Model Answer

2nd Mid Semester Examination - 2018

Mathematics - II [MA-1002]

Q.1(a)

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{m=0}^{\infty} a_m x^m$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots = \sum_{m=1}^{\infty} m a_m x^{m-1}$$

Given ODE $xy' = 4y + k$

$\Rightarrow x(a_1 + 2a_2 x + 3a_3 x^2 + \cdots) = 4(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) + k$

$\Rightarrow a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 = (4a_0 + 4a_1 x + 4a_2 x^2 + 4a_3 x^3 + \cdots) + k$

$\Rightarrow 4a_0 + k + (4a_1 - a_1)x + (4a_2 - 2a_2)x^2 + (4a_3 - 3a_3)x^3 + (4a_4 - 4a_4)x^4 = 0 = 0$

$\Rightarrow 4a_0 + k = 0 \Rightarrow a_0 = -\frac{k}{4}$

$a_1 = 0, \ a_2 = 0, \ a_3 = 0, \ a_4 =$ arbitrary.

$\therefore \boxed{y = -\frac{k}{4} + a_4 x^4}$

1(b) See the text book.

1(c) $n(n+1) = 12$

$\Rightarrow n = 3$

$\therefore \cancel{y = y_1(x)}$, ~~where~~

$\cancel{y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots}$

$\cancel{= 1 - 6x^2}$

$\therefore y = P_n(x) = P_3(x) = \frac{1}{2}[5x^3 - 3x]$

1(d) $(x^2 - x)y'' - xy' + y = 0$

$\Rightarrow x(x-1)y'' - xy' + y = 0$

$\Rightarrow xy'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0$

$\Rightarrow x^2 y'' - \frac{x}{x-1}xy' + \frac{x}{x-1}y = 0$

$b(x) = -\frac{x}{x-1} = x(1-x)^{-1}$

$= x(1 + x + x^2 + \cdots)$

$= x + x^2 + x^3 + \cdots$

$c(x) = \frac{x}{x-1} = -x(1-x)^{-1}$

$= -(x + x^2 + x^3 + \cdots)$

$\therefore b_0 = 0, \ c_0 = 0$

$r(r-1) + rb_0 + c_0 = 0$

$\Rightarrow r(r-1) = 0$

$\Rightarrow \boxed{r = 0, 1.}$

2

1(e) $\Gamma(\frac{3}{2}) = \Gamma(1+\frac{1}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\times\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$

Q.2

$$y'' + 2xy' + y = 0 \longrightarrow (1)$$

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1} + \cdots$$
$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + \cdots + n(n-1)a_nx^{n-2} + \cdots$$

From (1),

$$2a_2 + 6a_3x + 12a_4x^2 + \cdots + (n+2)(n+1)a_{n+2}x^n + \cdots$$
$$+ 2x\left[a_1 + 2a_2x + 3a_3x^2 + \cdots + (n+1)a_{n+1}x^n + \cdots\right]$$
$$+ a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots = 0$$

$$\Rightarrow (2a_2 + a_0) + (6a_3 + 2a_1 + a_1)x + (12a_4 + 4a_2 + a_2)x^2 + \cdots$$
$$+ \left[(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_n\right]x^n + \cdots = 0$$

$\Rightarrow 2a_2 + a_0 = 0, \quad 6a_3 + 3a_1 = 0, \quad 12a_4 + 5a_2 = 0$

$\Rightarrow a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{2}, \quad a_4 = -\frac{5}{12}\times -\frac{a_0}{2}$

$= \frac{5}{24}a_0$

$(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + a_n = 0$

$\Rightarrow a_{n+2} = -\frac{[(n+1)a_{n+1} + a_n]}{(n+1)(n+2)}$

$$a_5 = -\frac{(4a_4 + a_3)}{4.5} = -\frac{1}{20}\left(4\times\frac{5}{24}a_0 - \frac{a_1}{2}\right)$$
$$= -\frac{a_0}{24} + \frac{a_1}{40}$$

$$y = a_0 + a_1x - \frac{a_0}{2}x^2 - \frac{a_1}{2}x^3 + \frac{5}{24}a_0x^4 + \left(-\frac{a_0}{24} + \frac{a_1}{40}\right)x^5 + \cdots$$
$$= a_0\left(1 - \frac{x^2}{2} + \frac{5}{24}x^4 - \cdots\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} - \cdots\right)$$

3(a)

We know that $(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$ —— ①

Differentiating ① partially w.r.t t, we get

$$-\frac{1}{2}(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-1/2} = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow x\sum_{n=0}^{\infty} t^n P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} - 2x\sum_{n=0}^{\infty} n t^n P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x) \longrightarrow ②$$

Equating the coefficient of t^n,

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$ (proved)

Q. 3(b)

$$L\{t \sinh(2t)\}$$

$$= L\left\{\frac{e^{2t}-e^{-2t}}{2} t\right\}$$

$$= \frac{1}{2} L\{t e^{2t}\} - \frac{1}{2} L\{e^{-2t} t\}$$

$$= \frac{1}{2}\frac{1}{(s-2)^2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

