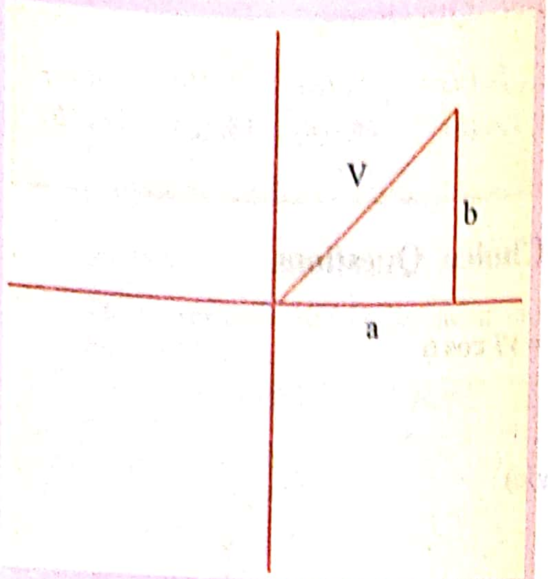




# 17

# Phasor Algebra

## INTRODUCTION

So far we used phasor diagrams to solve problems on a.c. circuits. A phasor diagram is a graphical representation of the phasors (*i.e.*, voltages and currents) of an a.c. circuit and may not yield quick results in case of complex circuits. Engineers have developed techniques to represent a phasor in an algebraic (*i.e.*, mathematical) form. Such a technique is known as *phasor algebra* or *complex algebra*. Phasor algebra has provided a relatively simple but powerful tool for obtaining quick solution of a.c. circuits. It simplifies the mathematical manipulation of phasors to a great extent. In this chapter, we shall discuss the various methods of representing phasors in a mathematical form and their applications to a.c. circuits.

## 17.1. NOTATION OF PHASORS ON RECTANGULAR CO-ORDINATE AXES

Consider a phasor **V** lying along *OX*-axis as shown in Fig. 17.1. If we multiply this phasor by −1, the phasor is reversed *i.e.*, it is rotated through 180° in the counterclockwise (*CCW*) direction. Let us see with what factor the phasor be multiplied so that it rotates through 90° in *CCW* direction. Suppose this factor is *j*. As shown in Fig. 17.1, multiplying the phasor by $j^2$ rotates the phasor through 180° in *CCW* direction. This

...ns that multiplying the phasor by $j^2$ ...the same as multiplying by ... It follows, therefore, that

$$j^2 = -1$$

or

$$j = \sqrt{-1}$$

We arrive at a very important ...nclusion that *when a phasor is ...plied by $j(=\sqrt{-1})$, the phasor is ...ated through 90° in CCW direction.* ...ch successive multiplication by $j$ ...ates the phasor through an additional ...° in the *CCW* direction. It is easy to ...e that multiplying a phasor by

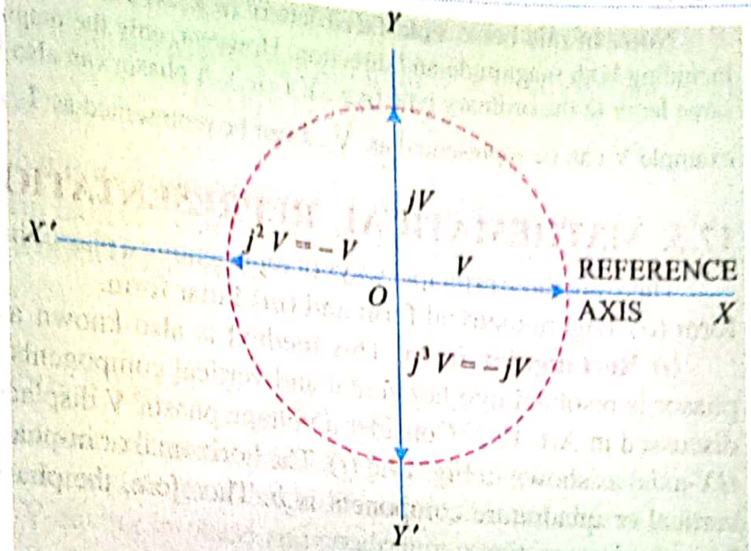

Fig. 17.1

$j = \sqrt{-1}$ ... 90° *CCW* rotation from *OX*-axis

$j^2 = -1$ ... 180° *CCW* rotation from *OX*-axis

$j^3 = j^2 . j = -j$ ... 270° *CCW* rotation from *OX*-axis

$j^4 = j^2 j^2 = 1$ ... 360° *CCW* rotation from *OX*-axis

Fig. 17.1 shows the effect of multiplying a phasor by $j$.

