
$$8y+6z=-6$$
$$-x+2y-3z=20$$
The Augmented matrix is
$$\tilde{A}=\left[\begin{array}{ccc|c}1 & 1 & -1 & 9\\ 0 & 8 & 6 & -6\\ -1 & 2 & -3 & 20\end{array}\right]$$
The pivot element is $a_{11}=1$, because it is not 0.

We have to element $a_{31}=-1$ by the row operation

$R_3 \to R_3 + R_1$
$$\tilde{A}=\left[\begin{array}{ccc|c}1 & 1 & -1 & 9\\ 0 & 8 & 6 & -6\\ 0 & 3 & -4 & 29\end{array}\right]$$
Now $a_{22}=8\neq 0$ is pivot element and we have to eliminate $a_{32}=3$ by using the row operation

$R_3 \to R_3 - \frac{3}{8}R_2$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 8 & 6 & -6 \\ 0 & 0 & -25/4 & 125/4 \end{array}\right]$$

The row equivalent linear system is $(-25/4)z = 125/4,\ \ 8y + 6z = -6,\ \ x + y - z = 9$

$\Rightarrow z = -5,\ \ y = 3$ and $x = 1.$

Here, $r(A) = r(\tilde{A}) = 3$

Case-II: Infinite number of solutions

If $r(A) = r(\tilde{A}) < n$ (number of variables) then the system has infinitely many solutions.

Example: Solve the linear system of equations by Gauss-elimination method.

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

Solution: The augmented matrix is

$$\tilde{A} = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right]$$

The pivot element is $a_{11} = 3 \neq 0.$

Now we have to eliminate $a_{21} = 0.6$ and $a_{31} = 1.2$ by the row operations

$R_2 \to R_2 - 0.2R_1,\ R_3 \to R_3 - 0.4R_1$

$$\tilde{A} = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right]$$

The pivot element is $a_{22} = 1.1$.

We have to eliminate $a_{32} = -1.1$ by the row operation

$R_3 \to R_3 + R_2$

$$\tilde{A} = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The row equivalent linear system is

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$1.1x_2 + 1.1x_3 - 4.4x_4 = 1.1$$

From the 2nd equation we get $x_2 = 1 - x_3 + 4x_4$

Putting x_2 in 1st equation we get $x_1 = 2 - x_4$

Since x_3 and x_4 remain arbitrary, so the given linear system of equations has infinitely many solutions.

Here, $r(A) = r(\tilde{A}) = 2 < 4$

Case-III: No solution

If $r(A) \neq r(\tilde{A})$, then the system is inconsistent means it has no solution.

Example: Solve the linear system of equations by Gauss-elimination method.

$$3x_1 + 2x_2 + x_3 = 3$$
$$2x_1 + x_2 + x_3 = 0$$
$$6x_1 + 2x_2 + 4x_3 = 6$$

Solution: The Augmented matrix is

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right]$$

The pivot element is $a_{11} = 3 \neq 0$.

$R_2 \to R_2 - \frac{2}{3}R_1,\ R_3 \to R_3 - 2R_1$

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

Now the pivot element is $a_{22} = -1/3 \neq 0$.

$R_3 \to R_3 - 6R_2$

$$\tilde{A} = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

From 3rd row we get 0=12, which is not possible.

Hence the given linear system of equations has no solution.

Here, $r(A) = 2 \neq r(\tilde{A}) = 3$

Example: Apply Gauss-elimination to find the solution of the following linear system of equations.

$$13x + 12y = -6$$
$$-4x + 7y = -73$$
$$11x - 13y = 157$$

Solution: The augmented matrix is

$$\tilde{A} = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ -4 & 7 & -73 \\ 11 & -13 & 157 \end{array}\right]$$

The pivot element is $a_{11} = 13$.

We have to eliminate $a_{21} = -4$ and $a_{31} = 11$ by using the row operations

$R_2 \to R_2 + \frac{4}{13}R_1,\ R_3 \to R_3 - \frac{11}{13}R_1$

$$\tilde{A} = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ 0 & 139/13 & -973/13 \\ 0 & -301/13 & 2107/13 \end{array}\right]$$

The pivot element is $a_{22} = 139/13$.

