- **Propositions:**

It is a declarative sentence which can be either true or false but not both.

Ex: P : 2+3 ⟶ True
↳ Variable

Truth table

P	~P
T	F
F	T

- **Negation (NOT)**

Let P be a proposition. Then negation of P is denoted by ~P or $\bar{P}$

⇒ "It is not the case that P"

T ⟶ 1
F ⟶ 0

- **Conjunctions (AND)**

Let P and q be two propositions. Conjunctions of p and q is denoted by $p \wedge q$ is a statement
"p and q"

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

When both are true
⇒ True

- **Disjunctions (OR)**

Let P and q are two propositions. Disjunction of p and q is denoted by $p \vee q$ is a statement
"p or q".

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

When any one is true
⇒ True

Q Find the truth table of $(p \wedge q) \vee r$

p	q	r	$p \wedge q$	$(p \wedge q) \vee r$	$(p \vee q) \oplus r$
T	T	T	T	T	F
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	T	T
F	T	F	F	F	F
F	F	T	F	T	T
F	F	F	F	F	F

Exclusive OR :

$p \oplus q$

↳ XOR

When both are different ⇒ true

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Conditional Statement

$p \to q$ ⇒ "if p then q"

$p \to q$ = false [if p = true, q = false]

$p \to q$ otherwise true

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Converse :

If $p \to q$ = Conditional

then $q \to p$ = converse

p	q	$p \to q$	$q \to p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Contrapositive :

If $p \to q$ is proposition

then $\sim p \to \sim p$ = Contrapositive

Inverse : $\sim p \to \sim q$ is inverse of $p \to q$

p	q	$p \to q$	$\sim p$	$\sim q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
T	T	T	F	F	T	T	T
T	F	F	F	T	T	T	F
F	T	T	T	F	F	F	T
F	F	T	T	T	T	T	T

- Biconditional Statements : (XNOR)

$p \leftrightarrow q \Rightarrow$ "p if and only if q"

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Q Construct TT for the following propositions

(i) $(p \vee q) \to (p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(ii) $(p \vee \sim q) \to q$

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \to q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

Logical equivalence

The notation $p \equiv q$ denotes that p and q are logically equivalent

- How to check two propositions are equivalent.

One way to determine whether two compound propositions are equivalent to use a truth table.

In particular, the compound propositions p and q are equivalent if and only if the columns are same.

p	$\sim p$	q	$\sim p \to q$
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	F

Q $p \to q \equiv \sim p \to q$

p	q	$p \to q$	$\sim p$	$\sim p \to q$
T	T	T	F	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

Not logically equivalent.

Q $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ ⟶ Distributive Law

p	q	r	$p \vee q$	$p \vee r$	$q \wedge r$	$(p \vee q) \wedge (p \vee r)$	$p \vee (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Q $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ ⟶ Distributive law

p	q	r	$p \wedge q$	$q \vee r$	$p \wedge r$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	T	F	F	F
F	T	F	F	T	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

Q $\sim(p \wedge q) \equiv \sim p \vee \sim q$ $\longrightarrow$ De Morgans Law

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$(\sim p \wedge \sim q)$
T	F	F	T	F	T	F
F	T	T	F	F	T	F

$\sim(p \vee q) \equiv \sim p \wedge \sim q$ $\longrightarrow$ De Morgans Law

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	F	F	T	T	F	F
F	T	T	F	T	F	F

Q $p \vee (p \wedge q) \equiv p$ $\longrightarrow$ Absorption Law

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	F	F	T
F	T	F	F

$p \wedge (p \vee q) \equiv p$ $\longrightarrow$ Absorption Law

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	F	T	T
F	T	T	F

- **Tautology**: A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it is called a tautology.

Ex: $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

- **Contradiction**: A compound proposition that is always false is called contradiction

Ex: $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Q. Check if the following are tautology or not ?

(a) $(p \wedge q) \to p$ (c) $\sim p \to (p \to q)$

(b) $p \to (p \vee q)$ (d) $(p \wedge q) \to (p \to q)$

(a)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Tautology

(b)

p	q	$\sim p$	$p \to q$	$\sim p \to (p \to q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Tautology

(c)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Tautology

(d)

p	q	$p \wedge q$	$p \to q$	$(p \wedge q) \to (p \to q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Tautology

Logical Equivalence

(a) $p \wedge T \equiv p$ (d) $p \vee T \equiv T$

(b) $p \vee F \equiv p$ (e) $p \vee p \equiv p$

(c) $p \wedge F \equiv F$ (f) $p \wedge p \equiv p$

Here,

T → Tautology

F → Contradiction

Standard logical equivalence

$\sim(\sim p) \equiv p$ → Double Negation Law

$p \vee q \equiv q \vee p$
$p \wedge q \equiv q \wedge p$ } Commutative Law

$(p \vee q) \vee r \equiv p \vee (q \vee r)$
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ } Associative Law

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ } Distributive law

$\sim(p \wedge q) \equiv \sim p \vee \sim q$
$\sim(p \vee q) \equiv \sim p \wedge \sim q$ } De Morgan's law

$p \vee (p \wedge q) \equiv p$
$p \wedge (p \vee q) \equiv p$ } Absorption Law

logical equivalences involving conditional statements

$$p \rightarrow q \equiv \sim p \vee q$$
$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$
$$p \vee q \equiv \sim p \rightarrow q$$
$$p \wedge q \equiv \sim(p \rightarrow \sim q)$$
$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$
$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$
$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$
$$(p \rightarrow r) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$
$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

Argument: An argument in propositional logic is a sequence of propositions.

Final proposition in the argument → Conclusion
All the other proposition in the argument → Premises

The argument is valid if the conclusion is true.

① First construct the compound proposition of the premises
$(p_1 \wedge p_2 \wedge \dots \wedge p_n)$

② The above mentioned argument will be true if
$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology
where 'q' is the conclusion.

