Expectation of jointly distribution

(iii) If X & Y are discrete with pmf $p(x,y)$ then expectation of the function of jointly distributed r.v. $h(X,Y)$ is

$$E[h(X,Y)] = \sum_y \sum_x h(x,y)\, p(x,y)$$

(a) If $h(X,Y) = X$, then

$$E(X) = \sum_y \sum_x x\, p(x,y)$$

(b) If $h(X,Y) = Y$, then

$$E(Y) = \sum_y \sum_x y\, p(x,y)$$

(c) If $h(X,Y) = X \pm Y$, then

$$E(X \pm Y) = \sum_y \sum_x (x \pm y)\, p(x,y)$$

$$= \sum_y \sum_x x\, p(x,y) \pm \sum_y \sum_x y\, p(x,y)$$

$$= E(X) \pm E(Y)$$

(d) If $h(X,Y) = XY$, then

$$E(XY) = \sum_y \sum_x xy\, p(x,y) \neq E(X)\,E(Y)$$ in general

Now if X & Y are independent, then

$$p(x,y) = p_X(x)\, p_Y(y)$$

and $E(XY) = \left(\sum_x x\, p_X(x)\right)\left(\sum_y y\, p_Y(y)\right) = E(X)\,E(Y)$

Covariance of X & Y

If (X, Y) is the jointly discrete r.v. with mean (μ_X, μ_Y) where $\mu_X = E(X)$ & $\mu_Y = E(Y)$ then covariance of X & Y is

$$Cov(X, Y) = \sum_y \sum_x (x-\mu_X)(y-\mu_Y)\, p(x,y)$$

Q. Prove that

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Proof

$$Cov(X,Y) = \sum_y \sum_x (x-\mu_X)(y-\mu_Y)\, p(x,y)$$

$$= \sum_y \sum_x xy\, p(x,y) - \mu_X \sum_y \sum_x y\, p(x,y) - \mu_Y \sum_y \sum_x x\, p(x,y) + \mu_X \mu_Y \sum_y \sum_x p(x,y)$$

$$\boxed{\because \sum_y \sum_x p(x,y) = 1}$$

$$= \sum_y \sum_x xy\, p(x,y) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y$$

$$= E(XY) - E(X)E(Y)$$

Note If X & Y are independent, then $E(XY) = E(X)E(Y)$, therefore

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

Q. Prove that $Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y)$

proof Variance of $X+Y = Var(X+Y)$

Now

$$Var(X+Y) = E(X+Y)^2 - [E(X+Y)]^2$$

$$= E[X^2 + 2XY + Y^2] - [E(X) + E(Y)]^2$$

$\boxed{\because Var(X) = E(X^2) - [E(X)]^2}$

$$= E(X^2) + 2E(XY) + E(Y^2) - \left([E(X)]^2 + 2E(X)E(Y) + (E(Y))^2 \right)$$

$$= \left[E(X^2) - (E(X))^2\right] + \left[E(Y^2) - (E(Y))^2\right] + 2\left[E(XY) - E(X)E(Y)\right]$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y) \quad \text{proved}$$

Corelation coefficient

If (X, Y) is any jointly distribution (distrete or continuous), then corelation coefficient of X & Y is defined by

$$corr(X,Y) = \frac{cov(X,Y)}{SD(X)\,SD(Y)}$$

where $SD(X)$ = standard deviation of X
$SD(Y)$ = " " " Y.

Notation

① $var(X)$ = variance of $X = \sigma_X^2$, $SD(X) = \sigma_X$
$var(Y)$ = variance of $Y = \sigma_Y^2$, $SD(Y) = \sigma_Y$

② $$Corr(X,Y) = \rho(X,Y) = \frac{cov(X,Y)}{\sigma_X\,\sigma_Y}$$

Note If X & Y are independent, then $\rho = 0$ as $cov(X,Y) = 0$

Q. Prove that

$$-1 \le \rho(X, Y) \le 1$$

proof correlation coefficient of X & Y

is
$$\rho(X, Y) = \mathrm{corr}(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sigma_X \, \sigma_Y}$$