We have to eliminate $a_{32} = -301/13$ by the row operation

$R_3 \to R_3 + \frac{301}{139} R_2$

$$\tilde{A} = \left[\begin{array}{cc|c} 13 & 12 & -6 \\ 0 & 139/13 & -973/13 \\ 0 & 0 & 0 \end{array}\right]$$

This is the last step of Gauss-elimination method.
Here, $r(A) = r(\tilde{A}) = 2$. Hence the system has unique solution.
The row equivalent linear system is

$$\frac{139}{13} y = -\frac{973}{13}$$

$$13x + 12y = -6$$

From the 1st equation we get $y = -7$ and from the 2nd equation we get $x = 6$.

Example: Solve the linear system of equation by Gauss-elimination method.

$$\begin{aligned} 4y + 3z &= 8 \\ 2x \quad - z &= 2 \\ 3x + 2y \quad &= 5 \end{aligned}$$

Solution: The augmented matrix is

$$\tilde{A} = \left[\begin{array}{ccc|c} 0 & 4 & 3 & 8 \\ 2 & 0 & -1 & 2 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Since $a_{11} = 0$, so it cannot be taken as pivot element. By interchanging rows we can make a_{11} nonzero.
Interchanging 1st row and 2nd row we get

$R_1 \leftrightarrow R_2$ $$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 3 & 2 & 0 & 5 \end{array}\right]$$

Now the pivot element is $a_{11} = 2$.

We have to eliminate $a_{31} = 3$ by the row operation

$R_3 \to R_3 - \frac{3}{2} R_1$ $$\tilde{A} = \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 2 & 3/2 & 2 \end{array}\right]$$

Now, the pivot element is $a_{22} = 4$.

We have to eliminate $a_{32} = 2$ by using the row operation

$R_3 \to R_3 - \frac{1}{2}R_2 \qquad \tilde{A} = \left[\begin{array}{ccc|c} 2 & 0 & -1 & 2 \\ 0 & 4 & 3 & 8 \\ 0 & 0 & 0 & -2 \end{array}\right]$

The given system has no solution.

Vector Space

A nonempty set V of elements $a,\ b,\cdots$ is called a real ***vector space*** and these elements are called ***vectors*** if in V there are defined two algebraic operations called vector addition and scalar multiplication as follows:

I. Vector Addition

There are five properties under ***vector addition***

i. Closure Property

If $a,\ b \in V$ then $a + b \in V$.

ii. Commutative Property

For any two elements $a,\ b \in V$, $a + b = b + a$.

iii. Associativity

For any three elements $a,\ b,\ c \in V$, $(a+b)+c = a+(b+c)$.

iv. Additive Identity Property

For every $a \in V$ there exists $0 \in V$ such that $0 + a = a$, where 0 is called additive identity element.

v. Additive Inverse Property

For every $a \in V$ there exists $-a \in V$ such that $-a + a = 0$, where $-a$ is called additive inverse of a.

II. Scalar Multiplication

The real numbers are called scalars. ***Scalar multiplication*** is the product of a scalar α and a vector a i.e. αa, where $a \in V$ and α is a scalar in V. There are four axioms which are satisfied by the scalar multiplication αa.

i. It is distributive over scalar multiplication

If $a,\ b \in V$ and α be any scalar then

$$\alpha(a+b) = \alpha a + \alpha b$$

ii. It is distributive over scalar addition

If $a \in V$ and $\alpha,\ \beta$ be any two scalars then

$$a(\alpha+\beta) = \alpha a + \beta a$$

iii. For all scalars α and β in V and for every $a \in V$

$$\alpha(\beta a) = (\alpha\beta)a$$

iv. For every $a \in V$, there exists $1 \in V$ such that $1 \cdot a = a$.

Linear Combination

Let $v_1,\ v_2,\cdots,v_m$ be m vectors of a vector space V and let $\alpha_1,\ \alpha_2,\cdots,\alpha_m$ be m scalars then the expression $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ is called a ***linear combination*** of $v_1,\ v_2,\cdots,v_m$.

Trivial linear combination

The linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ of m vectors $v_1,\ v_2,\cdots,v_m$ is called ***trivial*** if all scalars $\alpha_1,\ \alpha_2,\cdots,\alpha_m$ are zero.

Non-trivial linear combination

The linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$ is ***non-trivial*** if at least one of the scalars $\alpha_1,\ \alpha_2,\cdots,\alpha_m$ is not zero.

Linearly Independent Vector

A set of m vectors $\{v_1, v_2, \cdots, v_m\}$ is said to be ***Linearly Independent (L. I.)*** if there exists a trivial linear combination of $v_1, v_2, \cdots, v_m$ that equals the zero vector.

