The exponential distribution

X is said to have an exponential distribution with parameter $\lambda \, (\lambda > 0)$ if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Note: Exponential distribution is special case of Gamma distribution with condition

$$\frac{1}{\beta} = \lambda \quad \text{and} \quad \alpha = 1.$$

Q. Verify i) proper pdf ii) $E(X)$ iii) $V(X)$.

Sol^n

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_0^{\infty} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^{\infty}$$

$$= \lambda \left(0 - \left(-\frac{1}{\lambda}\right)\right) = 1 \,//$$

ii) $$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

Let $\lambda x = t \Rightarrow \lambda dx = dt$

$$= \frac{1}{\lambda} \int_0^{\infty} t e^{-t}\, dt$$

$$= \frac{1}{\lambda} \int_0^{\infty} t^{2-1} e^{-t}\, dt$$

$$= \frac{1}{\lambda} \Gamma(2) = \frac{1}{\lambda} \,//$$

Gamma function

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx \quad \text{for } \alpha > 0$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_0^\infty \lambda x^2 \cdot e^{-\lambda x}\, dx$$

Let $t = \lambda x$

$$= \frac{1}{\lambda} \int_0^\infty (\lambda x)^2 \, e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda} \times \frac{1}{\lambda} \int_0^\infty t^2 e^{-t}\, dt$$

$$= \frac{1}{\lambda^2} \Gamma(3) = \frac{2}{\lambda^2}$$

$$\therefore V(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Q. Cdf of the exponential distribution.

Soln:

$$F_X(x) = \int_0^x \lambda e^{-\lambda x}\, dx$$

$$= \lambda \cdot \left. \frac{e^{-\lambda x}}{-\lambda} \right|_0^x$$

$$= \lambda \times -\frac{1}{\lambda} \times \left(e^{-\lambda x} - 1\right)$$

$$= 1 - e^{-\lambda x}.$$

The Gamma Distribution.

A continious RV X is said to have a gamma distribution if the pdf of X is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the parameters $\alpha > 0, \beta > 0$.

If $\beta = 1$, $f(x;\alpha,1) = \begin{cases} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

is called the standard gamma distribution.

$$\int_0^\infty f(x;\alpha)\,dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

$$= \frac{1}{\Gamma(\alpha)} \times \Gamma(\alpha) = 1. \quad \text{(proper pdf).}$$

If $\alpha = 1$ and $\beta = \frac{1}{\lambda}$,

$$f(x;1,\tfrac{1}{\lambda}) = \begin{cases} \frac{1}{(1/\lambda)^{\alpha}\,\Gamma(\alpha)} x^{\alpha} e^{-x/(1/\lambda)} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \lambda \cdot e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad \text{(exponential distribution).}$$

i) $E(X) = \alpha\beta$

ii) $V(X) = \alpha\beta^2$

iii) cdf of gamma distribution.

$$F_X(x) = P(X \leq x) = \int_0^x \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}\,dx$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^x x^{\alpha-1} e^{-x/\beta}\,dx.$$

Let $\frac{x}{\beta} = t \Rightarrow \frac{dx}{\beta} = dt$

$$= \frac{1}{\cancel{\beta^\alpha}\,\Gamma(\alpha)} \int_0^x (\cancel{\beta} t)^{\alpha-1} e^{-t}\, d(\cancel{\beta} t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1.$$