4 Continuous random variable and probability distribution,

4.1 Probability density function

A random variable X is said to be continuous if :-

1. The possible values of X comprise a single ~~val~~ interval on a number line or union of disjoint intervals.
2. $P(X=c)=0$ for 'c' is a possible value of x

$P(1 \leq X \leq 2)$

$P(X \geq 1)$

3. The continuous random variable takes the values in an interval

4. The probability distribution function or density function in an interval [a,b] is a function $f(x)$ such that for any two values 'a' and 'b'

$$\left.\begin{array}{c} P(a<X<b) \\ \text{or} \\ P(a<x\leq b) \\ \text{or} \\ P(a\leq x<b) \\ \text{or} \\ P(a\leq x\leq b) \end{array}\right\} = \int_a^b f(x)\,dx$$

Area under the curve from a to b.

probability distribution function.

Here, $f(x)$ is a legitimate pdf if

1) $f(x) \geq 0 \quad \forall x$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

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Uniform distribution

A continuous random variable 'x' is said to have uniform distribution in [A, B] if the pdf is given by

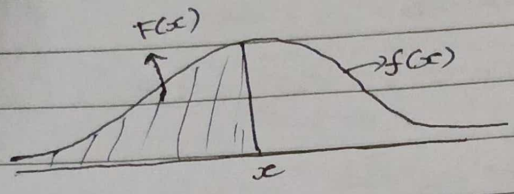
$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & \text{if } A \le X \le B \\ 0, & \text{otherwise} \end{cases}$$

Exc: 4.2

Theory start

Cumulative distribution function (cdf) (for continuous random variable)) 'X' is denoted as F(x) and is defined as $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

$f(x) \rightarrow$ pdf (Probability density function)



Note:-

1. If 'x' is a continuous random variable with pdf $f(x)$ and cdf $(F(x))$ then

i) For any number 'a', $p(x>a) = 1 - P(x \leq a)$
$= 1 - F(a)$

$P(x > 5) = 1 - P(X \leq 5)$

ii) For any two numbers a and b with $a < b$,

$P(a < x < b) = F(b) - F(a)$

$P(a \leq x < b) = F(b) - F(a)$

$P(a \leq x \leq b) = F(b) - F(a)$

$P(a < x \leq b) = F(b) - F(a)$

iii) $F'(x) = f(x)$ $\forall x$, $\rightarrow$ pdf is equal to the derivative of cdf where the derivatives of F(x) exist

eg: pdf of uniform distribution is given by

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A < x < B \\ 0, & \text{otherwise} \end{cases}$$

Find the cdf:-

x is a no. lying between A and B

Ans: We know, $F'(x) = f(x)$

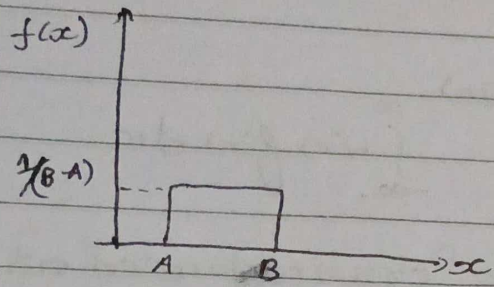
$$\Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{A} f(x)\,dx + \int_{A}^{x} f(x)\,dx$$

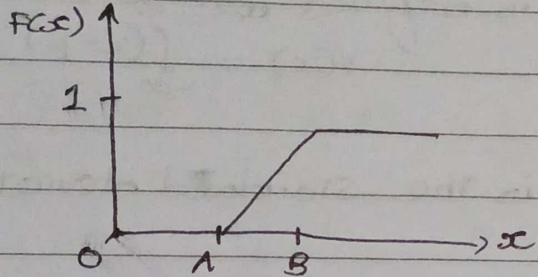
$$= 0 + \frac{1}{B-A}\int_{A}^{x} dx$$

$A \leq x \leq B$ or $x \geq A$

$$= \frac{x-A}{B-A}$$

$$\Rightarrow F(x) = \begin{cases} 0, & \text{for } x \le A \\ \frac{x-A}{B-A}, & A < x \le B \\ 1, & x \geq B \end{cases}$$



