Date - 11/08/23

Probability

i) The set of all possible outcome of an experiment is the Sample Space and to be denoted by S, Ω

ii) Any element of the sample space set is to be a sample.

iii) Any subject of the sample set be an event.

Examples / Experiment - 1

- Tossing a coin — Once, $S = \{H, T\}$, $|S| = 2$
- " " " — Twice, $S = \{HH, HT, TH, TT\}$, $|S| = 2^2 = 4$
- " " " — Three times, $S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$, $|S| = 2^3 = 8$

In general, for n times $\longrightarrow |S| = 2^n$

Experiment - 2

- Throwing a dice — once, $S = \{1, 2, 3, 4, 5, 6\}$
  $|S| = 6$

  " " " twice, $S = \{11, 12, 13, 14, 15, 16, 21, 22, \ldots, 64, 65, 66\}$
  $|S| = 6^2 = 36$

  " " " — n times $|S| = 6^n$

* A pack of card contains — Total 52 cards....
(of 4 diff suits)

Heart, Diamond → Red.

Club, Spade → Black.

Each suit contains 13 cards. (Heart, Diamond, Club, Spade)

There are,

4 Aces
4 Jacks
4 Kings
4 Queens

Total 12 face cards.

Combination :

* If $n \in \{0, 1, 2, 3, 4, \ldots\}$

then $n! = n(n-1)(n-2)(n-3) \ldots 1$

* $((-ve)\text{ integer})! = \infty$

$(-5)! = (-5)(-6)(-7) \ldots = \infty$

* In particular $0! = 1$

* If there are 'n' no of different things and we want to select 'r' no. of things at a time. It can be done $c(n, r)$ ways where $\underline{r \leq n}$.

$$c(n, r) = \frac{n!}{(n-r)!\, r!}$$

* In particular,

$$c(n, 0) = 1 = c(n, n)$$
$$c(n, 1) = n = c(n, n-1)$$
$$c(n, 2) = \frac{n(n-1)}{2!} = c(n, n-2)$$
$$c(n, 3) = \frac{n(n-2)(n-1)}{3!} = c(n, n-3)$$

Experiment - 3

Drawing 2 cards from a shuffled cards.

$$|S| = C(52, 2)$$

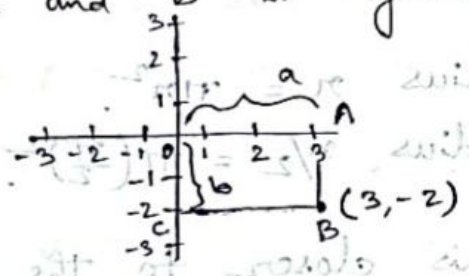
Drawing 3 cards, $|S| = C(52, 3)$

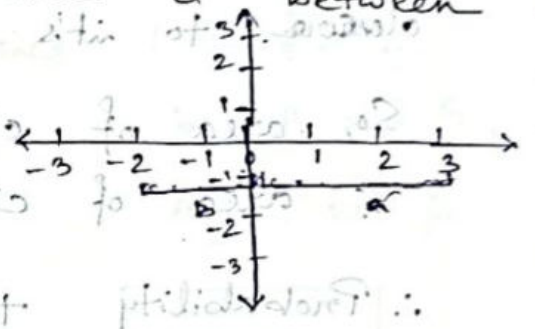
⋮

Drawing 52 cards, $|S| = C(52, 52)$

Q On the real line $\mathbb{R}$, points a and b are selected at random such that $-2 \leq b \leq 0$ and $0 \leq a \leq 3$. Find the probability that the distance 'd' between a and b is greater than 3.

Ans





Exp → Any point within the rectangle OABC

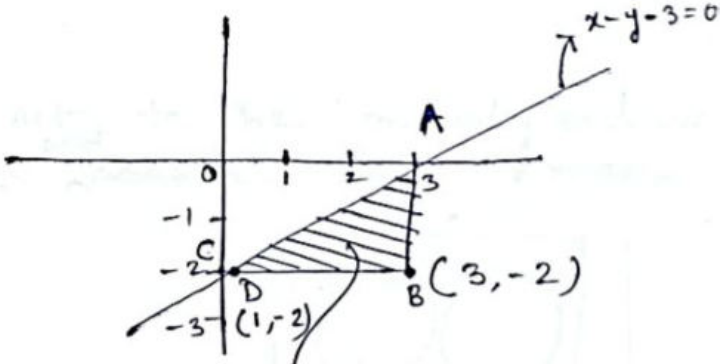
$$|S| = \text{Area of } OABC = 2 \times 3 = 6.$$

$x - y = 3$

or, $x - y - 3 = 0$

x- intersect $x = 3$

y-intersect $y = -3$



$x - y > 3$ region

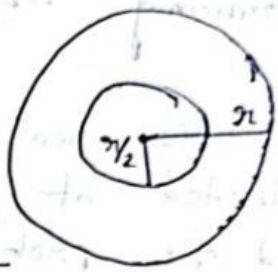
$$|A| = \text{Area of } \Delta ABD$$
$$= \frac{1}{2} \times 2 \times 2 = 2$$

∴ The probability will be $= \frac{2}{6}$.

Q A point is selected at random inside a circle. Find the probability that the point is closure to the center of the circle than it's circumference.

Ans let the radius of the circle be r and inside of that circle, let's consider another circle with radius $r/2$.

When you choose a point inside the inner circle the point will be closure to it's centre but when you choose a point outside of the inner circle, the point will be closure to it's circumference.



So, area of circle with radius $r = \pi r^2$.

area of circle with radius $r/2 = \pi\left(\frac{r}{2}\right)^2$.

$\therefore$ Probability that a point is closer to the centre than it's circumference $= \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$.

Definition of Probability:

Let S be the sample space set of an experiment 'A' be an event, then probability of A

$$= P(A) = \frac{|A|}{|S|}$$

Let S be the sample space set of an exp.

→ ϕ be the impossible event where $|\phi| = 0$,

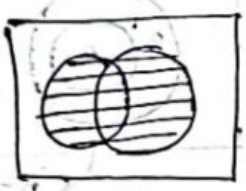
So ~~$P(\phi) = 0$~~ $P(\phi) = 0$.

→ The probability of any event lies b/w 0 and 1.

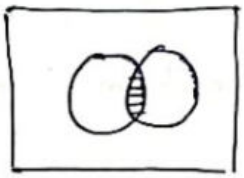
→ $A \cup B$ be the event of occurance of A ~~and~~ or B.

→ $A \cap B$ " " " " " " both A and B.

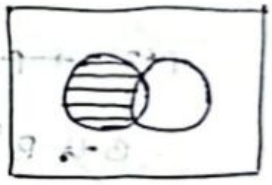
→ $A - B$ " " " " " " A but not B.



$A \cup B$



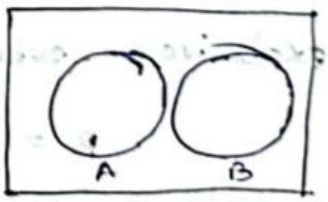
$A \cap B$



$A - B$

Mutually Exclusive Events:

Two events A and B are set to be mutually exclusive if they don't have any common point ~~section~~ that means $A \cap B = \phi$. $P(A \cap B) = 0$



$A \cap B = \phi$

Axioms of Probability:

Let S be the sample space set.

i) $0 \leq P(A) \leq 1$ [Let A be an event]

ii) $P(S) = 1$

iii) $P(A \cup B) = P(A) + P(B)$ [When A and B are mutually exclusive events]

iv) $P(A_1 \cup A_2 \cup \ldots \cup A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$ [if $A_1, A_2, \ldots, A_n$ are M.E.E

Theorems of Probability:

i) $P(A^c) = 1 - P(A)$

$S = \{1, 2, 3, 4, 5\}$

$A = \{1, 2\}$

$A^c = \{3, 4, 5\}$

Proof:

From the Ven diagram,

A and A^c are mutually exclusive events and also

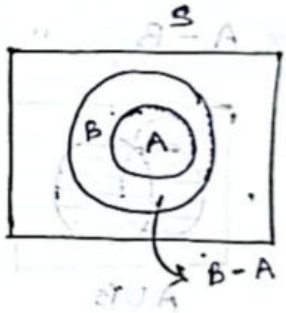
$A \cup A^c = S$

$P(A \cup A^c) = P(S)$ [From the axiom of probability]

or, $P(A) + P(A^c) = 1$

or, $P(A^c) = 1 - P(A)$

ii) If $A \subseteq B$ then $P(A) \leq P(B)$



~~$P(B) + P(B^c) = S$~~

~~$B \cup B^c = S$~~

$P(B \cup B^c) = P(S)$

or, ~~$P(B) + P(B^c) = 1 \Rightarrow P(B) = 1 - P(B^c)$~~

or, ~~$P(A) + P(A^c) = 1 \Rightarrow P(A) = 1 - P(A^c)$~~

From Ven diagram A and $(B - A)$ are mutually exclusive events.

$B = A \cup (B - A)$

By the axiom of Probability,

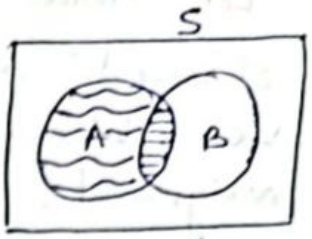
$P(B) = P(A) + P(B - A)$

Since for any event, it's probability lies in 0 and 1

So, $0 \leq P(B - A) \leq 1$

$\therefore P(B) \geq P(A)$

iii) $P(A-B) = P(A) - P(A \cap B)$



From the Ven diagram,

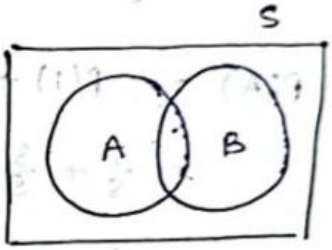
$(A-B)$ and $(A \cap B)$ are mutually exclusive events.

$$P(A) = P[(A-B) \cup (A \cap B)]$$ [By the axioms of probability]

or, $P(A) = P(A-B) + P(A \cap B)$

or, $P(A-B) = P(A) - P(A \cap B)$

iv) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



From the Ven diagram

$A \cup B = (A-B) + B$

Both $(A-B)$ and B are mutually exclusive.

$$P(A \cup B) = P(A-B) + P(B)$$ [By the axiom of probability]

or, $P(A \cup B) = P(A) - P(A \cap B) + P(B)$

or, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

v) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$\cancel{P(A) + P(B) + P(C) - P(A) - P(B) + P(A \cup B) - P(B) - P(C) + P(B \cup C) - P(C) - P(A) + P(C \cup A) + P(A \cap B \cap C)}$

$\therefore P(A \cup B \cup C) = P(A \cup (B \cup C))$

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - \{P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))\}$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Q Let three coins be tossed and the no of heads observed and the sample space set $(S) = \{0, 1, 2, 3\}$. We obtain the probability space by the following assignment. $P(0) = \frac{1}{8}$, $P(1) = \frac{3}{8}$, $P(2) = \frac{3}{8}$, $P(3) = \frac{1}{8}$.

i) Find the probability of the event that at least one head appears?

ii) Find the probability of the event that all heads and all tails appear.

Ans i) At least one head appears means there can be one head, two heads or three heads.

$A = \{1, 2, 3\}$ [Event of coming at least one head]

$$P(A) = P(1) + P(2) + P(3)$$
$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}$$
$$= \frac{7}{8}$$

or

$$= 1 - P(0)$$
$$= 1 - \frac{1}{8}$$
$$= \frac{7}{8}$$

ii) $B = \{0, 3\}$ [Event of coming all heads or all tails]

$$P(B) = P(0) + P(3)$$
$$= \frac{1}{8} + \frac{1}{8}$$
$$= \frac{2}{8}$$

Finite Probability Space :

Let S be a finite sample space set. $S = \{a_1, a_2, \dots a_n\}$

If $P(a_i) = P_i$ for $i = 1$ to n

then $P_1 + P_2 + P_3 + \dots + P_n = 1$

Infinite Sample Space :

Let S be the sample space set.

$$S = \{a_1, a_2, a_3, \dots\}$$

$$P(a_i) = P_i \geq 0$$

$$\sum_{i=1} P_i = 1$$

Q Consider the sample space $S = \{1, 2, 3, \dots\}$ of the experiment of tossing a coin till head appears [where n denotes the no of times the coin is to be tossed] Find the probability space ?

Ans $P(1) = \frac{1}{2}$ $\quad S = \{H, T\}$ $\quad P(H) = \frac{1}{2}$

$P(2) = \frac{1}{4}$ $\quad S = \{HH, HT, TH, TT\}$ $\quad P(TH) = \frac{1}{4}$

$P(3) = \frac{1}{8}$ $\quad S = \{HHH, HHT, HTH, HTT, TTH, THT, THH, TTT\}$ $\quad P(TTH) = \frac{1}{8}$

$P(\infty) = \frac{1}{2^n}$ $\quad S = \{\dots\}$ ....

$= 0$

Finite Equi Probable ~~Set~~ Space :

A finite probability space is said to be equiprobable ~~set~~ space or uniform space if all the sample points have same probability. If $|S| = n$ and P(any sample space set) $= \frac{1}{n}$. Then P(any event) where 'r' no of sample points occur. Then $P(A) = r \cdot \frac{1}{n}$!

$A = \{r$ no of sample points$\}$

Q Let a card be selected at random from a ordinary pack of cards. A be the event, {the card is a spade} B be the event, {the card is a face card}. Find the probability of A, B and $A \cap B$ if $|S| = n$ and $P(\text{any sample point}) = \frac{1}{n}$. Then the probability of the given

Ans

Ans $|S| = C(52, 1) = 52$

$P(\text{Any sample point}) = 1/52$

$P(A) = 13 \times \frac{1}{52} =$

$P(B) = 12 \times \frac{1}{52}$

$P(A \cap B) = 3 \times \frac{1}{52}$

Q Let two items ~~be~~ are choosen at random from a lot containing 12 items out of which 4 items are defective. A is the event, {both items are defective}. B is the event, {both items are non-defective}. C is the event, {at least one item is defective}. Find $P(A)$, $P(B)$, $P(C)$.

Ans. No of defective items = 4

No of non-defective items = $(12-4) = 8$

$|S| = C(12,2) = \frac{12!}{(12-2)!\,2!} = \frac{12 \times 11}{2} = 66$

$P(A) = \frac{6}{66} = \frac{C(4,2)}{C(12,2)}$ $\qquad C(4,2) = \frac{4!}{(4-2)!\,2!} = 6$

$P(B) = \frac{28}{66} = \frac{C(8,2)}{C(12,2)}$ $\qquad C(8,2) = \frac{8!}{6!\,2!} = 28$

$P(C) = \frac{24}{66} \frac{(^4C_1 \times {}^8C_1) + {}^4C_2}{^{12}C_2} = \frac{38}{66}$ $\qquad C(4,1) \times C(4,2) = 4 \times 6 = 24$

$P(C) = P(B^c)$

$= 1 - P(B)$

$= 1 - \frac{28}{66} = \frac{38}{66}$

Section - 2.1

1.

a) List of all possible outcomes.

$S = \{1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3124, 3142, \cancel{3421, 3412}, 3241, 3214, 4123, 4132, 4213, 4231\}$

b) Event A contains the outcomes where 1 wins the tournament, $A = \{1324, 1342, 1423, 1432\}$.

c) Event B contains the outcomes where 2 wins first or second place,

$B = \{2314, 2341, 2413, 2431, 3241, 3214, 4213, 4231\}$

d) $A \cup B = \{1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, 3241, 3214, 4213, 4231\}$

$A \cap B = \phi$, since 1 and 2 can't get into the championship.

$A' = S - A = \{2314, 2341, 2413, 2431, 3124, 3142, 3241, 3214, 4123, 4132, 4213, 4231\}$

2.

a) List of all outcomes in the event A that all three vehicles go in the same direction,

$A = \{RRR, LLL, SSS\}$

b) List of all outcomes in the event B that all three vehicles go in the different way,

$B = \{SRL, SLR, RSL, RLS, LRS, LSR\}$

c) List of all outcomes in the event C that all exactly two of the three vehicles turn right,

$C = \{SRR, RSR, RRS, LRR, RLR, RRL\}$ SRS, SLS, RSS, LSS}

d) $D = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL,$

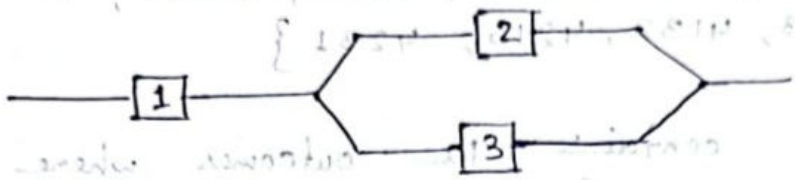
e) D' = List of outcomes where all cars either go to the same or different direction.

$= \{RRR, LLL, SSS, RLS, RSL, LSR, LRS, SRL, SLR\}$

$C \cup D = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL, SRS, SLS, RSS, LSS\}$

$C \cap D = \{RRL, RRS, RLR, RSR, LRR, SRR\}$

3.



a) List of outcomes in the event A that exactly two out of the three components function,

$A = \{SSF, SFS, FSS\}$

b) List of outcomes in the event B that at least two of the components function,

$B = \{SSS, SSF, SFS, FSS\}$

c) For event C, the system must have component 1 working (S in the first position) then at least one of the other two components must work (at least one S in the second or third position).

$C = \{SSS, SSF, SFS\}$.

d) $C' = \{SFF, FSS, FSF, FFS, FFF\}$

$A \cup C = \{SSS, SSF, SFS, FSS\}$

$A \cap C = \{SSF, SFS\}$

$B \cup C = \{SSS, SSF, SFS, FSS\}$

$B \cap C = \{SSS, SSF, SFS\}$

4.

a)

Total $2^4 = 16$ outcomes

Outcome	Home Mortgage Number 1	2	3	4
1	F	F	F	F
2	F	F	F	V
3	F	F	V	F
4	V	F	F	F
5	F	F	V	V
6	F	V	V	F
7	V	V	F	F
8	F	V	F	V
9	V	F	V	F
10	F	V	V	V
11	V	F	V	V
12	V	V	V	F
13	V	V	F	V
14	F	V	F	F
15	V	F	F	V
16	V	V	V	V

b) 2, 3, 4, 14 are the events that exactly three of the selected mortgages are fixed rates.

c) 1, 16 are in the event that all four mortgages are of the same type.

d) 1, 2, 3, 4, 14 are in the event that at most one of the four is a variable-rate mortgage.

e) Union of (c) and (d) event $\longrightarrow$ 1, 2, 3, 4, 14, 16.
Intersection of (c) and (d) event $\longrightarrow$ 1

f) Union of (b) and (c) event $\longrightarrow$ 1, 2, 3, 4, 14, 16
Intersection of (b) and (c) event $\longrightarrow$ ϕ

At most $\longrightarrow$ সর্বাধিক

At least $\longrightarrow$ কমপক্ষে

5.

a)

Outcome Number	Outcome
1	111
2	112
3	113
4	121
5	122
6	123
7	131
8	132
9	133
10	211
11	212
12	215
13	221
14	222
15	223
16	231
17	232
18	233
19	311
20	312
21	313
22	321
23	322
24	323
25	331
26	332
27	333

b) List of all outcomes in the event that all three members go to the same station $\longrightarrow$ 1, 14, 27 above.

c) List of all outcomes in the event that all members go to different stations $\longrightarrow$ 6, 8, 12, 16, 20, 22 above.

d) List of all outcomes in the event that no one goes to station 2 $\longrightarrow$ 1, 3, 7, 9, 19, 21, 25, 27 above.

6. a) List of all outcomes in S,

$S = \{123, 124, 125, 213, 214, 215, 13, 14, 15, 23, 24, 25, 3, 4, 5\}$

b) $A = \{3, 4, 5\}$

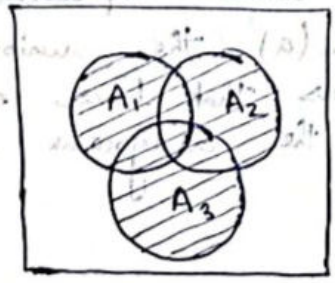
c) $B = \{125, 215, 15, 25, 5\}$

d) $C = \{23, 24, 25, 3, 4, 5\}$

7. a) S = {BBBAAAA, BBABAAA, BBAABAA, BBAAABA, BBAAAAB, BABBAAA, BABABAA, BABAABA, BABAAAB, BAABBAA, BAABABA, BAABAAB, BAAABBA, BAAABAB, BAAAABB, ABBBAAA, ABBABAA, ABBABAA, ABBAAAB, ABABAA, ABABAB, ABABAAB, ABAABBA, ABAABAB, ABAAABB, AABBBAA, AABABA, AABBABA, AABBABA, AABBAAB, AABABBA, AABABAB, AABAABB, AAABBBA, AAABBAB, AAABABB, AAAABBB}

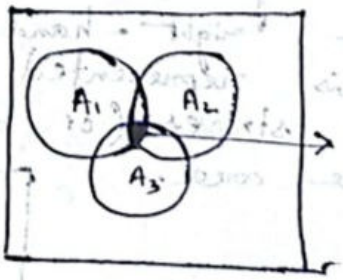
b) AAAABBB, AAABABB, AAABBAB, AABAABB, AABABAB.

8. a)



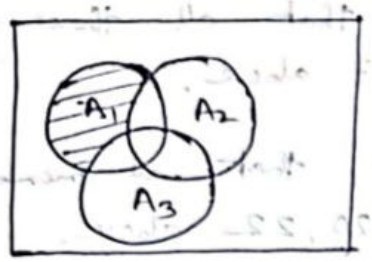
$A_1 \cup A_2 \cup A_3$

b)



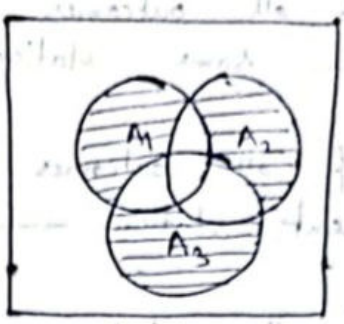
$\rightarrow A_1 \cap A_2 \cap A_3$

c)



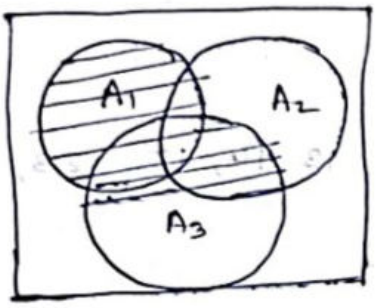
$A_1 \cap A_2' \cap A_3'$

d)



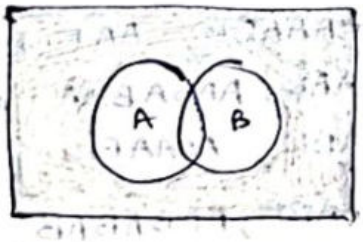
$(A_1 \cap A_2' \cap A_3') \cup (A_1' \cap A_2 \cap A_3') \cup (A_1' \cap A_2' \cap A_3)$

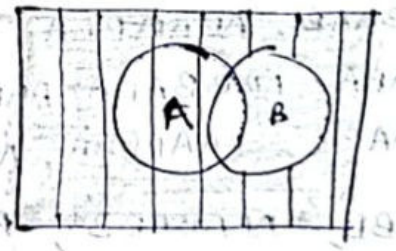
e)



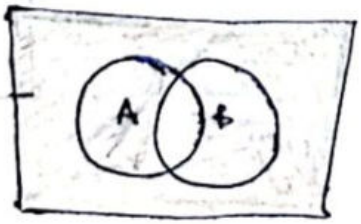
$A_1 \cup (A_2 \cap A_3)$

9. a) In the left diagram the shaded region shows the $(A \cup B)'$ area. On the right diagram, the shaded area is A' and the striped area is B'. The common region (intersection) of A' and B' is the same region as $(A \cup B)'$ where both the shaded and striped area is drawn.





b) In the diagram below, the shaded area represents $(A \cap B)'$. Using the right-hand diagram from (a), the union of A' and B' is represented by the areas that have either shading or stripes (or both). Both of the diagrams display the same area.



Section - 2.2

11. a) Probability of the selected individual owns shares in the balanced fund is $= \frac{7}{100} = 0.07$.

b) Probability of the individuals who owns shares in the bond fund $= \frac{15}{100} + \frac{10}{100} + \frac{5}{100} = \frac{30}{100} = 0.30$.

c) Let A be the event of individual who owns shares in the stock.

So, $P(A) = \frac{18}{100} + \frac{25}{100} = \frac{43}{100} = 0.43$

A' will be the event of individual who doesn't own any share in the stock.

$P(A') = 1 - 0.43 = 0.57$.

12. a) This is not possible. Since $(A \cap B)$ is contained within event B. In that case $P(A \cap B) \leq P(B)$. However $.5 > .4$

b) Probability of students having at least one of the cards $= P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.6 + 0.4 - 0.3$

$= 0.7$

c) Probability of students having neither type of card

$= P((A \cup B)') = 1 - P(A \cup B)$

$= 1 - 0.7$

$= 0.3$

d) Probability of students having a Visa card but not a Master card $= \cancel{P(A \cup B')}$

$= P(A \cap B')$

$= P(A) - P(A \cap B)$

$= 0.6 - 0.3$

$= 0.3$

e) Probability of students having exactly one of those two types of cards $= P(A \cup B) - P(A \cap B)$

$= 0.7 - 0.3$ $\quad P(\text{exactly one}) = P(\text{at least one}) - P(\text{both})$

$= 0.4$

13. $A_i = \{\text{awarded project } i\}$ $i = 1, 2, 3.$

$P(A_1) = 0.22$, $P(A_2) = 0.25$, $P(A_3) = 0.28$

$P(A_1 \cap A_2) = 0.11$ $P(A_1 \cap A_3) = 0.05$, $P(A_2 \cap A_3) = 0.07$

$P(A_1 \cap A_2 \cap A_3) = 0.01$

a) $(A_1 \cup A_2)$ = (Awarded either 1 or 2) or (Awarded both)

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$= 0.22 + 0.25 - 0.11$$
$$= 0.36$$

b) $(A_1' \cap A_2')$ = $(A_1 \cup A_2)'$ = Awarded neither 1 or 2

$$P(A_1' \cap A_2') = P((A_1 \cup A_2)') = 1 - P(A_1 \cup A_2)$$
$$= 1 - 0.36$$
$$= 0.64$$

c) $(A_1 \cup A_2 \cup A_3)$ = Awarded at least one of these three projects.

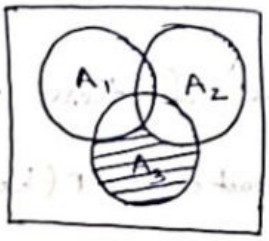
$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$
$$= 0.22 + 0.25 + 0.28 - 0.11 - 0.05 - 0.07 + 0.01$$
$$= 0.53$$

d) $(A_1' \cap A_2' \cap A_3')$ = Awarded none of these projects.

$$P(A_1' \cap A_2' \cap A_3') = 1 - P(A_1 \cup A_2 \cup A_3)$$
$$= 1 - 0.53$$
$$= 0.47$$

e) $(A_1' \cap A_2' \cap A_3)$ = Awarded 3 but neither 1 or 2

$$P(A_1' \cap A_2' \cap A_3) = P(A_3) + P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$$
$$= 0.28 + 0.01 - 0.07 - 0.05$$
$$= 0.17$$



f) $(A_1' \cap A_2') \cup A_3$ = (Awarded neither 1 or 2) or (Awarded 3)

$$P((A_1' \cap A_2') \cup A_3) = P(\text{none awarded}) + P(A_3)$$
$$= P(A_1' \cap A_2' \cap A_3') + P(A_3)$$
$$= 0.47 + 0.28$$
$$= 0.75$$

14. A = Event of consuming coffee for an adult

B = " " " soda " " "

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or, $0.70 = 0.55 + 0.45 - P(A \cap B)$

or, $P(A \cap B) = 1 - 0.70 = 0.30$

∴ Probability of consuming both coffee and soda of an adult is 0.30.

b) Two ways to solve ....

Does not consume at least one

$$P(A' \cap B') = P((A \cup B)')$$
$$= 1 - P(A \cup B)$$
$$= 1 - 0.70$$
$$= 0.30$$

There is at least one beverage an adult does not consume

$$P(A' \cup B') = P((A \cap B)')$$
$$= 1 - P(A \cap B)$$
$$= 1 - 0.30$$
$$= 0.70$$

15.

a) E = Event that at most one purchases an electric dyer.

E' = Event that at least two purchase an electric dyer.

∴ $P(E') = 1 - P(E) = 1 - 0.428 = 0.572$.

b) A = Event that all five purchase gas.

B = Event that all five purchase electric dyer.

∴ Probability of at least one of each type is purchased

$= 1 - [P(A) + P(B)] = 1 - [0.116 + 0.005] = 0.879$

16. a) All possible outcomes are,

$S = \{CDP, CPD, DCP, DPC, PCD, PDC\}$

Since, the same cola is in every glass, there six outcomes are likely to occur, and the probability assigned to each is $\frac{1}{6}$.

b) Probability of C ranking first $= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = 0.333$

c) Probability of C ranking first and D ranking last $= \frac{1}{6}$

17. a) $P(A) + P(B) \neq 1$ because there are other software packages besides SPSS and SAS for which requests could be made.

b) $P(A') = 1 - P(A) = 1 - 0.30 = 0.70$

c) Since A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B) = 0.30 + 0.50 = 0.80$$

d) $P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B) = 1 - 0.80 = 0.20$

18. There are $4 + 6 = 10$ non-₹10 bills and $4 + 6 + 5 = 15$ bills (total) in the wallet.

$\therefore$ The probability of this event $= \frac{10}{15} = \frac{2}{3}$.

19. Let A be that selected joint found defective by the inspector.

$$P(A) = \frac{724}{10,000}$$

Let B be analogous for inspector B.

$$P(B) = \frac{751}{10,000}$$

At least one of the inspection judged a joint to be defective is $(A \cup B)$.

$$P(A \cup B) = \frac{1159}{10,000}$$

a) $P(\text{neither A nor B}) = P(A' \cap B') = P((A \cup B)') = 1 - P(A \cup B)$

$$= 1 - \frac{1159}{10,000}$$

$$= \frac{8841}{10,000}$$

b) The desired event is $P(B \cap A') = P(B) - P(A \cap B)$

we know that,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or, $$\frac{1159}{10,000} = \frac{724}{10,000} + \frac{751}{10,000} - P(A \cap B)$$

or, $$P(A \cap B) = \frac{316}{10,000}$$

$$\therefore P(B \cap A') = P(B) - P(A \cap B)$$

$$= \frac{751}{10,000} - \frac{316}{10,000}$$

$$= \frac{435}{10,000}$$

20. a) let s_1, s_2, s_3 represent day, swing and night shifts respectively let c_1 and c_2 represent unsafe conditions and unrelated to conditions.

$$S = \{s_1c_1, s_1c_2, s_2c_1, s_2c_2, s_3c_1, s_3c_2\}$$

b) $P(C_1) = P(\{s_1c_1, s_2c_1, s_3c_1\}) = 0.10 + 0.08 + 0.05 = 0.23$

c) $P(S_1') = P(\{s_2c_1, s_2c_2, s_3c_1, s_3c_2\}) = 0.08 + 0.20 + 0.05 + 0.22$
$= 0.55$

21. The first letter refers to the auto deductible and the second letter refers to the homeowner's deductible.

a) $P(MH) = 0.10$

b) $P(\text{low auto deductible}) = P(\{LN, LL, LM, LH\})$
$= 0.04 + 0.06 + 0.05 + 0.03$
$= 0.18$

$P(\text{low homeowner's deductible}) = P(\{LL, LM, LH\})$
$= 0.06 + 0.10 + 0.03$
$= 0.19$

c) $P(\text{same deductible for both}) = P(\{LL, MM, HH\})$
$= 0.06 + 0.20 + 0.15 = 0.41$

d) P(deductibles are different)

$= 1 - P(\text{deductibles are same})$

$= 1 - 0.41$

$= 0.59$

e) P(at least one low deductible)

$= P(\{LN, LL, LM, LH, ML, HL\})$

$= 0.04 + 0.06 + 0.05 + 0.03 + 0.10 + 0.03$

$= 0.31$

f) P(neither deductible is low)

$= 1 - P(\text{at least one deductible is low})$

$= 1 - 0.31$

$= 0.69$

22. let A = motorist must stop at first signal.