## 17.2. SIGNIFICANCE OF OPERATOR j

Just as the symbol + indicates the operation of adding two numbers, similarly $j$ indicates an operation of rotating the phasor through 90° in *CCW* direction. The operator $j$ does not change the magnitude of the phasor. Consider a phasor **V** displaced θ° counter-clockwise from *OX*-axis as shown in Fig. 17.2. This phasor can be resolved into two rectangular components *viz.* the horizontal component '$a$' along *X*-axis and the vertical component '$b$' along *Y*-axis. It can be seen that vertical component is displaced 90° *CCW* from *OX*-axis. Therefore, mathematically, we can express this component as $jb$, meaning that component $b$ is displaced 90° *CCW* from component $a$ (*i.e.*, *OX*-axis).

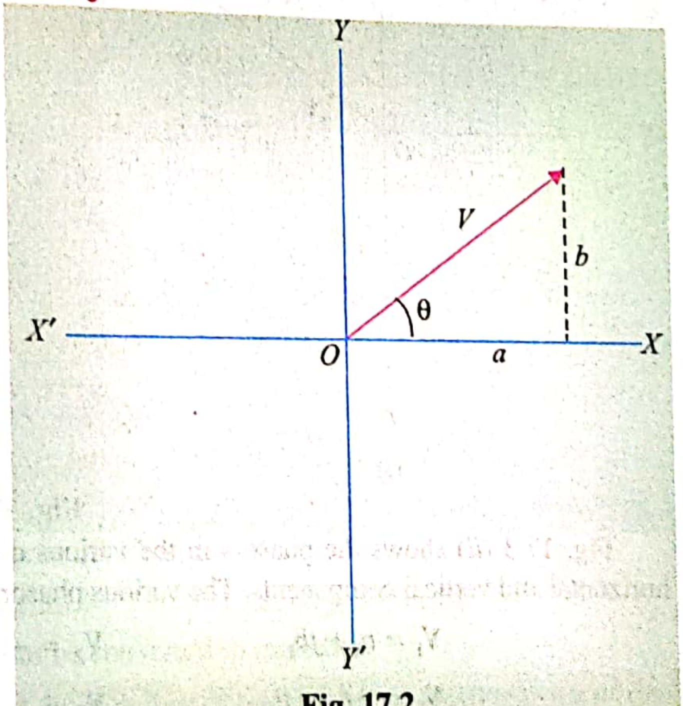

Fig. 17.2

$$\therefore \quad \mathbf{V} = a + jb$$

Magnitude of **V**, $\quad V = \sqrt{a^2 + b^2}$

Its angle with *OX*-axis, $\quad \theta = \tan^{-1}(b/a)$

The reader may note that $a + jb$ is the mathematical form of the phasor **V**.

(*i*) The quantity $a + jb$ is called a complex number or complex quantity.

(*ii*) The horizontal component (*i.e.*, $a$) is called the *in-phase* (or active) component while the vertical component is called *quadrature* (or reactive) component.

**Note.** In this book, bold-faced letters (*e.g.*, **V**, **I** *etc.*) will be used to represent the phasor completely, including both magnitude and direction. However, only the magnitude of the phasor will be represented by the same letter in the ordinary type (*e.g.*, *V*, *I etc.*). A phasor can also be represented by a dot under the symbol. For example **V** can be represented as $\underset{\cdot}{V}$, **I** can be represented as $\underset{\cdot}{I}$, etc.

## 17.3. MATHEMATICAL REPRESENTATION OF PHASORS

There are three principal ways of representing a phasor in the mathematical form *viz.* (*i*) Rectangular form (*ii*) Trigonometrical form and (*iii*) Polar form.

**(*i*) Rectangular form.** This method is also known as *symbolic notation.* In this method, the phasor is resolved into horizontal and vertical components and is expressed in the complex form as discussed in Art. 17.2. Consider a voltage phasor **V** displaced θ° *CCW* from the reference axis (*i.e.*, *OX*-axis) as shown in Fig. 17.3 (*i*). The horizontal or in-phase component of this phasor is *a* while the vertical or quadrature component is *b*. Therefore, the phasor can be represented in the rectangular form as:

