LESSON PLAN

Chapter 1 ⇒ Logic

1. Statement / Proposition, truth values
2. Composite statement, Tautology, contradiction
3. Construction of Truth Table
4. Basic laws
5. Simplification without using truth table
6. Rules of inference.
7. Predicates & quantifiers.
8. Method of Induction.

Chapter 2 ⇒ Set Theory, Relations & functions

1. Basic concepts of set & set operations.
2. Definition of ~~bin~~ binary & binary relation
3. Properties of binary relation (reflexive, irreflexive, symmetric, Anti-symmetric, Assymmetric, Transitive)
4. Equivalence relation, Equivalence class, Partition of set.
5. Matrix representation of binary relation
6. Matrix operations
7. Digraph
8. Closures of relation
9. Warshall's algorithm
10. Partial ordered relation, POSET, Hasse diagram
11. Lattice & their properties
12. Defination of functions & types of functions.
13. Composition of function & inverse of function
14. Permutation

—— X —— Mid Sem —— X ——

LectureNotes.in

Scanned by CamScanner

Chapter 3 ⇒ Recurrence Relation

① Concept of linear RR with constant coefficient
② Homogeneous solution
③ Particular solution
④ Concept of generating function
⑤ Solution of linear RR by using GF.

Chapter 4 ⇒ Algebraic Structure.

① Binary operation
② Semigraph, Monoid, group
③ Properties
④ Abelian & cyclic g
⑤ subgroup, , Normal subgroup
⑥ Lagrange's theorem
⑦ Ring, Field, Integral domain
⑧ Code & group code

Chapter 5 ⇒ Graph Theorem

① Basic terminology & types of graph
② Shortest path algorithm
③ Tree
④ M S Tree Algorithm

Boolean Algebra

LectureNotes.in

Scanned by CamScanner

LOGIC

☆ A sentence is said to be a statement if it infers to either True (T) or False (F)

Eg ⇒ Sun rises in the east ✓

$2+3=7$ ✓

Ram is a good boy ✗

☆ Generally, the statements are to be denoted by P, Q, R, S ---

eg : P : Sun rises in east

Q : $2+3=7$

☆ T & F be the truth value of a statement

☆

P
T
F

$2^1 = 2$

P	Q
T	T
T	F
F	T
F	F

$2^2 = 4$

In general, for n-statements, there are 2^n no. of truth values.

Connectives ⇒

① Negation ($\sim$, not)

② Conjuction ($\wedge$, and)

③ Disjunction ($\vee$, or)

④ Implication ($\rightarrow$, if then, implies) or (P only if Q)

⑤ Double Implication ($\leftrightarrow$ if and only if, implies & implied by)

Composite statement ⇒ When two or more statements are joined by the connectives to form a new statement then that statement is said to be composite statement.

Scanned by CamScanner

Eg ⇒ $P \wedge Q$, $P \vee Q$, $P \to Q$, $P \leftrightarrow Q$, $P \wedge (P \to Q)$
$P \vee (Q \vee R)$, $P \to (Q \vee R)$, $\bar{P} \vee \bar{Q}$ --- etc

P	$\bar{P}$	$\bar{\bar{P}}$
T	F	T
F	T	F

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

$P \vee Q \cong Q \vee P$

$P \equiv \bar{\bar{P}}$ $\quad P \wedge Q \cong Q \wedge P$

P	Q	$P \to Q$ (condition → consequence)	$Q \to P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

$(P \to Q) \neq (Q \to P)$

$$\boxed{P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)}$$

P	Q	$P \vee Q$	$\overline{P \vee Q}$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

☆ Tautology ⇒ A composite statement is said to be tautology if the truth value is true irrespective of the truth value of the individual statements.

☆ Contradiction ⇒ A composite statement is said to be contraction if its truth value is false irrespective of the truth values of individual statements.

☆ Contigent ⇒ A composite statement is neither a tautology nor a contradiction is said to be contigent.

9/7/16

① $P \vee \bar{P}$ be a tautology

P	$\bar{P}$	$P \vee \bar{P}$
T	F	T
F	T	T

② $(P \to Q) \leftrightarrow (\bar{Q} \to \bar{P})$ is also tautology

P	Q	$\bar{P}$	$\bar{Q}$	$P \to Q$	$\bar{Q} \to \bar{P}$	$(P \to Q) \leftrightarrow (\bar{Q} \to \bar{P})$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

③ $P \wedge \bar{P}$ be a contradiction

P	$\bar{P}$	$P \cap \bar{P}$
T	F	F
F	T	F

④ $P \to (P \wedge Q)$ be a contingent

P	Q	$P \wedge Q$	$P \to P \wedge Q$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

① $\boxed{P \wedge T \cong P}$

P	T	$P \wedge T$
T	T	T
F	T	F

② $\boxed{P \wedge F \cong F}$

P	F	$P \wedge F$
T	F	F
F	F	F

③ $\boxed{P \cup T \cong T}$

P	T	$P \cup T$
T	T	T
F	T	T

④ $\boxed{P \cup F \cong P}$

P	F	$P \vee F$
T	F	T
F	F	F

Scanned by CamScanner

9/7/16

(5) (i) $\boxed{\overline{(P \wedge Q)} \equiv \overline{P} \vee \overline{Q}}$

(ii) $\boxed{\overline{(P \vee Q)} \equiv \overline{P} \wedge \overline{Q}}$

(i)

P	Q	$P \wedge Q$	$\overline{P \wedge Q}$	$\overline{P}$	$\overline{Q}$	$\overline{P} \vee \overline{Q}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

(ii)

P	Q	$P \vee Q$	$\overline{P \vee Q}$	$\overline{P}$	$\overline{Q}$	$\overline{P} \wedge \overline{Q}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(6) $\boxed{P \to Q \equiv \overline{P} \vee Q}$ but, acc. to Jiwan's complete law. $P \to Q \equiv \overline{P} \vee Q$

P	Q	$P \to Q$	$\overline{P}$	$\overline{P} \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

NOTES ⇒

☆ If a statement is of the form $P \to Q$ then its converse, inverse & contrapositive will exist.

Converse of $P \to Q$ is $Q \to P$

Inverse of $P \to Q$ is $\overline{P} \to \overline{Q}$

Contrapositive of $P \to Q$ is $\overline{Q} \to \overline{P}$

Eg ⇒ (i) If $x = 1$ then $2x + 1 = 3$

$P : x = 1$

$Q : 2x + 1 = 3$

9/7/16

Converse

Inverse

i.e.

or

Contra

i.e.

(ii) Jin

" Su

Resta

the

Conv

Inve

Co

LectureNotes.in

Scanned by CamScanner

9/7/16

Converse : $Q \to P$

i.e. $2x+1=3 \Rightarrow x=1$

or if $2x+1=3$ then $x=1$

Inverse : $\overline{P} \to \overline{Q}$

i.e. $x \neq 1 \Rightarrow 2x+1 \neq 3$

or if $x \neq 1$ then $2x+1 \neq 3$

Contrapositive : $\overline{Q} \to \overline{P}$

i.e. $2x+1 \neq 3 \Rightarrow x \neq 1$

or if $2x+1 \neq 3$ then $x \neq 1$

(ii) Find the converse, inverse & contrapositive of "Sum of two odd integer is even"

Restate : if two integers are odd then their sum is even.

the

P : Two odd integers

Q : Sum of two integers even.

Converse : $Q \to P$

If sum of two integers is even then both are odd.

Inverse : $\overline{P} \to \overline{Q}$

If two integers are not odd then their sum is not even.

Contrapositive : $\overline{Q} \to \overline{P}$

If sum of two integers is not even then both are not odd.

LectureNotes.in

Scanned by CamScanner

9/7/16

Eg ⇒

(i) Two restaurants are front to each other.
One has the sign board "Good foods are not cheap."
Other has the sign board showing "Cheap foods are not good."
Are they saying same thing.

Ans ⇒ St 1 : Good foods are not cheap
St 2 : Cheap foods are not good.

Restate :

☆ St 1 : If food is good then food is not cheap
St 2 : If food is cheap then food is not good.

☆ P : Food is good | $\bar{P}$: Food is not good
Q : Food is cheap | $\bar{Q}$: Food is not cheap

☆ So symbolically

St 1 : $P \rightarrow \bar{Q}$

St 2 : $Q \rightarrow \bar{P}$

P	Q	$\bar{P}$	$\bar{Q}$	$P \rightarrow \bar{Q}$	$Q \rightarrow \bar{P}$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

i.e. $\boxed{P \rightarrow \bar{Q} \equiv Q \rightarrow \bar{P}}$

(ii) A : The capital is in the mountain or the road to the right goes to the capital.
B : The capital is in the mountain & the road to the right goes to the capital
A : that man ~~A~~ is liar
B : If the capital is in the mountain then the road to the

Scanned by CamScanner

9/7/16

right goes to the capital.

P : The capital is in the mountain

Q : The road to the right goes to the cap.

Symbolically (∵ A : that man A is liar is not ~~the~~ ~~its~~ of any use)

A : $P \vee Q$

B : $P \wedge Q$

B : $P \rightarrow Q$

P	Q	A : $P \vee Q$	B : $P \wedge Q$	B : $P \rightarrow Q$	Vas A is knight
T	T	T	T	T	
T	F	T	F	F	
F	T	T	F	T	
F	F	F	F	T	X

Basic Laws ⇒

① Idempotent Law ⇒ $P \wedge P \equiv P$
$P \cup P \equiv P$

② Identity Law ⇒ $P \wedge T \equiv P$
$P \cup F \equiv P$

③ Inverse Law ⇒ $P \wedge \bar{P} \equiv F$
$P \vee \bar{P} \equiv T$

④ Domination Law ⇒ $P \cup T \equiv T$
$P \wedge F \equiv F$

⑤ Commutative Law ⇒
$P \wedge q \equiv q \wedge P$
$P \vee q = q \vee P$

⑥ Double Negation ⇒

⑦ Associative Law ⇒ $P \wedge (q \wedge r) \equiv (P \wedge q) \wedge r$
$P \vee (q \vee r) \equiv (P \vee q) \vee r$

⑧ ~~Dissociative~~ Distributive Law ⇒ $P \wedge (q \vee r) \equiv (P \wedge q) \vee (P \wedge r)$
$P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r)$

Scanned by CamScanner

9/7/16

right goes to the capital.

P : The capital is in the mountain

Q : The road to the right goes to the cap.

Symbolically (∵ A : that man A is Liar is not ~~the~~ ~~us~~ of any use)

A : $P \vee Q$

B : $P \wedge Q$

B : $P \rightarrow Q$

P	Q	A : $P \vee Q$	B : $P \wedge Q$	B : $P \rightarrow Q$	V as A is knight
T	T	T	T	T	
T	F	T	F	F	
F	T	T	F	T	→ X
F	F	F	F	T	

Basic Laws ⇒

① Idempotent Law ⇒ $P \wedge P \equiv P$
$P \vee P \equiv P$

② Identity Law ⇒ $P \wedge T \equiv P$
$P \vee F \equiv P$

③ Inverse Law ⇒ $P \wedge \bar{P} \equiv F$
$P \vee \bar{P} \equiv T$

④ Domination Law ⇒ $P \vee T \equiv T$
$P \wedge F \equiv F$

⑤ Commutative Law ⇒
$P \wedge q \equiv q \wedge P$
$P \vee q = q \vee P$

⑥ Double Negation ⇒

⑦ Associative Law ⇒ $P \wedge (q \wedge r) \equiv (P \wedge q) \wedge r$
$P \vee (q \vee r) \equiv (P \vee q) \vee r$

⑧ ~~Dissociative~~ Distributive Law ⇒ $P \wedge (q \vee r) \equiv (P \wedge q) \vee (P \wedge r)$
$P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r)$

Scanned by CamScanner

11/7/16

(9) Demorgan's law ⇒ $(\overline{p \wedge q}) \equiv \bar{p} \vee \bar{q}$

$(\overline{p \vee q}) \equiv \bar{p} \wedge \bar{q}$

(10) Law of contrapositive ⇒ $(p \to q) \equiv \bar{q} \to \bar{p}$

(11) Implication & Conversion law ⇒ $p \to q \equiv \bar{p} \vee q$

LectureNotes.in

we have to choose higher precedence operation

$\to > \sim$

Eg ⇒ Simplify

$\sim(p \to q)$

$\equiv \sim(\bar{p} \vee q)$ [Imp & conv. law]

$\equiv \bar{\bar{p}} \wedge \bar{q}$ [De-morgan's law]

$= p \wedge \bar{q}$ [Double Negation]

OR

Without constructing truth table show that $(\overline{p \to q}) \equiv p \wedge \bar{q}$

LectureNotes.in

Ex - (1) $\sim(\sim p \wedge \sim q) \equiv p \vee q$

(2) $\sim(\sim p \vee q) \equiv p \wedge \bar{q}$

(3) $\sim(\sim p \vee \sim q) \equiv p \wedge q$

(4) $\sim(p \wedge \bar{q}) \equiv \bar{p} \vee q$

(5) $p \to \bar{q} \equiv \sim(p \wedge q)$

(1) $\sim(\sim p \wedge \sim q)$

$\equiv \sim(\overline{p \vee q})$ [Demorgans law]

$\equiv p \vee q$ [Double negation]

or $\equiv (\overline{\bar{p} \wedge \bar{q}})$

$\equiv \bar{\bar{p}} \vee \bar{\bar{q}}$ [DM]

$= p \vee q$ [D.N]

(2) $\sim(\sim p \vee q)$

$= \cancel{\sim(p \wedge q)}$

$= \sim(\overline{\bar{p} \vee q})$

$= (\bar{\bar{p}} \wedge \bar{q})$ [DM]

$= p \wedge \bar{q}$ [DN]

(3) $\sim(\sim p \vee \sim q)$

$\equiv (\overline{\bar{p} \vee \bar{q}})$

$\equiv (\bar{\bar{p}} \wedge \bar{\bar{q}})$ [DM]

$\equiv (p \wedge q)$ [DN]

Scanned by CamScanner

11/7/16

④ $\sim(p \wedge \bar{q})$

$\equiv \overline{(p \wedge \bar{q})}$

$= \bar{p} \vee \bar{\bar{q}}$ [DM]

$= \bar{p} \vee q$ [DN]

⑤ $p \to \bar{q}$

$\equiv \bar{p} \vee \bar{q}$ [Imp. & Conve

$= \overline{(p \wedge q)}$ [DM]

Ex → Simplify

① $p \wedge (p \wedge q)$

$(p \wedge q) \wedge p$

$(q \wedge p) \wedge p$

$\equiv (p \wedge p) \wedge q$ (Associative law)

$\equiv p \wedge q$ (Idempotent law)

② $p \vee (p \vee q)$

$\equiv (p \vee p) \vee q$ [Associative L]

$= p \vee q$ [Idempotent L]

③ $p \vee (\bar{p} \wedge q)$

$\equiv (p \vee \bar{p}) \wedge (p \vee q)$ [Distr. L

$= T \wedge (p \vee q)$ [Inver L

$= p \vee q$ [Identity L

Simplify ⇒

$(p \wedge \bar{q}) \vee q \vee (\bar{p} \wedge q)$ (as operation ($\vee$) is same) so we apply Associative

$\equiv ((p \wedge \bar{q}) \vee q) \vee (\bar{p} \wedge q)$ [Associative]

$= (q \vee (p \wedge \bar{q})) \vee (\bar{p} \wedge q)$ [Commutative]

$= ((q \vee p) \wedge (q \vee \bar{q})) \vee (\bar{p} \wedge q)$ [Distributive

$\equiv ((q \vee p) \wedge T) \vee (\bar{p} \wedge q)$ [Inverse

$= (q \vee p) \vee (\bar{p} \wedge q)$ [Idempotent

$= q \vee (p \vee (\bar{p} \wedge q))$ [Associative]

$= q \vee ((p \vee \bar{p}) \wedge (p \wedge q))$ [Distributive

$= q \vee (T \wedge (p \vee q))$ [Inverse

$= q \vee (p \vee q)$ [Idem

$= (p \vee q) \vee q$ [Comm

$= p \vee (q \vee q)$ [Assoc.

$= p \vee q$ [Idemp.

Scanned by CamScanner

Ex ⇒ ① $p \to p$

$\equiv \bar{p} \vee p$ [Imp & Con

$\equiv T$ [Inv

So $p \to p$ be a tautology

② $(p \wedge q) \to p$

$\equiv (\overline{p \wedge q}) \vee p$ (Imp & Con

$\equiv (\bar{p} \vee \bar{q}) \vee p$ (De Morga

$\equiv p \vee (\bar{p} \vee \bar{q})$ (Comm)

$\equiv (p \vee \bar{p}) \vee \bar{q}$ (Associ)

$\equiv T \vee \bar{q}$ (Inver)

$\equiv T$ (Domin)

③ $p \to p \vee q$

$\equiv \bar{p} \vee (p \vee q)$ (Imp & Cover

$\equiv (\bar{p} \vee p) \vee q$ (Associat

$\equiv T \vee q$ (Inverse

$\equiv T$ (Dominatile

④ $[p \wedge (p \to q)] \to q$

$\equiv (\overline{p \wedge (p \to q)}) \vee q$ (Imp & Con

$\equiv (\bar{p} \vee \overline{p \to q}) \vee q$ (De morgan's law)

$\equiv (\bar{p} \vee (\overline{\bar{p} \vee q})) \vee q$ (Imp & Conv)

$\equiv (\bar{p} \vee (p \wedge \bar{q})) \vee q$ (De mongans)

$\equiv ((\bar{p} \vee p) \wedge (\bar{p} \vee \bar{q})) \vee q$ (Distributive)

$\equiv (T \wedge (\bar{p} \vee \bar{q})) \vee q$ (Inverse)

$\equiv ((T \wedge p) \vee (T \wedge \bar{q})) \vee q$ (Distributive)

LectureNotes.in

Scanned by CamScanner

11/7/16

$= p \vee$

$\equiv (p \vee \bar{q})$

$\equiv q \vee (T \wedge (p \vee \bar{q}))$ [Commutative]

$\equiv (q \vee T) \wedge (q \vee (p \wedge \bar{q}))$ [Distributive

$\equiv T \wedge (q \vee (p \wedge \bar{q}))$ [Dominative)

$\equiv T \wedge ((q \vee p) \wedge (q \vee \bar{q}))$ [Distributive

$\equiv T \wedge ((q \vee p) \wedge T)$ [Inverse law)

$\equiv$

$\equiv (\bar{p} \vee \bar{q}) \vee q$ (Idempotent)

$\equiv \bar{p} \vee (\bar{q} \vee q)$ (Associat

$\equiv \bar{p} \vee T$ (Inverse

$\equiv T$

(5) $[(p \to q) \wedge \bar{q}] \to \bar{p}$

$\equiv (\overline{p \to q \wedge \bar{q}}) \vee \bar{p}$ [Imp & Con]

$\equiv (\overline{p \vee q} \vee q) \vee \bar{p}$ [DM & Imp & Con.

$\equiv ((p \wedge \bar{q}) \vee q) \vee \bar{p}$ (DM)

$\equiv q \vee (p \wedge \bar{q}) \vee \bar{p}$ [Comm.

$\equiv (q \vee p) \wedge (q \vee \bar{q}) \vee \bar{p}$ [Distr.

$\equiv (q \vee p) \wedge T \vee \bar{p}$ [Inverse

$\equiv (q \vee p) \vee \bar{p}$ [Identity

$\equiv q \vee (p \vee \bar{p})$ [Associative

$\equiv q \vee T$ [Inverse

$\equiv T$

(6) $[(p \vee q) \wedge \bar{p}] \to q$

$\equiv (\overline{(p \vee q) \wedge \bar{p}}) \vee q$ [Imp & Con

$\equiv ((\bar{p} \wedge \bar{q}) \vee p) \vee q$ [DM

$\equiv p \vee (\bar{p} \wedge \bar{q}) \vee q$ [Comm]

LectureNotes.in

Scanned by CamScanner

$\equiv (p \vee \bar{p}) \wedge (p \vee \bar{q}) \vee q$ [Distri

$\equiv T \wedge (p \vee \bar{q}) \vee q$ [Inverse

$\equiv (p \vee \bar{q}) \vee q$ [Identity

$\equiv p \vee (\bar{q} \vee q)$ [Asso

$\equiv p \vee T$ [Inve

$\equiv T$ [Dominative.

Rules of Inference ⇒

q is the conclusion of the permises $P_1, P_2, P_3 \ldots P_n$

iff $(P_1 \wedge P_2 \wedge P_3 \ldots \wedge P_n) \rightarrow q$

be a Tautology

i.e.

And $\wedge$ $\left\{ \begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_n \end{array} \right.$

$\overline{q}$ ⇒ i.e. q is the conclusion

$\therefore$ $p \rightarrow q$
p
$\overline{\therefore q}$

Rules ⇒ Tautological form.

① Addition

$\frac{p}{\therefore p \vee q}$

$p \rightarrow (p \vee q)$ be a T

② Simplification

$\frac{p \wedge q}{\therefore p}$ or q

$(p \wedge q) \rightarrow p \equiv T$

③ Modus Poren

p
$p \rightarrow q$
$\overline{\therefore q}$

$[p \wedge (p \rightarrow q)] \Rightarrow q \equiv T$

④ Modus Tollen

$p \rightarrow q$
$\overline{q}$

$[(p \rightarrow q) \wedge \bar{q}] \Rightarrow \bar{p} \equiv T$

Scanned by CamScanner

14/7/16

(5) Disjunctive Syllogism

$[(p \vee q) \wedge \bar{p}] \Rightarrow q \equiv T$

$$\begin{array}{l} p \vee q \\ \bar{p} \\ \hline \therefore q \end{array}$$

(6) Chain Rule

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

(7) Conjuction

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

(8) Resolution

$$\begin{array}{l} p \vee q \\ \bar{p} \vee r \\ \hline \therefore q \vee r \end{array}$$

Test Validation ⇒

Ex ⇒ (i) { If Socrates is a man, then Socrates is mortal
Socrates is a man
Thus, Socrates is mortal

P : Socrates is a man

Q : Socrates is mortal

So symbolically,

$$\begin{array}{l} p \to Q \\ p \\ \hline \therefore Q \end{array}$$

(1) $p \to Q$ [Given

(2) p ["

(3) Q [Modus Ponen with (1) & (2)]

So Q be the valid conclusion

LectureNotes.in

Scanned by CamScanner

(ii) If a baby is hungry then the baby cries
If the baby is not mad then he does not cry
If a baby B is mad then he has a red face.
Find the valid conclusion.

P : The baby is hungry
Q : The baby cries
R : The baby is mad
S : The baby has a red face.

In symbolic form

$$\begin{array}{l} P \to Q \\ \overline{R} \to \overline{Q} \\ R \to S \\ \hline \therefore ? \end{array}$$

(1) $P \to Q$ (Given
(2) $\overline{R} \to \overline{Q}$ ("
(3) $Q \to R$ (law of contrapositive in (2)
(4) $P \to R$ [Chain Rule in (1)(3)
(5) $R \to S$ [Given
(6) $P \to S$ [Chain Rule in (4)(5)

The valid conclusion is
i.e. If the baby is hungry then the baby has red face.

(iii) It is not sunny this afternoon & it is cooler than yesterday.
We will go for swimming only if it is sunny
If we do not go for swimming then we will take a long drive.
If we take a long drive then we will be home by sunset.
Thus we will be home by sunset.

Scanned by CamScanner

P: It is sunny this afternoon
Q: It is cooler than yesterday
R: We will go for swimming
S: We will take a long drive
T: We will be home by sunset.

In symbolic form ⇒

$\bar{P} \wedge Q$
$R \rightarrow P$
$\bar{R} \rightarrow S$
$S \rightarrow T$
$\therefore T$

① $\bar{P} \wedge Q$ [Given
② $\bar{P}$ [Simplification ① as Q is neither used after 1st statement]
③ $\bar{R} \rightarrow P$ [Given
④ $\bar{R}$ [Modus Tollen in ② & ③
⑤ $\bar{R} \rightarrow S$ [Given
⑥ S [Modus Ponen in ④ & ⑤
⑦ $S \rightarrow T$ [Given
⑧ T [Modus Ponen in ⑥ & ⑦

Valid conclusion is T.

(iv) Find the valid conclusion
If Clinton is not reelected then ~~Obama~~ Tulsa will loose its Airbase.
Clinton will be reelected if & only if Tulsa votes for him.
If Tulsa keeps its air base then Clinton will be reelected.

P: ~~Ar~~ Clinton is reelected
Q: Tulsa keeps its airbase
R: Tulsa votes for Clinton

Scanned by CamScanner

In symbolic form.

$\bar{P} \to \bar{Q}$

$P \leftrightarrow R$

$Q \to P$

?

① $\bar{P} \to \bar{Q}$ [Given

② $Q \to P$ [Law of contrapositive]

④ $P \leftrightarrow R \equiv (P \to R) \wedge (R \to P)$

$\equiv (P \to R \wedge)\ R \to P$

$\therefore P \to R$ [Simplification

③ $P \leftrightarrow R$ [Given

④ $P \to R$

⑤ $Q \to R$ [Chain Rule in ② & ④

then valid conclusion is $Q \to R$

i.e. If Julsa keeps its air base, then Julsa votes for clinton

☆ There are three type of faulty inferences such as

① The fallacy of affirming the consequent

$P \to Q$
P

$\therefore Q$

(Modus Ponen)

but

$P \to Q$
Q

$\therefore P$

This is valid conclusion

This is be a fallacy

Eg ⇒ ④ if the price of gold is rising then inflation is surely coming

inflation is surely coming thus the price of gold is rising.

P: Price of gold is rising

Q: Inflation is surely coming

Scanned by CamScanner

15/7/16

In symbolic form

$$\begin{array}{l} P \to Q \\ \underline{Q} \\ \therefore P \end{array} \qquad [(P \to Q) \wedge Q] \to P \text{ is not a Tautology}$$

This is be a faulty inference.

This type of inference is the fallacy of affirming the consequence.

LectureNotes.in

~~(b) Fall~~

(2) Fallacy of denying the antecedent

Since Modus Tollen is the valid conclusion

i.e. $\begin{array}{l} P \to Q \\ \underline{\bar{Q}} \\ \therefore \bar{P} \end{array}$ but $\begin{array}{l} P \to Q \\ \underline{\bar{P}} \\ \therefore \bar{Q} \end{array}$ is such fallacy

$[(P \to Q) \wedge \bar{Q}] \to \bar{P} \equiv T$ $\qquad$ $[(P \to Q) \wedge \bar{P}] \to \bar{Q} \not\equiv T$

(3) Non - sequitier fallacy

$$\begin{array}{l} P \\ \underline{} \\ \therefore Q \end{array} \qquad P \wedge Q \not\equiv T$$

Eg ⇒ PK Mohanty is a man

Then PKM is mortal.

LectureNotes.in

Mathematical Induction ⇒

Let P_n be the given statement (The ques^n which is to be proved)

LectureNotes.in

Scanned by CamScanner

Mathematical Induction ⇒

Let P_n be the given statement (The ques^n which is to be proved)

P_n is to be true for all values of n

iff

Basis of induction (BOI)

Either $P_0 / P_1 / P_{sp.}$ value is true

Inductive hypothesis

Assume P_k is true

Inductive step

Show that P_{k+1} is true

Eg ⇒ (i) $1+2+3+ \cdots +n = \frac{n(n+1)}{2}$

Let P_n : $1+2+3+ \cdots +n = \frac{n(n+1)}{2}$

(here we do not take $n=0$ as here n is no. of terms & no. of terms is not equal to zero.)

BOI $(n=1)$

LHS of $P_1 = 1$

RHS of $P_1 = \frac{1(1+1)}{2} = \frac{1\times 2}{2} = 1$

Inductive hypothesis

Assume P_k is true

i.e. $1+2+3+ \cdots +k = \frac{k(k+1)}{2}$

Inductive step

Claim - P_{k+1} is true

i.e. $1+2+3+ \cdots +(k+1) = \frac{(k+1)(k+2)}{2}$

Scanned by CamScanner

$$LHS = \underbrace{1+2+3+\cdots+k}+(k+1)$$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{(k+1)(k+2)}{2}$$

$$= RHS$$

So P_{k+1} is true

Thus P_n is true $\forall n$

(ii) $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$, $x \neq 1$

(Here there is no specified value as here $x \neq 1$ is given not $n \neq 1$ i.e. nothing is said about n)

Let P_n : $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$

B O I

$n = 0$

LHS of $P_0 = 1$

RHS of $P_0 = \dfrac{1-x^{0+1}}{1-x}$

$= 1$

So P_0 is true

$n = 1$

LHS of $P_1 = 1+x$

RHS of $P_1 = \dfrac{1-x^{1+1}}{1-x}$

$= 1+x$

Inductive hypothesis

Assume P_k is true

i.e. $1+x+x^2+\cdots+x^k = \dfrac{1-x^{k+1}}{1-x}$

Inductive step

claim → P_{k+1} is true

i.e. $1+x+x^2+\cdots+x^{k+1} = \dfrac{1-x^{(k+1)+1}}{1-x}$

$$LHS = \underbrace{1+x+x^2+\cdots+x^k}+x^{k+1}$$

$$= \frac{1-x^{k+1}}{1-x} + x^{k+1}$$

$$= \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x}$$

Scanned by CamScanner

$$= \frac{1 - 9^{k+2}}{1 - 9}$$

$= RHS$

so P_{k+1} is true

then P_n is true $\forall n$

(iii) $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

Let P_n: $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

BOI

$n = 1$

$LHS = P_1 = 1$

$RHS = P_1 = 1^2(2\times1^2-1)$

$= 1$

$LHS = RHS$

$\Rightarrow$ So P_1 is true

Inductive hypothesis

Assume P_k is true

i.e. $1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2-1)$

Inductive step

Claim - P_{k+1} is true

i.e. $1^3 + 3^3 + 5^3 + \cdots + (2(k+1)-1)^3 = (k+1)^2(2(k+1)^2-1)$

$LHS = \underbrace{1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3} + (2k+1)^3$

$= k^2(2k^2-1) + (2k+1)^3$

$= 2k^4 - k^2 + 8k^3 + 1 + 12k^2 + 6k$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1 = RHS$

$= \cancel{2k^3(k+4)}$

$= \cancel{2k^2(k^2+4k)}$

$= 2k^4 + 8k^3 + 9k^2 + \cancel{2k} + \cancel{2k^2+4k+1}$

$=$

$= (k+1)^2(2k^2+2+4k+1)$

$= (k+1)^2(2k^2+4k+1)$

$\cancel{= (k+1)^2(}$

$= 2k^4 + 8k^3 + 11k^2 + 6k + 1$

$= 2k^4 +$

$= 2k^4$

$= 2k$

$= (k$

$= (k$

$= ($

$=$

(iv)

Let

BO

Let

LHS

RH

LectureNotes.in

Scanned by CamScanner

$= 2k^4 + 8k^3 + 8k^2 + 3k^2 + 6k + 1$

$= 2k^4 + 2k^3 + 6k^3 + 6k^2 + 5k^2 + 5k + k + 1$

$= 2k^3(k+1) + 6k^2(k+1) + 5k(k+1) + k+1$

$= (k+1)(2k^3 + 6k^2 + 5k + 1)$

$= (k+1)(2k^3 + 2k^2 + 4k^2 + 4k + k + 1)$

$= (k+1)(2k^2(k+1) + 4k(k+1) + (k+1))$

$= (k+1)^2(2k^2 + 4k + 1)$

$= (k+1)^2(2(k+1)^2 - 1)$

$=$ RHS

So P_{k+1} is true

Then P_n is true $\forall n$

(iv) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Let $P_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$

BOI

Let $n = 1$

LHS of $P_1 = \frac{1}{1\times 2} = \frac{1}{2}$

RHS of $P_1 = \frac{1}{1+1} = \frac{1}{2}$

So P_1 is true

Inductive hypothesis

Assume P_k is true

i.e. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots \frac{1}{k(k+1)} = \frac{k}{k+1}$

Inductive step.

Claim - P_{k+1} is true

i.e. $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$

$LHS = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+2)(k+1)} = \frac{k+1}{k+2} =$ RHS

So P_{k+1} is true

Scanned by CamScanner

16/7/16

Eg ⇒ (i) $11^{n+2} + 12^{2n+1}$ is divisible by 133

Let P_n : $11^{n+2} + 12^{2n+1}$ is divisible by 133

BOI $(n=0)$

Since $11^2 + 12^1 = 121 + 12 = 133$ which is divisible by 133

So P_0 is true.

Inductive Hypothesis

Assume P_k is true

i.e. $11^{k+2} + 12^{2k+1}$ is divisible by 133.

ie. $11^{k+2} + 12^{2k+1} = 133m$ (m is +ve integer)

Inductive Step

Claim ⇒ P_{k+1} is true

ie. $11^{k+1+2} + 12^{2(k+1)+1}$ is divisible by 133

Now $11^{k+2} \cdot 11 + 12^{2k+1} (12)^2$

$= 11^{k+2}(11) + 12^{2k+1}(144)$

$= 11^{k+2}(11) + 12^{2k+1}(133+11)$

$= 11\{11^{k+2} + 12^{2k+1}\} + 133(12^{2k+1})$

$= 11(133m) + 133(12^{2k+1})$

$= 133(11m + 12^{2k+1})$

which is divisible by 133

So P_{k+1} is true

Thus P_n is true $\forall n$

(ii) $x-y$ is a factor of the polynomial $x^n - y^n$

Let P_n : $x-y$ is a factor of polynomial $x^n - y^n$

BOI $(n = 1)$

$x^n - y^n = x^1 - y^1$

$= x-y$ ⇒ ∴ $x-y$ is a factor of $(x-y)$

$x-y$ is a factor of polynomial $x^n - y^n$.

So P_0 is true.

Inductive Hypothesis ⇒

Assume P_k is true

Scanned by CamScanner

i.e. $(x-y)$ is factor of polynomial $x^k - y^k$

i.e. $x^k - y^k = (x-y) f(x,y)$ — (1)

Inductive Step ⇒

Claim ⇒ P_{k+1} is true i.e. $(x-y)$ is factor of $x^{k+1} - y^{k+1}$

i.e. $x^{k+1} - y^{k+1}$

$= x^k x - y^k \cdot y$

$= \{y^k + (x-y) f(x,y)\} x - y^{k+1}$ (from (1))

$= xy^k + x(x-y) f(x,y) - y^{k+1}$

$= (x-y) y^k + x(x-y) f(x,y)$

$= (x-y) \{ y^k + x \cdot f(x,y) \}$

i.e. $(x-y)$ is a factor of $x^{k+1} - y^{k+1}$

So P_{k+1} is true

Then P_n is true for all (+ve) Integers.

(iii) $6^{m+2} + 7^{2n+1}$ is divisible by 43 for each +ve integer n

Let P_n : $6^{n+2} + 7^{2n+1}$ is divisible by 43

BOI $(n=1)$

$P_1 = 6^{1+2} + 7^{2\times1+1} = 6^3 + 7^3$

$= \cancel{108+343}\ 216+343$

$= \cancel{379}\ 559$

which is divisible by 43

Inductive hypothesis

Assume P_k is true

i.e. $6^{k+2} + 7^{2k+1}$ is divisible by 43 for each +ve integer

i.e. $6^{k+2} + 7^{2k+1} = 43m$

Inductive step

Claim ⇒ P_{k+1} is true

i.e. $6^{(k+1)+2} + 7^{2(k+1)+1}$

$= 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 7^2$

$= 6^{k+2} \cdot 6 + 7^{2k+1} (6 + 43)$

$= 6 [6^{k+2} + 7^{2k+1}] + 7^{2k+1} \cdot 43$

$= 6 \times 43m + 43 \cdot 7^{2k+1}$

Scanned by CamScanner

$= 43\,(6m + 7^{2k+1})$

i.e. which is divisible by 43

So P_{k+1} is true

Thus P_n is true for each positive integer n.

(iv) $(x+y)$ is a factor of $x^{2n+1} + y^{2n+1}$

Let P_n : $(x+y)$ is a factor of $x^{2n+1} + y^{2n+1}$

BOI $(n=0)$

$x^{2n+1} + y^{2n+1} = x^{0+1} + y^{0+1}$
$= x + y$

i.e. $(x+y)$ is a factor of $x+y$

So P_0 is true

Inductive Hypothesis

Assume P_k is true

i.e. $(x+y)$ is a factor of $x^{2k+1} + y^{2k+1}$

i.e. $x^{2k+1} + y^{2k+1} = (x+y)\,f(x,y)$

Inductive step

Claim ⇒ P_{k+1} is true i.e. $(x+y)$ is a factor of

i.e. $x^{2(k+1)+1} + y^{2(k+1)+1}$

i.e. $x^{2(k+1)+1} + y^{2(k+1)+1}$

$= x^{2k+1} \cdot x^2 + y^{2k+3}$

$= (x+y)\,f(x,y) - y^{2k+1})\,x^2 + y^{2k+3}$

$= x^2(x+y)\,f(x,y) + y^{2k+1}(y^2 - x^2)$

$= (x+y)\,(x^2 f(x,y) + y^{2k+1}(y-x))$

$= (x+y)\,g(x,y)$

i.e. $(x+y)\,f(x,y)$ is a factor of $x^{2k+3} + y^{2k+3}$

So P_{k+1} is true

Thus P_n is true $\forall n$

$\exists x\ P(x) =$
$\forall x\ P(x)$

(v) $n(n$

Let

BOI

Since

Induc

Let

Ind

Clain

LectureNotes.in

Scanned by CamScanner

$\exists x \; P(x) = P(0) \vee P(1) \vee P(2)$ ---- = Existential quantifiers
$\forall x \; P(x) = P(0) \wedge P(1) \wedge P(2)$. ---- = Universal Quantifiers

(v) $n(n^2+5)$ is integer multiple of 6

Let P_n : $n(n^2+5)$ is integer multiple of 6

BOI ($n=0$)

since $0\cdot(0^2+5) = 0 = 0\times 6$

So P_0 is true

Inductive Hypothesis

Let P_k is true

i.e. $k(k^2+5)$ is integer multiple of 6

i.e. $k(k^2+5) = 6m$ (m is integer)

i.e. $k^3+5k = 6m$

Inductive step

Claim : P_{k+1} is true

i.e. $(k+1)((k+1)^2+5)$ is integer multiple of 6

i.e. $(k+1)((k+1)^2+5)$

$= (k+1)(k^2+2k+6)$

$= k^3+3k^2+8k+6$

$= k^3+5k+3k^2+3k+6$

$= \cancel{6m+3k(k^2+k+2)}$

$= 6m+3k(k+1)+6$

$= 6m+3\times 2m_1+6$

$= 6(m+m_1+6)$

$= 6m_2$

which is multiple of 6

$\because$ k & $k+1$ are two consecutive integer. so their product always be even.
$\Rightarrow k(k+1) = 2m_1$

So P_{k+1} is true

Thus P_n is true $\forall\, n \in \mathbb{Z}$ (~~+ve integer~~)

LectureNotes.in

Scanned by CamScanner

18/7/16 ☆ for some $x \in S$, $\forall = \exists_{x \in S}$; $\exists_{x \in S}$ $P(x)$

$\sim(\forall x\, P(x)) \equiv \exists_x \sim P(x)$ | quantifier predicates

Strong Induction ⇒

Q (i) $q_n : n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $n = 2, 3, 4 \ldots$
(Every integer is product of prime nos)

q_2 is true

Assume $\begin{cases} q_3 \text{ is true} \\ \vdots \\ q_n \text{ is true} \end{cases}$

To prove q_{n+1} is true

If $(n+1)$ is prime then q_{n+1} is true

If $(n+1)$ is not prime then let $n+1 = lm$, $1 < l, m < n+1$

q_l & q_m are true

$\therefore$ $l = p_1^{a_1} p_2^{a_2} \cdots p_i^{a_i}$

$m = p_1^{b_1} p_2^{b_2} \cdots p_i^{b_i}$

$(n+1) = l \cdot m$

$(n+1) = p_1^{a_1+b_1} p_2^{a_2+b_2} \cdots p_i^{a_i+b_i}$

$\Rightarrow q_{n+1}$ is true.

LectureNotes.in

Predicate Calculus ⇒

Every sentence (statement) has two parts ⇒ (i) Subject
(ii) Predicates.
Properties

Eg ⇒ Rama is a student

Here Rama is the subject

Student is the predicate

*) The subject is to be quantified. Generally there are two types of quantifiers such as

(i) Universal Quantifier ($\forall x$)

(ii) Existential Quantifier ($\exists_x$)

Scanned by CamScanner

30/7/16

*) Now we want to introduce the subject by a variable 'x'

For that variable 'x' we must have to define the Universe of discourse (domain of set)
(UOD)

*) To represent symbolically a statement we have to take the 1st letter of the predicate.

Eg ⇒ x is a student

It is to be denoted by

S(x): x is a student.

1st letter of predicate student

→ If the U.O.D is the set of all people then

S(PKM): PKM is a student.

Where truth value of S(PKM) is F.

*) We may have more than one subject in a statement.

Eg ⇒ x is greater than y

It is to be denoted by

G(x,y) : x>y

Here we have two subjects x & y.

U.O.D → By default the real nos.

G(1,2) : 1>2 → F

G(3,5) : 3>5 → F

G(5,2) : 5>2 → T

etc.

LectureNotes.in

Scanned by CamScanner

The variable must have universe of discourse (domain)

U.O.D ⇒ The set of complex nos.

$\forall x, y \in C$

$G(x, y): x > y$ is meaningless.

As two complex nos. are not comparable.

*) So we can ~~use~~ introduce any no. of variables.

suppose the eqn is $x + y = z$

$E(x, y, z): x + y = z$

$E(1, 2, 3): 1 + 2 = 3 \rightarrow T$

$E(2, 3, 4): 2 + 3 = 4 \rightarrow F$

etc--

Universal Quantifiers ⇒ It is to be used.

For all x

For each x

For every x

Everything x such that

Each thing x such that

Statement	Negation
$\forall x, P(x)$	$\exists x, \bar{P}(x)$
$\exists x, P(x)$	$\forall x, \bar{P}(x)$

Existential Quantifiers ⇒

For some x

Some x such that

There exists an x such that

There is atleast one x such that

Eg ⇒ (i) something is good

(ii) Everything is good

(iii) Nothing is good

(iv) something is not good

Scanned by CamScanner

For universal quantifier, if then is to be used
For existential quantifier and is to be used.

Representation:

$G(x)$: x is Good

$\bar{G}(x)$: x is not good

U.O.D ⇒ The set of all things

* $\exists x, G(x)$
* $\forall x, G(x)$
* $\forall x, \bar{G}(x)$
* $\exists x, \bar{G}(x)$

Eg ⇒ All monkey have tails

No monkey has a tail

Some monkeys have tails

Some monkeys have no tails

Representation:

U.O.D ⇒ The set of all animals.