B = motorist must stop at second signal.

$P(A) = 0.4$, $P(B) = 0.5$, $P(A \cup B) = 0.6$

a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or, $P(A \cap B) = 0.4 + 0.5 - 0.6$

$= 0.3$

b) $P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1$

c) P(stop at exactly one signal)

$= P(A \cup B) - P(A \cap B)$

$= 0.6 - 0.3$

$= 0.3$

or $= P[(A \cap B') \cup (A' \cap B)]$

$= P(A \cap B') + P(A' \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$

$= 0.4 - 0.3 + 0.5 - 0.3$

$= 0.3$

23. Assume that the computers are numbered 1-6 as described and that computers 1 and 2 are the two laptops.

There are 15 possible outcomes.

$$S = \{(1,2), (1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\}$$

a) $P(\text{both are laptops}) = P(\{(1,2)\}) = \frac{1}{15} = 0.067$

b) $P(\text{both are desktops}) = P(\{(3,4), (3,5), (3,6), (4,5), (4,6), (5,6)\})$

$= \frac{6}{15} = 0.40$

c) $P(\text{at least one desktop}) = 1 - P(\text{no desktops})$

$= 1 - P(\text{both are laptops})$

$= 1 - 0.067$

$= 0.933$

d) $P(\text{at least one of each type}) = 1 - P(\text{both are the same})$

$= 1 - [P(\text{both are laptops}) + P(\text{both are desktops})]$

$= 1 - [0.40 + 0.067]$

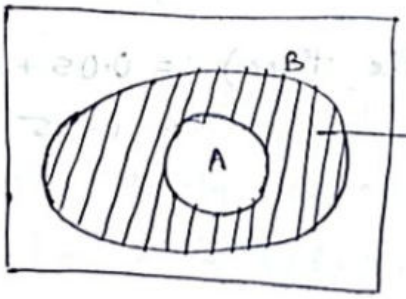
$= 0.533$

24. Since A is contained in B, we may write $B = A \cup (B \cap A')$, the union of two mutually exclusive events.

$P(B) = P(A \cup (B \cap A')) = P(A) + P(B \cap A') \longrightarrow$ Axiom 3

$P(B \cap A') \geq 0 \longrightarrow$ Axiom 1

$P(B) = P(A) + P(B \cap A') \geq P(A) + 0 = P(A)$



$\longrightarrow$ striped area $= (B \cap A')$

25. $P(A \cap B) = P(A) + P(B) - P(A \cap B)$

or, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= 0.40 + 0.55 - 0.63$

$= 0.32$

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$

or, $P(A \cap C) = P(A) + P(C) - P(A \cup C)$

$= 0.40 + 0.70 - 0.77$

$= 0.33$

$P(B \cup C) = P(B) + P(C) - P(B \cap C)$

or, $P(B \cap C) = P(B) + P(C) - P(B \cup C)$

$= 0.55 + 0.70 - 0.80$

$= 0.45$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

or, $-P(A \cap B \cap C) = 0.40 + 0.55 + 0.70 - 0.32 - 0.45 - 0.33 + 0.85$

or, $P(A \cap B \cap C) = 0.30$

a) $P(A \cup B \cup C) = 0.85$

b) $P(\text{none selected}) = 1 - P(\text{at least one selected})$

$= 1 - P(A \cup B \cup C)$

$= 1 - 0.85$

$= 0.15$

c) $P(\text{only automatic transmission selected}) = 0.22$

d) $P(\text{exactly one of the three}) = 0.05 + 0.08 + 0.22$

$= 0.35$

26. a) $P(A_1') = 1 - P(A_1) = 1 - 0.12 = 0.88$

b) $P(A \cup B) = P(A) + P(B) - P(A \cap B) \longrightarrow$ Addition rule

for A_1 and A_2,

$$P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$$
$$= 0.12 + 0.07 - 0.13$$
$$= 0.06$$

c) $P((A_1 \cap A_2) \cap A_3') = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$

$$= 0.06 - 0.01$$
$$= 0.05$$

d) P(at most two defects)

$= 1 - P$(all three defects)

$= 1 - P(A_1 \cap A_2 \cap A_3)$

$= 1 - 0.01$

$= 0.99$

27. There are 10 equally likely outcomes,

$\{A, B\}$ $\{A, Co\}$ $\{A, Cr\}$ $\{A, F\}$ $\{B, Co\}$ $\{B, Cr\}$ $\{B, F\}$
$\{Co, Cr\}$ $\{Co, F\}$, $\{Cr, F\}$

a) $P\{A, B\} = \frac{1}{10} = 0.1$

b) P(at least one C) $= P(\{A, Co\}$ or $\{A, Cr\}$ or $\{B, Co\}$, $\{B, Cr\}$ or $\{Co, Cr\}$ or $\{Co, F\}$ or $\{Cr, F\})$

$$= \frac{7}{10} = 0.7$$

c) Replacing each person with his/her years of experience,

P(at least 15 years) $= P(\{3, 14\}$ or $\{6, 10\}$ or $\{6, 14\}$ or $\{7, 10\}$ or $\{7, 14\}$ or $\{10, 14\})$

$$= \frac{6}{10} = 0.6$$

28. There are 27 equally likely outcomes.

a) P(all the same station) $= P((1,1,1)$ or $(2,2,2)$ or $(3,3,3))$

$$= \frac{3}{27} = \frac{1}{9}$$

b) P(at most 2 are assigned to the same station)

$= 1 - P$(all 3 are the same) $= 1 - \frac{1}{9} = \frac{8}{9}$

$2.4 \rightarrow 45, 47, 50, 54, 55, 56, 58, 59$

c) P(all different stations)

$= P((1,2,3) \text{ or } (1,3,2) \text{ or } (2,1,3) \text{ or } (2,3,1) \text{ or } (3,1,2) \text{ or } (3,2,1))$

$= \frac{6}{27} = \frac{2}{9}$

Q $S = \{a_1, a_2, a_3, a_4\}$

Which function defines a probability space on S?

i) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = \frac{1}{5}$

ii) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = -\frac{1}{4}$, $P(a_4) = \frac{1}{2}$

iii) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = \frac{1}{8}$, $P(a_4) = \frac{1}{8}$

iv) $P(a_1) = \frac{1}{2}$, $P(a_2) = \frac{1}{4}$, $P(a_3) = \frac{1}{4}$, $P(a_4) = 0$

Ans i) $\Sigma P_i = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \neq 1$

ii) $P(a_3) = -\frac{1}{4}$ (It can't be negative)

} This two can't be...

iii) $\Sigma P_i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 1$

iv) $\Sigma P_i = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + 0 = 1$

} Both function define a probability space on S.

Q A coin is weighted so that head is twice as likely to appear as a tail. Find probability of tail and probability of head.

Ans let,

$P(T) = P$

$P(H) = 2P(T) = 2P$

By definition,

$\Sigma P_i = 1$

or, $P(T) + P(H) = 1$

or, $P + 2P = 1$

or, $P = \frac{1}{3}$

$\therefore P(H) = \frac{2}{3}$

2.5 $\longrightarrow$ 71, 72, 78, 80, 81, 83, 87

Q $S = \{a_1, a_2, a_3, a_4\}$

i) $P(a_1) = ?$ if $P(a_2) = \frac{1}{3}$, $P(a_3) = \frac{1}{6}$, $P(a_4) = \frac{1}{9}$

Ans $1 - \left(\frac{1}{3} + \frac{1}{6} + \frac{1}{9}\right)$

$P(a_1) + P(a_2) + P(a_3) + P(a_4) = 1$

$= \frac{7}{18}$

$= P(a_1)$

Q i) $P(a_1) = ?$ if $P\{(a_2, a_3)\} = \frac{2}{3}$, $P\{(a_2, a_4)\} = \frac{1}{2}$, $P(a_2) = \frac{1}{3}$

Ans a_1, a_2, a_3, a_4 are mutually exclusive events.

$P\{(a_2, a_3)\} = \frac{2}{3}$

or, $P(a_2) + P(a_3) = \frac{2}{3}$

or, $P(a_3) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$P\{(a_2, a_4)\} = \frac{1}{2}$

or, $P(a_2) + P(a_4) = \frac{1}{2}$

or, $P(a_4) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$P(a_1) + P(a_2) + P(a_3) + P(a_4) = 1$

or, $P(a_1) = 1 - \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{6}\right) = 1 - \frac{5}{6} = \frac{1}{6}$

Q Three light bulbs are choosen at random from 15 ~~bulb~~ bulbs out of which 5 are defective. Find the probability that

i) None is defective

ii) Exactly one is defective

iii) At least one is defective

Ans ~~i) Let A be the event of choosing defective bulbs~~ = 5

~~Let B " " " " " non- " " = 10~~

~~i) $P((A \cup B)')$ iii) $P(A \cup B)$~~

$|S| = C(15, 3) = 455$

15 $\rightarrow$ 5 Defective

15 $\rightarrow$ 10 Non - defective

i) Let A be the event where all bulbs are non-defective.

$|A| = C(10, 3) = 120$ $\quad P(A) = \frac{120}{455}$

ii) exactly one is defective, (let B be the event)

$$|B| = C(5,1) \times C(10,2) = 5 \times 45 = 225$$

$$P(B) = \frac{225}{455}$$

iii) At least one is defective, (let C be the event)

$$|C| = C(5,1) \times C(10,2) + C(5,2) \times C(10,1) + C(5,3)$$

or $P(C) = P(A^c) = 1 - P(A) = 1 - \frac{120}{455} = \frac{335}{455}$.

Q $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{4}$; Find $P(A \cup B)$, $P(A^c)$, $P(B^c)$, $P(\bar{A} \cap \bar{B})$, $P(\bar{A} \cup \bar{B})$, $P(A \cap \bar{B})$, $P(B \cap \bar{A})$

Ans $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{3}{8} + \frac{1}{2} - \frac{1}{4}$$

$$= \frac{3+4-2}{8}$$

$$= \frac{5}{8}$$

$$\begin{array}{r|l} 2 & 8, 2, 4 \\ 2 & 4, 1, 2 \\ & 2, 1, 1 \end{array}$$

$$P(A^c) = 1 - P(A) = 1 - \frac{3}{8} = \frac{5}{8}$$

$$P(B^c) = 1 - P(B) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\bar{A} \cap \bar{B}) = P((A \cup B)^c) = 1 - P(A \cup B)$$

$$= 1 - \frac{5}{8}$$

$$= \frac{3}{8}$$

$$\cancel{P(\bar{A} \cup \bar{B}) = P(A) - P(A \cap B)}$$

$$\cancel{= \frac{3}{8} - \quad}$$

$$P(\bar{A} \cup \bar{B}) = P((A \cap B)^c) = 1 - P(A \cap B)$$

$$= 1 - \frac{1}{4}$$

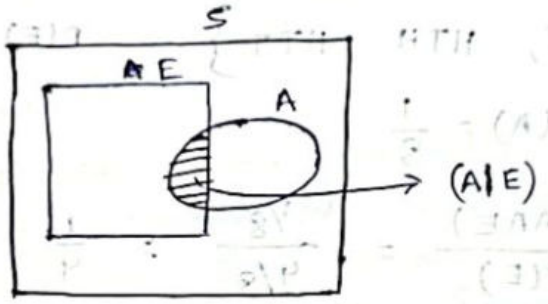
$$= \frac{3}{4}$$

$P(B \cap \bar{A}) = P(B) - P(A \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{3}{8} - \frac{1}{4} = \frac{1}{8}$

Conditional Probability

Let E be an arbitary event with sample space S and $P(E) > 0$. We want to find the probability of the event A that occures once E has occured.



So A|E be the conditional event where, ~~P(A|E)~~

$$P(A|E) = \frac{P(A \cap E)}{P(E)}$$

Q A pair of fair dice be tossed. If the sum is 6 then find the probability that one of the dice is 2 ?

Ans Let E be the arbitary event (sum is 6) and the sample space is S.

$|S| = 6^2 = 36 \longrightarrow$ Sample Space Cardinality.

$E = \{(1,5), (5,1), (2,4), (4,2), (3,3)\}$

$P(E) = \frac{5}{36}$

Let A be the event when one dice is 2.

$A = \{(1,2), (2,1), (2,2), (3,2), (2,3), (4,2), (2,4), (5,2), (2,5), (6,2), (2,6)\}$

$P(A) = \frac{11}{36}$

$\therefore P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{2/36}{5/36} = \frac{2}{5}$ Ans

Q Three fair coins are tossed. Find the probability that they are all heads if

i) The first coin is head.

ii) One of the coin is head.

Ans i) let E be the event of the first coin is head.
let A be " " " all both the coins are head.

$|S| = 2^3 = 8$

$E = \{HHH, HHT, HTH, HTT\}$ $P(E) = \frac{4}{8}$

$A = \{HHH\}$ $P(A) = \frac{1}{8}$

$\therefore P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{1/8}{4/8} = \frac{1}{4}$

ii) let E be the event where one of the coin is head.
let B be " " " all the coins are head.

$E = \{HHH, HHT, HTH, HTT, TTH, THT, THH\}$ $P(E) = \frac{7}{8}$

$B = \{HHH\}$ $P(B) = \frac{1}{8}$

$\therefore P(B|E) = \frac{1/8}{7/8} = \frac{1}{7}$.

Q A pair of fair dice are thrown. If the two numbers appearing are different then find the probability that

i) sum is 6 ii) sum is 4 or less iii) an ace (1) appears

Ans $|S| = 6^2 = 36$

~~E = {(1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (1,5), (5,1), (1,6), (6,1)}~~

~~E = {12, 21, 22~~

$E = \{$Both faces are different$\}$

$E = \{12, 13, 14, 15, 16, 21, 23, 24, 25, 26, 31, 32, 34, 35, 36, 41, 42, 43, 45, 46, 51, 52, 53, 54, 56, 61, 62, 63, 64, 65\}$

$P(E) = \frac{30}{36}$

i) let A be the event when sum is 6.

$A = \{15, 24, 33, 42, 51\}$ $P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{4/36}{30/36} = \frac{4}{30}$

ii) Let B be the event where the sum is 4 or less

$B = \{11, 12, 13, 21, 22, 31\}$

$$P(B|E) = \frac{P(B \cap E)}{P(E)} = \frac{4/36}{30/36} = \frac{4}{30}$$

iii) Let C be the event where one ace (1) appeares.

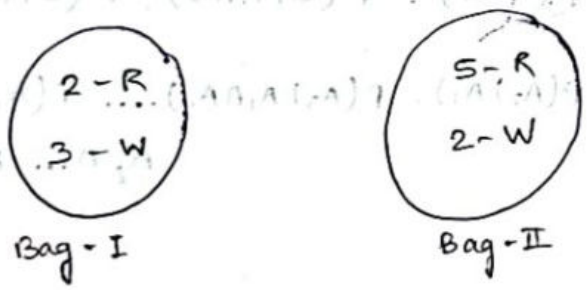
$C = \{11, 12, 13, 14, 15, 16, 21, 31, 41, 51, 61\}$

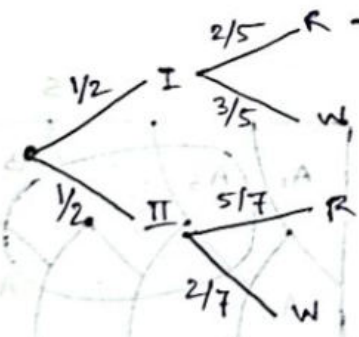
$$P(C|E) = \frac{P(C \cap E)}{P(E)} = \frac{10/36}{30/36} = \frac{10}{30}$$

Q

Stochastic Process:

An experiment occures in multiple phases. So, a finite sequence of experiments in which each experiment has a finite number of outcomes with given probabilities is said to be a. Stochastic Process (Finite).





$\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$

$\frac{1}{2} \times \frac{5}{7} = \frac{5}{14}$

'Tree Diagram'

$P(\text{Ball is red}) = \frac{1}{5} + \frac{5}{14} = \frac{39}{90}$

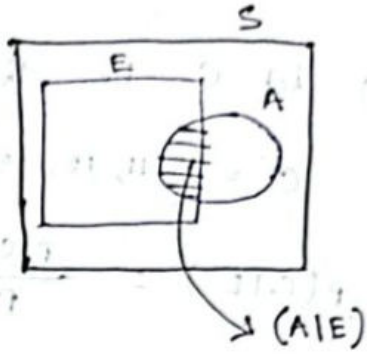
Multiplication Theorem of Conditional Probability

As we know $P(A|E) = \dfrac{P(A \cap E)}{P(E)}$

Again, $A \cap E = E \cap A$

So, $P(A|E) = \dfrac{P(E \cap A)}{P(E)}$

or, $P(E \cap A) = P(E) \cdot P(A|E)$



In general,

$$P(A \cap B) = P(A) \cdot P(B|A)$$

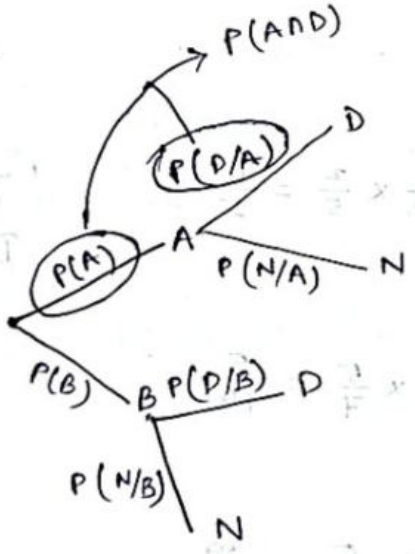
$$P(A \cap B \cap C) = P((A \cap B) \cap C)$$

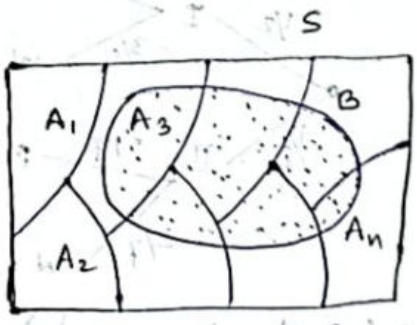
$$= P(A \cap B) \cdot P(C | A \cap B)$$

$$= P(A) \cdot P(B|A) \cdot P(C | A \cap B)$$

$$P(A \cap B \cap C \cap D) = P(A) \cdot P(B|A) \cdot P(C|A \cap B) \cdot P(D|A \cap B \cap C)$$

$$P(A_1 \cap A_2 \cap \dots A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \dots P(A_n|A_1 \cap A_2 \cap \dots A_{n-1})$$





③ B is any other arbitary event.

① Let $A_1, A_2, A_3, \dots A_n$ are the partition of the sample space set.

② Clearly $A_1, A_2, \dots A_n$ are mutually exclusive events.

$(A_1 \cup A_2 \cup \dots \cup A_n) = S$

$$B = S \cap B$$
$$= (A_1 \cup A_2 \cup \dots \cup A_n) \cap B$$
$$= (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup \dots \cup (A_n \cap B)$$

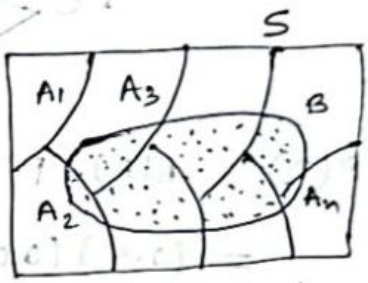
$\therefore$ The events $(A_1 \cap B)$, $(A_2 \cap B)$, ..., $(A_n \cap B)$ are also Mutually Exclusive Events.

So,
$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + \dots + P(A_n \cap B)$$
$$= P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \dots + P(A_n) \cdot P(B|A_n)$$

if A & B are M.E.E
$P(A \cup B) = P(A) + P(B)$

Baye's Theorem :

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



$$P(A_i \cap B) = P(B) \cdot P(A_i|B)$$
$$= P(A_i) \cdot P(B|A_i)$$

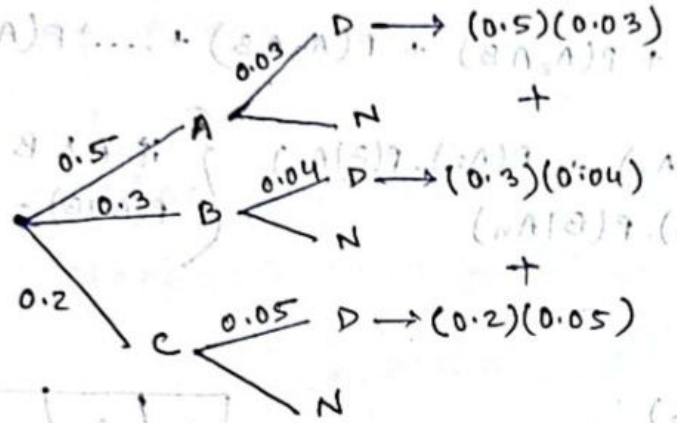
$$P(A_i \cap B) = \frac{P(A_i \cap B)}{P(B)}$$
$$= \frac{P(A_i) \cdot P(B|A_i)}{P(A_1) \cdot P(B|A_1) + \dots + P(A_n) \cdot P(B|A_n)}$$

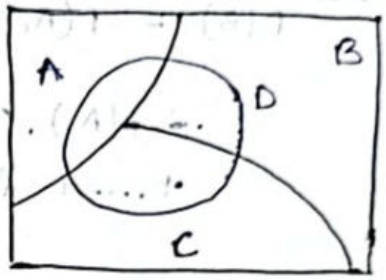
$B \rightarrow$ Defective item
$A_i \rightarrow$ Particular machine

$(A_i|B)$
$\downarrow$
"Defective item from a particular machine"

Q Three machines A, B and C produce 50%, 30% and 20% of total no of items of a factory respectively. The % of defective output of these machines are 3%, 4% and 5% respectively. If an item is selected at random, find the probability that the i) Item is defective. ii) The defective item is produced from machine A

Ans





i) $P(D) = P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) + P(D|C)$

$= (0.5)(0.03) + (0.3)(0.04) + (0.2)(0.05)$

$= 0.037 = \frac{37}{1000}$

ii) $P(A|D) = \frac{P(A) \cdot P(D|A)}{P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C)}$

$= \frac{(0.5)(0.03)}{(0.5)(0.03) + (0.3)(0.04) + (0.2)(0.05)}$

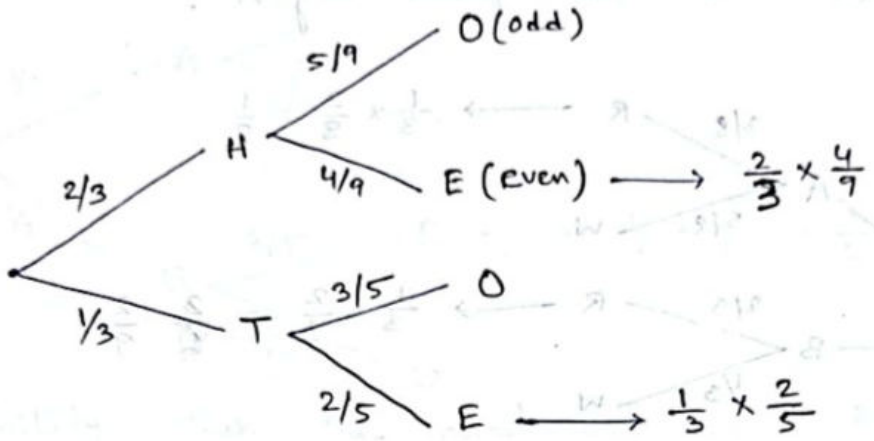
$= \frac{0.015}{0.037}$

$= 0.405$

$= \frac{15}{37}$

Q A biased coin is tossed and $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. If head appears then a no is selected at random from 1 to 9. If tail appears then a no is selected at random from 1 to 5. Find the probability that an even no is selected.

Ans



$$P = \left(\frac{2}{3} \times \frac{4}{9}\right) + \left(\frac{1}{3} \times \frac{2}{5}\right) = \frac{8}{27} + \frac{2}{15} = \frac{58}{135}$$

$$= 0.3\overline{11}$$

Q A box contains 3 coins. One is fair, one is two headed and one coin is weighted, so that $P(H) = 1/3$. A coin is selected at random and tossed. Find the probability that head appears.

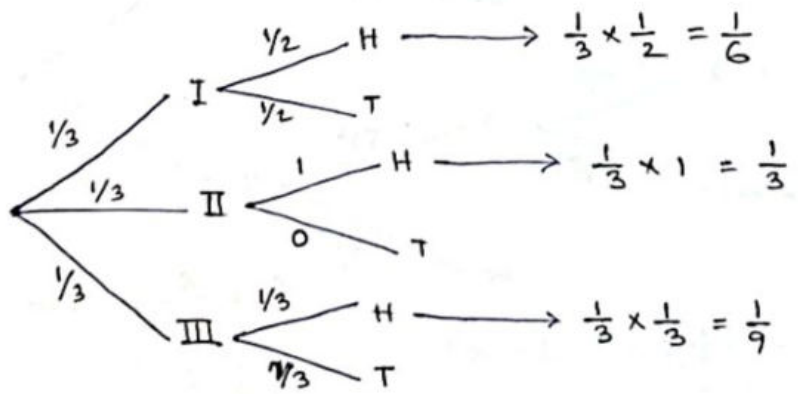
Ans

$$\left.\begin{aligned} P(H) &= \frac{1}{3} \\ P(T) &= 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}\right\} \text{weighted coin}$$

$P(\text{fair coin}) = \frac{1}{2} \longrightarrow P(H), P(T)$

$P(\text{2 headed}) = 1 \longrightarrow P(H), P(T) = 0$

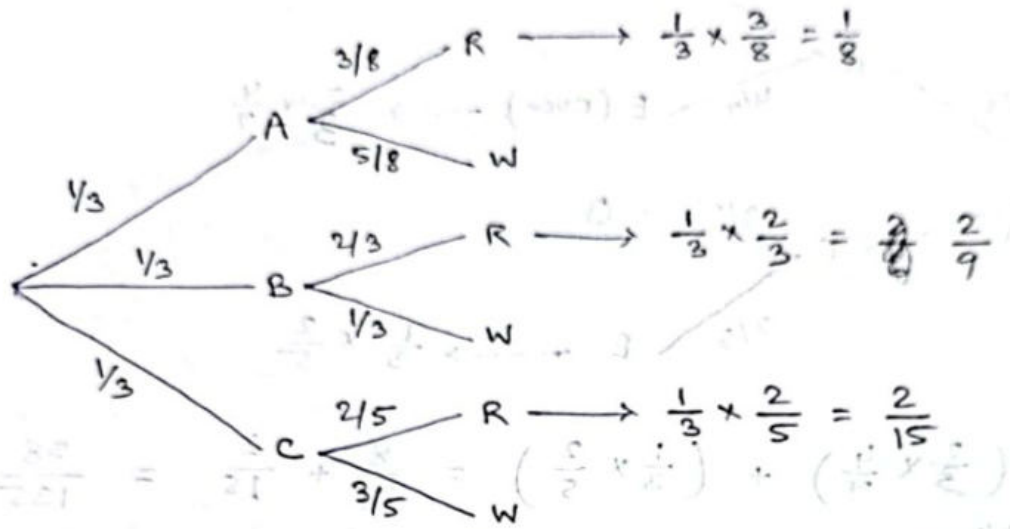
I. Fair coin
II. Two headed coin
III. Weighted coin



$\therefore$ Required probability that head appears $= \frac{1}{6} + \frac{1}{3} + \frac{1}{9} = \frac{11}{18}$

Q 3 bags are given. Bag A contains 3 red and 5 white marbles. Bag B contains 2 red and 1 white marbles and bag C contains 2 red and 3 white marbles. A bag is selected at random and a marble is drawn from the bag. If marble is red, then what is the probability that it is from bag A?

Ans



$\frac{1}{3} \times \frac{3}{8} = \frac{1}{8}$

$\frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$

$\frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

~~∴ Required probability that it (red marble) is from the bag A~~ $= \frac{1}{8} + \frac{2}{9} + \frac{2}{15}$ = Probability that the marble is red

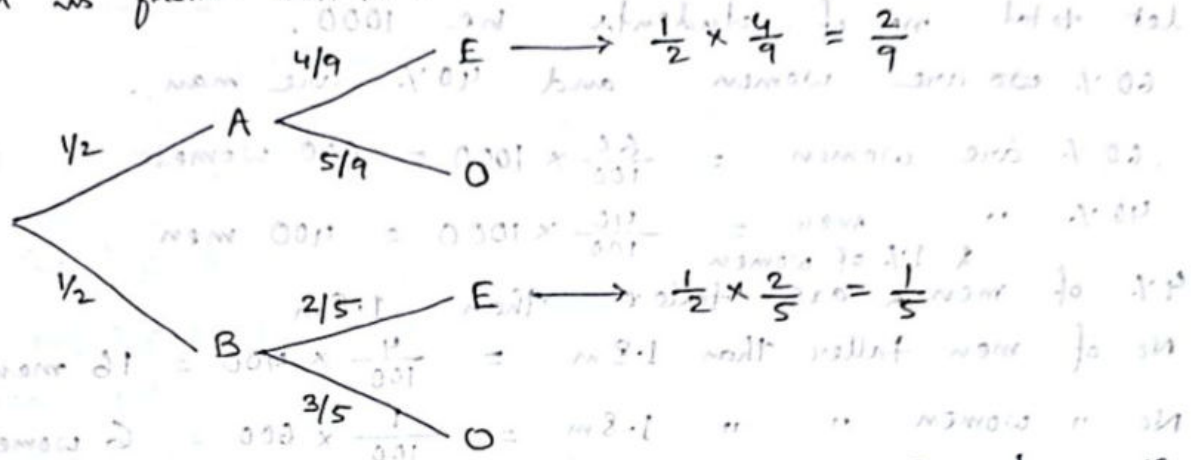
$= \frac{173}{360}$

∴ Required probability that the red marble is drawn from the bag A $= \frac{1}{8} \times \frac{360}{173} = \frac{45}{173}$

$= \frac{1/8}{173/360}$

Q Box A contains 9 cards numbered from 1 to 9. Box B contains 5 cards numbered from 1 to 5. A box is choosen at random and a card is drawn. If the number is even then find the probability that the card is from box A.