$$\mathbf{V} = a + jb$$

Magnitude of phasor, $\quad V = \sqrt{a^2 + b^2}$

Its angle *w.r.t.* *OX*-axis, $\quad \theta = \tan^{-1}(b/a)$

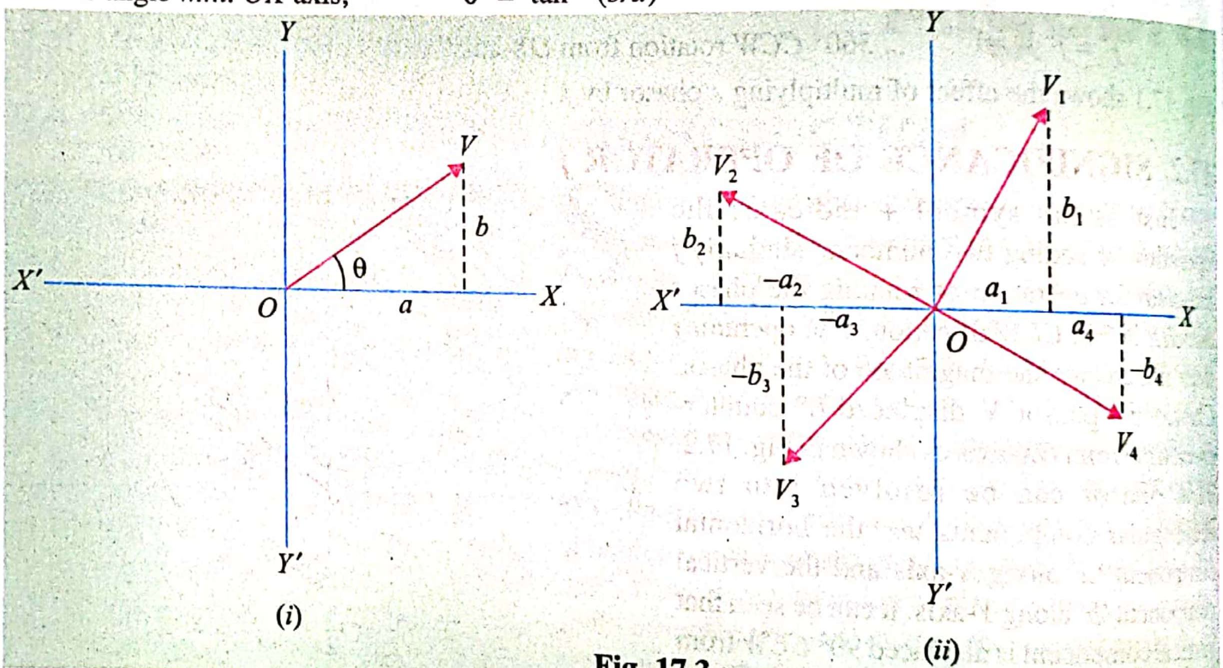


**Fig. 17.3**

Fig. 17.3 (*ii*) shows the phasors in the various quadrants. The phasors have been resolved into horizontal and vertical components. The various phasors can be represented in the rectangular form as:

$$\mathbf{V}_1 = a_1 + jb_1 \qquad \mathbf{V}_2 = -a_2 + jb_2$$

$$\mathbf{V}_3 = -a_3 - jb_3 \qquad \mathbf{V}_4 = a_4 - jb_4$$

The magnitude and phase angles of these phasors can be found out as explained above. Remember phase angles are to be measured from the reference axis *i.e.*, *OX*-axis. If the angle is measured *CCW* from this reference axis, it is assigned a +ve sign. If the angle is measured clockwise from the reference axis, it is assigned a –ve sign.

**(*ii*) Trigonometrical form.** It is similar to the rectangular form except that in-phase and quadrature components of the phasor are expressed in the trigonometrical form. Thus referring to Fig. 17.3 (*i*) above, $a = V\cos\theta$ and $b = V\sin\theta$ where *V* is the magnitude of the phasor **V**. Hence phasor **V** can be expressed in the trigonometrical form as:

$$\mathbf{V} = V(\cos\theta + j\sin\theta)$$

It may be noted that in this form, we express the phasor in terms of its magnitude and phase angle θ.

(iii) **Polar form.** It is a usual practice to write the trigonometrical form $\mathbf{V} = V(\cos\theta + j\sin\theta)$ in what is called polar form as:

$$*\mathbf{V} = V\angle\theta$$

where $V$ is the magnitude of the phasor and θ is its angle measured $CCW$ from the reference axis *i.e.*, $OX$-axis. A negative angle in the polar form indicates clockwise measurement of the angle from the reference axis. Hence, polar form can be written in general as:

$$\mathbf{V} = V\angle\pm\theta$$

## 17.4. CONVERSION FROM ONE FORM TO THE OTHER

The reader may see that the above three mathematical forms of representing a phasor convey the same information *i.e.*, magnitude of the phasor and its phase angle. Therefore, it is possible to convert one form to the other. Consider a phasor **V** having in-phase and quadrature components as 3 and 4 respectively as shown in Fig. 17.4 (*i*).