$$= \frac{1}{2}\left[\frac{1}{(s-2)^2} - \frac{1}{(s+2)^2}\right]$$

$$= \frac{4s}{(s^2-4)^2}$$

$L\{f(t)\} = F(s)$

$L\{e^{at} f(t)\} = F(s-a)$

$f(t) = t$

$F(s) = L\{f(t)\} = L\{t\}$

$= \frac{1}{s^2}$

Q.4

$$xy'' + 2y' - xy = 0 \quad \text{——— (1)}$$

$$y = x^r \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{r+m}$$

$$y' = \sum_{m=0}^{\infty} a_m (r+m)\, x^{r+m-1}$$

$$y'' = \sum_{m=0}^{\infty} a_m (r+m)(r+m-1)\, x^{r+m-2}$$

From (1),

$$x \sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-2} + 2 \sum_{m=0}^{\infty} a_m (r+m) x^{r+m-1} - x \sum_{m=0}^{\infty} a_m x^{r+m} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} a_m (r+m)(r+m-1) x^{r+m-1} + 2 \sum_{m=0}^{\infty} (r+m) x^{r+m-1} - \sum_{m=0}^{\infty} a_m x^{r+m+1} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [(r+m)(r+m+1)]\, a_m x^{r+m-1} - \sum_{m=0}^{\infty} a_m x^{r+m+1} = 0$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$s = m-1 \qquad\qquad s = m+1$$

$$\Rightarrow \sum_{s=-1}^{\infty} (r+s+1)(r+s+2)\, a_{s+1} x^{r+s} - \sum_{s=1}^{\infty} a_{s-1} x^{r+s} = 0$$

$$\Rightarrow r(r+1) a_0 x^{r-1} + (r+1)(r+2) a_1 x^r + \sum_{s=1}^{\infty} [(r+s+1)(r+s+2) a_{s+1} - a_{s-1}] x^{r+s} = 0$$

$$\Rightarrow r = 0, -1, \qquad (r+1)(r+2) a_1 = 0$$

$$\Rightarrow a_1 = 0 \text{ (when } r = 0),$$

For $r = 0$

$$(r+s+1)(r+s+2) a_{s+1} - a_{s-1} = 0$$

$$\Rightarrow a_{s+1} = \frac{a_{s-1}}{(s+1)(s+2)}$$

$$\therefore a_2 = \frac{a_0}{2 \cdot 3} = \frac{a_0}{6}, \quad a_3 = \frac{a_1}{(2+1)(2+2)} = 0$$

$$a_4 = \frac{a_2}{4 \cdot 5} = \frac{a_0}{6 \times 4 \times 5} = \frac{a_0}{120}$$

$$\therefore y = x^0 \left(a_0 + 0 + \frac{a_0}{6} x^2 + 0 + \frac{a_0}{120} x^4 + \cdots\right)$$

$$= a_0 \left(1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots\right)$$

$$\therefore y_1(x) = 1 + \frac{x^2}{6} + \frac{x^4}{120} + \cdots = \frac{1}{x}\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] = \frac{\sinh x}{x}$$

5(i)

$P_0(x) = 1$ $\Rightarrow$ $1 = P_0(x)$

$P_1(x) = x$ $\quad x = P_1(x)$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$ $\quad x^2 = \frac{1 + 2P_2(x)}{3} = \frac{P_0 + 2P_2}{3}$

$$\therefore x^2 + x + 2 = \frac{P_0(x) + 2P_2(x)}{3} + P_1(x) + 2P_0(x)$$

$$= \frac{1}{3}P_0(x) + 2P_0(x) + \frac{2}{3}P_2(x) + P_1(x)$$

$$= \frac{7}{3}P_0(x) + P_1(x) + \frac{2}{3}P_2(x)$$

5(ii) See the Text Book.

Q.6

$x = 4z$, $\quad y' = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{4}\frac{dy}{dz}$

$$y'' = \frac{d}{dx}\left(\frac{1}{4}\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{1}{4}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{1}{4}\frac{d^2y}{dz^2}\cdot\frac{1}{4} = \frac{1}{16}\frac{d^2y}{dz^2}$$

$$x^2y'' + xy' + \frac{1}{16}(x^2 - 1)y = 0$$

$$\Rightarrow 16z^2\cdot\frac{1}{16}\frac{d^2y}{dz^2} + 4z\cdot\frac{1}{4}\frac{dy}{dz} + \frac{1}{16}(16z^2 - 1)y = 0$$

$$\Rightarrow z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + \left(z^2 - \frac{1}{16}\right)y = 0$$

$$n = \frac{1}{4}$$

$$\therefore y = c_1 J_{\frac{1}{4}}(z) + c_2 J_{-\frac{1}{4}}(z)$$

$$= c_1 J_{\frac{1}{4}}\left(\frac{x}{4}\right) + c_2 J_{-\frac{1}{4}}\left(\frac{x}{4}\right)$$

Q. 7

$$y(t) - \int_0^t (t-\tau)\, y(\tau)\, d\tau = \sin 2t$$

$$y - y * t = \sin 2t \longrightarrow (1)$$

Let $L\{y(t)\} = Y(s)$

From (1), $L\{y\} - L\{y * t\} = L\{\sin 2t\}$

$$\Rightarrow Y(s) - L\{y\} * L\{t\} = \frac{2}{s^2+2^2}$$

$$\Rightarrow Y(s) - Y(s)\frac{1}{s^2} = \frac{2}{s^2+4}$$

$$\Rightarrow Y(s)\left[1 - \frac{1}{s^2}\right] = \frac{2}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{2s^2}{(s^2-1)(s^2+4)} = \frac{2}{5}\,\frac{5s^2}{(s^2-1)(s^2+4)}$$

$$= \frac{2}{5}\left[\frac{1}{s^2-1} + \frac{4}{s^2+4}\right]$$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{\sinh at\} = \frac{a}{s^2-a^2}$

$$\Rightarrow y(t) = \frac{2}{5}\left[L^{-1}\left\{\frac{1}{s^2-1}\right\} + 4L^{-1}\left\{\frac{1}{s^2+4}\right\}\right]$$

$$= \frac{2}{5}\left[\frac{\sinh t}{1} + 4\,\frac{\sin 2t}{2}\right]$$

$$= \frac{2}{5}(\sinh t + 2\sin 2t)$$

20/03/18

20/03/18

20.3.18