Ans



$\frac{1}{2} \times \frac{4}{9} = \frac{2}{9}$

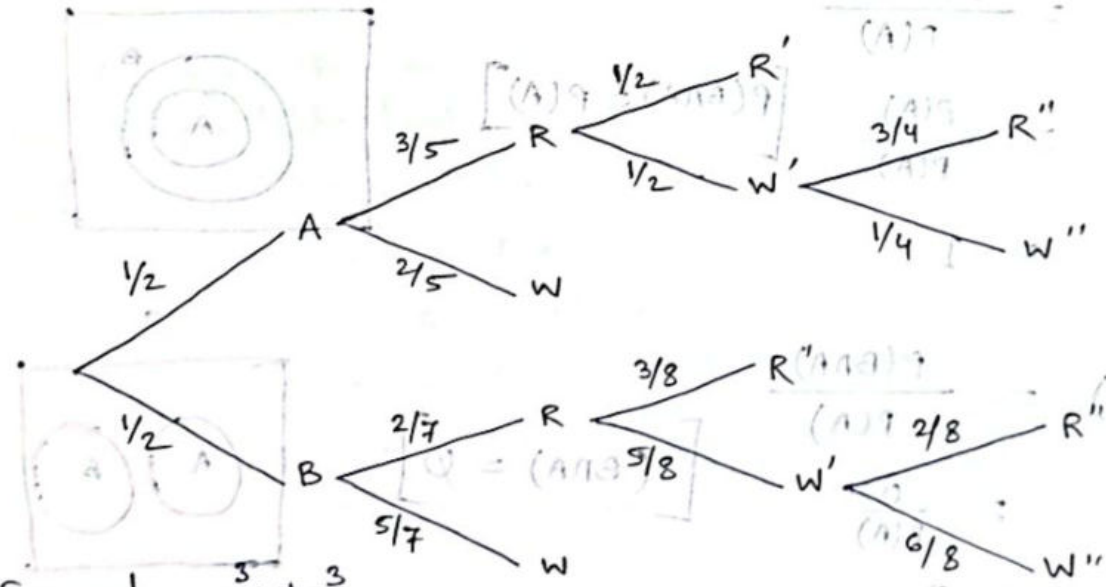
$\frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$

Probability that the card is even $= \frac{2}{9} + \frac{1}{5} = \frac{19}{45}$

$\therefore$ Required probability that the card is drawn from box A $= \frac{2}{9} \times \frac{45}{19} = \frac{10}{19}$

Q There are 2 bags. Bag A contains 3 red marbles and 2 white marbles where as bag B contains 2 red and 5 white marbles. One bag is selected at random. A marble is drawn and put into the other bag. Then a marble is drawn from the 2nd bag. Find the probability that both the marbles are of same colour.

Ans



For red $= \frac{3}{20} + \frac{3}{56}$

$= \frac{114}{560}$

For white $= \left(\frac{1}{2} \times \frac{2}{5} \times \frac{1}{4}\right) + \left(\frac{1}{2} \times \frac{5}{7} \times \frac{6}{8}\right)$

$= \frac{356}{1120}$

Q In a certain college 4% of the men and 1% of the women are taller than 1.8 ~~1.08~~ m. Further 60% of the students are women. Now if a student is selected at random and taller than 1.8 ~~1.08~~ m. then what is the probability that the student is a women.

Ans let total no of students be 1000.

60% ~~wo~~ are women and 40% are man.

60% are women $= \frac{60}{100} \times 1000 = 600$ women

40% " men $= \frac{40}{100} \times 1000 = 400$ men

4% of men & 1% of women are taller than 1.8 m

No of men taller than 1.8 m $= \frac{4}{100} \times 400 = 16$ men

No " women " " 1.8 m $= \frac{1}{100} \times 600 = 6$ women

one student is selected at random who is taller than 1.8 m

$\therefore$ Probability that it is a girl $= \frac{P(G > 1.8)}{P(G > 1.8) + P(B > 1.8)}$

$$= \frac{6/1000}{\frac{6}{1000} + \frac{16}{1000}} = \frac{6}{22}$$

$$= \frac{3}{11}$$

Q Find $P(B|A)$ if i) $A \subset B$ (Proper Subset) ii) A and B are M.E.E

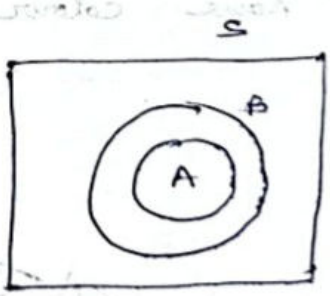
Ans if ~~A and B are mutually exclusive events~~ then,

~~$P(A \cap B) = 0$~~

i) $P(B|A) = \frac{P(B \cap A)}{P(A)}$

$= \frac{P(A)}{P(A)}$ $\quad [P(B \cap A) = P(A)]$

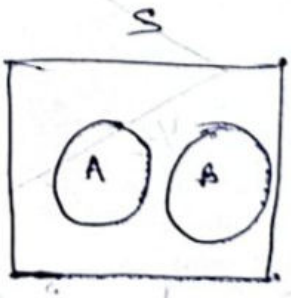
$= 1$



ii) $P(B|A) = \frac{P(B \cap A)}{P(A)}$

$= \frac{0}{P(A)}$ $\quad [(B \cap A) = \phi]$

$= 0$



Q. For any event A,

i) $0 \leq P(A|E) \leq 1$, $P(E) > 0$

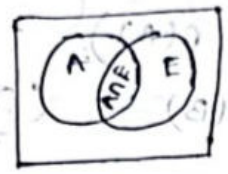
ii) $P(S|E) = 1$

iii) If A and B are mutually exclusive events,

$$P(A \cup B|E) = P(A|E) + P(B|E)$$

Ans i) $P(A|E) = \dfrac{P(A \cap E)}{P(E)}$

So, $(A \cap E) \subseteq E$



$P(A \cap E) \leq P(E)$

Since $P(E)$ is +ve

$\dfrac{P(A \cap E)}{P(E)} \leq \dfrac{P(E)}{P(E)}$

or, $P(A|E) \leq 1$

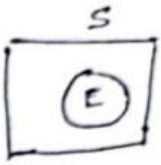
Again for any event it's prob, ≥ 0

$\therefore 0 \leq P(A|E) \leq 1$

ii) $P(S|E) = \dfrac{P(S \cap E)}{P(E)}$

$= \dfrac{P(E)}{P(E)}$

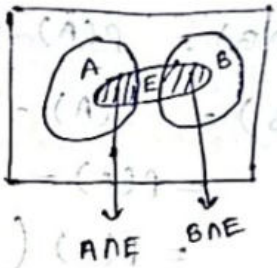
$= 1$



iii) $P((A \cup B)|E) = \dfrac{P((A \cup B) \cap E)}{P(E)}$

$\cancel{= \dfrac{P[(A \cup E) \cap (B \cup E)]}{P(E)}}$

$= \dfrac{P[(A \cap E) \cup (B \cap E)]}{P(E)}$



$= \dfrac{P(A \cap E)}{P(E)} + \dfrac{P(B \cap E)}{P(E)}$

$= P(A|E) + P(B|E)$

Independent Event:

Two events A and B are set to be independent event if $P(A \cap B) = P(A).P(B)$

if A and B are ~~the~~ independent then $\bar{A}$ and $\bar{B}$ are also independent. Proof that $P(\bar{A} \cap \bar{B}) = P(\bar{A}).P(\bar{B})$

Proof:

$P(A \cap B) = P(A).P(B)$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= 1 - P(A) - P(B) + P(A).P(B)$

$= 1(1 - P(A)) - P(B)(1 - P(B))$

$= (1 - P(A))(1 - P(B))$

$= P(\bar{A}).P(\bar{B})$ (Proved)

De Morgan's Law

$(\overline{A \cup B}) = \bar{A} \cap \bar{B}$

$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$

$P(\bar{A}) = 1 - P(A)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Q If A and B are independent then $\bar{A}$ and B are independent.

Ans

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

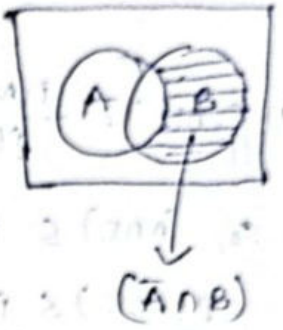
$$= P(B) - P(A) \cdot P(B)$$

$$\cancel{= P(A)(1 - P(B))}$$

$$\cancel{= P(A) \cdot P(\bar{B})}$$

$$= P(B)(1 - P(A))$$

$$= P(\bar{A}) \cdot P(B) \quad \text{(Proved)}$$



Q If A and B are independent then A and $\bar{B}$ are independent

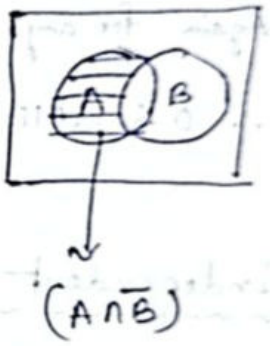
$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

$$= P(A) - P(A) \cdot P(B)$$

$$= P(A)(1 - P(B))$$

$$= P(A) \cdot P(\bar{B}) \quad \text{(Proved)}$$



★ If A, B, C three events are pairwise independent that means,

A and B are independent
B " C " "
A " C "

So we can say,

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

★ Independent events can't be shown in the Ven diagram

★ If A, B and C are pairwise mutually exclusive events,

$$P(A \cup B \cup C \mid E) = \frac{P[(A \cap E) \cup (B \cap E) \cup (C \cap E)]}{P(E)}$$

$$= \frac{P(A \cap E)}{P(E)} + \frac{P(B \cap E)}{P(E)} + \frac{P(C \cap E)}{P(E)}$$

$$= P(A \mid E) + P(B \mid E) + P(C \mid E)$$

Q The probability that A hits the target is 1/4. Probability that B hits the target is 2/5. What is the probability that the target will be hit if A and B each fires at the target?

Ans

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[As both the events are independent]

$$= P(A) + P(B) - P(A) \cdot P(B)$$

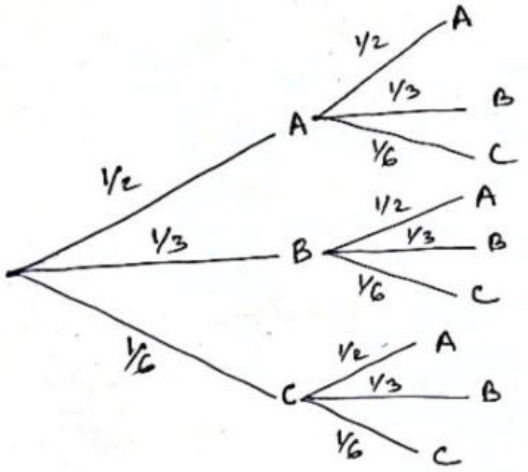
$$= \left(\frac{1}{4} + \frac{2}{5}\right) - \left(\frac{1}{4} \times \frac{2}{5}\right)$$

$$= \frac{13}{20} - \frac{1}{10}$$

$$= \frac{11}{20}$$

Q Three horses A, B and C race together. Their respective probabilities are 1/2, 1/3 and 1/6. Find the sample space set and probability of each sample if the horse's ~~rest~~ race twice?

Ans



$$S = \{AA, AB, AC, BA, BB, BC, CA, CB, CC\}$$

$$P(AA) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$P(BB) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

$$P(CC) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$$P(AB) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

$$P(BA) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P(CA) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$$

$$P(AC) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$$

$$P(BC) = \frac{1}{3} \times \frac{1}{6} = \frac{1}{18}$$

$$P(CB) = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$$

Q Let A, B be the event where,

A = { The family has atleast children of both sexes }

B = { The family has at most one boy }

i) Show that A and B are independent if the family has 3 children.

ii) Show that A and B are dependent, if the family has 2 children.

i) S = { bbb, bgb, bbg, gbb, ggb, gbg, bgg, ggg }

A = { bgb, bbg, gbb, ggb, gbg, bgg }

At most ⟶ Highest

B = { ggb, gbg, bgg, ggg }

P(A∩B) = ggb, gbg, bgg

ii) S = { bb, bg, gb, gg }

A = { bg, gb }

B = { bg, gb, gg }

P(A∩B) = bg, gb

Random Variable

Ex A coin is tossed 3 times. Find the probability distribution of getting no of heads.

Let X be the random variable of getting no of heads.

$X = \{0, 1, 2, 3\}$

$|S| = 2^3 = 8$

$S = \{HHH, HHT, HTH, HTT, TTH, THT, THH, TTT\}$

$P(HHH) = \frac{1}{8} = P(3\text{-Heads}) \longrightarrow HHH$

$P(2\text{-Heads}) = \frac{3}{8} \longrightarrow HHT, HTH, THH$

$P(1\text{-Head}) = \frac{3}{8} \longrightarrow HTT, THT, TTH$

$P(0\text{-Head}) = \frac{1}{8} \longrightarrow TTT$

So,

$X = x$	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Total $= \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$
$= 1$

Q A coin is weighted so that $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. Coin is tossed for 3 times. Then

i) Find the probability distribution of getting no of heads.

ii) " " " " " Largest no of successive heads.

Ans i) Let X be the random variable of getting no of heads.

$X = \{0, 1, 2, 3\}$

$S = \{HHH, HHT, HTH, HTT, THH, TTH, THT, TTT\}$

$P(HHH) = P(H).P(H).P(H) = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$

$P(HHT) = P(H).P(H).P(T) = \frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$

$P(HTH) = P(H).P(T).P(H) = \frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{27}$

$P(HTT) = P(H).P(T).P(T) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$

All of the events are individually independent

$P(TTT) = P(T).P(T).P(T) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

$P(TTH) = P(T).P(T).P(H) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$

$P(THT) = P(T).P(H).P(T) = \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$

$P(THH) = P(T).P(H).P(H) = \frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$

$\text{Total} = (P(HHH) + P(HHT) + \dots + P(THH)) = \frac{8}{27} + \frac{4}{27} + \dots + \frac{4}{27}$

$= 1$

$P(0-\text{Head}) = \frac{1}{27}$

$P(1-\text{Head}) = \frac{2}{27} + \frac{2}{27} + \frac{2}{27} = \frac{6}{27}$

$P(2-\text{Head}) = \frac{4}{27} + \frac{4}{27} + \frac{4}{27} = \frac{12}{27}$

$P(3-\text{Head}) = \frac{8}{27}$

$X = x$	0	1	2	3
$P(X=x)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

Probability Distribution Table of getting head

ii) $X = \{0, 1, 2, 3\}$

successive = back to back

$P(0-\text{Head}) = \{TTT\}$ $P_1 = \frac{1}{27}$

back to back only 1 head

$P(1-\text{Head}) = \{HTH, HTT, TTH, THT\}$ $P_2 = \left(\frac{4}{27} + \frac{2}{27} + \frac{2}{27} + \frac{2}{27}\right) = \frac{10}{27}$

$P(2-\text{Head}) = \{HHT, THH\}$ $P_3 = \left(\frac{4}{27} + \frac{4}{27}\right) = \frac{8}{27}$

back to back 2 heads

$P(3-\text{Head}) = \{HHH\}$ $P_4 = \frac{8}{27}$

back to back 3 heads

$X = x$	0	1	2	3
$P(X=x)$	$\frac{1}{27}$	$\frac{10}{27}$	$\frac{8}{27}$	$\frac{8}{27}$

Probability Distribution Table for getting successive no of heads

Q. A fair dice is tossed for two times. Find the probability distribution of getting maximum or largest of the faces.

Ans. $|S| = 6^2 = 36$

$S = \{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66\}$

$A = \{\text{max is } 1\} = \{11\} \rightarrow P_1 = \frac{1}{36}$

$B = \{\text{max is } 2\} = \{12, 21, 22\} \rightarrow P_2 = \frac{3}{36}$

$C = \{\text{max is } 3\} = \{13, 31, 23, 32, 33\} \rightarrow P_3 = \frac{5}{36}$

$D = \{\text{max is } 4\} = \{14, 41, 24, 42, 34, 43, 44\} \rightarrow P_4 = \frac{7}{36}$

$E = \{\text{max is } 5\} = \{15, 51, 25, 52, 35, 53, 45, 54, 55\} \rightarrow P_5 = \frac{9}{36}$

$F = \{\text{max is } 6\} = \{16, 61, 26, 62, 36, 63, 46, 64, 56, 65, 66\} \rightarrow P_6 = \frac{11}{36}$

So, X = Getting largest value $= \{1, 2, 3, 4, 5, 6\}$

$X = x$	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Q. Find the probability distribution of getting sum of the faces. (For tossing a fair dice two times)

Ans. $|S| = 6^2 = 36$

$A = \{\text{sum is } 2\} = \{11\} \rightarrow P_1 = \frac{1}{36}$

$B = \{\text{sum is } 3\} = \{12, 21\} \rightarrow P_2 = \frac{2}{36}$

$C = \{\text{sum is } 4\} = \{13, 31, 22\} \rightarrow P_3 = \frac{3}{36}$

$D = \{\text{sum is } 5\} = \{14, 41, 23, 32\} \rightarrow P_4 = \frac{4}{36}$

$E = \{\text{sum is } 6\} = \{15, 51, 24, 42, 33\} \rightarrow P_5 = \frac{5}{36}$

$F = \{\text{sum is } 7\} = \{16, 61, 25, 52, 34, 43\} \Rightarrow P_6 = \frac{6}{36}$

~~$G = \{\text{sum is } 8\} = \{11, 26, 62, 35, 53, 44\} \rightarrow P_7 = \frac{7}{36}$~~

~~$H = \{\text{sum is } 9\} = \{18, 81,$~~

$G = \{\text{sum is } 8\} = \{26, 62, 35, 53, 44\} \Rightarrow P_7 = \frac{5}{36}$

$H = \{\text{sum is } 9\} = \{36, 63, 45, 54\} \rightarrow P_8 = \frac{4}{36}$

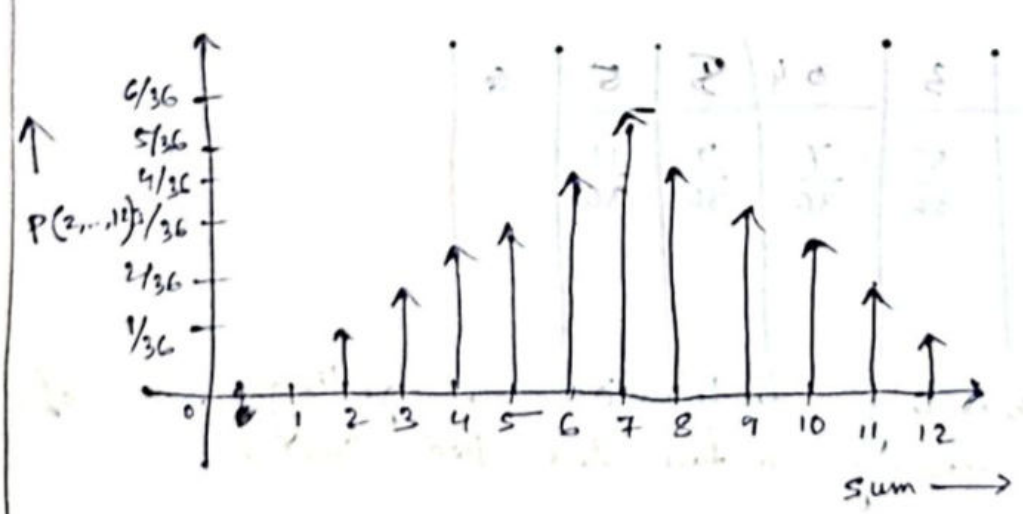
$I = \{\text{sum is } 10\} = \{46, 64, 55\} \rightarrow P_9 = \frac{3}{36}$

$J = \{\text{sum is } 11\} = \{56, 65\} \rightarrow P_{10} = \frac{2}{36}$

$K = \{\text{sum is } 12\} = \{66\} \rightarrow P_{11} = \frac{1}{36}$

$X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ = Getting sum of faces.

$X = x$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



Q A sample of 3 items are selected at random from a box containing 12 items, out of which 3 are defective. Construct the probability distribution of defective items.

Ans

$$12 \rightarrow D = 3, \quad ND = 9$$

X be the random variable of getting defective items.

$X = \{0, 1, 2, 3\}$

$|S| = C(12, 3) = 220$

$A = \{0 - \text{defective}\} \rightarrow |A| = C(9, 3) = 84 \qquad P(A) = \frac{84}{220}$

$B = \{1 - \text{defective} + 2 - \text{non defective}\} \rightarrow |B| = C(3,1) \times C(9,2) = 108 \qquad P(B) = \frac{108}{220}$

$C = \{2 - \text{defective}\}$

$= \{2D + 1ND\} \Rightarrow |C| = C(3,2) \times C(9,1) = 27 \qquad P(C) = \frac{27}{220}$

$D = \{3 - \text{defective}\}$

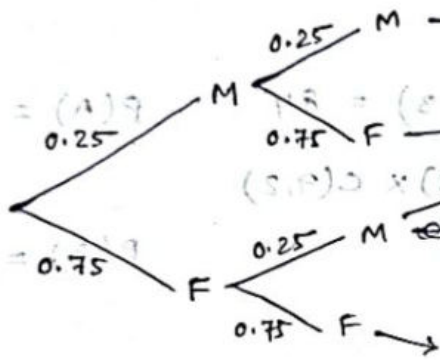
$= \{3D + 0ND\} \rightarrow |D| = C(3,3) \times C(9,0) = 1 \qquad P(D) = \frac{1}{220}$

$X = x$	0	1	2	3
$P(X = x)$	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

Q According to a survey in past 75% of all the teachers are female. Assume that the result hold true for the current population of all teachers in the city. Suppose 2 teachers are randomly selected from the population of all teachers in that city. Find the ~~probabili~~ possible list of female teachers.

Ans X be the random variable of female teacher.

$X = \{0, 1, 2\}$



$M \xrightarrow{0.25} M \longrightarrow P(MM) = P(0 \text{ female teacher}) = (0.25)(0.25) = \frac{1}{16}$

$M \xrightarrow{0.75} F \longrightarrow P(MF) = P(1 \text{ female teacher}) = (0.75)(0.25) = \frac{3}{16}$

$F \xrightarrow{0.25} M \longrightarrow P(FM) = P(1 \text{ female teacher}) = (0.75)(0.25) = \frac{3}{16}$

$F \xrightarrow{0.75} F \longrightarrow P(FF) = P(2 \text{ female teacher}) = (0.75)(0.75) = \frac{9}{16}$

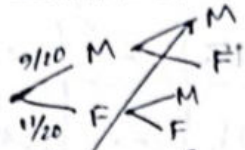
$A = \{0 - F \ldots T \ldots\} \rightarrow P(A) = \frac{1}{16}$

$B = \{1 - F \ldots T \ldots\} \rightarrow B = \{MF, FM\} = P(MF) + P(FM) = \frac{6}{16}$

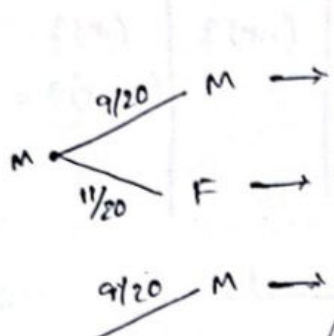
$C = \{2 - F \ldots T \ldots\} \rightarrow P(C) = \frac{9}{16}$

$X = x$	0	1	2
$P(X = x)$	$\frac{1}{16}$	$\frac{6}{16}$	$\frac{9}{16}$

Q A machine design class has 20 students and 11 of them are girls. Suppose 2 students are randomly selected from the class. Construct the probability distribution of girl students.



Ans



$M \to M \to P(MM) = P\{0-\text{female student}\} = \frac{81}{380} \quad \frac{9}{20} \times \frac{9}{19}$

$M \to F \to P(MF) = P\{1-\text{female student}\} = \frac{99}{400} = \frac{9}{20} \times$

$F \to M \to P(FM) = P\{1-\text{female student}\} = \frac{99}{400}$

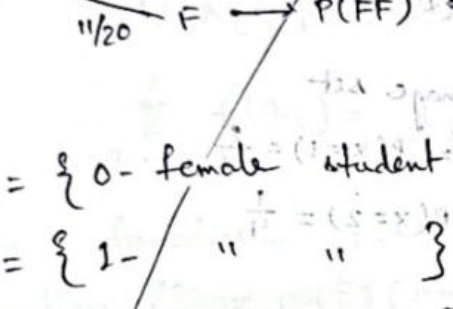
$F \to F \to P(FF) = P\{2-\text{female student}\} = \frac{121}{400}$

$A = \{0-\text{female student}\} = \frac{81}{400}$

$B = \{1- \quad " \quad " \quad \} = \frac{99}{400} + \frac{99}{400} = \frac{198}{400}$

$C = \{2- \quad " \quad " \quad \} = \frac{121}{400}$

Let X be the event of getting female student



$M \to M \to P(MM) = \frac{9}{20} \times \frac{8}{19} = \frac{72}{380} \to P(0-G)$

$M \to F \to P(MF) = \frac{9}{20} \times \frac{11}{19} = \frac{99}{380}$

$F \to M \to P(FM) = \frac{11}{20} \times \frac{9}{19} = \frac{99}{380}$

$\Big\} P(1-G)$

$F \to F \to P(FF) = \frac{11}{20} \times \frac{10}{19} = \frac{110}{380} \to P(2-G)$

$X=x$	0	1	2
$P(X=x)$	$\frac{72}{380}$	$\frac{198}{380}$	$\frac{110}{380}$

Discrete Random Variable :

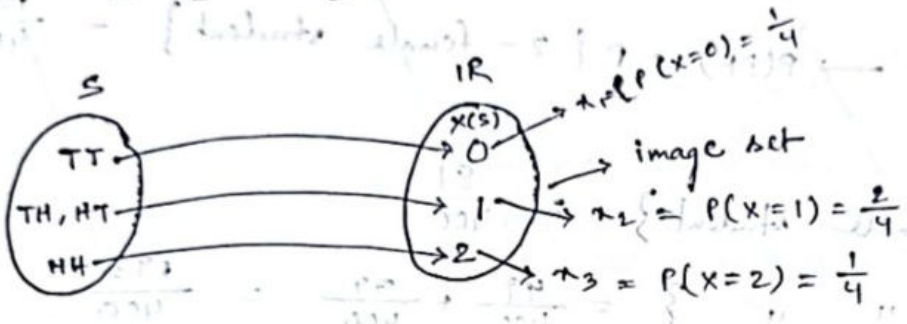
Let X be a random variable

i.e. $X : S \to \mathbb{R}$

Ex A coin is tossed 2 times.

X be the RV of getting H.

Ans $\therefore X = \{0, 1, 2\}$



If the image set $X(S)$ is finite or countably infinite then the random variable X is set to be discrete random variable.

If $X = \{x_1, x_2, \dots, x_n\}$

$\therefore f(x_i) = P(X = x_i)$

$X = x_i$	x_1	x_2	...	x_n
$f(x_i)$	$f(x_1)$	$f(x_2)$	....	$f(x_n)$

Let X be a discrete random variable with following probability distribution.

$(X=x)$	x_1	x_2	$\cdots$	x_n
$f(x) = P(X=x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

It is clear that,

$$\forall x_i, \quad 0 \leq P(x_i) \leq 1$$

$$\sum_{x=1}^{n} f(x_i) = 1$$

The function $f(x)$ is said ~~to~~ as probability mass function ~~(pmf pdf)~~ (pmf)

or probability density function (pdf)

Mean / Expectation :

For the discrete random variable X,

$E(X)$ denotes the expected value or expectation of X.

μ denotes for mean.

Also, $\mu = E(X)$

where, $\mu = x_1 f(x_1) + x_2 f(x_2) + \cdots + x_n f(x_n)$

$= \sum x_i f(x_i)$

$E(X^2) = x_1^2 f(x_1) + x_2^2 f(x_2) + \cdots + x_n^2 f(x_n)$

$= \sum x_i^2 f(x_i)$

$E(X^3) = x_1^3 f(x_1) + x_2^3 f(x_2) + \cdots + x_n^3 f(x_n)$

$= \sum x_i^3 f(x_i)$

$$E(x^n) = x_1^n f(x_1) + x_2^n f(x_2) + \dots + x_n^n f(x_n)$$

let X be the random variable and K be the constant

$$E(X+K) = E(X) + K$$

$$E(X+Y) = E(X) + E(Y)$$

Variance & Standard Deviation :

σ denotes the standard deviation of the random variable X.

σ^2/var(x) denotes the variance.

where, std deviation = $\sqrt{var(x)}$

$$\sigma^2 = (x_1 - \mu)^2 f(x_1) + (x_2 - \mu)^2 f(x_2) + \dots + (x_n - \mu)^2 f(x_n)$$

$$= \Sigma (x_i - \mu)^2 f(x_i)$$

Alternative formula,

$$\sigma^2 = \Sigma (x_i - \mu)^2 f(x_i)$$

$$= \Sigma \{x_i^2 + \mu^2 - 2x_i\mu\} f(x_i)$$

$$= \Sigma x_i^2 f(x_i) + \Sigma \mu^2 f(x_i) - \Sigma 2\mu x_i f(x_i)$$

$$= \Sigma x_i^2 f(x_i) + \mu^2 \Sigma f(x_i) - 2\mu \Sigma x_i f(x_i)$$

$$= \Sigma x_i^2 f(x_i) + \mu^2 \cdot 1 - 2\mu \cdot \mu$$

$$= \Sigma x_i^2 f(x_i) - \mu^2$$

or

$$\sigma^2 = E(x^2) - \mu^2$$

Ex

$X = x$	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

μ, $E(X) = (0 \times \frac{1}{8}) + (1 \times \frac{3}{8}) + (2 \times \frac{3}{8}) + (3 \times \frac{1}{8})$

$= \frac{12}{8} = \frac{3}{2}$

$\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = (0^2 \times \frac{1}{8}) + (1^2 \times \frac{3}{8}) + (2^2 \times \frac{3}{8}) + (3^2 \times \frac{1}{8})$

$= 3$

So, $\sigma^2 = 3 - (\frac{3}{2})^2$

$= 3 - \frac{9}{4}$

$= \frac{3}{4}$

$\therefore \sigma = \sqrt{\frac{3}{2}}$

variance $\rightarrow \sigma^2$

standard deviation $\rightarrow \sigma$

Ex

$X = x$	0	1	2	3
$f(x)$	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$

μ, $E(X) = (0 \times \frac{1}{27}) + (1 \times \frac{6}{27}) + (2 \times \frac{12}{27}) + (3 \times \frac{8}{27})$

$= \frac{54}{27} = 2$

$\sigma^2 = E(X^2) - \mu^2$

$E(X^2) = (0^2 \times \frac{1}{27}) + (1^2 \times \frac{6}{27}) + (2^2 \times \frac{12}{27}) + (3^2 \times \frac{8}{27})$

$= 4.66$

So, $\sigma^2 = 4.66 - 3$

$= 1.66$

or, $\sigma = 1.28$

Notes

*) Let X be a discrete Random variable with the probability distribution

$X = x$	x_1	x_2	$\cdots$	x_n
$f(x) = P(X=x)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

Let $f(x)$ be the probability mass function / density function.

$$f(x) = P(X=x)$$

*) $0 \leq f(x_i) \leq 1$

$\Sigma f(x_i) = 1$

Ex From the following function state which one represents probability density function for the random variable X.

