Pythagorean identities:

$\sin^2\theta+\cos^2\theta=1$

$\tan^2\theta+1=\sec^2\theta$

$1+\cot^2\theta=\csc^2\theta$

Reciprocal identities:

$\csc x=\dfrac{1}{\sin x}$

$\sec x=\dfrac{1}{\cos x}$

$\cot x=\dfrac{1}{\tan x}$

Even - odd identities:

$\sin(-x)=-\sin x$

$\cos(-x)=\cos x$

$\tan(-x)=-\tan x$

Co - function identities:

$\cos\left(\dfrac{\pi}{2}-x\right)=\sin x$

$\sin\left(\dfrac{\pi}{2}-x\right)=\cos x$

$\tan\left(\dfrac{\pi}{2}-x\right)=\cot x$

$\cot\left(\dfrac{\pi}{2}-x\right)=\tan x$

$\csc\left(\dfrac{\pi}{2}-x\right)=\sec x$

$\sec\left(\dfrac{\pi}{2}-x\right)=\csc x$

Periodicity identities:

$\sin(x\pm 2\pi)=\sin x$

$\cos(x\pm 2\pi)=\cos x$

$\tan(x\pm \pi)=\tan x$

$\cot(x\pm \pi)=\cot x$

$\sec(x\pm 2\pi)=\sec x$

$\csc(x\pm 2\pi)=\csc x$

Sum and difference formulas:

$\sin(x\pm y)=\sin x\cos y\pm\cos x\sin y$

$\cos(x\pm y)=\cos x\cos y\mp\sin x\sin y$

$\tan(x\pm y)=\dfrac{\tan x\pm\tan y}{1\mp\tan x\tan y}$

Product to sum formulas:

$\sin x\cdot\sin y=\dfrac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$

$\cos x\cdot\cos y=\dfrac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$

$\sin x\cdot\cos y=\dfrac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$

$\cos x\cdot\sin y=\dfrac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$

Half - angle formulas:

$\sin\left(\dfrac{x}{2}\right)=\pm\sqrt{\dfrac{1-\cos x}{2}}$

$\cos\left(\dfrac{x}{2}\right)=\pm\sqrt{\dfrac{1+\cos x}{2}}$

$\tan\left(\dfrac{x}{2}\right)=\dfrac{(1-\cos x)}{\sin x}$

Sum to product:

$\sin x\pm\sin y=2\sin\left(\dfrac{x\pm y}{2}\right)\cos\left(\dfrac{x\mp y}{2}\right)$

$\cos x+\cos y=2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\cos x-\cos y=-2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

Law of sines:

$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

Law of cosines:

$a^2=b^2+c^2-2bc\cos A$

$A=\cos^{-1}\left(\dfrac{b^2+c^2-a^2}{2bc}\right)$

Double - angle formulas:

$\sin 2\theta=2\cdot\sin\theta\cos\theta$

$\cos 2\theta=\cos^2\theta-\sin^2\theta=1-2\sin^2\theta=2\cos^2\theta-1$

$\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

Area of triangle:

$\dfrac{1}{2}ab\sin C$

$\sqrt{s(s-a)(s-b)(s-c)}$,

where $s=\dfrac{1}{2}(a+b+c)$

Hyperbolic Functions

Definition

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

Hyperbolic Identities

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$$

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}, x > 1$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}, |x| < 1$$

$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1 - x^2}}, x \neq 0$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1 - x^2}}, 0 < x < 1$$

$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}, |x| > 1$$

Differentiation Formulas:

1. $\frac{d}{dx}(x) = 1$
2. $\frac{d}{dx}(ax) = a$
3. $\frac{d}{dx}(x^n) = nx^{n-1}$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\tan x) = \sec^2 x$
7. $\frac{d}{dx}(\cot x) = -\csc^2 x$
8. $\frac{d}{dx}(\sec x) = \sec x \tan x$
9. $\frac{d}{dx}(\csc x) = -\csc x(\cot x)$
10. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
11. $\frac{d}{dx}(e^x) = e^x$
12. $\frac{d}{dx}(a^x) = (\ln a)a^x$
13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$

Integration Formulas:

1. $\int 1\,dx = x + C$
2. $\int a\,dx = ax + C$
3. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \sec^2 x\,dx = \tan x + C$
7. $\int \csc^2 x\,dx = -\cot x + C$
8. $\int \sec x(\tan x)\,dx = \sec x + C$
9. $\int \csc x(\cot x)\,dx = -\csc x + C$
10. $\int \frac{1}{x}\,dx = \ln|x| + C$
11. $\int e^x\,dx = e^x + C$
12. $\int a^x\,dx = \frac{a^x}{\ln a} + C\ a > 0, a \neq 1$
13. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
14. $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
15. $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$

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Curl of a Vector field.

$$V = v(x, y, z) = [v_1, v_2, v_3]$$

$$\text{Curl } V = \nabla \times v$$

$$= \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) - \hat{j}\left(\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right) +$$

$$\hat{K}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

$$\text{curl } V = \nabla \times V = \left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

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flow of compressible liquid/fluid.