$M(x)$: x is a monkey

$T(x)$: x has tail.

Restate ⇒

* If x is a monkey then x has a tail
* x is a monkey & x has a tail

* $\forall x, M(x) \rightarrow T(x)$
* $\forall x, M(x) \rightarrow \bar{T}(x)$
* $\exists x, M(x) \rightarrow T(x)$
* $\exists x, M(x) \rightarrow \bar{T}(x)$

Eg of negation ⇒ $\sim(\forall x, \bar{P}(x))$

$\equiv \exists x, \bar{\bar{P}}(x)$

$= \exists x, P(x)$

Scanned by CamScanner

★ for variable (general) statement rules of inference are not applicable.

Rules of Inference in predicate logic:

Universal specification ⇒ $\dfrac{\forall x, P(x)}{\therefore P(c), \text{ for all } c}$

Universal generalisation ⇒ $\dfrac{P(c), \text{ for all } c}{\therefore \forall x, P(x)}$

Existential specification ⇒ $\dfrac{\exists x, P(x)}{\therefore P(c_0), \text{ for some } c}$

Existential generalisation ⇒ $\dfrac{P(c_0), \text{ for some } c}{\exists x, P(x)}$

Test the validity.

Eg ⇒ Every living thing is a plant or an animal.
John's gold fish is alive & it is not a plant.
All animals have hearts.
Then John's gold fish has a heart.

Verification ⇒

U.O.D ≡ The set of all living things

$P(x)$: x is a plant

$A(x)$: x is an animal.

g : John's gold fish → It is particular one, so it is a single statement not a composite

$H(x)$: x has a heart

$\overline{P}(x)$: x is not a plant

$\overline{P}(g)$: John's gold fish is not a plant

$H(g)$: John's gold fish has a heart.

Symbolically.

$\forall x, P(x) \vee A(x)$

$\overline{P}(g)$

$\forall x, A(x) \rightarrow H(x)$

$\therefore H(g)$

Scanned by CamScanner

① $\forall x,\ P(x) \vee A(x)$ [Given]

(∵ it is general statement, so, here we do not use rule of interference, so we convert it into particular statement)

② $P(g) \vee A(g)$ [US]

③ $\bar{P}(g)$ [Given]

$$\begin{array}{l} p \vee q \\ \bar{p} \\ \hline \therefore q \end{array}$$

④ $A(g)$ [DS]

⑤ $\forall x,\ A(x) \rightarrow H(x)$ [Given]

⑥ $A(g) \rightarrow H(g)$ [G US]

⑦ $H(g)$ [MP in ④ & ⑥]

Thus the valid conclusion is $H(g)$

i.e. John's gold fish has a heart.

Strong Mathematical Induction ⇒

Let P_n be the given statement.

P_n will be true for all values of n or for specified values of n iff.

* $P_0 / P_1 / P_{sp.}$ value is true (BOI).
* Assuming P_i is true for $0 \le i \le k$ show that P_{k+1} is true

i.e. Assume $P_1, P_2, P_3 - - P_{k-2}, P_{k-1}, P_k$ are true show that P_{k+1} is true.

Eg ⇒ If $\underbrace{a_n = 5a_{n-1} - 6a_{n-2}}_{\text{recurrence relation}}$ for $n \ge 2$ & $a_0 = 12$, $a_1 = 29$ then $a_n = 5(3^n) + 7(2^n)$

Let P_n: $a_n = 5(3^n) + 7(2^n)$

BOI $(n = 2)$

∵ RHS of $P_2 = a_2 = 5a_1 - 6a_0 = \cancel{60 - 144} = \cancel{-84} = 73$

RHS of $P_2 = 5(3^2) + 7(2^2) = 73$

Scanned by CamScanner

1/8/16 ☆ By method of induction the solution of recurrence solⁿ is to be proved.

So P_2 is true.

Inductive Hypothesis ⇒

Assume $P_3, P_4 \ldots P_k$ are true

i.e. $P_k, P_{k-1}, P_{k-2} \ldots P_3$ are true

So, $P_k : a_k = (5 \times 3^k) + (7 \times 2^k)$

$P_{k-1} : a_{k-1} = (5 \times 3^{k-1}) + (7 \times 2^{k-1})$

$P_{k-2} : a_{k-2} = (5 \times 3^{k-2}) + (7 \times 3^{k-2})$

so on ...

Inductive Step

Claim → P_{k+1} is true

i.e. $P_{k+1} : a_{k+1} = (5 \times 3^{k+1}) + (7 \times 2^{k+1})$

$a_{k+1} = 5a_k - 6a_{k-1}$

$= 5(5 \times 3^k + 7 \times 2^k) - 6(5 \times 3^{k-1} + 7 \times 2^{k-1})$

$= 25 \times 3^k + 35 \cdot 2^k - 30 \cdot 3^{k-1} - 42 \cdot 2^{k-1}$

$= 25 \times 3^k + 35 \cdot 2^k - 10 \cdot (3^k) - 21(2^k)$

$= 15 \cdot (3^k) + 14(2^k)$

$= 5(3^{k+1}) + 7(2^{k+1})$

$=$ RHS

So P_{k+1} is true

Then P_n is true $\forall n \geq 2$

(2) Prove that the function $b(n) = 2(3^n) - 5$ is the unique fun define by i) $b(0) = -3$, $b(1) = 1$

ii) $b(n) = 4b(n-1) - 3b(n-2)$, for $n \geq 2$

Scanned by CamScanner

Let $P_n : b(n) = 2(3^n) - 5$

<u>BOI</u> $(n=2)$

LHS of $P_2 = b2 = 4b(2-1) - 3b(2-2)$

$= 4b(1) - 3b(0)$

$= 4+9$

$= 13$

RHS of $P_2 = 2(3^2) - 5$

$= 13$

So P_2 is true

<u>Inductive hypothesis</u>:

Assume $P_3, P_4 \ldots P_k$ are true

or $P_k, P_{k-1} \ldots$ are true.

$P_k : b(k) = 2(3^k) - 5$

$P_{k-1} : b(k-1) = 2(3^{k-1}) - 5$

$P_{k-2} : b(k-2) = 2b^{(k-2)} - 5$ so on -

<u>Inductive step</u>

Claim $\Rightarrow$ P_{k+1} is true

i.e. $P_{k+1} : b(k+1) = 2(3^{k+1}) - 5$

$P_{k+1} = b(k+1)$

$= 4b(k) - 3b(k-1)$

$= 4 \times \{2 \times 3^k - 5\} - 3\{2 \times 3^{k-1} - 5\}$

$= 8 \times 3^k - 20 - 2 \cdot 3^k + 15$

$= 6(3^k) - 5$

$= 2(3^{k+1}) - 5$

$=$ RHS is true

So P_{k+1}

Thus P_n is true $\forall\ n \geq 2$.

LectureNotes.in

Scanned by CamScanner

③ Prove that if F_n is the nth fibonacci series

then $F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}, n \geq 0$

$\because F_n = F_{n-1} + F_{n-2}$ with $F_0 = 1$ & $F_1 = 1$

Let $P_n : F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}$

LectureNotes.in

<u>BOI</u> $(n=0)$

LHS of $P_0 = F_0 = 1$

RHS of $P_0 = \frac{1}{\sqrt{5}}\left\{\frac{1+\sqrt{5}}{2} - \left(\frac{1-\sqrt{5}}{2}\right)\right\}$

$= \frac{1}{\sqrt{5}}\left\{\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$

$= \frac{1}{\sqrt{5}} \cdot \sqrt{5}$

$= 1$

So P_0 is true

<u>Strong Ind. hypo</u>

Assume $P_k, P_{k-1}, P_{k-2} \ldots P_1$ are true

$P_k : F_k = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}$

$P_{k-1} : F_{k-1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right\}$

$P_{k-2} = F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$

<u>Strong Inductive step</u> ⇒

Claim ⇒ P_{k+1} is true

$P_{k+1} : F_{k+1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right\}$

Scanned by CamScanner

$$LHS = F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\right\} + \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right\}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}+1\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$$

$$= RHS$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{2} = \frac{1}{4}(1+5+2\sqrt{5}) = \frac{3+\sqrt{5}}{2}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^{2} = \frac{1}{4}(1+5-2\sqrt{5}) = \frac{3-\sqrt{5}}{2}$$

So P_{k+1} is true

Thus P_n is true $\forall n \geq 0$

LectureNotes.in

Scanned by CamScanner

Set Theory, Relation & Functions

31/8/16

Set ⇒ (1) It is undefined but for practical purpose we define it as the collection of well defined objects or limits.

(2) ~~Seve~~ Generally the sets are denoted by English capital Alphabets such as A, B, C etc & describe in two ways:

(i) Tabular Method

(ii) Set Builder method.

* D_n be the set of all +ve divisors of n

① $N, Z^+, I^+ = \{1, 2, 3, 4, 5 \ldots\}$ ⇒ The set of counting nos/+ve integer /natural nos.

② $N^+, W = \{0, 1, 2, 3 \ldots\}$ ⇒ The set of all natural nos. (including 0.) /whole nos.

③ $Z, I = \{0, \pm 1, \pm 2, \ldots\}$ ⇒ The set of integer

④ $Q = \{x \mid x = \frac{p}{q}, \forall p, q \in Z, q \neq 0\}$ ⇒ The set of rational nos.

⑤ $Q' = \{x \mid x \notin Q\}$ ⇒ The set of irrational nos.

⑥ $R = Q \cup Q'$ ⇒ The set of real nos.

⑦ $C = \{z \mid z = x + iy, \forall x, y \in R\}$ ⇒ The set of complex nos.

(3) The total no. of elements contained in a set is its order or cordinality

if A be the set then

O(A) denotes the order of set

Card A, |A| denotes the cordinality of the set

Finite set ⇒ If the set

Infinite set ⇒ If is

Eg ⇒ i) A

ii) A

Similar set

Eg ⇒ (1) A =

B =

|A| = |B|

A

Empty se

Single

☆

S

LectureNotes.in

Scanned by CamScanner

Finite set ⇒ If the order of the set is finite then the set is said to be finite set.

Infinite set ⇒ If the order of the set is infinite then the set is said to be infinite set.

Eg ⇒ i) $A = \{a, e, i, o, u\} \rightarrow |A| = 5 \rightarrow A$ is finite set.

ii) $A = \{1, 1, 2, 3, 5, 8 - - -\} \rightarrow A$ is infinite set

Similar set ⇒ Two sets A & B is said to be similar iff

$$|A| = |B|$$

Eg ⇒ (i) $A = \{1, 2, 3, 4, 5\}$

$B = \{a, e, i, o, u\}$

$|A| = |B| = 5$

A & B are similar.

(ii) $A = \{1, 2, 3, 4, 5\}$

$B = \{a, b, c, d\}$

$|A| = 5$; $|B| = 4$

A & B is not similar.

Empty set ϕ ⇒ The set without limit is called as Empty set.

$$\phi = \{\}$$
$$= \{x \mid x \neq x\}$$

Single Term set ⇒ The set with one element is

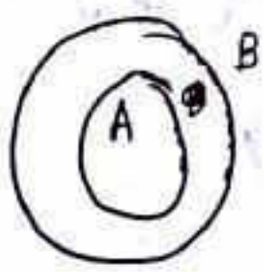
$$\{\phi\}$$

☆ ϕ & $\{\phi\}$ are different

Subset ⇒

$$A \subset B$$

i.e. $x \in A \Rightarrow x \in B$



Scanned by CamScanner

① ϕ is a subset of every set.

② Every set is a subset of itself.

③ Power set of A ⇒ The set of all possible subset of the set A is denoted by $P(A)$ called as Power set of A.

④ If $A \subset B$ & $B \subset A$ then $\boxed{A = B}$

LectureNotes.in

THEOREM ⇒

If $|A| = n$ then $P|A| = 2^n$

Since $|A| = n$

The total no. of subsets containing 0-element $= C(n,0)$

" " " " " " 1-element $= C(n,1)$

" " " " " " 2-element $= C(n,2)$

⋮

n-element $= C(n,n)$

Total subsets $= C(n,0) + C(n,1) + C(n,2) + \dots + C(n,n)$

$= 2^n$

LectureNotes.in

As $(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \dots + C(n,n)x^n$

For $x = 1$

$(1+1)^n = C(n,0) + C(n,1) + C(n,2) + \dots + C(n,n)$

$\Rightarrow 2^n = C(n,0) + C(n,1) + C(n,2) + \dots + C(n,n)$

LectureNotes.in

Eg ⇒ ① $A = \{\}$

$\Rightarrow |A| = 0$

$P(A) = \{\phi\} \Rightarrow P(A) = 1 = 2^0$

② $A = \{a\}$

$\Rightarrow |A| = 1$

$P(A) = \{\phi, \{a\}\} \Rightarrow P|A| = 2 = 2^1$

Scanned by CamScanner

③ $A = \{a, b\}$

$|A| = 2$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\Rightarrow |P(A)| = 4 = 2^2$

④ $A = \{a, b, c\}$

$\Rightarrow |A| = 3$

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a, b, c\}\}$

$\Rightarrow |P(A)| = 8 = 2^3$

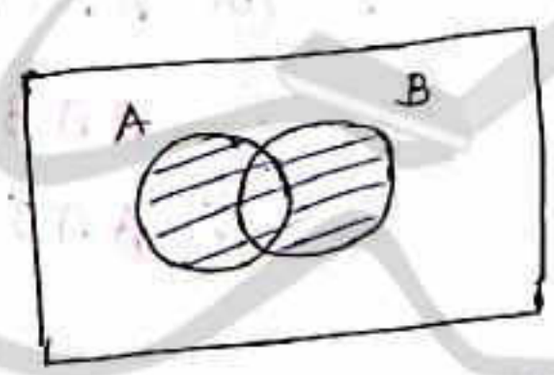
Operations On Set ⇒

① Union (U) ⇒

$$\boxed{A \cup B = \{x \mid x \in A \ \vee \ x \in B\}}$$

eg ⇒ ① $A = \{1, 2\}$, $B = \{3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$



② $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

Properties ⇒ ① $A \cup B = B \cup A$ [Commutative

② $A \cup A = A$ [Idempotent

③ $A \cup \phi = A$ [Identity

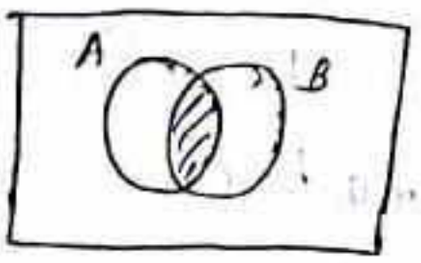
④ $A \cup (B \cup C) = (A \cup B) \cup C$ [Associative

⑤ $A \subseteq A \cup B$, $B \subseteq A \cup B$

Scanned by CamScanner

② Intersection ⇒

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$



Eg ⇒ (i) $A = \{1, 2\}$; $B = \{3, 4\}$

$A \cap B = \phi$

(ii) $A = \{1, 2, 3\}$; $B = \{3, 4, 5\}$

$A \cap B = \{3\}$

★ $A \cap B = \phi \Leftrightarrow A$ & B are disjoint set.

Properties ⇒
1. $A \cap B = B \cap A$
2. $A \cap A = A$
3. $A \cap \phi = \phi$
4. $A \cap (B \cap C) = (A \cap B) \cap C$
5. $A \cap B \subseteq A$
6. $A \cap B \subseteq B$

Theorem Proof ⇒

If $A \cup B = A \cap B$ then $A = B$

Since $A \subseteq (A \cup B)$

$\Rightarrow A \subseteq (A \cap B)$ $[\because A \cup B = A \cap B]$ — (I)

also we know

$A \cap B \subseteq A$ — (II)

from (I) & (II) we get

$\boxed{A = A \cap B}$ — (V)

Similary

$B \subseteq (A \cup B)$

$\Rightarrow B \subseteq (A \cap B)$ — (III) $[\because A \cup B = A \cap B]$

also we know $A \cap B \subseteq B$ — (IV)

Scanned by CamScanner

from (iii) & (iv) we get

$$\boxed{B = A \cap B} \quad \text{— (vi)}$$

from (v) & (vi)

$$A = B$$

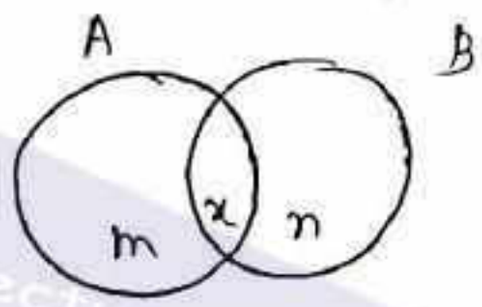
Imp

Theorem ⇒ For the finite sets A & B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof ⇒ by venn diagram

~~Let $|A| = m + x$~~



Let $|A| = m + x$

$|B| = n + x$

$|A \cap B| = x$

So $|A \cup B| = m + x + n$

$= (m+x) + (n+x) - x$

$= |A| + |B| - |A \cap B|$

$|A \cup B|$

★ If A & B are disjoint sets then $|A \cup B| = |A| + |B|$

eg ⇒ For the sets A, B, C, prove that by basic theory

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

$|A \cup B \cup C| = |A \cup (B \cup C)|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$

$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= |A| + \boxed{|B| + |C| - |A \cap B|} - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Scanned by CamScanner

☆ if A, B, C are disjoint sets

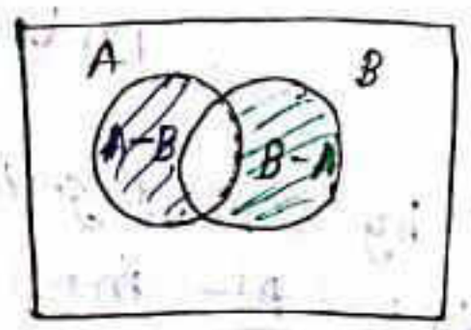
then

$|A \cup B \cup C| = |A| + |B| + |C|$

③ Difference ⇒

* $A - B = \{x \mid x \in A \wedge x \notin B\}$

$B - A = \{x \mid x \in B \wedge x \notin A\}$



Eg ⇒ ① $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

$A - B = \{1, 2\}$

$B - A = \{4, 5\}$

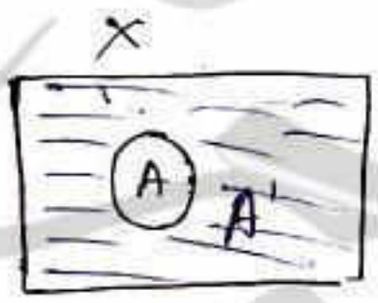
Properties ⇒ ① $A - B \neq B - A$

② $A - A = \phi$

③ $X - A = A'$

④ $A \cup A' = X$

⑤ $A \cap A' = \phi$



☆ (a, b) denotes an ordered pair

where $(a, b) \neq (b, a)$

☆ $(x, y) = (a, b) \Leftrightarrow x = a,\ y = b$

Cartesian Product ⇒

$A \times B = \{(a, b) \mid a \in A,\ b \in B\}$

Eg ⇒ ① $A = \{1, 2\}$; $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

☆ $A \times B \neq B \times A$

☆ If $|A| = m$, $|B| = n$ then $\boxed{|A \times B| = mn}$

LectureNotes.in

Scanned by CamScanner

5/8/16

Divisibility ⇒

Let $a, b, \mathbb{Z}^+$

$a|b$ reads as "a divides b"

OR "b is divisible by a"

OR "a is the divisor of b"

iff there exists an integer 'k' for which

$$\boxed{b = ka}$$

Eg ⇒ $2 \nmid 3$ as $3 = 1.5 \times 2$

$2 | 6$ $\quad 6 = 3 \times 2$

Properties ⇒ ① $a|a$, $a \neq 0$

② $a|b$ & $b|a$ ⇒ $b = \pm a$

$a, b \neq 0$

③ $a|b$ & $a|c$ ⇒ $a|b+c$

④ $a|b$ & $b|c$ ⇒ $a|c$

Proof of Property ②

$a|b \Rightarrow b = k_1 a \quad k_1 \in \mathbb{Z}$

$b|a = a = k_2 b \quad k_2 \in \mathbb{Z}$

$b = k_1(k_2 b)$

$b = k_1 k_2 b$

Scanned by CamScanner

ie. $k_1 k_2 = 1$

$\Rightarrow k_1 = k_2 = \pm 1$

$\Rightarrow b = \pm a$

Proof of Property (3) $\Rightarrow$

$a/b \Rightarrow b = k_1 a, \quad k_1 \in \mathbb{Z}$

$a|c \Rightarrow c = k_2 a, \quad k_2 \in \mathbb{Z}$

Now

$b + c = k_1 a + k_2 a$

$= (k_1 + k_2) a$

$b + c = k_3 a$

$\Rightarrow a/b+c$

Proof of Property (4)

$a|b \Rightarrow b = k_1 a, \quad k_1 \in \mathbb{Z}$

$b|c \Rightarrow c = k_2 b, \quad k_2 \in \mathbb{Z}$

Now

$c = k_2 (k_1 a)$

$= k_2 k_1 a$

$c = k_3 a$

$\Rightarrow a/c$

Prime & Composite No. $\Rightarrow$

Prime No. $\Rightarrow$ The Natural no. $P > 1$ is said to be prime no. if it has no divisor other then ± 1 & $\pm P$

The prime nos. are - 2, 3, 5, 7, 11, ——

Scanned by CamScanner

Composite Nos ⇒ The natural no. $p > 1$ is said to be composite no. iff it is not prime.

So the composite nos. are 4, 6, 8, 9, 10 - —

☆ 1 is a special no.

☆ Let $a, b \in N$

a & b are said to be Relatively Prime to each other iff their GCD is 1.

i.e. $\gcd(a, b) = 1 \Leftrightarrow$ a & b are relatively prime to each other.

Eg ⇒ * Any two prime nos. are relatively prime to each other.

* Any two consecutive natural nos. are relatively prime to each other.

Congruence Modulo ⇒

Let $a, b \in Z$, $n \in N$

$a \equiv b \pmod{n}$ reads as "a is congruent to be modulo n".

iff $n \mid b-a$

Eg ⇒ (i) $2 \not\equiv 3 \pmod{5}$

As $b-a = 3-2 = 1$ which is not divisible by 5.

(ii) $2 \equiv -3 \pmod{5}$

As $b-a = -3-2 = -5 = (-1)(5)$ which is divisible by 5.

Scanned by CamScanner

Properties ⇒

① Since 0 is divisible by every natural no. so for $n \in N$

$$n|0 \Rightarrow n|a-a$$
$$\Rightarrow a \equiv a \pmod{n}$$

② If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$

③ If $a \equiv b \pmod{n}$, $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$

Proof of Property ②

$$a \equiv b \pmod{n}$$
$$\Rightarrow n|b-a \qquad \left[\begin{array}{l} 2|6 \\ 2|-6 \end{array}\right.$$
$$\Rightarrow n|-(b-a)$$
$$\Rightarrow n|(a-b)$$
$$\Rightarrow b \equiv a \pmod{n}$$

Proof of Property ③ ⇒

$$a \equiv b \pmod{n} \Rightarrow n|b-a$$
$$b \equiv c \pmod{n} \Rightarrow n|c-b$$

As we know

$$a|b \ \& \ a|c \Rightarrow a|b+c$$

So

$$n|b-a \ \& \ n|c-b$$
$$\Rightarrow n|b-a+c-b$$
$$\Rightarrow n|c-a$$
$$\Rightarrow a \equiv c \pmod{n}$$

LectureNotes.in

Scanned by CamScanner

★ If R be a binary relⁿ defined on set A then
$R \subseteq A \times A$

$n C_r = \frac{n!}{r!(n-r)!}$

Binary Relation :

① Let a & b are two non empty sets.
Any subset of $(A \times B)$ is a binary relation from set A to set B

② R be a relation from set A to set B i.e. $R \subseteq A \times B$

③ we know if $|A| = m$, $|B| = n$ then $|A \times B| = mn$

So there will be $\boxed{2^{mn} \text{ no. of binary relⁿs}}$ for set A to set B

Eg : ① $A = \{1, 2\}$, $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

The total no. of binary relation $= 2^4 = 16$.

$R_1 = \phi$ → 1

$R_2 = \{(1,3)\}$
$R_3 = \{(1,4)\}$
$R_4 = \{(2,3)\}$
$R_5 = \{(2,4)\}$ } 4

$R_6 = \{(1,3), (1,4)\}$
$R_7 = \{(1,3), (2,3)\}$
$R_8 = \{(1,3), (2,4)\}$

$R_9 = \{(1,4), (2,3)\}$
$R_{10} = \{(1,4), (2,4)\}$
$R_{11} = \{(2,3), (2,4)\}$ } 6

$R_{12} = \{(1,3), (1,4), (2,3)\}$
$R_{13} = \{(1,3), (1,4), (2,4)\}$
$R_{14} = \{(1,4), (2,3), (2,4)\}$
$R_{15} = \{(1,3), (2,3), (2,4)\}$
$R_{16} = \{(1,3), (1,4), (2,3), (2,4)\}$ } 4.

0 — $C(4,0) = 1$
1 — $C(4,1) = 4$
2 — $C(4,2) = 6$
3 — $C(4,3) = 4$

* If R be

② Let $A = \{1, 2, 3, 4, 5, 6\}$

(i) $R = \{(a,b) \mid a \text{ divides } b, \forall a, b \in A\}$

Here $R \subseteq A \times A$

~~$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,4), (2,6), (3,6)\}$~~

(No. of binary relⁿ $= 2^{36}$)

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$\vdots$
$(6,1)\ (6,2), (6,3), (6,4), (6,5), (6,6)\}$

Scanned by CamScanner

$(a,b) \in R$ iff $a|b$

So
$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6)$
$(3,3), (3,6), (4,4), (5,5), (6,6)\}$

(ii) $R = \{(a,b) \mid a$ & b are relatively prime to each other, $\forall a,b \in$

then $R \subseteq A \times A$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,3), (2,5)$
$(3,1), (3,2), (3,4), (3,5)$
$(4,1), (4,3), (4,5)$
$(5,1), (5,2), (5,3), (5,4), (5,6)$
$(6,1), (6,5)\}$

(iii) $R = \{(a,b) \mid a+b = \text{Even} \quad \forall a,b \in A\}$

$R = \{(1,1), (1,3), (1,5)$
$(2,2), (2,4), (2,6)$
$(3,1), (3,3), (3,5)$
$(4,2), (4,4), (4,6)$
$(5,1), (5,3), (5,5)$
$(6,2), (6,4), (6,6)\}$

(iv) $R = \{(a,b) \mid (a-b)^2 \in A, \forall a,b \in A\}$
$R \subseteq A \times A$
$R = \{(1,2), (1,3), (2,1), (2,3), (2,4)$
$(3,1), (3,2), (3,4), (3,5)$
$(4,2), (4,3), (4,5), (4,6)$
$(5,3), (5,4), (5,6)$
$(6,4), (6,5)\}$

LectureNotes.in

Scanned by CamScanner

6/2/16

(v) $R = \{ (a,b) \mid a \equiv b \pmod 3, \forall a, b \in A \}$

i.e. $3 \mid b-a$

$R \subseteq A \times A$

$R = \{ (1,1), (1,4), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4)$
$(5,2), (5,5), (6,3), (6,6) \}$

Graphical Representation of Binary Relation ⇒

*) Let $G = (V, E)$ be a graph
where V be a finite set called Vertex set
E be a another finite set called as Edge set

*) (a, b) be the directed path from vertex 'a' to 'b'

$\{a, b\}$ be the path in b/w vertices a & b

*) (a, a) denotes a self loop.

*) Since $R \subseteq A \times B$
so for the graphical representation
A is to be the vertex set
B is to be the edge set.

☆ Diagraph ⇒ If all the edges contained in the graph are directed then the graph is said to be diagraph.

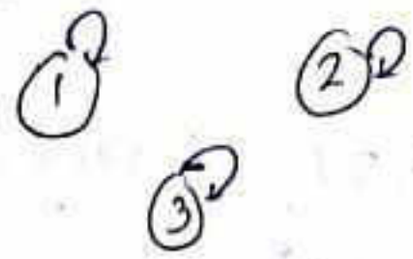
Non directed Path ⇒ If all the edges contained in the graph are non-directed then the graph is said to be N.D.P.

Scanned by CamScanner

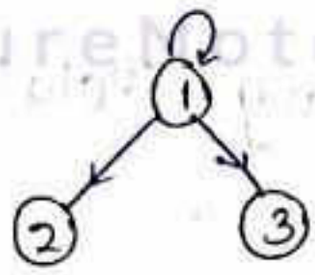
Eg ⇒ $A = \{1, 2, 3\}$

$R \subseteq A \times A$

① $R_1 = \{(1,1), (2,2), (3,3)\}$

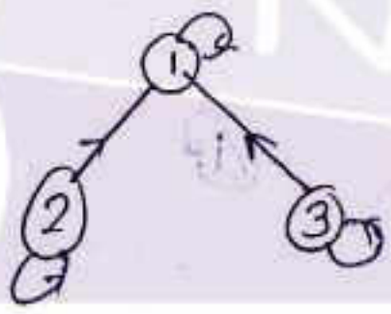


② $R_2 = \{(1,1), (1,2), (1,3)\}$



③ $R_3 = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

④ $R_4 = \{(1,1), (2,1), (2,2), (3,1), (3,3)\}$



Properties ⇒ Let $R \subseteq A \times A$

*) R is said to be reflexive iff $\forall a \in A, (a,a) \in R$

*) R is said to be irreflexive iff $\forall a \in A$, $(a,a) \notin R$

*) R is said to be symmetric iff for $(a,b) \in R \Rightarrow (b,a) \in R$

*) R is said to be antisymmetric iff for $(a,b) \in R$ & $(b,a) \in R$ $\Leftrightarrow a = b$

*) R is said to be transitive iff for $(a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$

*) R is said to be equivalence relation iff R is reflexive, symmetric & transitive.

*) R is so reflexive, antisymmetric & transitive then R be a Partial Order Relation (POR)

Scanned by CamScanner

*) If R be a POR defined on set A then set A is said to be POS (Partial ordered set)

*) In such case [A, R] is said to be a POSET

FROM DIAGRAPH ⇒

*) Reflexive ⇒ If self loop enters at each vertex then the relⁿ is reflexive.

*) Irreflexive ⇒ If self loop enters at any vertex then the relⁿ is said to be irreflexive.

*) Symmetric ⇒ If there exists a path from vertex 'a' to vertex 'b' & also the reverse path then R is symmetric.

*) Asymmetric ⇒ If there exists a path from vertex 'a' to vertex 'b' but reverse path is not present or even self loops are not present then R is asymmetric

*) Antisymmetric ⇒ If there exists a path from vertex 'a' to vertex 'b' & also the reverse path exists & even the self loop exists, then R is antisymmetric.

*) Transitive ⇒ If there exists path from 'a' to 'b' & vertex 'b' to 'c' & also from vertex 'a' to 'c' then R is transitive.

Eg ⇒ $A = \{1, 2, 3\}$

* $R_1 = \{(1,1), (2,2), (3,3)\}$

* $R_2 = \{(1,1), (1,2), (2,1), (2,2)\}$

→ R_1 is to be reflexive

→ R_1 is not Irreflexive

→ R_1 is symmetric

→ R_1 is not asymmetric

Scanned by CamScanner

8/8/16

→ R is not antisymmetric

→ R is transitive

$R_2 = \{(1,1), (1,2), (2,1), (2,2)\}$

R is not reflexive as $3 \in A$ but $(3,3) \notin R$

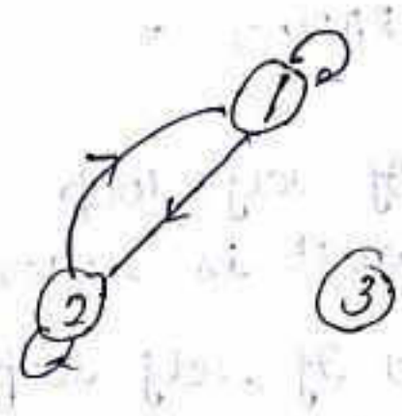
R is not irreflexive.

R is not symmetric

R is not assymetric

R is not antisymmetric

R is not transitive



$(1,1) \in R,\ (1,2) \in R \Rightarrow (1,2) \in R$

$(1,2) \in R,\ (2,1) \in R \Rightarrow (1,1) \in R$

$(1,2) \in R,\ (2,2) \in R \Rightarrow (1,2) \in R$

$(2,1) \in R,\ (1,1) \in R \Rightarrow (2,1) \in R$

$(2,1) \in R,\ (1,2) \in R \Rightarrow (2,2) \in R$

$(2,2) \in R,\ (2,1) \in R \Rightarrow (2,1) \in R$

*) Contrapositive of Transitive Relation

If $(a,c) \notin R$ then $(a,b) \in R$

$(b,c) \in R$

If $(a,b) \in R \wedge (b,c) \in R$ then $(a,c) \in R$

Eg ⇒ Let

① R =

R

where

A × A

(a,b

R

Dia

LectureNotes.in

Scanned by CamScanner

8/8/16

Eg ⇒ Let $A = \{1, 2, 3, 4, 5\}$

① $R = \{(a,b) \mid a \text{ divides } b, \forall a, b \in A\}$

$R \subseteq A \times A$

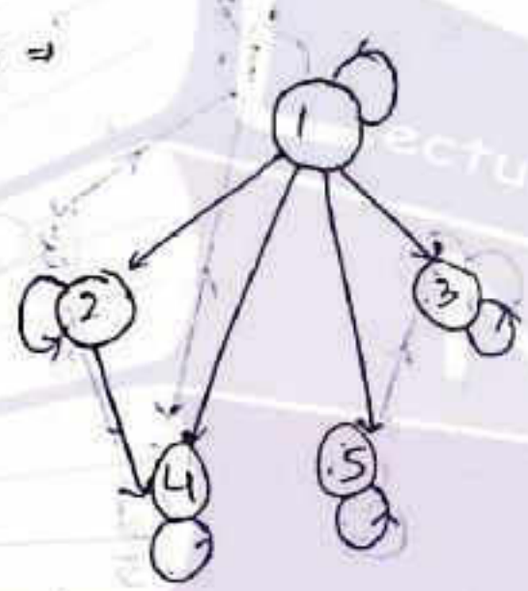
where

$A \times A = \{(1,1), (1,2), (1,3), (1,4), (1,5)$
$(2,1), (2,2), (2,3), (2,4), (2,5)$
$(3,1), (3,2), (3,3), (3,4), (3,5)$
$(4,1), (4,2), (4,3), (4,4), (4,5)$
$(5,1), (5,2), (5,3), (5,4), (5,5)\}$

$(a,b) \in R$ iff $a|b$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5)$
$(2,2), (2,4), (3,3), (4,4), (5,5)\}$

Diagram ⇒



R is reflexive

R is irreflexive

R is not symmetric as $(1,3) \in R$ but $(3,1) \notin R$

R is not assymetric as $(1,1), (2,2) \ldots \in R$

R is antisymmetric [as their is no opposite pair

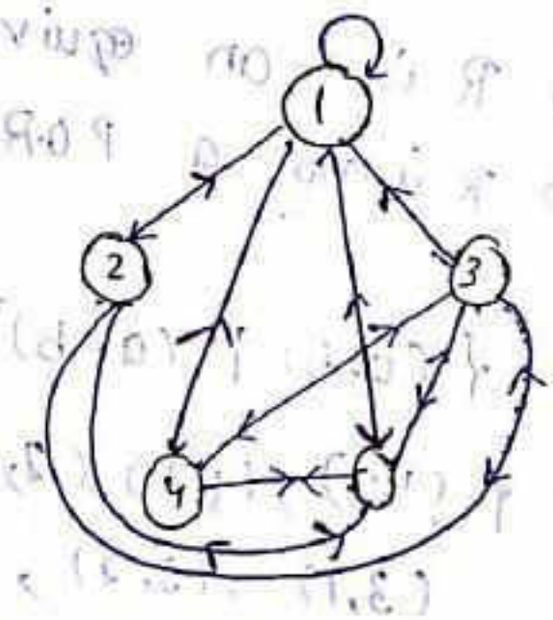
R is transitive

Thus R, is not an equivalence rel^n

R be a partial order relation (P.O.R)

② $R = \{(a,b) \mid a \,\&\, b \text{ are relatively Prime}, \forall a, b \in A\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5)$
$(2,1), (2,3), (2,5), (3,1), (3,2)$
$(3,4), (3,5), (4,1), (4,3), (4,5)$
$(5,1), (5,2), (5,3), (5,4)\}$



Scanned by CamScanner

☆ R is not reflexive, as for $2, 3, 4, 5 \in A$: $(2,2) \notin R$, $(3,3) \notin R$ - - -

☆ R is not irreflexive as $(1,1) \in R$

☆ R is symmetric

☆ R is not asymmetric

☆ R is not antisymmetric

☆ R is not transitive as $(2,1) \in R$ & $(1,2) \in R$ but $(2,2) \notin R$

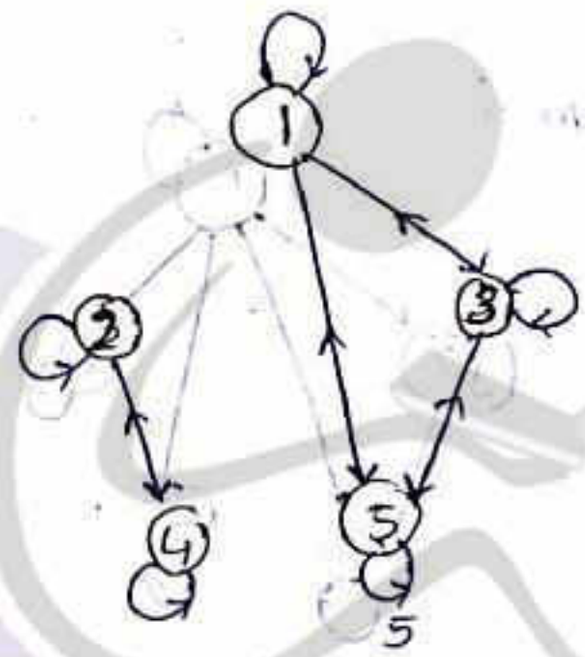
Hence R is not an equivalence relⁿ

R is not a P.O.R

③ $R = \{(a,b) \mid a \equiv b \pmod 2 ; \forall a, b \in A\}$

$\Rightarrow \quad 2 \mid b-a \Rightarrow b-a = k2$

$R = \{(1,1), (1,3), (1,5),$
$(2,2), (2,4), (3,1),$
$(3,3), (3,5), (4,2), (4,4)$
$(5,1), (5,3), (5,5)\}$



☆ R is reflexive

☆ R is not irreflexive

☆ R is symmetric

☆ R is not asymmetric

☆ R is not antisymmetric

☆ R is transitive

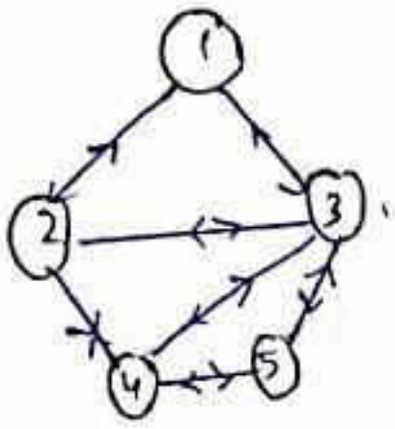
Hence
$\Rightarrow$ R is an equivalence relⁿ.
$\Rightarrow$ R is not a P.O.R

④ $R = \{(a,b) \mid (a-b)^2 \in A, \forall a, b \in A\}$

$R = \{(1,2), (1,3), (2,1), (2,3), (2,4),$
$(3,1), (3,2), (3,4), (3,5),$
$(4,2), (4,3), (4,5), (5,3), (5,4)\}$

Scanned by CamScanner

☆ R is not reflexive as $1, 2, 3, 4, 5 \in A$ but $(1,1) \notin R$,
$(2,2) \notin R$ --- $(5,5) \notin R$



☆ R is irreflexive

☆ R is symmetric

☆ R is not asymmetric

☆ R is not antisymmetric

☆ R is not transitive as $(1,2) \in R \wedge (2,1) \in R$ but $(1,1) \notin R$

Hence R is not an equivalence rel^n

R is not a P.O.R

Eg ⇒ <u>Infinite set</u>

(1) $R = \{(a,b) \mid a \text{ divides } b, \forall a, b \in \mathbb{Z}^+\}$

So $R \in \mathbb{Z}^+ \times \mathbb{Z}^+$

☆ Since $\forall a \in \mathbb{Z}^+$, Every +ve integer is divisible by itself

So, a divides itself ⇒ $(a,a) \in R$

⇒ R is reflexive

☆ R is not irreflexive

☆ Let $(a,b) \in R \Rightarrow a \mid b \Rightarrow b \neq a \Rightarrow (b,a) \notin R$ $\left\{ \begin{array}{l} 2 \mid 6 \\ 6 \neq 2 \end{array} \right.$

⇒ R is not symmetric

☆ R is not asymmetric

☆ Let $(a,b) \in R$ & $(b,a) \in R \Rightarrow a \mid b$ & $b \mid a$

$\Rightarrow b = a$ [as $b = \pm a$]

⇒ R is antisymmetric

☆ Let $(a,b) \in R$ & $(b,c) \in R$

$\Rightarrow a \mid b$ & $b \mid c$

$\Rightarrow a \mid c$

$\Rightarrow (a,c) \in R$ ⇒ R is transitive

Scanned by CamScanner

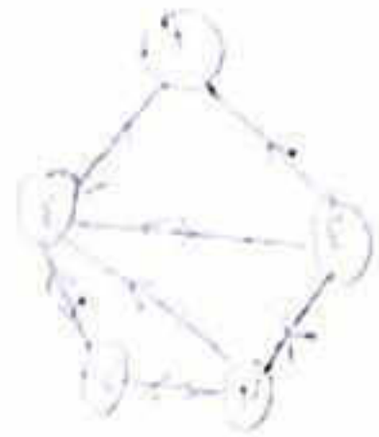
Thus

R is not an equivalence rel^n

R is a P.O.R defined on $\mathbb{Z}^+$

So $\mathbb{Z}^+$ be a P.O.S

$(\mathbb{Z}^+, |]$ be a POSET.