1) $f(x) = \frac{x-2}{2}$, $x = 0, 1, 2, 3, 4$

2) $f(x) = \frac{x^2}{4}$, $x = 1, 2, 3, 4$

3) $f(x) = \frac{3x+1}{4}$, $x = 0, 1$

Ans

1)

$X = x$	0	1	2	3	4
$f(x)$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1

As $f(0) < 0$, $f(1) < 0$ and $\Sigma f_i \neq 1$

So it's not a probability distribution function. (pdf).

2)

$X = x$	0	1	2	3	4
$f(x)$		$\frac{1}{4}$	1	$\frac{9}{4}$	4

$\because \Sigma f_i \neq 1$

So it's not a PDF.

3)

$X = x$	0	1
$f(x)$	$\frac{1}{4}$	1

$\because \sum f_i \neq 1$

$\therefore$ So, it's not a pdf.

*) $f(x) = P(X = x)$

*) $P(a \leq x \leq b)$

All the image points of x are to be taken from a to b.

$a = x_1, x_2, x_3, \ldots, x_{10} = b$

$P(a \leq x \leq b) = P(X = x_2) + P(X = x_3) + \ldots + P(X = x_{10})$

*) $P(x \leq a)$

All the images of points of x are to be taken from the 1st point upto 'a'.

$x_1, x_2, x_3, \ldots, x_5 = a$

$P(X \leq a) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_5)$

$P(X < a) = P(X = x_1) + P(X = x_2) + \ldots + P(X = x_4)$

$P(a < X < b) = P(X = x_3) + P(X = x_4) + \ldots + P(X = x_9)$

Ex A fair coin is tossed for 10 times and its lists of getting head is as follows.

$X = x$	0	1	2	3	4	5	6	7	8	9	10
$f(x)$	$C(10,0)(\frac{1}{2})^{10}$	$C(10,1)(\frac{1}{2})^{10}$	$C(10,2)(\frac{1}{2})^{10}$	$C(10,3)(\frac{1}{2})^{10}$	$C(10,4)(\frac{1}{2})^{10}$	$C(10,5)(\frac{1}{2})^{10}$	$C(10,6)(\frac{1}{2})^{10}$	$C(10,7)(\frac{1}{2})^{10}$	$C(10,8)(\frac{1}{2})^{10}$	$C(10,9)(\frac{1}{2})^{10}$	$C(10,10)(\frac{1}{2})^{10}$

1) Find the probability of getting exactly 6 heads.

2) " " " " " at least 5 heads.

3) " " " " " at most 7 heads.

Ans 1) $f(x) = P(X = x)$

$P(X = 6) = f(6) = C(10,6)\left(\frac{1}{2}\right)^{10}$

2) Probability of getting at least 5 heads

$P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$= C(10,5)\left(\frac{1}{2}\right)^{10} + C(10,6)\left(\frac{1}{2}\right)^{10} + C(10,7)\left(\frac{1}{2}\right)^{10} + C(10,8)\left(\frac{1}{2}\right)^{10} + C(10,9)\left(\frac{1}{2}\right)^{10} + C(10,10)\left(\frac{1}{2}\right)^{10}$

$=$

3) Probability of getting at most 7 heads

$P(X \leq 7) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$= C(10,0)\left(\frac{1}{2}\right)^{10} + C(10,1)\left(\frac{1}{2}\right)^{10} + C(10,2)\left(\frac{1}{2}\right)^{10} + C(10,3)\left(\frac{1}{2}\right)^{10} + C(10,4)\left(\frac{1}{2}\right)^{10} + C(10,5)\left(\frac{1}{2}\right)^{10} + C(10,6)\left(\frac{1}{2}\right)^{10} + C(10,7)\left(\frac{1}{2}\right)^{10}$

$=$

4) Probability of getting 3 to 5 heads

$P(3 \leq x \leq 5) = P(X=3) + P(X=4) + P(X=5)$

$= C(10,3)\left(\frac{1}{2}\right)^{10} + C(10,4)\left(\frac{1}{2}\right)^{10} + C(10,5)\left(\frac{1}{2}\right)^{10}$

$=$

5) Probability of getting the heads in b/w 3 and 5

$P(3 < x < 5) = P(X=4)$

$= C(10,4)\left(\frac{1}{2}\right)^{10}$

$=$

* $P(x \geq a) = 1 - P(x < a)$
* $P(x \leq a) = 1 - P(x > a)$

2) or $P(X \geq 5) = 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)\}$

3) or $P(X \leq 7) = 1 - \{P(X=8) + P(X=9) + P(X=10)\}$

~~Find the probability of~~

Cumulative Distribution Function:

$F(x)$ denotes for cumulative distribution function for the random variable X.

$X = x$	x_1	x_2	...	x_n
$f(x)$	$f(x_1)$	$f(x_2)$	...	$f(x_n)$

$F(x_1) = f(x_1)$, $x \leq x_1$

$F(x_2) = f(x_1) + f(x_2)$, $x \leq x_2$

$F(x_3) = f(x_1) + f(x_2) + f(x_3)$, $x \leq x_3$

$\vdots$

$F(x_n) = f(x_1) + f(x_2) + \ldots + f(x_n) = 1$, $x \leq x_n$

*) $P(a \leq X \leq b) = F(b) - F(a)$

Binomial Distribution: ($n \to$ small, $p \to$ large, with replacement)

let X be the discrete random variable where the experiment has exactly 2 outcomes called as success & failure.

There should be 'n' no of repeated trials.

let, 'p' be the probability of success.

'q' " " " " failure.

So, $(p + q) = 1$

*) Binomial Distribution can be applied when 'n' is not so large.

*) If n is large then Poison Distribution to be applied.

Binomial Distribution :

*) The distribution is with replacement.

*) This distribution can be applied when n is small and P is large $(0 < P < 1)$.

*) The probability distribution function for binomial distribution function is $f(x) = P(X = x)$

= Probability of getting x no of success

$= C(n, x) P^x q^{n-x}$ $\left[\begin{array}{l} q = 1 - P \\ P + q = 1 \end{array} \right]$

Q A coin is tossed for 3 times. Find the binomial probability distribution function of getting x no of heads..

Ans $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

So, $P = \frac{1}{2}$, $q = \frac{1}{2}$, $n = 3$

$f(x) = P(X = x) = C(3, x) \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}$ **

Find the probability of getting exactly 2 heads;

$P(X = 2) = C(3, 2) \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} =$

Find the probability of getting at least 1 head,

$P(X \geq 1) = 1 - P(X = 0)$

$= 1 - C(3, 0) \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{3-0} =$

Find the probability when $P(1 < x < 3)$

$= P(X = 2)$

$= C(3, 2) \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}$

Q A professional basketball player makes 80% of the free ~~throug~~ throws it rise. Assuming this percentage will hold true for future attempts. Find the probability that in the next 8 tries the no of the throws he will make ① exactly 8 ② exactly 5 ③ at most 6 ④ at least 2 ⑤ in set 3 to 6 ⑥ no free attempt

Ans

$P = \frac{80}{100}$, $q = \frac{20}{100}$, $n = 8$

$= 0.8$ $= 0.2$

$P(S) = 0.8$ (Success)

$P(F) = 0.2$ (Failure)

$$f(x) = P(X=x) = C(8,x)\left(\frac{80}{100}\right)^x\left(\frac{20}{100}\right)^{n-x}$$
$$= C(8,x)(0.8)^x(0.2)^{n-x}$$

① exactly 8 $= P(X=8)$
$= C(8,8)(0.8)^8(0.2)^{8-8}$

② exactly 5 $= P(X=5)$
$= C(8,5)(0.8)^5(0.2)^{8-5}$
$= C(8,5)(0.8)^5(0.2)^3$

③ at least 2 $= P(X \geq 2)$
$= P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$ or $1 - \{P(X=0) + P(X=1)\}$

④ at most 6 $= 1 - [P(X=7) + P(X=8)]$ or $P(X=0) + \ldots + P(X=6)$
$= P(X \leq 6)$ $=$

⑤ in set 3 to 6 $= P(X=4) + P(X=5)$

⑥ No free attempt $= P(X=0)$

Q Let X denotes the no of mechanical components that are defective in a testing process and assume that X is a binomial random variable with $p = 0.001$ If 1000 of these components are tested, find the probability density function?

Ans $n = 1000, \quad p = 0.001, \quad q = (1 - 0.001) = 0.999$

Here p is small and q is large.

$$f(x) = P(X = x) = C(1000, x)\,(0.001)^{x}\,(0.999)^{1000-x}$$

Here X is mentioned as RN that's why we are going with this method, otherwise we can calculate using any type of distribution.

Binomial Theorem:

*) $(P+Q)^n = C(n,0)P^nQ^0 + C(n,1)P^{n-1}Q^1 + C(n,2)P^{n-2}Q^2 + C(n,3)P^{n-3}Q^3 + \dots + C(n,n)P^0Q^n$

$$= P^n + nP^{n-1}Q + \frac{n(n-1)}{2!}P^{n-2}Q^2 + \frac{n(n-1)(n-2)}{3!}P^{n-3}Q^3 + \dots + Q^n$$

*) $(p+q)^{n-1} = ?$

*) $(p+q)^{n-2} = ?$

$$(p+q)^{n-1} = C(n-1,0)p^{n-1}q^0 + C(n-1,1)p^{n-2}q^1 + C(n-1,2)p^{n-3}q^2 + \dots + C(n-1,n-1)p^0q^{n-1}$$

$$= (q+p)^{n-1} = p^{n-1} + (n-1)p^{n-2}q + \frac{(n-1)(n-2)}{2!}p^{n-3}q^2 + \frac{(n-1)(n-2)(n-3)}{3!}p^{n-4}q^3 + \dots + q^{n-1}$$

$$(p+q)^{n-2} = p^{n-2} + (n-2)\,p^{n-3}q + \frac{(n-2)(n-3)}{2!}\,p^{n-4}q^2 + \frac{(n-2)(n-3)(n-4)}{3!}\,p^{n-5}q^3 + \dots + q^{n-2}$$

Mean & Variance of Binomial Distribution : ★★

$$\mu = \sum x\, f(x)$$

$$= \sum_{x=0}^{n} x\, c(n,x)\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!(n-x)!}\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n\,\cancel{x!}}{(x-1)!(n-x)!}\, p^x q^{n-x} \qquad \left[\frac{x}{x!} = \frac{x}{x(x-1)!} = \frac{1}{(x-1)!}\right]$$

$x = 0, 1, \dots, n$

$$= \cancel{\frac{n!}{(-1)!(n-0)!}\, p^0 q^{n-0}} + \frac{n!}{0!(n-1)!}\, p^1 q^{n-1} + \frac{n!}{1!(n-2)!}\, p^2 q^{n-2} + \dots + \frac{n!}{(n-1)!(n-n)!}\, p^n q^0$$

$$= \frac{n!}{(n-1)!}\, pq^{n-1} + \frac{n!}{(n-2)!}\, p^2 q^{n-2} + \frac{n!}{2!(n-3)!}\, p^3 q^{n-3} + \dots + \frac{n!}{(n-1)!}\, p^n$$

$$= npq^{n-1} + n(n-1)p^2 q^{n-2} + \frac{n(n-1)(n-2)}{2!}\, p^3 q^{n-3} + \dots + np^n$$

$$= np\left[q^{n-1} + (n-1)pq^{n-2} + \frac{(n-1)(n-2)}{2!}\, p^2 q^{n-3} + \dots + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1} \qquad [p+q=1]$$

$$= np$$

Mean

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \sum x^2 f(x) - \mu^2$$

$$= \sum x^2 f(x) - (np)^2$$

$$= \sum_{x=0}^{n} x^2 c(n,x) p^x q^{n-x} \qquad \frac{x^2}{x!} = \frac{x^2}{x(x-1)!} = \frac{x}{(x-1)!}$$

$$= \sum_{x=0}^{n} x^2 \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{x \cdot n!}{(x-1)!(n-x)!} p^x q^{n-x}$$

$x = 0, 1, \ldots, n$

$$= \frac{0 \cdot n!}{(-1)!(n-0)!} p^0 q^{n-0} + \frac{n!}{0!(n-1)!} p^1 q^{n-1} +$$

$$\frac{2n!}{1!(n-2)!} p^2 q^{n-2} + \ldots + \frac{n \cdot n!}{(n-1)!\,0!} p^n q^0$$

$$= \frac{n!}{(n-1)!} p q^{n-1} + \frac{2n!}{(n-2)!} p^2 q^{n-2} + \frac{3n!}{2!(n-3)!} p^3 q^{n-3}$$

$$+ \ldots + \frac{n \cdot n!}{(n-1)!} p^n$$

$$= npq^{n-1} + 2n(n-1) p^2 q^{n-2} + \frac{3}{2!}(n-1)(n-2) p^3 q^{n-3}$$

$$+ \ldots + n^2 p^n$$

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \sum x^2 f(x) - \mu^2$$

$$= \sum x^2 f(x) - (np)^2$$

$$= \sum_{x=0}^{n} x^2 c(n,x) p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x + x) c(n,x) p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (x^2 - x) c(n,x) p^x q^{n-x} + \sum_{x=0}^{n} x c(n,x) p^x q^{n-x} \to \mu$$

$$= \sum_{x=0}^{n} x(x-1) \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x} + \mu$$

$$= \sum_{x=0}^{n} \frac{n!}{(x-2)!(n-x)!} p^x q^{n-x} + \mu$$

$$= 0 + 0 + \frac{n!}{0!\,(n-2)!} p^2 q^{n-2} + \frac{n!}{1!\,(n-3)!} p^3 q^{n-3} + \frac{n!}{2!\,(n-4)!} p^4 q^{n-4}$$

$$+ \dots + \frac{n!}{(n-2)!\,0!} p^n q^0 + \mu$$

$$= n(n-1) p^2 q^{n-2} + n(n-1)(n-2) p^3 q^{n-3} + \frac{n(n-1)(n-2)(n-3)}{2!} p^4 q^{n-4} +$$

$$\dots + n(n-1) p^n + \mu$$

$$= n(n-1) p^2 \left[q^{n-2} + (n-2) p q^{n-3} + \frac{(n-2)(n-3)}{2!} p^2 q^{n-4} + \right.$$

$$\left. \dots + p^{n-2} \right] + \mu$$

$$= n(n-1) p^2 (q+p)^{n-2} + \mu$$

$$= n(n-1) p^2 + np$$

$$\sigma^2 = E(x^2) - \mu^2$$

$$= \Sigma x^2 f(x) - (np)^2$$

$$= n(n-1) p^2 + np - n^2 p^2$$

$$= n^2 p^2 - np^2 + np - n^2 p^2$$

$$= np(1-p)$$

$$= npq \quad \text{// Variance}$$

Q Find the μ and σ^2 for the worked out examples.

① Coin tossed 3 times,

$n = 3$, $p = \frac{1}{2}$, $q = \frac{1}{2}$

$\mu = np = \frac{3}{2}$ | $\sigma^2 = npq = \frac{3}{4}$

② $p = 0.8$, $q = 0.2$, $n = 8$

$\mu = np = (0.8 \times 8) =$

$\sigma^2 = npq = (0.8 \times 0.2 \times 8) =$

③ $p = 0.001$, $q = 0.999$, $n = 1000$

$\mu = np = (1000 \times 0.001) =$

$\sigma^2 = npq = (1000 \times 0.001 \times 0.999) =$

$n \rightarrow$ No of trials (to be small)

$p \rightarrow$ Pros of success (to be large)

$q \rightarrow$ Pros of failure ($q = 1 - p$)

pdf $\rightarrow$ $f(x) = c(n, x)\, p^x q^{n-x}$

This distribution is with replacement.

Poisson Distribution ($n \rightarrow$ large, $p \rightarrow$ small)

Let X be a discrete Random Variable of getting success.

There should be 'n' no of trials where n is so large.

p be the pros. of success where p is so small.

q " " " " failure " $q = 1 - p$

Pros. density function:

From binomial distribution we have,

$$f(x) = c(n, x)\, p^x q^{n-x}$$

$$= \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}$$

Term

$x + y$

+, − operator involved → Term

*, / operator involved → Factor

$x * y$

Factor

Now,

$$\frac{n!}{(n-x)!} = \frac{n(n-1)(n-2)\ldots(n-x+1)\cancel{(n-x)!}}{\cancel{(n-x)!}}$$

$$= n(n-1)(n-2)\ldots(n-x+1)$$ → Total x no of factors

$$\approx n.n\ldots\ldots x \text{ times}$$ [x is not to be large]

$$= n^x$$

$$q^x = (1-p)^x \approx 1^x = 1$$

$$q^{n-x} = \frac{q^n}{q^x} \approx q^n \quad [\because q^x = 1]$$

So, $$f(x) = \frac{n^x}{x!}\, p^x q^n = \frac{(np)^x}{x!}\,(1-p)^n$$

$$= \frac{(np)^x}{x!}\left\{c(n,0)1^n p^0 - c(n,1)1^n p^1 + c(n,2)1^n p^2 - c(n,3)1^n p^3 + \dots\right\}$$

$$= \frac{(np)^x}{x!}\left\{1 - np + \frac{n(n-1)p^2}{2!} - \frac{n(n-1)(n-2)p^3}{3!} + \dots\right\}$$

$$\approx \frac{(np)^x}{x!}\left\{1 - np + \frac{n.n.p^2}{2!} - \frac{n.n.n.p^3}{3!} + \dots\right\}$$

$$= \frac{(np)^x}{x!}\left\{1 - np + \frac{(np)^2}{2!} - \frac{(np)^3}{3!} + \dots\right\}$$

$$= \frac{(np)^x}{x!}.e^{-np}$$

$$= \frac{\lambda^x}{x!}.e^{-\lambda} \quad [\text{Let } \lambda = np]$$

$$\left\{\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots \end{aligned}\right.$$

So the probability density function of Poisson distribution will be,

$$f(x) = P(X = x)$$

$$\therefore f(x) = \frac{e^{-\lambda}.\lambda^x}{x!}$$

Q A coin is tossed for 10 times, Find the probability of getting 5 heads.

Ans $p = \frac{1}{2}$, $q = (1 - \frac{1}{2}) = \frac{1}{2}$, $n = 10$

Binomial Distribution,

$$f(x) = c(n,x)\, p^x q^{n-x} = c(10,5)\left(\tfrac{1}{2}\right)^5\left(\tfrac{1}{2}\right)^5$$

$$\cancel{= c(10,5)\left(\tfrac{1}{2}\right)^5 q^{10-5}} = 0.246$$

$$= \underline{24.6\%}$$ This ans is correct

Poisson Distribution,

$$\lambda = np = 10 \times \frac{1}{2} = 5$$

$$f(x) = P(X = x) = \frac{e^{-\lambda}.\lambda^x}{x!}$$

$$P(X = 5) = \frac{e^{-5}.5^5}{5!}$$

$$= 0.175 = \underline{17.5\%}$$ This ans is wrong bcz in this case n is not so large

Mean and Variance of Poisson Distribution:

$$\mu = \sum_{x=0}^{\infty} x \cdot f(x)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \lambda^{x}}{x!}$$

$$= e^{-\lambda} \left[\frac{0 \cdot \lambda^{0}}{0!} + \frac{1 \cdot \lambda^{1}}{1!} + \frac{2\lambda^{2}}{2!} + \frac{3\lambda^{3}}{3!} + \cdots \right]$$

$$= e^{-\lambda} \left[\lambda + \lambda^{2} + \frac{\lambda^{3}}{2} + \frac{\lambda^{4}}{6} + \cdots \right]$$

$$= \lambda e^{-\lambda} \left[1 + \lambda + \frac{\lambda^{2}}{2} + \frac{\lambda^{3}}{6} + \cdots \right]$$

$$= \lambda \cdot e^{-\lambda} \cdot e^{\lambda}$$

$$= \lambda \cdot e^{0}$$

$$= \lambda$$

$$= np \quad \text{mean}$$

$$\sigma = E(x^2) - \mu^2$$

$$= \sum_{x=0}^{\infty} x^2 f(x) - (np)^2$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda} \cdot \lambda^{x}}{x!} - (np)^2$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{x^2 \cdot \lambda^{x}}{x!} - (np)^2$$

$$\cancel{= e^{-\lambda} \left[\frac{0^2 \cdot \lambda^{0}}{0!} + \frac{1^2 \lambda^{1}}{1!} + \frac{2^2 \lambda^{2}}{2!} + \frac{3^2 \lambda^{3}}{3!} + \cdots \right] - (np)^2}$$

$$\cancel{= e^{-\lambda} \left[\lambda + 2\lambda^{2} + \frac{3}{2}\lambda^{3} + \right.}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(x^2 - x + x)\lambda^{x}}{x!} - (np)^2$$

$$= \left[e^{-\lambda} \sum_{x=0}^{\infty} \frac{(x^2 - x)\lambda^{x}}{x!} + e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \lambda^{x}}{x!} \right] - (np)^2$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{(x-2)!} + (np) - (np)^2$$

$$= e^{-\lambda}\left[\frac{\lambda^0}{(-2)!} + \frac{\lambda^1}{(-1)!} + \frac{\lambda^2}{0!} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \frac{\lambda^5}{3!} + \cdots\right] + np - (np)^2$$

$$= e^{-\lambda}\left[\lambda^2 + \lambda^3 + \frac{\lambda^4}{2!} + \frac{\lambda^5}{3!} + \cdots\right] + np - (np)^2$$

$$= \lambda^2 e^{-\lambda}\left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right] + np - (np)^2$$

$$= \lambda^2 . e^{-\lambda} . e^{\lambda} + np - (np)^2$$

$$= \lambda^2 + np - (np)^2$$

$$= n^2p^2 + np - n^2p^2$$

$$= np \quad // \quad \text{Variance}$$

Sampling

- With replacement
  - Bernouli's distribution
  - Binomial distribution
  - Geometric distribution
  - Negative Binomial distribution
- Without replacement
  - Hyper Geometric distribution

Bernouli Trial :

A trial or experiment having exactly 2 possible outcomes each called Bernouli Trial.

Ex i) In tossing a coin, getting head & tail are 2 possible outcomes.

ii) If we roll a dice, then getting a 6 not getting a 6 & not getting a 6. In this way we can get 2 outcomes for each success & failure.

Bernouli RV :

It is defined by X

$$\gamma = \begin{cases} 0 & \text{if } s = \text{failure} \\ 1 & \text{if } s = \text{success} \end{cases}$$

The mass function of Bernouli RV is defined by,

$$p(n) = \begin{cases} p & \text{if } n = 1 \\ q & \text{if } n = 0 \end{cases}$$

$\therefore$

$p + q = 1$

or, $q = 1 - p$

Q Toss a fair coin once & X = counts the no of heads (H)

Ans $\Omega = \{H, T\}$

$$X = \begin{cases} 0 & \text{if } S = H \\ 1 & \text{if } S = T \end{cases}$$

$$p(n) = \begin{cases} p = \frac{1}{2} & \text{if } n = 1 \\ p = \frac{1}{2} & \text{if } n = 0 \end{cases}$$

Q If rolling a dice S = getting a six and F = not getting a six.

Ans

$$p(x) = \begin{cases} \frac{1}{6} & \text{if } S = \text{getting a six} \\ \frac{5}{6} & \text{if } S = \text{not getting a six} \end{cases}$$

Mean :

$$\mu = \sum_{x \in X} x\, p(x)$$

$$= 0 \times p(0) + 1 \times p(1)$$

$$= p$$

$$= \Sigma(X)$$

K^{th} moment of X :

$$E(X^K) = \sum x^K p(x)$$

$$= 0^K . p(0) + 1^K . p(1) \quad [x \in X]$$

$$= p$$

Variance :

$$\sigma^2 = V(x) = E(x^2) - [E(x)]^2$$
$$= p - p^2$$
$$= p(1-p)$$
$$= pq$$

Binomial distribution is equivalent with 'n' independent Bernoulli trials with same probability of success.

If $x_1, x_2, \ldots, x_n$ are n independent Bernoulli trials with $P(S) = P$ $\Rightarrow \Sigma(x_1 + x_2 + \ldots + x_n)$

$$= \Sigma(x_1) + \Sigma(x_2) + \ldots + \Sigma(x_n)$$
$$= p + p + \ldots + n \text{ times } p$$
$$= np \quad (\text{Mean value of Binomial distribution})$$

Similarly,

$$V(x_1 + x_2 + \ldots + x_n)$$
$$= V(x_1) + V(x_2) + \ldots + V(x_n)$$
$$= pq + pq + \ldots + n \text{ times } pq$$
$$= npq$$

Hypergeometric Distribution : ($n \to$ small, $p \to$ large, without replacement)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = \begin{cases} P(B) \cdot P(A|B) & \text{if } P(B) > 0 \\ P(A) \cdot P(B|A) & \text{if } P(A) > 0 \end{cases}$$

$A \to 1^{st}$ ball is white

$B \to 2^{nd}$ ball is white

$P(\text{getting two white ball}) = P(A) \cdot P(B|A) = \frac{1}{5} \times \frac{1}{4} = \frac{1}{20}$

Total objects $= N$

Choose 'n' objects from N objects $p \cup q$ ways $({}^NC_n)$

→ No of type - I (defective) $= M$

→ No of ways to choose small x type objects from $M = ({}^MC_x)$

→ If there are two types of things then no of type 2 things $= (N-M)$

→ No of ways to choose $(n-x)$ type 2 things $\left({}^{N-M}C_{n-x}\right)$

→ P(getting x type 1 thing from n trials)

$$= \frac{({}^{M}C_{x})({}^{N-M}C_{n-x})}{({}^{N}C_{n})}$$

$$({}^{M}C_{x}) = \frac{M!}{x!\,(M-x)!} \Rightarrow x \geq 0 \;\&\; x \geq x$$

$$\Rightarrow 0 \leq x \leq M \longrightarrow (1)$$

$$\left({}^{N-M}C_{n-x}\right) = \frac{(N-M)!}{(n-x)!\,(N-M-n+x)!} \Rightarrow n \geq x$$

$$\& \; (N-M-n+x) \geq 0 \longrightarrow (2)$$

From (1) $0 \leq x \leq M$

From (2) $n+M-N \leq x \leq n$

$$\text{Max}\{0, n+M-N\} \leq x \leq \text{Min}\{M, n\}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(B).P(A|B), \quad P(B) > 0$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \Rightarrow P(A\cap B) = P(A).P(A|B), \quad P(A) > 0$$

Geometric Distribution (Negative Binomial Distribution)

$P(X=x)$ = Probability of 1st success in x^{th} trial.

experiments $\rightarrow$ $\underbrace{t_1, t_2, t_3, \ldots, t_{x-1}}_{\text{Failure}}, \underbrace{t_x}_{\text{Success}}$

$$P(X=x) = \{q.q.q \ldots (x-1) \text{ times } q\}.p$$
$$= q^{x-1}.p$$
$$= (1-p)^{x-1}.p$$

n is fixed in binomial distribution but not in geometric distribution.

$P(X=x)$ = Probability of rth success in x^{th} trial

$$= P(1^{st}(r-1) \text{ successes in } (x-1) \text{ trials}) \times P(r^{th} \text{ success in } x^{th} \text{ trial})$$
$$= b(r-1; x-1, p) \cdot p$$
$$= C(x-1, r-1) p^{r-1}.q^{x-r}.p$$
$$= C(x-1, r-1) p^{r}.q^{x-r}$$

$x = r, r+1, \ldots$

In general a geometric distribution can be thought of a negative binomial distribution with parameter $r = 1$.

Moment Generating Function

$G(t)$ = MGF with parameter t

If X is discrete rv then

$G(t) = \sum_x e^{tx} p(x)$, $p(x)$ = pmf for the rv X

$G'(t) = \sum x.e^{tx}.p(x)$

$G''(t) = \sum x^2 e^{tx}.p(x)$

$G'(0) = \sum x\,p(x) = E(X) = \mu$

$G''(0) = \sum x^2 p(x) = E(X^2)$

$$V(x) = E(x^2) - (E(x))^2$$

$$= G''(0) - [G'(0)]^2$$

MGF for the binomial distribution is $(pe^t + q)^n$.

Q Proof that the pmf for the binomial distribution is

$$p(x) = b(x; n, p) = \begin{cases} \binom{n}{x} p^x \cdot q^{n-x} & , x = 0, 1, \ldots n \\ 0 & , \text{otherwise} \end{cases}$$

The MGF is given by,

$$G(t) = \sum_x e^{tx} \cdot p(x) = \sum_{x=0} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0} \binom{n}{x} (pe^t)^x$$

$$= (pe^t + q)^n$$

$$G'(t) = n(pe^t + q)^{n-1} \cdot pe^t$$

$$G''(t) = n(n-1)(pe^t + q)^{n-2} \cdot pe^t \cdot pe^t + n(pe^t + q)^{n-1} \cdot pe^t$$

$$G'(0) = n(p+q)^{n-1} \cdot pe^0$$

$$= np$$

$$= E(x)$$

$$G''(0) = n(n-1)p^2 + np$$

$$= n^2p^2 - np^2 + np$$

$$V(x) = G''(0) - [G'(0)]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= npq$$

Q. Prove that mean and variance of Poisson distribution are equal using moment generating function.

Ans. Moment generating function for Poisson distribution,

$$G(t) = E(e^{tx})$$

Probability function for Poisson distribution is,

$$b(x;\mu) = \frac{e^{-\mu}.\mu^x}{x!} \quad x = 0, 1, 2, \ldots, n$$

$$G(t) = E(e^{tx}) = \sum e^{tx} f(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\mu}.\mu^x}{x!}$$

$$= e^{-\mu} \sum_{x=0}^{\infty} \frac{(\mu e^t)^x}{x!}$$

$$= e^{-\mu}.e^{\mu e^t}$$

The moment generating function for Poisson distribution is,

$$G(t) = e^{\mu e^t - \mu}$$

Mean:

$$\mu = G'(0) = \left.\frac{dG}{dt}\right|_{t=0}$$

$$\mu = \left[\frac{d}{dt} e^{\mu e^t - \mu}\right]_{t=0} = \left[e^{\mu e^t - \mu} \times \mu e^t\right]_{t=0} = \mu$$

Mean of the distribution $= \mu$

Variance:

$$V(x) = G''(0) - [G'(0)]^2$$

$$= G''(0) - \mu^2$$

$$G(t) = e^{\mu e^t - \mu}$$

$$\frac{dG}{dt} = \mu e^t . e^{\mu e^t - \mu}$$

or, $\frac{d^2G}{dt^2} = \mu e^t e^{\mu e^t - \mu} + (\mu e^t)^2 e^{\mu e^t - \mu} = G''(t)$

Now, $G''(0) = \mu + \mu^2$

$\therefore$ Variance $(V(x)) = G''(0) - \mu^2 = \mu + \mu^2 - \mu^2 = \mu$

$\therefore$ Hence mean & variance are the same

Determination of Mean & Variance from moment generating func

$$G(t) = E(e^{tx})$$

$$= E\left(1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right)$$

$$= E(1) + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \dots$$

$$\cancel{= E(1) +}$$

$$G(t) = 1 + \mu t + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \dots$$

$$\frac{dG}{dt} = \mu + tE(x^2) + \frac{t^2}{2}E(x^3) + \dots$$

$$\left.\frac{dG}{dt}\right|_{t=0} = \mu \qquad \frac{d^2G}{dt^2} = E(x^2) + tE(x^3) + \dots$$

$$\therefore \text{Mean} = \mu = G'(0) \qquad \left.\frac{d^2G}{dt^2}\right|_{t=0} = E(x^2)$$

$$\therefore \text{Variance} = E(x^2) = G''(0)$$

$$\sigma_x^2 = E(x^2) - [E(x)]^2$$

$$= E(x^2) - (\mu)^2$$

$$= G''(0) - [G'(0)]^2$$

From the MGF, mean and variance can be calculated by →

$$\mu = G'(0)$$

$$\sigma_x^2 = G''(0) - [G'(0)]^2$$

<u>Markov's Inequality</u>:

If the rv X takes non-negative values and mean(μ) of X is finite then for any $a > 0$

$$P(X \geq a) \leq \frac{E(x)}{a}$$ → This is called Markov's Inequality

<u>Note</u>:

① $P(x^k \geq a^k) \leq \frac{E(x^k)}{a^k}$

② $P((x-\mu)^k \geq a^k) \leq \frac{E(x-\mu)^k}{a^k}$

Chebyshev's Inequality :

$$P[(X-\mu)^2 \geq a^2] \leq \frac{E(X-\mu)^2}{a^2} = \frac{\sigma_x^2}{a^2}$$

$$\text{or, } P[|(X-\mu)| \geq a] \leq \frac{\sigma_x^2}{a^2}$$

$$a = K\sigma_x$$

$$\text{or, } P[|X-\mu| \geq K\sigma_x] \leq \frac{\sigma_x^2}{K^2\sigma_x^2} = \frac{1}{K^2} \quad K \geq 1$$

$$\text{or, } P[|X-\mu| \geq K\sigma_x] \leq \frac{1}{K^2}$$

This, is called Chebyshev's Inequality

Q $P[|X-\mu| \geq 3\sigma] \leq \frac{1}{9}$

Q Let μ & σ be mean and std. deviation of any rv X then prove that $P[|X-\mu| \geq K\sigma] \leq \frac{1}{K^2}$ $K \geq 1$

Q Prove that geometric distribution is a probability distribution

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Q The probability that an individual recovers from an illness in a 2 week period without medical treatment is 0.1 if 20 independent individuals suffering from this illness are treated with a medicine and ~~cp~~ two of ~~up~~ them recover in a 2 week period. If the medicine has no effect, what is the probability that 4 or more people recover in the 2 week period?

