Applications of Ordinary Differential Equations (Modelling Problems)

Modelling means setting up a mathematical model of a physical or other system. The model may be a function to be evaluated or plotted or a differential or other equation to be solved. In this section we consider systems that can be modelled in terms of a separable differential equation.

Definition:
Whenever a physical law involves a rate of change of a function such as velocity, acceleration etc. it will lead to a differential equation and that equation is known as ***Mathematical model*** of that physical process.

In this section we shall solve the following four types of Modelling problems:

Type-I: Growth/Decay Problem

In this case the amount $y(t)$ at time t is directly proportional to the rate of change of $y(t)$.

$$\frac{dy}{dt}\,\alpha\, y(t), \qquad \Rightarrow y' = ky(t)$$

The sign of k depends on growth and decay of the physical process.

Problem Set-1.1

Q.17) Half-Life (Time in which half of the given amount of radioactive substance disappears)

What is the half-life of ${}_{88}Ra^{226}$ (in years)?

Solution: Let $y(t)$ be the amount of radioactive substance at time t.

The rate of change w.r.to time is $\frac{dy}{dt}$.

By the physical law governing the process of radiation

$$\frac{dy}{dt}\,\alpha\, y(t), \qquad \Rightarrow y' = ky(t)$$

The constant $k < 0$ is a definite physical constant whose numerical value is known for various radioactive substance.

In case of Radium ${}_{88}Ra^{226}$, the value of $k \approx -1.4 \times 10^{-11}\ \text{sec}^{-1}$.

Let the initial amount of radioactive substance at time $t = 0$ be y_0.

Hence the initial condition is $y(0) = y_0$.

The initial value problem is

$$\frac{dy}{dt} = ky(t), \quad y(0) = y_0$$

Solving the differential equation by separating variables we get

$$\int \frac{dy}{y} = \int k\,dt, \qquad \Rightarrow \ln|y| = kt + C_1 \qquad \Rightarrow y(t) = Ce^{kt}$$

Using the initial condition we find $C = y_0$

Hence the solution is $y(t) = y_0 e^{kt}$

We have to find out the time during which half of the substance disappeared i.e. $y(t) = \frac{y_0}{2}$.

$$\Rightarrow \frac{y_0}{2} = y_0 e^{kt}, \qquad \Rightarrow t = \frac{\ln 2}{-k} = \frac{\ln 2}{1.4} \times 10^{11}\ \text{sec} = 1570\ \text{years}.$$

Q.18) Radium ${}_{88}Ra^{224}$ has a half-life of about 3.6 days.

a. Given 1 gram, how much will be still present after 1 day?
b. After 1 year?

Solution:

a. Initial amount is given 1 gram.
   In this case the initial value problem is

$$\frac{dy}{dt} = ky(t), \quad y(0) = 1$$

   Solving this we get $y(t) = e^{kt}$.

   It is given that the Radium ${}_{88}Ra^{224}$ has a half-life of about 3.6 days.

   That means in 3.6 days initial amount 1 gram becomes 0.5 gram.

$$\Rightarrow y(3.6) = 0.5$$

$$\Rightarrow 0.5 = e^{k(3.6)} \qquad \Rightarrow k = \frac{-\ln 2}{3.6}$$

   Hence $y(t) = e^{-(\ln 2/3.6)t}$ gram.

   After 1 day it becomes $y(1) = e^{-\ln 2/3.6} = 0.8248605943$ gram.

b. After 1 year=365 days it becomes $y(365) = e^{(-\ln 2/3.6)\times 365} = 3.01233465\times 10^{-31}$ gram.

Q.20) The efficiency of the engine of subsonic airplanes depends on air pressure and usually is maximum near about 35,000 ft. Find the air pressure $y(x)$ at this height. The rate of change $y'(x)$ is proportional to the pressure. At 18,000 ft. it is half its value $y_0 = y(0)$ at sea level.

Solution: Let $y(x)$ be the air pressure. It is given that $y'(x) \alpha\, y(x)$.

$$\Rightarrow y'(x) = ky$$

At sea level air pressure is y_0, the given initial condition is $y_0 = y(0)$.

Again it is given that at 18,000 ft. the pressure is decreased to half its value y_0.

Hence the 2nd initial condition is $y(18000) = y_0/2$.

The initial value problem is

$$y'(x) = ky, \qquad y(0) = y_0 \quad \text{and} \quad y(18000) = y_0/2$$

Solving the ODE we get

$$y(x) = Ce^{kx}$$

Using the initial condition $y(0) = y_0$ we find $C = y_0$

Hence $y(x) = y_0 e^{kx}$

Again using $y(18000) = y_0/2$ we get

$$\frac{y_0}{2} = y_0 e^{18000k} \qquad \Rightarrow k = -\ln 2/18000$$

$$\Rightarrow y(x) = y_0 e^{(-\ln 2/18000)x}$$

The pressure at 35,000 ft. height is

$$\Rightarrow y(35000) = y_0 e^{(-\ln 2/18000)\times 35000} \approx \frac{y_0}{4}.$$

Problem Set-1.3

Q.12) If the growth rate of the number of bacteria at any time t is proportional to the number present at t and doubles in 1 week, how many bacteria can be expected after 2 weeks? After 4 weeks?