LectureNotes.in

(2) $R = \{(a,b) \mid a \equiv b \pmod 5, \forall a,b \in \mathbb{Z}\}$

$R \subseteq \mathbb{Z} \times \mathbb{Z}$

$(a,b) \in R \Leftrightarrow a \equiv b \pmod 5$

i.e. $5 \mid b-a$

*) $\forall a \in \mathbb{Z}$, Since $5 \mid 0 \Rightarrow 5 \mid a-a$

$\Rightarrow$ ~~a is~~ $a \equiv a \pmod 5 \Rightarrow (a,a) \in R$

$\Rightarrow$ R is reflexive

*) R is not irreflexive

*) R is symmetric

Let $(a,b) \in R \Rightarrow a \equiv b \pmod 5 \Rightarrow 5 \mid b-a$

$\Rightarrow 5 \mid -(b-a)$

$\Rightarrow 5 \mid (a-b)$

$\Rightarrow b \equiv a \pmod 5$

$\Rightarrow (b,a) \in R$

*) R is not asymmetric

*) R is not antisymmetric

*) Let $(a,b) \in R$ & $(b,c) \in R$

$\Rightarrow a \equiv b \pmod 5$ & $b \equiv c \pmod 5$

$5 \mid b-a$ & $5 \mid c-b$

$5 \mid (c-b) + (b-a)$

$5 \mid c-a$

$\Rightarrow a \equiv c \pmod 5 \Rightarrow (a,c) \in R$

$\Rightarrow$ R is transitive

Thus R is an equivalence rel^n over $\mathbb{Z}^+$

R is not a P.O.R

Scanned by CamScanner

Asymmetric ⇒ neither reflexive nor symmetric

10/8/16 In antisymmetric ⇒ self loops are permitted but not symmetric

Ex ⇒ $R = \{x, y \mid x \leq y, \forall x, y \in N\}$

$R \subseteq N \times N$

$(x, y) \in R \Leftrightarrow x \leq y$

→ $\in x \in N$, $\because x = x$

$\Rightarrow (x, x) \in R$

R is reflexive.

→ R is not irreflexive.

→ Let $(x, y) \in R \Rightarrow x \leq y \Rightarrow y \nleq x$

Eg $(1, 3) \in R$ but $(3, 1) \notin R$

R is not symmetric

→ R is not asymmetric as $(x, x) \in R$

→ Let $(x, y) \in R$ & $(y, x) \in R$

$\Rightarrow x \leq y$ & $y \leq x$

$\Rightarrow x = y$

R is antisymmetric

→ Let $(x, y) \in R$ & $(y, z) \in R$

$x \leq y$ & $y \leq z$

$\Rightarrow x \leq z$

$\Rightarrow (x, z) \in R$

$\Rightarrow$ R is transitive

$\Rightarrow$ R is not an equivalence relation

$\Rightarrow$ R be a P.O.R

$\Rightarrow$ N be a P.O.S

$\Rightarrow$ So $[N, \leq]$ be a poset

Scanned by CamScanner

Eg :)

$R = \{ (x,y) \mid x-y = \text{even}, \forall x, y \in \mathbb{Z} \}$

→ $R \subseteq \mathbb{Z} \times \mathbb{Z}$

$(x,y) \in R \Leftrightarrow (x-y) = \text{even}$

→ $\forall x \in \mathbb{Z} \Rightarrow x - x = 0$ which is even

$\Rightarrow (x,x) \in R$

not in $\Rightarrow$ Reflexive

→ R is reflexive.

→ $\forall x, y \in \mathbb{Z}$, let $(x,y) \in R$

$\Rightarrow x - y = \text{even}$

$\Rightarrow -(y-x) = \text{Even}$

$\Rightarrow (y-x) = \text{Even}$

$\Rightarrow \cancel{(y-x \in R}$

$\Rightarrow (y,x) \in R$

$\Rightarrow$ R is symmetric

→ R is not asymmetric

→ R is not antisymmetric

→ Let $(x,y) \in R$ & $(y,z) \in R$

$\Rightarrow x - y = \text{even}$ & $y - z = \text{even}$

$x - y + y - z = \text{even} + \text{even}$

$x - z = \text{even}$

$\Rightarrow (x,z) \in R$

$\Rightarrow$ R is transitive

Thus R be an equivalence rel^n

So R is not a POR.

$R = \{ (x,y) \mid x = y^2, \forall x, y \in \mathbb{Z} \}$

$R \subseteq \mathbb{Z} \times \mathbb{Z}$

$R = \{(0,0), (1,1), (1,-1), (4,2), (4,-2), (9,3), (9,-3) \ldots \}$

Scanned by CamScanner

$(x,y) \in R \Leftrightarrow x = y^2$

$\rightarrow \forall x \in Z \Rightarrow$ ~~$x \neq y^2$~~ $\because x \neq x^2$ for $x = 2$ eg $\Rightarrow$ $2 \neq 2^2$

as $\sim 4 =$

$\Rightarrow (x,x) \notin R$

$\Rightarrow$ R is not reflexive

$\rightarrow$ R is not irreflexive as $(1,1) \in R$

$\rightarrow$ Let $(x,y) \in R$

$\Rightarrow x = y^2$

$\Rightarrow y \neq x^2$

as $(4,2) \in R$ but $(2,4) \notin R$

$\Rightarrow (y,x) \notin R$

$\rightarrow$ R is not symmetric

$\rightarrow$ R is not asymmetric as $(0,0) \in R$

$\rightarrow$ R is antisymmetric

$\rightarrow$ Let $(x,y) \in R$ & $(y,z) \in R$

$x = y^2$ & $y = z^2$

R is transitive as $(1,1) \in R$ & $(1,-1) \in R$

$\Rightarrow (1,-1) \in R$

as for another case $(x,y) \in R$ but there is no $(y,z) \in R$, so it is transitive

R is transitive

$\Rightarrow$ R is not equivalence rel^n

$\Rightarrow$ R is not POR.

Scanned by CamScanner

Eg ⇒ Imp Let A be any finite set

$R = \{(X, Y) \mid X \subseteq Y \in P(A)\}$

Here $R \subseteq P(A) \times P(A)$

$(X, Y) \in R \Leftrightarrow X \subseteq Y$

*) $\forall X \in P(A)$, Every set is subset of itself

So, $X \subseteq X \Rightarrow (X, X) \in R$

$\Rightarrow$ R is reflexive

*) R is not irreflixve.

*) Let $(X, Y) \in R \Rightarrow X \subseteq Y \Rightarrow Y \not\subseteq X$

$\Rightarrow (Y, X) \notin R$

$\Rightarrow$ R is not symmetric

*) R is not asymmetric (as $(X, X) \in R$)

*) Let $(X, Y) \in R$ & $(Y, X) \in R$

$\Rightarrow X \subseteq Y$ & $Y \subseteq X$

$\Rightarrow X = Y$

$\Rightarrow$ R is antisymmetric

*) Let $(X, Y) \in R$ & $(Y, Z) \in R$

$X \subseteq Y$ & $Y \subseteq Z$

$\Rightarrow X \subseteq Z$

$\Rightarrow (X, Z) \in R$

$\Rightarrow$ R is transitive

Thus R is not an equivalence relation

R is a P.O.R

P(A) be a P.O.S

So, $[P(A), \subseteq]$ be a POSET

LectureNotes.in

Scanned by CamScanner

$R = \{ (a,b) \mid a-b \in \mathbb{Z}, \forall a, b \in \mathbb{Q} \}$

$R \subseteq \mathbb{Q} \times \mathbb{Q}$

$(a,b) \in R \Leftrightarrow (a-b) \in \mathbb{Z}$

*) $\forall a \in \mathbb{Q}$, $a-a = 0 \in \mathbb{Z}$

$\Rightarrow (a,a) \in R$

$\Rightarrow R$ is reflexive

*) R is not irreflexive

*) Let $(a,b) \in R \Rightarrow a-b \in \mathbb{Z}$

$b-a \in \mathbb{Z}$

$(b,a) \in R$

$\Rightarrow R$ is symmetric

*) R is not asymmetric

*) R is not antisymmetric. as here $(b,a) \in R$

*) Let $(a,b) \in R$ & $(b,c) \in R$

i.e. $a-b \in \mathbb{Z}$ $\quad b-c \in \mathbb{Z}$

$a-b+b-c \in \mathbb{Z}+\mathbb{Z}$

$a-c \in \mathbb{Z}$

$(a,c) \in R$

$\Rightarrow R$ is transitive

$\Rightarrow R$ is equivalence rel^n

$\Rightarrow R$ is not a POR.

<u>NOTES</u> $\Rightarrow$ Let $R \subseteq A \times A$

So $\Delta = \{ (a,a) \mid \forall a \in A \}$

eg $\Rightarrow$ (i) $A = \{1\}$

$\Delta A = \{1,1\}$

(ii) $A = \{1,2\}$

$\Delta A = \{(1,1), (2,2)\}$

(iii) $A = \{1,2,3\}$

$\Delta A = \{(1,1), (2,2), (3,3)\}$

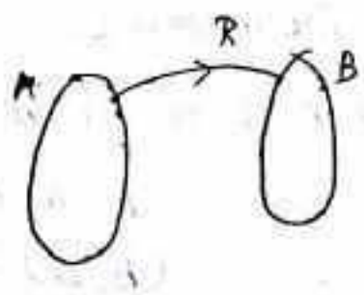
Scanned by CamScanner

*) The smallest equivalence rel^n over set A is to be
$\Delta = \{(a,a) \mid \forall a \in A\}$

*) The largest equivalence rel^n defined over set A is to be
$A \times A$

LectureNotes.in

Inverse of A Relation ⇒



*) Let $R \subseteq A \times B$
so $R^{-1} \subseteq B \times A$
where $R^{-1} = \{(b,a)\}$

Eg ⇒ (i) $A = \{1,2\}$, $B = \{a,b\}$ (ii) If $R \subseteq A \times A$
$R = \{(1,a), (1,b), (2,b)\}$
$R^{-1} = \{(a,1), (b,1), (b,2)\}$

*) If $R \subseteq A \times A$ then $R^{-1} \subseteq A \times A$

eeg ⇒ $A = \{1,2,3\}$
$R = \{(1,1), (1,2), \cancel{(1,3)}, (2,3), (3,1)\}$
$R^{-1} = \{(1,1), (2,1), \cancel{(3,1)}, (3,2) (1,3)\}$

LectureNotes.in

Domain & Range ⇒

LectureNotes.in

Let $R \subseteq A \times B$
$D(R)$ denotes the domain of R
$R(R)$ denotes the range of R
where $D(R) = \{a \mid \forall (a,b) \in R\}$
$R(R) = \{b \mid \forall (a,b) \in R\}$

Scanned by CamScanner

Eg ⇒ Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

① $R = \{(1,a), (1,b), (1,c)\}$

$D(R) = \{1\}$

$R(R) = \{a, b, c\}$

② $R = \{(1,a), (2,a), (3,a)\}$

$D(R) = \{1, 2, 3\}$

$R(R) = \{a\}$

③ $R = \{(1,a), (2,b), (3,c)\}$

$D(R) = \{1, 2, 3\}$

$R(R) = \{a, b, c\}$

So if $R \subseteq A \times B$

$D(R) \subseteq A$

$R(R) \subseteq B$

NOTE ⇒

V. Imp

POSET Diagram / Hasse Diagram ⇒ A pictorial / graphical representation of a POSET is POSET diagram / Hasse Diagram.

How to Construct ⇒

① Write down the relation set R in tabular form.

② Delete all the identity pairs (a,a) from R

③ Delete the pair (a,c) if $(a,b) \in R$ & $(b,c) \in R$

④ With the remaining elements of R draw the graph ~~from below to o~~ where edges are to be drawn from below to above without arrow mark.

Scanned by CamScanner

12/5/18

$\boxed{\text{*) Since } [\mathbb{Z}^+, \overset{\text{divide}}{|}] \text{ be a POSET}}$

for 4 marks

Eg: (i) Q) Is $[D_8, |]$ be a poset ?

If yes draw its hasse diagram

D_8 : The set of all positive divisors of 8

$= \{1, 2, 4, 8\}$

$R = \{(x, y) \mid x \text{ divides } y, \forall x y$

ie. $R \subseteq D_8 \times D_8$

$R = \{(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)\}$

*) R is reflexive

*) R is antisymmetric

*) R is transitive

$\therefore$ R be a P.O.R

So, D_8 be a P.O.S

$[D_8, |]$ be a POSET

*) $R = \{\cancel{(1,1)}, (1,2), \cancel{(1,4)}, \cancel{(1,8)}, \cancel{(2,2)}, (2,4), \cancel{(2,8)}, \cancel{(4,4)}, (4,8), \cancel{(8,8)}\}$

$= \{(1,2), (2,4), (4,8)\}$

Hess Diagram

Scanned by CamScanner

for 2 marks

Eg ⇒ (ii) Draw the hasse diagram of the POSET $[D_{12}, |]$

$\because D_{12} = \{1, 2, 3, 4, 6, 12\}$

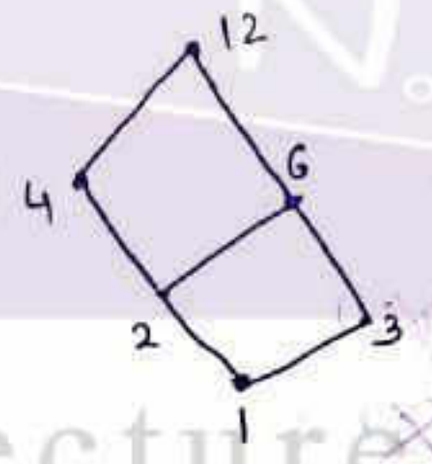
$R \subseteq D_{12} \times D_{12}$

$R = \{(x, y) \mid x \text{ divides } y, \forall x, y \in D_{12}\}$

$= \{\cancel{(1,1)}, (1,2), (1,3), (1,4), \cancel{(1,6)}, \cancel{(1,12)},$
$\cancel{(2,2)}, (2,4), (2,6), \cancel{(2,12)},$
$\cancel{(3,3)}, (3,6), \cancel{(3,12)},$
$\cancel{(4,4)}, (4,12),$
$\cancel{(6,6)}, (6,12),$
$\cancel{(12,12)}\}$

$= \{(1,2), (1,3), (2,4), (2,6), (3,6), (4,12), (6,12)\}$

Hess Diagram ⇒



Eg ⇒ (iii) $A = \{1, 2, 3, 4, 6, 9\}$

Draw the hess diagram of the POSET $[A, |]$.

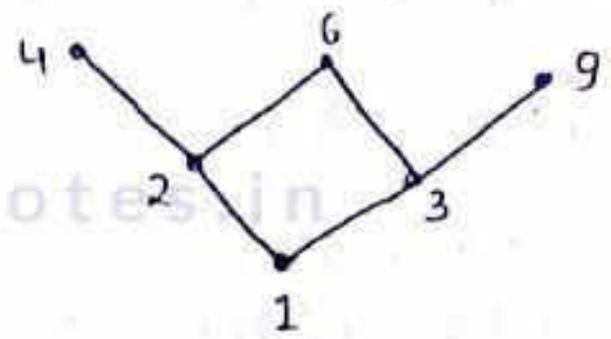
$R \subseteq A \times A$

$R = \{\cancel{(1,1)}, (1,2), (1,3), \cancel{(1,4)}, \cancel{(1,6)}, \cancel{(1,9)},$
$\cancel{(2,2)}, (2,4), (2,6),$
$\cancel{(3,2)}, (3,6), (3,9)$
$\cancel{(4,4)},$
$\cancel{(6,6)}$
$\cancel{(9,9)}\}$

Scanned by CamScanner

$\Rightarrow \{(1,2), (1,3), (2,4), (2,6), (3,6), (3,9)\}$

Hess diagram $\Rightarrow$



x) $R = \{(x,y) \mid x \le y, \forall x,y \in N\}$ be a P.O.R

So $[N, \le]$ be a POSET

Eg $\Rightarrow$ (i) $A = \{1,2,3,4,5\}$

Draw the hess diagram of the poset $[A, \le]$

$R \subseteq A \times A$

$R = \{(x,y) \mid x \le y, \forall x,y \in A\}$

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5)$
$(2,2), (2,3), (2,4), (2,5)$
$(3,3), (3,4), (3,5)$
$(4,4), (4,5)$
$(5,5)\}$

$\Rightarrow \{(1,2), (2,3), (3,4), (4,5)\}$

Scanned by CamScanner

Eg (ii) Draw the hess diagram of the POSET $[A, \geq]$

where $A = \{1, 2, 3, 4, 5\}$

$R \subseteq A \times A$

$R = \{(x,y) \mid x \geq y ; \forall x, y \in A\}$

$R = \{(\cancel{1,1}),$
$(\cancel{2,2}), (2,1),$
$(\cancel{3,3}), (3,2), (\cancel{3,1}),$
$(\cancel{4,4}), (4,3), (\cancel{4,2}), (\cancel{4,1})$
$(\cancel{5,5}), (5,4), (\cancel{5,3}), (\cancel{5,2}), (\cancel{5,1})\}$

$= \{(2,1), (3,2), (4,3), (\cancel{4,1}), (5,4), (\cancel{5,2})\}$

$= \{(2,1), (3,2), (4,3), (5,4)\}$

> * For any finite set A
> $[P(A), \subseteq]$ be a POSET

Eg ① For $A = \{a\}$

Find the hess diagram of $[P(A), \subseteq]$

$A = \{a\}$

$P(A) = \{\phi, \{a\}\}$

$R \subseteq P(A) \times P(A)$

$(x,y) \in R$ iff $x \subseteq y$

So $R = \{(\cancel{\phi,\phi}), (\phi, \{a\}), (\cancel{\{a\},\{a\}})\}$

$= \{(\phi, \{a\})\}$

LectureNotes.in

Scanned by CamScanner

② $A = \{a, b\}$

Draw the hasse diagram of $[P(A), \subseteq]$

$P(A) = 2^2 = 4 = \cancel{\{\phi, \{a,a\}, (a,b), (b,b)\}}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$ total 16 elements in $P(A) \times P(A)$

$R = \{(x,y) \mid x \subseteq y, \forall x, y \in P(A)\}$

$R = \{(\cancel{\phi,\phi}), (\phi, \{a\}), (\phi, \{b\}), (\phi, \cancel{\{a,b\}}),$

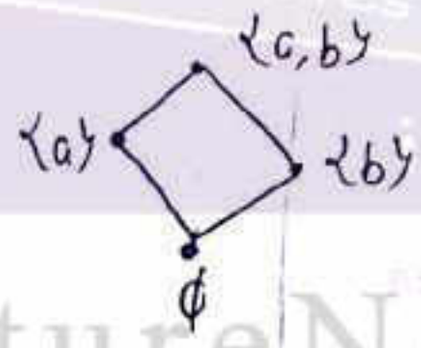
$(\cancel{\{a\},\{a\}}), (\{a\}, \{a,b\})$

$\cancel{(\{b\},\{b\})}\ (\cancel{\{b\},\{b\}})\ (\{b\}, \{a,b\})$

$(\{a,b\}, \{a,b\})\}$

$= \{(\phi, \{a\}), (\phi, \{b\}), (\{a\}, \{a,b\}), (\{b\}, \{a,b\}),$

$(\{a,b\}, \{a,b\})\}$



LectureNotes.in

Assignment :

① If $A = \{a, b, c\}$ then draw the hasee diagram of $[P(A), \subseteq]$

② If $A = \{2, 3, 5, 30, 60, 120, 180, 360\}$ then draw the hasee diagram of $[A, |]$

③ $A = \{2, 3, 4, 9, 12, 18\}$ then Draw the hasee diagram of $[A, |]$

④ $A = \{1, 2, 3, 4, \ldots 12\}$ then draw the hasee diagram of $[A, |]$

⑤ Draw the hasee diagram of $[D_6, |]$

Scanned by CamScanner

① $A = \{a, b, c\}$

$P(A) = 2^3 = 8$

$= \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

total 64 elementy

$R = \{(x, y) \mid x \subseteq y, \forall x, y \in P(A)\}$

$R = \{(\cancel{\phi, \phi}), (\phi, \{a\}), (\phi, \{b\}), (\phi, \{c\}), (\cancel{\phi, \{a, b\}}), (\cancel{\phi, \{b, c\}}), (\cancel{\phi, \{a, c\}}),$
$(\phi, \cancel{\{a, b, c\}})$

$(\cancel{\{a\}, \{a\}}), (\{a\}, \{a, b\}), (\{a\}, \{a, a\}), (\cancel{\{a\}, \{a, b, c\}})$

$(\cancel{\{b\}, \{b\}}), (\{b\}, \{a, b\}), (\{b\}, \{b, c\}), (\cancel{\{b\}, \{a, b, c\}})$

$(\cancel{\{c\}, \{c\}}), (\{c\}, \{b, c\}), (\{c\}, \{a, a\}), (\cancel{\{c\}, \{a, b, c\}})$

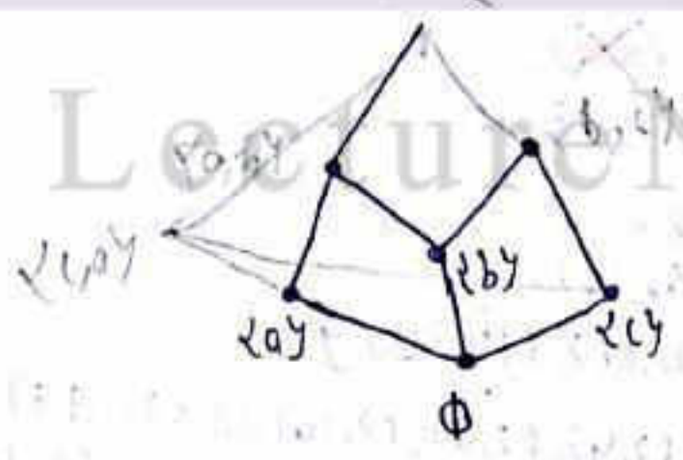
$(\cancel{\{a, b\}, \{a, b\}}), (\{a, b\}, \{a, b, c\})$

$(\cancel{\{b, c\}, \{b, c\}}), (\{b, c\}, \{a, b, c\})$

$(\cancel{\{a, a\}, \{a, a\}}), (\{a, a\}, \{a, b, c\})$

$(\cancel{\{a, b, c\}, \{a, b, c\}})\}$

$R = \{(\phi, \{a\}), (\phi, \{b\}), (\phi, \{c\}), (\{a\}, \{a, b\}), (\{a\}, \{a, a\}),$
$(\{b\}, \{a, b\}), (\{b\}, \{b, c\}), (\{c\}, \{b, c\}), (\{c\}, \{a, a\}),$
$(\{a, b\}, \{a, b, c\}), (\{b, c\}, \{a, b, c\}), (\{a, a\}, \{a, b, c\})\}$



② $A = \{2, 3, 5, 30, 60, 120, 180, 360\}$

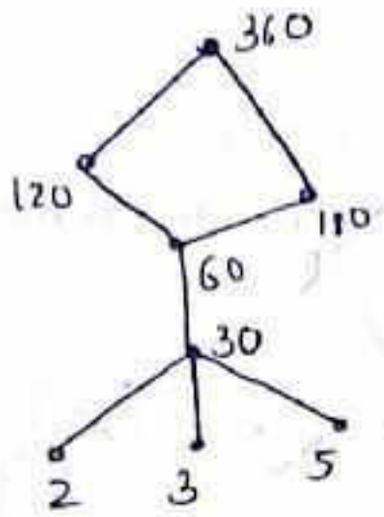
$R = \{(x, y) \mid x \text{ divides } y, \forall x, y \in A\}$

$R \subseteq A \times A$

$R = \{(\cancel{2, 2}), (2, 30), (\cancel{2, 60}), (\cancel{2, 120}), (\cancel{2, 180}), (\cancel{2, 360})$
$(\cancel{3, 3}), (3, 30), (\cancel{3, 60}), (\cancel{3, 120}), (\cancel{3, 180}), (\cancel{3, 360})$
$(\cancel{5, 5}), (5, 30), (\cancel{5, 60}), (\cancel{5, 120}), (\cancel{5, 180}), (\cancel{5, 360})$
$(\cancel{30, 30}), (30, 60), (\cancel{30, 120}), (\cancel{30, 180}), (\cancel{30, 360})$
$(\cancel{60, 60}), (60, 120), (60, 180), (\cancel{60, 360})$
$(\cancel{120, 120}), (120, 360)$
$(\cancel{180, 180}), (180, 360)$
$(\cancel{360, 360})\}$

Scanned by CamScanner

$R = \{(2,30), (3,30), (5,30), (30,60), (60,120), (60,180), (120,360), (180,360)\}$



(3) $A = \{2, 3, 4, 9, 12, 18\}$

$R = \{(2,\cancel{2}), (2,4), (2,\cancel{12}), (2,18),$
$(\cancel{3},\cancel{3}), (3,9), (3,12), (3,\cancel{8})$
$(\cancel{4},\cancel{4}), (4,12)$
$(\cancel{9},\cancel{9}), (9,18)$
$(\cancel{12},\cancel{12}), (\cancel{18},\cancel{18})\}$

$R = \{(2,4), (2,18), (3,9), (3,12), (4,12), (9,18)\}$

(4) $A = \{1, 2, 3, 4 - - 12\}$

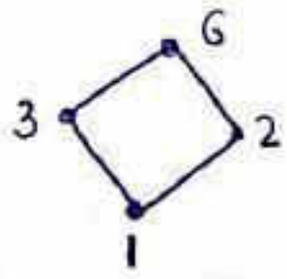
$R = \{(\cancel{1},\cancel{1}), (1,2), (1,3), (\cancel{1},\cancel{4}), (1,5), (\cancel{1},\cancel{6}), (1,7), (\cancel{1},\cancel{8}), (\cancel{1},\cancel{9}), (\cancel{1},\cancel{10}), (1,11), (\cancel{1},\cancel{12}),$
$(\cancel{2},\cancel{2}), (2,4), (2,6), (\cancel{2},\cancel{8}), (2,10), (\cancel{2},\cancel{12})$
$(\cancel{3},\cancel{3}), (3,6), (3,9), (\cancel{3},\cancel{12}),$
$(\cancel{4},\cancel{4}), (4,8), (4,12),$
$(\cancel{5},\cancel{5}), (5,10),$
$(\cancel{6},\cancel{6}), (6,12),$
$(\cancel{7},\cancel{7}), (\cancel{8},\cancel{8}), (\cancel{9},\cancel{9}), (\cancel{10},\cancel{10}), (\cancel{11},\cancel{11}), (\cancel{12},\cancel{12})\}$

$R = \{(1,2), (1,3), (1,5), (1,7), (1,11), (2,4), (2,6), (2,10), (3,6), (3,9), (4,8), (4,12), (5,10), (6,12)\}$

(5) $D_6 = \{1, 2, 3, 6\}$

$R = \{(\cancel{1},\cancel{1}), (1,2), (1,3), (\cancel{1},\cancel{6}),$
$(\cancel{2},\cancel{2}), (2,6), (\cancel{3},\cancel{3}), (3,6), (\cancel{6},\cancel{6})\}$

$R = \{(1,2), (1,3), (2,6), (3,6)\}$



Scanned by CamScanner

Matrix Representation ⇒

Let $R \subseteq A \times A$

The adjacency matrix of R is to be denoted by M_R whose

$$M_R = M_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{otherwise} \end{cases}$$

Eg ⇒ (i) $A = \{1, 2\}$, $B = \{a, b\}$

A × B	a	b
1	(1,a)	(1,b)
2	(2,a)	(2,b)

B × A	1	2
a	(a,1)	(a,2)
b	(b,1)	(b,2)

(ii) $A = \{1, 2, 3\}$

A \ A	1	2	3
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

(a) if $R = \{(1,1), (2,2), (3,3)\}$

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) if $R = \{(1,1), (1,2), (2,1), (3,3)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) If $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

then $R = \{(1,1), (1,3), (2,2), (3,1), (3,2), (3,3)\}$

Scanned by CamScanner

18/8/16

Closures ⇒

Let $R \subseteq A \times A$

$\Delta = \{(a,a) \mid \forall a \in A\}$

Eg ⇒ ① $A = \{1\}$

$\Delta = \{(1,1)\}$

② $A = \{1, 2\}$

$\Delta = \{(1,1), (2,2)\}$

③ $A = \{1, 2, 3\}$

$\Delta = \{(1,1), (2,2), (3,3)\}$

*) The reflexive closure of R is $R \cup \Delta$

*) The symmetric closure of R is $R \cup R^{-1}$

Eg ⇒ $A = \{1, 2, 3\}$

① $R = \{(1,1), (1,2), (1,3)\}$

Reflexive closure of R is

$R \cup \Delta = \{(1,1), (3,3), (2,2)\} \cup \{(1,1), (1,2), (1,3)\}$

$= \{(1,1), (2,2), (3,3), (1,2), (1,3)\}$

Symmetric closure of R is

$R \cup R^{-1} = \{(1,1), (1,2), (1,3)\} \cup \{(1,1), (2,1), (3,1)\}$

$= \{(1,1), (1,2), (2,1), (1,3), (3,1)\}$

② $R = \{(1,1), (2,2), (2,3), (3,3)\}$

Ref. closure is $= R \cup \Delta$

$= \{(1,1), (2,2), (2,3), (3,3)\} \cup \{(1,1), (2,2), (3,3)\}$

$= \{(1,1), (2,2), (2,3), (3,3)\}$

Symmetric closure $= R \cup R^{-1}$

$= \{(1,1), (2,2), (2,3), (3,3)\} \cup \{(1,1), (2,2), (3,2), (3,3)\}$

$= \{(1,1), (2,2), (2,3), (3,2), (3,3)\}$

Scanned by CamScanner

V V V Imp

Transitive Closure ⇒ (Transitive extension of rel^n)

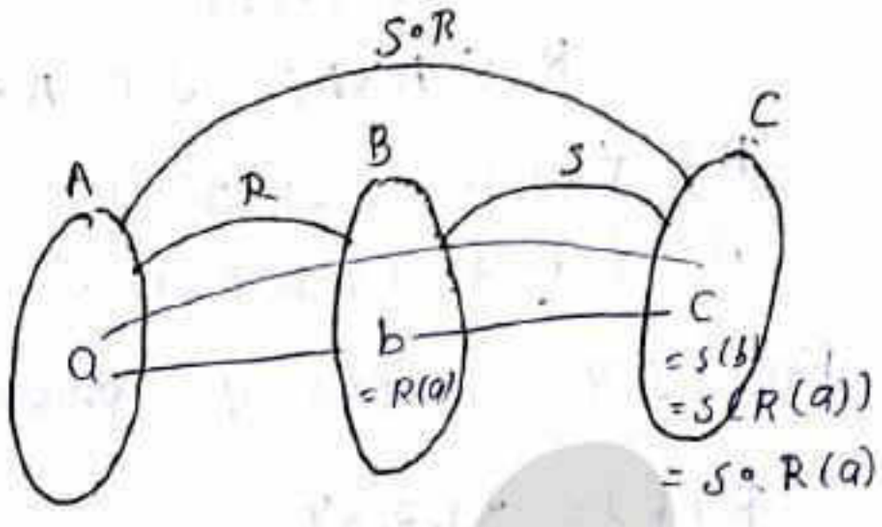
The transitive closure of R is R^*

where $R^* = R \cup R^2 \cup R^3 \cup R^4 \ldots$

$R^2 = R \circ R$, $R^3 = R^2 \circ R$, $R^4 = R^3 \circ R \ldots$

Composition ⇒

Let $R \subseteq A \times B$, $S \subseteq B \times C$ | $S \circ R \subseteq A \times C$



If $(a,b) \in R$ & $(b,c) \in S$

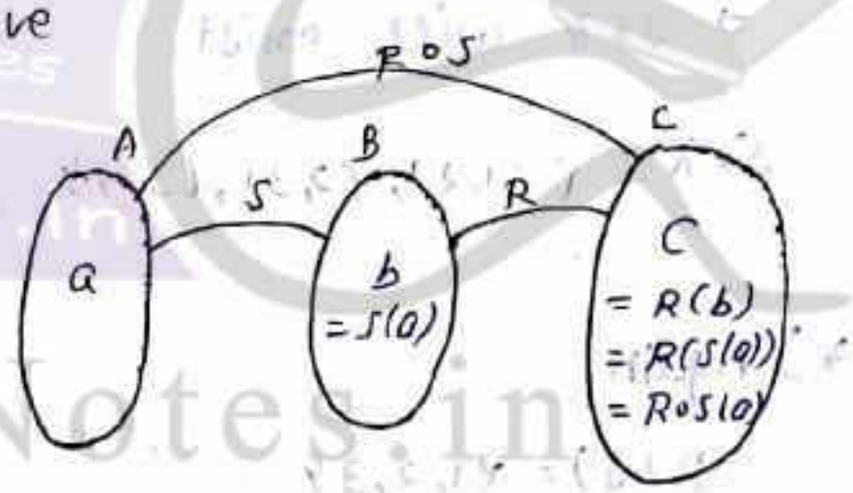
then $(a,c) \in S \circ R$

*) To define $S \circ R$, we must have

$R(R) = D(S)$

*) To define $R \circ S$ we must have

$R(S) = D(R)$



where

$S \subseteq A \times B$

$R \subseteq B \times C$

then $R \circ S \subseteq A \times C$

If $(a,b) \in S$ & $(b,c) \in R$

then $(a,c) \in R \circ S$

*) $\boxed{R \circ S \neq S \circ R}$

Eg ⇒ Let $A = \{1,2,3\}$, $B = \{a,b,c\}$, $C = \{x,y,z\}$

$R \subseteq A \times B$ & $S \subseteq B \times C$.

where $R = \{(1,a), (2,b), (3,c)\}$

$S = \{(a,x), (b,y), (c,z)\}$

Find $R \circ S$ & $S \circ R$ if possible.

$R(R) = \{a,b,c\}$

$D(S) = \{a,b,c\}$

$\because R(R) = D(S)$

$R(S) = \{x,y,z\}$

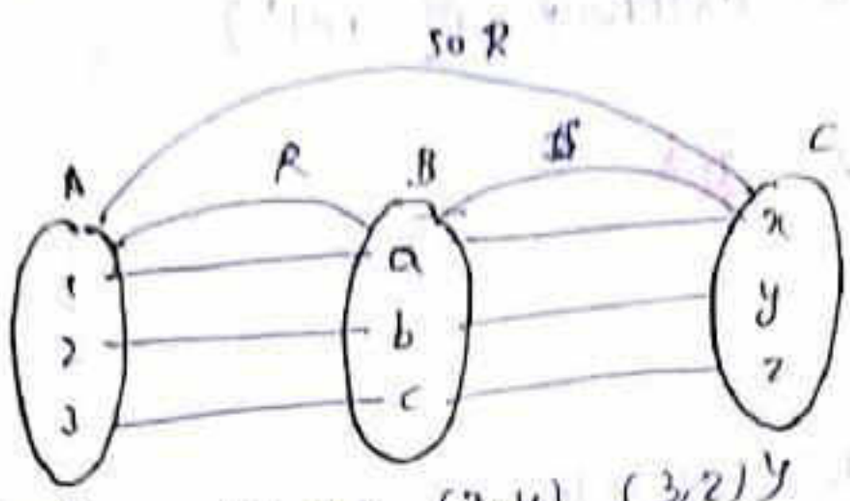
$D(R) = \cancel{\{a,b,c\}}\ \{(1,2,3)\}$

$\because R(S) \neq D(R)$

$\Rightarrow$ $S \circ R$ does not exist

Scanned by CamScanner

So $S \circ R$ can exist



$S \circ R = \{(1,x), (2,y), (3,z)\}$

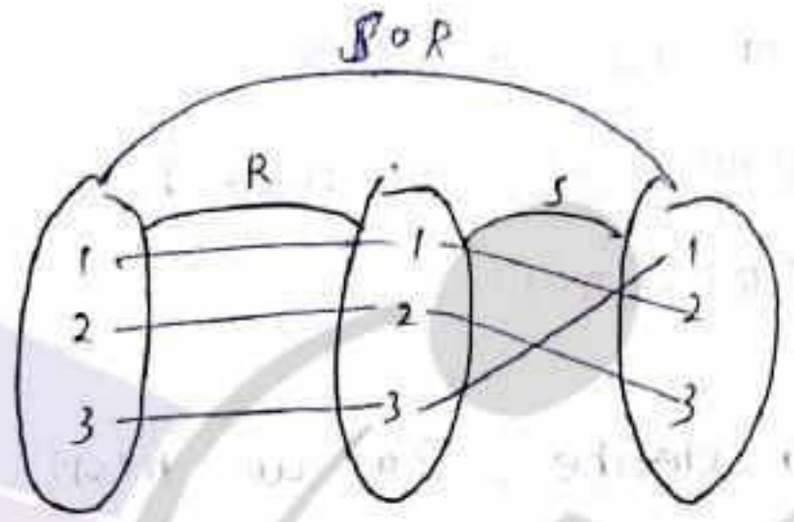
Eg ⇒ Let $A = \{1,2,3\}$

$R \subseteq A \times A$, $S \subseteq A \times A$

$R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,2), (2,3), (3,1)\}$

Find $S \circ R$ & $R \circ S$ if possible.



$R(R) = \{1,2,3\}$

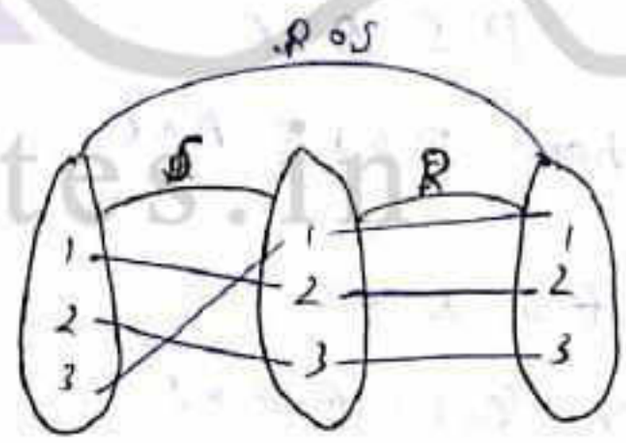
$D(S) = \{1,2,3\}$

So $R(R) = D(S)$

⇒ $S \circ R$ will exist

$S \circ R = \{(1,2), (2,3), (3,1)\}$

*) Again



$R(S) = \{1,2,3\}$

$D(R) = \{1,2,3\}$

$R(S) = D(R)$ ⇒ $R \circ S$ will exist

$R \circ S = \{(1,2), (2,3), (3,1)\}$

Here $R \circ S = S \circ R$

Eg ⇒ $A = \{1,2,3\}$

$R \subseteq A \times A$; $S \subseteq A \times A$

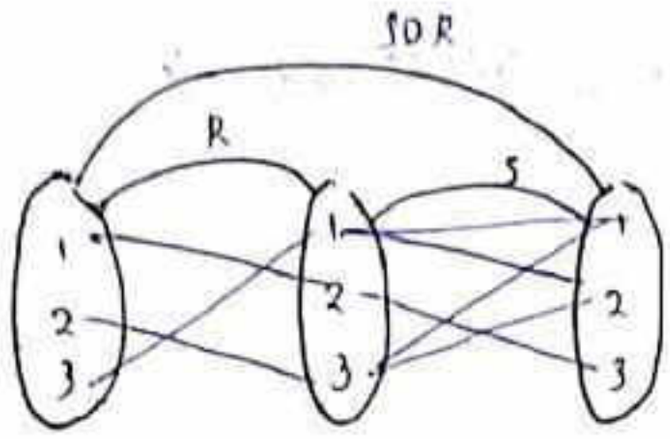
$R = \{(1,2), (2,3), (3,1)\}$

$S = \{(1,1), (1,2), (2,3), (3,1), (3,2)\}$

$D(R) = \{1,2,3\} = R(S)$

⇒ $S \circ R$ will exist

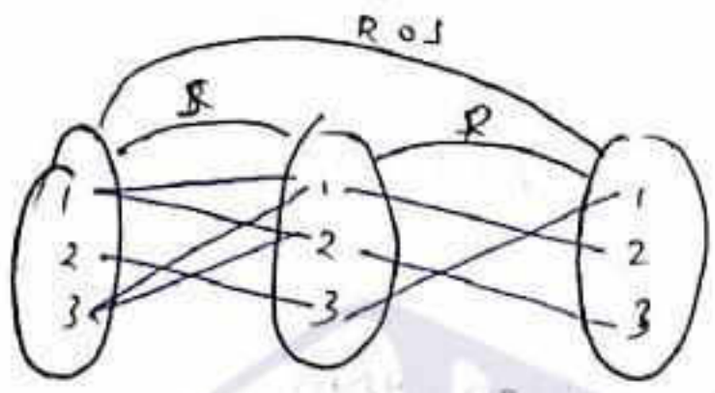
Scanned by CamScanner



$S \circ R = \{(1,3), (2,1), (2,2), (3,1), (3,2)\}$

*) $R(S) = D(R) = \{1,2,3\}$

$\Rightarrow$ $R \circ S$ exist



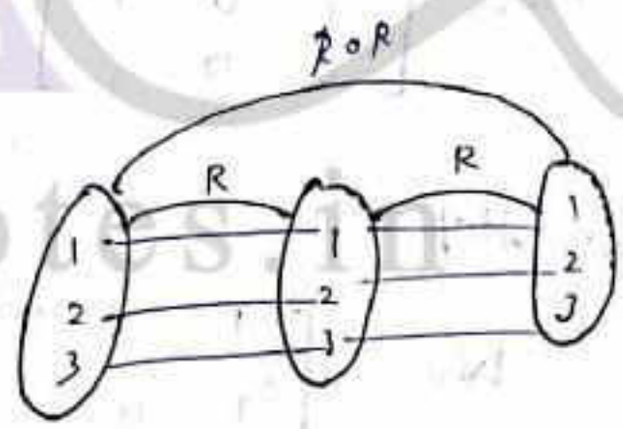
$R \circ S = \{(1,2), (1,3), (2,1), (3,2), (3,3)\}$

$\Rightarrow R \circ S \neq S \circ R.$

Eg $\Rightarrow$ Let $A = \{1,2,3\}$

(1) $R = \{(1,1), (2,2), (3,3)\}$

Find R^2, R^3, R^4 - —



$R^2 = R \circ R$

$= \{(1,1), (2,2), (3,3)\}$

Here $R = R^2, R^3, = R^4$ - —

So the transitive closure of R is

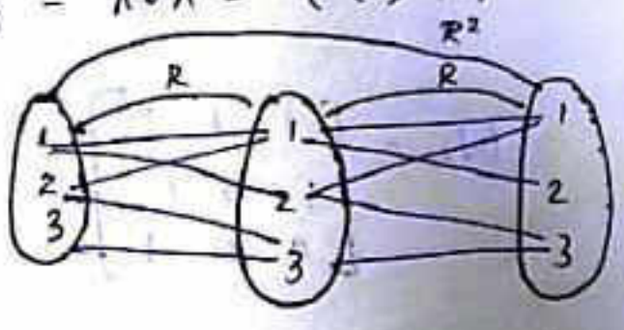
$R^* = R \cup R^2 \cup R^3$ - —

$= R$

$= \{(1,1), (2,2), (3,3)\}$

Eg $\Rightarrow$ $R = \{(1,1), (1,2), (2,1), (2,3), (3,3)\}$

$R^2 = R \circ R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,3)\}$

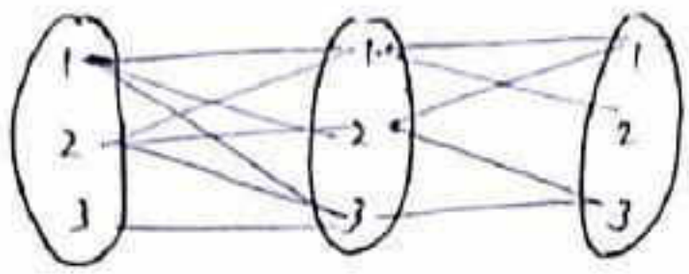


$\cancel{R^2 = \{(1,1), (1,2), (1,3), (2,1)}$

LectureNotes.in

Scanned by CamScanner

$R^3 = R^2 \circ R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,3)\}$



$R^2 = R^3 = R^4 = \cdots$

So transitive closure of R will be

$R^* = R \cup R^2 \cup R^3 \cup \cdots$

$= R \cup R^2$

$= \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,3)\}$

v. imp for 4 marks

Transitive Closure by Warshall's Algorithm.