Q. Check if the argument is valid or not.
$(p \rightarrow q) \wedge (\sim p \rightarrow r) \wedge (r \rightarrow s) \longrightarrow (\sim q \rightarrow s)$

p	q	r	s	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim p \rightarrow r$	$r \rightarrow s$	$\sim q \rightarrow s$	$(p\rightarrow q)\wedge(\sim p\rightarrow r) \wedge (r \rightarrow s)$	Ca…
0	0	0	0	1	1	T	F	T	F	F	T
0	0	0	1	1	1	T	F	T	T	F	T
0	0	1	0	1	1	T	T	F	F	F	T
0	0	1	1	1	1	T	T	T	T	T	T
0	1	0	0	1	0	T	F	T	T	F	T
0	1	0	1	1	0	T	F	T	T	F	T
0	1	1	0	1	0	T	T	F	T	F	[illegible]
0	1	1	1	1	0	T	T	T	T	T	[illegible]
1	0	0	0	0	1	F	T	T	F	F	
1	0	0	1	0	1	F	T	T	T	F	
1	0	1	0	0	1	F	T	F	F	F	
1	0	1	1	0	1	F	T	T	T	F	
1	1	0	0	0	0	T	T	T	T	T	
1	1	0	1	0	0	T	T	T	T	T	
1	1	1	0	0	0	T	T	F	T	F	
1	1	1	1	0	0	T	T	T	T	T	

Predicate and Quantifiers

Predicates: Statements containing one or more than one unknown variables. All the variables $\in R$
↳ Domain

"$x \geq 3$", "$x = y+3$", "$x+y=z$", "$x+1=5$"

For particular value of a predicate, that predicate will be a proposition such as $x=1$, $z=3$

$P(x)$ — Predicate or propositional function if $P(x)$ is a ~~populatio~~ proposition for each value of x.

Once a value has been assigned to a variable x, the statement $P(x)$ becomes a proposition and has a truth value.

Quantifiers: In English the words like all, some, many, none, few are used for quantification.

Quantification
↙ Universal Quantification
↘ Existential Quantification

$x+1 \leq 5$
Universal Quantification

$x = \{0, 1, 2\}$ → All the values satisfy the above equation

$x = \{2, 3, 6, 7\}$ → Some of the values (not all) satisfy the above condition
↳ Existential Quantification

The universal quantification $P(x)$ is the statement
"$P(x)$ for all values of x in the domain"

The notation $\forall x\, P(x)$ denotes universal quantification of $P(x)$

"for all $x\, P(x)$" or "for every $x\, P(x)$"

When all the elements in the domain can be listed - say $x_1, x_2 \dots x_n$ → it follows that the Universal Quantification $\forall x P(x)$ is the same as the conjunction.

$$P(x_1) \land P(x_2) \land \dots \land P(x_n)$$

become the conjunction is true if and only if $P(x_1), P(x_2), \dots P(x_n)$ are all true

The Existential Quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$"

We use the notation $\exists x P(x)$ for the existential Quantification of $P(x)$

$\exists$ → Existential Quantifier

Rules of inference

Rules of inference are the templates or formulas of the proposition for constructing a valid argument or tautology

$(p \to q) \land p \to q$

p	q	$p \to q$	$(p \to q) \land p$	$(p \to q) \land p \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	~~F~~ T
F	F	T	F	T

↳ Tautology
Hence a valid argument

Modus Tollens

$$\begin{array}{l} \sim q \\ p \to q \\ \hline \therefore \sim p \end{array}$$

$(\sim q) \wedge (p \rightarrow q) \rightarrow (\sim p)$

p	q	~p	~q	p → q	(~q) ∧ (p → q)	(~q) ∧ (p → q) → (~p)
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Valid argument because of tautology

• <u>Hypothetical Syllogism</u>

$p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$

p	q	r	p → q	q → r	p → r	(p → q) ∧ (q → r)	(p → q) ∧ (q → r) → (p → r)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Valid argument because its tautology

• <u>Disjunctive Syllogism</u>

$p \vee q$

$\sim p$

$\therefore q$

$(p \vee q) \wedge (\sim p) \rightarrow q$

p	q	~p	p ∨ q	(p ∨ q) ∧ (~p)	(p ∨ q) ∧ (~p) → q
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Valid argument because its tautology.

Addition

$$\frac{p}{\therefore p \vee q} \qquad p \rightarrow (p \vee q)$$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Simplification

$$\frac{p \wedge q}{\therefore p} \qquad (p \wedge q) \rightarrow p$$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q} \qquad p \wedge q \longrightarrow (p \wedge q)$$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Resolution

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array}$$

$(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$

p	q	r	$\sim p$	$p \vee q$	$q \vee r$	$\sim p \vee r$	$(p \vee q) \wedge (\sim p \vee r)$	$(p \vee q) \wedge (\sim p \vee r) \rightarrow (q \vee r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	T	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T	T
F	F	T	T	F	T	T	F	T
F	F	F	T	F	F	T	F	T

- $(p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \longrightarrow (p \vee q \vee r)$

p	q	r	$\sim p$	$\sim q$	$\sim r$	$(p \vee \sim q \vee r)$	$(\sim p \vee q \vee r)$	$(p \vee q \vee \sim r)$	$(p \vee \sim q \vee r) \wedge (\sim p \vee q \vee r) \wedge (p \vee q \vee \sim r) \rightarrow (p \vee q \vee r)$
T	T	T	F	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T	T
T	F	T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	F	T	F
F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	F	T	T	F
F	F	T	T	T	F	T	T	F	F
F	F	F	T	T	T	T	T	T	T

$p \wedge T \equiv p$ $\quad p \wedge F \equiv F$ $\quad p \wedge p \equiv p$

$p \vee T \equiv T$ $\quad p \vee F \equiv p$ $\quad p \vee p \equiv p$

$\sim p \vee p \equiv T$ $\quad \sim p \wedge T \equiv \sim p$ $\quad \sim p \wedge F \equiv F$

$\sim p \vee T \equiv T$ $\quad \sim p \wedge p \equiv F$ $\quad \sim p \vee F \equiv \sim p$

■ Prove the following without truth table

(i) $(p \wedge q) \to (p \vee q)$
(ii) $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$

} are tautology

(i) $(p \wedge q) \to (p \vee q)$

We know $p \wedge q \equiv \sim p \vee q$

So, $(p \wedge q) \to (p \vee q)$

$\equiv (\sim p \vee q) \to (p \vee q)$

$\equiv (\sim p \vee \sim q) \to (p \vee q)$

$\equiv ((\sim p \vee \sim q) \vee p) \vee q$

$\equiv ((\sim q \vee \sim p) \vee p) \vee q$ → Commutative law

$\equiv (\sim q \vee (\sim p \vee p)) \vee q$ → Associative law

$\equiv (\sim q \vee \text{tautology}) \vee q$

$\equiv \text{tautology} \vee q$

$\equiv \text{tautology}$

$\left.\begin{array}{l} \sim(p \vee q) \equiv \sim p \wedge \sim q \\ \sim(p \wedge q) \equiv \sim p \vee \sim q \end{array}\right\}$

$\left.\begin{array}{l} p \vee (q \vee r) \equiv (p \vee q) \vee r \\ p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \end{array}\right\}$