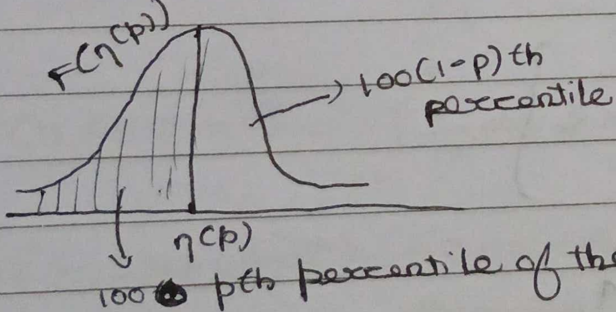


$100 \times p = F(\eta(p))$

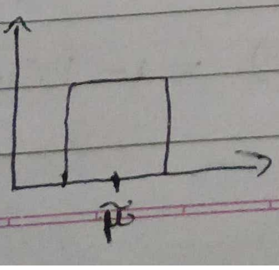
Note

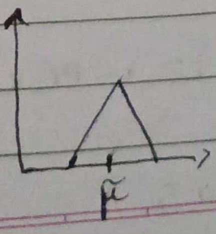
2) If 'p' is a number between 0 and 1, then the 100 pth percentile of a distribution for a continuous random variable is denoted as $F(\eta(p))$ and is defined as

$$p = F\{\eta(p)\} = \int_{-\infty}^{\eta(p)} f(y)\,dy$$



3) The median of a continuous distribution is denoted by $\tilde{\mu}$, is the 50th percentile i.e, $\hat{f}(\tilde{\mu}) = 0.5$





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Note

4) The mean value / expected value of a continuous distribution for a crv 'x' is denoted as $E(X)$ or μ_x or μ and is defined as $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$\sum_{x \in D} x\, p(x)$

sinx

For any function $h(x)$

$$E(h(x)) = \int_{-\infty}^{\infty} h(x)\, f(x)\, dx$$

5) The variance for the crv is denoted as σ_x^2 or $V(x)$ or var X or σ^2 and is defined as

$$V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

6) σ_x or σ $= \sqrt{V(x)}$ is the standard deviation.

7) i) E_{ax+b} or $\mu_{ax+b} = E(ax+b) = aE(x)+b$

ii) $V(ax+b) = a^2 V(x)$

iii) $\sigma_{ax+b} = |a|\sigma_x$

iv) $V(x) = E(X^2) - \{E(X)\}^2$

A.3 <u>Normal distribution</u>

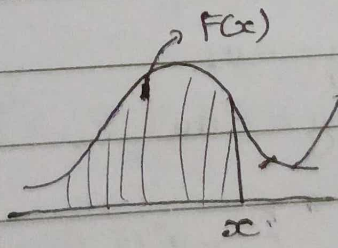
A crv 'x' is said to have normal distribution with mean 'μ' ~~and standard deviation σ~~ $(-\infty < \mu < \infty)$ and standard deviation "σ" $(\sigma > 0)$ if the pdf of X is

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

A random variable which has normal distribution is usually denoted as $X \sim N(\mu, \sigma)$

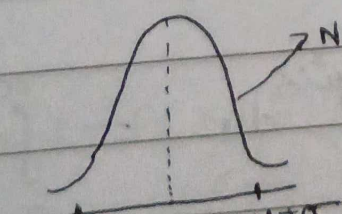
The cdf of 'X' is given by

$$P(X \le x) = F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{\left(-1/2\left(\frac{x-\mu}{\sigma}\right)^2\right)} dx$$



$$\frac{x-\mu}{\sigma} = z$$

$$\Rightarrow \frac{dx}{\sigma} = dz$$



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Standard normal distribution

A normal distribution with mean 'μ (= 0)' & "σ (= 1)" is called the standard normal distribution.

The random variable which has standard normal distribution is called as standard normal random variable on is called as standard normal random variable.

It is usually denoted as 'Z'

random variable (capital letter)

The pdf of Z is given by

$$f(z; 0, 1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

small z

The cdf is given by $P(Z \le z) = \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$.

Let, $\frac{x-\mu}{\sigma} = z$

$\Rightarrow \frac{1}{\sigma} \times dx = dz$

Here, pdf is obtained from cdf

Note

i) $\phi(-z) = 1 - \phi(z)$ Proof will be asked.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy \quad \text{— (i)}$$

$$\Rightarrow \phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-y^2/2} dy$$

Let, $y = -u$

$dy = -du$

u	∞	z
y	$-\infty$	$-z$

$$\Rightarrow \phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{z} e^{-u^2/2} (-du)$$

$$\Rightarrow \phi(-z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\mu^2/2} du = \int_{z}^{\infty} f(u) du \quad \text{— (ii)}$$