We have

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{cov}(X, Y)$$

So
$$\begin{aligned}
\mathrm{Var}(aX + bY) &= \mathrm{Var}(aX) + \mathrm{Var}(bY) + 2\,\mathrm{cov}(aX, bY) \\
&= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{cov}(X, Y)
\end{aligned}$$

Taking $a = \frac{1}{\sigma_X}$, $b = \frac{1}{\sigma_Y}$, we get

$$\begin{aligned}
0 \le \mathrm{Var}\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) &= \frac{1}{\sigma_X^2}\,\mathrm{Var}(X) + \frac{1}{\sigma_Y^2}\,\mathrm{Var}(Y) + 2 \cdot \frac{1}{\sigma_X} \cdot \frac{1}{\sigma_Y}\,\mathrm{cov}(X, Y) \\
&= 1 + 1 + 2\,\mathrm{corr}(X, Y) \\
&= 2\left[1 + \rho(X, Y)\right]
\end{aligned}$$

$\Rightarrow -1 \le \rho(X, Y)$

Similarly by taking $a = \frac{1}{\sigma_X}$, $b = -\frac{1}{\sigma_Y}$, we get

$$0 \le \mathrm{Var}\left(\frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}\right) = 2\left[1 - \rho(X, Y)\right]$$

$\Rightarrow \rho(X, Y) \le 1$. Hence $-1 \le \rho(X, Y) \le 1$

proved

Q From the jointly discrete distribution table,

$X \downarrow$ \ $Y \rightarrow$	0	1	2
0	1/8	1/8	0
1	1/8	2/8	1/8
2	0	1/8	1/8

find $\rho(X,Y)$

Sol^n $\text{corr}(X,Y) = \rho(X,Y) = \dfrac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$\sigma_X^2 = E(X^2) - (E(X))^2$, $\sigma_Y^2 = E(Y^2) - (E(Y))^2$

x	y	$p(x,y)$	x^2	y^2	xp	yp	x^2p	y^2p	xy	xyp
0	0	1/8	0	0	0	0	0	0	0	0
0	1	1/8	0	1	0	1/8	0	1/8	0	0
0	2	0	0	4	0	0	0	0	0	0
1	0	1/8	1	0	1/8	0	1/8	0	0	0
1	1	1/4	1	1	1/4	1/4	1/4	1/4	1	1/4
1	2	1/8	1	4	1/8	1/4	1/8	1/2	2	1/4
2	0	0	4	0	0	0	0	0	0	0
2	1	1/8	4	1	1/4	1/8	1/2	1/8	2	1/4
2	2	1/8	4	4	1/4	1/4	1/2	1/2	4	1/2
					$E(X) = 1$	$E(Y) = 1$	$E(X^2) = \frac{3}{2}$	$E(Y^2) = \frac{3}{2}$		$E(XY) = 5/4$

$$\sigma_x^2 = E(x^2) - (E(x))^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow \sigma_x = \frac{1}{\sqrt{2}}$$

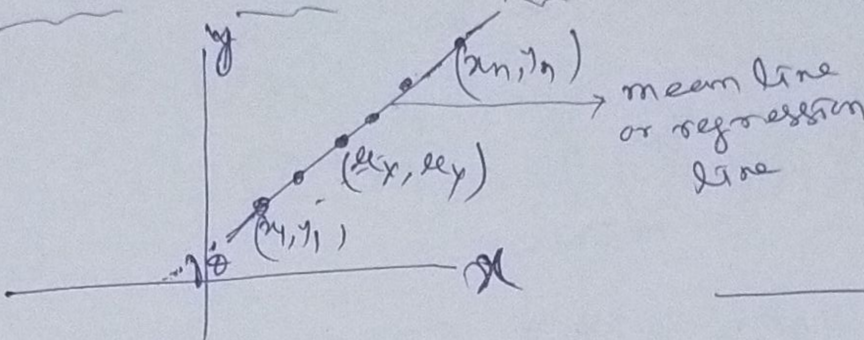
$$\sigma_y^2 = E(y^2) - (E(y))^2 = \frac{3}{2} - 1 = \frac{1}{2} \Rightarrow \sigma_y = \frac{1}{\sqrt{2}}$$

$$\text{cov}(x, y) = E(xy) - E(x)\,E(y)$$
$$= \frac{5}{4} - 1 = \frac{1}{4}$$

$$\rho(x, y) = \frac{\text{cov}(x, y)}{\sigma_x \sigma_y} = \frac{1/4}{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} = \frac{1/4}{1/2} = 1/2 = 0.5$$