Eg ⇒

$A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,1), (2,3), (3,3)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Initial step ⇒

$$W_0 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 1 ⇒

Col 1	Row 1	Transitive Pair
(1,1)	(1,1)	(1,1) ✓
(2,1)	(1,2)	(1,2) ✓
		(2,1) ✓
		$\boxed{(2,2)}$

$$W_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 2 ⇒

Col 2	Row 2	Transitive Pair
(1,2)	(2,1)	(1,1) ✓
(2,2)	(2,2)	(1,2) ✓
	(2,3)	$\boxed{(1,3)}$
		(2,1) ✓
		(2,2) ✓, (2,3) ✓

$$W_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

LectureNotes.in

Scanned by CamScanner

Step 3 →

Col 3	Row 3	Transitive Pair
(1, 3)	(3, 3)	(1, 3) ✓
(2, 3)		(2, 3) ✓
(3, 3)		(3, 3) ✓

No change

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,3)\}$

LectureNotes.in

LectureNotes.in

LectureNotes.in

Scanned by CamScanner

(3) $W_0 = M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Col 1	Row 1	Transitive Pair
(1,1)	(1,1)	(1,1) ✓
(4,1)	(1,4)	(1,4) ✓
		(4,1) ✓
		$\boxed{(4,4)}$

$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Col 2	Row 2	T.P
(2,2)	(2,2)	(2,2) ✓
(5,2)		(5,2) ✓

$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Col 3	Row 3
	(3,4)
	(3,5)

$W_3 = W_2 = W_1$

$R^* = \{(1,1), (1,4), (2,2), (3,4), (3,5), (4,1), (4,4), (5,2), (5,5)\}$

(9) $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(2,1)	(1,4)	(1,4) ✓
		(2,1) ✓
		$\boxed{(2,4)}$

$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Col 2	Row 2	T.P
(2,2)	(2,1)	(2,1) ✓
	(2,2)	(2,2) ✓
	(2,4)	(2,4) ✓

$W_2 = W_1$

LectureNotes.in

Scanned by CamScanner

$R^* = \{(1,1),(1,4),(2,1),(2,2),(2,4),(3,3),(4,4)\}$

(10) $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(2,1)	(1,2)	(1,2) ✓
		(2,1) ✓
		$\boxed{(2,2)}$

$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Col 2	Row 2	T.P
(1,2)	(2,1)	(1,1) ✓
(2,2)	(2,2)	(1,2) ✓
		(2,1) ✓
		(2,2) ✓

$W_2 = W_1$

$R^* = \{(1,1),(1,2),(2,1),(2,2),(4,3)\}$

(11) $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(4,1)	(1,4)	(1,4) ✓
		(4,1) ✓
		$\boxed{(4,4)}$ ✓

$W_1 = W_0$

$R^* = \{(1,1),(1,4),(2,2),(2,3),(3,2),(3,3),(4,1),(4,4)\}$

(12) $M_R = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(2,1)	(1,4)	(2,4) ✓

$W_1 = W_0$

$R^* = \{(1,4),(2,1),(2,4),(3,2),(3,4),(4,3)\}$

LectureNotes.in

Scanned by CamScanner

Eg 2 ⇒ $M_R = W_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Col 1	Row 1	T.P
(2,1)	(1,2)	(2,2)

$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Col 2	Row 2	T.P
~~(1,1)~~	(2,1)	$\boxed{(1,1)}$
~~(2,2)~~	(2,2)	(1,2) ✓
(1,2)	(2,3)	$\boxed{(1,3)}$
(2,2)		(2,1) ✓
		(2,2) ✓
		(2,3) ✓

$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Col 3	Row 3	T.P
(1,3)	(3,4)	$\boxed{(1,4)}$
(2,3)		$\boxed{(2,4)}$

$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Col 4	Row 4	T.P
(1,4)		
(2,4)		
(3,4)		

$W_3 = W_2 = W_1$

⇒ $R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,4)\}$

Eg 3 ⇒ $M_R \cup M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(2,1)	(1,2)	(1,2) ✓
		(2,1) ✓
		(2,2) ✓

$W_1 = W_0$

LectureNotes.in

Scanned by CamScanner

Col 2	Row 2	T.P
(1,2)	(2,1)	(1,1) ✓
(2,2)	(2,2)	(1,2) ✓
		(2,1) ✓
		(2,2) ✓

$W_2 = W_1 = W_0$

Col 3	Row 3	T.P
(3,3)	(3,3)	(3,3) ✓
(4,3)	(3,4)	(3,4) ✓
		(4,3) ✓
		(4,4) ✓

$W_3 = W_2 = W_1 = W_0$

Col 4	Row 4	T.P
(3,4)	(3,3)	(3,3) ✓
(4,4)	(3,4)	(3,4) ✓
(5,4)	(3,5)	(3,5) ✓
		(4,3) ✓
		(4,4) ✓
		$\boxed{(4,5)}$
		$\boxed{(5,3)}$
		(5,4) ✓
		(5,5) ✓

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

① ⓐ $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,3), (1,3), (3,1), (3,2)\}$

$$M_R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = W_0$$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(3,1)	(1,2)	(1,2) ✓
	(1,3)	(1,3) ✓
		(3,1) ✓
		(3,2) ✓
		$\boxed{(3,3)}$

$$W_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

LectureNotes.in

Scanned by CamScanner

Col 2	Row 2	T.P
(1,2)	(2,3)	(1,3) ✓
(3,2)		(3,3) ✓

$W_2 = W_1$

Col 3	Row 3	T.P
(1,3)	(3,1)	(1,1)
(2,3)	(3,2)	(1,2)
(3,3)	(3,3)	(1,3)
		(2,1)
		(2,2)
		(2,3)
		(3,1)
		(3,2)
		(3,3)

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

(2) $M_R \cup M_S = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
	(1,2)	(1,2) ✓
	(1,4)	(1,4) ✓

$W_1 = W_0$

Col 2	Row 2	T.P
(1,2)	(2,2)	(1,2) ✓
(2,2)		(2,2) ✓
(3,2)		(3,2) ✓
(4,2)		(4,2) ✓

$W_2 = W_1 \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Col 3	Row 3	T.P
(3,3)	~~(3,1)~~	~~(3,1)~~
(4,3)	(3,2)	(3,2) ✓
	(3,3)	(3,3) ✓
		~~(4,1)~~
		(4,2) ✓
		~~(4,4)~~

$W_3 = W_2$

LectureNotes.in

Scanned by CamScanner

Col 4	Row 4	T.P
(1,4)	~~(4,1)~~	~~(1,1)~~
(4,4)	(4,2)	(1,2) ✓
	(4,3)	$\boxed{(1,3)}$
	(4,4)	(1,4) ✓
		~~(4,1)~~
		(4,2) ✓
		(4,3) ✓
		(4,4) ✓

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$R^* = \{(1,1), (1,2), (1,3), (1,4), (2,2); (3,2), (3,3), (4,2), (4,3), (4,4)\}$

(15) $M_R \cup M_S = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = W_0$

Col 1	Row 1	T.P
(1,1)	(1,1)	(1,1) ✓
(2,1)	(1,2)	(1,2) ✓
(3,1)	(1,3)	(1,3) ✓
		(2,1) ✓
		(2,2) ✓
		(2,3) ✓
		(3,1) ✓
		(3,2) ✓
		(3,3) ✓

$W_1 = W_0$

Col 2	Row 2	T.P	
(1,2)	(2,1)	(1,1) ✓	$\boxed{(4,1)}$
(2,2)	(2,2)	(1,2) ✓	(4,2) ✓
(3,2)	(2,3)	(1,3) ✓	(4,3) ✓
(4,2)	(2,4)	$\boxed{(1,4)}$	(4,4) ✓
		(2,1) ✓	
		(2,2) ✓	
		(2,3) ✓	
		$\boxed{(2,4)}$ ✓	
		(3,1) ✓	
		(3,2) ✓	
		(3,3) ✓	
		(3,4) ✓	

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

LectureNotes.in

Scanned by CamScanner

Col 3	Row 3	T.P
(1,3)	(3,1)	(1,1)
(2,3)	(3,2)	(1,2)
(3,3)	(3,3)	(1,3)
(4,3)	(3,4)	(1,4)
		⋮

$W_3 = W_2$

Col 4	Row 4	T.P
(1,4)	(4,1)	(1,5)
(2,4)	(4,2)	(2,5)
(3,4)	(4,3)	(3,5)
(4,4)	(4,4)	(4,5)
(5,4)	(4,5)	(5,1)
		(5,2)
		(5,3)
		(5,4) ✓

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Col 5	Row 5	T.P
(4,5)	(5,1)	(4,1)
(5,5)	(5,2)	(4,2)
	(5,3)	⋮
	(5,4)	(4,5)
	(5,5)	(5,1)
		(5,2)
		⋮
		(5,5)

$W_5 = W_4$

$R^* = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

Equivalence Class ⇒

Let R be an equivalence rel^n defined over set A

$\forall a \in A$, $[a]$ denotes the equivalence class of 'a'

where $[a] = \{ b \mid \forall (a,b) \in R \}$

Since R is an equivalence rel^n

So R is reflexive

i.e. $(a,a) \in R$

i.e. $a \in [a]$

Scanned by CamScanner

* Any two equivalence class either identical or disjoint.
* Union of all distinct equivalent classes is the set where all

11/12/16

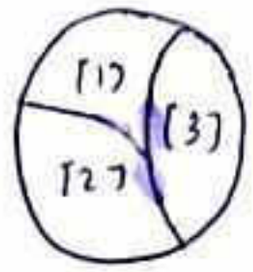
Eg ⇒ $A = \{1, 2, 3\}$

* $R = \{(1,1), (2,2), (3,3)\}$

$[1] = \{1\}$

$[2] = \{2\}$

$[3] = \{3\}$



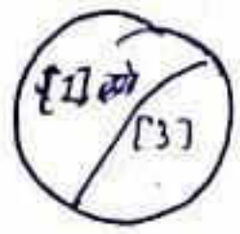
* $R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}$

$[1] = \{1, 2\}$

$[2] = \{1, 2\}$

$[3] = \{3\}$

$[1] = [2]$



* $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$[1] = \{1, 2, 3\}$

$[2] = \{1, 2, 3\}$

$[3] = \{1, 2, 3\}$

Here

$[1] = [2] = [3]$

Eg ⇒ $R = \{(x, y) \mid x \equiv y \pmod 2, \forall x, y \in \mathbb{Z}\}$ be an equivalence rel^n

i.e. $(x, y) \in R \Leftrightarrow x \equiv y \pmod 2$

$2 \mid y - x \Leftrightarrow y - x = 2k$

$y = x + 2k, \; k \in \mathbb{Z}$

$\mathbb{Z} = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$[x] = \{y \mid (x, y) \in R\}$

$[0] = \{y \mid y = 2k, k \in \mathbb{Z}\} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

$[1] = \{y \mid y = 2k+1, k \in \mathbb{Z}\} = \{\ldots -5, -3, -1, 1, 3, 5, 7 \ldots\}$

$[2] = \{y \mid y = 2k+2, k \in \mathbb{Z}\} = \{\ldots -4, -2, 0, 2, 4, 6 \ldots\}$

$[3] = \{y \mid y = 2k+3, k \in \mathbb{Z}\} = \{\ldots -3, -1, 3, 5, 7, 9 \ldots\}$

$[-1] = \{y \mid y = 2k-1, k \in \mathbb{Z}\} = \{\ldots -5, -3, -1, 1, 3, 5, 7 \ldots\}$

$[-2] = \{y \mid y = 2k-2, k \in \mathbb{Z}\} = \{\ldots -6, -4, -2, 0, 2, 4, 6 \ldots\}$

Here $[0] = [2] = [4] = [6] = \ldots = [-2] = [-4] = [-6] = \ldots [2m]$

$[1] = [3] = [5] = \ldots = [-1] = [-3] = [-5] = \ldots = [2m+1]$

i.e. $[0] \cup [1] = \mathbb{Z}$

Scanned by CamScanner

Eg ⇒ $R = \{(x,y) \mid x \equiv y \pmod 3, \forall x,y \in \mathbb{Z}\}$ be an equivalence rel^n

i.e. $(x,y) \in R \Leftrightarrow x \equiv y \pmod 3$

$3 \mid y - x \Leftrightarrow y - x = 3k$

$y = x + 3k, \; k \in \mathbb{Z}$

$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$

$[x] = \{y \mid (x,y) \in R\}$

$[0] = \{y \mid y = 3k, k \in \mathbb{Z}\} = \{\ldots -6, -3, 0, 3, 6, 9 \ldots\}$

$[1] = \{y \mid y = 3k+1, k \in \mathbb{Z}\} = \{\ldots -5, -2, 1, 4, 7 \ldots\}$

$[2] = \{y \mid y = 3k+2, k \in \mathbb{Z}\} = \{\ldots -4, -1, 2, 5, 8 \ldots\}$

$[3] = \{y \mid y = 3(k+1), k \in \mathbb{Z}\} = \{\ldots -3, 0, 3, 6, 9 \ldots\}$

$[-1] = \{y \mid y = 3k-1, k \in \mathbb{Z}\} = \{\ldots -7, -4, -1, 2, 5 \ldots\}$

$[-2] = \{y \mid y = 3k-2, k \in \mathbb{Z}\} = \{\ldots -8, -5, -2, 1, 4 \ldots\}$

$[-3] = \{y \mid y = 3k-3, k \in \mathbb{Z}\} = \{\ldots -3, 0, 3, 6, 9, \ldots\}$

$[4] = \{y \mid y = 3k+4, k \in \mathbb{Z}\} = \{\ldots -5, -2, 1, 4, 7 \ldots\}$

$[5] = \{y \mid y = 3k+5, k \in \mathbb{Z}\} = \{\ldots -4, -1, 2, 5, 8 \ldots\}$

Here $[0] = [3] = [-3] = [6] = [-6] = \ldots = [3n]$

$[1] = [4] = [7] = [10] = \ldots = [-2] = [-5] = \ldots = [3n+1]$

$[2] = [5] = [8] = \ldots = [3n+2]$

Here we have 3 distinct equivalence classes such as $[0], [1], [2]$ or $[3n], [3n+1], [3n+2]$

NOTES

*) For the rel^n $x \equiv y \pmod n$ we have n - no. of distinct equivalent classes such as $[0], [1], [2] \ldots [n-1]$.

For eg ⇒ $n = 2$
Equivalence class = $[0]$ & $[1]$

For eg = $n = 3$
Equivalence class = $[0], [1], [2]$

For eg ⇒ $n = 4$
Equivalence class = $[0]$ $[1], [2], [3]$

Scanned by CamScanner

Definition I ⇒ Let S be any given set

$$A = \{A_1, A_2, \ldots A_m\}$$

where each A_i = for $i = 1, 2, \ldots m$. is a subset of S.

& $\bigcup_{i=1}^{m} A_i = S$

In such case the set A is said to be <u>Covering of</u> S.

& the sets $A_1, A_2 \ldots A_m$ are said to be cover of S.

Eg ⇒ $R = \{(x, y) \mid x \equiv y \pmod 2, \forall x, y \in \mathbb{Z}\}$

here $S = \mathbb{Z}$

$A = \{[0], [1]\}$

We can say A is covering of $S(\mathbb{Z})$

$[0]$ & $[1]$ is covers of $\mathbb{Z}$

Definition II ⇒

Partition ⇒ Let A be any non empty set & $A_1, A_2, A_3 \ldots A_m$ are subjets of A.

The set denoted by π is said to be i.e. $\pi = \{A_1, A_2 \ldots A_k\}$ a partition of the set A if

for $i = j$, $A_i \cap A_j = \phi$

&

$\bigcup_{i=1}^{m} A_i = A$

Eg = Let $A = \{a, b, c, d, e, f, g, h, i\}$

Let $A_1 = \{a, b, c, d, e\}$

$A_2 = \{f, g, h\}$

$A_3 = \cancel{\{a, b, c, d\}}$ $\{i, j\}$

$A_4 =$ ~~let~~ $\{a, b, c, d\}$

$A_5 = \{e\}$

e

* > Is $A \cdot \pi = \{A_1, A_2, A_3\}$ be a partition of A?

Verification

$A_1 \cap A_2 = \phi$

$A_1 \cap A_3 = \phi$

$A_2 \cap A_3 = \phi$

$A_1 \cup A_2 \cup A_3 = \{a, b, c, d, e\} \cup \{f, g, h\} \cup \{i, j\}$

$= \{a, b, c, d, e, f, g, h, i, j\}$

$= A$

Then π be a partition set of A

ii) is $\pi = \{A_1, A_4, A_5\}$ be a partion?

then $A_1 \cap A_4 = \{a, b, c, d, e\} \cap \{a, b, e, d\}$

$= \{a, b, e, d\}$

$\neq \phi$

So π is not a partition of set A.

iii) $\pi = \{A_3, A_4\}$ Is it a part?-

$A_3 \cap A_4 = \phi$

$A_3 \cup A_4 = \{a, b, e, d, i\}$

$\neq A$

So π is not a partition of A

Eg ⇒ List all partitions of $A = \{1, 2, 3\}$

where $A_1 = \{1\}$, $A_4 = \{1, 2\}$

$A_2 = \{2\}$, $A_5 = \{2, 3\}$

$A_3 = \{3\}$, $A_6 = \{1, 3\}$

The partitions of A are

① $\pi_1 = \{A_1, A_2, A_3\}$

$A_1 \cap A_3 = \phi$

as $A_1 \cap A_2 = \phi$

$A_2 \cap A_3 = \phi$

& $A_1 \cup A_2 \cup A_3 = A$

② $\pi_2 = \{A_2, A_6\}$

as $A_2 \cap A_6 \neq \phi$

$A_2 \cup A_6 = A$

③ $\pi_3 = \{A_1, A_5\}$

④ $\pi_4 = \{A_3, A_4\}$

Scanned by CamScanner

20/8/16

NOTES

*> If $\pi = \{A_1, A_2, A_3, \ldots A_n\}$ be a partition of the set A then A_1, A_2, A_3 —— A_n are called as Block

*> Each element of a block corresponds with other elements of that block.

i.e. If $A_1 = \{a, b, c\}$

then $(a,a) \in R$, $(a,b) \in R$, $(a,c) \in R$, $(b,a) \in R$, $(b,b) \in R$, $(b,c) \in R$, $(c,a) \in R$, $(c,b) \in R$, $(c,c) \in R$.

Eg ⇒ Let $A = \{1,2,3,4\}$

$P = \{\{1,2,3\}, \{4\}\}$ be a partition of A. Find the equivalence rel^n determined by P.

So $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\}$

$[1] = \{1,2,3\}$

$[2] = \{1,2,3\}$

$[3] = \{1,2,3\}$

$[4] = \{4\}$

Matrix operations ⇒

We know

$1 \cup 1 = 1$ $\quad$ $1 \cap 1 = 1$

$1 \cup 0 = 1$ $\quad$ $1 \cap 0 = 0$

$0 \cup 1 = 1$ $\quad$ $0 \cap 1 = 0$

$0 \cup 0 = 0$ $\quad$ $0 \cap 0 = 0$

*> If M_R & M_S be the adjacency matrix of two binary relations R & S.

then $M_{R \cup S} = M_R \cup M_S$

$M_{R \cap S} = M_R \cap M_S$

Eg ⇒ Let $A = \{a, b, c\}$

$R \subseteq A \times A$ & $S \subseteq A \times A$

where $R = \{(a,a), (a,c), (b,a), (c,b)\}$

$S = \{(a,a), (a,c), (b,b), (b,c), (c,a)\}$

Scanned by CamScanner

Find the adjacency matrix of $R \cup S$ & $R \cap S$.

$\Rightarrow R \cup S = \{(a,a), (a,c), (b,a), (b,b), (b,c), (c,b), (c,a), (c,b)\}$

$$M_{R \cup S} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$R \cap S = \{(a,a), (a,c)\}$

$$M_{R \cap S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Alt way

since $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ & $M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$M_R \cup M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cup \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cup 1 & 0 \cup 0 & 1 \cup 1 \\ 1 \cup 0 & 0 \cup 1 & 0 \cup 1 \\ 0 \cup 1 & 1 \cup 0 & 0 \cup 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_R \cap M_S = \begin{bmatrix} 1 \cap 1 & 0 \cap 0 & 1 \cap 1 \\ 1 \cap 0 & 0 \cap 1 & 0 \cap 1 \\ 0 \cap 1 & 1 \cap 0 & 0 \cap 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

*) If M_R be the adjacency matrix of binary rel^n R then M_R^{-1} be the adjacency matrix of binary rel^n R^{-1}.

where $\boxed{M_R^{-1} = [M_R]^T}$

Eg $\Rightarrow$ If $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$M_R^{-1} = M_R^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Scanned by CamScanner

$\wedge$ = Product
$\vee$ = Addition

*) If $M_R (= M_{ij})$ be the adjacency matrix of R then $M_{\bar{R}}$ adjacent matrix of complement of R.

where,

$$M_{\bar{R}} = (\bar{M}_{ij})$$

$$\bar{M}_{ij} = \begin{cases} 1, & M_{ij} = 0 \\ 0, & M_{ij} = 1 \end{cases}$$

Eg ⇒ $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$M_{\bar{R}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Boolean Product ⇒

*) If A & B are two boolean matrices then their product will be

$$A \odot B \text{ or } B \odot A$$

Eg ⇒ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$$A \odot B = \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

$$B \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 \vee 0 & 0 \vee 1 \vee 0 \\ 0 \vee 0 \vee 1 & 0 \vee 1 \vee 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \odot B \neq B \odot A$$

Scanned by CamScanner

*) Let M_R & M_S be the adjacency matrices of two binary relations A & B

$$M_{S \circ R} = M_R \odot M_S$$

*) $M_{R^2} = M_R \odot M_R$

$$M_{R^3} = M_{R^2} \odot M_R$$

*) The transitive closure of R is

$$R^* = M_R \cup M_{R^2} \vee M_{R^3} \vee M_{R^4} \text{ - - - -}$$

Eg ⇒ $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $M_S = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Find $M_{S \circ R}$ & $M_{R \circ S}$

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 0 \vee 0 & 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 \\ 0 \vee 1 \vee 0 & 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 \end{bmatrix}$$

$$M_{S \circ R} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_{R \circ S} = M_S \odot M_R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \vee 0 \vee 1 & 0 \vee 0 \vee 0 & 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 0 & 0 \vee 0 \vee 0 & 1 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 & 0 \vee 1 \vee 0 & 0 \vee 1 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Eg ⇒ Find the transitive closure of R, whose adjacency matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_R^* = M_R \vee M_{R^2} \vee M_{R^3} \vee M_{R^4} \text{ ———}$$

Scanned by CamScanner

25/8/16

$$M_{R^2} = M_R \overset{\text{composition}}{\odot} M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 1 \vee 0 \vee 0 \vee 1 & 0 \vee 0 \vee 1 & 1 \vee 0 \vee 0 \vee 0 \\ 0 \vee 0 \vee 0 \vee 0 & 0 \vee 1 \vee 1 \vee 0 & 0 \vee 0 \vee 0 \vee 0 \\ 1 \vee 0 \vee 0 \vee 0 & 0 \vee 1 \vee 1 \vee 0 & 1 \vee 0 \vee 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$M_{R^3} = M_{R^2} \odot \cancel{M_R} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$M_R \odot M_{R^2}$

$$M_{R^2} = M_{R^3} = M_{R^4} = \cdots$$

$$\text{So } M_{R^*} = M_R \vee M_{R^2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Logical Questions ⇒

Q ① Let R be a symmetric & transitive rel^n on set A. Show that if for every $a \in A$, $\exists$ (there exist) $b \in A$ s.t. $(a,b) \in R$ then R is an equivalence rel^n.

Given R be a symmetric & transitive rel^n on set A.

$\forall a \in A, \exists b \in A$ such that $(a,b) \in R$

Claim → R is an equivalence rel^n.

i.e. only we have to verify R is reflexive.

since by assumption

$\forall a \in A, \exists b \in A$ s.t $(a,b) \in R$

$\Rightarrow (b,a) \in R$ [Given R is symmetric]

Now $(a,b) \in R$ & $(b,a) \in R$

As R is transitive. So $(a,a) \in R$

Hence the proof.

Q ② Let R be a transitive & reflexive rel^n on set A. T be rel^n on A s.t. $(a,b) \in T$ iff $(a,b) \in R$ & $(b,a) \in R$ show that T is an equivalence relation.

Given R be a transitive & reflexive rel^n on set A.

Scanned by CamScanner

$(a,b) \in T$ iff $(a,b) \in R$ & $(b,a) \in R$

$\because (a,b) \in R$ $(b,a) \in R$

$\Rightarrow$ R is symmetric

(b,a) is also $\in T$ as here also if $(b,a) \in R$ then $(a,b) \in R$ (acc. to assumption)

$\Rightarrow$ T is symmetric.

OR!

Given - $R \subseteq A \times A$

R is transitive & reflexive

$T \subseteq A \times A$.

$(a,b) \in T$ iff $(a,b) \in R$ & $(b,a) \in R$

Claim - T is an equivalence rel^n

i.e. T is reflexive, symmetric & transitive

*) Let $(a,b) \in T$

$\Leftrightarrow (a,b) \in R$ & $(b,a) \in R$

$\Leftrightarrow (b,a) \in R \wedge (a,b) \in R$ $[P \wedge Q \equiv Q \wedge P]$

$\Leftrightarrow (b,a) \in T$

$\Rightarrow$ T is symmetric

*) Let $(a,b) \in T$ & $(b,c) \in T$

$\Leftrightarrow (a,b) \in R \wedge (b,a) \in R$, $(b,c) \in R \wedge (c,b) \in R$

As $(a,b) \in R$ & $(b,c) \in R$ $\Rightarrow (a,c) \in R$ [Given R is transitive]

$(c,b) \in R$ & $(b,a) \in R$; $(c,a) \in R$ [" " " "]

Now $(a,c) \in R$ & $(c,a) \in R$

$\Leftrightarrow (a,c) \in T$

So T is transitive

*) Since R is reflexive

So $\forall a \in A$, $(a,a) \in R$

As $P \wedge P \equiv P$

Thus $(a,a) \in R \wedge (a,a) \in R$

$\Leftrightarrow (a,a) \in T$

$\Rightarrow$ T is reflexive

$\Rightarrow$ T is an equivalence rel^n

Scanned by CamScanner

③ Let R be a binary relⁿ on set A
$S = \{(a,b) \mid (a,c) \in R \ \& \ (c,b) \in R$, for some $c\}$
Show that if R is an equivalence relⁿ then S is also an equivalence relⁿ.

$R \subseteq A \times A$; $S \subseteq A \times A$

$(a,b) \in S \Leftrightarrow (a,c) \in R \ \& \ (c,b) \in R$ for some 'c'

Assume R is an equivalence relⁿ

Claim ⇒ S is also an equivalence relⁿ

*) Let $(a,b) \in S$

$\Leftrightarrow (a,c) \in R \ \& \ (c,b) \in R$

$\Leftrightarrow (c,a) \in R \wedge (b,c) \in R$ [R is symmetric]

$\Leftrightarrow (b,c) \in R \wedge (c,a) \in R$ [$P \wedge Q = Q \wedge P$]

$\Leftrightarrow (b,a) \in S$

S is symmetric

*) Let $(a,b) \in S$ & $(b,d) \in S$

$\Leftrightarrow (a,c) \in R \wedge (c,b) \in R$; $(b,c) \in R \wedge (c,d) \in R$

$\Leftrightarrow (a,c) \in R \wedge (c,d) \in R$

$\Leftrightarrow$ ~~$(a,d) \in R$ [∵ R is transitive]~~

$\Leftrightarrow (a,d) \in S$

S is transitive

*) Since R is reflexive

$\forall a \in A$, $(a,a) \in R$

$\Rightarrow (a,a) \in R \wedge (a,a) \in R$ [$P \wedge P \equiv P$]

$\Rightarrow (a,a) \in S$

$\Rightarrow$ S is reflexive

$\therefore$ S is an equivalence relⁿ

(4)

(4) Let R be a reflexive relⁿ on set A.
Show that R is an equivalence (1) if & only if $(a,b) \in R$ & $(a,c) \in R \Rightarrow (b,c)$ (2)

we have to proof both ie. if R is equivalence then proof (2)
& also if 2 is given then proof (1).

Scanned by CamScanner

Given that R be a reflexive rel^n on set A

*) Suppose R be an equivalence rel^n.

Claim $\Rightarrow$ $(a,b) \in R$ & $(a,c) \in R \Rightarrow (b,c) \in R$

Now as $(a,b) \in R$ & $(a,c) \in R$

as R is symmetric so $(a,b) \in R \Rightarrow (b,a) \in R$

As $(b,a) \in R$ & $(a,c) \in R$

$\Rightarrow$ $\underline{(b,c) \in R}$ since R is transitive.

Hence proved

<u>Converse</u>

Let $(a,b) \in R$ & $(a,c) \in R \Rightarrow (b,c) \in R$

Claim $\Rightarrow$ R be an equivalence rel^n

i.e. only remains to show R is symmetric & transitive.

*) Let $(a,b) \in R$

As R is reflexive, so $(a,a) \in R$

Now $(a,b) \in R$ & $(a,a) \in R$

$\Rightarrow (b,a) \in R$

R is symmetric.

*) Let $(a,b) \in R$ & $(b,c) \in R$

$\because$ R is symmetric as $(a,b) \in R \Rightarrow (b,a) \in R$

$\Rightarrow (b,a) \in R$ & $(b,c) \in R$

$\Rightarrow (a,c) \in R$ (By assumption)

$\Rightarrow$ R is transitive

Hence R is an equivalence rel^n.

Scanned by CamScanner

NOTES :

Let $|A| = n$

So $|A \times A| = n^2$

$|P(A \times A)| = 2^{n^2}$

Total no. of possible relations $= 2^{n^2}$

Total no. of reflexive rel^n. $= 2^{n(n-1)}$

Total no. of irreflexive rel^{n}s $= 2^{n(n-1)}$

Total no. of symmetric rel^{n}s $= 2^{\frac{n(n+1)}{2}}$

Total no. of asymmetric rel^{n}s $= 3^{\frac{n(n-1)}{2}}$

Total no. of antisymmetric rel^{n}s $= 2^n \cdot 3^{\frac{n(n-1)}{2}}$

Total no. of reflexive & symmetric rel^{n}s $= 2^{\frac{n(n-1)}{2}}$

Total no. of neither reflexive nor irreflexive $= 2^{n^2} - 2 \cdot 2^{n(n-1)}$

LectureNotes.in

Scanned by CamScanner

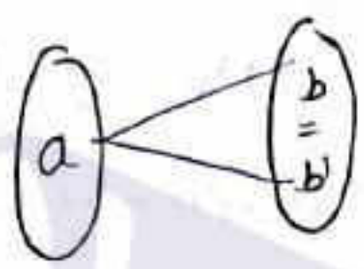
8/9/11

FUNCTION

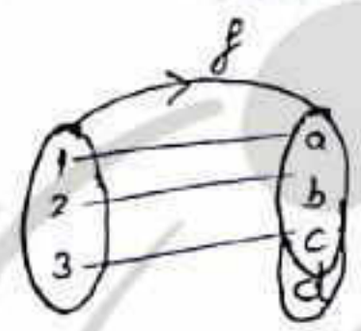
*) Let A & B are two non-empty sets.
f is said to be a function from A to B
iff

① $f \subseteq A \times B$
② $D(f) = A$
③ $R(f) \subseteq B$
④ If $(a,b) \in f$ & $(a,b') \in f$ then $b = b'$

i.e. f is not to be one many



Eg ⇒ $A = \{1,2,3\}$, $B = \{a,b,c,d\}$



① $f = \{(1,a), (2,b), (3,c)\}$

Here $f \subset A \times B$

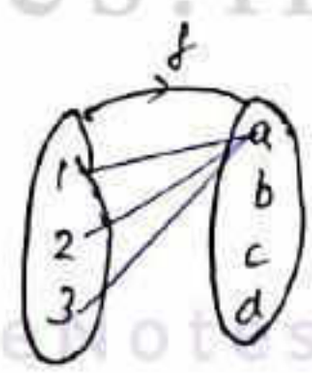
$D(f) = \{1,2,3\} = A$

$R(f) = \{a,b,c\} \subset B$

f is not one many

So f be a function from A to B.

② $f = \{(1,a), (2,a), (3,a)\}$



Here $f \subset A \times B$

$D(f) = \{1,2,3\} = A$

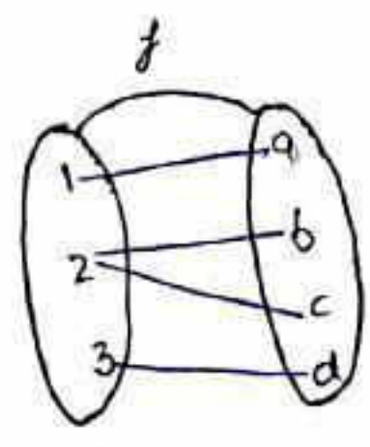
$R(f) = \{a\} \subset B$

f is not one many.

Their is no pair of the form (a,b) & (a,b') in f

So f be a function from A to B

③ $f = \{(1,a), (2,b), (2,c), (3,d)\}$



Here $f \subset A \times B$

$D(f) = \{1,2,3\} = A$

$R(f) = \{a,b,c,d\} = B$

But here we have

Scanned by CamScanner

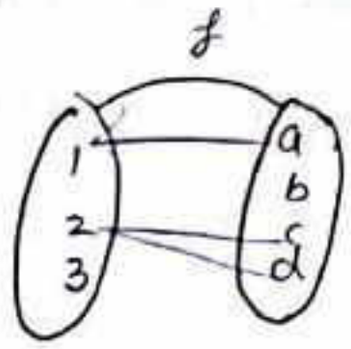
$(2,b) \in f$ & $(2,c) \in f$

such that $b \neq c$

So f is one many.

f is not a func^n from A to B

(4) $f = \{(1,a),(2,b),(2,c)\}$



Here $f \subset A \times B$

$D(f) = \{1,2\} \neq A$

So f is not a func^n from A to B.

NOTE ⇒

(1) Every functions are relations but every relations are not functions

(2) f is a function from A to B

i.e. $f : A \to B$ be a function where A be domain set of f

B be codomain set of f

$f(A)$ be the range/Image set of f.

Always

$f(A) \subseteq B$

i.e. $\boxed{\text{Image /Range of } f. \subseteq \text{codomain of } f}$

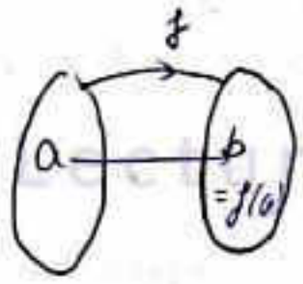
*) If $A = \{a_1, a_2, a_3 - ___\}$

then $f(A) = \{f(a_1), f(a_2), f(a_3), - ___\}$

*) $f : A \to B$ be a func^n

$(a,b) \in f$

i.e. $b = f(a)$



where b is the Image of a

& a is the preimage of b

Eg ⇒ $A = \{1,2,3\}$; $B = \{a,b,c,d\}$

$f = \{(1,a),(2,b),(3,c)\} \Rightarrow \left| \begin{array}{l} f(1) = a \\ f(2) = b \\ f(3) = c \end{array} \right.$

Here $f(A) = \{f(1), f(2), f(b)\}$

$= \{a,b,c\}$

Scanned by CamScanner

Set of natural no. & real no. is infinite but natural no. is countable but real no. are not.

* Let $f : A \to B$ be a func^n

⇒ It is said to be ONE-ONE iff

$$f(a_1) = f(a_2) \Leftrightarrow a_1 = a_2$$

⇒ It is said to be ONTO iff

$$R(f) = \text{Codomain}(f).$$

⇒ If $R(f) \subset$ Codomain f.

then f is INTO

⇒ The func^n which is both one-one & onto is said to be Bijective.

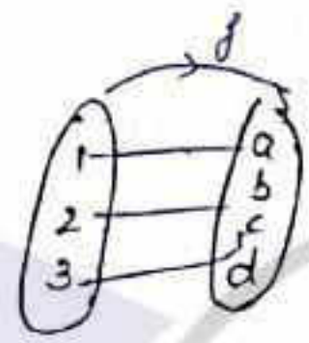
⇒ If $f : A \to B$ is BIJECTIVE

then $|A| = |B|$

Eg ⇒ ① $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$

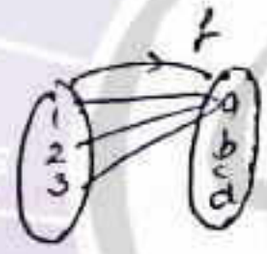
→ $f = \{(1,a), (2,b), (3,c)\}$

f is one-one & Into



→ $f = \{(1,a), (2,a), (3,a)\}$

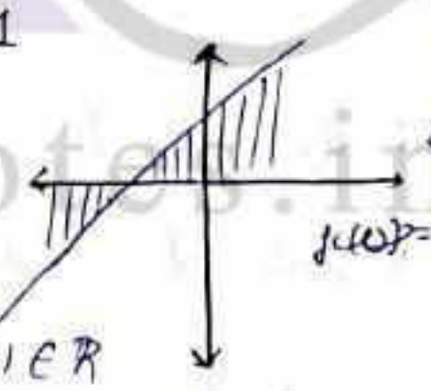
f is many-one & into



② $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+1$

$D(f) = x$-axis

$Cod(f) = y$-axis



$f(0) = 1$
$f(1) = 3$
$f(2) = 5$
$f(-1) = -1$

* Here f is one-one

Let $f(x_1) = f(x_2)$

$\Rightarrow 2x_1 + 1 = 2x_2 + 1$

$\Rightarrow x_1 = x_2$

$x \in R \Rightarrow f(x) = 2x+1 \in R$

$x - 1 \in R \Rightarrow f(x-1) = 2x-1 \in R$

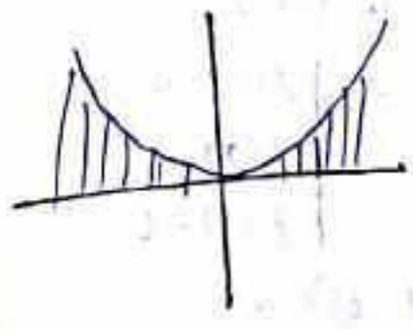
$x + 1 \in R \Rightarrow f(x+1) = 2x+3 \in R$

f is onto

→ f is bijective

③ $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2$$



* $f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 = \pm x_2$

f is not one-one

Scanned by CamScanner

Here f is many one

Again f is not onto

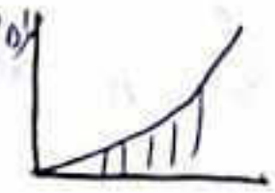
f is into

$\forall x \in \mathbb{R}$, $f(x) = x^2 \geq 0$

$R(f) = \mathbb{R}^+ \cup \{0\} \subset \mathbb{R}$

*) If we will take

$f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$



or

$f : [0, \infty) \to [0, \infty)$

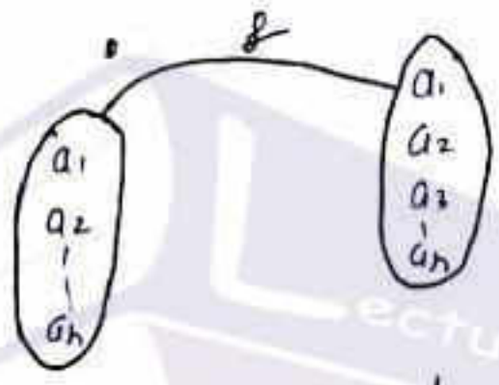
then f is said to be bijective

THEOREMS FROM BOOK

Bijective Function On single Set ⇒

*) Let $A = \{a_1, a_2, a_3 --- a_n\}$

$f : A \to A$ be a ~~finite~~ bijective func^n.