No ... diff... betw... and

where $f(u) = \frac{1}{\sqrt{2\pi}} \times e^{-u^2/2}$

we also know, $\int_{-\infty}^{\infty} f(u)\,du = 1$

$$\Rightarrow \int_{-\infty}^{z} f(u)\,du + \int_{z}^{\infty} f(u)\,du = 1$$

$$\Rightarrow \int_{z}^{\infty} f(u)\,du = 1 - \int_{-\infty}^{z} f(u)\,du$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-u^2/2}\,du = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\,du$$

By (i) and (ii)

$$\phi(-z) = 1 - \phi(z)$$

<u>Non-standard normal distributions</u>

If 'X' has a normal distribution with mean (μ) and standard deviation (σ) then $Z = \left(\frac{x-\mu}{\sigma}\right)$ has standard normal distribution. Thus,

1. $P(a \le x \le b) = P\left(\frac{a-\mu}{\sigma} \le \frac{x-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right)$

$$= P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

2. $P(X \le a) = P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right)$

$$= P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$P(X \le x) = F(x)$

$$= \phi\left(\frac{a-\mu}{\sigma}\right)$$

no difference between $<, \le$ and $>, \ge$ when crv

3. $P(X \ge b) = 1 - P(X \le b)$

$$= 1 - P\left(Z \le \frac{b-\mu}{\sigma}\right)$$

$$= 1 - \phi\left(\frac{b-\mu}{\sigma}\right)$$

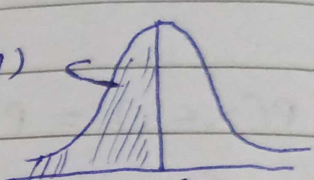
Then the 100 pth perce...

Continuing classwork

4.3

Percentiles of the standard normal distribution :-

For any p lying between 0 and 1, the 100 pth percentile of the standard normal distribution can be obtained by using appendix table A3 or calculator.



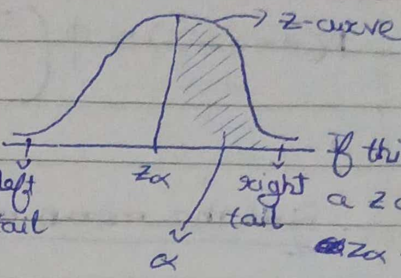
$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\,dy$$

$$p = \phi(\eta(p))$$

$$\phi(z) = P(Z \le z)$$

$$\therefore P(Z \le \eta(p))$$

Z_α notation for z critical values



If this is a z axis, then Z_α is a value within this

Z_α denotes a value on the z-axis for which α of the area under the z-curve lies to the right of Z_α

i.e, $P(Z \geq Z_\alpha) = \alpha$

$P(Z \geq Z_\alpha) = \alpha$

Percentiles of arbitrary normal distribution

(100 pth percentile of a normal distribution with mean 'μ' & standard deviation 'σ') = μ + (100p^{th} percentile of the standard normal distribution) σ

we apply the normal distribution in place of binomial distribution if

Approximation of binomial distribution

Consider a binomial experiment of n trials with probability of success = p

Let 'X' be the random variable with $\mu = np$ and $\sigma = \sqrt{npq}$

If $np \geq 10$ and $nq \geq 10$ then

$$P(X \leq x) = \phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

$$\Rightarrow B(x; n, p) = \phi\left(\frac{x + 0.5 - np}{\sqrt{npq}}\right)$$

$$= \phi\left(\frac{x + 0.5 - \mu}{\sigma}\right)$$

If fraction (a^-) If integer ($a-1$)

$$P(a \leq x \leq b) = \phi\left(\frac{b + 0.5 - \mu}{\sigma}\right) - \phi\left(\frac{a^- + 0.5 - \mu}{\sigma}\right)$$

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$$P(4 \le X \le 5) = \phi\left(\frac{5.5-\mu}{\sigma}\right) - \phi\left(\frac{3.5-\mu}{\sigma}\right)$$ → Here, a-1 because a is a integer.

Find the mean and the variance for the normal distribution

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad \to f(x;\mu,\sigma)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let, $\frac{x-\mu}{\sigma} = z$

$\Rightarrow x = \sigma z + \mu$

$\Rightarrow dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-z^2/2} dz$$

x	$-\infty$	∞
z	$-\infty$	∞

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z e^{-z^2/2} dz \quad \to \text{odd function}$$

$$+ \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz \quad \to \text{even function.}$$

$$= 0 + \frac{2\mu}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$$ → How to do this?