$\sim p \vee p \equiv \text{tautology}$

(ii) $\sim(p \vee q) \vee (\sim p \wedge q) \vee p$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee p$

$\equiv \sim p \wedge (\sim q \vee q) \vee p$

$\equiv (\sim p \wedge \text{tautology}) \vee p$

$\equiv p \vee (\sim p \wedge \text{tautology})$

$\equiv p \vee \sim p$

$\equiv \text{tautology}$

■ Without truth table proof that

(i) $p \vee (p \wedge q) \equiv p$
(ii) $\sim(\sim(p \vee q) \vee (\sim p \wedge q)) \equiv p$
(iii) $(p \wedge q) \vee (p \wedge \sim q) \equiv p$
(iv) $p \to (q \vee R) \equiv (p \to q) \vee (p \to R)$

(i) $p \vee (p \wedge q)$

$p \equiv p \wedge \text{tautology}$

$\equiv (p \wedge \text{tautology}) \vee (p \wedge q)$

$\equiv p \wedge (\text{tautology} \vee q)$

$\equiv p \wedge \text{tautology}$

$\equiv p$

(iii) $\sim(\sim(p \vee q) \vee (\sim p \wedge q))$

$\equiv \sim((\sim p \wedge \sim q) \vee (\sim p \wedge q))$

$\equiv \sim(\sim p \wedge (\sim q \vee q))$

$\equiv \sim(\sim p \wedge \text{tautology})$

$\equiv \sim(\sim p)$

$\equiv p$

(iiii) $(p \wedge q) \vee (p \wedge \sim q)$

$\equiv p \wedge (q \vee \sim q)$

$\equiv p \wedge \text{tautology}$

$\equiv p$

■ Show that $(p \leftrightarrow q)$ and $(p \vee q) \rightarrow (p \wedge q)$ are logically equivalent without truth table

$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$

$\equiv ((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))$

$\equiv ((\sim p \wedge \sim q) \vee \text{contradiction}) \vee (\text{contradiction} \vee (q \wedge p))$

$\equiv (\sim(p \vee q) \vee \text{contradiction}) \vee (\text{contradiction} \vee (q \wedge p))$

$\equiv \sim(p \vee q) \vee (q \wedge p)$

$\equiv \sim(p \vee q) \vee (p \vee q)$

$\equiv (p \vee q) \rightarrow (p \wedge q)$

■ Show that $(p \leftrightarrow q)$ and $(p \wedge q) \vee (\sim p \wedge \sim q)$ are logically equivalence without truth table

$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv ((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee ((\sim p \wedge p) \vee (q \wedge p))$

$\equiv (\sim(p \vee q) \vee \text{contradiction}) \vee (\text{contradiction} \vee (q \wedge p))$

$\equiv \sim(p \vee q) \vee (q \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee (p \wedge q)$

$\equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

* Show that $\sim(p \leftrightarrow q)$ and $(p \leftrightarrow \sim q)$ are logically equivalent (without truth table)

$$\sim(p \leftrightarrow q) \equiv \sim((p \rightarrow q) \wedge (q \rightarrow p))$$
$$\equiv \sim(p \rightarrow q) \vee \sim(q \rightarrow p)$$
$$\equiv \sim(\sim p \vee q) \vee \sim(\sim q \vee p)$$
$$\equiv (\sim(\sim p) \wedge (\sim q)) \vee (\sim(\sim q) \wedge \sim p)$$
$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

$$(p \leftrightarrow \sim p) \equiv (p \rightarrow \sim q) \wedge (\sim q \rightarrow p)$$
$$\equiv (\sim p \vee \sim q) \wedge (\sim(\sim q) \vee p)$$
$$\equiv (\sim p \vee \sim q) \wedge (q \vee p)$$
$$\equiv ((\sim p \vee \sim q) \wedge q) \vee ((\sim p \vee \sim q) \wedge p)$$
$$\equiv ((\sim p \wedge q) \vee (\sim q \wedge q)) \vee ((\sim p \wedge p) \vee (\sim q \wedge p))$$
$$\equiv ((\sim p \wedge q) \vee \text{contradiction}) \vee (\text{contradiction} \vee (\sim q \wedge p))$$
$$\equiv (\sim p \wedge q) \vee (\sim q \wedge p)$$
$$\equiv (q \wedge \sim p) \vee (p \wedge \sim q)$$
$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

Propositional functions

Mathematical Induction

$P(n): 1+2+3+\dots+n = \frac{n(n+1)}{2}$ where n is a +ve integer.

Step 1: Basis Step

We show that the statement is true for a +ve integer

Step 2: Inductive Step

let $P(k)$ is true where $k \geq b$

then we show that $P(k+1)$ is true.

For $n=1$, $\frac{1(1+1)}{2} = 1$

For $n=2$, $\frac{2(2+1)}{2} = 3$

Both are true

let $P(k)$ is true, for $n = k > 2$

We've to prove,

$P(k+1)$: $\underbrace{1+2+3+\dots+k}_{\text{Sum of first } k \text{ natural numbers}}+(k+1) = \frac{k(k+1)}{2} + (k+1)$

$= (k+1)(k/2 + 1)$

$= \frac{(k+1)(k+2)}{2}$

$P(n)$ is true for all +ve integers

Q. Use mathematical induction to prove that $(n^3 - n)$ is divisible by 3

$P(n)$: $(n^3 - n)$ is divisible by 3

For $n=1$, $(n^3-n) = 0$ is divisible by 3

" $n=2$, $(n^3-n) = 6$ " " "

" $n=3$, $(n^3-n) = 24$ " " "

let $P(k)$ is true for $n = k > 2$

$P(k+1) = (k+1)^3 - (k+1) = (k+1)((k+1)^2 - 1)$

$= (k^3 + 3k^2 + 3k + 1) - (k+1)$

$= (k^3 - k) + 3(k^2 + k)$

Now for $K = 1, 2, 3, \dots K$

$P(K+1)$ is divisible by 3

Since $(K^3 - K)$ is divisible by 3 and 3 is a factor of $3(K^2+K)$

So, $(K^3-K) + 3(K^2+K)$ is divisible by 3

$\Rightarrow (K+1)^3 - (K+1)$ " "

$\Rightarrow P(K+1)$ is true

Q Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57

For $n=1$ $P(1) = 7^3 + 8^3 = 855 / 57 = 15$

For $n=2$ $P(2) = 7^4 + 8^4 = 35169 / 57 = 617$

Let $P(K)$ is true when $n = K \geq 2$

$P(K+1)$: $7^{(K+1)+2} + 8^{2(K+1)+1}$

$= 7(7^{K+2}) + (8^2 \times 8^{2K+1})$

$= 7 \times 7^{K+2} + (7+57) \times 8^{2K+1}$

$= (7 \times 7^{K+2}) + (7 \times 8^{2K+1}) + (57 \times 8^{2K+1})$

$= 7(7^{K+2} + 8^{2K+1}) + (57 \times 8^{2K+1})$

Q If $a_n = 2^n + 3^n$, $n \in Z^+$ and $a_0 = 2$ then prove that $a_n = 5a_{n-1} - 6a_{n-2}$ $\forall n \geq 2$ by mathematical induction.