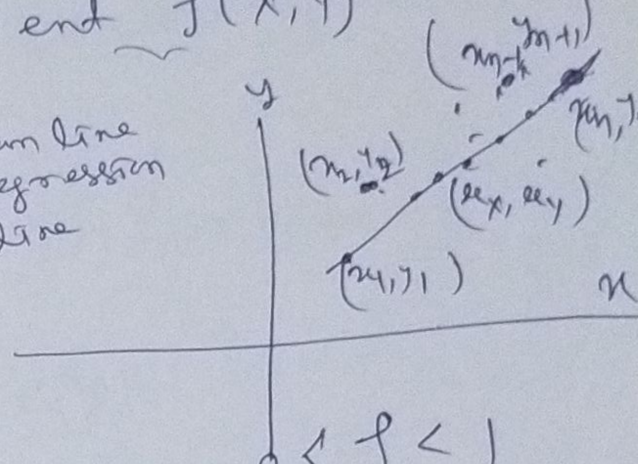
Note: The straight line passing through mean pt (μ_x, μ_y) is called regression line

Graphical interpretation of correlation coefficient $\rho(x, y)$



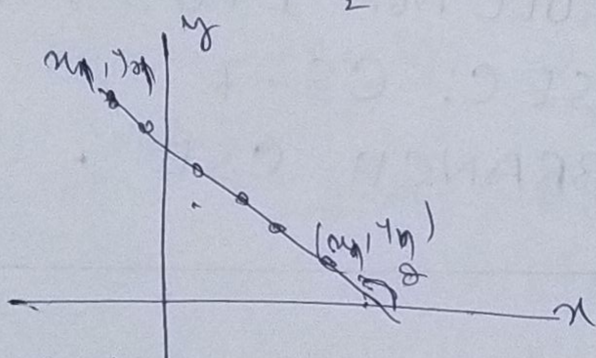
$\rho = 1$

(All points (x_i, y_i) lie on the mean line passing through (μ_x, μ_y) & slope making acute angle) $0 < \theta < \frac{\pi}{2}$

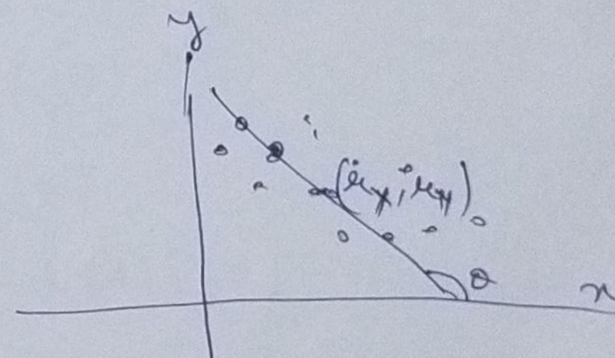


$0 < \rho < 1$

(Some points lie on the mean line)



All pts (x_i, y_i) lie on mean line which making obtuse angle $(\frac{\pi}{2} < \theta < \pi)$



Some points lie on mean line

If (X, Y) is jointly continuous r.v. with pdf $f(x,y)$, then expectation of the function $h(X,Y)$ is

$$E(h(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy$$

(a) for $h(X,Y) = X$,

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$$

(b) for $h(X,Y) = Y$,

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$$

(c) for $h(X,Y) = X \pm Y$,

$$E(X \pm Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x \pm y) f(x,y)\, dx\, dy$$

$$= E(X) \pm E(Y)$$

(d) for $h(X,Y) = XY$,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\, dx\, dy \neq E(X) E(Y)$$ in general