Ans $p = 0.1$ $\quad P(X \geq 4) = ?$ $\quad q = (1-0.1) = 0.9$

$n = 20$

Binomial Distribution

$$f(x) = P(X=x) = c(20,x)\,p^x q^{(20-x)}$$

$$= 1 - P(X<4)$$

$$= 1 - \{P(X=0) + P(X=1) + P(X=2) + P(X=3)\}$$

$$= 1 - [c(20,0)\,p^0 q^{20} + c(20,1)\,p^1 q^{19} + c(20,2)\,p^2 q^{18} + c(20,3)\,p^3 q^{17}]$$

$$= 1 - (0.122 + 0.270 + 0.285 + 0.190) = 0.133$$

$p = 0.1 \quad q = 0.9 \quad n = 20$

$$f(x) = P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$\lambda = np = 20 \times 0.1 = 2$$

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$\cancel{= 1 - \left[\frac{e^{-2}\lambda^{0}}{0!} + \frac{e^{-2}\lambda^{1}}{1!} + \frac{e^{-2}\lambda^{2}}{2!} + \frac{e^{-2}\lambda^{3}}{3!}\right]}$$

$$= 1 - \left[\frac{e^{-2}2^{0}}{0!} + \frac{e^{-2}\cdot 2^{1}}{1!} + \frac{e^{-2}\cdot 2^{2}}{2!} + \frac{e^{-2}\cdot 2^{3}}{3!}\right]$$

$$= 1 - [0.135 + 0.270 + 0.270 + 0.180]$$

$$= 0.1428$$

here $n \to$ large, $p \to$ small so apply PD.

Q A newspaper contains an average of 1.2 typographical error on any page. Selecting a page randomly what should be the probability of getting first bit exactly 3 typographical error? more than 3 i.e.? no error?

Ans When there's only mean value is given we have only one choice that is $\to$ PD

$$\lambda = 1.2$$

$$f(x) = P(X=x) = \frac{e^{-1.2}\cdot(1.2)^{x}}{x!}$$

i) $P(X=3) = \dfrac{e^{-1.2}\cdot(1.2)^{3}}{3!}$

ii) $P(X > 3) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - P(X \leq 3) = 1 - \left[\dfrac{e^{-1.2}\cdot(1.2)^{0}}{0!} + \dfrac{e^{-1.2}\cdot(1.2)^{1}}{1!} + \dfrac{e^{-1.2}\cdot(1.2)^{2}}{2!} + \dfrac{e^{-1.2}\cdot(1.2)^{3}}{3!}\right]$

$=$

iii) $P(X=0) = e^{-1.2}$

Q. A survey found that the traffic flowing through an intersection with an average of 3 cars for 30 seconds. i) Find the probability of 5 or more cars through the intersection within 30 sec. ii) Find the minimum no of cars through the intersection so that the probability of this no or fewer cars in 30 sec is at least 90%. iii) if the varience occurs per minute through the intersection is 20 then PD is appropriate?

Ans

$\lambda = 3$

$$P(X=x) = \frac{e^{-3} \cdot 3^x}{x!}$$

i) $P(X \geq 5) = 1 - P(X < 5)$

$$= 1 - [P(X=0) + \dots + P(X=4)]$$

$$= 1 - \left[\frac{e^{-3} \cdot 3^0}{0!} + \frac{e^{-3} \cdot 3^1}{1!} + \frac{e^{-3} \cdot 3^2}{2!} + \frac{e^{-3} \cdot 3^3}{3!} + \frac{e^{-3} \cdot 3^4}{4!}\right]$$

$$= 1 - [0.049 + 0.149 + 0.224 + 0.224 + 0.168]$$

$$= 0.186$$

ii) $P(X \leq x) > \cancel{0.09}\ 0.9$

$$\cancel{\text{or, } \frac{e^{-3} \cdot 3^x}{x!} > 0.09}$$

or, $P(X=0) + P(X=1) + \dots + P(X=x) > \cancel{0.09}\ 0.9$

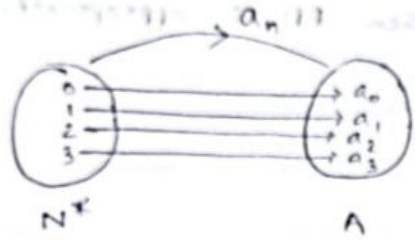
or, $\underbrace{0.049 + 0.149 + 0.224 + 0.224 + 0.168}_{0.834} + \dots > 0.9$ → This is not true

iii) When $\sigma^2 = 20$ then $p \to$ small, $n \to$ large then PD is not appropriate.

Generating Function :

let (a_n) be a sequance. It is a bijective function defined from N^* to A, $A \subseteq \mathbb{R}$

$N^* \rightarrow$ Whole number set $= \{0, 1, 2, 3, 4, \ldots\}$



So, a_0, a_1, a_2, a_3 are the terms of the sequance.

Ex

1) $a_n = \dfrac{n}{n+1}$ It's terms are $0, \frac{1}{2}, \frac{2}{3}, \frac{4}{5}, \ldots \rightarrow 1$

↳ Increasing Sequance

2) $a_n = \dfrac{1}{n+1}$ It's terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \rightarrow 0$

↳ Decresing Sequance

3) $a_n = (-1)^n$ It's terms are $1, -1, 1, -1, 1, -1, \ldots$

↳ Oscilatory Sequance

4) $a_n = 2^n$ It's terms are $1, 2, 2^2, 2^3, \ldots \rightarrow \infty$

↳ Divergent Sequance

① & ② $\rightarrow$ Convergent Sequance also!

let $a_1, a_2, a_3, a_4, \ldots$ are the terms of a sequance. These terms generate a power series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

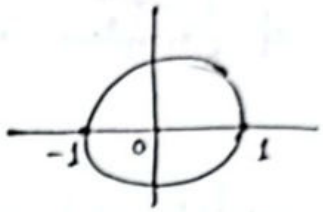
If this series is convergent and the corresponding function is set to be generating function and to be denoted by $A(x)$.

* $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots, \quad |x| < 1$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

$x = 1 \rightarrow$ the value of $\frac{1}{1-x}$ goes to ∞. So the singular point of $\frac{1}{1-x}$ is $\underline{x = 1}$.

Start from drawing a circle by considering the center at $\underline{x = 0}$ and draw it such that $\underline{x = 1}$ doesn't go inside the circle.... So the convergence lies b/w -1 to 1



Ex-1

$a^n = 2^n$

It's terms are $\underset{a_0}{1}, \underset{a_1}{2}, \underset{a_2}{2^2}, \underset{a_3}{2^3}, \underset{a_4}{2^4}, \dots$

Power Series $= 1 + 2x + 2^2x^2 + 2^3x^3 + 2^4x^4 + \dots$

$= 1 + (2x) + (2x)^2 + (2x)^3 + (2x)^4 + \dots$

$= \frac{1}{1-2x}, \quad \underline{-\frac{1}{2} < x < \frac{1}{2}}$

Ex-2

$a_n = (-1)^n$

It's terms are $1, -1, 1, -1, 1, -1, \dots$

$a_0, a_2, a_3, a_4, a_5, a_6, \dots$

$A(x) = 1 - x + x^2 - x^3 + \dots$

$= \frac{1}{1+x}, \quad -1 < x < 1$

Ex-3

$X=x$	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Terms of the finite sequance

1) Pdf of discrete random variable X denotes by,

$$f(x) = P(X=x)$$

$$\mu = E(X) = \sum x_i f(x_i)$$

$$E(X^2) = \sum x_i^2 f(x_i)$$

New Notation

$$a_i = P(X=i) = P_i$$

$X=i$	0	1	2	3	....
$P_i = P(X=i)$	P_0	P_1	P_2	P_3	....

New

$X=x$	x_0	x_1	x_2	...
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	...

old

Probability Generating Function (PGF):

let $a_i = P(X=i)$

$= P_i$ be the Pdf of the discrete RV of X.

So, it's terms are $P_0, P_1, P_2, \ldots$

Those terms generate a Power Series of the term

$$P_0 + P_1 s + P_2 s^2 + \ldots = \sum_{n=0}^{\infty} P_n s^n$$

If the series is convergent, in same interval then the corresponding is the PGF and to be denoted by $G(S)$.

Moment Generating Function:

let X be a discrete RV with pdf, $P(X=x) = P_n$

The Probability Generating Function is $G(s)$

where, $G(s) = P_0 + P_1 s + P_2 s^2 + P_3 s^3 + \dots$

Put $s = e^t$,

$$G(e^t) = P_0 + P_1 e^t + P_2 e^{2t} + P_3 e^{3t} + \dots$$

$$= \sum_{n=0}^{\infty} P_n e^{tn}$$

Now the moment generating function of the RV X with pdf $P(X=n) = P_n$ is denoted has $M_X(t)$ where

$$M_X(t) = G(e^t)$$

$$= \sum_{n=0}^{\infty} P_n e^{tn}$$

$$= E(e^{tn})$$

$E(X) = \sum x f(x)$

$E(X^2) = \sum x^2 f(x)$

$\vdots$

$E(X^r) = \sum x^r f(x)$

$M_X(t) = E(e^{tx})$

$= \sum f(x) e^{tx}$

Ex let X be a discrete RV with pdf $P(X=n) = q^{n-1} \cdot P$ $n = 1, 2, 3, \dots$ Find it's Moment Generating Function?

Ans. $M_X(t) = E(e^{tn})$

$$= \sum_{n=1}^{\infty} q^{n-1} \cdot P \cdot e^{tn}$$

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$\frac{1}{1-x} - 1 = x + x^2 + x^3 + \dots$

here, n is not finite

$$= \frac{P}{q} \sum_{n=1}^{\infty} q^n e^{tn}$$

$$= \frac{P}{q} \sum_{n=1}^{\infty} (qe^t)^n$$

$$= \frac{P}{q} \left\{ (qe^t) + (qe^t)^2 + (qe^t)^3 + \dots \right\}$$

$$= \frac{P}{q} \times \left[\frac{1}{1-qe^t} - 1 \right]$$

$$= \frac{P}{q} \left[\frac{1 - 1 + qe^t}{1 - qe^t} \right] = \frac{P}{q} \times \frac{qe^t}{1-qe^t} = \frac{pe^t}{1-qe^t}$$

Ex Find the moment generating function for binomial distribution.

Ans $P(X=x) = C(n,x)\, p^x q^{n-x}$ $\quad x = 0, 1, 2, \dots n$

n → no of trials → It's finite

$$M_x(t) = E(e^{tx})$$
$$= \sum_{x=0}^{n} C(n,x)\, p^x q^{n-x} \cdot e^{tx}$$
$$= \sum_{x=0}^{n} C(n,x)\, (pe^t)^x q^{n-x}$$
$$= C(n,0)(pe^t)^0 q^n + C(n,1)(pe^t)^1 q^{n-1} + C(n,2)(pe^t)^2 q^{n-2} + \dots + C(n,n)(pe^t)^n q^0$$
$$= (pe^t + q)^n$$

Ex Find the mgf for Poisson Distribution?

Ans $f(x) = P(X=x) = \dfrac{e^{-\lambda} \cdot \lambda^x}{x!}$ $\quad x = 0, 1, 2, \dots, \infty$

$$M_x(t) = E(e^{tx})$$
$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \cdot e^{tx}$$
$$= e^{-\lambda}\left[\frac{\lambda^0 e^0}{0!} + \frac{\lambda^1 e^t}{1!} + \frac{\lambda^2 e^{2t}}{2!} + \dots + \frac{\lambda^\infty e^\infty}{\infty!}\right]$$
$$= e^{-\lambda} \cdot e^{\lambda e^t}$$
$$= e^{\lambda(e^t - 1)}$$

Ex $P(X=n) = \dfrac{1}{2^n}$, $n = 1, 2, 3, \dots \infty$ Find the mgf.

Ans $f(x) = P(X=x) = \dfrac{1}{2^x}$

$$M_x(t) = E(e^{tx})$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot e^{tn}$$
$$= \sum_{n=1}^{\infty} \left(\frac{e^t}{2}\right)^n$$
$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \left(\frac{1}{1-\frac{e^{t/2}}{2}} - 1\right) = \left(\frac{1}{1-\frac{e^{t}}{2}} - 1\right)$$

$$= \frac{2-1+e^{t/2}}{2-e^{t/2}} = \frac{e^{t}}{2-e^{t}}$$

$$= \frac{1+e^{t/2}}{2-e^{t/2}}$$

Mean & Variance from Moment Generating Function.

$$E(x) = M_x'(0)$$

$$E(x^2) = M_x''(0)$$

$$E(x^3) = M_x'''(0)$$

$$E(x^r) = M_x^{(r)}(0) \longrightarrow r^{th} \text{ derivative}$$

Ex Find μ & σ^2 of Binomial distribution from mgf ?

Ans

$$M_x(t) = (q+pe^t)^n$$

$$M_x'(t) = n(q+pe^t)^{n-1} \cdot pe^t = npe^t(q+pe^t)^{n-1}$$

$$\therefore \mu = E(X) = M_x'(0) = npe^0(q+pe^0)^{n-1} = np.(q+p)^{n-1}$$

$$= np \qquad [\because q+p=1]$$

$$\sigma^2 = E(x^2) - \{E(x)\}^2$$

$$M_x''(t) = np\left[e^t(q+pe^t)^{n-1} + e^t(n-1)(q+pe^t)^{n-2}pe^t\right]$$

$$M_x''(0) = np[1.1 + 1.(n-1).1.p.1]$$

$$= np(1+np-p)$$

$$= np(q+np)$$

$$= npq + n^2p^2$$

$$\sigma^2 = M_x''(0) - \{M_x'(0)\}^2$$

$$= npq + n^2p^2 - (np)^2$$

$$= npq$$

Hypergeometric Distribution :

let X be a discrete random variable. P be the probability of success, n be the no of trials.

*) n is to be small without replacement.

*) Given there are n different things out of which K are defective.

So, There are N different things.
$(N-K)$ are non-defective.

n no of things are to be drawn

we want to find the probability of x items as defective

$$N \longrightarrow n \longrightarrow C(N, n)$$

$$\longrightarrow K \longrightarrow x \longrightarrow C(K, x)$$

$$\longrightarrow N-K \longrightarrow n-x \longrightarrow C(N-K, n-x)$$

Here X be the RV of getting defective items

So, the pdf will be,

$$f(x) = P(X=x) = \frac{C(K,x)\, C(N-K, n-x)}{C(N,n)}$$

$$P = \frac{K}{N}$$

Mean $(\mu) = np$

Variance $(\sigma^2) = np(1-P)\left(\frac{N-n}{N-1}\right)$

Q. A machine design class has 20 students. Out of the 20 students 11 of them are girls. 2 students are randomly selected. Calculate the pdf ?

Ans

$N = 20 \qquad N - K = 9$

$K = 11 \qquad n = 2$

$$f(x) = P(X = x) = \frac{C(K, x)\, C(N-K, n-x)}{C(N, n)}$$

$$= \frac{C(11, x)\, C(9, 2-x)}{C(20, 2)}$$

$X = x$	0	1	2
$f(x)$			

$$P(X=0) = \frac{C(11,0)\, C(9,2)}{C(20,2)} = \frac{18}{95}$$

$$P(X=1) = \frac{C(11,1)\, C(9,1)}{C(20,2)} = \frac{99}{190}$$

$$P(X=2) = \frac{C(11,2)\, C(9,0)}{C(20,2)} = \frac{11}{38}$$

Q. Suppose a box contains 5 red balls and 10 blue balls. If 7 balls are selected at random without replacement, find the probability that at least 3 red balls will be obtained ?

Ans X be the RV of getting red balls.

$N = 5 + 10 = 15$

$K = 5 \qquad N - K = 10$

$n = 7$

$$f(x) = P(X = x) = \frac{C(K, x)\, C(N-K, n-x)}{C(N, n)}$$

$$= \frac{C(5, x)\, C(10, 7-x)}{C(15, 7)}$$

$$P(X \geq 3) = 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

or $P(X=3) + P(X=4) + P(X=5)$

$$P(X=3) = \frac{c(5,3)\, c(10,4)}{c(15,7)}$$

$$P(X=4) = \frac{c(5,4)\, c(10,3)}{c(15,7)}$$

$$P(X=5) = \frac{c(5,5)\, c(10,2)}{c(15,7)}$$

$x = r$	0	1	2	3	4	5
$f(r)$						

$$\text{Mean}(\mu) = np = 7 \times \frac{1}{3} \qquad p = \frac{K}{N} = \frac{5}{15} = \frac{1}{3}$$

$$\text{Variance}(\sigma^2) = np(1-p)\left(\frac{N-n}{N-1}\right)$$

$$= 7 \times \frac{1}{3} \times \frac{2}{3} \times \left(\frac{15-7}{15-1}\right)$$

$$= \frac{8}{9}$$

Q: Out of 50 manufactured steel rods by a company it is found that 12 rods are defective. If 10 rods are selected at random for inspection. Then find the probability of exactly 3 rods are defective out of these 10 selected rods? Find the mean & variance of the distribution also?

Ans

$N = 50 \qquad N - K = 38$

$K = 12$

$n = 10$

$$f(r) = P(X=r) = \frac{c(12,r)\, c(38, 10-r)}{c(50,10)}$$

$$P(X=3) = \frac{c(12,3)\, c(38,7)}{c(50,10)}$$

$$p = \frac{12}{50}$$

$$\mu = 10 \times \frac{12}{50} = \frac{12}{5}$$

$$\sigma^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$$

$$= 10 \times \frac{12}{50} \times \left(1 - \frac{12}{50}\right) \times \left(\frac{50-10}{50-1}\right)$$

$$= \frac{1824}{1225} = 1.489$$

Continuous Random Variable

let X be a Continuous Random Variable with pdf $f(x)$.

So, i) $f(x) \geq 0 \quad \forall x$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Q Find the value of K for which the following func will represent a pdf.

$$f(x) = \begin{cases} Kx^2, & 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Ans

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{1} f(x)\,dx}^{\,0} + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \cancel{\int_{3}^{\infty} f(x)\,dx}^{\,0}$$

$$\Rightarrow \int_{1}^{3} f(x)\,dx = 1$$

$$\Rightarrow \frac{K}{3}\left(x^3\right)_1^3 = 1$$

$$\Rightarrow \frac{K}{3} \times 26 = 1$$

$$\Rightarrow K = \frac{3}{26}$$

Continuous Random Variable

Let X be a Continuous Random Variable with pdf $f(x)$.

So,

i) $f(x) \geq 0 \quad \forall x$

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Q Find the value of k for which the following func will represent a pdf.

$$f(x) = \begin{cases} kx^2, & 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}$$

Ans

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{1} f(x)\,dx}^{\,0} + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \cancel{\int_{3}^{\infty} f(x)\,dx}^{\,0}$$

$$\Rightarrow \int_{1}^{3} f(x)\,dx = 1$$

$$\Rightarrow \frac{k}{3}\left(x^3\right)_1^3 = 1$$

$$\Rightarrow \frac{k}{3} \times 26 = 1$$

$$\Rightarrow k = \frac{3}{26}$$

- The cummulative distribution function is $F(x)$ where $F(x) = \int_{-\infty}^{\infty} f(t)\,dt$ if the pdf is given.
- If cdf of $F(x)$ is given, then the pdf of $f(x)$ will be found out by $f(x) = \frac{d}{dx} F(x)$
- For the continuous RV, X, the probability at a point is to be always 0.

i.e. $P(X = a) = 0$

But,

$$P(a < x < b) = \begin{cases} \int_a^b f(t)\,dt \\ F(b) - F(a) \end{cases}$$

- $P(a < x < b) = P(a < x < b)$
  $= P(a < x \leq b)$
  $= P(a \leq x \leq b)$

- Also, $F(x) = P(X \leq x)$

$$\mu = E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(x^3) = \int_{-\infty}^{\infty} x^3 f(x)\, dx$$

$$E(x^k) = \int_{-\infty}^{\infty} x^k f(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

(or)

$$\int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

(or)

$$E(x^2) - [E(x)]^2$$

Q The pdf of the RV X is given by,

$$f(x) = \begin{cases} \dfrac{e^{-x/4}}{4} & , x \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

1. Find the prob. of $P(2 \leq x \leq 4)$
2. Find the cdf.
3. Determine the μ and σ^2.

Ans 2. $F(x) = \int_{-\infty}^{x} f(t)\, dt$

$$= \cancel{\int_{-\infty}^{0} f(t)\, dt}^{\,0} + \int_{0}^{x} f(t)\, dt$$

$$= \int_{0}^{x} \frac{e^{-t/4}}{4}\, dt$$

$$= \frac{1}{4}\left[\frac{-e^{-t/4}}{(-1/4)}\right]_0^x$$

$$= -\left(e^{-x/4}\right)_0^x$$

$$= -\left(e^{-x/4} - 1\right)$$

$$= 1 - e^{-x/4}$$

1. $P(2 \leq x \leq 4)$

<u>Using pdf</u>,

$$P(2 \leq x \leq 4) = \int_2^4 f(x)\,dx$$

$$= \int_2^4 \frac{e^{-x/4}}{4}\,dx$$

$$= -\left(e^{-x/4}\right)_2^4$$

$$= -\left(e^{-1} - e^{-1/2}\right)$$

$$= e^{-1/2} - e^{-1}$$

<u>Using cdf</u>,

$$P = F(4) - F(2)$$

$$= \left(1 - e^{-1}\right) - \left(1 - e^{-1/2}\right)$$

$$= e^{-1/2} - e^{-1}$$

3. $$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \cancelto{0}{\int_{-\infty}^{0} x \cdot \frac{e^{-x/4}}{4}\,dx} + \int_0^{\infty} x \frac{e^{-x/4}}{4}\,dx$$

$$= \frac{1}{4}\int_0^{\infty} x \cdot e^{-x/4}\,dx$$

$$= \frac{1}{4}\left[(x)\left(\frac{e^{-x/4}}{-1/4}\right) - (1)\left(\frac{e^{-x/4}}{1/16}\right)\right]_0^{\infty}$$

$$= -\left(xe^{-x/4}\right)_0^{\infty} - 4\left(e^{-x/4}\right)_0^{\infty}$$

$$= -(0-0) - 4(0-1)$$

$$= 4$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - [4]^2$$

$$= \frac{1}{4}\int_0^{\infty} x^2 \cdot e^{-x/4}\,dx - [4]^2$$

$$= \frac{1}{4}\left[(x^2)\left(\frac{e^{-x/4}}{-1/4}\right) - (2x)\left(\frac{e^{-x/4}}{+1/16}\right) + (2)\left(\frac{e^{-x/4}}{-1/64}\right)\right]_0^{\infty}$$

$$= -\left(x^2 e^{-x/4}\right)_0^\infty - \left(8x e^{-x/4}\right)_0^\infty - \left(32 x e^{-x/4}\right)_0^\infty$$

$$= 0 - 0 - 32(0-1)$$

$$\sigma^2 = 32 - 16$$

$$= 16$$

* $\int 1st \times 2nd = \int uv$ → differentiate, → Integrate

$$= uv^2 - u'v'' + u''v''' - u'''v''''$$

$$\int x^2 \sin 3x\,dx = x^2\left(-\frac{\cos 3x}{3}\right) - (2x)\left(-\frac{\sin 3x}{9}\right) + 2\left(\frac{\cos 3x}{27}\right)$$

Q $f(x) = \begin{cases} \frac{3x^2}{2}, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

Ans $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \int_{-1}^{x} \frac{3t^2}{2}\,dt$$

$$= \left(\frac{t^3}{2}\right)_{-1}^{x}$$

$$= \frac{1}{2}\left(x^3 - (-1)^3\right)$$

$$= \frac{x^3+1}{2}$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-1}^{1} x \cdot \frac{3x^2}{2}\,dx = \frac{3}{2}\left(\frac{x^4}{4}\right)_{-1}^{1} = \frac{3}{8}(1-1) = 0$$

$$\sigma^2 = E(x^2) - \cancel{[E(x)]^2}^{\,0}$$

$$= E(x^2)$$

$$= \int_{-1}^{1} x^2 \cdot \frac{3x^2}{2}\,dx$$

$$= \frac{3}{2} \cdot \int_{-1}^{1} x^4\,dx$$

$$= \frac{3}{2} \cdot 2\int_{0}^{1} x^4\,dx = \frac{3}{5}\left(x^5\right)_0^1 = \frac{3}{5}$$

Normal Distribution:

Gamma Function:

*) For any real value n except $n = 0, -1, -2, -3, \ldots$

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$$

*) In particular,

For $n = 0, 1, 2, 3, \ldots$ $\Gamma(n+1) = n!$

$\Gamma(n+1) = n\Gamma(n)$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

*) $\int_0^\infty x^{1/2} \cdot e^{-x} dx = \Gamma\left(\frac{3}{2}\right)$ $\quad \left[\begin{array}{l} n-1 = \frac{1}{2} \\ n = \frac{3}{2} \end{array}\right]$

$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{2} \cdot \sqrt{\pi}$

*) $\int_0^\infty e^{-x^2} dx$ $\quad \left[\begin{array}{l} x^2 = z \\ 2x dx = dz \\ dx = \frac{dz}{2x} = \frac{dz}{2\sqrt{z}} = \frac{z^{-\frac{1}{2}} dz}{2} \end{array}\right.$

$= \int_0^\infty \frac{e^{-z} \cdot z^{-\frac{1}{2}} dz}{2}$

$= \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)$

$= \frac{\sqrt{\pi}}{2}$

*) Let x be a continuous RV. It is to be normal when it's pdf is $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Ans $\int_{-\infty}^{\infty} f(x) dx = 1$

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} \cdot \sigma dz = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$$

$\frac{x-\mu}{\sigma} = z$

$x = \mu + \sigma z$

$dx = \sigma dz$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^\infty e^{-z^2/2}\, dz$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-p} \cdot \frac{dp}{\sqrt{2p}}$$

$$= \frac{1}{\pi} \int_0^\infty e^{-\frac{1}{2}} e^{-p}\, dp$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$$\Gamma(n) = \int_0^\infty e^{-p} \cdot p^{n-1}\, dp$$

$$n-1 = \frac{1}{2}$$

$$n = \frac{3}{2}$$

<u>odd func</u>

$$f(-x) = -f(x)$$

<u>Even func</u>

$$f(-x) = f(x)$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0 \\ 2\int_0^a f(x)\,dx \end{cases}$$

$$\frac{z^2}{2} = p$$

$$\text{or, } z^2 = 2p$$

$$\text{or, } dz = \frac{dp}{\sqrt{2p}}$$

<u>Mean</u>:

$$\mu \text{ or } E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-z^2/2}\, dz \cdot \sigma$$

$$\frac{x-\mu}{\sigma} = z$$

$$x = \mu + \sigma z$$

$$dx = \sigma\, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu e^{-z^2/2}\, dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma z e^{-z^2/2}\, dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz + \frac{\sigma}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} z e^{-z^2/2}\, dz}_{\text{odd}} \to 0$$

$$= \frac{\mu}{\sqrt{2\pi}} 2 \int_0^\infty e^{-z^2/2}\, dz$$

$$= \mu \cdot \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-z^2/2}\, dz$$

$$= \mu \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$= \mu$$

$$f(z) = z e^{-z^2/2}$$

$$f(-z) = -z \cdot e^{-(-z)^2/2}$$

$$= -z \cdot e^{-z^2/2}$$

$$= -f(z)$$

Variance

$$E(x^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu+\sigma z)^2 \cdot e^{-z^2/2} \cancel{\sigma}\, dz$$

$$= \frac{\mu^2}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz + \frac{2\mu\sigma}{\cancel{\sigma}\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} \cancel{z e^{-z^2/2}}}_{\text{odd} \to 0} dz$$

$$+ \frac{\sigma^2}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \cdot e^{-z^2/2} dz$$

$\frac{x-\mu}{\sigma} = z$

or, $x = \mu + \sigma z$

or, $dx = \sigma dz$

$$= \frac{\mu^2}{\sqrt{2\pi}} 2\int_0^{\infty} e^{-z^2/2} dz + \frac{\sigma^2}{\sqrt{2\pi}} 2\int_0^{\infty} z^2 \cdot e^{-z^2/2} dz$$

$$= \mu^2 \cdot \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-z^2/2} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$= \mu^2 \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}} + \sigma^2 \cdot \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{2}}$$

$$= \mu^2 + \sigma^2$$

$$\therefore Var(x) \cancel{\text{ or } \sigma^2} = E(x^2) - \mu^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

If X is the Normal RV, then the area under the normal curve = 1 $\longrightarrow \int_{-\infty}^{\infty} f(x)\, dx = 1$

For the purpose of problem solving, if X is the Normal RV, we have to convert it in form of standard ~~stan~~ normal random variable (Z)

$$Z = \frac{(x-\mu)}{\sigma}$$

And for standard normal RV, $\mu = 0$ & $\sigma^2 = 1$

Moment Generating Function :

Let X be a continuous Random variable with pdf $f(x)$. So the MGF will be,

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\,dx$$

X be a discrete Random Variable.

$$M_x(t) = E(e^{tx})$$

$$= \sum e^{tx} f(x)$$

$$\mu'_r = \frac{d^r M_x(0)}{dt^r}$$

$M_x(t)$ for Normal Distribution :

Let X be the normal RV with pdf

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} \cdot e^{-z^2/2} \sigma . dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\mu + t\sigma z} \cdot e^{-z^2/2} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma z t - \frac{z^2}{2}} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2\sigma z t)} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\frac{1}{2}\left\{(z^2 - 2\sigma z t + \sigma^2 t^2) - \sigma^2 t^2\right\} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\frac{1}{2}(z - \sigma t)^2 e^{\frac{\sigma^2 t^2}{2}} dz$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \cdot e^{\sigma^2 t^2/2} \int_{-\infty}^{\infty} e^{-1/2(z-\sigma t)^2} dz$$

$$= \frac{e^{t\mu + (\sigma^2 t^2)/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-p^2/2} dp.$$

$$\boxed{\begin{aligned} z - \sigma t &= p \\ dz &= dp \end{aligned}}$$

$$= \frac{e^{t\mu + (\sigma^2 t^2)/2}}{\sqrt{2\pi}} \cdot 2 \cdot \sqrt{\frac{\pi}{2}} = e^{t\mu + (\sigma^2 t^2)/2}$$

Q. Find the Mean & Variance of Normal Distribution from its moment generating function.

Ans We know,

$$\mu = M_x'(0)$$

$$\sigma^2 = M_x''(0) - \{M_x'(0)\}^2$$

$$\text{Mean } (\mu) = M_x'(0) = \frac{d\left(e^{t\mu + (\sigma^2 t^2)/2}\right)}{dt} = \mu$$

$$M_x'(t) = \left(\mu + \frac{2\sigma^2 t}{2}\right) e^{\left(t\mu + \sigma^2 t^2/2\right)}$$

$$M_x''(t) = \sigma^2\left(e^{t\mu + \sigma^2 t^2/2}\right) + (\mu + \sigma^2 t)(\mu + \sigma^2 t) e^{(t\mu + \sigma^2 t^2/2)}$$

$$= \sigma^2\left(e^{t\mu + \sigma^2 t^2/2}\right) + (\mu + \sigma^2 t)^2 e^{(t\mu + \sigma^2 t^2/2)}$$

$$M_x''(0) = \sigma^2\left(e^{(0 + \sigma^2(0)/2)}\right) + (\mu + \sigma^2(0))^2 + e^{(0+0)}$$

$$= \sigma^2(e^0) + \mu^2(e^0)$$

$$= \sigma^2 + \mu^2$$

$$M_x'(0) = \mu.(e^0) = \mu$$

$$\sigma^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$$

For Problem Solving:

Let X be the n.m.v (Normal Random Variable) with mean μ and standard deviation σ.