By strong induction,

here, $a_n = 2^n + 3^n$, $n \in Z^+$

$a_0 = 2$, $a_1 = 5$

$P(n)$: $a_n = 5a_{n-1} - 6a_{n-2}$

for $n=3$, $a_3 = 5a_2 - 6a_1$

$P(2)$ is true since $a_2 = 13 = 2^2 + 3^2$

Let $P(n)$ is true for $n = 3, 4, 5 \dots$

then $P(3), P(4) \dots P(K-2), P(K-1), P(K)$ are all true;

$a_K = 5a_{K-1} - 6a_{K-2}$ ⟶ ②

Now, $5a_{(K+1)-1} - 6a_{(K+1)-2}$

$= 5a_K - 6a_{K-1}$

$= 5(5a_{K-1} - 6a_{K-2}) - 6a_{K-1}$

$= 19a_{K-1} - 30a_{K-2}$

$= 19(2^{K-1} + 3^{K-1}) - 30(2^{K-2} + 3^{K-2})$

$= 19 \cdot 2^{K-1} + 19 \cdot 3^{K-1} - 15 \cdot 2^{K-1} - 10 \cdot 3^{K-1}$

$= 4 \cdot 2^{K-1} + 9 \cdot 3^{K-1}$

$= 2^{K+1} + 3^{K+1}$

$= a_{K+1}$

Q Given that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_{n+3} = a_{n+2} + a_{n+1} + a_n$ $\forall n \in Z^+$ prove that $a_n < 2^n$ by strong induction.

$a_1 = 1$, $a_2 = 2$, $a_3 = 3$

$n = 1$

$a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6 < 2^4$

$P(4)$ is true

let $P(5), P(6) \dots P(k-2), P(k-1), P(k)$ are true.

[$a_{k+1} < 2^{k+1}$]

$a_{k+1} = a_{k-2+3} = a_{k-2+2} + a_{k-2+1} + a_{k-2}$

$= a_k + a_{k-1} + a_{k-2}$

$= < 2^k + < 2^{k-1} + < 2^{k-2}$

$= 2^k(1 + 1/2 + 1/4)$

$= 2^k \cdot 7/4$

We know, $7/4 < 2$ $\quad < 2^k \cdot 2$

$\therefore$ $P(k+1)$ is true

Q Given $a_0 = 2$, $a_1 = 7$ and $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, then prove that $a_n = 3^{n+1} - 2^n$ $\forall n \in N$ by strong induction

$a_0 = 2$, $a_1 = 7$ and $a_n = 5a_{n-1} - 6a_{n-2}$ — ①

$a_n = 3^{n+1} - 2^n$

$P(1) = 3^2 - 2^1 = 9 - 2 = 7$ is true for $n = 1$

when $P(n) = 3^{n+1} - 2^n$

let $P(n)$ true for $n = 2, 3, 4, 5 \dots$ then $P(3), P(4), P(5) \dots P(k-1), P(k)$ all are true

Let $P(n)$ be propositional func

$P(n)$: $a_n = 3^{n+1} - 2^n$ $\forall x \in N$ — ②

Since $p(k)$ and $p(k-1)$ are true so

$$\left.\begin{aligned} a_k &= 3^{k+1} - 2^k \\ a_{k-1} &= 3^k - 2^{k-1} \end{aligned}\right\} — ③$$

Putting $n = k+1$ in ①, we get

$a_{k+1} = 5a_{(k+1)-1} - 6a_{(k+1)-2}$

$= 5a_k - 6a_{k-1}$

$= 5(3^{k+1} - 2^k) - 6(3^k - 2^{k-1})$

$= 5\cdot 3^{k+1} - 5\cdot 2^k - 6\cdot 3^k + 6\cdot 2^{k-1}$

$= 5\cdot 3^{k+1} - 5\cdot 2^k - 2\cdot 3\cdot 3^k + 6\cdot 2^{k-1}$

$= 5\cdot 3^{k+1} - 5\cdot 2^k - 2\cdot 3^{k+1} + 3\cdot 2^k$

$= 3\cdot 3^{k+1} - 2\cdot 2^k = 3^{k+2} - 2^{k+1}$

Now we can say that $P(k+1)$ is true

Therefore $P(n)$ is true $\forall n \in N$

$a_n = 3^{n+1} - 2^n$

Functionality Complete set of Connectives

Normally we know the connectives $\{\wedge, \vee, \sim\}$
Set of connectives $\{\vee\}, \{\vee, \wedge\}, \{\vee, \sim\} \ldots$
A set of connectives is set to be functionally complete if a compound proposition can be expressed as a logically equivalent compound proposition by the said set of connectives.
Ex: $\{\wedge, \vee\}$, $\{\sim, \wedge\}$ and $\{\sim, \vee\}$

Satisfiable

A compound proposition is said to be satisfiable if there exist some truth values of the variables that makes its true.

Ex: $(p \vee \sim q) \wedge (q \vee \sim r) \wedge (r \vee \sim p)$ is satisfiable

p	q	r	$\sim p$	$\sim q$	$\sim r$	$p \vee \sim q$	$q \vee \sim r$	$r \vee \sim p$	$(p \vee \sim q) \ldots (r \wedge \sim p)$
T	T	T	F	F	F	T	T	T	T
T	T	F	F	F	T	T	T	F	F
T	F	T	F	T	F	T	F	T	F
T	F	F	F	T	T	T	T	F	F
F	T	T	T	F	F	F	T	T	F
F	T	F	T	F	T	F	T	T	F
F	F	T	T	T	F	T	F	T	F
F	F	F	T	T	T	T	T	T	T

There exist some truth values. not all the truth values of the variable

Q $(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$(p \vee q \vee r)$	$(\neg p \vee \neg q \vee \neg r)$	$(p \vee q \vee r) \wedge (\ldots)$
T	T	T	F	F	F	T	F	F
T	T	F	F	F	T	T	T	T
T	F	T	F	T	F	T	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	T	T	T
F	F	F	T	T	T	F	T	F

Unsatisfiable

A compound proposition is said to be unsatisfiable if there are no truth values of the variables that make it true.