$\cancel{\frac{2\mu}{\sqrt{\pi}} \times e^{-z^2/2}}$

$$= \mu\sqrt{\frac{2}{\pi}} \times \sqrt{\frac{\pi}{2}}$$ (learn gamma function)

$$= \mu$$

$$V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x;\mu,\sigma)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let, $\frac{x-\mu}{\sigma} = z$

$\Rightarrow x = \sigma z + \mu$

$\Rightarrow dx = \sigma dz$

$$\Rightarrow \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-1/2 \times z^2} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-1/2\, z^2} dz \quad \to \text{even function}$$

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$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty z^2 e^{-z^2/2} dz$$

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Inverse

$\int (uv) dx = u \int v dx - \int \left(\frac{du}{dx} \int v dx \right) dx$
I II

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \times \Big[$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty z^2 e^{-z^2/2} dz$$

Let, $\frac{z^2}{2} = t$

$\Rightarrow z^2 = 2t$

$\therefore z = \sqrt{2t}$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty \sqrt{2t}\, e^{-t} dt$$

t	0	∞
z	0	0

$\frac{2z}{2} dz = dt$

$\therefore z = \sqrt{2t}$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty \sqrt{t}\, e^{-t} dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{3/2 - 1} dt$$

$\Gamma(n+1) = n\Gamma(n)$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

$$= \sigma^2 \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) = \sigma^2 \times \frac{\sqrt{\pi}}{\sqrt{\pi}} = \sigma^2$$

So, $V(x) = \sigma^2$.

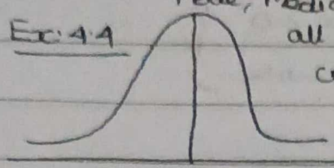
Ex: 4.3

(All are calculated from

In case of normal distribution, curve is bell shaped

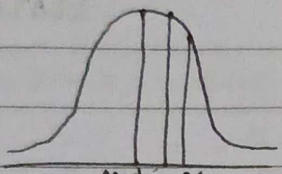
~~Ex 4.14~~ Ex 4.4

Mean, Mode, Median



all coincides with each other.

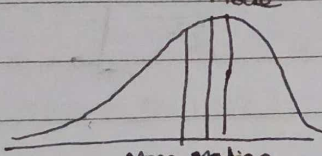
Symmetric distribution



Mode | Mean
Median

(+ve skew) Mean and median is right of the mode.
(Right skewed distribution)

Mode



Mean Median

-ve skew (Mean and median is Left of the mode)
(Left skewed distribution)

The density curve corresponding to normal distribution is bell shaped and hence symmetric. Sometimes our variable of interest may correspond to the skewed distributions like exponential distribution, gamma distribution, etc.

Exponential distribution:-

A continuous random variable X is said to have an exponential distribution with parameter 'λ' if the pdf of X is given by

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \ \cancel{\text{is greater than}} \\ 0, & \text{otherwise} \end{cases}$$

cdf of x is denoted as $= F(x;\lambda) = \int_0^x \lambda e^{-\lambda y}\,dy$

$$= \lambda \int_0^x e^{-\lambda y}\,dy$$

$$= \frac{\lambda}{-\lambda}\left[e^{-\lambda y}\right]_0^x$$

$$= -1\left[e^{-\lambda x} - 1\right]$$

$$= 1 - e^{-\lambda x}, \quad x \geq 0$$

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$$F(x;\lambda) = \begin{cases} 1-e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E(x) = \int_{-\infty}^{\infty} x \overbrace{f(x)}^{\to f(x;\lambda)} dx$$

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$$= \int_{0}^{\infty} x \lambda e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x} dx$$

$$= \lambda \left[x \int_{0}^{\infty} e^{-\lambda x} dx - \int_{0}^{\infty} \left(\int e^{-\lambda x} dx \right) dx \right]$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \lambda \times \frac{1}{\lambda^2} = \frac{1}{\lambda}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} dx$$

$$= \lambda \int_{0}^{\infty} x^2 e^{-\lambda x} dx$$

$$= \lambda \left[x^2 \frac{e^{-\lambda x}}{-\lambda} - 2x \frac{e^{-\lambda x}}{\lambda^2} - 2 \frac{e^{-\lambda x}}{\lambda^3} \right]_0^{\infty}$$

$$= \lambda \left[\frac{2}{\lambda^3} \right] = \frac{2}{\lambda^2}$$

$$V(x) = E(x^2) - \{E(x)\}^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

$$\sigma_x = \frac{1}{\lambda} = \sqrt{V(x)}$$