8.3. Symmetric, Skew-Symmetric, and Orthogonal Matrices

Symmetric, Skew-Symmetric, and Orthogonal Matrices.

A real square matrix $A = [a_{jk}]$ is called

1. **symmetric** if transposition leaves it unchanged,
$$A^T = A, \qquad \text{thus} \qquad a_{kj} = a_{jk},$$

2. **skew-symmetric** if transposition gives the negative of A,
$$A^T = -A, \qquad \text{thus} \qquad a_{kj} = -a_{jk},$$

3. **orthogonal** if transposition gives the inverse of A,
$$A^T = A^{-1}.$$

Example: The matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

are symmetric, skew-symmetric, and orthogonal, respectively.

Note:

1. The determinant of an orthogonal matrix has the value +1 or −1.
2. Every square matrix can be expressed as the addition of symmetric and skew-symmetric matrix.

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

Eigenvalues of Symmetric and Skew-Symmetric and Orthogonal Matrices:

(a) The eigenvalues of a symmetric matrix are real.
(b) The eigenvalues of a skew-symmetric matrix are pure imaginary or zero.
(c) The eigenvalues of an orthogonal matrix **A** are real or complex conjugates in pairs and have absolute value 1.

Example:

The matrix $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ is symmetric and has eigenvalues 2 and 8.

The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric and has eigenvalues $-i$ and i.

Example: The orthogonal matrix

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

has eigenvalues -1, $(5+i\sqrt{11})/6$ and $(5-i\sqrt{11})/6$, which have absolute value 1.

8.5. Complex Matrices and Different Forms.

Complex Conjugate Matrix

$\bar{A} = [\bar{a}_{jk}]$ is obtained from $A = [a_{jk}]$ by replacing each entry $a_{jk} = \alpha + i\beta$ (α, β real) with its complex conjugate $\bar{a}_{jk} = \alpha - i\beta$. Also, $\bar{A}^T = [\bar{a}_{kj}]$ is the transpose of $\bar{A}$, hence the conjugate transpose of A.

Hermitian, Skew-Hermitian, and Unitary Matrices.

A square matrix $A = [a_{kj}]$ is called

1. **Hermitian** if $\bar{A}^T = A$, that is, $\bar{a}_{kj} = a_{jk}$
2. **skew-Hermitian** if $\bar{A}^T = -A$, that is, $\bar{a}_{kj} = -a_{jk}$
3. **unitary** if $\bar{A}^T = A^{-1}$.

Note:

1. Let A be a unitary matrix. Then its determinant has absolute value one, that is, $|\det A| = 1$.
2. Every complex square matrices can be expressed as the addition of Hermitian and Skew-Hermitian matrices.

$$A = \frac{A + (\overline{A})^T}{2} + \frac{A - (\overline{A})^T}{2}$$

Eigenvalues of Hermitian and Skew-Hermitian and Unitary Matrices.

(a)The eigenvalues of a Hermitian matrix (and thus of a symmetric matrix) are real.

(b)The eigenvalues of a skew-Hermitian matrix (and thus of a skew-symmetric matrix) are pure imaginary or zero.

(c)The eigenvalues of a unitary matrix (and thus of an orthogonal matrix) have absolute value 1.

Example:

The matrix $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ is Hermitian and has eigenvalues 9 and 2.

The matrix $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is skew-Hermitian and has eigenvalues 4i and -2i.

The matrix $\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$ is unitary and has eigenvalues $(\sqrt{3}+i)/2$ and $(-\sqrt{3}+i)/2$, which have absolute value 1.