Example: Prove that the vectors (1 0 1), (1 1 0) and (1 1 -1) are L. I.

Proof: Let $v_1 = (1\ 0\ 1),\ v_2 = (1\ 1\ 0)$ and $v_3 = (1\ 1\ -1)$ be three vectors and $\alpha,\ \beta$ and γ be three scalars.

$\Rightarrow \alpha v_1 + \beta v_2 + \gamma v_3 = \alpha(1\ 0\ 1) + \beta(1\ 1\ 0) + \gamma(1\ 1\ -1) = (0\ \ 0\ \ 0)$

$\Rightarrow (\alpha\ \ 0\ \ \alpha) + (\beta\ \ \beta\ \ 0) + (\gamma\ \ \gamma\ \ -\gamma) = (0\ \ 0\ \ 0)$

$\Rightarrow (\alpha + \beta + \gamma\quad \beta + \gamma\quad \alpha - \gamma) = (0\ \ 0\ \ 0)$

$\alpha + \beta + \gamma = 0,\ \ \beta + \gamma = 0,\ \ \alpha - \gamma = 0$

Since $\beta + \gamma = 0$, so from 1st equation we get $\alpha = 0$.

$\alpha - \gamma = 0 \qquad \Rightarrow \alpha = \gamma = 0$ and $\beta = -\gamma = 0$

Hence $\alpha = \beta = \gamma = 0$

This proves that the linear combination of $v_1,\ v_2$ and v_3 is trivial. Therefore, the given vectors are L. I.

Linearly Dependent vector

The set of vectors $\{v_1, v_2, \cdots, v_m\}$ is said to be ***linearly dependent (L. D.)*** if there exists a non-trivial linear combination of $v_1, v_2, \cdots, v_m$ that equals the zero vector.

Note:

1. If in the set of given vectors, one vector can be expressed as the linear combination of other vectors then the vectors are L. D.
   Example: The vectors $(2\ \ 3\ \ 4),\ (4\ \ 5\ \ 6),\ (6\ \ 8\ \ 10)$ are L. D. because
   $(6\ \ 8\ \ 10) = 1(2\ \ 3\ \ 4) + 1(4\ \ 5\ \ 6)$
2. If in the set of vectors, at least one of the vector is a zero vector then the vectors are L. D.
   Example: The vectors $(2\ \ 3\ \ 4),\ (0\ \ 0\ \ 0),\ (6\ \ 8\ \ 10)$ are L. D.
3. Let $p =$ number of given vectors and $n =$ number of components in each vectors.
   If $n < p$ then the vectors are L. D.
   Example: The vectors $(4\ \ -1\ \ 3),\ (0\ \ 8\ \ 1),\ (1\ \ 3\ \ -5),\ (2\ \ 6\ \ 1)$ are L. D. because $p = 4$ and $n = 3$. Here $n < p$.
4. If the determinant value of the matrix whose rows are individual given vectors, is zero then the given vectors are L. D.
   Example: Let the vectors be $(2\ \ 3\ \ 4),\ (4\ \ 5\ \ 6),\ (7\ \ 8\ \ 9)$.
   Now, $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2(45-48) - 3(36-42) + 4(32-35) = -6 + 18 - 12 = 0$
   Hence the given vectors are L. D.

Determination of L. I. /L. D. vectors by using Gauss-elimination

We can determine the given vectors are L. I. or L. D. by using Gauss-elimination process as follows:

I. Write all the components of the individual vectors in a row and find a matrix which may be square or rectangular.

II. Apply Gauss-elimination to the matrix.
   Here two cases arise

   Case-I: If there is at least one zero row in the last step of Gauss-elimination then the vectors are L. D.

Case-II: If there is no zero row at all in the last step of Gauss-elimination then the vectors are L. I.

Example: Verify whether the vectors (1 0 1), (1 1 0) and (-1 0 -1) are L. I. or L. D.

Solution: Let the matrix be

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 + R_1 \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since in the last step of Gauss-elimination, the matrix has a zero row, so the given vectors are L. D.

Example: Verify whether the vectors (1 0 1), (1 1 0) and (1 1 -1) are L. I. or L. D.

Solution: Let the matrix be

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1 \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$R_3 \to R_3 - R_2 \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Since in the last step of Gauss-elimination, there is no zero row in the matrix, so the given vectors are L. I.

Span of a set

The ***Span*** of a set is the set of all linear combinations of given vectors $v_1, v_2, \cdots, v_m$ with same number of components. Span of a set S is denoted by $[S]$.