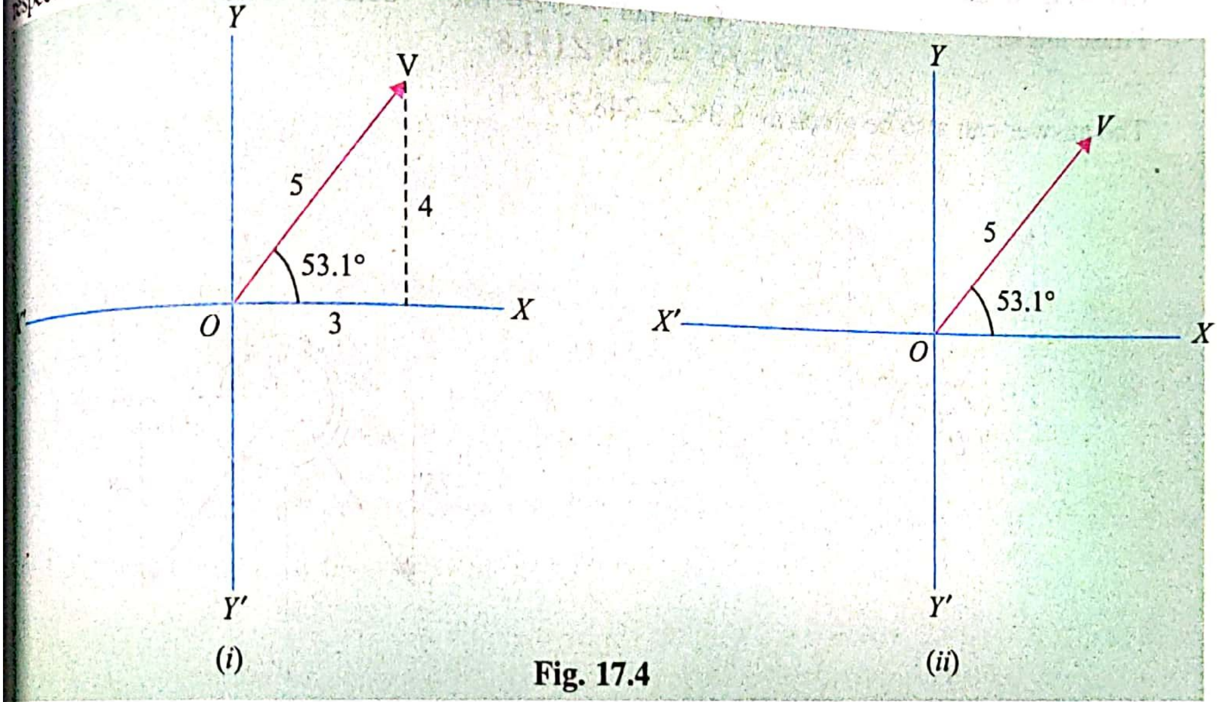


**Fig. 17.4**

| | |
|---|---|
| Magnitude of the phasor, | $V = \sqrt{3^2 + 4^2} = 5$ |
| Its angle w.r.t. $OX$-axis, | $\theta = \tan^{-1} 4/3 = 53.1°$ |
| In rectangular form, | $\mathbf{V} = 3 + j4$ |
| In trigonometrical form, | $\mathbf{V} = 5(\cos 53.1° + j\sin 53.1°)$ |
| In polar form, | $\mathbf{V} = 5\angle 53.1°$ [ See Fig. 17.4 (*ii*)] |

This numerical illustrates how one form can be converted to the other.

**Example 17.1.** *Express the polar form of voltage* $\mathbf{V} = 50\angle 36.87°$ *in trigonometrical and rectangular forms.*

**Solution.**

**Trigonometrical form**

$$\mathbf{V} = V(\cos\theta + j\sin\theta)$$
$$= 50(\cos 36.87° + j\sin 36.87°) \quad \textbf{Ans.}$$

**Rectangular form**

In-phase component $= V\cos\theta = 50 \times \cos 36.87° = 40$

* meaning **V** equals $V$ at angle θ (in $CCW$ direction). There is mathematical explanation for this form. It is a shorthand form of trigonometrical form.

Quadrature component $= V \sin\theta = 50 \times \sin 36.87° = 30$

$\mathbf{V} = 40 + j30$ **Ans.**

**Example 17.2.** *Express the following in polar form* (i) $3 + j7$ (ii) $-2 + j5$ (iii) $-50 - j75$ and (iv) $6 -$

**Solution.**

(i) Magnitude $= \sqrt{3^2 + 7^2} = 7.62$

Phase angle, $\theta = \tan^{-1}(7/3) = 66.8°$

$\therefore \quad 3 + j7 = \mathbf{7.62\angle 66.8°}$

Note that $3 + j\,7$ lies in the first quadrant. As seen in Fig. 17.5 (i), $\theta$ measured in CCW direction from $OX$-axis is 66.8° and is, therefore, positive. If $\theta$ is measured in clockwise direction from axis, its value is = 360° − 66.8° = 293.2° and this angle is negative. Hence the above answer can be given as $7.62\angle -293.2°$.