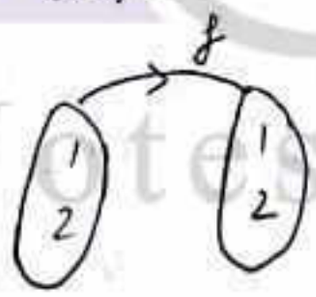
The total no. of such bijectives function will be $= n!$

$= n(n-1)(n-2) -- 1$

$= n!$

Eg ⇒ (1) List all the bijective func^n defined on $A = \{1, 2\}$

$f_1 = \{(1,1), (2,2)\}$

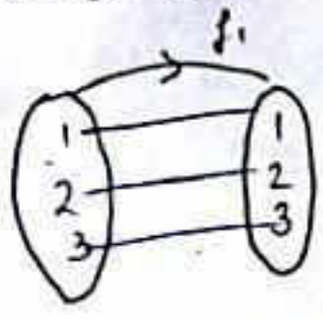


$f_2 = \{(1,2), (2,1)\}$

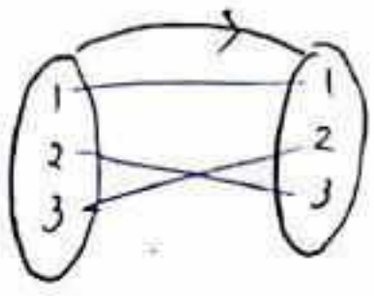
Eg ⇒ $A = \{1, 2, 3\}$

Total no. of bijective func^n will be $3! = 6$

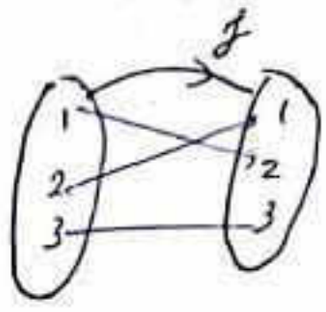
Those are



$f_1 = \{(1,1), (2,2), (3,3)\}$



$f_2 = \{(1,1), (2,3), (3,2)\}$



$f_3 = \{(1,2), (2,1), (3,3)\}$

$f_4 = \{(1,2), (2,3), (3,1)\}$

$f_5 = \{(1,3), (2,1), (3,2)\}$

$f_6 = \{(1,3), (2,2), (3,1)\}$

Scanned by CamScanner

(2) $A = \{1, 2, 3, 4\}$

Total no. of bijective function will be $4! = 24$

$f_1 = \{(1,1), (2,2), (3,3), (4,4)\}$

$f_2 = \{(1,1), (2,3), (3,2), (4,4)\}$

$f_3 = \{(1,1), (2,2), (3,4), (4,3)\}$

$f_4 = \{(1,1), (2,4), (3,3), (4,2)\}$

$f_5 = \{(1,1), (2,4), (3,2), (4,3)\}$

$f_6 = \{(1,1), (2,3), (3,4), (4,3)\}$

$f_7 = \{(1,2), (2,3), (3,4), (4,1)\}$

$f_8 = \{(1,2), (2,4), (3,2), (4,1)\}$

$f_9 = \{(1,2), (2,3), (3,1), (4,3)\}$

$f_{10} = \{(1,2), (2,4), (3,1), (4,3)\}$

$f_{11} = \{(1,2), (2,1), (3,4), (4,3)\}$

$f_{12} = \{(1,2), (2,1), (3,3), (4,4)\}$

$f_{13} = \{(1,3), (2,1), (3,2), (4,4)\}$

$f_{14} = \{(1,3), (2,1), (3,4), (4,2)\}$

$f_{15} = \{(1,3), (2,4), (3,2), (4,1)\}$

$f_{16} = \{(1,3), (2,2), (3,4), (4,1)\}$

$f_{17} = \{(1,3), (2,2), (3,1), (4,4)\}$

$f_{18} = \{(1,3), (2,4), (3,1), (4,2)\}$

$f_{19} = \{(1,4), (2,1), (3,2), (4,3)\}$

$f_{20} = \{(1,4), (2,3), (3,2), (4,1)\}$

$f_{21} = \{(1,4), (2,1), (3,3), (4,2)\}$

$f_{22} = \{(1,4), (2,3), (3,1), (4,2)\}$

$f_{23} = \{(1,4), (2,2), (3,3), (4,1)\}$

$f_{24} = \{(1,4), (2,2), (3,1), (4,3)\}$

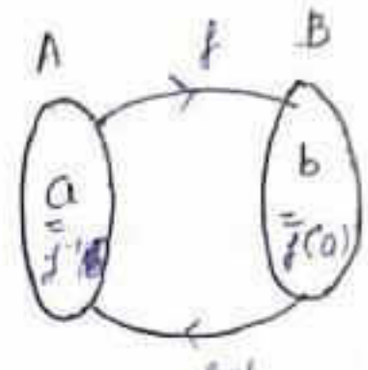
Scanned by CamScanner

9/9/16

☆ Invertible Function → If func^n $f: A \to B$ is bijective then, f is invertible.

i.e. f^{-1} will exist, where $f^{-1}: B \to A$

$b = f(a) \iff a = f^{-1}(b)$



Eg → ① $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $f = \{(1,a), (2,b), (3,c)\}$

Clearly f is bijective.

So, $f^{-1} = \{(a,1), (b,2), (c,3)\}$

② $f: R \to R$ defined by $f(x) = 2x+1$. Clearly (verified) & is bijective.

So f^{-1} will exist.

$y = f(x) \iff x = f^{-1}(y)$

i.e. $y = 2x+1 \Rightarrow x = \frac{y-1}{2}$

i.e. $f^{-1}(y) = \frac{y-1}{2}$

$\Rightarrow f^{-1}(x) = \frac{x-1}{2}$

PERMUTATION

☆ Every bijective function defined on a finite set itself is a permutation.

i.e. Let A be a finite set

f be a bijective func^n i.e. $f: A \to A$

So, f be a permutation.

⇒ The permutations are denoted by P_1, P_2, P_3 - - - - -

⇒ If $|A| = n$, then there will be $n!$ number of permutations defined for A to A.

⇒ If $A = \{a_1, a_2, a_3, ----, a_n\}$

then $P = \begin{pmatrix} a_1 & a_2 & a_3 & ---- & a_n \\ P(a_1) & P(a_2) & P(a_3) & ---- & P(a_n) \end{pmatrix}$

↓ image of a_1

Scanned by CamScanner

Eg ⇒ List all the permutations defined on set A where

① $A = \{1, 2\}$

Since the bijective func^n is defined for set A to A, then

$f_1 = \{(1,1), (2,2)\}$

$f_2 = \{(1,2), (2,1)\}$

So the permutations are

$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$; $P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

② $A = \{1, 2, 3\}$

Since the bijective func^n is defined for set A to A, then

$f_1 = \{(1,1), (2,2), (3,3)\}$

$f_2 = \{(1,1), (2,3), (3,2)\}$

$f_3 = \{(1,2), (2,1), (3,3)\}$

$f_4 = \{(1,2), (2,3), (3,1)\}$

$f_5 = \{(1,3), (2,1), (3,2)\}$

$f_6 = \{(1,3), (2,2), (3,1)\}$

So the permutations are :→

$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$; $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$; $P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$; $P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$; $P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Eg ⇒

① $A = \{1, 2, 3, 4\}$

$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

since $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

$P(1) = 2$, $P(2) = 1$, $P(3) = 3$, $P(4) = 4$

$P^{-1}(2) = 1$, $P^{-1}(1) = 2$, $P^{-1}(3) = 3$, $P^{-1}(4) = 4$

$$P^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \underset{P^{-1}(1)}{2} & \underset{P^{-1}(2)}{1} & \underset{P^{-1}(3)}{3} & \underset{P^{-1}(4)}{4} \end{pmatrix}$$

LectureNotes.in

Scanned by CamScanner

(2) $P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

$P^{-1}(3) = 1$; $P^{-1}(1) = 2$; $P^{-1}(2) = 3$, $P^{-1}(4) = 4$

$P^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

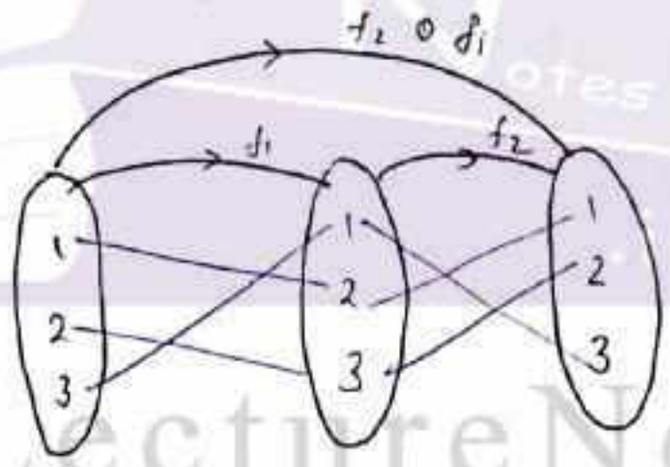
LectureNotes.in

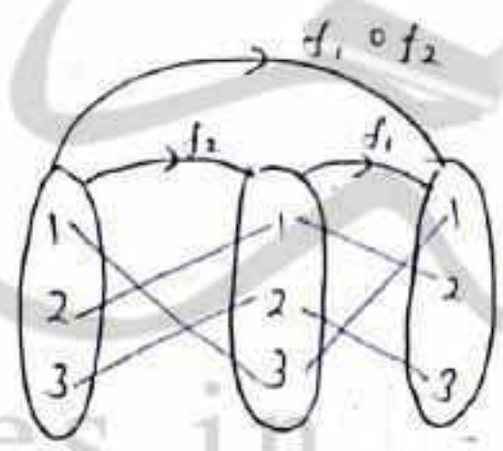
<u>Composition</u> ⇒

Eg ⇒ $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

Find $P_1 \circ P_2$ & $P_2 \circ P_1$

$f_1 = \{(1,2), (2,3), (3,1)\}$ $\qquad$ $f_2 = \{(1,3), (2,1), (3,2)\}$





$f_2 \circ f_1 = \{(1,1), (2,2), (3,3)\}$ $\qquad$ $f_1 \circ f_2 = \{(1,1), (2,2), (3,3)\}$

Here $f_1 \circ f_2 = f_2 \circ f_1$

Alternative Way ⇒

$P_1 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

$P_2 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

Eg ⇒ $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

$P_2 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ $\qquad$ $P_1 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$P_1 \circ P_2 \neq P_2 \circ P_1$

Scanned by CamScanner

The permutation

$$P = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}$$ is said to be Identity permutation & denoted by I_n.

⇒ If $|A| = n$. then S_n be the set of all permutations defined on set A to itself

LectureNotes.in

$S_2 = \{P_1, P_2\}$

$S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

Eg ⇒ (1) Construct the composition table of S_2

Since $A = \{1, 2\}$

$S_2 = \{P_1, P_2\}$ where

$$P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} ; \quad P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Composition

$\circ$	P_1	P_2
P_1	$P_1 \circ P_1$	$P_1 \circ P_2$
P_2	$P_2 \circ P_1$	$P_2 \circ P_2$

LectureNotes.in

$$P_1 \circ P_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = P_1$$

$$P_1 \circ P_2 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = P_2$$

LectureNotes.in

$$P_2 \circ P_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = P_2$$

$$P_2 \circ P_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = P_1$$

(2) Construct Composition table of S_3

$A = \{1, 2, 3\}$

$S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

Scanned by CamScanner

$$P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} ; \quad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} ;$$

$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} ; \quad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} ; \quad P_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$\circ$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	$P_1 \circ P_1$	$P_1 \circ P_2$	$P_1 \circ P_3$	$P_1 \circ P_4$	$P_1 \circ P_5$	$P_1 \circ P_6$
P_2	$P_2 \circ P_1$	$P_2 \circ P_2$	$P_2 \circ P_3$	$P_2 \circ P_4$	$P_2 \circ P_5$	$P_2 \circ P_6$
P_3	$P_3 \circ P_1$	$P_3 \circ P_2$	$P_3 \circ P_3$	$P_3 \circ P_4$	$P_3 \circ P_5$	$P_3 \circ P_6$
P_4	$P_4 \circ P_1$	$P_4 \circ P_2$	$P_4 \circ P_3$	$P_4 \circ P_4$	$P_4 \circ P_5$	$P_4 \circ P_6$
P_5	$P_5 \circ P_1$	$P_5 \circ P_2$	$P_5 \circ P_3$	$P_5 \circ P_4$	$P_5 \circ P_5$	$P_5 \circ P_6$
P_6	$P_6 \circ P_1$	$P_6 \circ P_2$	$P_6 \circ P_3$	$P_6 \circ P_4$	$P_6 \circ P_5$	$P_6 \circ P_6$

$$P_1 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = P_1$$

$$P_2 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_2$$

$$P_3 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_3$$

$$P_4 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_4$$

$$P_5 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5$$

$$P_6 \circ P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_6$$

$$P_1 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_2$$

$$P_2 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = P_1$$

Scanned by CamScanner

$$P_3 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_4$$

$$P_4 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_3$$

$$P_5 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_6$$

$$P_6 \circ P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5$$

$$P_1 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = P_3$$

$$P_2 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_6$$

$$P_3 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5$$

$$P_4 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_2$$

$$P_5 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = P_1$$

$$P_6 \circ P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_4$$

$$P_1 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = P_4$$

$$P_2 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = P_5$$

$$P_3 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = P_6$$

$$P_4 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = P_1$$

$$P_5 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_2$$

$$P_6 \circ P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} =$$

$$P_1 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$P_2 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$P_3 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$P_4 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$P_5 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$P_6 \circ P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

LectureNotes.in

Scanned by CamScanner

Cyclic Permutation ⇒

Let $b_1, b_2, b_3 \ldots b_n$ are n- distinct elements of $A = \{a_1, a_2, \ldots a_n\}$ for $r \leq n$.

If $P(b_1) = b_2$, $P(b_2) = b_3$, $P(b_3) = b_4 \ldots$ $P(b_n) = b_1$ & for remaining elements, $P(x) = x$ then P is said to be cyclic Permutation of length 'r'.

Eg ⇒ $A = \{1, 2, 3\}$, $P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1,2)$ → cycle of length 2.

$P = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2,3) \rightarrow$ cycle of length 2

$P = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1,2,3)$ → (cycle of length 3)

Eg ⇒ $A = \{1, 2, 3, 4\}$

$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} = (1,2,3) \rightarrow$ cycle of length 3

$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = (2,4)$ → cycle of length 2

☆ Every cycle of length 2 is said to be a transposition

☆ A cyclic permutation can be written as product of transposition.

i.e. $(b_1, b_2, b_3 \ldots b_n) = (b_1, b_n) \circ (b_1, b_{n-1}) \circ (b_1, b_{n-2}) \ldots \circ (b_1, b_2)$

Eg ⇒ $(2 \quad 3 \quad 5 \quad 6 \quad 4)$

$(2,4) \circ (2,6) \circ (2,5) \circ (2,3)$

☆ Even permutation ⇒ If there are even no. of transposition then the permutation is said to be even permutation

☆ Odd permutation ⇒ If there are odd no of transposition then the permutation is said to be odd permutation

Scanned by CamScanner

RECURRENCE

13/9/16

☆ If $|A| = n$

then there are

$\frac{n!}{2}$ no. of even permutation

$\frac{n!}{2}$ no. of odd permutation

Recurrence Relation ⇒

☆ The linear recurrence relation with constant coefficient is of the form

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$$

⇒ It is of order k.

⇒ If $f(n) = 0$ then the above R.R is said to be homogeneous otherwise non homogeneous.

⇒ The above R.R can be written as

$$a_n = A_1 a_{n-1} + A_2 a_{n-2} + A_3 a_{n-3} + \cdots + A_k a_{n-k} + f(n)$$

Eg ⇒ $a_n = a_{n-1} + a_{n-2}$ → homogeneous of order 2.

$a_n = 3a_{n-1} + n$ → non homogeneous & order 1

$a_n = a_{n-1} + 2a_{n-2} - 3a_{n-3}$ → homogeneous & order 3.

Solution of Homogeneous R.R ⇒

The R.R is

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0 \rightarrow \text{Order is } k.$$

Let the trial sol^n is $a_n = \alpha^n$

$a_{n-1} = \alpha^{n-1}$, $a_{n-2} = \alpha^{n-2}$, $a_{n-3} = \alpha^{n-3}$ ---- $a_{n-k} = \alpha^{n-k}$

Putting in the given R.R

$$C_0 \alpha^n + C_1 \alpha^{n-1} + C_2 \alpha^{n-2} + \cdots + C_k \alpha^{n-k} = 0$$

$$\alpha^{n-k}\{C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k\} = 0$$

since $\alpha^{n-k} \neq 0$ ⇒ $\boxed{C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k = 0}$

⇓

This is called as Characteristic Equation.

LectureNotes.in

Scanned by CamScanner

RELATION

Solving this eq^n we will get k-no. of values of α.
Such as $\alpha_1, \alpha_2, \alpha_3 - - - \alpha_k$.

Case I ⇒ If all the roots are real & distinct.

$$\boxed{a_n^{(h)} = A_1 \alpha^n + A_2 \alpha_2^n + A_3 \alpha_3^n + - - - + A_k \alpha_k^n}$$

Case II ⇒ Real & Equal Roots

* If $\alpha_1 = \alpha_2$ & all remaining are distinct.

$$a_n^{(h)} = (A_1 + A_2 n)\alpha_1^n + A_3 \alpha_3^n + - - - + A_k \alpha_k^n$$

* If $\alpha_1 = \alpha_2 = \alpha_3$ & all remainings are distinct.

$$a_n^{(h)} = (A_1 + A_2 n + A_3 n^2)\alpha_1^n + A_4 \alpha_4^n + A_5 \alpha_5^n + - - - + A_k \alpha_k^n$$

* If $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$ & all other are distinct

$$a_n^{(h)} = (A_1 + A_2 n + A_3 n^2 + A_4 n^3)\alpha_1^n + A_5 \alpha_5^n + A_6 \alpha_6^n + - - + A_k \alpha_k^n$$

* If $\alpha_1 = \alpha_2$ & $\alpha_3 = \alpha_4$ but all other are distinct.

$$a_n^{(h)} = (A_1 + A_2 n)\alpha_1^n + (A_3 + A_4 n)\alpha_3^n + A_5 \alpha_5^n + - - - + A_k \alpha_k^n$$

Case III ⇒ Complex roots

$$\alpha_1 = \alpha + i\beta$$
$$\alpha_2 = \alpha - i\beta$$

$$\boxed{a_n^{(h)} = e^{\alpha n} = (C_1 \cos\beta n + C_2 \sin\beta n)}$$

Eg ⇒ ① $a_n - 3a_{n-1} = 0$

It is homogeneous & order = 1

⇒ C. Eq^n will be

$\alpha - 3 = 0$

$\alpha = 3$

$a_n = \alpha^n$
$a_{n-1} = \alpha^{n-1}$
so $\alpha^n - 3\alpha^{n-1} = 0$
$\alpha^{n-1}(\alpha - 3) = 0$
$\alpha - 3 = 0, \ \alpha^{n-1} \neq 0$

Required eq^n

$a_n^{(h)} = A_1 3^n$

LectureNotes.in

Scanned by CamScanner

(2) $a_n + 3a_{n-1} + 2a_{n-2} = 0$

Homogeneous & order = 2

$$\left(\begin{array}{l} a_n = \alpha^n \\ \text{So } a_{n-1} = \alpha^{n-1},\ a_{n-2} = \alpha^{n-2} \\ \alpha^n + 3\alpha^{n-1} + 2\alpha^{n-2} = 0 \\ \alpha^{n-2}(\alpha^2 + 3\alpha + 2) = 0 \\ \alpha^{n-2} \neq 0,\ \alpha^2 + 3\alpha + 2 = 0 \end{array} \right)$$

Ch. Eq^n is

$\alpha^2 + 3\alpha + 2 = 0$

$(\alpha+1)(\alpha+2) = 0$

$\alpha = -1, -2$

$a_n^{(h)} = c_1(-1)^n + c_2(-2)^n$

LectureNotes.in

(3) $a_n - 7a_{n-1} + 12a_{n-2} = 0$

It is homogeneous & order = 2

Ch. eq^n will be

$\alpha^2 - 7\alpha + 12 = 0$

$(\alpha - 3)(\alpha - 4) = 0$

$\alpha = 3 \,\&\, 4$

$a_n^{(h)} = c_1(3^n) + c_2(4^n)$

(4) $a_n + 4a_{n-1} + 4a_{n-2} = 0$

Homogeneous & order = 2

Ch. eq^n is

$\alpha^2 + 4\alpha + 4 = 0$

$(\alpha + 2)^2 = 0$

$\alpha = -2, -2$

LectureNotes.in

Required sol^n will be

$a_n^{(h)} = (c_1 + c_2 n)(-2)^n$

(5) $a_n = a_{n-1} + a_{n-2}$

where $a_0 = 1$; $a_1 = 1$

LectureNotes.in

$a_n - a_{n-1} - a_{n-2} = 0$

homogeneous & order = 2

Ch. eq^n is

$\alpha^2 - \alpha - 1 = 0$

$\alpha = \dfrac{1 \pm \sqrt{1+4}}{2}$

$\alpha = \dfrac{1+\sqrt{5}}{2},\ \dfrac{1-\sqrt{5}}{2}$

$a_n^{(h)} = c_1\left(\dfrac{1+\sqrt{5}}{2}\right)^n + c_2\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Scanned by CamScanner

Since $a_0 = 1 \Rightarrow 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 \Rightarrow c_1 + c_2 = 1$ — (I)

$a_1 = 1 \Rightarrow 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow$

$\Rightarrow (1+\sqrt{5})c_1 + (1-\sqrt{5})c_2 = 2$ — (II)

$$(1+\sqrt{5})c_1 + (1-\sqrt{5})c_2 = 2$$
$$(1+\sqrt{5})c_1 + (1+\sqrt{5})c_2 = 1+\sqrt{5}$$
$$(1-\sqrt{5}-1-\sqrt{5})c_2 = 1-\sqrt{5}$$
$$(-2\sqrt{5})c_2 = 1-\sqrt{5}$$
$$c_2 = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)$$

$$c_1 = 1 - c_2$$
$$= 1 + \frac{1-\sqrt{5}}{2} = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

Req^{ed} sol^n is

$$a_n = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$
$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right\}$$

(6) $a_n - 7a_{n-1} + 10a_{n-2} = 0$ $\quad a_0 = 0,\ a_1 = 3$

Homogeneous & Order = 2

Ch. Eq^n is

$\alpha^2 - 7\alpha + 10 = 0$

$(\alpha-5)(\alpha-2) = 0$

$\alpha = 2, 5$

$\Rightarrow a_n^{(h)} = c_1 2^n + c_2 5^n$

$a_0 = c_1 2^0 + c_2 5^0 = 0$ & $a_1 = c_1 2^1 + c_2 5^1 = 3$

$\Rightarrow c_1 + c_2 = 0$ $\quad \Rightarrow 2c_1 + 5c_2 = 3$

$$2c_1 + 5c_2 = 3$$
$$2c_1 + 2c_2 = 0$$
$$3c_2 = 3$$
$$c_2 = 1$$

& $c_1 = -c_2$

$c_1 = -1$

Req' sol^n is

$$\boxed{a_n = -2^n + 5^n}$$

LectureNotes.in

Scanned by CamScanner

(7) $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

Homogeneous & order = 3

Ch. eqn is

$\alpha^3 - 6\alpha^2 + 12\alpha - 8 = 0$

$\cancel{\alpha^3 - 8\alpha^2 + 6\alpha +}$

$\alpha^3 - 6\alpha^2 + 6\alpha + 6\alpha - 6 = 0$

$\alpha^2(\alpha - 6) +$

$(\alpha - 2)^3 = 0$

$\cancel{\alpha = \sqrt{2}, \sqrt{2},}$

$\alpha = 2, 2, 2$

$a_n^{(h)} = (c_1 + c_2 n + c_3 n^2) 2^n$

(8) $a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$

Homogeneous & order = 3

Ch. eqn is

$\alpha^3 - 7\alpha^2 + 16\alpha - 12 = 0$

$(\alpha - 2)^2(\alpha - 3) = 0$

$\alpha = 2, 2, 3$

$a_n^{(h)} = (c_1 + c_2 n) 2^n + c_3 3^n$

Solution of Linear Non-homogeneous RR with Constant Coffecients

★ The linear non homogenous R.R with cons. cof. is of the form

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

★ $a_n^{(T)}$ be its general / total solution

where $\boxed{a_n^{(T)} = a_n^{(h)} + a_n^{(P)}}$

$a_n^{(h)}$ be the homogeneous sol^n

$a_n^{(P)}$ be the particular sol^n.

LectureNotes.in

Scanned by CamScanner

15/9/16

How to find Particular Solution ?

Rule (1) If $f(n)$ = Polynomial of degree k & 1 is not a characteristic root

then assume

$a_n = b_0 + b_1 n + b_2 n^2 + \dots + b_k n^k$

Sol. Particular

If $f(n)$ = const $\longrightarrow a_n = P$

$f(n)$ = polynomial of degree 1 $\longrightarrow a_n = An + B$

i.e. $a_n + b$

or

a_n

$f(n)$ = poly. of degree 2

i.e. $a_n^2 + bn$

or

$a_n^2 + bn \longrightarrow a_n = An^2 + Bn + C$

or

$a_n^2 + c$

or

a_n^2

Eg. (1) $a_n = 5a_{n-1} + 6a_{n-2} + 5$

To get homo. sol^n

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

Characteristic eq^n is

$r^2 - 5r + 6 = 0$

$r = 2, 3$

$$\boxed{a_n^{(h)} = c_1(2)^n + c_2(3^n)}$$

To get particular sol^n

Here $f(n) = 5 =$ const.

Assume $a_n^{(p)} = P$

So $a_{n-1} = P$

$a_{n-2} = P$

Putting in given R.R

$P - 5P + 6P = 5$

$P = 5/2$

$$\boxed{a_n^{(p)} = \frac{5}{2}}$$

So required sol^n is

$a_n^{(r)} = a_n^{(h)} + a_n^{(p)}$

$$\boxed{a_n^{(r)} = c_1 2^n + c_2 3^n + \frac{5}{2}}$$

LectureNotes.in

Scanned by CamScanner

(2) $a_n - 7a_{n-1} + 12a_{n-2} = 3n$

To get H.S

$a_n - 7a_{n-1} + 12a_{n-2} = 0$

Ch. Eq^n is

$\alpha^2 - 7\alpha + 12 = 0$

$\alpha = 3, 4$

$$\boxed{a_n^{(h)} = c_1 3^n + c_2 4^n}$$

To get P.S

$f(n) = 3n$ - poly of degree 1

$\Rightarrow$ Put $a_n = An + B$

$a_{n-1} = A(n-1) + B$

$a_{n-2} = A(n-2) + B$

Putting in RR.

$(An + B) - 7\{A(n-1) + B\} + 12\{A(n-2) + B\} = 3n$

$An + B - 7An + 7A - 7B + 12An - 24A + 12B = 3n$

$6An + (6B - 17A) = 3n$

Comparing

$A = 1/2$ $\quad 6B - 17A = 0$

$B = \frac{17}{12}$

$$\boxed{a_n^{(p)} = \frac{1}{2}n + \frac{17}{12}}$$

$$\therefore \boxed{a_n^{(T)} = c_1 3^n + c_2 4^n + \frac{n}{2} + \frac{17}{12}}$$

(3) $a_n - 4a_{n-1} + 4a_{n-2} = 2n^2$

H.S

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

Ch. Eq^n

$\alpha^2 - 4\alpha + 4 = 0$

$(\alpha - 2)^2 = 0$

$\alpha = 2, 2$

$$\boxed{a_n^{(h)} = (c_1 + c_2 n)2^n}$$

P.S

$f(n) = 2n^2$ - degree 2

$\Rightarrow$ Put $a_n^{(p)} = An^2 + Bn + C$

$a_{n-1} = A(n-1)^2 + B(n-1) + C$

$a_{n-2} = A(n-2)^2 + B(n-2) + C$

Putting in eq^n.

$An^2 + Bn + C - 4\{\overset{n^2+1-2n}{A(n-1)^2 + B(n-1) + C}\} + 4\{\overset{n^2+4-4n}{A(n-2)^2 + B(n-2) + C}\} = 2n^2$

$An^2 + Bn + C - 4An^2 - 4A + 8An - 4Bn + 4B - 4C + 4An^2 + 16A - 16An + 4Bn - 8B + 4C = 2n^2$

$An^2 - 8An + Bn + C + 12A - 4B = 2n^2$

$An^2 + (B - 8A)n + 12A - 4B + C = 2n^2$

$A = 2$; $\quad B - 8A = 0$

$B = 16$

$12A - 4B + C = 0$

$24 - 64 + C = 0$

$C = 40$

Scanned by CamScanner

$\Rightarrow \boxed{a_n^{(p)} = 2n^2 + 16n + 40}$

$\therefore \boxed{a_n^{(T)} = (c_1 + c_2 n)2^n + 2n^2 + 16n + 40}$

(4) $a_n - 7a_{n-1} + 12a_{n-2} = 3n + 5$

$\boxed{a_n^{(h)} = c_1 3^n + c_2 4^n}$ from ① ②

& also.

$(An + B) - 7\{A(n-1) + B\} + 12\{A(n-2) + B\} = 3n + 5$

$6An + (6B - 17A) = 3n + 5$

comparing

$6A = 3$ $\quad$ $6B - 17A = 5$

$A = \frac{1}{2}$ $\quad$ $6B = 5 + \frac{17}{2}$

$26B = \frac{27}{2}$ $\Rightarrow$ $B = \frac{9}{4}$

$\Rightarrow \boxed{a_n^{(p)} = \frac{n}{2} + \frac{9}{4}}$

$\therefore \boxed{a_n^{(T)} = c_1 3^n + c_2 4^n + \frac{n}{2} + \frac{9}{4}}$

(5) $a_n - 3a_{n-1} + 2a_{n-2} = 5$

H.S

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

Ch. Eq^n is

$\alpha^2 - 3\alpha + 2 = 0$

$\alpha = 1, 2$

$\boxed{a_n^{(h)} = c_1 1^n + c_2 2^n}$

P.S

$f(n) = 5 = \text{const.}$

Put $a_n = p \Rightarrow a_{n-1} = p$ & $a_{n-2} = p$

$\Rightarrow p - 3p + 2p = 5$

$0 = 5$

LectureNotes.in

Scanned by CamScanner

So if characteristic root is 1 then

Rule 2 → If $f(n)$ = (Poly. of degree k) $\times \beta^n$

→ where β is not a ch. root then

$$\boxed{a_n = (A_0 + A_1 n + A_2 n^2 + \dots + A_k n^k)\beta^n}$$

how many times β occurs

→ where β is a characteric root of multiplicity m

then $$\boxed{a_n = (A_0 + A_1 n + \dots + A_k n^k)\beta^n \cdot n^m}$$

So in previous ques^n

① P.S

$f(n) = 5 = 5 \times 1^n$

where 1th is a chara. root of multiplicity 1

$\Rightarrow \beta = 1, \quad m = 1$

so, $a_n = P.1^n.n^1$

$a_n = Pn$

$a_{n-1} = P(n-1)$

$a_{n-2} = P(n-2)$

Putting

$Pn - 3P(n-1) + 2P(n-2) = 5$

$P = -5$

$$\boxed{a_n^{(P)} = -5n}$$

$$\therefore \boxed{a_n^{(T)} = C_1 2^n + C_2 . 1^n - 5n}$$

Q ② $a_n - 4a_{n-1} + 4a_{n-2} = 3^n$

$a_n^{(h)} = (C_1 + C_2 n)\, 2^n$ same as Q 3.

P.S ; $\beta = 3$; not a characteristic root.

$f(n) = 3^n = 1 \times 3^n$

(Polynomial of degree 0) 3^n

So, $a_n = P.3^n$

$a_{n-1} = P.3^{n-1}$

$a_{n-2} = P.3^{n-2}$

Scanned by CamScanner

Putting in eq^n

$$P.3^n - 4.P3^{n-1} + 4.P3^{n-2} = 3^n$$

$$3^{n-2}\{P.3^2 - 4.P.3 + 4.P\} = 3^n.3^2$$

$$(9P - 12P + 4P) = 9$$

$$P = 9$$

So $\quad a_n^{(P)} = 9.3^n$

Required sol^n

$$\boxed{a_n^{(T)} = (C_1 + C_2 n)2^n + 9.3^n}$$

③ $a_n - 4a_{n-1} + 4a_{n-2} = n2^n$

Ans

$a_n - 4a_{n-1} + 4a_{n-2} = 0$

Ch. eq^n

$\alpha^2 - 4\alpha + 4 = 0$

$\alpha = 2, 2$

$$\boxed{a_n^{(h)} = (C_1 + C_2 n)2^n}$$

P.S

$f(n) = n2^n =$

$= (\text{Poly of degree } 1)\, 2^n$

$\beta = 2, \quad m = 2$

2 is characteristic root of multiplicity 2.

So $\quad a_n^{(P)} = (An + B)2^n.n^2$

$a_n^{(P)} = (An^3 + Bn^2)2^n$

$a_{n-1} = (A(n-1)^3 + B(n-1)^2)2^{n-1}$

$a_{n-2} = (A(n-2)^3 + B(n-2)^2)2^{n-2}$

Putting in eq^n

$$(An^3 + Bn^2)2^n - 4\{An^3 - 3An^2 + 3nA - A + Bn^2 - 2Bn + B\}2^{n-1}$$
$$+ 4\{An^3 - 6An^2 + 12nA - 8A + Bn^2 - 4Bn + 4B\}2^{n-2} = n2^n$$

$$2^{n-2}[\cancel{4An^3} + \cancel{4Bn^2} - \cancel{8An^3} + \cancel{24An^2} - 24nA + 8A - \cancel{8Bn^2} + \cancel{16Bn} - 8B$$
$$+ \cancel{4An^3} - \cancel{24An^2} + 48nA - 32A + \cancel{4Bn^2} - \cancel{16Bn} + 16B\} = n2^n$$

$$\frac{1}{2^2}[24nA - 24A + 8B] = n.$$

$$\cancel{8nA + 24A + 16B = 4n}$$

$\Rightarrow \quad \cancel{8A = -4}$

$\cancel{A = -1/2}$

$24An + (8B - 24A) = 4n$

$24A = 4$

$A = \frac{1}{6} \quad ; \quad B = 3A = \frac{1}{2}$

$$\therefore \boxed{a_n^{(P)} = \left(\frac{1}{6}n + \frac{1}{2}\right)2^n.n^2}$$

Scanned by CamScanner

16/9/16

(4) $a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$

H.S

$a_n - 5a_{n-1} + 6a_{n-2} = 0$

Ch. Eq^n

$\alpha^2 - 5\alpha + 6 = 0$

$\alpha = 2, 3$

$$\boxed{a_n^{(1)} = C_1 2^n + C_2 3^n}$$

P.S

$f(n) = 2^n + n$

$= (\text{pol. of deg. } 0)\, 2^n + (\text{Pol. of deg. } 1)$

here 2 is ch. root of multiplicity 1.

So, $a_n^{(P)} = P.\, 2^n . n^1 + (An + B)$

$a_{n-1} = P(n-1)\, 2^{n-1} + A(n-1) + B$

$a_{n-2} = P(n-2)\, 2^{n-2} + A(n-2) + B$

Putting in Eq^n

$\{Pn\, 2^n + An + B\} - 5\{P(n-1)2^{n-1} + A(n-1) + B\} + 6\{P(n-2)\, 2^{n-2} + A(n-2) + B\} = 2^n +$

$\{Pn\, 2^n - 5P(n-1)2^{n-1} + 6P(n-2)\, 2^{n-2}\} + \{An + B - 5A(n-1) - 5B + 6A(n-2) + 6B) =$

$2^{n-2}\{4Pn - 10P(n-1) + 6P(n-2)\} + \{An + B - 5An + 5A - 5B + 6An - 12A + 6B\} = 2^n$

$2^{n-2}\{\cancel{4Pn} - \cancel{10Pn} + 10P + \cancel{6Pn} - 12P\} + \{2An + (2B - 7A)\} = 2^n + n$

$\cancel{-2P} = 2^{n-2} . (-2P) = 2^n$

$-\frac{2P}{4} . 2^n = 2^n$

$P = -2$

$2An + (2B - 7A) = n$

$2A = 1$; $2B - 7A = 0$

$A = 1/2$ $B = \frac{7}{2}A$

$B = \frac{7}{4}$

$$\therefore \boxed{a_n^{(P)} = -2 . 2^n . n + \frac{n}{2} + \frac{7}{4}}$$

$$\therefore \boxed{a_n^{(T)} = C_1 2^n + C_2 3^n - 2n . 2^{n+1} + \left(\frac{n}{2} + \frac{7}{4}\right)}$$

Scanned by CamScanner

Binomial Theorem ⇒

$C(n,r) = \frac{n!}{r!(n-r)!}$ $(r \le n)$

$(a+x)^n = C(n,0)a^n + C(n,1)a^{n-1}x + C(n,2)a^{n-2}x^2 + \dots + C(n,n)x^n$

① $(a+x)^n = C(n,0)a^n + C(n,1)a^{n-1}x + C(n,2)a^{n-2}x^2 + \dots + C(n,n)x^n$

ⓐ $(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + \dots + C(n,n)x^n$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots x^n$

ⓑ $(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots + (-1)^n x^n$

$|x| < 1$

ⓒ $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots \infty$

ⓓ $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots \infty$

Eg ⇒

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

General term	Coefficient of G.T
$C(n,r)a^{n-r}x^r$	$C(n,r)$
$\frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$	$\frac{n(n-1)\dots(n-r+1)}{r!}$
$\frac{(-1)^r n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$	$\frac{(-1)^r n(n-1)\dots(n-r+1)}{r!}$
$\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r$	$\frac{n(n+1)(n+2)\dots(n+r-1)}{r!}$
x^r	1
$(r+1)x^r$	$r+1$

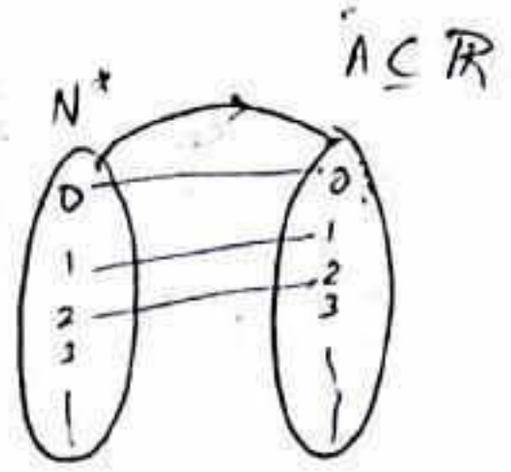
Scanned by CamScanner

x) Let a_n be a sequence

So its terms are $a_0, a_1, a_2, a_3 - - -$

The power series generated by the terms of the sequence (a_n)

is $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + - -$



Bijective Func^n

LectureNotes.in

Generating Function ⇒

x) If this power series can be converted in terms of a func^n then that function is said to be generating function & denoted by $A(x)$

So $A(x) = a_0 + a_1 x + a_2 x^2 + - -$

Eg ⇒ ① $a_n = n$

Its term's are 0, 1, 2, 3

So $A(x) = 0 + 1x + 2x^2 + 3x^3 + -$

$= x\,(1 + 2x + 3x^2 + 4x^3 + - -)$

$= x \cdot \frac{1}{(1-x)^2}$

② $a_n = n+1$

It's terms are 1, 2, 3, 4 - -

$A(x) = 1 + 2x + 3x^2 + - -$

$= \frac{1}{(1-x)^2}$

③ $a_n = (-1)^n$

Its terms are $1, -1, 1, -1, 1 - -$

$A(x) = 1 - x + x^2 - x^3 + x^4 + -$

$= \frac{1}{1+x}$

④ $a_n = 1^n$

It's terms are 1, 1, 1, 1, 1 - -

$A(x) = 1 + x + x^2 + x^3 + -$

$= \frac{1}{1-x}$

⑤ $a_n = a^n$

It's terms are $1, a, a^2, a^3 - -$

$A(x) = 1 + ax + a^2x^2 + a^3x^3 + -$

$= \frac{1}{1-ax}$

⑥ $a_n = (-a)^n$ or $(-1)^n a^n$

It's terms are

$(-1)^0 a^0, (-1)^1 a, (-1)^2 a^2, (-1)^3 a^3, - -$

$= 1, -a, a^2, -a^3, - -$

$A(x) = 1 - ax + a^2x^2 - a^3x^3 + -$

$A(x) = \frac{1}{1+ax}$

Scanned by CamScanner

So, in general

$$\boxed{a^n = \frac{1}{1-ax}}$$

&

$$\boxed{(-a)^n = \frac{1}{1+ax}}$$

In particular,

$2^n \longrightarrow \frac{1}{1-2x}$

$3^n \longrightarrow \frac{1}{1-3x}$

$4^n \longrightarrow \frac{1}{1-4x}$

In particular

$(-2)^n \longrightarrow \frac{1}{1+2x}$

$(-3)^n \longrightarrow \frac{1}{1+3x}$

$(-4)^n \longrightarrow \frac{1}{1+4x}$

Sequence	Generating Function
① $C(k,n)$	$(1+x)^k$
② 1	$\frac{1}{1-x}$
③ $(-1)^n$	$\frac{1}{1+x}$
④ a^n	$\frac{1}{1-ax}$
⑤ $(-a)^n$	$\frac{1}{1+ax}$
⑥ $C(k-1+n,n)$	$\frac{1}{(1-x)^k}$
⑦ $C(k-1+n,n)\,a^n$	$\frac{1}{(1-ax)^k}$
⑧ $C(k-1+n,n)(-a)^n$	$\frac{1}{(1+ax)^k}$
⑨ $n+1$	$\frac{1}{(1-x)^2}$
⑩ n	$\frac{x}{(1-x)^2}$
⑪ $(n+2)(n+1)$	$\frac{2}{(1-x)^3}$

LectureNotes.in

Scanned by CamScanner

17/9/16

(12) $(n+1)n$ $\qquad \frac{2x}{(1-x)^3}$

(13) n^2 $\qquad \frac{x(1+x)}{(1-x)^3}$

(14) $(n+3)(n+2)(n+1)$ $\qquad \frac{6}{(1-x)^4}$

(15) $(n+2)(n+1)n$ $\qquad \frac{6x}{(1-x)^4}$

(16) $(n+1)a^n$ $\qquad \frac{1}{(1-ax)^2}$

(17) na^n $\qquad \frac{ax}{(1-ax)^2}$

(18) n^2a^n $\qquad \frac{ax(1+ax)}{(1-ax)^3}$

(1) $A(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{k-1}x^{k-1} + a_kx^k + a_{k+1}x^{k+1} + \cdots = \sum_{n=0}^{\infty} a_nx^n$

(2) $A(x) - a_0 - a_1x - a_2x^2 \cdots - a_{k-1}x^{k-1} = a_kx^k + a_{k+1}x^{k+1} + \cdots = \sum_{n=k}^{\infty} a_nx^n$

(3) $x\{A(x) - a_0 - a_1x - a_2x^2 - \cdots - a_{k-2}x^{k-2}\} = x\{a_{k-1}x^{k-1} + a_kx^k + \cdots\}$
$= a_{k-1}x^k + a_kx^{k+1} + a_{k+1}x^{k+2} + \cdots = \sum_{n=k}^{\infty} a_{n-1}x^n$

(4) $x^2\{A(x) - a_0 - a_1x - a_2x^2 \cdots - a_{k-3}x^{k-3}\} = x^2\{a_{k-2}x^{k-2} + a_{k-1}x^{k-1} + a_kx^k + \cdots\}$
$= a_{k-2}x^k + a_{k-1}x^{k+1} + a_kx^{k+2} + \cdots = \sum_{n=k}^{\infty} a_{n-2}x^n$

(5) $x^kA(x) = x^k\{a_0 + a_1x + a_2x^2 + \cdots\} = a_0x^k + a_1x^{k+1} + a_2x^{k+2} = \sum_{n=k}^{\infty} a_{n-k}x^n$