If X & Y are independent, then

$\Rightarrow$ $$f(x,y) = f_X(x) f_Y(y)$$

implying $$E(XY) = E(X) E(Y)$$

$$\therefore \operatorname{cov}(X,Y) = E(XY) - E(X)E(Y) = 0$$

Q. $\operatorname{cov}(aX+b, cY+d) = ac \operatorname{cov}(X,Y)$. Prove

Proof $\operatorname{cov}(aX+b, cY+d) = E[(aX+b)(cY+d)] - E(aX+b) E(cY+d)$

$= ac[E(XY) - E(X)E(Y)] = ac \operatorname{cov}(X,Y)$

Q. $\text{Corr}(aX+b, cY+d) = \begin{cases} \text{corr}(X,Y) & \text{if } ac>0 \\ -\text{corr}(X,Y) & \text{if } ac<0 \end{cases}$

Proof

$$\text{corr}(aX+b, cY+d) = \frac{\text{cov}(aX+b, cY+d)}{SD(aX+b)\,SD(cY+d)}$$

Now

$$\text{var}(aX+b) = a^2\,\text{var}(X) = a^2\sigma_X^2$$

$$\text{var}(cY+d) = c^2\,\text{var}(Y) = c^2\sigma_Y^2$$

implying $SD(aX+b) = |a|\sigma_X$, $SD(cY+d) = |c|\sigma_Y$

Hence

$$\text{corr}(aX+b, cY+d) = \frac{ac\,\text{cov}(X,Y)}{|a||c|\,\sigma_X\sigma_Y}$$

$$= \begin{cases} \dfrac{\text{cov}(X,Y)}{\sigma_X\sigma_Y} & \text{if } ac>0 \\ -\dfrac{\text{cov}(X,Y)}{\sigma_X\sigma_Y} & \text{if } ac<0 \end{cases}$$

$$= \begin{cases} \text{corr}(X,Y) & \text{if } ac>0 \\ -\text{corr}(X,Y) & \text{if } ac<0 \end{cases}$$

proved

Note

① X & Y are independent, then
$\rho = \text{corr}(X,Y) = 0$
but not conversely.

② $\rho = 1$ or -1 if and only if $Y = aX+b$,
for some numbers a, b; $a \neq 0$

Q. The jointly pdf of the r.v (X, Y) is

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $\operatorname{corr}(X, Y)$.

Solution

$$\operatorname{corr}(X,Y) = \rho = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X^2 = \operatorname{Var}(X) = E(X^2) - (E(X))^2$$

$$\sigma_Y^2 = \operatorname{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$\operatorname{cov}(X,Y) = E(XY) - E(X)\,E(Y)$$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 x \cdot \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= 3/5$$

$$E(X^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f(x,y)\,dx\,dy = \int_0^1\int_0^1 x^2 \cdot \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= 13/30$$

$$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 y \cdot \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= 3/5$$

$$E(Y^2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f(x,y)\,dx\,dy = \int_0^1\int_0^1 y^2 \cdot \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= 11/25$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy \cdot \frac{6}{5}(x+y^2)\,dx\,dy$$

$$= 7/20$$

$\sigma_x^2 = E(X^2) - (E(X))^2 = \frac{13}{30} - \left(\frac{3}{5}\right)^2 = \frac{11}{150}$

$\Rightarrow \sigma_x = 0.5204$

$\sigma_y^2 = E(Y^2) - (E(Y))^2 = \frac{11}{25} - \left(\frac{3}{5}\right)^2 = \frac{2}{25}$

$\sigma_y = 0.2828$

$\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{7}{20} - \frac{3}{5} \cdot \frac{3}{5} = -\frac{1}{100} = -0.01$

$\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_x \sigma_y} = \frac{-0.01}{0.5204 \times 0.2828}$

$= -0.0679$

Ans

Q. The jointly pdf of (X,Y) is

$$f(x,y) = \begin{cases} Kxy, & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) Find K