Note: A span is also a vector space.

Basis of a vector space

A nonempty subset B of a vector space V is called ***a basis*** for V if B will satisfy the following two conditions:

i. All the elements of a basis set are L. I. vectors

ii. $[B] = V$. That means B generates V. It means every element of V can be expressed as the linear combination of vectors that are elements of B.

Dimension of a vector space

The ***dimension*** of a vector space is the number of elements of a basis set. Dimension of a vector space V is denoted by $\dim V$.

Example: Find a basis and dimension of the vector space of all ordered pairs.

Solution: Basis:

$$(a,\ b) = a(1,\ 0) + b(0,\ 1)$$

$$B = \{(1,\ 0),\ (0,\ 1)\}$$

Dimension: $\dim V = 2 =$ No. of elements in basis set

Example: Find dimension and basis of the set of all 2×2 matrices.
Solution: The set of all 2×2 matrices is a vector space.
Dimension

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

There are four independent components in each element of V.
So $\dim V = 4$
Basis

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Every element of V can be expressed as the linear combination of these four matrices.
Row space
The span of the row vectors of a matrix A is called the row space of A.
Column space
The span of the column vectors of a matrix A is called the column space of A.
Note: $\dim$ row space of matrix $A = \dim$ column space of matrix $A = r(A)$
Inverse of a Matrix

Inverse of a square matrix $A = [a_{ij}]_{n\times n}$ is a square matrix A^{-1} of order n.

$$\Rightarrow AA^{-1} = A^{-1}A = I$$

Here I is an identity matrix of order n.
Results

1. Inverse of a matrix $A = [a_{ij}]_{n\times n}$ exists if and only if $|A| \neq 0$ means A is non-singular.
2. Inverse of a matrix is unique.
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

   Example: Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ $\qquad |A| = 12 - 2 = 10 \neq 0$

   $$\Rightarrow A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

5. For a diagonal matrix $D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$, $D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & \cdots & 0 \\ 0 & 1/d_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $\Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$

6. $(A^{-1})^T = (A^T)^{-1}$

Proof: We have $AA^{-1} = I$
Taking transpose on both sides we get
$(AA^{-1})^T = (A^{-1})^T A^T = I^T = I = (A^T)^{-1} A^T$

By right cancellation we have $(A^{-1})^T = (A^T)^{-1}$.

7. $(A^{-1})^{-1} = A$

Proof: We have $AA^{-1} = I$
Taking inverse on both sides we get
$(AA^{-1})^{-1} = (A^{-1})^{-1} A^{-1} = I^{-1} = I = AA^{-1}$

By right cancellation we have $(A^{-1})^{-1} = A.$

8. $(A^{-1})^2 = (A^2)^{-1}$

Proof: $(A^2)^{-1} = (AA)^{-1} = (A^{-1})(A^{-1}) = (A^{-1})^2.$

Inverse of a Matrix by Gauss-Jordan Elimination

The inverse of a matrix A can be determined by using Gauss-Jordan elimination as follows:

Procedure

I. Take a matrix of the form $[A|I]$.

II. Apply Gauss-elimination to the matrix and reduce the matrix A to upper triangular matrix $[U|H]$.

III. Perform other row operations to reduce the upper triangular matrix U to Identity matrix I $\Rightarrow [I|K]$.

The matrix $K = A^{-1}$.

Example: Calculate the inverse of the following matrix by Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 11 \end{bmatrix}$

Applying Gauss-elimination to the matrix we get:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 4 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 2R_1 \qquad \left[\begin{array}{ccc|ccc} 1 & 2 & 5 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_2 \to (-1)R_2, \quad R_1 \to R_1 + 2R_2 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 9 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$R_1 \to R_1 - 9R_3, \quad R_2 \to R_2 + 2R_3 \qquad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 19 & 2 & -9 \\ 0 & 1 & 0 & -4 & -1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 19 & 2 & -9 \\ -4 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$$

7.9 Linear Transformation

Consider a set of n linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

This set of n equations can be represented by

$$Y = AX$$

This is called ***linear transformation*** which transforms X into Y.

Where $Y = [y_1,\ y_2, \cdots, y_n]^T$ and $X = [x_1,\ x_2, \cdots, x_n]^T$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ This is called matrix of the transformation.

The ***inverse transformation*** of $Y = AX$ is

$$X = A^{-1}Y$$

END