Different Forms

1. **Quadratic Form**

   Quadratic form is an expression of the form

   $$Q = x^T A x$$

   Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

   $$\Rightarrow Q = x^T A x = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + a_{21}x_1x_2 + a_{22}x_2^2 + \cdots + a_{n1}x_1x_n + \cdots + a_{nn}x_n^2$$

   $$\Rightarrow Q = x^T A x = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + \cdots + (a_{1n} + a_{n1})x_1x_n$$

   In particular

   Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

   $$\Rightarrow Q = x^T A x = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12} + a_{21})x_1x_2 + (a_{13} + a_{31})x_1x_3 + (a_{23} + a_{32})x_2x_3$$

Determination of Symmetric matrix from Quadratic form

From the given Quadratic form $Q = x^T A x$ we can obtain symmetric matrix as follows:

I. Place coefficients of x_i^2 diagonally.

II. Divide the coefficients of $x_i x_j, i \neq j$ equally and place equal values at a_{ij} and a_{ji} places respectively.

Example: Find the symmetric matrix from the following quadratic form

$$Q = 2x_1^2 + 4x_2^2 + 3x_3^2 + 10x_1x_2 + 12x_2x_3 + 8x_3x_1$$

Solution: In case of symmetric matrix $a_{ij} = a_{ji}$

$$\Rightarrow Q = 2x_1^2 + 4x_2^2 + 3x_3^2 + (5+5)x_1x_2 + (6+6)x_2x_3 + (4+4)x_3x_1$$

The symmetric matrix is

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 5 & 4 & 6 \\ 4 & 6 & 3 \end{bmatrix}$$

2. **Hermitian Form**

   Hermitian form H is given by

$$H = \bar{x}^T A x$$

   Where A is a Hermitian matrix and $x$is a column matrix.

Note: Hermitian form is always real.

Example: Is the following matrix Hermitian or Skew-Hermitian matrix? Find the Hermitian form or Skew-Hermitian form.

$$A = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix}, x = \begin{bmatrix} -2i \\ 1+i \end{bmatrix}$$

Solution: $\bar{A} = \begin{bmatrix} 4 & 3+2i \\ 3-2i & -4 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} = A$

Since $\bar{A}^T = A$, so A is Hermitian.

Hermitian form is $H = \bar{x}^T A x$

Here $x = \begin{bmatrix} -2i \\ 1+i \end{bmatrix} \quad \Rightarrow \bar{x} = \begin{bmatrix} 2i \\ 1-i \end{bmatrix}$

$$\Rightarrow H = \bar{x}^T A x = [2i \quad 1-i] \begin{bmatrix} 4 & 3-2i \\ 3+2i & -4 \end{bmatrix} \begin{bmatrix} -2i \\ 1+i \end{bmatrix} = [7i+5 \quad 10i] \begin{bmatrix} -2i \\ 1+i \end{bmatrix} = 4$$

3. **Skew-Hermitian Form**

   Skew-Hermitian form S is given by

$$H = \bar{x}^T A x$$

   Where A is a Skew-Hermitian matrix and x is a column matrix.

Note: Skew-Hermitian form is always 0 or purely imaginary.

Example: Is the following matrix Hermitian or Skew-Hermitian matrix? Find the Hermitian form or Skew-Hermitian form.

$$A = \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix}, x = \begin{bmatrix} i \\ 4 \end{bmatrix}$$

Solution: $\bar{A} = \begin{bmatrix} -i & -2-3i \\ 2-3i & 0 \end{bmatrix}$, $\bar{A}^T = \begin{bmatrix} -i & 2-3i \\ -2-3i & 0 \end{bmatrix} = -\begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} = -A$

Since $\bar{A}^T = -A$, so A is Skew-Hermitian.

Skew-Hermitian form is $S = \bar{x}^T A x$. Here $x = \begin{bmatrix} i \\ 4 \end{bmatrix} \quad \Rightarrow \bar{x} = \begin{bmatrix} -i \\ 4 \end{bmatrix}$

$$\Rightarrow S = \bar{x}^T A x = [-i \quad 4] \begin{bmatrix} i & -2+3i \\ 2+3i & 0 \end{bmatrix} \begin{bmatrix} i \\ 4 \end{bmatrix} = [9+12i \quad 2i+3] \begin{bmatrix} i \\ 4 \end{bmatrix} = 17i$$