(ii) Find $E(X)$, $var(X)$, σ_x

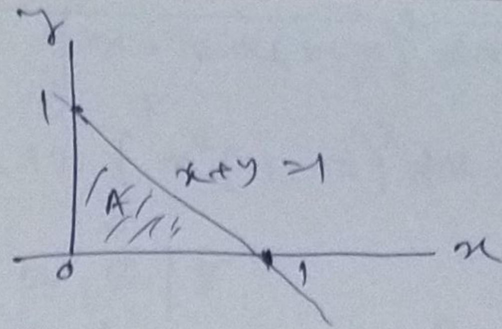
(iii) Find $E(Y)$, $var(Y)$, σ_y

(iv) Find $E(XY)$, $cov(X,Y)$

(v) Find $corr(X,Y)$

(vi) X & Y are independent or not

Solution:



(i) By unit property of jointly density function,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \iint_A kxy\,dx\,dy = 1$$

$$\Rightarrow \int_0^1 \int_0^{1-x} kxy\,dy\,dx = 1$$

$$\Rightarrow k = 24$$

$\because x+y=1$
$\Rightarrow y = 1-x$
Since $y \geq 0$,
$0 \leq y \leq 1-x$

We have

$$f(x,y) = \begin{cases} 24xy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 24xy & \text{for } (x,y) \in A \\ 0 & \text{otherwise} \end{cases}$$

where $A = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1\}$

(ii) Marginal density of X

$$= f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{1-x} 24xy\,dy$$

$$= \begin{cases} 12x(1-x)^2, & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x \cdot 12x(1-x)^2\, dx$$
$$= 12\int_0^1 x^2(1-x)^2\, dx$$
$$= 2/5$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2 \cdot 12x(1-x)^2\, dx$$
$$= 12\int_0^1 x^3(1-x)^2\, dx$$
$$= 1/5$$

$$\mathrm{var}(X) = E(X^2) - (E(X))^2 = \frac{1}{5} - \frac{4}{25} = \frac{1}{25}$$
$$\Rightarrow \sigma_X = \sqrt{\mathrm{var}(X)} = 1/5$$

(iii) Marginal density of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^{1-y} 24xy\, dx$$
$$= \begin{cases} 12y(1-y)^2, & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

~~(vi) Since $f(x,y) \neq f_X(x) f_Y(y)$, X & Y are not independent, implying $\mathrm{cov}(X,Y) \neq 0$~~

Since $f_X(x)$ & $f_Y(y)$ are symmetrical, so

$E(Y) = 2/5$, $E(Y^2) = 1/5$

$\mathrm{var}(Y) = 1/25$ & $\sigma_Y = \frac{1}{5}$

(vi) $f_X(x) f_Y(y) \neq f(x,y)$ so X & Y are not independent

implying $\mathrm{cov}(X,Y) \neq 0$

(iv) $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$

$= \iint_A xy \cdot 24xy\, dx\, dy$

$= 24 \int_0^1 \int_0^{1-x} x^2 y^2\, dy\, dx$

$= 24 \int_0^1 x^2 \frac{(1-x)^3}{3}\, dx = \frac{2}{15}$

$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$= \frac{2}{15} - \frac{2}{5} \cdot \frac{2}{5} = -\frac{2}{75}$

(v) $\text{corr}(X,Y) = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y}$

$= \frac{-2/75}{(1/5)(1/5)} = -\frac{2}{3}$

$= -0.6667$

Ans

Conditional Distribution

Let (X, Y) be jointly random variable, then

ⓐ jointly pmf is

$$p(x,y) = P(X=x, Y=y)$$
$$= P[X=x \cap Y=y]$$

ⓑ jointly pdf is

$$f(x,y) = P[X=x, Y=y]$$

Conditional probability ~~density~~ mass function (cpmf) of Y with given $X=x$ is defined by

$$p_{Y|X}(y|x) = p(y|x)$$
$$= \frac{p(x,y)}{p_X(x)} \quad \text{if } p_X(x) > 0$$

Conditional probability density function (cpdf) of Y with given $X=x$ is defined by

$$f_{Y|X}(y|x) = f(y|x)$$
$$= \frac{f(x,y)}{f_X(x)} \quad \text{if } f_X(x) > 0$$