In particular

since $\boxed{\sum_{n=k}^{\infty} a_{n-k}x^n = x^kA(x)}$

Scanned by CamScanner

$*)\ \sum_{n=1}^{\infty} a_{n-1} x^n = x A(x)$

$$\sum_{n=1}^{\infty} a_{n-1} x^n = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$
$$= x\,[a_0 + a_1 x + a_2 x^2 + \cdots]$$
$$= x\,A(x)$$

$*)\ \sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$
$$= x^2 \{ a_0 + a_1 x + a_2 x^2 + \cdots \}$$
$$= x^2 A(x).$$

$*)\ \sum_{n=3}^{\infty} a_{n-3} x^n = x^3 A(x)$

$*)\ \sum_{n=4}^{\infty} a_{n-4} x^n = x^4 A(x)$

$*)\ \sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
$$= A(x) - a_0$$

$*)\ \sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$
$$= A(x) - a_0 - a_1 x$$

$*)\ \sum_{n=3}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - a_2 x^2$

$*)\ \sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots$
$$= x \{ a_1 x + a_2 x^2 + a_3 x^3 + \cdots \}$$
$$= x \{ A(x) - a_0 \}$$

$*)\ \sum_{n=3}^{\infty} a_{n-1} x^n = a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots$
$$= x \{ a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \}$$
$$= x \{ A(x) - a_0 - a_1 x \}$$

$*)\ \sum_{n=3}^{\infty} a_{n-2} x^n = x^2 \{ a_1 x + a_2 x^2 + a_3 x^3 + \cdots \}$
$$= x^2 \{ A(x) - a_0 \}$$

LectureNotes.in

Scanned by CamScanner

Sequence of $A(x)$ is a_n

Solution of RR using generating function ⇒

① $a_n - 5a_{n-1} + 6a_{n-2} = 0$

Homog. order = 2 $(n \geq 2)$

$$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = 0 \times \sum_{n=2}^{\infty} x^n$$

as order is 2

$$\{A(x) - a_0 - a_1 x\} - 5x\{A(x) - a_0\} + 6x^2 A(x) = 0$$

$$A(x)[1 - 5x + 6x^2] = a_0 + a_1 x - 5a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - 5a_0 x}{1 - 5x + 6x^2} = \frac{a_0 + a_1 x - 5a_0 x}{(1-2x)(1-3x)}$$

$$A(x) = \frac{c_1}{1-2x} + \frac{c_2}{1-3x}$$ (Partial fraction)

Write its sequence.

$$\boxed{a_n = c_1(2^n) + c_2(3^n)}$$

② $a_n - 7a_{n-1} + 10a_{n-2} = 0$, $a_0 = 1$, $a_1 = 1$

Homo., order 2 $(n \geq 2)$

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$A(x) - a_0 - a_1 x - 7x\{A(x) - a_0\} + 10x^2 A(x) = 0$$

$$(1 - 7x + 10x^2) A(x) = a_0 + a_1 x - 7a_0 x$$

$$= 1 + x - 7x$$

$$= 1 - 6x$$

$$A(x) = \frac{1-6x}{1-7x+10x^2}$$

$$\frac{1-6x}{1-7x+10x^2} = \frac{1-6x}{(1-5x)(1-2x)} = \frac{A}{1-5x} + \frac{B}{1-2x}$$

$$A(1-2x) + B(1-5x) = 1-6x$$

$$(A+B) - (2A+5B)x = 1-6x$$

$$A + B = 1, \quad 2A + 5B = 6$$

$$2A + 5B = 6$$
$$2A + 2B = 2$$
$$3B = 4 \Rightarrow B = \frac{4}{3}$$

Scanned by CamScanner

$A = 1 - B$

$A = \frac{-1}{3}$

$\Rightarrow A(x) = \frac{1-6x}{1-7x+10x^2}$

$A(x) = \frac{-1/3}{1-5x} + \frac{4/3}{1-2x}$

Sequence

$$\boxed{a_n = -\frac{1}{3}(5^n) + \frac{4}{3}(2^n)}$$

(3) $a_n + a_{n-1} = n+1, \quad a_0 = 1$

non homo., order = 1 $(n \geq 1)$

$\sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} (n+1) x^n$

$A(x) - a_0 + x A(x) = 2x + 3x^2 + 4x^3 + \cdots$

$(1+x)A(x) - a_0 = \{1 + 2x + 3x^2 + \cdots\} - 1$

$= \frac{1}{(1-x)^2} - 1$

$A(x) = \frac{a_0}{1+x} + \frac{1}{(1-x)^2 (1+x)} - \frac{1}{1+x} = \cancel{\frac{1}{1+x}} + \frac{1}{(1+x)(1-x)^2} - \cancel{\frac{1}{1+x}}$

$A(x) = \frac{1}{(1-x^2)(1+x)}$ ——— (1)

$\frac{1}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$

$= \frac{A(1-x^2) + B(1+x)(1-x) + C(1+x)}{(1+x)(1-x)^2}$

$1 = A(1-x)^2 + B(1-x^2) + C(1+x)$

$\Rightarrow x = 1, \quad 2C = 1 \Rightarrow \boxed{C = 1/2}$

$x = -1, \quad 4A = 1 \Rightarrow \boxed{A = 1/4}$

$\frac{1}{4}(1-x)^2 + B(1-x^2) + \frac{1}{2}(1+x) = 1$

$\frac{1}{4}(1 - 2x + x^2) + B - Bx^2 + \frac{1}{2} + \frac{x}{2} = 1$

$\frac{1}{4} - \cancel{\frac{1}{2}x} + \frac{x^2}{4} + B - Bx^2 + \frac{1}{2} + \cancel{\frac{x}{2}} = 1$

$\left(\frac{1}{4} - B\right)x^2 + \left(\frac{1}{4} + B + \frac{1}{2}\right) = 1$

Scanned by CamScanner

$\frac{1}{4} - B = 0$, $\frac{3}{4} + B = 1$

$\Rightarrow \boxed{B = 1/4}$

$$\frac{1}{(1+x)(1-x^2)} = \frac{1}{4}\frac{1}{1+x} + \frac{1}{4}\frac{1}{1-x} + \frac{1}{2}\frac{1}{(1-x)^2}$$

on taking sequence of ①

Solve ques from back

$$a_n = \frac{1}{4}(-1)^n + \frac{1}{4}(1)^n + \frac{1}{2}(n+1)$$

Abstract Algebra -

LectureNotes.in

Scanned by CamScanner

19/9/16

ALGEBRAIC STRUCTURE

Abstract Algebra ⇒

Let $*$ be an operator

S be any non-empty set

① Closer axiom ⇒ $\forall a, b \in S \Rightarrow a * b \in S$

② Associative axiom ⇒ $\forall a, b, c \in S$

$$a * (b * c) = (a * b) * c$$

③ Identity axiom ⇒ $\forall a \in S, \exists e \in S$ for which

$$a * e = a = e * a$$

then 'e' be the Id element of S w.r.t $*$

④ Inverse axiom ⇒ $\forall a \in S, \exists b \in S$ for which

$$a * b = e = b * a$$

'b' is the Inverse of a

⑤ Commutative Axiom ⇒ $\forall a, b \in S$, $a * b = b * a$

Algebraic Structure ⇒ The non empty set S w.r.t the operator $*$ said to be an Algebraic structure if closer axioms holds in S.

In such case we will say $(S, *)$ be an Algebraic Structure

SemiGroup ⇒ The non-empty set S w.r.t the operator $*$ is said to be semigroup iff

→ Closer axiom

→ Associative axiom holds in S.

Scanned by CamScanner

Monoid ⇒ The non empty set S wrt the operator * is said to be a monoid iff.

→ Closer axiom

→ Associative axiom

→ Identity axiom holds in S.

Group ⇒ The non empty set G is said to be a group wrt * iff

→ Closer axiom

→ Associative axiom

→ Identity axiom

→ Inverse axiom holds in G.

Abelian Group ⇒ The group (G, *) is said to be Abelian
or
Commutative Group. iff G satisfies Commutatative axiom wrt *

Algebraic structure → Closer axiom

Semigroup → Closer + Associative

Monoid → Closer + Associative + Identity

Group → Closer + Associative + Identity + Inverse

~~Grp~~ →

Group ~~when~~ called Monoid if it satisfies Monoid + Inverse axiom

i.e. Group → Monoid + Inverse axiom

→ semigp + Identity axiom + Inverse axiom

→ Alg. struct + Assoc. axiom + Identity axiom + Inverse

Semigp → Alg. struct + Associative

Monoid → semigp + Id axiom

→ Alg. Struct + Associative + Identity axiom.

LectureNotes.in

Scanned by CamScanner

To see Identity element, two rows are same
To see Commutative, Matrix is symmetric ($R_i = C_i$)

Eg ⇒ ① $G = \{1, -1\}$ under '×'

×	1	-1
1	1	-1
-1	-1	1

Identity

⇒ $\forall a, b \in G \Rightarrow a \times b \in G \Rightarrow$ Closer axiom holds

⇒ $\forall a, b, c \in G \Rightarrow a \times (b \times c) = (a \times b) \times c \Rightarrow$ Asso. "

Real No. always follows Associative axiom

⇒ $\forall a \in S, \exists e = 1 \in S$ for which $a.1 = 1.a$

Id. axioms holds with Identity element 1

×	1	-1
1	1	-1
-1	-1	1

⇒ Inverse axiom holds as Inverse of 1 is 1
-1 is -1
(as every row contain Identity element 1)

⇒ $\forall a, b \in S \Rightarrow a.b = b.a \Rightarrow$ Commutative axiom holds

So $(G, \times)$ be an abelian gp.

② $G = \{1, \omega, \omega^2\}$ under ordinary multiplication

×	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	$\omega^3 = 1$
ω^2	ω^2	$\omega^3 = 1$	$\omega^4 = \omega$

⇒ Identity

*) $\forall a, b \in G \Rightarrow a.b \in G \Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in G \Rightarrow a.(b.c) = (a.b).c \Rightarrow$ Associative axiom holds

*) $\forall a \in G, \exists e = 1 \in G$ s.t $a.1 = a = 1.a \Rightarrow$ Identity axiom holds with Identity element 1.

*) Inverse axiom holds as Invers of 1 is 1
ω is ω^2
ω^2 is ω

*) $\forall a, b \in G \Rightarrow a.b = b.a \Rightarrow$ Commutative axiom holds

So, $(G, \cdot)$ be an abelian gp.

③ $G = \{1, -1, i, -i\}$ under ×

×	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

→ Identity

Scanned by CamScanner

*) $\forall a, b \in G \Rightarrow a \cdot b \in G \Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in G \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c \Rightarrow$ Associative axiom hold

Complex no. satisfies associative axiom.

*) $\forall a \in G \; \exists e = 1 \in G$ s.t. $a \cdot 1 = a = 1 \cdot a$

Id. axiom holds with Id element 1.

*) Inverse axiom holds as Inverse of 1 is 1
-1 is -1
i is -i
-i is i

*) $\forall a, b \in G \Rightarrow a \cdot b = b \cdot a \Rightarrow$ Commutative axiom holds

So, $(G, \cdot)$ be an abelian gp

(4) $G = \{i, -i\}$ under multiplicn

$\times$	i	$-i$
i	-1	1
$-i$	1	-1

*) $\forall a, b \in G \Rightarrow a \cdot b \notin G$

So, G is not an Alg. str wrt $\times$.

LectureNotes.in

Assignment

(1) $S = \{0, 1\}$ under $\wedge$

(2) $S = \{0, 1\}$ under $\vee$

(3) S_2 under $\circ$ where $S_2 = \{P_1, P_2\}$

(4) S_3 under $\circ$ where $S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

(1)

$\wedge$	0	1
0	0	0
1	0	1

$\Rightarrow \forall a, b \in S \Rightarrow a \wedge b \in G \Rightarrow$ Closer axiom holds

*) $\forall a, b, c, \in S \Rightarrow a \wedge (b \wedge c) = (a \wedge b) \wedge c \Rightarrow$ Ass. holds

*) $\forall a, \in S \; \exists e = 1 \in S$ st. $a \wedge 1 = a = 1 \wedge a$

Scanned by CamScanner

Id. axiom holds with Id. element 1

*) Inverse axiom does not holds as ~~Inverse of 1 is 1~~
every row does not contain 1

*) $\forall a, b \in S \Rightarrow a \wedge b = b \wedge a \Rightarrow$ Commutative axiom holds

So, $(S, \wedge)$ ~~be~~ not an abelian gp., it is commutative monoid

② $S = \{0, 1\}$ under $\vee$

$\vee$	0	1
0	0	1
1	1	1

$\Rightarrow \forall a, b \in S \Rightarrow a \vee b \in S \Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in S \Rightarrow a \vee (b \vee c) = (a \vee b) \vee c \Rightarrow$ Ass. holds.

*) $\forall a, \in S \ \exists e = 0 \in S$, st $a \vee 0 = a = 0 \vee a$.

Id. axiom holds with Id. element 0.

*) Inverse axiom does not holds as ~~Inverse of 0 is 0~~
each row does not contain 0.

*) $\forall a, b \in S \Rightarrow a \vee b = b \vee a \Rightarrow$ Commutative axiom holds

So, $(S, \vee)$ be not an abelian gp., it is monoid.

③ $S_2 = \{P_1, P_2\}$

$\circ$	P_1	P_2
P_1	$P_1 \circ P_1 = P_1$	$P_1 \circ P_2 = P_2$
P_2	$P_2 \circ P_1 = P_2$	$P_2 \circ P_2 = P_1$

$\circ$	P_1	P_2
P_1	P_1	P_2
P_2	P_2	P_1

*) $\forall A, B \in S_2 \Rightarrow A \circ B \in S_2 \Rightarrow$ Closer axiom holds

*) $\forall A, B, C \in S_2 \Rightarrow (A \circ B) \circ C = A \circ (B \circ C) \Rightarrow$ Associativety holds

Scanned by CamScanner

Composition function is always associative

*) $\forall A \in S_2 \; \exists e = P_1 \in S_2$ st. $A \circ P_1 = A = P_1 \circ A$

$\Rightarrow$ Id. axiom holds with Identity element P_1.

*) Inverse axiom holds as Inverse of P_1 is P_1

" " P_2 is P_2

*) $\forall A, B \in S_2 \Rightarrow A \circ B \neq B \circ A \Rightarrow$ Commutative holds

So, $(S_2, \circ)$ be an abelian gp.

(d) $S_3 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$

$\circ$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_4	P_3	P_6	P_5
P_3	P_3	P_5	P_1	P_6	P_2	P_4
P_4	P_4	P_6	P_2	P_5	P_1	P_3
P_5	P_5	P_3	P_6	P_1	P_4	P_2
P_6	P_6	P_4	P_5	P_2	P_3	P_1

$\Rightarrow \forall A, B \in S \Rightarrow A \circ B \in S \Rightarrow$ Closer axiom holds

$\Rightarrow \forall A, B, C \in S \Rightarrow (A \circ B) \circ C = A \circ (B \circ C) \Rightarrow$ Ass. axiom holds

$\Rightarrow \forall A \in S \; \exists e = P_1$, st. $A \circ P_1 = A = P_1 \circ A$

$\Rightarrow$ Id. axiom holds with Identity element P_1.

$\Rightarrow$ Inverse axiom holds as Inverse of P_1 is P_1

P_2 is P_2

P_3 is P_3

P_4 is P_5

P_5 is P_4

P_6 is P_6

LectureNotes.in

Scanned by CamScanner

*) $\forall A, B \in S \Rightarrow A \circ B \neq B \circ A$

$\Rightarrow$ ~~Do~~ Does not hold commutative axiom

$\Rightarrow$ It is not an abelian group.

~~It is~~ So $(S_3, \circ)$ is a group.

Eg $\Rightarrow$ (1) $S = \{1, 2, 3, 6, 12\}$, where $a * b = GCD(a, b)$

GCD	1	2	3	6	12
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
6	1	2	3	6	6
12	1	2	3	6	12

*) $\forall a, b \in S \Rightarrow a * b \in S \Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in S \Rightarrow (a * b) * c = a * (b * c) \Rightarrow$ Asso. holds

OR

*) $\forall a \in S, \exists$

$a * (b * c) = a * GCD(b, c) = GCD(a, b, c)$

$(a * b) * c = GCD(a, b) * c = GCD(a, b, c)$

$a * (b * c) = (a * b) * c$

$\Rightarrow$ holds

*) $\forall a \in S, \exists e = 12 \in S$ for which

$a * e = a * 12 = GCD(a, 12) = a$

$e * a = 12 * a = GCD(12, a) = a$

Identity axiom holds with Id. element 12

*) Inverse axiom fails as there is no Inverse of 1, 2, 3, 6

*) $\forall a, b \in S$, $a * b = GCD(a, b) = GCD(b, a) = b * a$

Comm. axiom holds.

Thus $(S, *)$ be a Commutative Monoid.

Scanned by CamScanner

(2) $S = \{1, 2, 3, 6, 9, 18\}$

$a * b = LCM(a, b)$

	1	2	3	6	9	18
1	1	2	3	6	9	18
2	2	2	6	6	18	18
3	3	6	3	6	9	18
6	6	6	6	6	18	18
9	9	18	9	18	9	18
18	18	18	18	18	18	18

*) $\forall a, b \in S \Rightarrow a * b = LCM(a, b) \in S$

$\Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in S \Rightarrow a * (b * c) = a * LCM(b, c) = LCM(a, b, c)$

$(a * b) * c = LCM(a, b) * c = LCM(a, b, c)$

$\Rightarrow a * (b * c) = (a * b) * c$

$\Rightarrow$ Associ. axiom holds

*) $\forall a \in S, \exists e = 1 \in S$, for which

$a * e = a * 1 = LCM(a, 1) = a$

$e * a = 1 * a = LCM(1, a) = a$

Id. axiom holds with Id. element 1

*) Inverse axiom ~~holds as~~ fails as there is no inverse of (2, 3, 6, 9, 18)

*) $\forall a, b \in S$, $a * b = LCM(a, b) = LCM(b, a) = b * a$

Comm. holds

$\Rightarrow$ $\boxed{\text{Thus } (S, *) \text{ be a Commutative Monoid.}}$

Scanned by CamScanner

Type II - Problem (Infinite Set) ⇒ i.e. set is infinite & operation is not basic

Eg ⇒ ① On $\mathbb{Z}$, $a * b = a + b - ab$

$\mathbb{Z} = \{0, \pm 1, \pm 2 - \text{———} \}$

*) $\forall a, b \in \mathbb{Z} \Rightarrow a+b \in \mathbb{Z}$ & $ab \in \mathbb{Z}$

$\Rightarrow a + b - ab \in \mathbb{Z}$

$\Rightarrow a * b \in \mathbb{Z}$

Closer axiom holds.

*) $\forall a, b, c \in \mathbb{Z}$

$a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - ac - bc + abc$

$(a * b) * c = (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - bc - ac + abc$

$\therefore a * (b * c) = (a * b) * c$

$\Rightarrow$ Ass. axiom holds

*) Let the identity axiom holds

$\forall a \in \mathbb{Z}, \exists e \in \mathbb{Z}$ s.t $a * e = a = e * a$

$a * e = a$

$a + e - ae = a$

$e(1 - a) = 0$

$e = 0 \in \mathbb{Z}$

$e * a = a$

$e + a - ea = a$

$e(1 - a) = 0$

$e = 0 \in \mathbb{Z}$

So identity axiom holds with id: element 0.

*) Let inverse axiom holds

$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ s.t $a * b = e = b * a$

$a * b = 0 = b * a$

Scanned by CamScanner

$a * b = 0$

$a + b - ab = 0$

$b(1-a) = -a$

$b = \frac{a}{a-1} \notin Z$

as $\left[\text{if } a = 3 \quad b = \frac{3}{3-1} = \frac{3}{2} \notin Z \right]$

$\Rightarrow$ Inverse axiom fails.

$*)$ $\forall a, b \in Z$, $a * b = a + b - ab$

$= b + a - ba$

$a * b = b * a$

$\Rightarrow$ Commutative axiom holds

$\boxed{\text{Thus } (Z, *) \text{ be a commutative Monoid.}}$

(2) On $\mathbb{R}$, $a * b = a + b + 2$

(3) On $\mathbb{R} - \{-1\}$, $a * b = a + b + ab$

(4) On Z, $a * b = a^b$

(5) On $\mathbb{R}$, $a * b = \frac{ab}{2}$

(6) On N, $a * b = a\sqrt{b}$

(2) on $\mathbb{R}$, $a * b = a + b + 2$

$*)$ $\forall a, b \in \mathbb{R}$, $a * b = a + b + 2 \in \mathbb{R}$

$= a + b + 2 \in \mathbb{R}$

$= a * b \in R$

$\Rightarrow$ Closure axiom holds

$*)$ $\forall a, b, c \in \mathbb{R}$, $a * (b * c) = a * (b + c + 2)$

$= a + b + c + 2$

$(a * b) * c = (a * b + 2) * c$

$= a + b + 2 + c$

Scanned by CamScanner

$a * (b * c) = (a * b) * c$

$\Rightarrow$ Ass. holds.

*) Let the Identity axiom holds

$\Rightarrow a * e = a = e * a$.

$a * e = a$ $\qquad$ $e * a = a$

$a + e + 2 = a$ $\qquad$ $e + a + 2 = a$

$e = -2 \in \mathbb{R}$ $\qquad$ $e = -2 \in \mathbb{R}$

$\Rightarrow$ Id. axiom holds with identity element -2.

*) Let Inverse axiom holds

$\Rightarrow a * b = e = b * a$

$a * b = -2$ $\qquad$ $b * a = e$

$a + b + 2 = -2$ $\qquad$ $b + a + 2 = -2$

$a + b = -4$ $\qquad$ $b + a = -4$

$b = -(4 + a) \in \mathbb{R}$ $\qquad$ $b = -(4 + a) \in \mathbb{R}$

$\Rightarrow$ Inverse axiom holds

*) $\forall a, b \in \mathbb{R}$; $a * b \in \mathbb{R}$

$= a + b + 2 \in \mathbb{R}$

$= b + a + 2 \in \mathbb{R}$

$a * b = b * a \in \mathbb{R}$

$\Rightarrow$ Commutative axiom holds

Thus $(\mathbb{R}, *)$ is an abelian group.

(3) On $\mathbb{R} - \{-1\}$, $a * b = a + b + ab$

*) $\forall a, b \in \mathbb{R} - \{-1\}$

$\cancel{a + b \in \mathbb{R} - \{-1\}}$

$\cancel{ab \in \mathbb{R} - \{-1\}}$

$\Rightarrow a + b + ab \in \mathbb{R} - \{-1\}$

$\Rightarrow$ Closure axiom holds as $a * b \in \mathbb{R} - \{-1\}$

Scanned by CamScanner

*) $\forall a, b, c \in \mathbb{R}-\{-1\}$

$$a * (b * c) = a * (b+c+bc)$$
$$= a+b+c+bc+a(b+c+bc)$$
$$= a+b+c+ab+bc+ac+abc$$

$$(a*b)*c = (a+b+ab)*c$$
$$= a+b+ab+c+(a+b+ab)c$$
$$= a+b+c+ab+bc+ac+abc$$

$\Rightarrow a*(b*c) = (a*b)*c$

$\Rightarrow$ Associativity holds.

*) Let the identity axiom holds $\forall a \in \mathbb{R}-\{-1\}$

$\exists e \in \mathbb{R}-\{-1\}$ s.t $a*e = a = e*a$

$$a*e = a$$
$$a+e+ae = a$$
$$e(1+a) = 0$$
$$\cancel{e = 0 \in \mathbb{R}-\{-1\}}$$
$$e = \frac{0}{1+a} \in \mathbb{R}-\{-1\}$$

$$e*a = a$$
$$e+a+ea = a$$
$$e(1+a) = 0$$
$$e = \frac{0}{1+a} \in \mathbb{R}-\{-1\}$$

$\Rightarrow$ Identity axiom holds with identity element 0.

*) Let the inverse axiom holds $\forall a \in \mathbb{R}-\{-1\}$, $\exists b \in \mathbb{R}-\{-1\}$

s.t $a*b = e = b*a$

$$a*b = 0$$
$$a+b+ab = 0$$
$$b(1+a) = -a$$
$$b = \frac{-a}{1+a} \in \mathbb{R}-\{-1\}$$

$$b*a = 0$$
$$b+a+ba = 0$$
$$b(1+a) = -a$$
$$b = -\frac{a}{1+a} \in \mathbb{R}-\{-1\}$$

if $a=0$

$\Rightarrow$ Inverse axiom holds

*) $\forall a, b \in \mathbb{R}-\{-1\}$, $a*b = a+b+ab$

$$= b+a+ba$$
$$a*b = b*a$$

$\Rightarrow$ Commutative axiom holds

Thus $(\mathbb{R}-\{-1\}, *)$ be an abelian group.

LectureNotes.in

Scanned by CamScanner

(4) On $\mathbb{Z}$, $a * b = a^b$

*) $\forall\, a, b \in \mathbb{Z}$, $a^b \notin \mathbb{Z}$

$$\left[\begin{array}{l} \text{as for } a = 2,\ b = -1 \\ a^b = 2^{-1} = \frac{1}{2} \notin \mathbb{Z} \end{array}\right]$$

$\Rightarrow a * b \notin \mathbb{Z}$

$\Rightarrow$ Closer axiom does not holds

(as closer does not holds so we do not go on next steps)

$\Rightarrow$ $\mathbb{Z}$ is not an algebric structure.

23.4

*) $\forall\, a, b, c \in \mathbb{Z}$, $a * (b * c) = a * (b^c)$

$= a^{b^c}$

$(a * b) * c = (a^b) * c$

$= (a^b)^c$

$= a^{bc}$

$a^{b^c} \neq a^{bc}$

$\Rightarrow$ Associativy axiom does not holds

*) Let the Identity axiom holds.

i.e. $\forall\, a \in \mathbb{Z}$, $\exists\, e \in \mathbb{Z}$ s.t $a * e = a = e * a$

$a * e = a$

$a^e = a$

$\Rightarrow e = 1 \in \mathbb{Z}$

$e * a = a$

$e^a = a$

Identity axiom does not holds

LectureNotes.in

Scanned by CamScanner

⑤ on $\mathbb{R}$, $a * b = \frac{ab}{2}$

*) Let $a, b \in \mathbb{R}$, $ab \in \mathbb{R}$

$\frac{ab}{2} \in \mathbb{R}$

$\Rightarrow a * b \in \mathbb{R}$

$\Rightarrow$ Closer axiom holds.

*) Let $a, b, c \in \mathbb{R}$

$a * (b * c) = a * \frac{bc}{2}$

$= \frac{abc}{4}$

$(a * b) * c = \frac{ab}{2} * c$

$= \frac{abc}{4}$

$a * (b * c) = (a * b) * c$

$\Rightarrow$ Ass. axiom holds

*) Let Identity axiom holds.

i.e. $\forall a \in \mathbb{R}, \exists e \in \mathbb{R}$ s.t

$a * e = a = e * a$

$a * e = a$

$\frac{ae}{2} = a$

$e = 2 \in \mathbb{R}$

$e * a = a$

$\frac{ea}{2} = a$

$e = 2 \in \mathbb{R}$

$\Rightarrow$ Identity axiom holds with identity element $e = 2$.

*) Let Inverse axiom holds

i.e $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ s.t $a * b = e = b * a$

$a * b = 2$

$\frac{ab}{2} = 2$

$b = \frac{4}{a} \notin \mathbb{Z}$ $\left[\begin{array}{l} \text{as for } a = 0 \\ b = \frac{4}{0} \notin \mathbb{Z} \end{array}\right]$

Scanned by CamScanner

$\Rightarrow$ Inverse axiom does not holds

$\cdot\times$) $\forall a, b \in \mathbb{Z}$, $a * b = \frac{ab}{2}$

$= \frac{ba}{2}$

$a * b = b * a$

$\Rightarrow$ Commutative axiom holds.

$\Rightarrow$ Thus, $(\mathbb{Z}, *)$ be a Commutative Monoid.

(6) On N, $a * b = a\sqrt{b}$

$*$) $\forall a, b \in N$ $\quad a \in N$, $\sqrt{b} \notin N$ $\quad$ [as for $b = 2$, $\sqrt{2} \notin N$]

$\Rightarrow a\sqrt{b} \notin N$

$\Rightarrow$ Closer axiom does not holds

$\Rightarrow$ So, $(N, *)$ is not an Algebraic structure w.r.t $*$

$*$) $\forall a, b, c \in N$

$a * (b * c) = a * b\sqrt{c}$ $\quad$ 1, 4, 9 / 1, 9, 16 / $\sqrt{3 \times 4}$

$= a\sqrt{b\sqrt{c}}$

$(a * b) * c = (a\sqrt{b}) * c$

$= a\sqrt{b}\sqrt{c}$

for $a = 1$, $b = 9$, $c = 16$

$a * (b * c) = a\sqrt{b\sqrt{c}} = 1\sqrt{9\sqrt{16}} = \sqrt{9 \times 4} = 6$

$(a * b) * c = a\sqrt{b}\sqrt{c} = 1 \times \sqrt{9}\sqrt{16} = 1 \times 3 \times 4 = 12$

$\therefore$ $a * (b * c) \neq (a * b) * c$

$\Rightarrow$ Asso. does not holds

$*$) Let Identy axiom holds

i.e. $\forall a \in N$, $\exists e \in N$ s.t $a * e = a = e * a$

$a * e = a$ $\quad$ $e * a = a$

~~$a + e + ae$~~

$a\sqrt{e} = a$ $\quad$ $e\sqrt{a} = a$

$e = 1$ $\quad$ $e^2 a = a^2$

$e = \sqrt{a}$

$\Rightarrow$ Identity axiom does not holds.

Scanned by CamScanner

*) $\forall\, a, b \in N$, $a * b = a\sqrt{b}$

$\neq b\sqrt{a}$

$\neq b * a$

$\left[\begin{array}{l} \text{for } a = 4,\ b = 9 \\ a\sqrt{b} = 4 \times 3 = 12 \\ b\sqrt{a} = 9 \times 2 = 18 \end{array}\right]$

$\Rightarrow$ Commutative Axiom does not holds.

(7) On $\mathbb{R}$, $a * b = a^b$

*) $\forall\, a, b \in \mathbb{R}$, $a * b = a^b \notin R$

as $\left[\begin{array}{l} a = -1,\ b = 1/2 \\ a * b = (-1)^{1/2} = \sqrt{-1} \end{array}\right]$

(8) S be set of all 2×1 matrices.

i.e $S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \ \forall\, a, b \in R \right\}$

$A * B = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{bmatrix} a+c \\ b+d+1 \end{bmatrix}$

Let $A = \begin{pmatrix} a \\ b \end{pmatrix}$; $B = \begin{pmatrix} c \\ d \end{pmatrix}$, $C = \begin{pmatrix} e \\ f \end{pmatrix}$

*) $\forall\, A, B \in S$

$\Rightarrow A * B = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d+1 \end{pmatrix} \in S$

$\Rightarrow$ Closer axiom holds

*) $\forall\, A, B, C \in S$

$A * (B * C) = A * \begin{pmatrix} c \\ d \end{pmatrix} * \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c+e \\ d+f+1 \end{pmatrix} = \begin{pmatrix} a+c+e \\ b+d+f+2 \end{pmatrix}$

Scanned by CamScanner

$$(A*B)*C = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} * C = \begin{pmatrix} a+c \\ b+d+1 \end{pmatrix} * \begin{pmatrix} e \\ f \end{pmatrix}$$

$$= \begin{pmatrix} a+c+e \\ b+d+f+2 \end{pmatrix}$$

$$\text{or} \quad = A * (B * C)$$

So associativity holds.

*) Let identity element exist

s.t. $\forall A \in S$, $\exists E = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in S$ for which

$$A * E = A = E * A$$

$$A * E = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = A$$

$$\Rightarrow \begin{pmatrix} a+e_1 \\ b+e_2+1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$\Rightarrow a+e_1 = a \Rightarrow e_1 = 0$

$b+e_2+1 = b \Rightarrow e_2 = -1$

$$\Rightarrow E = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in S$$

$$E * A = A$$

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} * \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} e_1+a \\ e_2+b+1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$e_1+a = a \Rightarrow e_1 = 0$

$e_2+b+1 = b \Rightarrow e_2 = -1$

$$\Rightarrow E = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in S$$

$\Rightarrow$ Id. axiom holds with Id. element $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

*) Let inverse axiom holds

so. $\forall A \in S$, $\exists B \in S$ for which

$$A * B = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = B * A$$

$$A * B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$a+c = 0$; $b+d+1 = -1$

$c = -a$ $\quad$ $d = -2-b$

$$B * A = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} c \\ d \end{pmatrix} * \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$c+a = 0$; $d+b+1 = -1$

$c = -a$ $\quad$ $d = -2-b$

Scanned by CamScanner

i.e. $B = \begin{pmatrix} -a \\ -2-b \end{pmatrix} \in S$ | $B = \begin{pmatrix} -a \\ -2-b \end{pmatrix} \in S$

$\Rightarrow$ Inverse axiom holds.

Inverse of $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\begin{pmatrix} -a \\ -b-2 \end{pmatrix}$

$*$> $\forall A, B \in S$, $A * B = \begin{pmatrix} a \\ b \end{pmatrix} * \begin{pmatrix} c \\ d \end{pmatrix}$

$$= \begin{pmatrix} a+c \\ b+d+1 \end{pmatrix}$$

$$= \begin{pmatrix} c+a \\ d+b+1 \end{pmatrix}$$

$$= \begin{pmatrix} c \\ d \end{pmatrix} * \begin{pmatrix} a \\ b \end{pmatrix}$$

$A * B = B * A$

$\Rightarrow$ Comm. holds.

Thus $(S, *)$ be an abelian gp.

Some Type 1 Problem

Eg ① Let $A = \{a, b\}$

(i)

$*$	a	b
a	a	b ✓
b	a	a

(ii)

$*$	a	b ✓
a	a	b ✓
b	b	b

What axiom holds.

(i)

$a * a = a$

$a * b = b$

$b * a = a$

$b * b = a$

Scanned by CamScanner

Closer

$*>$ $\forall\ x, y \in A$ $\qquad \forall\ x, y \in A$

$x * y \in A$ $\qquad x * y \in A$

$\Rightarrow$ Closer axiom holds.

Associativity

$*>$ $\begin{cases} a * (a * a) = a * a = a \\ (a * a) * a = a * a = a \end{cases}$

$3! = 6$ way

$*)$ $\begin{cases} a * (a * b) = a * b = b \\ (a * a) * b = a * b = b \end{cases}$

$*>$ $\begin{cases} a * (b * a) = a * a = a \\ (a * b) * a = b * a \quad a \end{cases}$

$*)$ $\begin{cases} b * (b * b) = b * a = a \\ (b * b) * b = a * b \quad b \end{cases}$

Ass. fails.

Thus $(A, *)$ be an algebraic structure

(ii) $a * a = a$

$a * b = b$

$b * a = b$

$b * b = b$

Closer

$\forall\ x, y \in A$

$x * y \in A$

$\Rightarrow$ Closer axiom holds

Associativity

$*)$ $\begin{cases} a * (a * a) = a * a = a \\ (a * a) * a = a * a = a \end{cases}$

$*)$ $\begin{cases} a * (a * b) = a * b = b \\ (a * a) * b = a * b = b \end{cases}$

LectureNotes.in

Scanned by CamScanner

*) $\begin{cases} a * (b * a) = a * b = b \\ (a * b) * a = b * a = b \end{cases}$

*) $\begin{cases} b * (b * b) = b * b = b \\ (b * b) * b = b * b = b \end{cases}$

*) $\begin{cases} b * (a * b) = b * b = b \\ (b * a) * b = b * b = b \end{cases}$

*) $\begin{cases} b * (b * a) = b * b = b \\ (b * b) * a = b * a = b \end{cases}$

$\Rightarrow$ Associativity holds.

*) <u>Identity</u>

Let Id. axiom holds.

i.e. $\forall a, b \in A$, $\exists e \in A$ s.t

$a * b = e = b * a$

$a * b = e$ $\quad$ $b * a = e$

$b = e$ $\quad$ $b = e$

Identity axiom holds with identity element a as $\forall x \in A$. $\exists e = a \in A$ for which $x * a = x = a * x$

*) Inverse axiom does not holds for b

i.e. Inverse axiom fails.

*) $\forall x, y \in A$

$x * y = y * x$

(as matrix is symmetric)

$\Rightarrow$ commutative axiom holds

Thus $(A, *)$ be a commutative Monoid.

Scanned by CamScanner

Type 3 ⇒ Infinite set Basic operator

Eg ⇒ (1) (i) $(N, +)$ (ii) $(N, -)$, (iii) $(N, \times)$, (iv) $(N, \div)$

(2) (i) $(N^*, +)$ (ii) $(N^*, -)$, (iii) $(N^*, \times)$ (iv) $(N^*, \div)$

(3) (i) $(Z, +)$ (ii) $(Z, -)$ (iii) $(Z, \times)$ (iv) $(Z, \div)$

(4) (i) $(Q, +)$ (ii) $(Q, -)$ (iii) $(Q, \times)$ (iv) $(Q, \div)$

(5) (i) $(R, +)$ (ii) $(R, -)$ (iii) $(R, \times)$ (iv) $(R, \div)$

(6) (i) $(C, +)$ (ii) $(C, -)$ (iii) $(C, \times)$ (iv) $(C, \div)$

(1) (i) $(N, +)$

*) $\forall a, b \in N$, $a+b \in N$

$\Rightarrow a*b \in N$

$\Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in N$,

$a*(b*c) = a*b+c = a+b+c$

$(a*b)*c = (a+b)*c = a+b+c$

$\Rightarrow$ Ass. holds

*) Let Iden. axiom holds.

i.e. $\forall a \in N, \exists e \in N$ s.t. $a*e = a = e*a$

$a*e = a$

$a+e = a$

$e = 0 \notin N$

$e*a = a$

$e+a = a$

$e = 0 \notin N$

$\Rightarrow$ Identity does not holds.

$\boxed{(N, +) \text{ be a semigroup}}$

(ii) $(N, -)$

$\forall a, b \in N$, $a-b \notin N$

as for $a=1$, $b=2$

$a-b = -1 \notin N$

$\Rightarrow$ Closer does not holds

$\Rightarrow$ $\boxed{(N, -) \text{ is not an algebraic structure.}}$

Scanned by CamScanner

(iii) $(N, \times)$

*) $\forall a, b \in N$, $a \times b \in N$

$\Rightarrow a \times b \in N$

$\Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in N$, $a \times (b \times c) = a \times (b \times c)$

$= a \times b \times c$

$(a \times b) \times c = (a \times b) \times c$

$= a \times b \times c$

$\Rightarrow a \times (b \times c) = (a \times b) \times c$

$\Rightarrow$ Ass. holds

*) Let Identity axioms holds $a * e = a = e * a$

i.e. $\forall a \in N$, $\exists e \in N$ s.t.

$a * e = a$ $\quad e \times a = a$

$a \times e = a$ $\quad e \times a = a$

$e = 1$ $\quad e = 1$

Identity axiom holds with identity element 1.

(iv) $(N, \times)$

*) Let Inverse axiom holds $a * b = e = b * a$

i.e. $\forall a \in N$, $\exists b \in N$ s.t.

$a * b = e$

$a \times b = 1$

$b = \frac{1}{a} \notin N$

$\left(\begin{array}{l} \text{for } a = 2 \in N \\ b = 1/2 \notin N \end{array} \right)$

$\Rightarrow$ Inverse axiom does not holds.

*) $\forall a, b \in N$, $a * b \in N$

$a * b = a * b$

$= b * a$ $\Rightarrow$ Commut. axiom holds

$a * b = b * a \in N$

$\Rightarrow$ $\boxed{(N, *) \text{ be commutative Monoid}}$

Scanned by CamScanner

(iv) $(N, \div)$

*) $\forall a, b \in N$, $a \div b \notin N$

as for $a = 1$, $b = 2$

$\frac{a}{b} = \frac{1}{2} \notin N$

$\Rightarrow$ Closer axiom does not holds

$\Rightarrow (N, \div)$ ~~be~~ is not an abelian ~~gp.~~ abelian.

LectureNotes.in

(2) (i) $(N^*, +)$

*) $\forall a, b \in N^*$, $a + b \in N$

$a * b \in N$

$\Rightarrow$ closer axiom holds.

*) $\forall a, b, c \in N^*$, $a * (b * c) = a * (b + c)$

$= a + b + c$

$(a * b) * c = (a + b) * c$

$= a + b + c$

$a * (b * c) = (a * b) * c$

$\Rightarrow$ Ass. axiom holds

LectureNotes.in

*) Let identity axiom holds.

i.e. $\forall a \in N^*$, $\exists e \in N^*$ s.t. $a * e = a = e * a$

$a * e = a$

$a + e = a$

$e = 0 \in N^*$

$e * a = a$

$e + a = a$

$e = 0 \in N^*$

LectureNotes.in

Identity axiom holds with identity element 0.

*) Let inverse axiom holds.

i.e. $\forall a \in N^*$, $\exists b \in N^*$ s.t. $a * b = e = b * a$

$a * b = e$

$a + b = 0$

$b = -a \notin N^*$

$b * a = e$

$b + a = 0$

$b = -a$

$\left\{ \begin{array}{l} \text{for } a = 1 \\ b = -1 \in N \end{array} \right\}$ $\Rightarrow$ Inverse axiom does not holds

Scanned by CamScanner

*) $\forall a, b \in N^*$ $a+b \in N^*$

$= b+a$

$a+b = b+a \in N^*$

Commutative holds.

$\Rightarrow$ $\boxed{(N^*, +) \text{ be a monoid. Commutative Monoid}}$

(ii) $(N^*, -)$

*) $\forall a, b \in N^*$, $a-b \notin N^*$

for $a = 1$, $b = 2$

$a-b = -1 \notin N^*$

$\Rightarrow$ Closer axiom does not holds

$\boxed{(N^*, -) \text{ is not an algebraic structure.}}$

(iii) $(N^*, \times)$

*) $\forall a, b \in N^*$, $a \times b \in N^*$

$a \times b \in N^*$

$\Rightarrow$ Closer axiom holds

*) $\forall a, b, c \in N^*$

$a*(b*c) = a*(b*c)$

$= a \times b \times c$

$(a*b)*c = (a \times b)*c$

$= a \times b \times c$

$\Rightarrow$ Ass. axiom holds

*) Let Iden. axiom holds

same as Q (i) (iii)

$\Rightarrow$ Identity axiom holds with identity element 1

*) Inverse does not holds same as Q 1 (iv)

*) $\forall a, b \in N^*$, $a \times b \in N^*$

$b \times a \in N^*$ $\Rightarrow$ Com. axiom holds

$a*b = b*a \in N^*$

$\Rightarrow$ $\boxed{(N^*, \times) \text{ be a commutative Monoid.}}$

Scanned by CamScanner

(iv) $(N^+, \div)$

*) $\forall a, b \in N^+$

$a \div b \notin N^+$

as for $a = 1$, $b = 2$

$\frac{1}{2} \notin N^+$

$\Rightarrow$ Closer axiom does not holds.