(ii) Magnitude $= \sqrt{(-2)^2 + (5)^2} = 5.39$

Phase angle, $\theta = \tan^{-1} - 5/2 = 180° - 68.2° = 111.8°$

$\therefore \quad -2 + j\,5 = \mathbf{5.39\angle 111.8°}$

The answer can also be given as $5.39\angle -248.2°$.

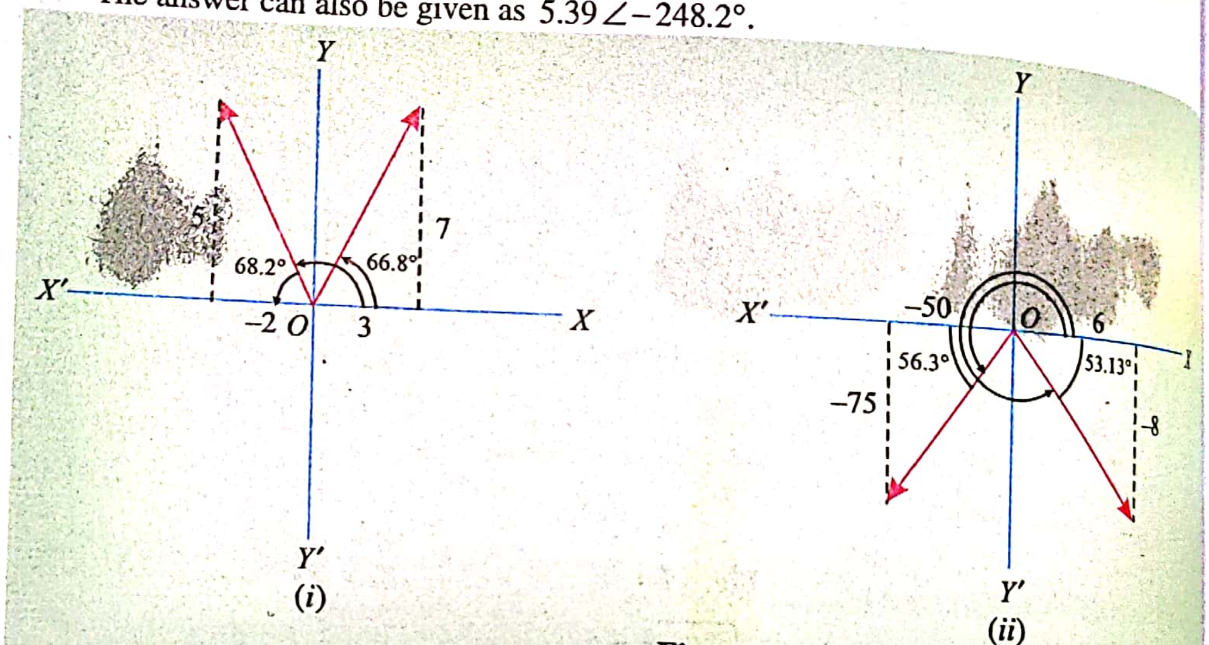


**Fig. 17.5**

(iii) Magnitude $= \sqrt{(-50)^2 + (-75)^2} = 90.1$

Phase angle, $\theta = \tan^{-1} - 75/-50 = 180° + 56.3° = 236.3°$

$\therefore \quad -50 - j\,75 = \mathbf{90.1\ \angle\ 236.3°}$

The answer can also be given as $90.1\angle -123.7°$.

(iv) Magnitude $= \sqrt{6^2 + (-8)^2} = 10$

Phase angle, $\theta = \tan^{-1} - 8/6 = 360° - 53.13° = 306.87°$

$\therefore \quad 6 - j\,8 = \mathbf{10\ \angle\ 306.87°}$

The answer can also be given as $10\ \angle\ \mathbf{-53.13°}$.

## 17.5. ADDITION AND SUBTRACTION OF PHASORS

The rectangular form is best suited for addition or subtraction of phasors. If the phasors are given in polar form, they should be first converted to rectangular form and then addition or subtraction be carried out.

*(i)* **Addition.** For the addition of phasors in the rectangular form, the in-phase components are added together and the quadrature components are added together. Consider two voltage phasors represented as:

$$\mathbf{V}_1 = a_1 + jb_1 \; ; \quad \mathbf{V}_2 = a_2 + jb_2$$

Then resultant,
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (a_1 + jb_1) + (a_2 + jb_2)$$
$$= (a_1 + a_2) + j(b_1 + b_2)$$

Magnitude of resultant,
$$V = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

Its angle from *OX*-axis,
$$\theta = \tan^{-1}\frac{(b_1 + b_2)}{a_1 + a_2}$$

*(ii)* **Subtraction.** Like addition, the subtraction of phasors is done by using ordinary rules of phasor algebra.

$$\mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2 = (a_1 + jb_1) - (a_2 + jb_2)$$
$$= (a_1 - a_2) + j(b_1 - b_2)$$

$\therefore$
$$V = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$$

Phase angle,
$$\theta = \tan^{-1}\left(\frac{b_1 - b_2}{a_1 - a_2}\right)$$

**Example 17.3.** *Two current phasors are given in the rectangular form as* $\mathbf{I}_1 = 15 + j10$ *and* $\mathbf{I}_2 = 12 + j6$. *Perform the operation (i)* $\mathbf{I}_1 + \mathbf{I}_2$ *and (ii)* $\mathbf{I}_1 - \mathbf{I}_2$.