Solution: Let $y(t)$ be the number of bacteria present at time t.

It is given that the growth rate of the number of bacteria at any time t is proportional to the number present at t.

$$\frac{dy}{dt} \alpha\, y(t), \qquad \Rightarrow y' = ky(t)$$

Let y_0 be the number of bacteria initially present at time $t = 0$. After 1 week it is doubled.
Hence initial conditions are $y(0) = y_0, \quad y(1) = 2y_0$

Solving the ODE $y' = ky$ we obtain

$$y(t) = Ce^{kt}$$

Using the initial condition $y(0) = y_0$ we find $C = y_0$

The solution becomes $y(t) = y_0 e^{kt}$.

Again using $y(1) = 2y_0$ we obtain

$$2y_0 = y_0 e^k, \qquad \Rightarrow \ln 2 = k$$

Thus the number of bacteria present at time t is

$$y(t) = y_0 e^{(\ln 2)t}$$

The expected number of bacteria after 2 weeks be

$$y(2) = y_0 e^{2(\ln 2)} = 4y_0$$

After 4 weeks $y(4) = y_0 e^{4(\ln 2)} = 16y_0$.

Q.20) If the birth rate and death rate of the number of bacteria are proportional to the number of bacteria present, what is the population as a function of time?

Solution: Let $y(t)$ be the individual present at time t.

Let $B(t)$ be the individual birth at time t and $D(t)$ be the individual death at time t.

It is given that

$$\frac{dB}{dt} \alpha\, y(t) \quad \text{and} \quad \frac{dD}{dt} \alpha\, y(t)$$

$$\Rightarrow \frac{dB}{dt} = k_1 y \quad \text{and} \quad \frac{dD}{dt} = k_2 y$$

$$\Rightarrow \frac{dB}{dt} - \frac{dD}{dt} = (k_1 - k_2)y$$

$$\Rightarrow \frac{dy}{dt} = (k_1 - k_2)y$$

Solving the ODE we get $y(t) = Ce^{(k_1 - k_2)t}$.

If $k_1 - k_2 > 0$ then population increase and if $k_1 - k_2 < 0$ then population decrease.

Q.27) (Dryer) If a wet sheet in a dryer loses its moisture at a rate proportional to its moisture content and if it loses half of its moisture during first 10 minutes, when will it have lost 99% of its moisture?

Solution: Let $y(t)$ be the moisture content in the wet sheet.

It is given that $\frac{dy}{dt} \alpha\, y(t), \qquad \Rightarrow y' = ky(t)$

Let y_0 be the initial moisture present at time $t = 0$. Again it is given that the wet sheet loses half of its moisture during 1st 10 minutes.

Hence initial conditions are $y(0) = y_0, \quad y(10) = y_0 / 2$

Solving the ODE $y' = ky(t)$ we find:

$$y(t) = Ce^{kt}$$

Using $y(0) = y_0$ we get $C = y_0$

$$\Rightarrow y(t) = y_0 e^{kt}$$

Again using $y(10) = y_0 / 2$ we get

$$\frac{y_0}{2} = y_0 e^{10k} \quad \Rightarrow k = -\frac{\ln 2}{10}$$

$$\Rightarrow y(t) = y_0 e^{\left(-\frac{\ln 2}{10}\right)t}$$

We have to calculate the time in which 99% of moisture will be lost that means $0.01y_0$ moisture will be present.

$$\Rightarrow 0.01y_0 = y_0 e^{\left(-\frac{\ln 2}{10}\right)t}$$

$$t = \ln(0.01) \times \frac{10}{\ln 2} \cong 66\,\text{min}.$$

Problems For Practice

Q.21 (Problem Set-1.3)

Q. If in a culture of yeast the rate of growth is proportional to the amount $y(t)$ present at time t, and if $y(t)$ doubles in 1 day, how much can be expected after 3 days at the same rate of growth? After 1 week?

Q. If the growth rate of a culture of bacteria is proportional to the number of bacteria present and after 1 day is 1.25 times the original number, within what interval of time will the number of bacteria (a) double, (b) triple?

Type-II: Newton's Law of Cooling

Let $T(t)$ be the temperature in a body and T_s be the temperature in the surrounding medium. By Newton's law of cooling

$$\frac{dT}{dt} \alpha\ (T - T_s), \qquad \Rightarrow \frac{dT}{dt} = k(T - T_s)$$

Problem Set-1.3

Q.25) A thermometer, reading $5^\circ C$, is brought into a room whose temperature is $22^\circ C$. One minute later the thermometer reading is $12^\circ C$. How long does it take until the reading is practically $22^\circ C$, say $21.9^\circ C$?