$\Rightarrow$ $\boxed{(N^+, \div) \text{ is not an algebraic structure.}}$

(3) (i) $(\mathbb{Z}, +)$

*) Closer holds same as Q (1) (i)

*) Ass. axiom holds

*) Id. axiom holds with identity element 0. same as Q2 (i)

*) Let Inverse axiom holds

i.e. $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$ s.t $a * b = 0 = b * a$

$a + b = 0$

$b = -a \in \mathbb{Z}$

$b + a = 0$

$b = -a \in \mathbb{Z}$

$\Rightarrow$ Inverse axiom holds, Inverse of a is $-a$

*) Commutative axiom holds as in Q (1) (i)

$\boxed{\text{Thus } (\mathbb{Z}, +) \text{ be an abelian group.}}$

(ii) $(\mathbb{Z}, -)$

*) $\forall a, b \in \mathbb{Z}$, $a - b \in \mathbb{Z}$

$a * b \in \mathbb{Z}$

Closer axiom holds.

*) $\forall a, b, c \in \mathbb{Z}$, $a * (b * c) = a * (b - c)$

$= a - b - c$

$(a * b) * c = (a - b) * c$

$= a - b - c$

$\Rightarrow$ Ass. axiom holds.

Scanned by CamScanner

* Let Identity element axiom holds.

i.e. $\forall a \in \mathbb{Z}$, $\exists e \in \mathbb{Z}$ s.t. $a * e = a = e * a$

$a * e = a$

$a - e = a$

$e = 0$

$e * a = a$

$e - a = a$

$e = 2a$

$\Rightarrow$ Identity axiom does not holds.

$\Rightarrow$ $\boxed{(\mathbb{Z}, -) \text{ be a semigroup.}}$

(iii) $(\mathbb{Z}, *)$

*) Closer axiom holds

*) Associativity holds

*) Identity axiom holds with Identity element 1.

*) Let Inverse axiom holds

i.e. $\forall a \in \mathbb{Z}$, $\exists b \in \mathbb{Z}$ s.t. $a * b = 1 = b * a$

$a * b = 1$

$a \times b = 1$

$b = \frac{1}{a}$

for $a = 2$, $b = \frac{1}{2} \notin \mathbb{Z}$

$\Rightarrow$ Inverse axiom does not holds.

*) Commutative axiom holds.

$\Rightarrow$ $\boxed{(\mathbb{Z}, *) \text{ be a commutative monoid}}$

(iv) $(\mathbb{Z}, \div)$

*) $\forall a, b \in \mathbb{Z}$, $\frac{a}{b} \notin \mathbb{Z}$

as for $a = 1$, $b = 2$, $\frac{1}{2} \notin \mathbb{Z}$

$\Rightarrow$ Closer axiom ~~holds~~ does not holds

$\Rightarrow$ $\boxed{\mathbb{Z} \text{ is not an algebraic structure on } \div}$

Scanned by CamScanner

(4) (i) $(Q, +)$

*) Closer axiom holds

*) Ass. axiom holds

*) Identity axiom holds with identity element 0.

*) Inverse axiom holds, ~~and~~ Inverse of a is $-a$

*) Commutative axiom holds.

$\Rightarrow$ $\boxed{(Q,+) \text{ be an abelian gp.}}$

(ii) $(Q, -)$

*) Closer axiom holds.

*) Ass. axiom holds.

*) Identity does not holds

$\Rightarrow$ $\boxed{(Q, -) \text{ be a semigroup.}}$

(iii) $(Q, *)$

*) Closer axiom holds

*) Ass. axiom holds

*) Identity axiom holds with identity element 1

*) Let Inverse axiom holds.

i.e. $\forall a \in Q, \exists b \in Q$ s.t $a*b = 1 = b*a$

$a*b = 1$

$a \times b = 1$

$b = \frac{1}{a} \notin Q$

as for $a = 0$

$b = \frac{1}{0} \notin Q$

$\Rightarrow$ Inverse axiom does not holds.

*) Commutative axiom holds

$\Rightarrow$ $\boxed{(Q, *) \text{ be a commutative monoid}}$

Scanned by CamScanner

(ii) $(\mathbb{Q}, \div)$

*) $\forall a, b \in \mathbb{Q}$, $\frac{a}{b} \notin \mathbb{Q}$

as for $b = 0$

$\frac{a}{0} \notin \mathbb{Q}$

$\Rightarrow$ Closer axiom does not holds

$\Rightarrow$ $\boxed{\mathbb{Q} \text{ is not an algebraic structure on } \div}$

5 (i) $(\mathbb{R}, +)$

*) Closer axiom holds

*) Ass. axiom holds

*) Identity axiom holds with identity element 0.

*) Inverse axiom holds with Inverse of a is -a.

*) Commutative axiom holds.

$\Rightarrow$ $\boxed{(\mathbb{R}, +) \text{ be an abelian group}}$

(ii) $(\mathbb{R}, -)$

*) Closer axiom holds

*) Asso. axiom holds

*) Identity axiom does not holds

$\boxed{(\mathbb{R}, -) \text{ be a semigroup}}$

(iii) $(\mathbb{R}, *)$

*) Closer axiom holds

*) Ass. axiom holds

*) Identity axiom holds with identity element 1.

*) Inverse axiom does not holds as $\left(\frac{1}{a} \notin \mathbb{R} \text{ for } a = 0\right)$

*) Commutative axiom

$\Rightarrow$ $\boxed{(\mathbb{R}, *) \text{ be commutative Monoid}}$

LectureNotes.in

Scanned by CamScanner

(ii) $(\mathbb{R}, \div)$

*) Closer axiom does not holds as $\frac{a}{0} \notin \mathbb{R}$

$\Rightarrow$ $(R, \div)$

$\Rightarrow$ $\boxed{\text{R is not an algebraic structure on } \div}$

(6) (i) $(\mathbb{Z}, +)$

*) $\forall a, b \in \mathbb{Z}$,

LectureNotes.in

Scanned by CamScanner

24/9/16

MODULO OPERATION ⇒

① Addition Modulo ⇒ Let $Z_m = \{0, 1, 2, \ldots m-1\}$

$\oplus_n$ denotes for "Addition modulo 'n'".

where

$$a \oplus b_n = \begin{cases} a+b, & \text{if } a+b < n \\ a+b-n, & \text{if } a+b \geq n \end{cases}$$

② Multiplication Modulo ⇒

$\otimes_n$ denotes for "Multiplication modulo n"

where

$a \otimes b_n$ = The remainder obtained when ab is divided by n

Eg ⇒

① $Z_2 = \{0, 1\}$

$\oplus_2$	0	1
0	0	1
1	1	0

$\otimes_2$	0	1
0	0	0
1	0	1

② $Z_3 = \{0, 1, 2\}$

$\oplus_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\otimes_3$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

③ $Z_4 = \{0, 1, 2, 3\}$

$\oplus_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\otimes_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

LectureNotes.in

Scanned by CamScanner

Verification →

Eg → (i) $(Z_2, \oplus_2)$

→ $\forall a, b \in Z_2 \Rightarrow a * b = a \oplus b_2 \in Z_2$

Closer axiom holds.

→ $\forall a, b, c \in Z_2$

$\Rightarrow (a \oplus_2 b) \oplus_2 c = a \oplus_2 (b \oplus_2 c)$

Ass. axiom holds

→ $\forall a \in Z_2, \exists e = 0 \in Z_2$ s.t

$a \oplus_2 0 = a = 0 \oplus_2 a$

Id. axiom holds with Id. element '0'.

→ Inverse axiom holds as Inverse of 0 is 0
Inverse of 1 is 1.

→ $\forall a, b \in Z_2$, $a \oplus b_2 = b \oplus a_2$

Comm. axiom holds.

$\boxed{\text{Thus } (Z_2, \oplus_2) \text{ be an abelian group}}$

(2) $(Z_2, \otimes_2)$

→ $\forall a, b \in Z_2 \Rightarrow a * b = a \otimes_2 b \in Z_2$

Closer axiom holds.

→ $\forall a, b, c \in Z_2 \Rightarrow (a \otimes_2 b) \otimes_2 c = a \otimes_2 (b \otimes_2 c)$

Ass. axiom holds.

→ $\forall a \in Z_2$, $\exists e = 1 \in Z_2$ st.

$a \otimes_2 1 = a = 1 \otimes_2 a$

Id. axiom holds with Id. element 1.

→ Inverse axiom ~~ho~~ fails, there is no Inverse of 0.

→ $\forall a, b \in Z_2 \Rightarrow a \otimes_2 b = b \otimes_2 a$ → commutative axiom holds

$\boxed{\text{Thus } (Z_2, \otimes_2) \text{ be a commutative monoid.}}$

Scanned by CamScanner

Group Properties ⇒

Let $(G, \circ)$ be a gp.

$\forall\ a, b, c \in G$

① Cancellation laws

$$\boxed{\begin{array}{ll} \text{(i)}\ a \circ b = a \circ c \Rightarrow b = c & [\text{L.C.L}] \\ \text{(ii)}\ a \cdot b = c \cdot b \Rightarrow a = c & [\text{R.C.L} \end{array}}$$

Proof ⇒

(i) Since $(G, \circ)$ be a group

⇒ Inverse axiom holds

ie. $\forall\ a \in G \Rightarrow a^{-1} \in G$

Given $a \circ b = a \circ c$

$a^{-1}(a \circ b) = a^{-1}(a \circ c)$ [Pre operating a^{-1}]

$(a^{-1} \cdot a) \cdot b = (a^{-1} \cdot a) \cdot c$ [Ass. Axiom]

$e \cdot b = e \cdot c$ [Inverse

$b = c$ [Identity

(ii) Sin $(G, \circ)$ be a gp.

ie. $\forall\ b \in G \Rightarrow b^{-1} \in G$

Given $a \cdot b = c \cdot b$

$(a \cdot b) b^{-1} = (c \cdot b) b^{-1}$ [Post operating b^{-1}]

$a \cdot (b \cdot b^{-1}) = c \cdot (b \cdot b^{-1})$ [Associativity

$a \cdot e = c \cdot e$ [Inverse]

$a = c$ [Identity.

LectureNotes.in

Scanned by CamScanner

② $\boxed{(a.b)^{-1} = b^{-1}.a^{-1}}$

Proof Since $(G,\circ)$ be a gr.

So by closer axiom

$\forall\ a,b \in G \Rightarrow a.b \in G$

from inverse axiom

$(a.b)\ (a.b)^{-1} = e$

$a^{-1}\ [(a.b).(a.b)^{-1}] = a^{-1}e$ [Pre operating a^{-1}

$(a^{-1}.a).[b.(ab)^{-1}] = a^{-1}e$ [Ass.

$e.[b.(ab)^{-1}] = a^{-1}e$ [Inverse

$b\ (ab)^{-1} = a^{-1}$. [Identity]

$b^{-1}.[b.(a.b)^{-1}] = b^{-1}a^{-1}$ [Pre operating b^{-1}]

$(b^{-1}.b).(a.b)^{-1} = b^{-1}a^{-1}$ [Ass.

$e.(a.b)^{-1} = b^{-1}a^{-1}$ [Inverse

$(a.b)^{-1} = b^{-1}a^{-1}$ [Identity

Hence proved

③ $\boxed{(a^{-1})^{-1} = a}$

Proof $\forall\ a \in G, \exists\ a^{-1} \in G$ s.t. $a.a^{-1} = e = a^{-1}.a$

Since $a^{-1} \in G \Rightarrow (a^{-1})^{-1} \in G$ s.t. $a^{-1}(a^{-1})^{-1} = e = (a^{-1})^{-1}.a^{-1}$

As $(a^{-1})^{-1}a^{-1} = e$

$[(a^{-1})^{-1}.a^{-1}].a = e.a$ [Post operating a

$(a^{-1})^{-1}.(a^{-1}.a) = e.a$ [Ass.

$(a^{-1})^{-1}\ e = ea$ [Inverse

$(a^{-1})^{-1} = a$ [Identity]

Hence Proved

LectureNotes.in

Scanned by CamScanner

(4) The Identity element 'e' of the gp $(G, \cdot)$ is unique.

Proof ⇒

If possible there exists two id. elements, e_1 & e_2 in G.

So, $\forall a \in G \Rightarrow$ $a \cdot e_1 = a = e_1 \cdot a$

$a \cdot e_2 = a = e_2 \cdot a$

$a \cdot e_1 = a \cdot e_2$ & $e_1 \cdot a = e_2 \cdot a$

$e_1 = e_2$ [L.C.L & $e_1 = e_2$ [R.C.L

$\Rightarrow e_1 = e_2$

(5) Inverse element of 'a' is unique.

Proof ⇒ If possible there exists two Inverse elements b & e of a.

So $a \cdot b = e$ & $a \cdot c = e$

also $a \cdot b = a \cdot c$

$b = c$ [LCL

Order of a Grp ⇒

Let $(G, \circ)$ be a group

The total no. of elements contained in G is its order & to be denoted by $O(G)$.

*) If $O(G)$ is finite then the gp. is a finite gp.

*) If $O(G)$ is infinite then the gp. is an Infinite gp.

Eg ⇒ (1) $G = \{1, -1\}$ be a gp. under ordinary multiplic^n

Here $O(G) = 2$

so it is a finite gp.

LectureNotes.in

Scanned by CamScanner

(11) $G = \mathbb{Z}$ be a gp under +

Here $O(G)$ = Infinite.

So $(\mathbb{Z}, +)$ be an Infinite Gp.

Order of an Element in a Group ⇒

→ Let $(G, \circ)$ be a gp with Id. element 'e' $\forall a \in G$, if $a^n = e$, for least positive Integer 'n' then $O(a) = n$

→ If there is no positive Integral of 'n', $a^n = e$ then $O(a)$ is zero or Infinity

→ Identity element 'e' is the only element of Order 1 in a gp.

Eg ⇒ (1) Let $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$

we have $O(a) = 8$ since $a^8 = e$

$O(a^2) = 4$ as $(a^2)^4 = a^8 = e$

$O(a^4) = 2$ as $(a^4)^2 = a^8 = e$

$O(a^3) = 0$ as there is no integer 'n' for which $(a^3)^n = e$

$O(a^5) = 0$ as there is no integer 'n' for which $(a^5)^n = e$

$O(a^6) = 0$ as there is no " " " " $(a^6)^n = e$

$O(a^7) = 0$ " " " " " $(a^7)^n = e$

(2) $G = \{\ldots a^{-4}, a^{-3}, a^{-2}, a^{-1}, a^0 = e, a, a^2, a^3, a^4 \ldots\}$ be a gp.

Here the order of every element is '0' or Infinite except a^0.

(3) $G = \{1, -1\}$ Be a gp. under multiplication. Here Id. element is 1.

So, $O(1) = 1$ as $1^1 = 1$

$O(-1) = 2$ as $(-1)^2 = 1$

Scanned by CamScanner

21/9/16

(4) $G = \{1, \omega, \omega^2\}$ be a gp. under multiplicⁿ. Here 3rd element is

So, $o(1) = 1$ as $1^1 = 1$

$o(\omega) = 2$ as $\omega^3 = 1$

$o(\omega^2) = 3$ as $(\omega^2)^3 = 1$

(5) $G = \{1, -1, i, -i\}$ be a gp. under multiplicⁿ with identity element

So, $o(1) = 1$ as $1^1 = 1$

$o(-1) = 2$ as $(-1)^2 = 1$

$o(i) = 4$ as $i^4 = 1$

$o(-i) = 4$ as $(-i)^4 = 1$

NOTE :-

*) The order of any element in a finite gp. is finite whereas some or all elements of an infinite gp. may be of finite order.

*) Order of 1st. element of every gp (finite of infinite) is One

Cyclic Group :- Let $(G, \circ)$ be a gp. with identity element 'e' for any $a \in G$, if all other elements can be written in form of 'a' then G is said to be a cyclic gp. generated by 'a' i.e. 'a' is to be the generator of G.

Eg :- (i) $G = \{1, -1\}$ be a gp. under multiplicⁿ.

Here $G = \{-1, (-1)^2 = 1\}$

So, $(G, \circ)$ be a ~~gp~~ cyclic gp. generated by (-1)

i.e. -1 is ~~its~~ its generator

Scanned by CamScanner

② $G = \{1, \omega, \omega^2\}$ be a gp. under ordinary multiplicⁿ.

Here $G = \{\omega, \omega^2, \omega^3 = 1\}$

So $(G, \circ)$ be a cyclic group generated by 'ω'.

③ $G = \{1, -1, i, -i\}$ be a gp. under multiplicⁿ.

Here $G = \{i, i^2 = 1, i^3 = -i, i^4 = 1\}$

So $(G, \circ)$ be a cyclic gp. generated by 'i'.

④ $(\mathbb{Z}, +)$ be a gp.

$\mathbb{Z} = \{\ldots -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$

Here it is a cyclic gp. generated by 1.

Sub Group ⇒

*> Let $(G, \circ)$ be a gp.

$\boxed{H \subseteq G \ \& \ H \neq \phi}$

If $(H, \circ)$ satisfies all the gp. axioms then H is said to be a Subgroup of the gp. G under $\circ$

*> $H = \{e\} \subseteq G$ & $H = G \subseteq G$ are two subgroups of $(G, \circ)$ & those two subgroups are said to be trivial subgroup

*) If there exists any other subgroup of $(G, \circ)$ except those two are said to be Non-trivial gp.

*) Any gp $(G, \circ)$ must have atleast 2 gps.

Eg ⇒ ① $G = \{1, -1\}$ be a gp. under multiplicⁿ with Iden. element 1.

$H_1 = \{1, 1\}$ → $\begin{array}{c|c} \circ & 1 \\ \hline 1 & 1 \end{array}$ ✓

$H_2 = \{-1\}$ → $\begin{array}{c|c} \circ & -1 \\ \hline -1 & 1 \end{array}$ ✗

$H_3 = \phi$

$H_4 = \{1, -1\}$

So H_1 & H_4 are the subgroups of G under $\circ$ called as trivial sgs.

But H_2 & H_3 does not satisfy gp. axioms.

So H_2 & H_3 are not subgroups of G.

There is no non-trivial sgs.

Scanned by CamScanner

Q Find all the subgps of the gp $G = \{1, \omega, \omega^2\}$ under ordinary multiplicn.

identity element 1.

Subsets = 8

$H_1 = \{1\}$

$H_2 = \{\omega\}$ → $\begin{array}{c|c} & \omega \\ \hline \omega & \omega^2 \end{array}$ → Closer axioms fails

$H_3 = \{\omega^2\}$ → $\begin{array}{c|c} & \omega^2 \\ \hline \omega^2 & \omega^4 = \omega \end{array}$ → "

$H_4 = \{1, \omega\}$ → $\begin{array}{c|cc} & 1 & \omega \\ \hline 1 & 1 & \omega \\ \omega & \omega & \omega^2 \end{array}$ → "

$H_5 = \{1, \omega^2\}$ → $\begin{array}{c|cc} & 1 & \omega^2 \\ \hline 1 & 1 & \omega^2 \\ \omega^2 & \omega^2 & \omega^4 = \omega \end{array}$ → "

$H_6 = \{\omega, \omega^2\}$ → $\begin{array}{c|cc} & \omega & \omega^2 \\ \hline \omega & \omega^2 & \omega^3 = 1 \\ \omega^2 & \omega^3 = 1 & \omega^4 = \omega \end{array}$ → " "

$H_7 = \{1, \omega, \omega^2\}$

$H_8 = \phi$ → X

It's subgps are $H = \{1\}$ & $H = \{1, \omega, \omega^2\}$ only the trivial subgps exists.

There is no non-trivial subg.

(2) $G = \{1, -1, i, -i\}$ be a gp. under multiplicn.

→ It's trivial subgps. are $H_1 = \{1\}$, $H_2 = \{1, -1, i, -i\}$.

→ This gp. have only one Non-trivial sg. i.e. $H_3 = \{1, -1\}$

Eg (1) $(\mathbb{Z}, +)$ be a gp.

It's trivial subgps are $H_1 = \{0\}$, $H_2 = \mathbb{Z}$

It's non-trivial sgs are $H_3 = \{0, \pm 2, \pm 4, \pm 6 \ldots\}$

$H_4 = \{0, \pm 3, \pm 6, \pm 9 \ldots\}$

$H_5 = \{0, \pm 4, \pm 8, \pm 12 \ldots\}$

etc

Scanned by CamScanner

(2) Intersection of two sgs is a sg.

Proof ⇒ Let $(G, \circ)$ be a gp.

$H \subseteq G$ & $H \neq \phi$; $K \subseteq G$ & $K \neq \phi$ are two sgs of G under $\circ$.

Clearly $H \cap K \neq \phi$ & $H \cap K \subseteq G$

Claim ⇒ $(H \cap K, \circ)$ be a sg. of $(G, \circ)$

i.e. $H \cap K$ will satisfy all the gp. axiom.

*) $\forall a, b \in H \cap K$

$\Rightarrow \forall a, b \in H$ & $\forall a, b \in K$

As H & K are sgs. so closer axiom holds there

~~So $a * b =$~~

So $a \cdot b \in H$ & $a \cdot b \in K$

$\Rightarrow a \cdot b \in H \cap K$

⇒ Closer axiom holds.

*) $\forall a, b, c \in H \cap K$

$\Rightarrow \forall a, b, c \in G$ $[H \cap K \subseteq G]$

G be a gp w.r.t. $\circ$

Ass. holds in G w.r.t. $\circ$

So $a \circ (b \circ c) = (a \circ b) \circ c$

*) $\forall a \in H \cap K \Rightarrow \forall a \in H \wedge \forall a \in K$

H & K are sgs, so Inverse axiom holds then.

$\forall a \in H$, $\exists a^{-1} \in H$ such that $a \cdot a^{-1} = e = a^{-1} \cdot a$.

$\forall a \in K$, $\exists a^{-1} \in K$ such that $aa^{-1} = e = a^{-1} \cdot a$.

for $\forall a \in H \cap K \Rightarrow a^{-1} \in H$ & $a^{-1} \in K$

$\Rightarrow a^{-1} \in H \cap K$

⇒ Inverse axiom holds.

Scanned by CamScanner

→ Since the id. element of a gp. is unique.

So $\forall a \in H \cap K$

$\Rightarrow \forall a \in G, \exists e \in G$ for which

$a.e = a = e.a$

Eg ⇒ Union of two subgps is not necessarily a subgp.

Proof ⇒ We will prove it by taking some concrete example

Let $G = \mathbb{Z}$ be a gp under +

$H = \{0, \pm 2, \pm 4 \ldots\}$ be a subgp of $(\mathbb{Z}, +)$

$K = \{0, \pm 3, \pm 6 \ldots\}$ " subgp of "

$H \cup K = \{0, \pm 2, \pm 3, \pm 4, \pm 6 \ldots\}$

Here closer axiom holds fails as $2 \in H \cup K$, $3 \in H \cup K$

but $2+3 = 5 \notin H \cup K$.

i.e. $H \cup K$ is not a sg of $(\mathbb{Z}, +)$.

Theorem ⇒

The necessary & sufficient condition that a non-empty subset H of a group G forms a subgroup is.

$a, b \in H \Rightarrow a.b^{-1} \in H$

OR

$(H, \circ)$ is a subgroup of the gp $(G, \circ)$ iff

$\forall a, b \in H \Rightarrow a \circ b^{-1} \in H$

Proof ⇒

Necessary Part ⇒

Let $(H, \circ)$ be the sg of the gp $(G, \circ)$.

Scanned by CamScanner

11/10/16

Claim ⇒ $\forall\, a, b \in H \Rightarrow a.b^{-1} \in H$

Since H be a sg of the gp G. H satisfies all the gp axiom.
So by Inverse axiom

$$\forall\, b \in H \Rightarrow b^{-1} \in H$$

Again by closer axiom.

for $\forall\, a, b \in H \Rightarrow a, b^{-1} \in H$
$\Rightarrow a \circ b^{-1} \in H$

<u>Sufficient Part</u> ⇒

Assume $\forall\, a, b \in H \Rightarrow a \circ b^{-1} \in H$

Claim ⇒ $(H, \circ)$ be a sg of the gp $(G, \circ)$
i.e. H satisfies gp. axiom.

→ Since $\forall\, a, b \in H \Rightarrow a.b^{-1} \in H$
By replacing b by a

$a.a^{-1} \in H$ $\left[\begin{array}{l} H \subset G \\ \forall\, a \in H \Rightarrow a \in G \\ \text{so } a.a^{-1} = e = a^{-1}a \end{array}\right]$

$\Rightarrow e \in H$ (identity element of gp & sg is same & unique)

Id. axiom holds.

→ Replacing a by e, $e.b^{-1} \in H$
$\Rightarrow b^{-1} \in H$

i.e. $\forall\, b \in H \Rightarrow b^{-1} \in H$ s.t. $b.b^{-1} = e$

Inverse axiom holds.

→ Replacing b by b^{-1}

$\forall\, a, b \in H \Rightarrow a(b^{-1})^{-1} \in H$ $\left[\begin{array}{l} H \subset G \\ b \in H \Rightarrow b \in G \\ \text{In } G, (b^{-1})^{-1} = b \end{array}\right]$

$\Rightarrow a.b \in H$

Closer axiom holds

Scanned by CamScanner

~~As H ⊂ G~~

→ $\forall\ a, b, c \in H$

$\Rightarrow \forall\ a, b, c \in G$ as $H \subset G$

In G, ass. axiom holds

So $a \circ (b \circ c) = (a \circ b) \circ c$

Hence the proof.

LectureNotes.in

Coset Decomposition ⇒

Let $(H, \circ)$ be a subgroup of the gp $(G, \circ)$.

For any element $a \in G$.

→ $a \circ H$ be the set of left coset of H in G, where

$a.H = \{a.b \mid \forall\ b \in H\}$

→ $H.a$ be the set of right coset of H in G where

$H.a = \{h.a \mid \forall\ h \in H\}$

Eg ⇒ ① $H = \{1, -1\}$ be the sg of the gp $G = \{1, -1, i, -i\}$ under multiplicn.

Left Cosets

$1.H = \{1.1, 1.-1\} = \{1, -1\}$

$-1.H = \{-1.1, -1.-1\} = \{-1, 1\}$

$i.H = \{i.1, i.(-1)\} = \{i, -i\}$

$-i.H = \{-i.1, -i.(-1)\} = \{-i, i\}$

Right Coset

$H.1 = \{1.1, -1.1\} = \{1, -1\}$

$H.(-1) = \{1.(-1), (-1).(-1)\} = \{-1, 1\}$

$H.i = \{1.i, (-1).i\} = \{i, -i\}$

$H.(-i) = \{1.(-i), (-1).(-i)\} = \{-i, i\}$

② $H = \{\ldots -4, -2, 0, 2, 4, 6 \ldots\}$ be the sg of the gp $(\mathbb{Z}, +)$

Since $\mathbb{Z}^+ = \{0, \pm 1, \pm 2 \ldots\}$

Left cosets ⇒

$0 + H = \{\ldots 0 + (-4), 0 + (-2), 0 + 0, 0 + 2, 0 + 4 \ldots\}$

$= \{\ldots -4, -2, 0, 2, 4 \ldots\}$

Scanned by CamScanner

$1+H = \{ \ldots 1+(-4),\ 1+(-2),\ 1+0,\ 1+2,\ 1+4 \ldots \}$

$= \{ \ldots -3, -1, 1, 3, 5 \ldots \}$

$-1+H = \{ \ldots (-1)+(-4),\ (-1)+(-2),\ (-1)+0,\ (-1)+2,\ (-1+4) \ldots \}$

$= \{ \ldots -5, -3, -1, 0, 1, \ldots \}$

etc.

LectureNotes.in

Right cosets ⇒

$H+0 = \{ \ldots -4+0,\ -2+0,\ 0+0,\ 2+0,\ 4+0 \ldots \} = \{ \ldots -4, -2, 0, 2, 4 \ldots \}$

$H+1 = \{ \ldots (-4)+1,\ -2+1,\ 0+1,\ 2+1,\ 4+1 \ldots \} = \{ \ldots -3, -1, 1, 3, 5 \ldots \}$

$H+(-1) = \{ \ldots (-4)+(-1),\ (-2)+(-1),\ 0+(-1),\ 2+(-1),\ 4+(-1) \} = \{ \ldots -5, -3, -1, 1, 3 \ldots \}$

NOTES ⇒

*> $H.h = H = h.H \quad \forall h \in H$

*> $H.e = H = e.H$

*> If $a \neq b$, where $a, b \in G$

then $a.H = b.H \iff a.b^{-1} \in H$

*> Any two left coset (right coset) of H in G are either disjoint or identical.

*> Union of all the distinct left (right) cosets of H in G is G

i.e. $\bigcup_{i=1}^{m} a_i.H = G = \bigcup_{i=1}^{n} H.a_i$

i.e. $(a_1.H) \cup (a_2.H) \cup (a_3.H) \cup \ldots = G$

$(H.a_1) \cup (H.a_2) \cup (H.a_3) \cup \ldots = G$

Scanned by CamScanner

Lagrange's Theorem ⇒

If H be a subgroup of the finite gp. G then $O(H) \mid O(G)$

Proof ⇒

Given that G be a finite gp.

Let $O(G) = n$ i.e. $G = \{a_1, a_2, \ldots a_n\}$

Any nonempty subset of finite set is always finite.

As given that H is a sg of G

so $H \subseteq G$, $H \neq \phi$ & H is finite

Let $O(H) = m$ i.e. $H = \{h_1, h_2, h_3 \ldots h_n\}$

Now

$a.H$ is the the set of left coset of H in G

As we know any two left coset are either disjoint or identical

Let us say there are k no of distinct left cosets of H in G

s.t. $a_1.H, a_2.H, a_3.H \ldots a_k.H$.

Union of those distinct left coset is G.

$$(a_1.H) \cup (a_2.H) \cup (a_3.H) \cup \ldots \cup (a_k.H) = G$$

⇒ Order of

$$O((a_1.H) \cup (a_2.H) \cup (a_3.H) \ldots \cup (a_k.H) = O(G)$$

$$O(a_1.H) + O(a_2.H) + \ldots O(a_k.H) = O(G)$$

(for disjoint set $|A \cup B| = |A| + |B|$)

$$m + m + \ldots + m \text{ k times} = n$$

$$km = n$$

$$\Rightarrow m \mid n$$

$$\Rightarrow O(H) \mid O(G)$$

divides.

Scanned by CamScanner

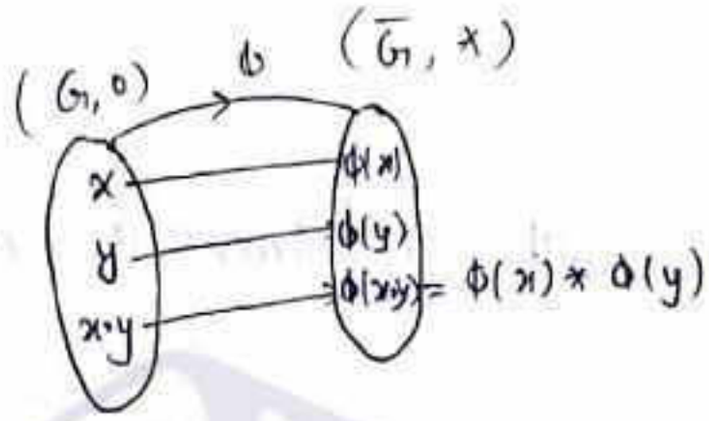
3/10/16

HOMOMORPHISM OF GROUPS ⇒

Let $(G, \circ)$ & $(\bar{G}, *)$ are two groups.

The mapping $\phi : G \to \bar{G}$ is said to be a homomorphism iff

$\forall\, x, y \in G$, $\phi(x \cdot y) = \phi(x) * \phi(y)$



Theorem ⇒ If $\phi : G \to \bar{G}$ be a homomorphism of groups & if e & e' be the identities in G & $\bar{G}$ resp. then

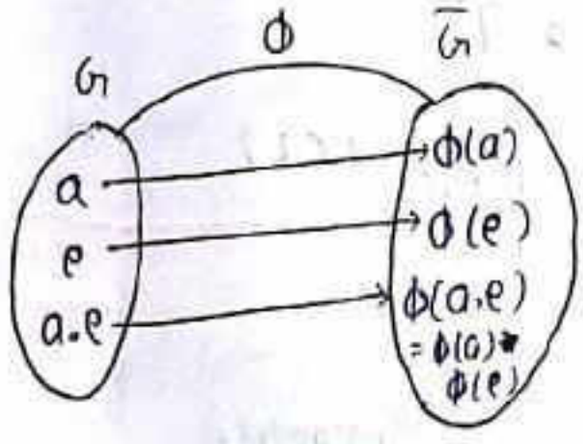
(i) $\phi(e) = e'$

(ii) $\phi(a^{-1}) = \{\phi(a)\}^{-1}$, $\forall a \in G$

Proof ⇒ G & $\bar{G}$ are two gps.

i) Given that e & e' are the id. element

$\phi : G \to \bar{G}$ be a homomorphism



$\forall a \in G$ & $e \in G \Rightarrow a.e \in G$ & $a.e = a$

such that $\phi(a) = \phi(a.e)$

$$\phi(a) = \phi(a) \cdot \phi(e) \quad \text{——— (1)}$$

As $a \in G \Rightarrow \phi(a) \in G$

Scanned by CamScanner

Again $e' \in \bar{G}$ so $\phi(a).e' \in \bar{G}$

such that

$$\phi(a) = \phi(a).\bar{e} \quad \text{—— (2)}$$

from (1) & (2)

$$\phi(a).\phi(e) = \phi(a).\bar{e}$$

By L.C.L

$$\boxed{\phi(e) = \bar{e}}$$

(ii) As G & $\bar{G}$ are two gps. with Id. elements e & e' respectively.

Show also $\phi : G \to \bar{G}$ be a homomorphism.

So, $\forall\ a \in G \Rightarrow a^{-1} \in G$ st. $a.a^{-1} = e$

Also $\phi(a) \in \bar{G}$

$\Rightarrow [\phi(a)]^{-1} \in \bar{G}$ st. $\phi(a).[\phi(a)]^{-1} = e'$ —— (1)

As $a.a^{-1} = e$

$\phi(a.a^{-1}) = \phi(e)$

$\phi(a).\phi(a^{-1}) = \phi(e)$ [$\because \phi$ be a homomorphism]

$\phi(a).\phi(a^{-1}) = e'$ —— (2) [$\because \phi(e) = e'$]

from (1) & (II)

$\phi(a).\phi(a^{-1}) = \phi(a).[\phi(a)]^{-1}$

$$\boxed{\phi(a^{-1}) = [\phi(a)]^{-1}}$$ [By LCL]

Eg ⇒ (1) $G = \mathbb{Z}$ be the gp. under +

$\bar{G}$ = The additive gp. of even integers.

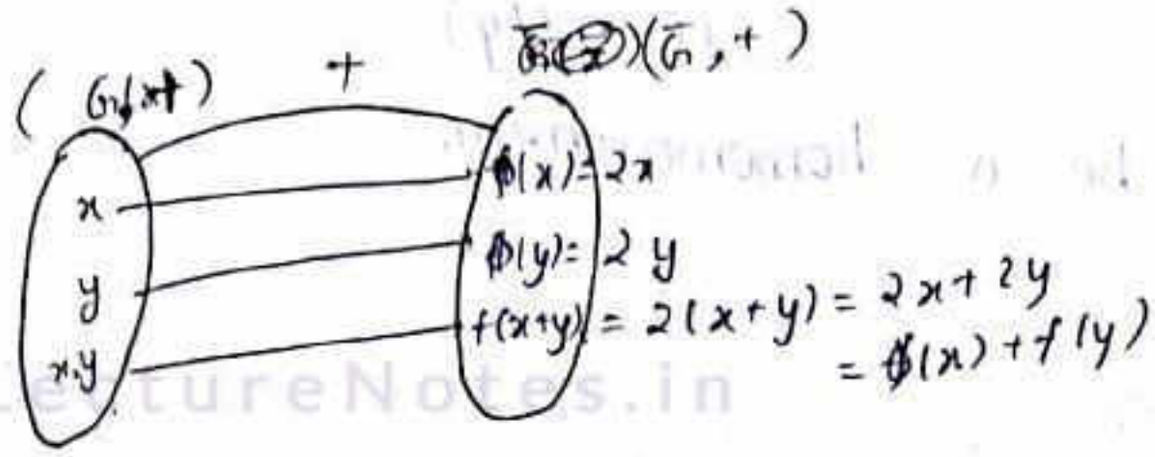
$f : G \to \bar{G}$ is given by $f(x) = 2x$

Is it a homomorphism ?

Scanned by CamScanner

$G = \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3 \ldots\}$ be the gp. under $+$

$\bar{G} = \{0, \pm 2, \pm 4, \pm 6 \ldots\}$ " " " " $+$



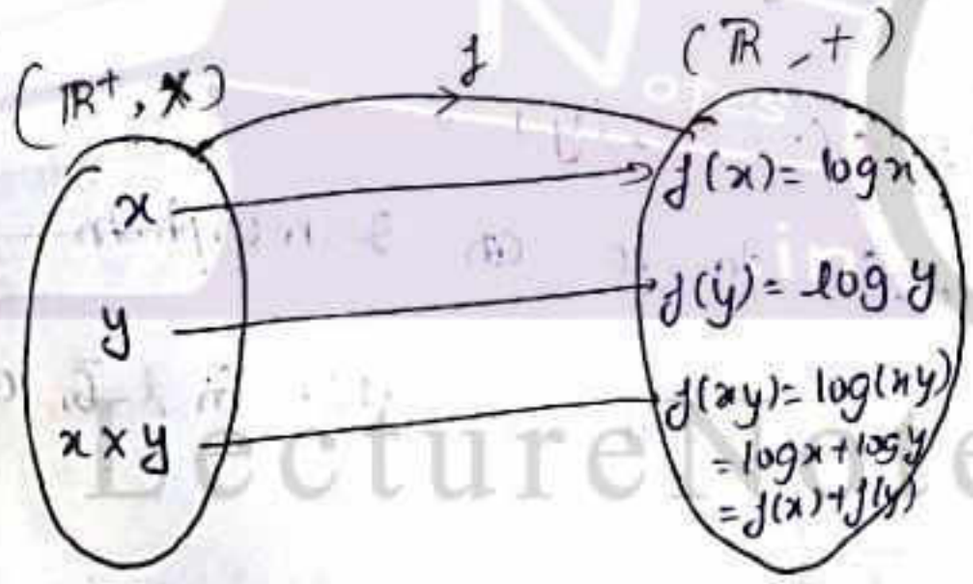
Hence f is a homomorphism

(2) $G = \mathbb{R}^+$ be a gp. under multiplicn

$\bar{G} = \mathbb{R}$ be a gp. under addition

$f : G \to \bar{G}$ is defined by $f(x) = \log x$.

Is it a hom. ?



Hence f is a homomorphism

(3) $G = \mathbb{R}$ be the gp. under $+$

$\bar{G} = \mathbb{R}^+$ be the gp under $\times$

$f : G \to \bar{G}$ is given by $f(x) = e^x$

Is it a homo ~~geneo~~ ?

Claim $\Rightarrow$ $f(x+y) = f(x) \cdot f(y)$

$\forall x \in G \Rightarrow f(x) = e^x \in \bar{G}$

$\forall y \in G \Rightarrow f(y) = e^y \in \bar{G}$

Scanned by CamScanner

Also

$$x+y \in G \Rightarrow f(x+y) = e^{x+y}$$
$$= e^{x} \cdot e^{y}$$
$$= f(x) \cdot f(y)$$

So f be a homomorphism.

Isomorphism $\Rightarrow$

Let $(G, \circ)$ & $(\bar{G}, *)$ are two gps.

The homomorphism $\phi : G \to \bar{G}$ is said to be an Isomorphism iff ϕ is bijective.

OR

Let $(G, \circ)$ & $(\bar{G}, *)$ are two gps.

If there exists a bijective func^n $\phi : G \to \bar{G}$ such that

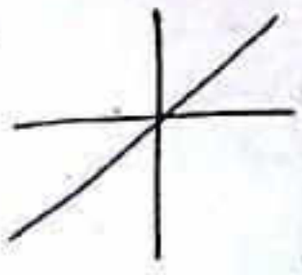
$$\phi(x \circ y) = \phi(x) * \phi(y)$$

then ϕ is said to be an isomorphism

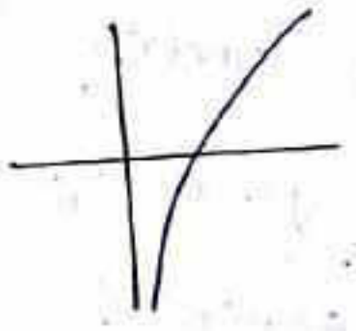
*) In such case we can say that G & $\bar{G}$ are isomorphic

$$\boxed{G \cong \bar{G}}$$

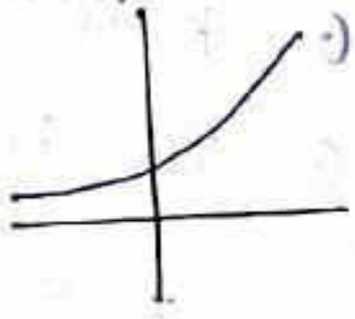
① $f(x) = 2x$



② $f(x) = \log x$



③ $f(x) = e^{x}$



in previous 3 examples $f(x)$ are bijective & also all those are homomorphism

$\Rightarrow$ All three are isomorphism.

Scanned by CamScanner

Automorphism ⇒
An isomorphism of a gp. on to itself is said to be an Automorphism.

i.e. If $\phi : G \to G$ be an isomorphism
then ϕ is an automorphism.

Endomorphism ⇒ A homomorphism of a gp. on to itself is said to be Endomorphism.

If $\phi : G \to G$ be a homomorphism
then ϕ is called Endomorphism.

Normal Subgroup ⇒ Let $(G, \circ)$ be a gp.
$N \subseteq G$, $N \neq \phi$
$(N, \circ)$ be a subgroup of the group $(G, \circ)$. N is said to be a normal subgroup iff
$\forall g \in G, \forall n \in N, \boxed{g n g^{-1} \in N}$

Eg ⇒ Every proper subgroup of an abelian gp.