Q Find formula of the expression and prove it by mathematical induction.

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots + \frac{1}{n(n+1)}$$

$$\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{n+1-1}{n(n+1)} = \frac{n}{n+1}$$

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots + \frac{1}{n(n+1)}$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \ldots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

$$= \frac{n}{n+1}$$

$P(n) = n/n+1$

$n=1,\ P(1) = 1/2$

$n=2,\ P(2) = 2/3$

Now let $P(k)$ is true for $k = n > 2$

Now for $P(k+1)$

$$Sum = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots + \frac{1}{(k+1)(k+2)}$$

$$\text{Sum} = \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$

$$= \frac{1}{K+1}\left(K + \frac{1}{K+2}\right) = \frac{K(K+2)}{(K+1)(K+2)} = \frac{K}{K+1}$$

$$\text{Sum} = \frac{K}{K+1} + \frac{1}{K+1} - \frac{1}{K+2} = 1 - \frac{1}{K+2} = \frac{K+2-1}{K+2} = \frac{K+1}{K+2}$$

which is in the form for $\frac{K+1}{K+1+1}$

OR $P(n) : \frac{n}{n+1}$

For $n=1$, $P(1) = 1/1+1 = 1/2$

$P(1)$ is true

Let $P(K)$ is true for $K = n \geq 1$

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \cdots + \frac{1}{K(K+1)} = \frac{K}{K+1}$$

Now,

$$\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{(K+1)(K+2)}$$

$$= \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \cdots + \frac{1}{K(K+1)} + \frac{1}{(K+1)(K+2)}$$

$$= \frac{K}{K+1} + \frac{1}{(K+1)(K+2)}$$

$$= \frac{1}{K+1}\left(K + \frac{1}{K+2}\right)$$

$$= \frac{1}{K+1}\left(\frac{K^2+2K+1}{K+2}\right)$$

$$= \frac{(K+1)^2}{(K+1)(K+2)}$$

$$= \frac{K+1}{K+2}$$

$P(n)$ is true $\forall\, n \in N$

$\therefore$ $P(K+1)$ is true

Q. Show that the condition statement $(p \wedge q) \rightarrow (p \rightarrow q)$ is tautology without truth table

$$(p \wedge q) \rightarrow (p \rightarrow q)$$
$$\equiv (p \wedge q) \rightarrow (\sim p \vee q)$$
$$\equiv \sim(p \wedge q) \vee (\sim p \vee q)$$
$$\equiv (\sim p \vee \sim q) \vee (\sim p \vee q)$$
$$\equiv (\sim p \vee \sim q \vee q) \vee \sim p$$
$$\equiv (\sim p \vee \text{tautology}) \vee \sim p$$
$$\equiv (\text{tautology} \vee \sim p)$$
$$\equiv \text{tautology}$$

Q. Check whether the following compound proposition

$(p \wedge q) \rightarrow (r \vee \sim p)$

and $(\sim p \vee r) \rightarrow (p \wedge \sim q)$

are logically equivalence or not by truth table

$(p \wedge q) \rightarrow (r \vee \sim p)$

p	q	r	$\sim p$	$(p \wedge q)$	$(r \vee \sim p)$	$(p \wedge q) \rightarrow (r \vee \sim p)$
T	T	T	F	T	T	T
T	T	F	F	T	F	F
T	F	T	F	F	T	T
T	F	F	F	F	F	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	T	F	T	T

$(\sim p \vee r) \rightarrow (p \wedge \sim q)$

p	q	r	$\sim p$	$\sim q$	$\sim p \vee r$	$p \wedge \sim q$	$(\sim p \vee r) \rightarrow (p \wedge \sim q)$
T	T	T	F	F	T	F	F
T	T	F	F	F	F	F	T
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	F	T	F	F
F	T	F	T	F	T	F	F
F	F	T	T	T	T	F	F
F	F	F	T	T	T	F	F

So, both the compound propositions are not logically equivalent.

De Morgans law of the Quantification

$\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

(i) $\sim \forall x\, P(x) \equiv \exists x (\sim P(x))$

(ii) $\sim \exists x\, P(x) \equiv \forall x (\sim P(x))$

Rules for inference for the quantification:

(i) Universal Instantiation

$\forall x\, P(x)$

P(c) where c is the element of the domain

↳ true.

(ii) Universal generalization

If p(c) is true, c is an arbitrary value of x, then $\forall x\, P(x)$ is true

(iii) Existential Instantiation

$\exists x\, P(x)$

↳ P(c) is true for some values of x

(iv) Existential generalization

If P(c) is true for some values of x

↳ $\exists x\, P(x)$

Ex: Show that the premises "Everyone in this discrete mathematics class taken a course in computer science" and "Maria is a student in this class" imply the conclusion.

"Maria has taken a course in Computer Science".

let

c(x): x is a student in discrete mathematics class

D(x): x has taken a course in computer science

Premises are:

(i) $\forall x\, (c(x) \rightarrow D(x))$

(ii) c(Maria)

Conclusion

D(Maria)

Step I: $\forall x\,(C(x) \rightarrow D(x)) \rightarrow C(\text{Maria}) \rightarrow D(\text{Maria})$

$$\frac{\forall x\,(C(x) \rightarrow D(x))}{C(\text{Maria}) \rightarrow D(\text{Maria})}$$ By Universal Instantiation

Step II: $C(\text{Maria})$

$$\frac{C(\text{Maria}) \rightarrow D(\text{Maria})}{D(\text{Maria})}$$ by Modus Ponens rule

Ex: Show that the premises "A student in this class has not read a book" and "Everyone in this class passed the first exam" implies the conclusion "Someone who passed the first exam has not read the book".

Let $C(x)$: x is in this class
$B(x)$: x has read a book
$\sim B(x)$: x has not read a book
$P(x)$: x has passed the first exam.

Premises are
(1) $\exists x\,(C(x) \wedge \sim B(x))$
(2) $\forall x\,(C(x) \rightarrow P(x))$

Conclusion
$\exists x\,(P(x) \wedge \sim B(x))$

Step 1: $\exists x\,(C(x) \wedge \sim B(x))$
$C(a) \wedge \sim B(a)$ where a is a student in the class

Step 2: $$\frac{C(a) \wedge \sim B(a)}{C(a)}$$ By simplification

Step 3: $$\frac{C(a) \wedge \sim B(a)}{\sim B(a)}$$ By simplification

Step 4: $$\frac{\forall x\,(C(x) \rightarrow P(x))}{C(a) \rightarrow P(a)}$$ By universal instantiation rule

Step 4 : $c(a) \longrightarrow P(a)$

$$\frac{c(a)}{P(a)}$$ by modus ponens

Step 5 : $P(a) \wedge \neg B(a)$, by conjunction rule

$\exists x(P(x) \wedge \neg B(x))$ by existential generalization

Set, Relation and Function

A set is a collection of well defined distinct objects.

