CHAPTER-1
First Order Differential Equations

1.1 Basic Concepts

A **differential equation** is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables. Differential equations are of widespread interest because of their connection with phenomena in the physical world. Many physical laws and relations can be expressed mathematically in the form of differential equations. These are useful tools for describing the natural phenomena of science and engineering models.

Based on the number of independent variables involved in the equation, differential equations are classified as Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs).

Ordinary Differential Equations (ODEs)

An ordinary differential equation is one that contains one or more functions of one independent variable and the derivatives of those functions. All the derivatives occurring in such differential equations are ordinary derivatives.

Example:

$$\frac{dy}{dx} = \cos x \tag{1.1}$$

$$\frac{d^2y}{dx^2} + 9y = e^{-2x} \tag{1.2}$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y = 0 \tag{1.3}$$

Partial Differential Equations (PDEs)

A PDE is one in which the dependent variable (say u) depends on two or more independent variables (say x, t, y…). In this case, the derivative occurring in the differential equation are partial derivatives.

Example:

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

PDEs have important engineering applications, but they are more complicated than ODEs.

Order: The order of a differential equation is the order of the highest order derivative appearing in the equation.

An ODE is said to be of **order** n if the nth order derivative of the unknown function y is the highest ordered derivative of y in the equation. The concept of order gives a useful classification into ODEs such as first-order, second-order, and so on.

Hence, eq. (1.1) is of the first-order ODE, eq. (1.2) is of second-order, and eq. (1.3) is of third-order ODE.

Degree: Degree of a differential equation is the degree of the highest order derivative (when the derivatives are cleared of radicals and fractions).

Linear Differential Equations:

An nth order ODE in the dependent variable y is said to be **linear** in y if

i) y and all its derivatives are of degree one.

ii) No product terms of y and/or any of its derivatives are present.

iii) No transcendental functions of y and/or its derivatives occur.

An ODE which failed to be linear is called **Nonlinear ODE**.

First-Order ODEs

The general form of a First-order differential equation is given by

$$F(x, y, y') = 0 \qquad (1.4)$$

or

$$y' = f(x, y). \qquad (1.5)$$

Eq. (1.5) is called the *explicit form*, whereas eq. (1.4) is known as the *implicit form*.

Solution of a differential equation:

When we say $x = 1$ is a solution of the algebraic equation $x^2 - 1 = 0$, we mean that when $x = 1$ is substituted in the equation the equality will hold. Similarly, we say that $y = x^2$ is a solution of the differential equation $\frac{dy}{dx} = 2x$ since if we put $y = x^2$ in the above equation the equality holds.

Thus, we give the formal definition of the solution of a general ordinary differential equation of first order

$$F(x, y, y') = 0$$

as follows

Definition 1.1 A function $y = h(x)$ is called a **solution** of a given ODE (1.4) on some open interval $a < x < b$ if $h(x)$ is defined and differentiable throughout the interval and is such that the equation becomes an identity if y and y' are replaced with h and h', respectively. The curve (the graph) of h is called a **solution curve**.

General Solution:

A general solution of an nth order ordinary differential equation is one that involves n necessary arbitrary constants.

Particular Solution:

The particular solution is a solution obtained from the general solutions by assigning particular values to the arbitrary constants.

Initial Value Problem (IVP):

An Initial Value Problem (IVP) consists of

1. A first order differential equation $y' = f(x, y)$, and
2. An initial condition of the form $y(x_0) = y_0$.

Example: The differential equation

$$y' = y \text{ with } y(0) = 3$$

is an example of IVP.

Note: Every IVP has exactly one solution.

Procedure to solve IVP:

Given an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

This can be solved as follows:

1. Find the general solution to the given differential equation, involving an arbitrary constant C.
2. Substitute $x = x_0$ and $y = y_0$ in the general solution to find the value of C.
3. Substitute the value of C in the general solution and find a particular solution to the given initial value problem.

Example: Verify that the given function y is a solution of ODE. Determine from y the particular solution of the IVP.

$$yy' = 4x, \quad y^2 - 4x^2 = c(y > 0), \; y(1) = 4.$$

Solution:

Verification:

Given function is $y^2 - 4x^2 = c$

Differentiating w.r.to x we find:

$$2yy' - 8x = 0, \quad \Rightarrow yy' = 4x$$

It is verified that $y^2 - 4x^2 = c$ is a general solution of $yy' = 4x$.