So pdf is,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$$

It's cummulative distribution function is,

$$F(x) = f(x) = \int_{-\infty}^{\infty} f(t)\, dt$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

• $F(X) = P(X \leq x)$ when the normal random variable when $\mu = 0$ & $\sigma = 1$ when the normal random variable x is said to be standard normal variable and to be denoted by 'z' where $z = \frac{x-\mu}{\sigma}$

• $F(z) = P(Z \leq z)$ will be converted to $\phi(z)$ i.e;

$\phi(z) = P(Z \leq z)$

• $P(Z > z) = 1 - P(Z \leq z) = 1 - \phi(z)$

$P(a < Z < b) = \phi(b) - \phi(a)$

Ex If 'X' be the normal random variable with mean 1 and standard deviation 2. Find 1) $P(X \leq 3)$ 2) $P(X > 1.5)$ 3) $P(X = 1)$ 4) $P(2 < X < 5)$

Sol^n: $z = \frac{x-\mu}{\sigma}$ $[\mu = 1 ; \sigma = 2]$

$z = \frac{x-1}{2}$ so,

1) $P(X \leq 3) = P(Z \leq \frac{3-1}{2})$

$= P(Z \leq 1)$

$= \phi(1)$

2) $P(X > 1.5) = P(Z > 0.25)$

$= 1 - P(Z \leq 0.25)$

$= 1 - \phi(0.25)$

3) $P(X = 1) = 0$

4) $P(2 < X < 5) = P(0.5 < Z < 2)$

$= \phi(2) - \phi(2.5)$

Normal Approximation to Binomial Distribution:

let X be a discrete random variable with Binomial Distribution where n is large and neither p nor q is small i.e., np & $nq \geq 10$. In such case the binomial distribution (bd) is supposed to be approximated by Normal Distribution.

Geometric Distribution :

Suppose we perform an experiment and concerned only about the occurance of the no of success or failure. As in case of binomial distribution the experiment is repeated for many times so that the repeatations are independent and each repeatation gives probability of success as p and failure as q. But unlike binomial distribution where we repeat the experiment for a fixed no of times irrespective of the outcome. Let us repeat the experiment until the success occurs for first time.

Let X be the discrete RV which denotes the no of repeatations required upto and including the first occurance of success. Thus X will assume the values $1, 2, 3, \ldots$

If $X = k$ if and only the kth occurance is success and the first $(k-1)$ occurance are failures.

So the pdf will be $P(X=n) = q^{n-1} p$

$$\text{Mean} = \frac{1}{p}$$

$$\text{Variance} = \frac{q}{p^2}$$

$$M_x(t) = \frac{pe^t}{1-qe^t}$$

Q You play a game of chance that you can either win or loss (no other possibilities). You play this game until you loss. Your probability of loosing the game is 0.57. What is the probability that it takes 5 games until you loose?

Ans

$$q\,p = 0.57 \qquad n = 5$$

$$p\,q = (1 - 0.57) = 0.43$$

$$P(X=5) = (0.43)^{5-1} \cdot (0.57)^{5-1} =$$

$$\text{Mean} = \frac{1}{0.57} \quad \frac{1}{0.43}$$

$$\text{Variance} = \frac{0.43}{(0.57)^2} \quad \frac{0.57}{(0.43)^2}$$

p = prob. of success
↓
lose

$$q = 0.57 \quad \Big| \quad n = 5$$
$$p = 0.43$$

$$\therefore P(X=5) = p^{n-1} \cdot q = (0.43)^{5-1} \cdot (0.57)$$

$$\text{Mean} = \frac{1}{0.57}$$

$$\text{Variance} = \frac{0.43}{(0.57)^2}$$

$$M_x(t) = \frac{pe^t}{1 - qe^t}$$

Negative Binomial Distribution

The experiment consists of a sequance of independent trials. Each trial can result in two ways such as success and failure. The probability of success is constant from trial to trial. The experiment continues until 'x' no of success is observed.

p = prob of success

'x' - no of success are to be observed.

The pdf is to be denoted by Negative Binomial Distribution $\longrightarrow$ $nb(x; r, p)$

$$= \left(\frac{x+r-1}{r-1}\right) p^r (1-p)^x$$

$$\mu = \frac{rq}{p}$$

$$\sigma^2 = \frac{rq}{p^2}$$

Q If a boy is throwing stones at a target, what is the probability that 10th throw will be his 5th success if the prob. of hitting the target is $1/2$.

Ans $p = \frac{1}{2}$, $x + r = 10$, $x = 5$, $r = 10 - 5 = 5$

$$p(x=5) = nb(5; 5+10; \tfrac{1}{2})$$

$$= \cancel{\left(\frac{5+10-1}{10-1}\right)\left(\frac{1}{2}\right)^{10}\left(1-\frac{1}{2}\right)^5}$$

$$= \left(\frac{5+5-1}{5-1}\right)(0.5)^5(1-0.5)^5$$

$$=$$

In general binomial distribution Mean > Variance

" Negative " " Mean < Variance

Uniform Distribution :

Let X be a continuous RV defined in the interval $[a, b]$ where it's pmf or pdf is,

$$P(X = x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Continuous RV :

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$\mu = \cancel{\int_{-\infty}^{a} x f(x)\, dx}^{\,0} + \int_{a}^{b} x f(x)\, dx + \cancel{\int_{b}^{\infty} x f(x)\, dx}^{\,0}$$

$$= \int_{a}^{b} x f(x)\, dx$$

$$= \frac{1}{b-a} \left(\frac{x^2}{2} \right)_a^b$$

$$= \frac{b^2 - a^2}{2} \times \frac{1}{b-a}$$

$$= \frac{b+a}{2}$$

$$E(x^2) = \int_{-\infty}^{a} x^2 f(x)\, dx + \int_{a}^{b} x^2 f(x)\, dx + \int_{b}^{\infty} x^2 f(x)\, dx$$

$$= \int_{a}^{b} x^2 f(x)\, dx$$

$$= \frac{1}{b-a} \left(\frac{x^3}{3} \right)_a^b$$

$$= \frac{1}{3(b-a)} (b^3 - a^3)$$

$$= \frac{b^2 + ba + a^2}{3}$$

$$\sigma^2 = E(x^2) - [E(x)]^2$$

$$= \frac{b^2 + ba + a^2}{3} - \left(\frac{b+a}{2} \right)^2 = \frac{a^2 + b^2 + ab}{3} - \frac{a^2 + b^2 + 2ab}{4}$$

Q. Suppose the reaction temperature X in a certain chemical process has a uniform distribution.

If $a = -5$, $b = 5$ then compute 1) $P(X < 10)$

2) $P(-2.5 < X < 2.5)$

3) $P(-2 \leq X \leq 3)$

4) For k satisfying $-5 < k < k+4 < 5$ compute $P(k < X < k+4)$

Ans

$$pdf = f(x) = \begin{cases} \frac{1}{b-a} = \frac{1}{+5+5}, & -5 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0.1, & -5 < x < 5 \\ 0, & \text{otherwise} \end{cases}$$

1) $$P(X<10) = \int_{-\infty}^{10} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{-5} f(x)\,dx}^{\,0} + \int_{-5}^{5} f(x)\,dx + \cancel{\int_{5}^{10} f(x)\,dx}^{\,0}$$

$$= \int_{-5}^{5} f(x)\,dx$$

$$= (0.1).[x]_{-5}^{5}$$

$$= 0.1 \times 10$$

$$= 1$$

2) $$P(-2.5 < x < 2.5) = \int_{-2.5}^{2.5} f(x)\,dx$$

$$= (0.1).[x]_{-2.5}^{2.5}$$

$$= 0.1 \times 5$$

$$= 0.5$$

3) $P(-2 \leq x \leq 3) = \int_{-2}^{3} f(x)\,dx$

$= (0.1) . [x]_{-2}^{3}$

$= 0.1 \times 5$

$= 0.5$

4) $-5 < K < K+4 < 5$

$P(K < x < K+4) = \int_{K}^{K+4} f(x)\,dx$

$= (0.1)\,[x]_{K}^{K+4}$

$= (0.1)\,(K+4-K)$

$= 0.1 \times 4$

$= 0.4$

Q let X be the time lapses between the end of the hour and the end of the lecture. If pdf is $f(x) = \begin{cases} Kx^2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

i) Find the value of K

ii) Find the prob. of the lecture ends within 1 minute of the end of the hour.

iii) Find the prob. of the lecture continues beyond the hour for b/w 60 to 90 seconds

iv) Find the prob. of the lecture continuing for at least 90 sec. beyond the end of the hour.

Ans i) pdf $= \int_{-\infty}^{\infty} f(x)\,dx = 1$

or, $\cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_{0}^{2} f(x)\,dx + \cancel{\int_{2}^{\infty} f(x)\,dx}^{\,0} = 1$

or, $\int_0^2 f(x)\,dx = 1$

or, $\int_0^2 kx^2\,dx = 1$

or, $\frac{k}{3}\cdot\left[x^3\right]_0^2 = 1$

or, $\frac{k}{3}(8-0) = 1$

or, $k = \frac{3}{8}$

So, $f(x) = \begin{cases} \frac{3}{8}x^2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$

ii) $P(X \leq 1) = \int_{-\infty}^{1} f(x)\,dx$

$= \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_0^1 f(x)\,dx$

$= \int_0^1 f(x)\,dx$

$= \int_0^1 \frac{3}{8}x^2\,dx$

$= \frac{3}{8}\left(\frac{x^3}{3}\right)_0^1$

$= \frac{1}{8}$

iii) $P(1 < x < 1.5) = \int_1^{1.5} f(x)\,dx$

$= \int_1^{1.5} \frac{3}{8}x^2\,dx$

$= \frac{3}{8\times 3}\left(x^3\right)_1^{1.5}$

$= \frac{1}{8}\times 2.375$

$= 0.297$

iv) $P(x > 1.5)$ beyond than

$= 1 - P(X \leq 1.5) = 1 - \int_{-\infty}^{1.5} f(x)\,dx = 1 - \left[\cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_0^{1.5} f(x)\,dx\right]$

$= 1 - \int_0^{1.5} \frac{3}{8}x^2\,dx$

$= 1 - 0.296 = 0.704$

Exponential Distribution :

let X be a continuous RV. It posses exponential distribution when, it's PGF is,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Area under exponential distribution curve is,

$$\int_{-\infty}^{\infty} f(x)\,dx = \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_{0}^{\infty} f(x)\,dx$$

$$= \int_{0}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \cdot \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

$$= -\left[e^{-\lambda x}\right]_0^{\infty}$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -\left[0 - 1\right]$$

$$= 1$$

Mean (μ) :

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x f(x)\,dx + \int_{0}^{\infty} x f(x)\,dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \int \frac{e^{-\lambda x}}{-\lambda}\,dx \right]_0^{\infty}$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^{\infty}$$

$$= \left[-x e^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda} \right]_0^{\infty}$$

$$= \frac{1}{\lambda}$$

Variance (σ^2)

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$= \int_{0}^{\infty} x^2 \cdot \lambda e^{-\lambda x}\,dx - \mu^2$$

$$= \lambda\left[x^2 \cdot \frac{e^{-\lambda x}}{-\lambda} - 2x\frac{e^{-\lambda x}}{\lambda^2} + \frac{2e^{-\lambda x}}{-\lambda^3}\right]_0^{\infty} - \mu^2$$

$$= \left[-x^2 e^{-\lambda x} - 2x \cdot \frac{e^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}\right]_0^{\infty} - \mu^2$$

$$= \left(\frac{2}{\lambda^2} - \frac{1}{\lambda^2}\right)$$

$$= \frac{1}{\lambda^2}$$

CDF :

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = P(X \le x)$$

The CDF of exponential distribution is denoted by,

$$F(x,\lambda) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \int_{-\infty}^{0} \cancel{f(t)\,dt}^{\,0} + \int_{0}^{x} f(t)\,dt$$

$$= \int_{0}^{x} \lambda e^{-\lambda t}\,dt$$

$$= 1 - e^{-\lambda x}$$

$$F(x,\lambda) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

$P(a \leq x \leq b) = F(b;\lambda) - F(a;\lambda)$

$P(x \leq a) = F(a,\lambda)$

Q. If X be a continuous RV having exponential distribution with mean$(\mu) = 6$. Find ~~prob~~ $P(x \leq 10)$, $P(5 \leq x \leq 10)$, $P(x > 5)$.

Ans $\mu = \frac{1}{\lambda} = 6 \Rightarrow \lambda = \frac{1}{6}$

$P(x \leq 10) = 1 - e^{-\frac{1}{6}\times 10}$

(CDF) $= 0.811$

or

$P(x \leq 10) = \int_{-\infty}^{10} \lambda e^{-\lambda x} = \int_{-\infty}^{10} \frac{1}{6} e^{-\frac{1}{6}\times 10}$

(PDF) $= 0.811$

$P(5 \leq x \leq 10) = P(10,\lambda) - P(5,\lambda)$

$= \left(1 - e^{-\frac{1}{6}\times 10}\right) - \left(1 - e^{-\frac{1}{6}\times 5}\right)$

$= 0.811 - 0.565$

$= 0.246$

$P(x > 5) =$

No difference b/w '<' and '≤' in continuous RV

Q. Let X be the time between two successive arrivals at the drive-up window for a local train. If X has exponential distribution with $\lambda = 1$, then compute the following:

i) Expected ~~time~~ value ii) Standard deviation

iii) $P(x \leq 4)$ iv) $P(2 < x < 5)$

Ans Expected value $= 1$

Std. deviation $= 1$

$P(x \leq 4) = (1 - e^{-4})$

$P(2 \leq x \leq 5) = F(5,\lambda) - F(2,\lambda) = (1 - e^{-5}) - (1 - e^{-2}) =$

Central Limit Theorem :

Many commonly used statistical method depends on the CLT. This theorem states that if a large enough sample is drawn from a population. Then the distribution of the sample is approximately normal. No matter from what population the sample is drawn. If the variable is normally distributed then its mean ($\bar{X}$)

Let $x_1, x_2, x_3, \ldots x_n$ be a simple random sample from the population with mean (μ) and std. deviation(σ) From statistics we know,

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

As in Normal Distribution to solve the problem, we are converting ~~std.~~ normal ~~varient~~ varience to standard normal varience

$$Z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}/\sqrt{n}}$$

~~σ or $\sigma_{\bar{x}}$~~

Q A Normal population has mean 80 and varience 25. ~~How longer must be the random sample~~. If there is a standard error of 1.5 then what should be the size of the random sample ?

Ans

$$\left.\begin{aligned} \mu &= 80 \\ \sigma^2 &= 25 \\ \text{or, } \sigma &= 5 \end{aligned}\right\}$$

$\sigma_{\bar{x}} = 1.5$

or, $\frac{\sigma}{\sqrt{n}} = 1.5$

or, $\frac{5}{\sqrt{n}} = 1.5$

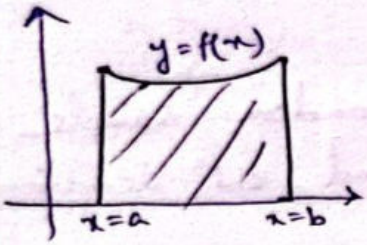
or, $n = \frac{2.5}{2.25} =$

let X be the normal RV with normal samples $x_1, x_2, \dots, x_n$.

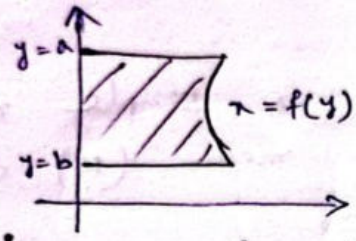
Denote the mean as $\mu_{\bar{x}}$ and std. deviation as $\sigma_{\bar{x}}$

where $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ [n is the size of the population]

Multiple Integral (Volume Integral)



Area under the curve $y = f(x)$ upon x-axis from $x = a$ to $x = b$ $= \int_a^b f(x)\,dx$



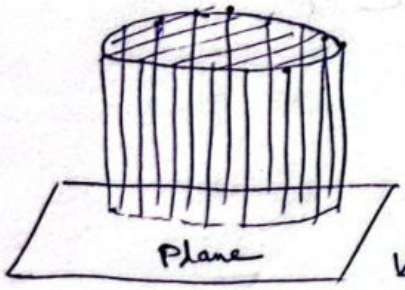
Area under the curve $x = f(y)$ upon y axis from $y = a$ to $y = b$ $= \int_a^b f(y)\,dy$

- When a path is traced in 2D, it is called curve.
- When a path is traced in 3D, it is called surface.
- Plain ⇒ Curve (2D)
  Dependent, $y = f(x)$
- Surface → (3D)
  $z = f(x, y)$ ← Dependent

When a surface is projected over a plane, it is known as Multiple Integral.

Double Integral:

$dR = dx\,dy$ or $dy\,dx$



Formula ⇒ $\iint_R f(x, y)\,dR$

Form-I

$$\int_a^b \int_{y_1}^{y_2} f(x, y)\,dy\,dx$$

$$\int_a^b \int_{\phi(x)}^{f(x)} f(x, y)\,dy\,dx$$

Inner limit → $f(y) = x$
Inner variable → y
Outer " → x

Form-II

$$\int_a^b \int_{x_1}^{x_2} f(x, y)\,dx\,dy$$

$$\int_a^b \int_{\phi(y)}^{f(y)} f(x, y)\,dx\,dy$$

Inner limit → $f(x) = y$
Inner variable → x
Outer " → y

Form - I

First integrate w.r.t 'y' Keeping 'x' constant

Form - II

First integrate w.r.t 'x' Keeping 'y' constant.

Q

$$\int_0^1 \int_0^x (x^2+y^2)\, dA = \int_0^1 \int_0^x (x^2+y^2)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)_0^x dx$$

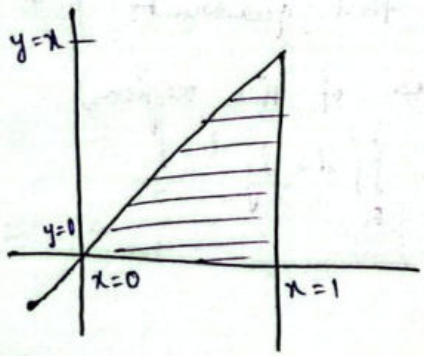
$$= \int_0^1 \left(x^2(x-0) + \frac{1}{3}(x-0)^3\right) dx$$

$$= \int_0^1 \left(x^3 + \frac{x^3}{3}\right) dx$$

$$= \frac{4}{3}\int_0^1 x^3\, dx$$

$$= \frac{4}{3} \times \frac{x^4}{4}$$

$$= \frac{1}{3}$$



Limit → $y \to 0$ to x

$y = 0$ to $y = x$

$x = 0$ to $x = 1$

$$\iint_R f(x,y)\, dR \quad ; \quad Z = f(x,y)$$

$\boxed{R}$

$f(x,y) = 1$ $\quad dx\,dy \to$ Area

Ex

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\, dy$$

Here $f(x,y) = 1$

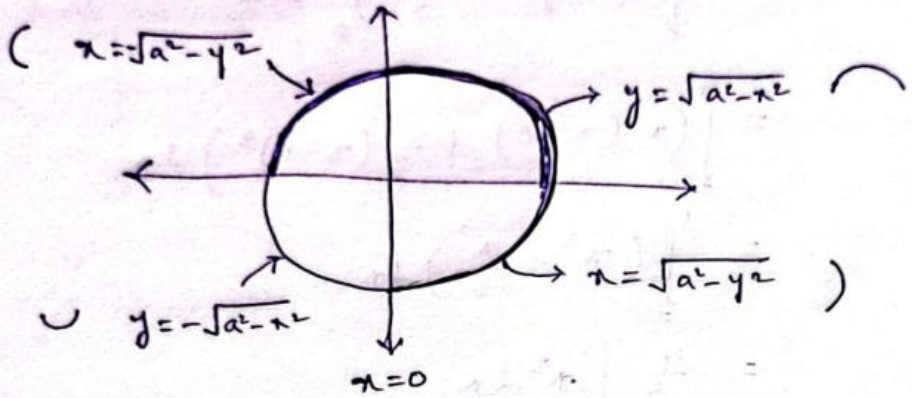
so it's area not volume

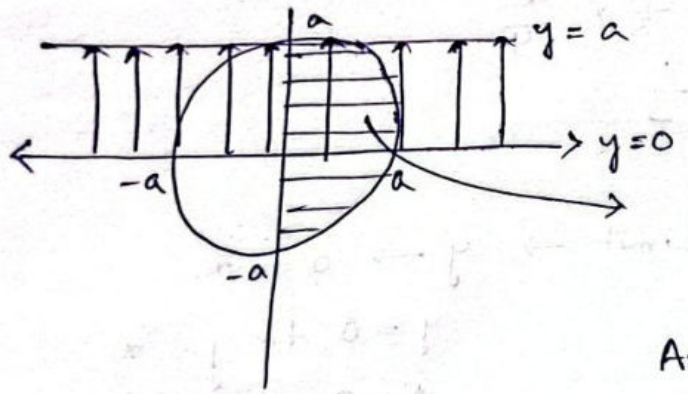
Here,

$x \to 0$ to $\sqrt{a^2-y^2}$ i.e. $x=0$ to $x=\sqrt{a^2-y^2}$

or, $x^2+y^2=a^2$

$y \to 0$ to a





Only first quadrant

(Area of this region)

$$\text{Area} = \iint_R dx\, dy$$

Evaluation :

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} dx\, dy$$

Inner Variable $\to x$

Outer Variable $\to y$

$$= \int_0^a (x)_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \sqrt{a^2-y^2}\, dy$$

$$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta} \,.\, a\cos\theta\, d\theta$$

$y = a\sin\theta$

$dy = a\cos\theta\, d\theta$

$y=0,\ \theta=0$

$y=a,\ \theta=\frac{\pi}{2}$

$$= \int_0^{\pi/2} \sqrt{a^2\cos^2\theta}\; a\cos\theta \, d\theta$$

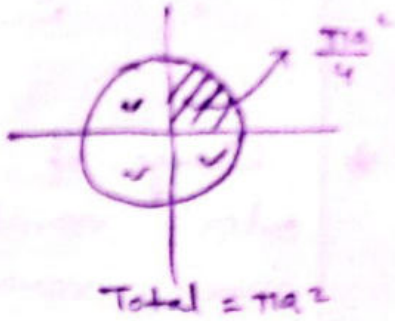
$$= \int_0^{\pi/2} a^2\cos^2\theta \, d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{1+\cos 2\theta}{2} \, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left\{\left(\frac{\pi}{2}-0\right) + \frac{1}{2}\left(\sin\pi - \sin 0\right)\right\}$$

$$= \frac{\pi a^2}{4}$$



2. $\int_0^a \int_0^{\sqrt{ay}} xy \, dx \, dy$ → Here $f(x,y) \neq 1$ → <u>Volume</u>

$x = 0$ to $x = \sqrt{ay}$ ⇒ $x^2 = ay$

$y = 0$ to $y = a$

$$= \int_0^a \left(\frac{x^2}{2}\right)_0^{\sqrt{ay}} y \, dy$$

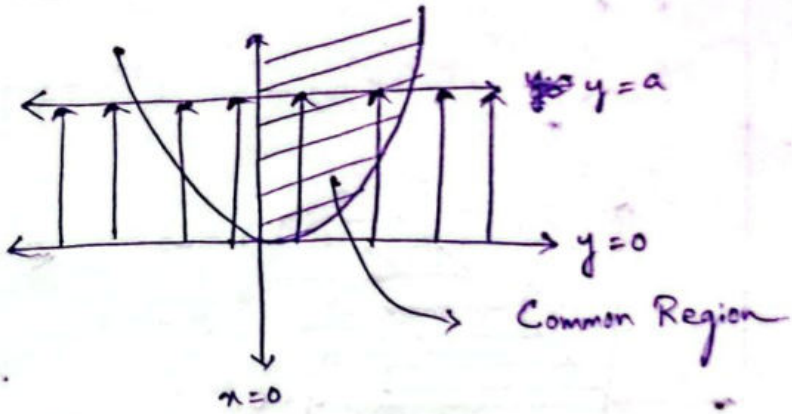
$$= \int_0^a y\left(\frac{ay}{2}\right) dy$$

$$= \frac{a}{2}\int_0^a y^2 \, dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a$$

$$= \frac{a}{6}\left(a^3\right)$$

$$= \frac{a^4}{6}$$



x will always move from left → right (Horizontally)

y will always move from bottom → top (Vertically)

3. $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy \, dx$

$= \int_0^\infty \int_0^x x e^{x} \cdot e^{-\frac{1}{y}} \, dy \, dx$

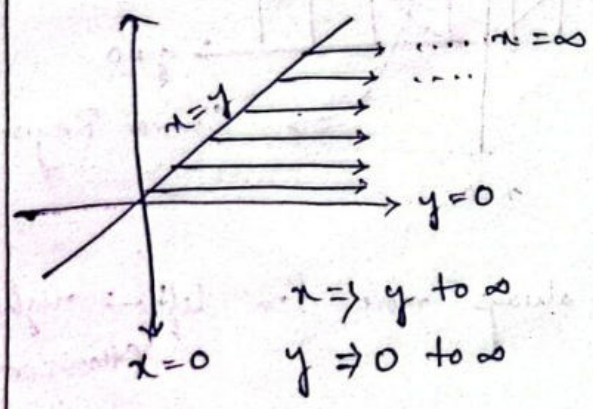
$= \int_0^\infty x e^{x} \underbrace{\int_0^x e^{-\frac{1}{y}} \, dy}_{} \, dx$ we can't integrate this part

★ Change of Order means if inner variable is 'y' and outer variable is 'x' then by changing its order inner variable will be 'x' and outer variable will be 'y' accordingly the limit of both variables will be changed without affecting the region and vice versa.