$A = \{a, b, c\}$ $\quad a \neq b \neq c$

$A = \{x : x \in \mathbb{R}\}$

let A is a set then B is said to be a subset of A if elements of B are elements of A but all elements aren't present

$B \subseteq A$

Power set : No. of subsets of a set. 2^n, $n \rightarrow$ cardinality

Cardinality : No. of elements of a set is called the cardinality of a set or the size of the set

$|A|$ or $n(A) = 3$ if $A = \{a, b, c\}$

Null set : Set having cardinality 0

ϕ, $\{ \}$

Power set of a set A is a set of all subset of A. Let S is a set, then power set of S is denoted by P(S) $\quad |S| = n$

$|P(S)| = 2^{|S|} = 2^n$

$A = \{2, 3, 4\}$

$|A| = 8 = \{\{2\} \{3\} \{4\} \{2,3\} \{3,4\} \{2,4\} \{\phi\} \{2,3,4\}\}$

Power set of A

Cardinality of set A is $|A| = 8$

$\{\phi\}$ is the power set of null set ϕ

Operation of sets : Union, Intersection, Complement, Difference

- **Union of two sets**

let A and B be sets. The union of the sets A and B is denoted by $A \cup B$ is the set that contains those elements that are either in A or B or in both.

$A \cup B = \{x \mid x \in A \vee x \in B\}$

Intersection of two sets

Let A and B be sets. The intersection of the sets A and B denoted by $A \cap B$ is the set containing those elements in both A and B. An element x belongs to the intersection of the sets A and B if and only if x belongs to A and x belongs to B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Difference of sets

Let A and B be sets. The difference of A and B denoted by $A-B$ is a set containing those elements that are in A but not in B.

$$A-B = \{x \mid x \in A \wedge x \notin B\}$$

Complement of a set

Let U be the universal set. The complement of the set A denoted by $\bar{A}$, the complement of A wrt U. Therefore, the complement of the set A is $U-A$

$$\bar{A} = \{x \in U \mid x \notin A\}$$

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

Suppose $x \in \overline{A \cap B}$

By defination of complement, $x \notin A \cap B$

By defination of intersection, proposition $\neg((x \in A) \wedge (x \in B))$ is true

Applying De Morgans law for propositions, we see that

$$\neg(x \in A) \text{ or } \neg(x \in B)$$

By defination of negation of propositions, $x \notin A$ or $x \notin B$

By defination of complement of set, $x \in \bar{A}$ or $x \in \bar{B}$

By defination of union, $x \in \bar{A} \cup \bar{B}$

$$\therefore \overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$$

Suppose $x \in \bar{A} \cup \bar{B}$

By defination of union, $x \in \bar{A}$ or $x \in \bar{B}$

By defination of complement, $x \notin A$ or $x \notin B$

The proposition $\neg(x \in A) \vee \neg(x \in B)$ is true

By De Morgans law of proposition $\neg(x \in A) \vee \neg(x \in B)$ is true

By defination of intersection, $\neg(x \in A \cap B)$

$\therefore$ By defination of complement, $x \in \overline{A \cap B}$

$$\therefore \quad \bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$$

Principle of inclusion exclusion

If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways"

This subtraction rule is also known as the principle of inclusion-exclusion, especially when it is used to count the number of elements in the union of two sets. Suppose that A_1 and A_2 are sets. Then there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2.

The number of ways to select an element from A_1 or from A_2 that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element that is both A_1 and A_2

Because there are $|A_1 \cup A_2|$ ways to select an element in either A_1 or in A_2 and $|A_1 \cap A_2|$ ways to select an element common to both sets

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Q How many positive integers not exceeding 100 are divisible either by 4 or by 6 ?

Let A be the set of positive integers not exceeding 100 that are divisible by 4

and B be the set of positive integers not exceeding 100 that are divisible by 6

Then $A \cap B$ is the set of positive integers not exceeding 100 that are divisible by 4 and 6

$\Rightarrow$ $A \cap B$ is the set of positive integers not exceeding 100 that are divisible by 12

Then $A \cup B$ is the set of positive integers not exceeding 100 are divisible either by 4 or by 6.

$|A| = [100/4] = 25$

$|B| = [100/6] = 16$

$|A \cap B| = [100/12] = 8$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$= 25 + 16 - 8$$
$$= 33$$

Therefore, there are 33 positive integers not exceeding 100 are divisible by 4 and 6

CARTESIAN PRODUCTS

Let A and B be non empty sets. The cartesian product of A and B denoted by $A \times B$, is a set of all orderd pairs (a, b) where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$$

Q Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$

$A \times B = \{(1,a), (1,b) (1,c) (2,a) (2,b) (2,c)\}$

Relation

Any subset of a cartesian product of finite number of sets is known as a relation.

Binary relation

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

Q Consider the following relations on $\{1,2,3,4\}$

$R_1 = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,1)(4,4)\}$

$R_2 = \{(1,1)(1,2)(2,1)\}$

$R_3 = \{(1,1)(1,2)(1,4)(2,1)(2,2)(3,3)(4,1)(4,4)\}$

$R_4 = \{(2,1)(3,1)(3,2)(4,1)(4,2)(4,3)\}$

$R_5 = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,3)(2,4)(3,3)(3,4)(4,4)\}$

$R_6 = \{(3,4)\}$

Which of these relations are reflexive, symmetric, antisymmetric and transitive?

R_2 and R_5 are reflexive because they both contain all pairs of the form (a,a) namely $(1,1)(2,2)(3,3)$ and $(4,4)$. The other relations are not reflexive because they do not contain all of these ordered pairs.

R_2 and R_3 are symmetric because in each case (b,a) belongs to the relation whenever (a,b) does.

R_4, R_5 and R_6 are all antisymmetric. For each of these relations there is no pair of elements a and b with $a \neq b$ such that both (a,b) and (b,a) belong to the relation.

R_4, R_5 and R_6 are transitive. For each of these relations, we can show it transitive by verifying that if (a,b) and (b,c) belong to the relation then (a,c) also does.