Solution of IVP:

It is given that $y = 4$ when $x = 1$

Substituting $y = 4$ and $x = 1$ in the general solution we get:

$$4^2 - 4 = 12 = c$$

The particular solution is $y^2 - 4x^2 = 12$.

<u>Formation of a Differential Equation</u>

Let

$$f(x, y, c_1, c_2, c_3, \cdots, c_n) = 0 \tag{1.6}$$

be an equation containing n arbitrary constants $c_1, c_2, c_3, \cdots, c_n$.

Differential equations are formed as follows:

Step-I: Differentiate the Equation (1.6) as many times as the number of arbitrary constants in the equation.

Differentiating Equation (1.6) w. r. to x successively n times, we obtain

$$\left.\begin{aligned} &f\left(x, y, c_1, c_2, c_3, \cdots, c_n, \frac{dy}{dx}\right) = 0 \\ &f\left(x, y, c_1, c_2, c_3, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \\ &f\left(x, y, c_1, c_2, c_3, \cdots, c_n, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0 \end{aligned}\right. \tag{1.7}$$

Step-II: Eliminate the arbitrary constants $c_1, c_2, c_3, \cdots, c_n$ from Equations (1.6) and (1.7) and find

$$\varphi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$$

This is the required differential equation.

Note: If an equation contains n number of arbitrary constants, then we obtain a differential equation of nth order. Hence nth order differential equation has exactly n arbitrary constants in its general solution.

<u>Illustrative Examples</u>

Eliminate the arbitrary constants from the following equations and obtain the differential equation.

(a). $y = cx + c^2$

(b). $y = Ae^x + Be^{-x} + C$

(c). Find the differential equation of all circles passing through the origin and having centers on the axis of x.

Solution:

(a). Given equation is

$$y = cx + c^2 \tag{1.8}$$

It has only one parameter c.

Differentiating Equation (1.8) w. r. to x we get

$$\frac{dy}{dx} = c \tag{1.9}$$

Using (1.9) in (1.8) we obtain

$$x\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2-y=0$$

This is the required differential equation.

(b). Given equation is

$$y=Ae^x+Be^{-x}+C \tag{1.10}$$

It has three parameters A, B and C.

Differentiating (1.10) thrice w. r. to x we get

$$\frac{dy}{dx}=Ae^x-Be^{-x}$$

$$\frac{d^2y}{dx^2}=Ae^x+Be^{-x}$$

$$\frac{d^3y}{dx^3}=Ae^x-Be^{-x}=\frac{dy}{dx}$$

The required differential equation is $\frac{d^3y}{dx^3}-\frac{dy}{dx}=0$

(c). Let the center of the circle be at $(a,0)$.

The equation of the circle passing through the origin and having centers on the axis of x is

$$(x-a)^2+y^2=a^2$$

$$x^2-2ax+y^2=0 \tag{1.11}$$

It involves only one arbitrary constant a.

Differentiating (1.11) w. r. to x, we find

$$2x-2a+2y\frac{dy}{dx}=0$$

$$x+y\frac{dy}{dx}=a \tag{1.12}$$

Using (1.12) in (1.11) we find

$$x^2-2x\left(x+y\frac{dy}{dx}\right)+y^2=0$$

The required differential equation is

$$\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$$

Questions for Practice:

Eliminate the arbitrary constants from the following equations and obtain the differential equation.

1. $x=A\cos(nt+\alpha)$
2. $e^{2y}+2axe^y+a^2=0$
3. $y=A\cos 2t+B\sin 2t$

Find differential equation of

4. All circles of radius r whose centers lie on the x-axis.
5. All parabolas with x-axis as the axis and $(a, 0)$ as focus.
6. All parabolas whose axes are parallel to y-axis.
7. All circles of radius a.

Answers

1. $\frac{d^2x}{dt^2}+n^2x=0$ 2. $(1-x^2)\left(\frac{dy}{dx}\right)^2+1=0$ 3. $\frac{d^2y}{dt^2}+4y=0$

4. Equation: $(x-a)^2+y^2=r^2$, ODE: $y^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=r^2$

5. Equation: $y^2=4ax$, ODE: $\frac{dy}{dx}=\frac{y}{2x}$

6. Equation: $(x-h)^2=4a(y-k)$, ODE: $\frac{d^3y}{dx^3}=0$

7. Equation: $(x-h)^2+(y-k)^2=a^2$, ODE: $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$

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