Note

(i) If (X,Y) is discrete, then

$$p_X(x) = \sum_y p(x,y) \;\&\; p_Y(y) = \sum_x p(x,y)$$

(ii) If (X,Y) is continuous then

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \;\&\; f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Ex-1

$x \backslash y$	0	1	2	$p_X(x)$
0	1/8	1/8	0	$\frac{2}{8} = p_X(0)$
1	1/8	2/8	1/8	$\frac{4}{8} = p_X(1)$
2	0	1/8	1/8	$\frac{2}{8} = p_X(2)$
$p_Y(y)$:	$\frac{2}{8} = p_Y(0)$	$\frac{4}{8} = p_Y(1)$	$\frac{2}{8} = p_Y(2)$	

(a) $P(Y=1 \mid X=1) = \dfrac{p(1,1)}{p_X(1)} = \dfrac{2/8}{4/8} = \dfrac{1}{2}$

(b) $P(X=0 \mid Y=2) = \dfrac{p(0,2)}{p_Y(2)} = \dfrac{0}{2/8} = 0$

(c) $P[Y=y \mid X=0] = \dfrac{p(0,y)}{p_X(0)} = \dfrac{2}{8}\,p(0,y)$
$= \dfrac{1}{2}\,p(0,y), \quad y = 0,1,2$

(d) $P[Y>0 \mid X=1] = P[Y=1 \mid X=1] + P[Y=2 \mid X=1]$
$= \dfrac{p(1,1)}{p_X(1)} + \dfrac{p(1,2)}{p_X(1)} = \dfrac{1}{4/8}\left[\dfrac{2}{8} + \dfrac{1}{8}\right]$
$= 3/4$

Ex-2

The jointly pdf is

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2), & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where

$f_X(x) = \frac{6}{5}\left(x + \frac{1}{3}\right), \quad 0 \le x \le 1$

$f_Y(y) = \frac{6}{5}\left(\frac{1}{2} + y^2\right), \quad 0 \le y \le 1$

(a) $f(y \mid 0.4) = \dfrac{f(0.4, y)}{f_X(0.4)} = \dfrac{\frac{6}{5}(0.4 + y^2)}{\frac{6}{5}\left(0.4 + \frac{1}{3}\right)}$
$= \frac{15}{11}(0.4 + y^2), \quad 0 \le y \le 1$

(b) $P[Y \le 0.5 \mid X = 0.4] = \int_{-\infty}^{0.5} f(y \mid 0.4)\,dy = \int_0^{0.5} \frac{15}{11}(0.4 + y^2)\,dy$
$= 0.3298$

(c) $E[Y \mid X = 0.4] = \int_{-\infty}^{1} y\, f(y \mid 0.4) = \int_0^1 y \cdot \frac{15}{11}(0.4 + y^2)\,dy = 0.6136$

Note

① If X & Y are independent r.v.s, then $cov(X,Y)=0$, implying

ⓐ $$Var(X \pm Y) = var(X) + var(Y) \pm 2 cov(X,Y)$$
$$= var(X) + var(Y)$$

ⓑ $$Var(aX \pm bY) = a^2 Var(X) + b^2 var(Y)$$

② If X & Y are any r.v.s, then expectation of conditional probabilities are defined by

ⓐ $$E[Y|X=a] = \sum_y y\, p(y|a)$$
$$= \sum_y y \frac{p(x,y)}{p_X(a)} \quad \text{if } (X,Y) \text{ is jointly discrete}$$

Similarly

$$E(X|Y=b) = \sum_x x\, p(x|b)$$
$$= \sum_x x \frac{p(x,y)}{p_Y(b)} \quad \text{if } (X,Y) \text{ is jointly discrete}$$

ⓑ If (X,Y) is jointly continuous r.v. then

$$E[Y|X=a] = \int_{-\infty}^{\infty} y f(y|a)\, dy$$
$$= \int_{-\infty}^{\infty} y \frac{f(x,y)}{f_X(a)}\, dy$$

& $$E[X|Y=b] = \int_{-\infty}^{\infty} x f(x|b) = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f_Y(b)}\, dx$$