$\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy \, dx$

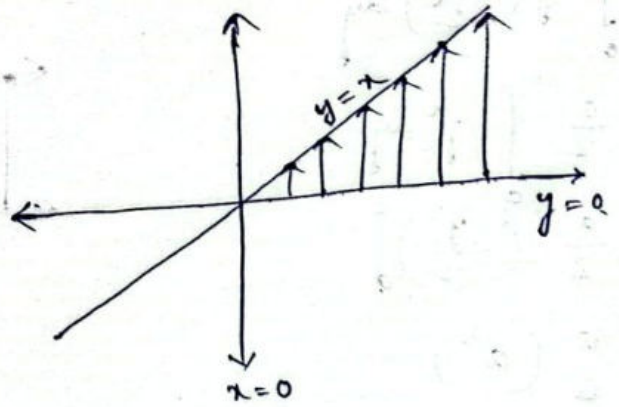
Inner Variable → y $(y = 0 \text{ to } x)$

Outer Variable → x $(x = 0 \text{ to } \infty)$



$x \Rightarrow y$ to ∞

$y \Rightarrow 0$ to ∞



Given Region

Change of Order

$\int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} \, dx \, dy$

$= \int_y^\infty e^{-z} \cdot \frac{y}{2} \, dz$

$= \frac{y}{2} \left(-e^{-z}\right)_y^\infty$

$= -\frac{y}{2} \left(e^{-\infty} - e^{-y}\right) = \frac{ye^{-y}}{2}$

$\frac{x^2}{y} = z$

or, $x^2 = yz$

or, $2x\,dx = y\,dz$

or, $x\,dx = \frac{y}{2} dz$

$x = y, \; z = y$

$x = \infty, \; z = \infty$

$$\int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}}\, dx\, dy = \int_0^\infty \frac{y e^{-y}}{2}\, dy$$

$$= \frac{1}{2}\left[(y)(-e^{-y}) - (1)(e^{-y})\right]_0^\infty$$

$$= -\frac{1}{2}\left(\cancel{y e^{-y}}\right)_0^\infty - \frac{1}{2}\left(e^{-y}\right)_0^\infty$$

$$= -\frac{1}{2}(0-1)$$

$$= -\frac{1}{2}$$

<u>Ex – 10.3</u> (1st Year Book)

Ex $\int_0^\infty \int_0^x y e^{-xy}\, dy\, dx$

$$= -\int_0^\infty \left\{ e^{-x^2} + \frac{e^{-x^2}}{x^2} - \frac{1}{x^2} \right\} dx$$

$$= -\frac{\sqrt{\pi}}{2} + \sqrt{\pi} + \infty \rightarrow \times$$

$\int y e^{-yx}\, dy$ $\quad [x = \text{const.}]$

$$= (y)\cdot\left(\frac{e^{-xy}}{-x}\right) - (1)\left(\frac{e^{-xy}}{x^2}\right)$$

$$= \frac{-y e^{-xy}}{x} - \frac{e^{-xy}}{x^2}$$

$$\int_0^x y e^{-xy}\, dy = -\frac{1}{x}\left(y e^{-xy}\right)_{y=0}^{x} - \frac{1}{x^2}\left(e^{-xy}\right)_{y=0}^{x}$$

$$= -\frac{1}{x}\left\{x e^{-x^2} - 0\right\} - \frac{1}{x^2}\left\{e^{-x^2} - 1\right\}$$

$$= -e^{-x^2} - \frac{e^{-x^2}}{x^2} + \frac{1}{x^2}$$

$$* \int_0^\infty \frac{1}{x^2}\, dx = -\left(\frac{1}{x}\right)_0^\infty$$

$$= -\left(\frac{1}{\infty} - \frac{1}{0}\right)$$

$$= \infty$$

$$* \int_0^\infty e^{-x^2}\, dx = \int_0^\infty e^{-t} \cdot \frac{dt}{\sqrt{t}\cdot 2}$$

$x^2 = t$

or, $2x\,dx = dt$

or, $dx = \frac{dt}{2x}$

$= \frac{dt}{2\sqrt{t}}$

$$= \frac{1}{2}\int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}$$

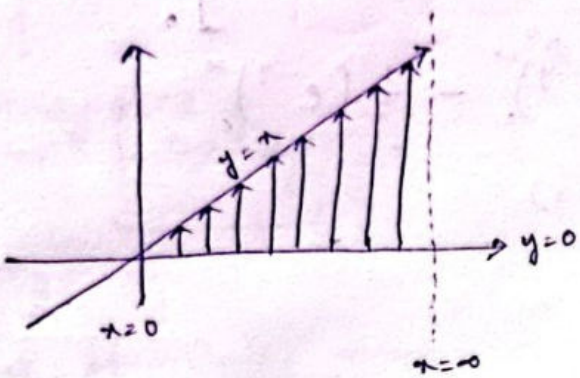
$$* \int_0^\infty \frac{e^{-x^2}}{x^2}\, dx = \int_0^\infty \frac{e^{-t}}{t} \cdot \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2}\int_0^\infty t^{-3/2} \cdot e^{-t}\, dt = \frac{1}{2}\Gamma\left(-\frac{1}{2}\right) = \sqrt{\pi}$$

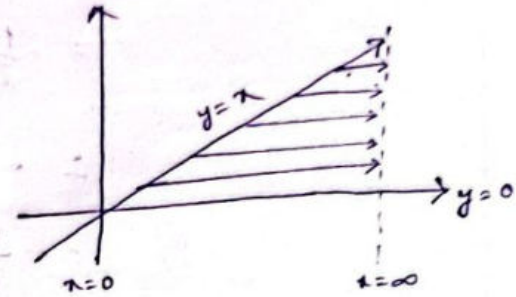
Order Reverse :

Given $y \to 0$ to x

$x \to 0$ to ∞



Given Region



Change of Order

Here $x \to y$ to ∞

$y \to 0$ to ∞

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} y e^{-xy} \, dx \, dy$$

$$\int_{y=0}^{\infty} \int_{x=y}^{\infty} y e^{-xy} \, dx \, dy$$

$$= \int_{0}^{\infty} y \left(\int_{y}^{\infty} e^{-xy} \, dx \right) dy$$

$$= \int_{0}^{\infty} y \left(\frac{e^{-xy}}{-y} \right)_{x=y}^{\infty} dy$$

$$= \int_{0}^{\infty} -\left(e^{-y^2} - e^{-\infty} \right) dy$$

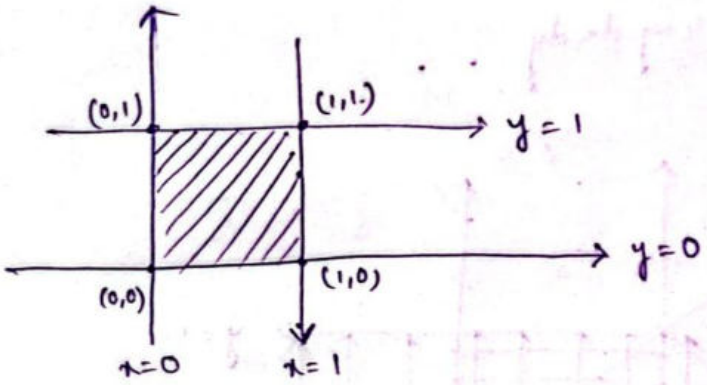
$$= \int_{0}^{\infty} e^{-y^2} \, dy$$

$$= \frac{\sqrt{\pi}}{2}$$

Ex $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$



$y \to 0$ to 1

$x \to 0$ to 1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\,dy\,dx$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy \qquad \text{(Order Reverse)}$$

$$= \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= \int_0^1 \frac{6}{5}\left(\frac{x^2}{2} + y^2 x\right)_0^1 dy$$

$$= \int_0^1 \frac{6}{5}\left(\frac{1}{2} + y^2\right) dy$$

$$= \frac{6}{5}\left(\frac{y}{2} + \frac{y^3}{3}\right)_0^1$$

$$= \frac{6}{5}\left(\frac{1}{2} + \frac{1}{3}\right)$$

$$= 1$$

X be a continuous RV with pdf of $f(x)$

*) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$f(x,y) = \begin{cases} 24xy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Evaluate $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 24xy\,dx\,dy$$

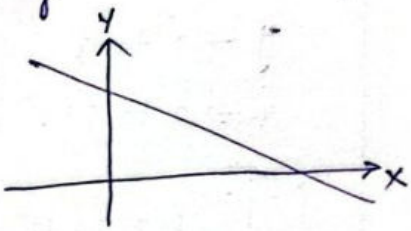
$$\int_{0}^{1}\int_{0}^{1} 24xy\,dx\,dy$$

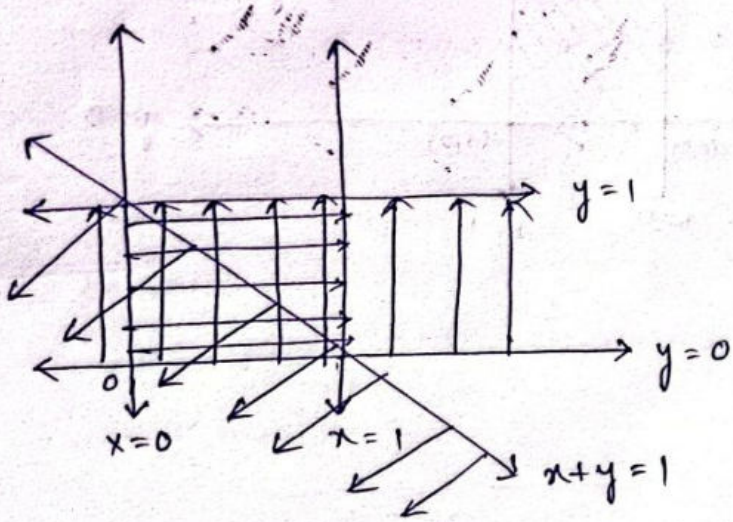
<u>Straight Line</u>

$Ax + By + C = 0$

x intersection $= -C/A$

y " $= -C/B$

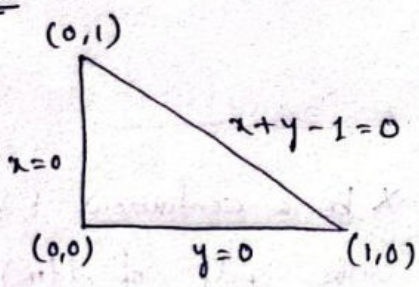




$x + y - 1 = 0$

x-intersect $= 1$ $\quad$ y-intersect $= 1$

<u>Case - I</u>



Inner Variable $\rightarrow$ x (left $\rightarrow$ right)

Outer Variable $\rightarrow$ y (lower $\rightarrow$ upper)

So, $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1-y} 24xy\,dx\,dy$$

$$= \int_{0}^{1} 24y \int_{0}^{1-y} x\,dx\,dy$$

$$= \int_{0}^{1} 24y \cdot \left(\frac{x^2}{2}\right)_{0}^{1-y} dy$$

$$= \int_{0}^{1} 24y \cdot \left[-\frac{(1-y)^2}{2}\right] dy$$

$$= -24\int_0^1 y\left(\frac{(1-y)^2}{2} - 2y\right) dy$$

$$= -12\int_0^1 (y + y^3 - 2y^2)\, dy$$

$$= -12\left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3}\right]_0^1$$

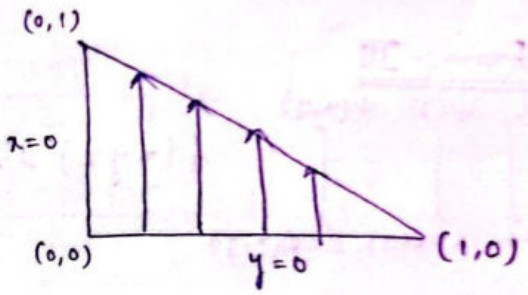
$$= -12\left[-\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)\right]$$

$$= \left(\frac{6+3-8}{12}\right) \times 12$$

$$= 1$$

Case - II

Order Reversed



Inner Variable $\rightarrow y$

Outer Variable $\rightarrow x$

$y \rightarrow 0$ to $(1-x)$

$x \rightarrow 0$ to 1

$$\int_0^1 \int_0^{1-x} 24xy\, dy\, dx$$

$$= \int_0^1 24x \cdot \left(\frac{y^2}{2}\right)_0^{1-x} dx$$

$$= \int_0^1 24x \cdot \left[-\frac{(1-x)^2}{2}\right] dx$$

$$= -12\int_0^1 x(1 + x^2 - 2x)\, dx$$

$$= -12\int_0^1 (x + x^3 - 2x^2)\, dx$$

$$= -12\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1$$

$$= +12\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right)$$

$$= 1$$

Triple Integration

*) **Form - I**

$$\int_{z=a}^{b} \int_{y=\phi(z)}^{\psi(z)} \int_{x=\phi(y,z)}^{\psi(y,z)} f(xyz)\, dx\, dy\, dz$$

*) **Form - II**

$$\int_{x=a}^{b} \int_{z=\phi(x)}^{\psi(x)} \int_{y=\phi(z,x)}^{\psi(z,x)} f(xyz)\, dy\, dz\, dx$$

*) **Form - III**

$$\int_{y=a}^{b} \int_{x=\phi(y)}^{\psi(y)} \int_{z=\phi(x,y)}^{\psi(x,y)} f(xyz)\, dz\, dx\, dy$$

$\iiint f(xyz)\, dx\, dy\, dz \longrightarrow$ Volume of Solid

$\iiint_S dx\, dy\, dz \longrightarrow$ Volume of the surface 'S'

Q

$$\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz = \int_{-1}^{1}\int_{-2}^{2} [x]_{-3}^{3}\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-2}^{2} 6\,dy\,dz$$

$$= 6\int_{-1}^{1} [y]_{-2}^{2}\,dz$$

$$= 24\int_{-1}^{1} dz = 24\times[z]_{-1}^{1} = 48$$

Q

$$\int_{0}^{4}\int_{0}^{x}\int_{0}^{x+y} z\,dz\,dy\,dx = \int_{0}^{4}\int_{0}^{x} \left[\frac{z^2}{2}\right]_{0}^{x+y} dy\,dx$$

$$= \int_{0}^{4}\int_{0}^{x} \frac{(x+y)^2}{2}\,dy\,dx$$

$$= \int_{0}^{4}\int_{0}^{x} \frac{x^2+2xy+y^2}{2}\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{4}\int_{0}^{x} (x^2+y^2+2xy)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{4} \left[x^2y + \frac{y^3}{3} + \frac{2xy^2}{2}\right]_{0}^{x} dx$$

$$= \frac{1}{2}\int_{0}^{4} -\left(-\frac{x^3}{3} - x^3\right) dx$$

$$= \frac{1}{2}\int_{0}^{4} \left(x^3 + \frac{x^3}{3}\right) dx$$

$$= \frac{1}{2}\int_{0}^{4} \frac{4x^3}{3}\,dx$$

$$= \frac{2}{3}\int_{0}^{4} x^3\,dx$$

$$= \frac{2}{3}\times\left[\frac{x^4}{4}\right]_{0}^{4}$$

$$= \frac{128}{3}$$

Q. $f(xyz) = \begin{cases} xy(1-z), & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1,\ x+y+z=1 \\ 0, & \text{otherwise} \end{cases}$

Ans.

$z \to 0$ to $1-x-y$

$y \to 0$ to $1-x$

$x \to 0$ to 1

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(xyz)\,dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xy(1-z)\,dz\,dy\,dx$$

$$= \int_{0}^{1} xy \int_{0}^{1-x} \left(z - \frac{z^2}{2}\right)_0^{1-x-y} dy\,dx$$

$$= \int_{0}^{1} xy \int_{0}^{1-x} -\left[(1-x-y) - \frac{(1-x-y)^2}{2}\right] dy\,dx$$

$$= \int_{0}^{1} xy \int_{0}^{1-x} (-1+x+y)\,dy\,dx$$

Q. Let X and Y are two discrete RV. Their joint prob. mass function is $f(x,y)$ where $f(x,y) = P(X=x, Y=y)$

$$\begin{cases} f(x,y) \geq 0 \\ \Sigma\Sigma f(x,y) = 1 \end{cases}$$

$F(x,y)$ be, its Joint prob. distribution func.

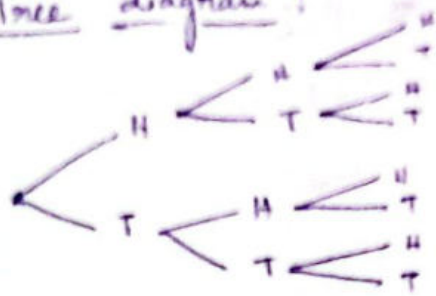
where $F(x,y) = P(X \le x, Y \le y)$

It's marginal pdf are $f_X(x)$ or $f_Y(y)$ or $P_X(x)$ or $P_Y(y)$

Q. Let a coin is tossed 3 times. X be the RV of getting no of heads in first two tosses and Y be another RV of getting no of tails in last two tosses

A coin is tossed 3 times $|S| = 2^3 = 8$

Tree diagram :



$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

S.S	X	Y	(x,y)	Prob.
HHH	2	0	(2,0)	1/8 = P(2,0)
HHT	2	1	(2,1)	1/8 = P(2,1)
HTH	1	1	(1,1)	1/8
HTT	1	2	(1,2)	1/8 = P(1,2)
THH	1	0	(1,0)	1/8 = P(1,0)
THT	1	1	(1,1)	1/8
TTH	0	1	(0,1)	1/8 = P(0,1)
TTT	0	2	(0,2)	1/8 = P(0,2)

P(1,1) = 2/8

X \ Y	0	1	2	
0	P(0,0)=0	P(0,1) = 1/8	P(0,2) = 1/8	$P_X(0) = \frac{2}{8}$
1	P(1,0) = 1/8	P(1,1) = 2/8	P(1,2) = 1/8	$P_X(1) = \frac{4}{8}$
2	P(2,0) = 1/8	P(2,1) = 1/8	P(2,2) = 0	$P_X(2) = \frac{2}{8}$
	$P_Y(0) = \frac{2}{8}$	$P_Y(1) = \frac{4}{8}$	$P_Y(2) = \frac{2}{8}$	

P(0,0), P(2,2) → Impossible events

Short Method

$F(0,0) = P(X=0, Y=0) = 0$

$F(0,1) = P(X \leq 0, Y \leq 1) = P(0,0) + P(0,1) = 0 + \frac{1}{8} = \frac{1}{8}$

$F(0,2) = P(X \leq 0, Y \leq 2) = P(0,0) + P(0,1) + P(0,2) = 0 + \frac{1}{8} + \frac{1}{8} = \frac{2}{8}$

$F(1,0) = P(X \leq 1, Y \leq 0) = P(0,0) + P(1,0) = 0 + \frac{1}{8} = \frac{1}{8}$

$F(1,1) = P(X \leq 1, Y \leq 1) = P(0,0) + P(0,1) + P(1,0) + P(1,1)$

$= 0 + \frac{1}{8} + \frac{1}{8} + \frac{2}{8} = \frac{4}{8}$

$F(1,2) = P(X \leq 1, Y \leq 2) = P(0,0) + P(0,1) + P(0,2) + P(1,0) + P(1,1) + P(1,2)$

$F(2,0) = P(X \leq 2, Y \leq 0) =$

$F(2,1) = P(X \leq 2, Y \leq 1) =$

$F(2,2) = P(X \leq 2, Y \leq 2) = 0$

Let X and Y are two continuous RVs. where joint prob. distribution func. is $f(x,y)$ where, $f(x,y) \geq 0$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

*) The joint prob. distribution func is $f(x,y)$

where, $$f(x,y) = \int_{-\infty}^{x}\int_{-\infty} f(x,y)\, dy\, dx$$

*) Marginal prob. density func. are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

*) $$P(a \leq X \leq b,\ c \leq X \leq d) = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\, dy\, dx$$

*) $$P(x \leq a) = \int_{-\infty}^{a} f_Y(y)\, dy$$

$$P(X \leq a) = \int_{-\infty}^{a} f_X(x)\, dx$$

Ex $f(x,y) = \begin{cases} k(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

① Find value of k for which $f(x,y)$ will be joint prob. distribution.

Ans $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_0^1\int_0^1 k(x+y^2)\,dx\,dy = 1$

$\Rightarrow k = 6/5$

② Find the joint prob. distribution func.

Ans $F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)\,dy\,dx = \int_0^x\int_0^y \frac{6}{5}(x+y^2)\,dy\,dx$

$= \frac{6}{5}\left(\frac{x^2y}{2} + \frac{xy^3}{3}\right)$

③ Find the marginal pdf of X

Ans $f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{6}{5}(x+y^2)\,dy$

$\downarrow$

$P_x(x)$

$= \frac{6}{5}\left(x+\frac{1}{3}\right)$

④ Find the marginal pdf of Y

Ans $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{6}{5}(x+y^2)\,dx$

$\downarrow$

$P_y(y)$

$= \frac{6}{5}\left(\frac{1}{2}+y^2\right)$

⑤ Find $P(0 \le x \le \frac{1}{4},\ 0 \le y \le \frac{1}{4})$

Ans $\int_0^{1/4}\int_0^{1/4} f(x,y)\,dx\,dy = \int_0^{1/4}\int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy$

$\left(\frac{1}{4} \le X \le \frac{3}{4}\right)$

Ans $P\left(\frac{1}{4} \le X \le \frac{3}{4}\right) = \int_{1/4}^{3/4} P_X(x)\,dx = \int_{1/4}^{3/4} \frac{6}{5}\left(x + \frac{1}{3}\right) dx = \frac{6}{5}\left(\frac{x^2}{2} + \frac{x}{3}\right)_{1/4}^{3/4}$

$=$

⑦ $P\left(\frac{1}{4} \le X \le \frac{3}{4}\right)$

Ans $P\left(\frac{1}{4} \le Y \le \frac{3}{4}\right) = \int_{1/4}^{3/4} P_Y(Y)\,dy = \int_{1/4}^{3/4} \frac{6}{5}\left(\frac{1}{2} + y^2\right) dy$

$= \frac{6}{5}\left(\frac{y}{2} + \frac{y^3}{3}\right)_{1/4}^{3/4}$

$=$

Expectation :

let X and Y are two discrete RV with joint Prob. distribution func. of $f(x,y)$

So, $E[h(x,y)] = \sum\sum h(x,y)\, f(x,y)$

In particular

$$E(X) = \sum\sum x\, f(x,y)$$

$$E(Y) = \sum\sum y\, f(x,y)$$

$$E(X \pm Y) = E(X) \pm E(Y)$$

$E(x)$ and $E(Y)$ are also denoted by $\overset{\mu_x}{\cancel{\mu(x)}}$ and $\overset{\mu_y}{\cancel{\mu(y)}}$

and the covarience of $f(x,y) = \sum\sum (x-\mu_x)(y-\mu_y) f(x,y)$

$$\text{cov}(X,Y) = E(XY) - E(X).E(Y)$$

let X and Y are two continuous RV with just joint Prob. distribution of $f(x,y)$

So, $E[h(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)$

In particular,

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} x\, f(x,y)\, dx\, dy$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\, dx\, dy$$

$E(X \pm Y) = E(X) \pm E(Y)$

Independent RV :

X and Y are said to be independent RV if

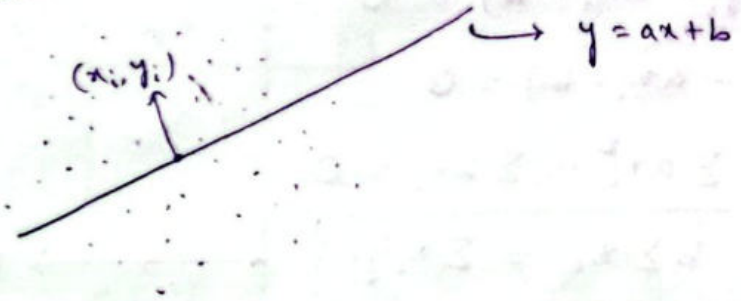
$$f(x,y) = P_X(X)\, P_Y(Y) \quad \text{for each pair of } (x,y)$$

Method of Least Square / Regration Line

Aim → To fit a straight line.

Their are multiple points and we want to fit a straight line of the form $y = mx + c$

i.e., the points (x_i, y_i), $i = 0, 1, 2, \ldots$ are given. Our aim is to fit a straight line of the form $y = ax + b$. It does not matter that the points will lie on that line or not but the distance of each line from the line is minimum.



let d_i be the distance

So,

$$d_i = y_i - (ax_i + b)$$

$$\Sigma d_i^2 = \Sigma (y_i - ax_i - b)^2 = R(\text{say})$$

$y = f(x)$

<u>Max / Min</u>

$$\frac{dy}{dx} = 0$$

1. Solve the eq^n
2. Find $\frac{d^2y}{dx^2}$
3. $\frac{d^2y}{dx^2} > 0$, Min $x = c_1$
4. At $x = c_1$, if $\frac{d^2y}{dx^2} < 0$ then Max at $x = c$

$$R = \Sigma (y_i - ax_i - b)^2$$

To find the minimum,

$$\frac{\partial R}{\partial a} = 0 \quad \& \quad \frac{\partial R}{\partial b} = 0$$

where,

$$\frac{\partial R}{\partial a} = \frac{\partial \Sigma (y_i - ax_i - b)^2}{\partial (y_i - ax_i - b)} \cdot \frac{\partial (y_i - ax_i - b)}{\partial a}$$

$$= \Sigma\, 2(y_i - ax_i - b)(-x_i)$$

As,

$$\frac{\partial R}{\partial a} = 0$$

or, $\Sigma -2x_i (y_i - ax_i - b) = 0$

or, $\Sigma x_i (y_i - ax_i - b) = 0$

or, $\Sigma x_i y_i - \Sigma a x_i^2 - \Sigma b x_i = 0$

or, $\boxed{a \Sigma x_i^2 + b \Sigma x_i = \Sigma x_i y_i} \longrightarrow (1)$

Similarly,

$$\frac{\partial R}{\partial b} = 0$$

or, $\Sigma\, 2(y_i - ax_i - b)(-1) = 0$

or, $\Sigma (-2y_i + 2ax_i + 2b) = 0$

or, $\Sigma (y_i - ax_i - b) = 0$

or, $\Sigma y_i - a \Sigma x_i - b \Sigma 1 = 0$

or, $\boxed{\Sigma y_i = a \Sigma x_i + nb} \longrightarrow (2)$ (n = no of points

To fit the line of the form $y = ax+b$

*) Given no of points = n

The formal eq^n are,

$$a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$a\sum x_i + nb = \sum y_i$$

Solving these 2 eq^{n}s we will get the value of a and b.

x	y	x^2	xy
⋮			
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum x_i y_i$

Q

x	1	2	3	4	5
y	14	27	40	55	68

Fit a straight line using the above data

Ans

$n = 5$

Normal eq^{n}s are,

$$a\sum x_i^2 + b\sum x_i = \sum x_i y_i$$

$$a\sum x_i + 5b = \sum y_i$$

x	y	x^2	xy
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

$\sum x_i = 15$

$\sum y_i = 204$

$\sum x_i^2 = 55$

$\sum x_i y_i = 748$

$55a + 15b = 748$

$15a + 5b = 204$

$a = 13.6$

$b = 0$

$y = ax+b$

$\therefore$ The straight line is $y = 13.6x$

Fit a straight line of the form $y = a + bx$. Using the data.

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

$n = 5$

The normal eq^n are,

$$b\sum x_i^2 + a\sum x_i = \sum x_i y_i$$

$$b\sum x_i + 5a = \sum y_i$$

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2

$\sum x_i = 10$

$\sum y_i = 16.9$

$\sum x_i^2 = 30$

$\sum xy = 47.1$

$30b + 10a = 47.1$

or, $10b + 5a = 16.9$

$10a + 30b = 47.1$

or, $5a + 10b = 16.9$

$$\left.\begin{aligned} a &= 0.72 \\ b &= 1.33 \end{aligned}\right\} \qquad y = 0.72 + 1.33x$$

$\underline{y = ax + b}$

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 x \cdot \frac{6}{5}(x+y^2)\, dx\, dy$$

$$= \int_0^1 \frac{6}{5}\int_0^1 (x^2 + y^2 x)\, dx\, dy$$

$$= \frac{6}{5}\int_0^1 \left[\frac{x^3}{3} + \frac{y^2 x^2}{2}\right]_0^1 dy$$

$$= \frac{6}{5}\int_0^1 \left(\frac{1}{3} + \frac{y^2}{2}\right) dy$$

$$= \frac{6}{5}\left[\frac{y}{3} + \frac{y^3}{6}\right]_0^1$$

$$= \frac{6}{5}\left[\frac{1}{3} + \frac{1}{6}\right]$$

$$= 0.6$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy = \int_0^1\int_0^1 y \cdot \frac{6}{5}(x+y^2)\, dx\, dy$$

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\, dx\, dy = \int_0^1\int_0^1 x^2 \cdot \frac{6}{5}(x+y^2)\, dx\, dy$$

$$E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\, dx\, dy = \int_0^1\int_0^1 y^2 \cdot \frac{6}{5}(x+y^2)\, dx\, dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\, dx\, dy = \int_0^1\int_0^1 xy \cdot \frac{6}{5}(x+y^2)\, dx\, dy$$

<u>check</u>

$$\text{cor}(XY) = E(XY) - E(X)E(Y)$$

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$\sigma_Y^2 = Var(Y) = E(Y^2) - [E(Y)]^2$$

**

$$\text{Correlation of } (X,Y) = \frac{\text{cor}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

---

let X and Y are two discrete RV with joint probability function $f(x,y)$ or $P(x,y)$.

For the func $h(x,y)$, $E[h(x,y)] = \Sigma\Sigma\, h(x,y)\, P(x,y)$

In particular,

$$\mu_x = E(X) = \Sigma\Sigma\, x \cdot P(x,y)$$

$$\mu_y = E(Y) = \Sigma\Sigma\, y \cdot P(x,y)$$

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2$$

$$\sigma_Y^2 = Var(Y) = E(Y^2) - [E(Y)]^2$$

$E(X^2) = \sum\sum x^2 p(x,y)$

$E(Y^2) = \sum\sum y^2 p(x,y)$

$E(XY) = \sum\sum xy\, p(x,y)$

$cov(XY) = E(XY) - E(X)E(Y)$

$\rho(x,y) = cor(x,y) = \dfrac{cov(x,y)}{\sigma_x \cdot \sigma_y}$

x	y	(x,y)	$p(x,y)$	$x.p$	$y.p$	$x^2.p$	$y^2.p$	$xy.p$
			1	$E(X)$	$E(Y)$	$E(X^2)$	$E(Y^2)$	$E(XY)$

Ex From the jointly discrete distribution table,

$x \backslash y$	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

Find $f(x,y)$

$E(X) = \sum x.p = 1$

$E(Y) = \sum y.p = 1$

x	y	$p(x,y)$	$x.p$	$y.p$	$x^2.p$	$y^2.p$	$xy.p$
0	0	1/8	0	0	0	0	0
0	1	1/8	0	1/8	0	1/8	0
0	2	0	0	0	0	0	0
1	0	1/8	1/8	0	1/8	0	0
1	1	2/8	2/8	1/4	2/8	2/8	2/8
1	2	1/8	1/8	1/4	1/8	4/8	2/8
2	0	0	0	0	0	0	0
2	1	1/8	1/4	1/8	4/8	1/8	2/8
2	2	1/8	1/4	1/4	4/8	4/8	4/8

$E(X^2) = \frac{12}{8}$ $\quad E(XY) = \frac{10}{8}$

$E(Y^2) = \frac{12}{8}$

$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$

$\sigma_Y^2 = Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{3}{2} - 1^2 = \frac{1}{2}$

$Cov(X,Y) = E(XY) - E(X).E(Y)$

$= \frac{5}{4} - 1.1$

$= \frac{1}{4}$

$\sigma_X = sd(X) = \frac{1}{\sqrt{2}}$ $\quad cor(X,Y) = \frac{cov(X,Y)}{\sigma_X . \sigma_Y}$

$\sigma_Y = sd(Y) = \frac{1}{\sqrt{2}}$ $\quad = \frac{1/4}{\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}} = \frac{1}{2} = 0.5$

<u>Alternative Formula for Continuous RV x and y</u>

let x and y are two continuous RVs with joint prob. func. $f(x,y)$ or $P(x,y)$

So it's marginal pdf will be,

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$E(X) = \int_{-\infty}^{\infty} x . f_x(x)\, dx$ $\quad E(X^2) = \int_{-\infty}^{\infty} x^2 . f_x(x)\, dx$

$E(Y) = \int_{-\infty}^{\infty} y . f_y(y)\, dy$ $\quad E(Y^2) = \int_{-\infty}^{\infty} y^2 . f_y(y)\, dy$

$E(XY) \rightarrow$ Cannot be found out by marginal pdf $\quad E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

Q

$$f(x,y) = \begin{cases} 24xy, & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Ans

$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy = 12x + 12x^3 - 24x^2$$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{1-y} 24xy\,dx = 12y + 12y^3 - 24y^2$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_x(x)\,dx = \int_0^1 x \cdot (12x + 12x^3 - 24x^2)\,dx$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_x(x)\,dx = \int_0^1 x^2 \cdot (12x + 12x^3 - 24x^2)\,dx$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f_y(y)\,dy = \int_0^1 y \cdot (12y + 12y^3 - 24y^2)\,dy$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 \cdot f_y(y)\,dy = \int_0^1 y^2 \cdot (12y + 12y^3 - 24y^2)\,dy$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} xy \cdot 24xy\,dx\,dy$$

Point Estimation (Parameter Estimation)

Let X_1 be a RV with pdf $f(x)$. Clearly $f(x)$ contains one or more parameter.