**Solution.**

*(i)* Resultant current, $\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = (15 + j10) + (12 + j6) = \mathbf{27 + j16}$

The magnitude of resultant current is $= \sqrt{27^2 + 16^2} = 31.38$ and it makes an angle $\theta = \tan^{-1} 16/27 = 30.65°$ with *OX*-axis as shown in Fig. 17.6 (*i*).

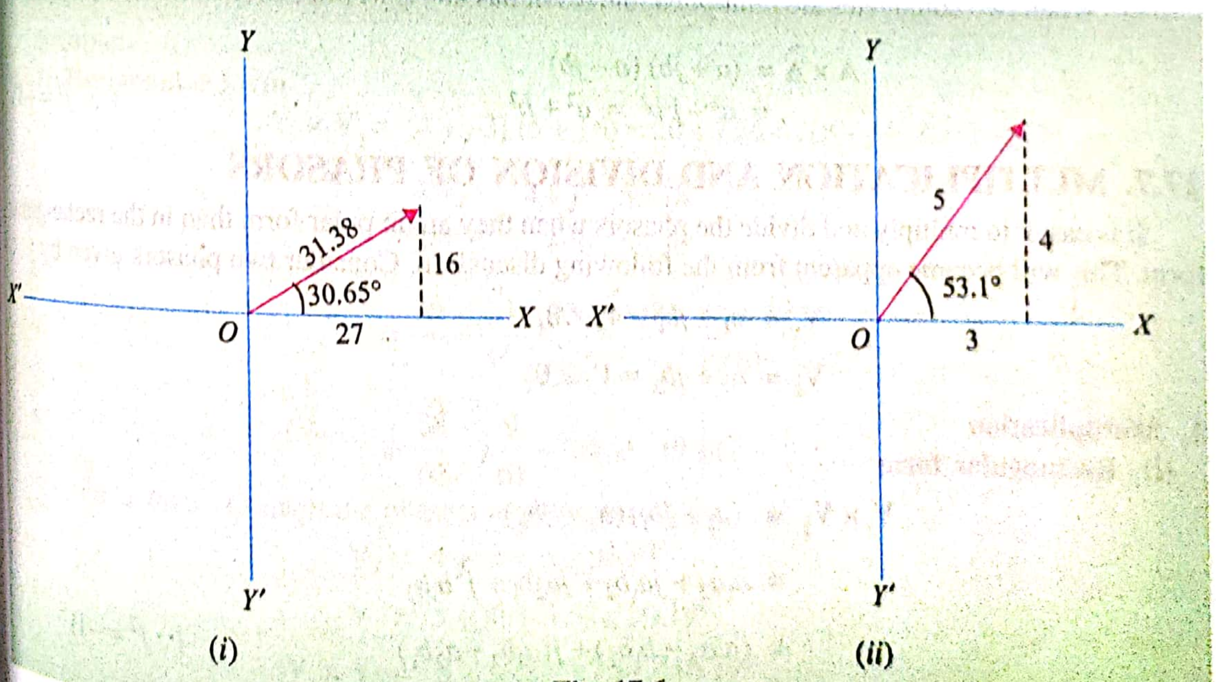


**Fig. 17.6**

*(ii)* Resultant current, $\mathbf{I} = \mathbf{I}_1 - \mathbf{I}_2 = (15 + j\,10) - (12 + j\,6) = \mathbf{3 + j4}$

The magnitude of resultant is $=\sqrt{3^2+4^2}=5$ and it makes an angle $\theta = \tan^{-1} 4/3 = 53.1°$ w
*OX*-axis as shown in Fig. 17.6 (*ii*).

**Example 17.4.** *Determine the resultant voltage of two sinusoidal generators in series wh voltages are* $\mathbf{V}_1 = 25\angle 15°\text{V}$ and $\mathbf{V}_2 = 15\angle 60°\text{V}$.