$$v(x, y, z) = [v_1, v_2, v_3]$$

$$\text{div } v = 0$$

div $V = 0$

V is also referred to as solenoidal

Eg: The flow with velocity vector $v = y\hat{i}$ is incompressible as, div $v = 0$

Eg: $\vec{V} = [0, 3z^2, 0]$

Laplace operator or Laplacian of a scalar fn.

$$\text{grad } f(x,y,z) = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\text{div}(\text{grad } f) = \nabla \cdot \text{grad } f$$

$$= \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{div}(\text{grad } f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$$

The differential operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator and

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

is called the Laplacian of f.

$$\text{div}(\text{grad } f) = \nabla^2 f$$

Divergence of a vector field.

$$v(x, y, z) = [v_1, v_2, v_3]$$

$$\text{div } v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$\nabla \cdot v = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [v_1, v_2, v_3]$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \text{div } v.$$

—— x ——

Chapter 9

1. $\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

Eg: $f(x,y,z) = x^3y + xyz^2$

$$\frac{\partial f}{\partial x} = \frac{\partial(x^3y + xyz^2)}{\partial x}$$

$$= 3x^2y + yz^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial(x^3y + xyz^2)}{\partial y}$$

$$= x^3 + xz^2$$

$$\frac{\partial f}{\partial z} = \frac{\partial(x^3y + xyz^2)}{\partial z}$$

$$= 2xyz$$

$$\text{grad } f = \nabla f = [3x^2y + yz^2, x^3 + xz^2, 2xy$$

6.6

* Differentiation and integration of Laplace transforms

$$L(f(t)) = F(s)$$

$$\Rightarrow L(tf(t)) = -F'(s)$$

$$L^{-1}(F'(s)) = -tf(t).$$

$$\Rightarrow L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(\bar{s})\,d\bar{s}$$

$$\Rightarrow L^{-1}\left(\int_s^\infty F(\bar{s})\,d\bar{s}\right) = \frac{f(t)}{t}$$

6.5

Convolution

$$L(f(t)\,g(t)) \neq L(f(t))\,L(g(t))$$

$$L(f(t) * g(t)) = L(f(t))\,L(g(t))$$

$$f(t) * g(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$F(s) = L(f(t))$

$G(s) = L(g(t))$

$$L^{-1}(F(s)G(s)) = f(t) * g(t)$$

2. Second Shifting theorem (t-Shifting):

$$\tilde{f}(t) = f(t-a)\,u(t-a) = \begin{cases} 0 & t < a \\ f(t-a) & t > a \end{cases}$$

$$L\{f(t-a)\,u(t-a)\} = e^{-as}F(s)$$

$$f(t-a)\,u(t-a) = L^{-1}\{e^{-as}F(s)\}$$

6.3

1. Unit step function (Heaviside function).

$$u(t-a) = \begin{cases} 0 & ; t < a \\ 1 & ; t > a \end{cases}$$

Laplace transform of $u(t-a)$ is

$$L\{u(t-a)\} = \int_0^{\infty} e^{-st} u(t-a)\, dt$$

$$= \int_0^{a} e^{-st} u(t-a)\, dt + \int_a^{\infty} e^{-st} u(t-a)\, dt$$

$$= \int_a^{\infty} e^{-st}\, dt$$

$$= \frac{1}{s} e^{-sa}$$

$$\therefore L\{u(t-a)\} = \frac{1}{s} e^{-sa}$$

- Laplace transform of Integrals:

Let $L\{f(t)\} = F(s)$

$$L\left\{\int_0^t f(\rho)\,d\rho\right\} = \frac{1}{s}F(s)$$

and $$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t L^{-1}(F(s))\,d\rho = \int_0^t f(\rho)\,d\rho$$

: $y'' - y = t \;;\; y(0) = 1 \,,\; y'(0) = 1 .$

Taking Laplace transform on both sides:

$L\{y\} = Y$

6.2.

- Laplace transform of Differentials:

* $L\{f'(t)\} = sL\{f(t)\} - f(0)$

* $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

* $L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - sf'(0) - f''(0)$

⋮

* $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$

First Shifting Theorem:

$L\{f(t)\} = F(s)$, $s > K$.

$L\{e^{at} f(t)\} = F(s-a)$ where $s-a > K$

$L^{-1}\{F(s-a)\} = e^{at} L^{-1}\{F(s)\} = e^{at} f(t)$

Eg: $L\{e^{-2t} \cos 2t\} = \dfrac{s+2}{(s+2)^2+4}$

$L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\} = e^{3t}\, t$.

6.1

1. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

2. $L^{-1}\{F(s)\} = f(t)$

3. $L\{1\} = \frac{1}{s}$

4. $L\{t\} = \frac{1}{s^2}$

5. $L\{t^n\} = \frac{n!}{s^{n+1}}$

6. $L\{e^{at}\} = \frac{1}{s-a}$

7. $L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$

8. $L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

9. $L\{\sinh at\} = \frac{a}{s^2-a^2}$

10. $L\{\cosh at\} = \frac{s}{s^2-a^2}$

11. $L\{e^{at}\sin \omega t\} = \frac{\omega}{(s-a)^2+\omega^2}$

12. $L\{e^{at}\cos \omega t\} = \frac{s-a}{(s-a)^2+\omega^2}$