Q $A = \{1,2,3\}$, $B = \{1,2,3,4\}$

$R_1 = \{(1,1)(2,2)(3,3)\}$

$R_2 = \{(1,1)(1,3)(1,2)(1,4)\}$

$R_1 \cup R_2 = \{(1,1)(1,2)(1,3)(1,4)(2,2)(3,3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2)(3,3)\}$

$R_2 - R_1 = \{(1,2)(1,3)(1,4)\}$

Q let A and B be the set of all students and the set of all courses at a school respectively. Suppose that R_1 consists of all ordered pairs (a,b) where a is a student who has taken course b and R_2 ~~cour~~ consists of all ordered pairs (a,b) where a is a student who requires course b to graduate. What are the relations $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 \oplus R_2$, $R_1 - R_2$, $R_2 - R_1$.

$R_1 \cup R_2$ consists of all ordered pairs (a,b) where a is a student who either has taken course b or needs course b to graduate.

$R_1 \cap R_2$ consists of all ordered pairs (a,b) where a is a student who has taken course b and needs this course to graduate.

$R_1 \oplus R_2$ consists of all ordered pairs (a,b) where student a has taken course b but doesn't need it to graduate or needs course b to graduate but has not taken it.

$R_1 - R_2$ is the set of ordered pairs (a,b) where a has taken course b but doesn't need it to graduate. that is b is an elective course that a has taken.

$R_2 - R_1$ is a set of all ordered pairs (a,b) where b is a course that a needs to graduate but has not taken by a.

Q $R_1 = \{(x,y) \mid x < y\}$ $R_2 = \{(x,y) \mid x > y\}$

$R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$

$(x,y) \in R_1 \cup R_2$ iff $(x,y) \in R_1$ or $(x,y) \in R_2$

$(x,y) \in R_1 \cup R_2$ iff $x < y$ or $y < x$ because the condition $x < y$ or $x > y$ is same as the condition $x \neq y$

$$R_1 \cup R_2 = \{(x,y) \mid x \neq y\}$$

It is impossible for a pair (x,y) to belong to both R_1 and R_2 because it is impossible that $x < y$ and $x > y$.

$$R_1 \cap R_2 = \phi$$

$$R_1 - R_2 = R_1$$

$$R_2 - R_1 = R_2$$

$$R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2) = \{(x,y) \mid x \neq y\}$$

Q What is the composite of the relations R and S where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1)(1,4)(2,3)(3,1)(3,4)\}$ and S is a relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with $S = \{(1,0)(2,0)(3,1)(3,2)(4,1)\}$

$S \circ R$ is constructed using all ordered pairs in R and ordered pairs in S, where the second element of the ordered pair in R agrees with the first element of the ordered pair in S.

For example, the ordered pair $(2,3) \in R$ and $(3,1) \in S$ produce the ordered pair $(2,1) \in S \circ R$

Computing all the ordered pairs in the composite, we find

$$S \circ R = \{(1,0)(1,1)(2,1)(2,2)(3,0)(3,1)\}$$

■ The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

Suppose $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$

In particular $R^2 \subseteq R$

If $(a,b) \in R$ and $(b,c) \in R$, then by definition of composition $(a,c) \in R^2$

Because $R^2 \subseteq R$, this means $(a,c) \in R$. Hence R is transitive

Assume $R^n \subseteq R$ where n is a positive integer. This is inductive hypothesis

Assume $(a,b) \in R^{n+1}$

Because $R^{n+1} = R^n \cdot R$, there is an element x with $x \in A$ such that $(a,x) \in R$ and $(x,b) \in R^n$

Since R is transitive, $(a,x) \in R$ and $(x,b) \in R$ it follows that $(a,b) \in R$.

$\therefore R^{n+1} \subseteq R$

■ Inverse relation

$R = \{(a,b) \mid a < b\}$

$R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a < b\} = \{(b,a) \mid b > a\}$

$\bar{R} = \{(a,b) \mid (a,b) \notin R\} = \{(a,b) \mid a \nless b\} = \{(a,b) \mid a \geq b\}$

Q $R = \{(a,b) \mid a \text{ divides } b\}$ be the relation on the set of positive integers

then $R^{-1} = \{(b,a) \mid (a,b) \in R\} = \{(b,a) \mid a \text{ divides } b\}$

$= \{(b,a) \mid b \text{ is divisible by } a\}$

$\bar{R} = \{(a,b) \mid (a,b) \notin R\}$

$= \{(a,b) \mid a \text{ doesn't divides } b\}$

Q Let R and S be the relations with $R \subseteq S$, then $R^{-1} \subseteq S^{-1}$

let R and S be two relations, then

$(R \cap S)^{-1} = R^{-1} \cap S^{-1}$,

$(R \cup S)^{-1} = R^{-1} \cup S^{-1}$

$(S \cdot R)^{-1} = R^{-1} \cdot S^{-1}$

- On a set of n elements,

  No. of reflexive relations $= 2^{n(n-1)}$

  No. of symmetric relations $= 2^{\frac{n(n+1)}{2}}$

  No. of reflexive and symmetric relations $= 2^{\frac{n(n-1)}{2}}$

Representing relations using matrices

A relation b/w finite sets can be represented using a zero-one matrix.
Suppose R is a relation from $A = \{a_1, a_2 \ldots a_m\}$ to $B = \{b_1, b_2 \ldots b_n\}$
The relation R can be represented by

matrix $M_R = [m_{ij}]$ where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

$1 \longrightarrow$ when a_i is related to b_j

$0 \longrightarrow$ when a_i is not related to b_j

Q $A = \{1, 2, 3\}$, $B = \{1, 2\}$. Let R be the relation from A to B containing (a,b) if $a > b$. What is matrix representing R?

$R = \{(2,1)(3,1)(3,2)\}$

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- $$b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

■ Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ zero one matrices. Then the join of A and B is the ~~join~~ zero one matrix with (i,j)th entry $a_{ij} \vee b_{ij}$. The join of A and B is given by $A \vee B$. The meet of A and B is the zero-one matrix with (i,j)th entry $a_{ij} \wedge b_{ij}$. The meet of A and B is denoted by $A \wedge B$.