**Solution.** We shall first convert polar form to rectangular form and then carry out the additi

$$\mathbf{V}_1 = 25(\cos 15° + j\sin 15°) = (24.15 + j6.47)\ \text{V}$$

$$\mathbf{V}_2 = 15\ (\cos 60° + j\sin 60°) = (7.5 + j\ 12.99)\text{V}$$

$\therefore$ $$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 = (24.15 + j6.47)\text{V} + (7.5 + j\ 12.99)\text{V} = (31.65 + j19.4$$

$\therefore$ $$V = \sqrt{(31.65)^2 + (19.46)^2} = 37.15$$

Phase angle, $$\theta = \tan^{-1}\ (19.46/31.65) = 31.6°$$

$\therefore$ $$\mathbf{V} = 37.15\ \angle\ 31.6°\ \text{V}$$

## 17.6. CONJUGATE OF A COMPLEX NUMBER

Two complex numbers (or phasors) are said to be conjugate if they differ only in the algebr sign of their quadrature components. Thus, the conjugate of $2 + j\,3$ is $2 - j\,3$. The conjugate of $-4$ $j3$ is $-4 - j3$. In the polar form, conjugate of $5\ \angle\ 30°$ is $5\ \angle -30°$ and conjugate of $10\ \angle -40°$ $10\ \angle\ 40°$. It is a usual practice to use an asterik (*)to indicate the conjugate.

Consider a complex number $\mathbf{A} = a + jb = \text{A}\ \angle\ \theta$. Then its conjugate will be $\mathbf{A}^* = a - jb = \text{A}\angle$ The conjugate numbers have the following properties:

(*i*) The sum of two conjugate numbers results in in-phase component only (*i.e.*, no *j* part).

$$\mathbf{A} + \overset{*}{\mathbf{A}} = (a + jb) + (a - jb) = 2a$$

(*ii*) The difference of two conjugate numbers results in quadrature component only.

$$\mathbf{A} - \overset{*}{\mathbf{A}} = (a + jb) - (a - jb) = j\,2b$$

(*iii*) When two conjugates are multiplied, the result has no *j*-part (*i.e.*, no quadrature compone

$$\mathbf{A} \times \overset{*}{\mathbf{A}} = (a + jb)\ (a - jb)$$

$$= a^2 - j^2b^2 = a^2 + b^2 \qquad (\because j^2 = -1$$

## 17.7. MULTIPLICATION AND DIVISION OF PHASORS

It is easier to multiply and divide the phasors when they are in polar form than in the rectangul form. This will become apparent from the following discussion. Consider two phasors given by;

$$\mathbf{V}_1 = a_1 + jb_1 = V_1\angle\theta_1$$

$$\mathbf{V}_2 = a_2 + jb_2 = V_2\angle\theta_2$$

**1. Multiplication**

(*i*) **Rectangular form**

$$\mathbf{V}_1 \times \mathbf{V}_2 = (a_1 + jb_1)(a_2 + jb_2)$$

$$= a_1a_2 + ja_1b_2 + ja_2b_1 + j^2b_1b_2$$

$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1) \qquad (\because j^2 = -1)$$

Magnitude of resultant $$= \sqrt{(a_1a_2 - b_1b_2)^2 + (a_1b_2 + a_2b_1)^2}$$

Its angle *w.r.t.* *OX*-axis, $\theta = \tan^{-1}\left(\dfrac{a_1b_2 + a_2b_1}{a_1a_2 - b_1b_2}\right)$

(*ii*) **Polar form.** To multiply the phasors that are in polar form, multiply their magnitudes and add the angles (algebraically).

$$\mathbf{V}_1 \times \mathbf{V}_2 = V_1\angle\theta_1 \times V_2\angle\theta_2 = V_1V_2\angle\theta_1 + \theta_2$$

The reader may see that multiplication of phasors becomes easier when they are expressed in polar form.

**2. Division**

(*i*) **Rectangular form**

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{a_1 + jb_1}{a_2 + jb_2}$$

$$= \frac{a_1 + jb_1}{a_2 + jb_2} \times \frac{a_2 - jb_2}{a_2 - jb_2} \qquad \left[\begin{matrix}\text{Rationalising} \\ \text{the denominator}\end{matrix}\right]$$

$$= \frac{(a_1a_2 + b_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

$$= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + j\frac{(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2}$$

(*ii*) **Polar form.** To divide the phasors that are in polar form, divide the magnitude of phasors and subtract the denominator angle from the numerator angle.

$$\therefore \quad \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{V_1\angle\theta_1}{V_2\angle\theta_2} = \frac{V_1}{V_2}\angle\theta_1 - \theta_2$$

Practically without exception, division is the easiest in the polar form.