Solution: Let $T(t)$ be the temperature in the thermometer and T_s be the temperature room temperature. By Newton's law of cooling

$$\frac{dT}{dt} \alpha\ (T - T_s), \qquad \Rightarrow \frac{dT}{dt} = k(T - T_s)$$

The room temperature is given $22^\circ C$ i.e. $T_s = 22^\circ C$

Hence the ODE is $\frac{dT}{dt} = k(T - 22)$

Initially thermometer reading was $5^\circ C$.

First initial condition is $T(0) = 5$

After one minute reading is $12^\circ C$, the second initial condition is $T(1) = 12$.

The initial value problem is

$$\frac{dT}{dt} = k(T - 22), \qquad T(0) = 5,\ T(1) = 12.$$

Solution of ODE:

$$\frac{dT}{dt} = k(T - 22)$$

$$\int \frac{dT}{T - 22} = k\int dt \qquad \Rightarrow \ln|T - 22| = kt + C_1$$

$\Rightarrow T(t) = 22 + Ce^{kt}$

Using the initial condition $T(0) = 5$ we get

$5 = 22 + C \qquad \Rightarrow C = -17$

Hence the solution becomes $T(t) = 22 - 17e^{kt}$

Again using $T(1) = 12$ we find

$12 = 22 - 17e^{k} \qquad \Rightarrow k = \ln\left(\dfrac{10}{17}\right) \cong -0.5306$

$\Rightarrow T(t) = 22 - 17e^{(-0.5306)t}$

Now we have to calculate t so that $T(t) = 21.9^{\circ}C$

$\Rightarrow 21.9 = 22 - 17e^{(-0.5306)t} \qquad \Rightarrow e^{(-0.5306)t} = \dfrac{0.1}{17} \cong 0.0059$

$\Rightarrow t = \dfrac{\ln(0.0059)}{-0.5306} \cong 9.7$ minute.

Problems For Practice

Q. A metal bar whose temperature is $20\,^{o}C$ is placed in boiling water. How long does it take to heat the bar to practically $100\,^{o}C$ say, to $99.9\,^{o}C$, if the temperature of the bar after 1 min of heating is 51.5°C?

Type-III: Mixing Problem

In a mixture (well stirring) if we add and remove salt quantity, then the changes occur. In this case the rate of change of salt quantity is equal to the difference between salt inflow and outflow rate.

If $y(t)$ be the amount of salt then mathematical formulation of this kind of problem is

$$\frac{dy}{dt} = \text{Salt inflow rate} - \text{Salt outflow rate}$$

Problem Set-1.3

Q.24) A tank contains 400 gal of brine in which 100 lb. of salt are dissolved. Fresh water runs into the tank at a rate of 2 gal/min. The mixture, kept practically uniform by stirring, runs out at the same rate. How much salt will there be in the tank at the end of 1 hour?

Solution: Let $y(t)$ be the amount of salt present in the tank at time t.

We have

$$\frac{dy}{dt} = \text{Salt inflow rate} - \text{Salt outflow rate}$$

Salt inflow rate:

Since fresh water is coming to the tank at a rate of 2gal/min, so there is no salt present.

Hence, Salt inflow rate=0

Salt outflow rate:

Since 400 gal contains $y(t)$ amount of salt, so 1 gal contains $\dfrac{y}{400}$ lb. salt.

Hence 2 gal contains $\dfrac{y}{200}$ lb. of salt.

Thus salt outflow rate $= y/200$

Hence the differential equation is

$$\frac{dy}{dt} = -\frac{y}{200}$$

Solving this ODE we get $\ln|y| = -\dfrac{t}{200} + C_1 \qquad \Rightarrow y(t) = Ce^{-t/200}$

Initially there is 100lb salt was present in the brine.

The initial condition is $y(0) = 100$.
Using the initial condition $y(0) = 100$ in the general solution we get $C = 100$

Hence the solution becomes $y(t) = 100e^{-t/200}$

At the end of 1 hour i.e. after 60 min. the amount of salt will be

$$y(60) = 100e^{-60/200} = 74.08 \text{ lb}$$

Problems For Practice

Q.1) The tank contains 200 gal of water in which 40lb of salt are dissolved. Five gal of brine, each containing 2lb of dissolved salt, run into the tank per minute, and the mixture kept uniform by stirring, runs out at the same rate. Find the amount of salt $y(t)$ in the tank at any time t.

Q.2) The tank contains 1000 gal of water in which 200lb of salt are dissolved. Fifty gal of brine, each containing $(1+\cos t)$ lb. of dissolved salt, run into the tank per minute. The mixture, kept uniform by stirring, runs out at the same rate. Find the amount of salt $y(t)$ in the tank at any time t.

Q.3) The tank contains 80 lb. of salt dissolved in 500 gal of water. The inflow per minute is 20 lb. of salt dissolved in 20 gal of water. The outflow is 20 gal/min of the uniform mixture. Find the time when the salt content in the tank reaches 95% of its limiting value (as $t \to \infty$).

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