Q Find the join and meet of the zero one matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1\vee 1 & 0 \vee 0 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 1\wedge 1 & 0 \wedge 0 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

■ Let $A = [a_{ij}]$ be an $m \times k$ zero one matrix and $B = [b_{ij}]$ be a $k \times n$ zero one matrix. Then the boolean product of A and B denoted by $A \odot B$ is the ~~matrix~~ $m \times n$ matrix with (i,j)th entry c_{ij} where

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{ik} \wedge b_{kj})$$

Q Find the boolean product of A and B where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} (1\wedge 1)\vee(1\wedge 1) & (1\wedge 0)\vee(1\wedge 1) & (1\wedge 1)\vee(1\wedge 0) \\ (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) & (0\wedge 1)\vee(1\wedge 0) \\ (0\wedge 1)\vee(0\wedge 1) & (0\wedge 0)\vee(0\wedge 1) & (0\wedge 1)\vee(0\wedge 0) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 1 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 0 \vee 1 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 0 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Let R be a relation on set A. Then $(a,b) \in R^n$ iff there is a path of length n from a to b, where n is a positive integer.

There is a path from a to b of length one iff $(a,b) \in R$, so the theorem is true when $n=1$

Assume that the theorem is true for positive integer n

This is inductive hypothesis

There is a path of length $n+1$ from a to b iff there is an element $c \in A$ such that there is a path of length one from a to c so $(a,c) \in R$ and a path of length n from c to b that is $(c,b) \in R^n$

Consequently by inductive hypothesis there is a path of length $n+1$ from a to b iff there is an element c with $(a,c) \in R$ and $(c,b) \in R^n$

But there is such an element iff $(a,b) \in R^{n+1}$. Therefore, there is a path of length $n+1$ from a to b iff $(a,b) \in R^{n+1}$

- Reflexive Closure

Let R be a relation on a set A and S is a reflexive closure of R. Then S is a reflexive relation containing R and if T is a reflexive relation containing R then $S \subseteq T$. The reflexive closure of R can be formed by adding all pairs of the form (a,a) with $a \in A$ to R

Reflexive closure of R equals $R \cup \Delta$ where $\Delta = \{(a,a) \mid a \in A\}$

Q. Reflexive closure of $R = \{(a,b) \mid a < b\}$ on a set of integers.

$R \cup \Delta = \{(a,b) \mid a < b\} \cup \{(a,a) \mid a \in \mathbb{Z}\}$

$= \{(a,b) \mid a \leq b\}$

Symmetric closure

Let R be a relation on a set A and S is symmetric closure of R. Then S is a symmetric relation containing R and if T is a symmetric relation containing R, then $S \subseteq T$. The symmetric closure of a relation can be constructed by taking the union of a relation with its inverse that is $R \cup R^{-1}$ is a symmetric closure of R where R^{-1}

$R^{-1} = \{(b,a) \mid (a,b) \in R\}$

Q Symmetric closure of $R = \{(a,b) \mid a > b\}$ on set of positive integers.

$R \cup R^{-1} = \{(a,b) \mid a > b\} \cup \{(b,a) \mid a > b\}$

$= \{(a,b) \mid a \neq b\}$

Transitive closure

The transitive closure of the relation R is the smallest relation R_t such that $R \subset R_t$ and R_t is transitive on the set A with n elements. The relation R_t is obtained by simply including all pairs that belong to the relations R, $R^2 = R \cdot R$ and

$R^n = R^{n-1} \cdot R$

In other words, the transitive closure of R is $R \cup R^2 \cup \ldots R^n$

Q Consider the relation $R = \{(1,2)(2,3)(3,3)\}$ on the set $A = \{1,2,3\}$. Determine the reflexive, symmetric and transitive closures of the relation R

~~$R_R = R \cup \{(1,1)(2,2)(3,3)\} = \{(1,2)(2,1)(2,3)(3,2)(3,3)\}$~~

~~$R_S = R \cup \{(2,1)(3,2)\}$~~

$R_r = R \cup \{(1,1)(2,2)(3,3)\} = \{(1,1)(1,2)(2,2)(2,3)(3,3)\}$

$R_s = R \cup \{(2,1)(3,2)\} = \{(1,2)(2,1)(2,3)(3,2)(3,3)\}$

$R_t = R \cup R^2 \cup R^3$ $[\because n = 3]$

$R^2 = R \cdot R = \{(1,3)(2,3)(3,3)\}$ $\longrightarrow$ $R_t = R \cup R^2 \cup R^3$

$R^3 = R^2 \cdot R = \{(1,3)(2,3)(3,3)\}$ $= \{(1,2)(1,3)(2,3)(3,3)\}$

Warshalls Algorithm

let $W_K = [W_{ij}^{[K]}]$ be the zero one matrix that has 1 in its (i,j) th position iff there is a path from v_i to v_j with interior vertices from the set $\{v_1, v_2, \dots, v_K\}$. Then

$$W_{ij}^{[K]} = W_{ij}^{[K-1]} \vee (W_{iK}^{[K-1]} \wedge W_{Kj}^{[K-1]})$$

wherever i, j, K are positive integers not exceeding n

Warshalls algorithm computes M_{R^*} by efficiently counting $W_0 = M_R$, $W_1, W_2 \dots W_n = M_{R^*}$.

Q By using Warshalls Algorithm, find the transitive closure of the relation $R = \{(2,1)(2,3)(3,1)(3,4)(4,1)(4,3)\}$ on the set $A = \{1,2,3,4\}$

$$R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}$$

Since there are 4 elements in set A. Therefore 4 steps are required in order to find the transitive closure of relation R

Step 1 : We will consider 1st column and 1st row of above matrix i.e. C_1 and R_1

Write all positions where 1 is present in column 1

$C_1 = \{2, 3, 4\}$

Also write all position where 1 is present in row 1

$R_1 = \phi$

Now take cross product of C_1 and R_1 $\Rightarrow C_1 \times R_1 = \phi$

$\therefore$ No new additions.

Step 2 : We will consider 2nd row and 2nd column of the above matrix

$C_2 = \phi$ $\quad R_2 = \{1, 3\}$

$C_2 \times R_2 = \phi$

$\therefore$ No new additions

Step 3 : We will consider 3rd row and 3rd column

$C_3 = \{2, 4\}$ $\qquad$ $R_3 = \{1, 4\}$

$C_3 \times R_3 = \{(2,1)(2,4)(4,1)(4,4)\}$

$\therefore$ New additions

New matrix :

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}$$

Step 4 : We will consider 4th row and 4th column

$C_4 = \{1, 3, 4\}$ $\qquad$ $R_4 = \{2, 3, 4\}$

$C_4 \times R_4 = \{(2,1)(2,3)(2,4)(3,1)(3,3)(3,4)(4,1)(4,3)(4,4)\}$

$\therefore$ New additions

New matrix :

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array} = R_t^+$$

$R_t^+ = \{(2,1)(2,3)(2,4)(3,1)(3,3)(3,4)(4,1)(4,3)(4,4)\}$

R_t^+ is the transitive closure of relation R