**Example 17.5.** *Two phasors are given in the following form:*

$\mathbf{V}_1 = 4 + j3$; $\qquad \mathbf{V}_2 = 5 + j6$

*Evaluate* $\mathbf{V}_1 \times \mathbf{V}_2$ and $\mathbf{V}_1/\mathbf{V}_2$ *in (i) rectangular form (ii) polar form.*

**Solution.**

(*i*) **Rectangular form**

$$\mathbf{V}_1 \times \mathbf{V}_2 = (4 + j3)(5 + j6) = 20 + j24 + j15 + j^2 18 = \mathbf{2 + j\,39}$$

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{4 + j3}{5 + j6} = \frac{4 + j3}{5 + j6} \times \frac{5 - j6}{5 - j6}$$

$$= \frac{20 - j24 + j15 - j^2 18}{5^2 - j^2 6^2} = \frac{38 - j9}{61}$$

$$= \frac{38}{61} - j\frac{9}{61} = \mathbf{0.623 - j0.147}$$

(*ii*) **Polar form.** Convert the phasors to polar form.

$$\mathbf{V}_1 = 4 + j3 = 5\angle 36.87°$$

$$\mathbf{V}_2 = 5 + j6 = 7.81\angle 50.19°$$

$$\therefore \quad \mathbf{V}_1 \times \mathbf{V}_2 = 5\angle 36.87° \times 7.81\angle 50.19°$$

$$= 5 \times 7.81\angle 36.87° + 50.19° = \mathbf{39.05\angle 87.06°}$$

$$\frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{5\angle 36.87^\circ}{7.81\angle 50.19^\circ} = \frac{5}{7.81}\angle 36.87^\circ - 50.19^\circ = 0.64\angle -13.32^\circ$$

The negative angle means that the phasor is below *OX*-axis.

**Example 17.6.** *The following three phasors are given:*

$$\mathbf{A} = 5 + j5; \quad \mathbf{B} = 50\angle 40^\circ; \quad \mathbf{C} = 4 + j0$$

*Perform the following indicated operations:*

(*i*) $\dfrac{\mathbf{AB}}{\mathbf{C}}$ (*ii*) $\dfrac{\mathbf{BC}}{\mathbf{A}}$

**Solution.** Expressing the three phasors in polar form, we have, $\mathbf{A} = 5 + j5 = 7.07\angle 45^\circ$; $\mathbf{B} = 50\angle 40^\circ$; $\mathbf{C} = 4 + j0 = 4\angle 0^\circ$

(*i*) $$\frac{\mathbf{AB}}{\mathbf{C}} = \frac{7.07\angle 45^\circ \times 50\angle 40^\circ}{4\angle 0^\circ} = \mathbf{88.37\angle 85^\circ}$$

(*ii*) $$\frac{\mathbf{BC}}{\mathbf{A}} = \frac{50\angle 40^\circ \times 4\angle 0^\circ}{7.07\angle 45^\circ} = \mathbf{28.29\angle -5^\circ}$$

**Example 17.7.** *The applied voltage in a series circuit is given by* $\mathbf{V} = 200\angle 0^\circ$ *volts. The impedance offered by the circuit is* $\mathbf{Z} = 83.3\angle 40^\circ\,\Omega$. *Find the magnitude of current. Is the circuit inductive or capacitive ?*

**Solution.**

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{200\angle 0^\circ}{83.3\angle 40^\circ} = \mathbf{2.4\angle -40^\circ\,A}$$

The voltage phasor lies along *OX*-axis. The negative angle of circuit current means that current phasor is below *OX*-axis. Hence the circuit is **inductive.**

## 17.8. POWERS AND ROOTS OF PHASORS

Powers and roots of complex numbers can be found very conveniently in polar form. If the complex number is not in polar form, it is always advisable to convert the number to this form and then carry out these algebraic operations.

**(*i*) Powers.** Suppose it is required to find the cube of the phasor $2\angle 10^\circ$. For this purpose, the phasor has to be multiplied by itself three times.

$$\therefore \quad (2\angle 10^\circ)^3 = 2 \times 2 \times 2\ \angle 10^\circ + 10^\circ + 10^\circ = 8\angle 30^\circ$$

In general, $$\mathbf{A}^n = \mathrm{A}^n\angle n\theta$$

**(*ii*) Roots.** Let us proceed backward with the above given example. Suppose we are to find the cube root of $8\angle 30^\circ$. It is clear that :

$$\sqrt[3]{8\angle 30^\circ} = 2\angle 10^\circ$$

In general, $$\sqrt[n]{\mathbf{A}} = \sqrt[n]{\mathrm{A}}\angle\frac{\theta}{n}$$

Thus the square root of $64\angle 40^\circ$ is $= \sqrt[2]{64}\angle\dfrac{40^\circ}{2} = 8\angle 20^\circ$

## 17.9. PHASOR ALGEBRA APPLIED TO R-L SERIES CIRCUIT

Fig. 17.7 shows an *R-L* series circuit and its phasor diagram. Since the circuit current is taken as the reference phasor (*i.e.*, along *OX*-axis), we have,

$$\mathbf{I} = I + j0$$