For BD, $f(x) = {}^nC_x \cdot p^x \cdot (1-p)^{n-x}$

Here the parameter is p which is prob. of success

For PD, $f(x) = \dfrac{e^{-\mu} \cdot \mu^x}{x!}$

Here the parameter is μ which is mean

For ND, $f(x) = \dfrac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

Here the parameters are μ (mean) and σ (std deviation)

The principle or the method of estimating the parameter or parameters is known as 'Maximum Likelyhood Method or Principle'

↳ Let $X_1, X_2, X_3, \ldots, X_n$ are n number of independent sample points with probability function $f(x)$. Suppose the parameter is θ. So we can write,

$$f(x) = f(x, \theta)$$

Thus,
$$f(x_1) = f(x_1, \theta)$$
$$f(x_2) = f(x_2, \theta)$$
$$\vdots$$
$$f(x_n) = f(x_n, \theta)$$

Let $L = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot \ldots \cdot f(x_n)$

$$\ln(L) = \ln f(x_1) + \ln f(x_2) + \ln f(x_3) + \ldots + \ln f(x_n)$$

The principle says, $\ln(L)$ is to be maximum

So, $\dfrac{\partial \ln(L)}{\partial \theta} = 0$ $\quad \partial \rightarrow$ Partial derivation

Solving $\frac{\partial \ln(L)}{\partial \theta} = 0$ we will get,

$$\theta = g(x_1, x_2, x_3, \ldots, x_n)$$

If there are multiple parameters, such as $\theta_1, \theta_2, \theta_3, \ldots, \theta_n$ then the Maximum Likelyhood conditions are to be

$$\frac{\partial \ln(L)}{\partial \theta_1} = 0, \quad \frac{\partial \ln(L)}{\partial \theta_2} = 0, \ldots, \frac{\partial \ln(L)}{\partial \theta_n} = 0$$

Solving these eq^{n}s we will get,

$$\theta_1 = g(x_1, x_2, \ldots, x_n)$$
$$\theta_2 = g(x_1, x_2, \ldots, x_n)$$
$$\vdots$$
$$\theta_n = g(x_1, x_2, \ldots, x_n)$$

<u>For BD</u>

Let X be the discrete RV possessing BD with pros. func. $f(x) = {}^nC_x \cdot p^x \cdot (1-p)^x$

here, p is the parameter and we want to estimate it

Suppose the random samples are,

$$x_1, x_2, x_3, \ldots, x_n$$

Corresponding pros. functions are,

$$f(x_1) = {}^nC_{x_1} \cdot p^{x_1} \cdot (1-p)^{n-x_1}$$
$$f(x_2) = {}^nC_{x_2} \cdot p^{x_2} \cdot (1-p)^{n-x_2}$$
$$f(x_n) = {}^nC_{x_n} \cdot p^{x_n} \cdot (1-p)^{n-x_n}$$

$$L = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot \ldots f(x_n)$$

So, $\ln(L) = \ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n)$

$$= \ln \cdot {}^nC_{x_1} \cdot p^{x_1} \cdot (1-p)^{n-x_1} + \ln \cdot {}^nC_{x_2} \cdot p^{x_2} \cdot (1-p)^{n-x_2}$$
$$+ \ln \cdot {}^nC_{x_3} \cdot p^{x_3} \cdot (1-p)^{n-x_3} + \ldots + \ln {}^nC_{x_n} \cdot p^{x_n} \cdot (1-p)^{n-x_n}$$

$$= \ln {}^nC_{x_1} + \ln p^{x_1} + \ln (1-p)^{n-x_1} + \ln {}^nC_{x_2} + \ln p^{x_2} + \ln (1-p)^{n-x_2}$$
$$+ \ln {}^nC_{x_3} + \ln p^{x_3} + \ln (1-p)^{n-x_3} + \dots + \ln {}^nC_{x_m} + \ln p^{x_m} + \ln (1-p)^{n-x_m}$$

$$= \left(\ln {}^nC_{x_1} + \ln {}^nC_{x_2} + \dots + \ln {}^nC_{x_m}\right) + \left(\ln p^{x_1} + \ln p^{x_2} + \dots + \ln p^{x_m}\right)$$
$$+ \left(\ln (1-p)^{n-x_1} + \ln (1-p)^{n-x_2} + \dots + \ln (1-p)^{n-x_m}\right)$$

$$= \left(\ln {}^nC_{x_1} + \ln {}^nC_{x_2} + \dots + \ln {}^nC_{x_m}\right) + \left(x_1 \ln p + x_2 \ln p + \dots + x_m \ln p\right)$$
$$+ \left((n-x_1)\ln(1-p) + (n-x_2)\ln(1-p) + \dots + (n-x_m)\ln(1-p)\right)$$

$$= \sum_{i=1}^{m} {}^nC_{x_i} + \ln p \sum_{i=1}^{m} x_i + \ln(1-p) \sum_{i=1}^{m} (n - x_i)$$

$$= \sum_{i=1}^{m} {}^nC_{x_i} + \ln p \sum_{i=1}^{m} x_i + \ln(1-p)\left[\sum_{i=1}^{m} n - \sum_{i=1}^{m} x_i\right]$$

$$= \sum_{i=1}^{m} {}^nC_{x_i} + \ln p \sum_{i=1}^{m} x_i + \ln(1-p)\left[nm - \sum_{i=1}^{m} x_i\right]$$

Now, $\dfrac{\partial \ln(L)}{\partial p} = 0 \longrightarrow$ (Extreme point condition)

or, $0 + \sum_{i=1}^{m} x_i \cdot \dfrac{1}{p} + \left[nm - \sum_{i=1}^{m} x_i\right] \cdot \dfrac{-1}{1-p} = 0$

or, $\dfrac{(1-p)\sum_{i=1}^{m} x_i - p\left(nm - \sum_{i=1}^{m} x_i\right)}{p(1-p)} = 0$

or, $(1-p)\sum_{i=1}^{m} x_i - pnm + p\sum_{i=1}^{m} x_i = 0$

or, $\sum_{i=1}^{m} x_i - \cancel{p\sum_{i=1}^{m} x_i} - pnm + \cancel{p\sum_{i=1}^{m} x_i} = 0$

or, $pnm = \sum_{i=1}^{m} x_i$

or, $p = \dfrac{\sum_{i=1}^{m} x_i}{nm} = \dfrac{1}{n} \cdot \dfrac{\sum_{i=1}^{m} x_i}{m} = \dfrac{\bar{x}}{n}$

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \dots + x_m}{m}$$

↳ Mean Value of random samples

If there are 10 no of sample points, then estimate $B(50, p)$

Given 10 samples such as $x_1, x_2, x_3, \ldots, x_3$

So,

$$m = 10$$

$$n = 50$$

$$p = \frac{\bar{x}}{n} = \frac{x_1 + x_2 + \ldots + x_{10}}{50 \times 10}$$

$$= \frac{x_1 + x_2 + \ldots + x_{10}}{500}$$

<u>For PD</u>

The parameter is μ

Prob. func. is $f(x) = \dfrac{e^{-\mu} \cdot \mu^{x}}{x!}$

let there are n no of independent samples such as

$$x_1, x_2, x_3, \ldots, x_n$$

$$f(x_1) = \frac{e^{-\mu} \cdot \mu^{x_1}}{x_1!}$$

$$f(x_2) = \frac{e^{-\mu} \cdot \mu^{x_2}}{x_2!}$$

$$f(x_n) = \frac{e^{-\mu} \cdot \mu^{x_n}}{x_n!}$$

$$L = f(x_1) \cdot f(x_2) \cdot f(x_3) \cdot \ldots f(x_n)$$

$$\ln(L) = \ln f(x_1) + \ln f(x_2) + \ldots + \ln f(x_n)$$

$$= \ln \frac{e^{-\mu} \cdot \mu^{x_1}}{x_1!} + \ln \frac{e^{-\mu} \cdot \mu^{x_2}}{x_2!} + \ldots + \ln \frac{e^{-\mu} \cdot \mu^{x_n}}{x_n!}$$

$$= (\ln e^{-\mu} + \ln \mu^{x_1} - \ln x_1!) + (\ln e^{-\mu} + \ln \mu^{x_2} - \ln x_2!)$$

$$+ \ldots + (\ln e^{-\mu} + \ln \mu^{x_n} - \ln x_n!)$$

$$= -n\mu + (x_1 + x_2 + \ldots + x_n) \ln(\mu) - (\ln x_1! + \ln x_2! + \ldots + \ln x_n!)$$

Principle says,

$$\frac{\partial \ln(L)}{\partial \mu} = 0$$

or, $-n + (x_1 + x_2 + \dots + x_n) \cdot \frac{1}{\mu} + 0 = 0$

or, $\frac{(x_1 + x_2 + \dots + x_n)}{\mu} = n$

or, $\mu = \frac{x_1 + x_2 + \dots + x_n}{n}$

$\hookrightarrow$ Sample Mean

<u>For ED</u>

The pdf of exponential distribution is,

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $\lambda > 0$ is the parameter.

For n independent samples $x_1, x_2, \dots, x_n$ we have,

$$f(x_i) = \lambda e^{-\lambda x_i}, \quad i = 1, 2, \dots, n$$

Likelyhood function is,

$$L = f(x_1) f(x_2) f(x_3) \dots f(x_n)$$

$$= (\lambda e^{-\lambda x_1})(\lambda e^{-\lambda x_2})(\lambda e^{-\lambda x_3}) \dots (\lambda e^{-\lambda x_n})$$

$$= \lambda^n \cdot e^{-\lambda(x_1 + x_2 + x_3 + \dots + x_n)}$$

or, $\ln(L) = n \ln(\lambda) - \lambda (x_1 + x_2 + x_3 + \dots + x_n)$

$$\frac{\partial \ln(L)}{\partial \lambda} = 0$$

or, $\frac{n}{\lambda} - (x_1 + x_2 + x_3 + \dots + x_n) = 0$

or, $\lambda = \frac{n}{x_1 + x_2 + \dots + x_n} = \frac{1}{\bar{x}}$

For ND

The pdf of Normal distribution is,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where μ and σ are the parameters

For n independent samples we have,

$$f(x_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right] \quad i = 1, 2, \ldots, n$$

Likelyhood function is,

$$L = f(x_1) f(x_2) f(x_3) \ldots f(x_n)$$

$$= \frac{1}{\sigma^n (\sqrt{2\pi})^n} \exp\left[-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2 - \frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2 - \ldots - \frac{1}{2}\left(\frac{x_n-\mu}{\sigma}\right)^2\right]$$

$$= \frac{1}{\sigma^n (\sqrt{2\pi})^n} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\right]$$

$$\ln(L) = -n\ln(\sigma) - \frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2$$

$$\frac{\partial \ln(L)}{\partial \mu} = 0$$

or, $-\frac{1}{2}\sum_{i=1}^{n} 2\left(\frac{x_i-\mu}{\sigma}\right)\left(-\frac{1}{\sigma}\right) = 0$

or, $\sum_{i=1}^{n}(x_i - \mu) = 0$

or, $\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\mu = 0$

or, $\sum_{i=1}^{n} x_i - n\mu = 0$

or, $\mu = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

For GD

The pdf of Geometric distribution is,

$$f(x) = p(1-p)^{x-1}$$

Likelyhood function is,

$$l = f(x_1) f(x_2) f(x_3) \dots f(x_n)$$

$$= p(1-p)^{x_1-1} \cdot p(1-p)^{x_2-1} \cdot p(1-p)^{x_3-1} \dots p(1-p)^{x_n-1}$$

$$\ln(l) = \ln\left(p(1-p)^{x_1-1}\right) + \ln\left(p(1-p)^{x_2-1}\right) + \ln\left(p(1-p)^{x_3-1}\right) + \dots + \ln\left(p(1-p)^{x_n-1}\right)$$

$$= \ln p + (x_1-1)\ln(1-p) + \ln p + (x_2-1)\ln(1-p) + \dots + (x_n-1)\ln(1-p) + \ln p$$

$$= n\ln p + \ln(1-p) \cdot \sum_{i=1}^{n} (x_i - 1)$$

Now, $\dfrac{\partial \ln(l)}{\partial p} = 0$ (Extreme Point Condition)

or, $\dfrac{n}{p} + \sum_{i=1}^{n} (x_i - 1) \cdot \dfrac{-1}{(1-p)} = 0$

or, $\dfrac{n}{p} + \sum_{i=1}^{n} x_i \cdot \dfrac{-1}{(1-p)} + \sum_{i=1}^{n} (-1) \cdot \dfrac{-1}{(1-p)} = 0$

or, $\dfrac{n}{p} + \dfrac{n}{1-p} = \sum_{i=1}^{n} x_i \cdot \dfrac{1}{(1-p)}$

or, $n\left(\dfrac{1}{p(1-p)}\right) = \dfrac{\sum_{i=1}^{n} x_i}{(1-p)}$

or, $\dfrac{n}{p} = \sum_{i=1}^{n} x_i$

or, $p = \dfrac{n}{\sum_{i=1}^{n} x_i} = \dfrac{1}{\bar{x}}$

Mean, Sample Mean, Variance, Sample Variance

Let there are n- samples such as $x_1, x_2, x_3, \ldots, x_n$.
It's mean is denoted by μ, where $\mu = \frac{x_1 + x_2 + \ldots + x_n}{n}$

or, $\frac{\sum_{i=1}^{n} x_i}{n}$

It's sample mean is denoted by $\bar{x}$ where $\bar{x} = \frac{\sum x_i}{n}$

It means mean & sample mean are same.

Variance is denoted by σ^2 where $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$

Sample variance " " " s^2 where $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$

Always sample variance > variance

$$s^2 > \sigma^2$$

Ex Find mean, sample mean, variance, sample variance, std. deviation, sample std. deviation from the following samples: 30, ~~30~~ 42, 40, 34, 48, 50

$$\text{Mean } (\mu) = \frac{\sum x_i}{n} = \frac{30 + 42 + 40 + 34 + 48 + 50}{6} = 40.67$$

Sample Mean $(\bar{x}) = 40.67$

$$\text{Variance } (\sigma^2) = \frac{1}{n} \sum (x_i - \mu)^2$$

$$= \frac{1}{6} \left[(30 - 40.67)^2 + (42 - 40.67)^2 + (40 - 40.67)^2 + (34 - 40.67)^2 + (48 - 40.67)^2 + (50 - 40.67)^2 \right] = \frac{1}{6} \times 301.33 = 50.22$$

$$\text{Sample Variance } (s^2) = \frac{1}{6-1} \times \left[(30 - 40.67)^2 + \ldots + (50 - 40.67)^2 \right]$$

$$= \frac{1}{5} \times 301.33 = 60.27$$

$s^2 > \sigma^2$

std. deviation $(\sigma) = \sqrt{50.22} =$

Sample std. deviation $(s) = \sqrt{60.27} =$

Alternative Formula For Sample Variance

$$s^2 = \frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right] \longrightarrow \text{Final Formula}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left\{x_i^2 - 2\bar{x}x_i + \bar{x}^2\right\}$$

$$= \frac{1}{n-1}\left\{\sum x_i^2 - 2\bar{x}\sum x_i + \bar{x}^2\sum 1\right\}$$

$$= \frac{1}{n-1}\left\{\sum x_i^2 - 2.\frac{\sum x_i}{n}.\sum x_i + \left(\frac{\sum x_i}{n}\right)^2\right\}$$

$$= \frac{1}{n-1}\left\{\sum x_i^2 - \frac{2}{n}\left(\sum x_i\right)^2 + \frac{1}{n}\left(\sum x_i\right)^2\right\}$$

$$= \frac{1}{n-1}\left\{\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2\right\}$$

x	x^2
30	900
42	1364
40	1600
32	1156
48	2304
50	2500
$\sum x_i = 244$	$\sum x_i = 10224$

$$s^2 = \frac{1}{n-1}\left[\sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2\right]$$

$$= \frac{1}{6-1}\left[10224 - \frac{1}{6}(224)^2\right]$$

$$= 60.27$$

Confidence Interval :

Let there are 'n' no of samples such as $x_1, x_2, x_3, \ldots, x_n$. θ be an unknown parameter.
↓
(such as μ and σ)

γ be the confidence level where $[0 \leq \gamma \leq 1]$

$\gamma \to 1$ means the confidence level is high with less prob. error.

$$\gamma = P(\theta_2 \leq \theta \leq \theta_2)$$

Generally, $\gamma = 90\%$ or 95% or 99%.

The confidence interval of the unknown parameter is denoted by $Conf_\gamma(\theta_1 \leq \theta \leq \theta_2)$ where θ_2 is the upper confidence limit and θ_1 is the lower confidence limit. Our aim is to find the confidence interval for μ of normal distribution. Two cases will arise.

<u>Case-I</u> : Known Variance (σ^2)

<u>Case-II</u> : Unknown Variance (s^2)

Similarly for

I. Known Mean ( )

II. Unknown Mean ( )

Confidence Interval of Mean for normal distribution with Known Variance (σ^2)

1. Chose a confidence level (95% or 99% or ....)

or

Choose a confidence level from the given data

2. Find the value of 'c' from ND table where

$\phi(c) = \frac{1+\gamma}{2}$

i.e. confidence level = 90%, $\gamma = 0.9$, $c = 1.645$

$\cancel{\phi(c) = \frac{1+0.9}{2}}$

" " = 95%, $\gamma = 0.95$, $c = 1.96$

" " = 99%, $\gamma = 0.99$, $c = 2.576$

3. Find the sample mean ($\bar{x}$) where $\bar{x} = \frac{\Sigma x_i}{n}$

4 Compute 'K' where $K = \frac{c\sigma}{\sqrt{n}}$

5. Now the require confidence interval will be,

$$\text{conf}\left(\bar{x} - K \leq \mu \leq \bar{x} + K\right)$$

The length of the interval $= 2K$

$$L = 2K$$

If the length of confidence interval (CI) is known then the size will be $L = \frac{2c\sigma}{\sqrt{n}}$

$$\text{or, } n = \left(\frac{2c\sigma}{L}\right)^2$$

Q Find a 95% CI for Mean (μ) of a normal population with std. deviation 4 from the sample 39, 51, 49, 43, 57, 59.

Ans $\gamma = 95\% = 0.95$

$\phi(c) = \frac{1+\gamma}{2} \Rightarrow c = 1.96$

$\bar{x} = \frac{\Sigma x_i}{n} = \frac{39+51+49+43+57+59}{6} = \cancel{56.83}\ 49.67$

Given, $\sigma = 4$

$K = \frac{c\sigma}{\sqrt{n}} = \frac{1.96 \times 4}{\sqrt{6}} = 3.2$

Required CI,

$$\text{Conf}\ (\bar{x} - K \le \mu \le \bar{x} + K)$$

$$= \text{conf}\ (\overset{49.67}{\cancel{56.83}} - 3.2 \le \mu \le \overset{49.67}{\cancel{56.83}} + 3.2)$$

$$\cancel{= \text{conf}\ (\overset{49.67}{53.63} \le \mu \le 60.03)}$$

$$= \text{conf}\ (46.67 \le \mu \le 52.87)$$

$\therefore$ length of the interval $\cancel{60.03 - 53.63}$

$$= (52.87 - 46.67)$$

$$= 6.4$$

Q Find the sample size m to obtain 95% confidence interval of length $l = 0.3$ of μ with $\sigma = 4$ for normal population.

Ans $\gamma = 95\% = 0.95$

$C = 1.96$, $\sigma = 4$, $L = 0.3$

$$K = \frac{C\sigma}{\sqrt{n}} \cancel{= 1.96 \times 4} \Rightarrow n = \left(\frac{2C\sigma}{L}\right)^2$$

$$= \left(\frac{2 \times 1.96 \times 4}{0.3}\right)^2$$

$L = 2K$

$$= 2731.8$$

$$\approx 2732$$

t - distribution :

The density function is

$$f(z) = K_m \left(1 + \frac{z^2}{m}\right)^{-(m+1)} , \quad -\infty < z < \infty$$

where $K_m = \dfrac{\Gamma\left(\frac{m+1}{2}\right)}{\sqrt{m\pi} \cdot \Gamma(m/2)}$

$\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1} dx , n > 0$

For $n = 0, 1, 2, 3, \ldots$

$\Gamma(n+1) = n!$

$\Gamma(n+1) = n\Gamma(n)$

Again,

$\Gamma(n) = \dfrac{\Gamma(n+1)}{n} , n > 0$

In particular,

$\Gamma(1/2) = \sqrt{\pi}$

The distribution function is,

$$f(z) = K_m \int_{-\infty}^{z} \left(1 + \frac{t^2}{m}\right)^{-(m+1)} dt$$

If $x_1, x_2, x_3, \ldots x_n$ are 'n' no of individual RVs with same mean and same variance then,

$$\bar{x} = \frac{\sum x_i}{n} \quad ; \quad s^2 = \frac{1}{(n-1)} \sum (x_i - \bar{x})^2$$

Now the RV $\dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ is to be denoted by 'T' which will posses a 'T - distribution' having $(n-1)$ degree of freedom.

Working rule for finding CI of mean with unknown variance for t-distribution :

(i) Choose the confidence level from the given data.

(ii) Find the sample mean and sample variance.

(iii) Find the value of 'c'

$$F_T(c) = \frac{1+\Gamma}{2}$$

(iv) Compute 'K' where,

$$K = \frac{c \cdot s}{\sqrt{n}}$$

(v) So, CI $(\bar{x} - K \leq \mu \leq \bar{x} + K)$ = Confidence Interval

Note

length of CI,

$$L = 2k = \frac{2cs}{\sqrt{n}}$$

size of the sample,

$$n = \left(\frac{2cs}{L}\right)^2$$

Q Find a 95% CI from for mean μ from the sample and the length of the sample.

30, 42, 40, 34, 48, 50

Ans

$\gamma = 95\% = 0.95$

$F_T(c) = \frac{1+\gamma}{2} = \frac{1+0.95}{2} = 0.975$ (From Appendix Table A-9)

$c = 2.57$

$$\bar{x} = \frac{30+42+40+34+48+50}{6}$$

$$= \frac{244}{6} = 40.67$$

$s^2 = 60.266$

$s = 7.76$

$$k = \frac{cs}{\sqrt{n}} = \frac{(2.57 \times 7.76)}{\sqrt{6}}$$

$$= \frac{2.57 \times 7.76}{2.44}$$

$$= 8.14$$

$$CI = (40.67 - 8.14 \leq \mu \leq 40.67 + 8.14)$$

$$= 32.53 \leq \mu \leq 48.81$$

$$L = 2K = (2 \times 8.14) = 16.28$$

Q Find the sample size of mean having t-distribution with length 'L' = 0.3 and having 95% confidence.

Ans

$L = 0.3$

$$n = \left(\frac{2cs}{L}\right)^2 = \frac{2 \times 2.57}{0.3}$$

Data insufficient (Another ques.)

Chi-Square Distance (χ^2 distance) :

The density func. is,

$$f(z) = C_m e^{-z/2} \cdot z^{\frac{m-2}{2}} \quad , \quad z \geq 0$$

$m = 1, 2, 3, \ldots$ are the no of degree of freedom.

Confidence Interval of variance for a normal population :

The sample mean $\bar{x}$ need not to be known.

(i) Choose the confidence level (T) from the given data.

(ii) From the chi-dist table (App 10 → Krez), find 'c1' and 'c2' where,

$$F(C_1) = \frac{1-\gamma}{2} \quad , \quad F(C_2) = \frac{1+\gamma}{2}$$

(iii) From the given 'n' no of samples $x_1, x_2, x_3, \ldots, x_n$ find the sample variance s^2.

(iv) Compute 'K_1' and 'K_2' where,

$$K_1 = \frac{(n-1)s^2}{C_2} \quad ; \quad K_2 = \frac{(n-1)s^2}{C_1}$$

(v) $CI = Conf_\gamma (K_1 \leq \sigma^2 \leq K_2)$.

Q Find a 95% CI for variance of a normal population from the sample,

30, 42, 40, 34, 48, 50

Ans $\gamma = 0.95$, $n = 6$

$s^2 = 60.26$

$c_1 = ?$ and $c_2 = ?$

$$F(C_1) = \frac{1-\gamma}{2} = \frac{0.05}{2} = 0.025 \qquad F(C_2) = \frac{1+\gamma}{2} = \frac{1.95}{2} = 0.975$$

So, from table (A-10)

$$\left.\begin{aligned} c_1 &= 0.83 \\ c_2 &= 12.83 \end{aligned}\right\}$$

$$K_1 = \frac{(n-1)s^2}{C_2} = \frac{5 \times 60.26}{12.83} = 23.484$$

$$K_2 = \frac{(n-1)s^2}{C_1} = \frac{5 \times 60.26}{0.83} = 363.01$$

$$CI = (23.484 \leq \sigma^2 \leq 363.01)$$

$$L_{(K_2 - K_1)} = 339.526$$

---

let x and y are bivariate variables with n samples $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

So, $\bar{x}$ = Sample mean of x

$\bar{y}$ = Sample mean of y

s_x^2 = Sample Variance of x

s_y^2 = Sample Variance of y

S_{xy} denotes the sample covariance

where $S_{xy} = \frac{1}{n-1}\left[\Sigma (x_i - \bar{x})(y_i - \bar{y})\right]$

<u>Alternative formula for S_{xy}</u>

$$S_{xy} = \frac{1}{n-1} \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{n-1} \Sigma \{ x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y} \}$$

$$= \frac{1}{n-1} \left[\Sigma x_i y_i - \Sigma x_i \bar{y} - \Sigma \bar{x} y_i + \Sigma \bar{x}\bar{y} \right]$$

$$= \frac{1}{n-1} \left[\Sigma x_i y_i - (\Sigma x_i) \cdot \left(\frac{\Sigma y_i}{n}\right) - \left(\frac{\Sigma x_i}{n}\right)(\Sigma y_i) + \bar{x}\bar{y} \Sigma 1 \right]$$

$$= \frac{1}{n-1} \left[\Sigma x_i y_i - \frac{1}{n}(\Sigma x_i)(\Sigma y_i) - \frac{1}{n}(\Sigma x_i)(\Sigma y_i) + n\bar{x}\bar{y} \right]$$

$$= \frac{1}{n-1} \left[\Sigma x_i y_i - \frac{2}{n}(\Sigma x_i)(\Sigma y_i) + \cancel{n}\left(\frac{\Sigma x_i}{\cancel{n}}\right)\left(\frac{\Sigma y_i}{n}\right) \right]$$

$$= \frac{1}{n-1} \left[\Sigma x_i y_i - \frac{1}{n}(\Sigma x_i)(\Sigma y_i) \right]$$

x	y	xy
Σx	Σy	Σxy

Ex Find the Sample covariance by both formula

x	1	3	4	6	8	10
y	5	9	7	11	25	5

Using the definition $(n=6)$,

$$\bar{x} = \frac{1+3+4+6+8+10}{6} = 5.33$$

$$\bar{y} = \frac{5+9+7+11+25+5}{6} = 10.33$$

$$S_{xy} = \frac{1}{n-1} \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

$$= \frac{1}{6-1} \left[(x_1 - \bar{x})(y_1 - \bar{y}) + (x_2 - \bar{x})(y_2 - \bar{y}) + (x_3 - \bar{x})(y_3 - \bar{y}) + (x_4 - \bar{x})(y_4 - \bar{y}) + (x_5 - \bar{x})(y_5 - \bar{y}) + (x_6 - \bar{x})(y_6 - \bar{y}) \right]$$

$$= \frac{1}{5} \left[(1-5.33)(5-10.33) + (3-5.33)(9-10.33) + (4-5.33)(7-10.33) + (6-5.33)(11-10.33) + (8-5.33)(25-10.33) + (10-5.33)(5-10.33) \right]$$

$$= \frac{1}{5} \left[9.66 + 3.0989 + 4.4289 + 1.7689 + 51.0489 + 28.4089 \right]$$

$$= 9.07$$

Alternative

x	y	xy
1	5	5
3	9	27
4	7	28
6	11	66
8	25	200
10	51	50
$\Sigma x_i = 32$	$\Sigma y_i = 62$	$\Sigma x_i y_i = 376$

$$S_{xy} = \frac{1}{n-1}\left[\Sigma x_i y_i - \frac{1}{n}(\Sigma x_i)(\Sigma y_i)\right]$$

$$= \frac{1}{5}\left[376 - \frac{1}{6} \times 32 \times 62\right]$$

$$= 9.07$$

Regression Analysis

Linear Regression:

If the variables are in a bivariate distribution are related we will find these points in a scattered diagram will cluster around some curve and that curve is said as Curve of regression. If the curve is a straight line then it is said to be line of regression and these is said to be the linear regression between the variables x and y.

For n sample points (x_i, y_i) for $i = 1, 2, 3, \ldots, n$

$\bar{x}$ and $\bar{y}$ are in sample mean.

So the line of regression will be the form,

$$y - \bar{y} = k(x - \bar{x})$$

where, $$k = \frac{S_{xy}}{S_{x^2}} = \frac{S.Cov(x, y)}{S.var(x)}$$

Ex The following data represents the ht(y) and yF(x) of 100 students.

$\Sigma x = 15000$ $\Sigma x^2 = 2272500$

$\Sigma y = 6500$ $\Sigma y^2 = 463025$

$\Sigma xy = 1022250$

Find the line of regression.

x	y	x^2	xy

Ans Here, $n = 100$, $\bar{x} = \frac{\Sigma x}{n} = \frac{15000}{100} = 150$

$$\bar{y} = \frac{\Sigma y}{n} = \frac{6800}{100} = 68$$

$$S_{x^2} = \frac{1}{n-1}\left[\Sigma x_i^2 - \frac{1}{n}(\Sigma x_i)^2\right]$$

$$= \frac{1}{99}\left[2272500 - \frac{1}{100}(15000)^2\right]$$

$$= 227.27$$

$$S_{xy} = \frac{1}{n-1}\left[\Sigma x_i y_i - \frac{1}{n}(\Sigma x_i)(\Sigma y_i)\right]$$

$$= \frac{1}{99}\left[1022250 - \frac{1}{100}(15000)(6500)\right]$$

$$= 22.78$$

$$K = \frac{S_{xy}}{S_{x^2}} = \frac{227.27}{22.78} = \frac{1}{10}$$

Required line of regression will be,

$$(y - \bar{y}) = K(x - \bar{x})$$

or, $(y - 68) = \frac{1}{10}(x - 150)$

or, $y = \frac{1}{10}(x - 150) + 68$

Correlation :

It is concerned with the relation between x and y in a bivariate distribution.

If there are n samples (x_i, y_i) for $i = 1(1)n$ then the correlation coefficient is denoted by ρ/r where (ρ or r)

$$r = \frac{S_{xy}}{S_x \cdot S_y}$$

Q Find the correlation coefficient of Previous question.

Ans $\bar{X} = 150$, $\bar{Y} = 68$, $S_x^2 = 227.27 \Rightarrow S_x = 15.07$

$$S_Y^2 = \frac{1}{n-1}\left[\Sigma y_i^2 - \frac{1}{n}(\Sigma Y_i)^2\right]$$

$$= \frac{1}{99}\left[463025 - \frac{1}{100}(6500)^2\right]$$

$$= 6.313$$

$S_y = 2.5125$

$S_{xy} = 22.73$

$$r = \frac{22.73}{(15.07)(2.5125)}$$

$$= 0.6$$