*) The subgroup N is said to be a normal subgroup of the gp. G iff
$\boxed{aN = Na}$
i.e. Left Coset = Right Coset

Ring ⇒ Let R be any non-empty set.
Two operations * & × are defined in R. So, the algebraic structure $(R, +, \times)$ is said to be a ring iff
→ $(R, +)$ is an abelian group
→ $(R, \times)$ is a semigroup
→ Both left & right distributive axiom holds,
i.e. $a.(b+c) = a.c + a.c$
& $(a+b).c = a.c + b.c$

Scanned by CamScanner

16/10/16

Commutative Ring ⇒ The ring $(R, +, \times)$ is said to be a commutative ring iff.

the commutative axiom under $\times$ holds in R

i.e. $\forall a, b \in R \Rightarrow a.b = b.a$

→ $(R, +)$ is an abelian gp.

→ $(R, \times)$ is a commutative semigroup

→ Both left & right distributive axiom holds.

Ring with unity ⇒ The ring $(R, +, \times)$ is said to be a ring with unity iff the identity axiom holds in under $\times$, i.e.

$\forall a \in R \; \exists \; I = e \in R$, for which $a.e = a = e.a$

$(R, +, \times)$ be a ring with unity iff

→ $(R, +)$ be an abelian gp.

→ $(R, \times)$ be a monoid

→ Both distributive axiom holds

Commutative Ring with Unity ⇒ The non-empty set R under $+$ & $\times$ is said to be a commutative ring with unity iff

→ $(R, +)$ is an abelian gp.

→ $(R, \times)$ is a commutative semigroup.

→ Both distributive axiom hold.

Eg ⇒ $(Z_3, \oplus, \otimes)$ be a ring, $Z_3 = \{0, 1, 2\}$

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Clearly (shown earlier)

→ $(Z_3, \oplus)$ is an abelian gp.

→ $(Z_3, \otimes)$ is a commutative monoid

Scanned by CamScanner

→ Both distributive axiom hold.

Hence $(Z_3, \oplus, \otimes)$ is a commutative ring within unity.

Field ⇒ Let F be any non-empty set

$(F, +, \times)$ is said to be a field iff.

→ $(F, +)$ is an abelian gp.

→ $(F - \{0\}, \times)$ is an abelian gp.

→ Both distributive axiom holds

Eg ⇒ ① $(Q, +, \times)$ is a field

② $(R, +, \times)$ is a field

③ $(C, +, \times)$ is a field

Zero Eliviser / Proper divisor of zero ⇒

If $a \neq 0$, $b \neq 0$ are two elements of the ring R such that $a.b = 0$, then we will say that a & b are the proper divisors of zero.

Eg ⇒ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$

but, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Integral Domain ⇒ A commutative ring with unity elements & having at least two elements without zero divisor is said to be an integral Domain.

Eg ⇒ $(Z, +, \times)$ is an integral domain.

Scanned by CamScanner

20/10/16

V.V. Imp

Ex ⇒ (i) If R is ring where $\forall x \in R$, $\boxed{x^2 = x}$ → Boolean Ring

then

i) $a + a = 0$, $\forall a \in R$

ii) $a + b = 0 \Rightarrow a = b$, $\forall a, b \in R$

iii) R is a commutative Ring

Proof

(i) As $\forall a \in R$, $a + a \in R$ [Closer axiom holds under $+$]

By given hypothesis $(a+a)^2 = a+a$

$(a+a)(a+a) = a+a$ [$x^2 = x.x$

$a.(a+a) + a.(a+a) = a+a$ [Distributive

$(a.a + a.a) + (a.a + a.a) = a+a$ [Distributive

$(a^2 + a^2) + (a^2 + a^2) = a+a$

$(a+a) + (a+a) = a+a$ [$x^2 = x$]

$(a+a) + (a+a) = (a+a) + 0$ [Identity axiom

$\boxed{a+a = 0}$ [By Left Cancellation Law]

(ii) Given that $a + b = 0$, $\forall a, b \in R$

As we have $a + a = 0$, $\forall a \in R$

so $a + b = a + a$

$\Rightarrow b = a$ [L.C.L]

(iii) To prove commutative ring we have to prove commutative law within $\times$ i.e. $ab = ba$.

$(ab)^2 = ab$

$\forall a, b \in R \Rightarrow a + b \in R$

$(a+b)^2 = (a+b)$ [Hypothesis]

$(a+b)(a+b) = a+b$ [$x^2 = x$

$a.(a+b) + b.(a+b) = a+b$ [Distributive

$(a.a + a.b) + (b.a + b.b) = a+b$ [Distributive

$(a^2 + ab) + (ba + b^2) = a+b$ [$x.x = x^2$

$(a + ab) + (ba + b) = a+b$ [$x^2 = x$

$(a+b) + (ab + ba) = a+b$ [Associative

$(a+b) + (ab+ba) = (a+b) + 0$ [Identity

$ab + ba = 0$ [By L.C.L]

$ab = ba$ [By 2nd hypothesis i.e. $a+b=0 \Rightarrow a=b$

$\Rightarrow R$ be a Commutative Ring

Scanned by CamScanner

② If R is a ring & for any $x, y \in R$ such that $(x+y)^2 = x^2 + 2xy + y^2$ then show that R is a Commutative Ring.

To Proof ⇒ $xy = yx$

Given ⇒ $(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)(x+y) = x^2 + 2xy + y^2$ [$x^2 = x.x$]

$x(x+y) + y(x+y) = x^2 + 2xy + y^2$ [Distributive Law

$(x^2 + xy) + (yx + y^2) = x^2 + 2xy + y^2$ [Distributive Law

$x^2 + (xy + yx) + y^2 = x^2 + 2xy + y^2$ [Associative

$xy + yx = 2xy$ [By LCL & RCL

$xy + yx = xy + xy$ [$x + x = 2x$

$yx = xy$ [By LCL

Questions On Right Coset & left Coset ⇒

Eg ⇒

① If a, b are any two element of a gp G & H is a subgroup of the gp G.

then $\boxed{Ha = Hb \text{ iff } ab^{-1} \in H}$

$\boxed{aH = bH \text{ iff } a^{-1}b \in H}$

Proof

Necessary Part ⇒

Given that H be the subgp of the gp. G.

Let $Ha = Hb$

Proof Clearly for $h \in H$, $ha = hb$ [$\because b \in G \Rightarrow b^{-1} \in G$

$(ha)b^{-1} = (hb)b^{-1}$

$h(ab^{-1}) = h(bb^{-1})$ [Associative

$h(ab^{-1}) = he$ [Inverse

$h(ab^{-1}) = h$ [Identity

$ab^{-1} = h^{-1}h$ [$\because h \in H$, $h^{-1} \in H$] $\Rightarrow h^{-1}h \in H$

$\boxed{ab^{-1} \in H}$

Scanned by CamScanner

$\boxed{\text{if } x \in A \ \& \ x \in B \Rightarrow A \subset B}$

Sufficient Part $\Rightarrow$

Let for any $a, b \in G$, $ab^{-1} \in H$

Claim $\Rightarrow$ $Ha = Hb$

As $(ab^{-1}) \in H \Rightarrow (ab^{-1})^{-1} \in H$

$(b^{-1})^{-1} a^{-1} \in H$ [Reversal law

$ba^{-1} \in H$ [Involution law

$$\left.\begin{aligned} \text{Let } h_1 &= ab^{-1} \\ h_1^{-1} &= ba^{-1} \end{aligned}\right\} \text{ — (i)}$$

for $h_i \in H$

$\Rightarrow h_i b \in Hb$ [Right coset

$h_i b = (h_i b) e$ [Identity

$= h_i b (a^{-1} a)$ [Inverse

$= h_i (ba^{-1}) a$ [Associative

$= h_i h_1^{-1} a$ [Using eqn (i)

$= (h_i h^{-1}) a$ [Associative

$h_i b \in Ha$ [as $h_i \in H$ & $h^{-1} \in H$ $\Rightarrow h_i h^{-1} \in H$

$\therefore h_i b \in Hb$ & $h_i b \in Ha$ — (ii)

$\Rightarrow Hb \subset Ha$ — (iii)

Similarly, we can show that $Ha \subset Hb$

from (ii) & (iii)

$\boxed{Ha = Hb}$

(2) If H be a subgroup of the gp G, then any two left cosets of H in G are either disjoint or identical

Given that H is a subgp of the gp G.

for $a, b \in G$, aH & bH are any two left cosets of H in G

Scanned by CamScanner

21/10/16

Suppose $aH \cap bH \neq \phi$

i.e. $e \in (aH) \cap (bH)$

$\Rightarrow e \in aH \wedge e \in bH$

Let $ah_1 \in aH$ st $e = ah_1$

$bh_2 \in bH$ st $e = bh_2$

Thus $ah_1 = bh_2$

$h_1 = a^{-1} b h_2$

$h_1 h_2^{-1} = a^{-1} b$

As $h_1, h_2 \in H \Rightarrow h_1, h_2^{-1} \in H$

$\Rightarrow h_1 . h_2^{-1} \in H$

So $a^{-1} b \in H$

$\Rightarrow \boxed{aH = bH}$

Congruence Relation $\Rightarrow$ Let H be a subgroup of the group G.

*) For any two elements $a, b \in G$ if $ab^{-1} \in H$ then we can say "a is congruent to b mod H" i.e. $a \equiv b \pmod H$

*) $a \equiv b \pmod H \Leftrightarrow ab^{-1} \in H$

Eg $\Rightarrow$

① Congruence Relation is an equivalence Relation.

Proof ~~*) Since $e = aa^{-1}$~~

Here H is the subgp. of gp. G.

The identity element of the gp & its subgps be same.

*) Let it be 'e'

Scanned by CamScanner

i.e. $e \in H$

Again $\forall a \in G$, $e = aa^{-1}$

So $aa^{-1} \in H$

$\Rightarrow a \equiv a \pmod H$

$\Rightarrow$ Reflexive holds

x) Let $a \equiv b \pmod H$

$ab^{-1} \in H$

$(ab^{-1})^{-1} \in H$

$(b^{-1})^{-1}a^{-1} \in H$

$ba^{-1} \in H$

$b \equiv a \pmod H$

$\Rightarrow$ symmetric holds

x) Let $a \equiv b \pmod H$ & $b \equiv c \pmod H$

$ab^{-1} \in H$ & $bc^{-1} \in H$

By closer axiom

$(ab^{-1})(bc^{-1}) \in H$

$a(b^{-1}b)c^{-1} \in H$ [Associativity

$(ae)c^{-1} \in H$ [~~Identity~~ Inverse Law

$ac^{-1} \in H$ [Identity

$a \equiv c \pmod H$

$\Rightarrow$ Transitivity holds

$\Rightarrow$ Congruence Relⁿ is an equivalence relⁿ.

Scanned by CamScanner

GRAPH

Graph Theory ⇒

*) ⇒ Let $G = (V, E)$ be a graph

where V be the non empty set called as Vertex set

E be the non empty set called as Edge set.

*) The Edge is the Path in between two vertices.

*) There are two types of edges

i) directed edge

ii) non-directed edge

*) $a \xrightarrow{e} b$ be the directed edge

Here $e = (a, b)$

*) $a \overset{e}{—} b$ be the non directed edge

$e = \{a, b\}$

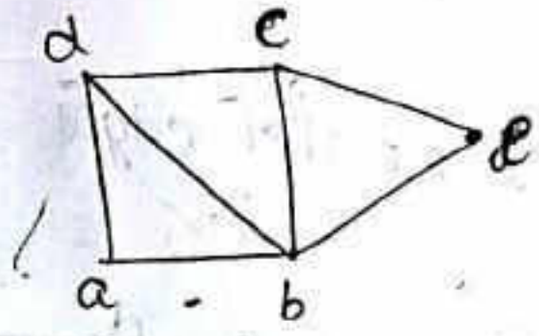
Diagraph ⇒

*) If all the edges contained in a graph are directed then the graph is said to be Directed Graph or Diagraph

*) Non Directed Graph ⇒ If all the edges contained in a graph are non-directed then the graph is said to be non-directed graph.

Eg ⇒

G :



This be a non directed graph.

The vertex set is $V = \{a, b, c, d, e\}$

The edge set is $E = \{\{a,b\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{b,e\}, \{c,e\}\}$

Scanned by CamScanner

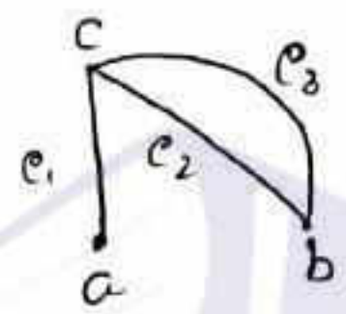
*> For the edge $e = \{a, b\}$ the vertices a & b are the end points

NOTE ⇒

*> Let G be a graph, If two or more edges of G have the same end vertices then these edges are said to be Parallel Edges.

Eg ⇒

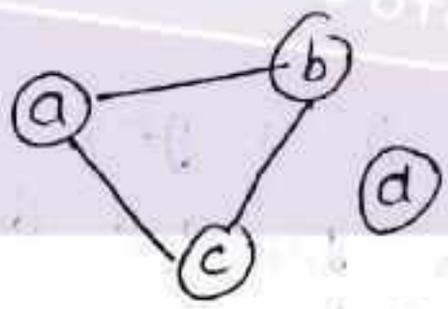
G :



Here e_2 & e_3 are Parallel Edges.

*> A vertex in the graph G which is not the end point of any edge is called as Isolated Vertex

Eg ⇒

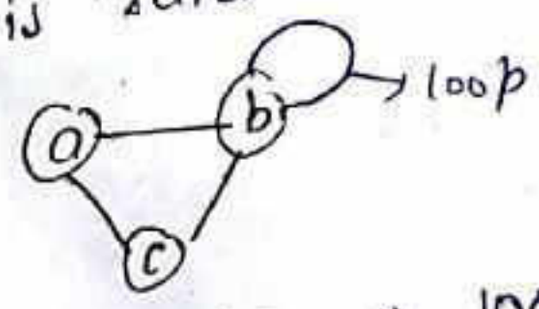


Here d is the isolated vertex.

*> Two vertices which are joined by an edge are said to be neighbour or Adjacent.

*> If there exist an edge at a vertex then that is said to be a Loop

Eg ⇒

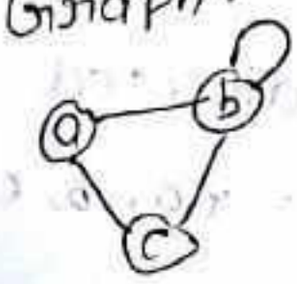


*> A graph without loop & parallel edges is said to be simple Graph.

G_1 :

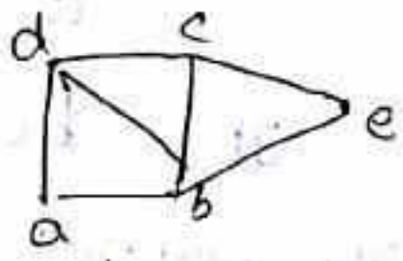
Not a simple Graph

G_2 :



Not a simple Graph

G_3 :



↓

Simple Graph

Scanned by CamScanner

*) A graph without any edge is said to Trivial Graph or Empty Graph

*) A graph is said to be a Complete Graph if it is simple & each pair of distinct vertices are joined by an edge.

Generally, Complete Graphs are denoted by K_n

n = no. of ~~edges~~ vertices

Eg ⇒ K_1 :

K_2 :

K_3 :

K_4 :

V. Imp

*) If a complete Graph if there are n no. of vertices then there will be $\frac{n(n-1)}{2}$ no. of edges.

Eg ⇒ In previous

K_1 : $|V| = 1$, $|E| = 0 \rightarrow n = 1$, $\frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$

K_2 : $|V| = 2$, $|E| = 1$, $\longrightarrow n = 2$, $\frac{n(n-1)}{2} = \frac{2(2-1)}{2} = 1$

K_4 = $|V| = 4$, $|E| = 6 \longrightarrow n = 4$, $\frac{n(n-1)}{4} = \frac{4(4-1)}{2} = 6$

~~*) The no. of~~

*) Let 'v' be a vertex of graph G, degree of v i.e. $d(v)$ or $\deg(v)$ be the no. of edges where end point is v

OR

An edge 'e' be the edge of Graph G is said to incident with vertex v if v is an end vertex of the edge e.

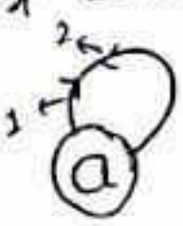
Scanned by CamScanner

In this case we also say that the vertex v is incident with the edge e.

*) Two edges 'e_1' & 'e_2' which are incident with a common vertex are said to be <u>adjacent</u>.

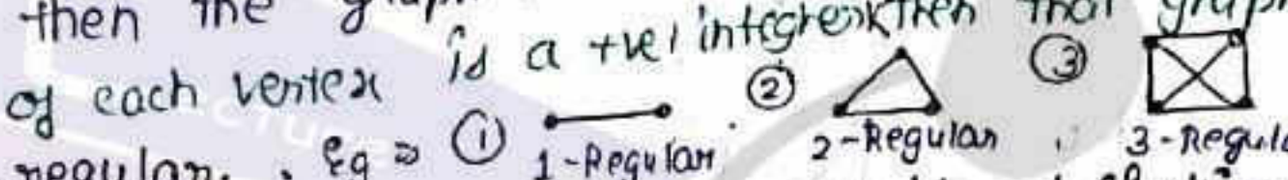
*) A vertex is said to be 'even' or 'odd' depending upon whether its degree is even or odd.

*) The degree of the vertex containing self loop is to be always 2



*) If the degree of each vertex in a graph are equal then the graph is said to be a <u>Regular Graph</u>

*) If degree of each vertex is a +ve integer k then that graph is said to be <u>k-regular</u>. Eg ⇒

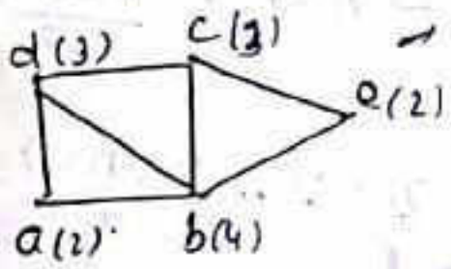


*) 1st Theorem of Graph Theory OR Hand Shaking Theorem

In Graph G,

$$\boxed{\sum_i \deg(v_i) = 2|E|}$$

Eg ⇒



$$\sum d(v_i) = 2+4+3+3+2$$
$$= 14$$
$$= 2 \times 7$$
$$= 2 \times |E|.$$

Each edge since it has two end vertices contributes precisely two to the sum of degree.

i.e. when the degrees of the vertices are added each edge is counted twice.

Scanned by CamScanner

21/10/16

Corollary ⇒

⇒ In any Graph G, there is an even no. of odd vertices

Proof ⇒

Let $G = (V, E)$ be the graph.

By 1st theorem on graph then,

$$\sum_{v_i \in V} \deg(v_i) = 2|E| \quad \text{—— (I)}$$

Let us split the vertext set V into two distinct (disjoint sets U & W

where U be the set of vertices with odd degree.

W be the set of vertices with even degree

So, $\sum_{v_i \in V} \deg(v_i) = \sum_{v_i \in U} \deg(v_i) + \sum_{v_i \in W} \deg(v_i)$ —— (II)

From (I) & (2)

$$\sum_{v_i \in V} \deg(v_i) + \sum_{v_i \in N} \deg(v_i) = 2|E|$$

$$\Rightarrow \sum_{v_i \in V} \deg(v_i) + \text{Even} = \text{Even}$$

$$\sum_{v_i \in V} \deg(v_i) = \text{Even} - \text{Even}$$

$$= \text{Even}$$

As we know sum of even no. of odds is Even.

Hence there are even no. of vertices with odd degrees

Eg ⇒ Is there any graph with degree sequence 4, 4, 4, 3

No, as there is single (odd) vertex with odd degree.

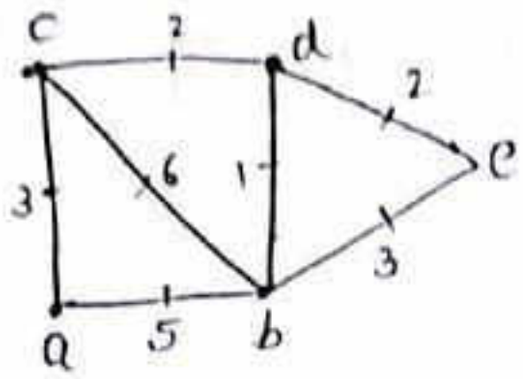
Weighted Graph ⇒

If in a graph G, all the edges are assigned by some value then the graph is weighted.

That value is to be the weight of the edge.

Scanned by CamScanner

Eg ⇒



Here $\omega(\{a, b\}) = 5$

$\omega(\{a, c\}) = 3$

$\omega(\{b, c\}) = 6$

$\omega(\{b, d\}) = 1$

$\omega(\{b, e\}) = 3$

$\omega(\{c, d\}) = 2$

$\omega(\{d, e\}) = 2$

LectureNotes.in

NOTES ⇒

*) If v be the vertical vertex of a graph then $\lambda(s)$ be the length/weight of the shortest path of vertex s from initial vertex

*) $\lambda(v) = 0$ if v is the initial vertex

Eg ⇒ In the graph,

$\lambda(a) = 0$, as a is the initial vertex

$\lambda(b) = \lambda(a) + \omega(\{a, b\}) = 0 + 5 = 5$

$\lambda(c) = \lambda(a) + \omega(\{a, c\}) = 0 + 3 = 3$

$\lambda(d) = \lambda(c) + \omega(\{c, d\}) = 3 + 2 = 5$

OR

$\lambda(b) + \omega(\{b, d\}) = 5 + 1 = 6$

$\Rightarrow \lambda(d) = 5$

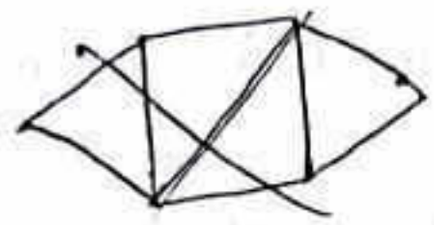
Q. How to learn the theorem?

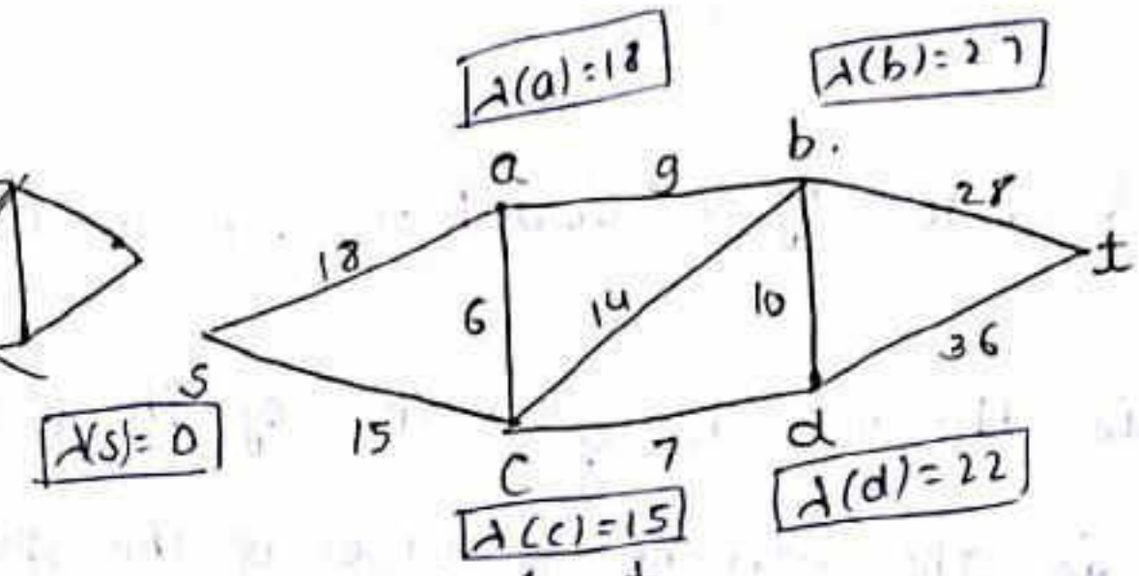
Dijkstra's Algorithm ⇒

1. Set $\lambda(s) = 0$, for all other vertices $v \neq s$, $\lambda(v) = \infty$. Set $T = V$, the vertex set of G.
2. Let u be the vertex of T for which $\lambda(u)$ is minimum.
3. If $u = t$, stop.
4. For every edge $e = uv$ incident with u, if $v \in T$ & $\lambda(v) > \lambda(u) + \omega(e)$, change the value of $\lambda(v)$ to $\lambda(u) + \omega(e)$
5. Change T by $T - \{u\}$ and go to step 2

Scanned by CamScanner

Eg ⇒ ①



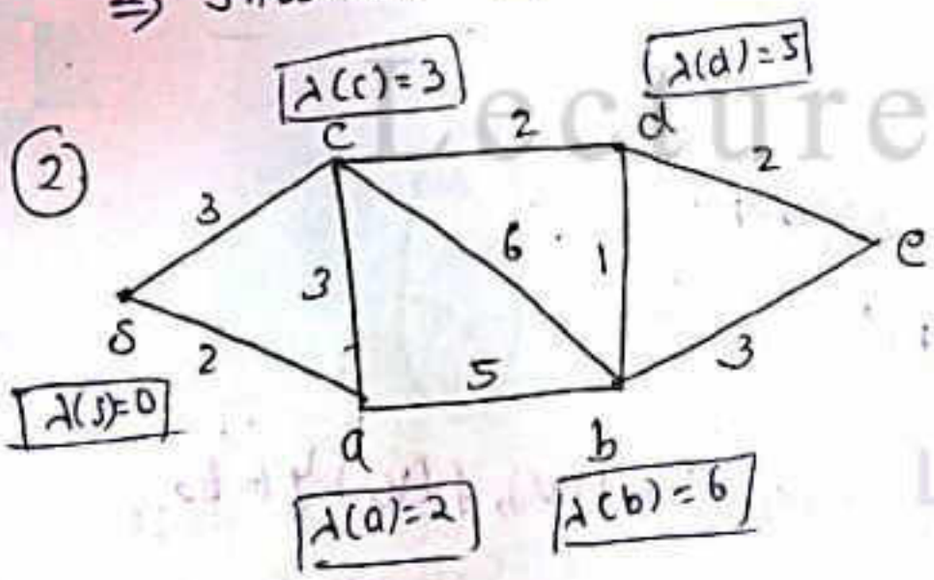


Find the shortest path from s to t.

	s	a	b	c	d	t	T
	0	∞	∞	∞	∞	∞	{s, a, b, c, d, t}
		18	∞	15	∞	∞	{a, b, c, d, t}
λ(v)		18 (as 21>18)	29		22	∞	{a, b, d, t}
			27		22	∞	{b, d, t}
			27 (32>27)			58	{b, t}
						55	{t}

⇒ shortest path from s to t is 55.

②



Find the shortest path from s to e

	s	a	b	c	d	e	T
	0	∞	∞	∞	∞	∞	{s, a, b, c, d, e}
		2	∞	3	∞	∞	{a, b, c, d, e}
λ(v)			7	3	∞	∞	{b, c, d, e}
			7		5	∞	{b, d, e}
			6			7	{b, e}
						7	{e}

⇒ shortest path from s to e is 7

LectureNotes.in

Scanned by CamScanner

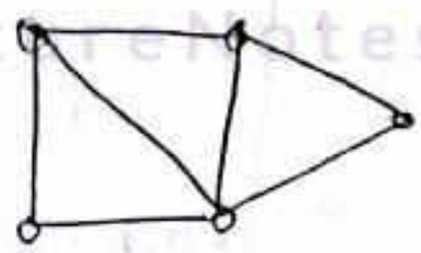
Syllabus Tree is connected acyclic graph.

Order & Size of a Graph ⇒ Let $G = (V, E)$ be a graph

⇒ $|V|$ is the total no. of vertices of the graph is its Order.

⇒ $|E|$ is the total no. of edges of the graph is its Size.

Eg ⇒



Order = 5
Size = 7

VV-Imp
Graph Isomorphism ⇒

Let $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are two graphs. Both graphs are said to be Isomorphic to each other iff.

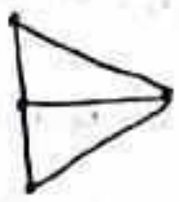
① Both have same order
i.e. $|V_1| = |V_2|$

② Both have same size
i.e. $|E_1| = |E_2|$

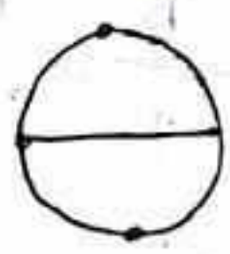
③ Both have same ~~order~~ degree sequence.

④ There exists a bijective func^n
$f : V_1 \rightarrow V_2$
such that if $\{v_1, v_2\} \in E_1$ then $\{f(v_1), f(v_2)\} \in E_2$

Eg ⇒ ① G_1 :



G_2 :



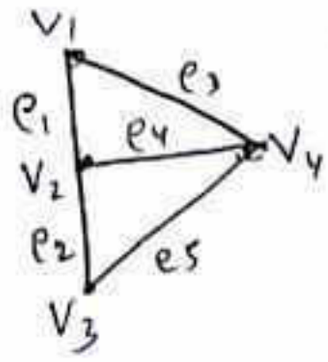
Whether Both graphs are isomorphic or not?

Verification

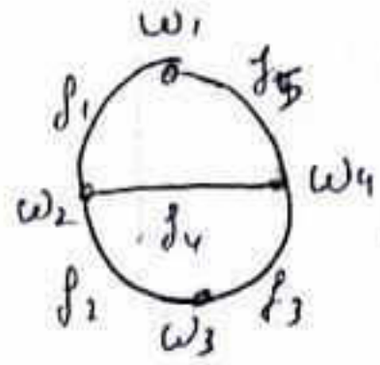
⇒ Let us name the vertices & edges.

Scanned by CamScanner

G_1 :



G_2 :



$V_1(G_1) = \{v_1, v_2, v_3, v_4\}$

$E_1(G_1) = \{e_1, e_2, e_3, e_4, e_5\}$

$V_2(G_2) = \{w_1, w_2, w_3, w_4\}$

$E_2\{G_2\} = \{f_1, f_2, f_3, f_4, f_5\}$

① $|V_1(G_1)| = 4 = |V_2(G_2)|$

Both graph have same order

② $|E_1(G_1)| = 5 = |E_2(G_2)|$

Both graphs have same size.

③ The degree sequence are

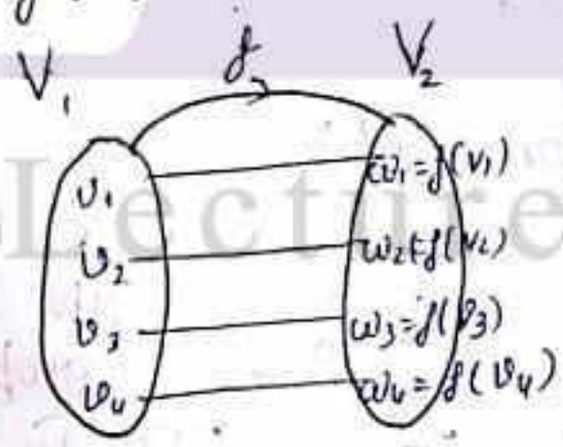
$G_1 : 2, 3, 2, 3 \equiv 3, 3, 2, 2$

$G_2 : 2, 3, 2, 3 \equiv 3, 3, 2, 2$

Both have same degree sequence

④ Let us define a bijective funct.

$f : V_1 \to V_2$



v_1 corresponds to either w_1 or w_3 as both have same degree

v_2 corresponds to either w_2 or w_4 as both v_2 & w_2 (or w_4) have same degree

Now.

$e_1 = \{v_1, v_2\} \in E_1(G_1) \Rightarrow \{f(v_1), f(v_2)\} = \{w_1, w_2\} \in E_2$

$e_2 = \{v_2, v_3\} \in E_1(G_1) \Rightarrow \{f(v_2), f(v_3)\} = \{w_2, w_3\} \in E_2$

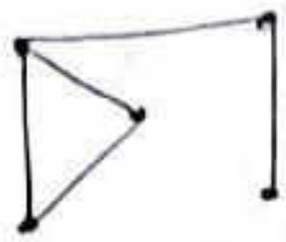
$e_3 = \{v_1, v_4\} \in E_1(G_1) \Rightarrow \{f(v_1), f(v_4)\} = \{w_1, w_4\} \in E_2$

$e_4 = \{v_2, v_4\} \in E_1 \Rightarrow \{f(v_2), f(v_4)\} = \{w_2, w_4\} \in E_2$

$e_5 = \{v_3, v_4\} \in E_1 \Rightarrow \{f(v_3), f(v_4)\} = \{w_3, w_4\} \in E_2$

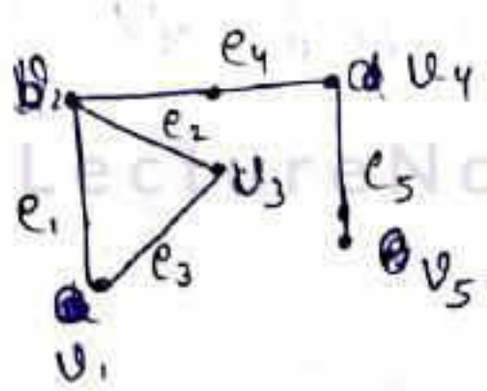
$\Rightarrow$ These both graphs are isomorphic.

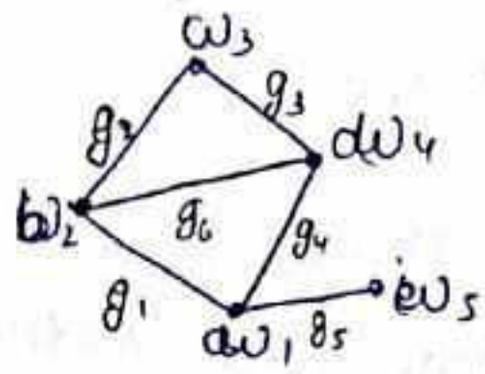
Scanned by CamScanner

(2) G_1 : 

G_2 :

*) Let us name the vertices & edges

G_1: 

G_2 

$V_1(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$; $E_1(G_1) = \{e_1, e_2, e_3, e_4, e_5\}$

$V_2(G_2) = \{$ ~~g_1, g_2, g~~ $w_1, w_2, w_3, w_4, w_5\}$; $E_2(G_2) = \{g_1, g_2, g_3, g_4, g_5, g_6\}$

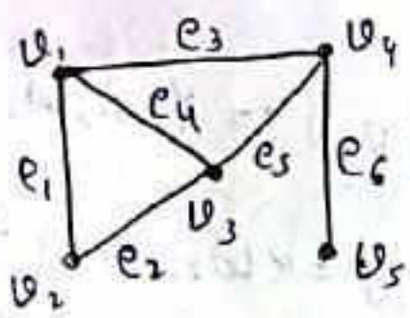
(1) $|V_1(G_1)| = |V_2(G_2)|$

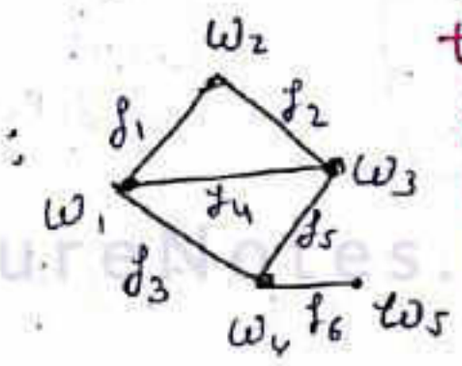
Both graphs have same vert order.

(2) $|E_1(G_1)| \neq |E_2(G_2)|$

$\Rightarrow$ Both graphs have not same size.

$\Rightarrow$ These graphs are not isomorphic

(3) G_1 : 

G_2 : 

We have to name the vertices of 2nd graph, according to 1st graph (i.e. to see that which vertex has same degree) & then given the corresponding vertices name.

$V_1(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$; $V_2(G_2) = \{w_1, w_2, w_3, w_4, w_5\}$

$E_1(G_1) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$; $E_2(G_2) = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

*) $|V_1(G_1)| = |V_2(G_2)|$

$\Rightarrow$ Both have same order.

*) $|E_1(G_1)| = |E_2(G_2)|$

$\Rightarrow$ Both have same size

Scanned by CamScanner

(3) The degree sequence are

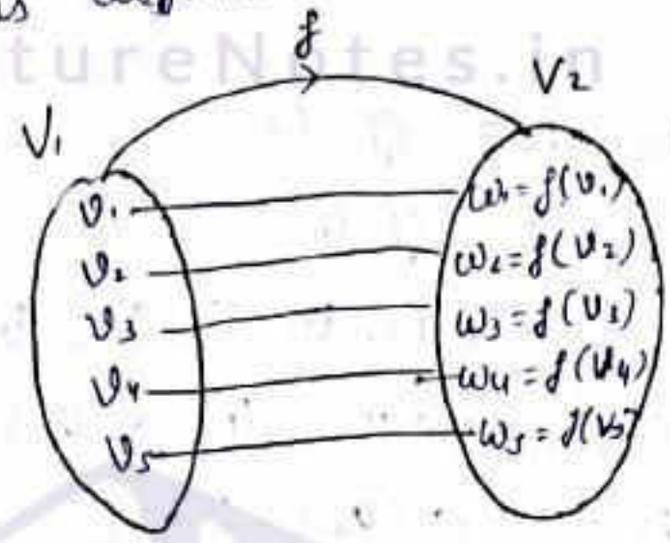
$G_1 : 3, 2, 3, 3, 1 \equiv 3, 3, 3, 2, 1$

$G_2 : 3, 2, 3, 3, 1 \equiv 3, 3, 3, 2, 1$

Both have same degree sequence

Try to solve problem

x) Let us define a bijective func^n.



$e_1 = \{v_1, v_2\} \in E_1(G_1) \Rightarrow \{f(v_1), f(v_2)\} = \{w_1, w_2\} \in E_2$

$e_2 = \{v_2, v_3\} \Rightarrow \{f(v_2), f(v_3)\} = \{w_2, w_3\} \in E_2$

$e_3 = \{v_1, v_4\} \Rightarrow \{f(v_1), f(v_4)\} = \{w_1, w_4\} \in E_2$

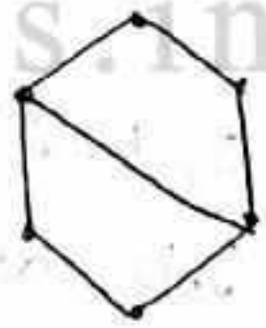
$e_4 = \{v_1, v_3\} \Rightarrow \{f(v_1), f(v_3)\} = \{w_1, w_3\} \notin E_2$

$e_5 = \{v_3, v_4\} \Rightarrow \{f(v_3), f(v_4)\} = \{w_3, w_4\} \in E_2$

$e_6 = \{v_4, v_5\} \Rightarrow \{f(v_4), f(v_5)\} = \{w_4, w_5\} \in E_2$

(4) G_1 :

G_2 :

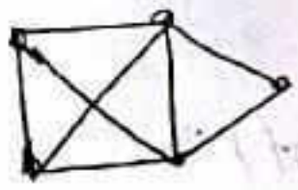


Both graphs are not isomorphic as both have different orders

$|V_1(G_1)| = 5, \quad |V_2(G_2)| = 6$

Not isomorphic.

(5) G_1 :



G_2 :

x) $|V_1(G_1)| = 5 = |V_2(G_2)|$

x) $|E_1(G_1)| = 8, \quad |E_2(G_2)| = 7$

Both have different size.

So graphs are not isomorphic

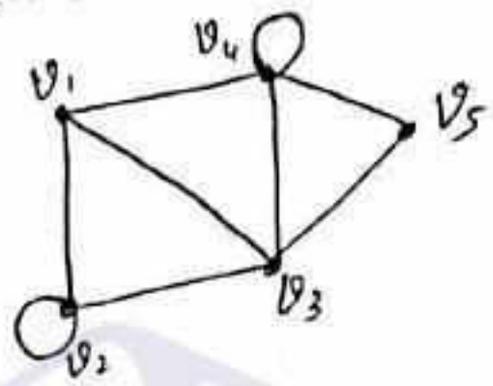
Scanned by CamScanner

27/10/16

Degree Sequence ⇒

Let $G = (V, E)$ be a graph. V be the vertex set with vertices $v_1, v_2, v_3 \ldots v_n$ having degrees $d_1, d_2, d_3 \ldots d_n$ respectively. Writting the degrees either in ascending or descending order it forms a degree sequence.

Eg ⇒



the order of $G = 5$

Size of $G = 9$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

$d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 4$, $d(v_4) = 5$, $d(v_5) = 2$

The degree sequence is

5, 4, 4, 3, 2

OR

2, 3, 4, 4, 5

<u>Notes</u> ⇒ *) A degree sequence $d_1, d_2, d_3 \ldots d_n$ is said to be GRAPHIC / GRAPHICAL if there exists a simple graph with the given degree sequence.

Eg ⇒ (1) 5, 4, 3, 3, 2, 2, 2

This degree sequence is graphical or not?

This degree sequence is not graphic

<u>Reason</u> ⇒ From first theorem of graph theory, we have the no. of odd vertices should be even, but here 3 vertices with odd degree.

Havel - Hakimi Theorem (Without Proof) ⇒

The degree sequence

$s, t_1, t_2, d_3 \ldots t_s, d_1, d_2, d_3 \ldots d_n$ is graphic

LectureNotes.in

Scanned by CamScanner

(iii) $d_1 - 1, d_2 - 1, d_3 - 1 \ldots d_s - 1, d_1, d_2, d_3 \ldots d_n$ is GRAPHI

Eg ⇒ (1) Is it graphic?

6, 6, 6, 6, 4, 3, 3, 0.

Using Havel - Hakimi Theorem

(1) Here the no. of odd vertices is even

(2) So now we have to verify for Havel - Hakimi Theorem.

6, 6, 6, 6, 4, 3, 3, 0

5, 5, 5, 3, 2, 2, 0

4, 4, 2, 1, 1, 0

3 1, 0, 0, 0

0, -1, -1, -1

It is not graphic.

As there are two vertices with (-ve) degrees.

That is not possible.

(2) 6, 5, 5, 4, 3, 3, 2, 2, 2. Is it graphic?

No. of odd vertices is 4 i.e. even

Now

6, 5, 5, 4, 3, 3, 2, 2, 2

4, 4, 3, 2, 2, 1, 2, 2 ⇒ This is not a degree sequence so first make degree sequence

≡ 4, 4, 3, 2, 2, 2, 2, 1

3, 2, 1, 1, 2, 2, 1 ⇒ Not a degree sequence

≡ 3, 2, 2, 2, 1, 1, 1

1, 1, 1, 1, 1, 1

It is graphical

Scanned by CamScanner

Bipartite Graph ⇒ Let $G=(V,E)$ be a graph.
It is said to be Bipartite Graph iff.

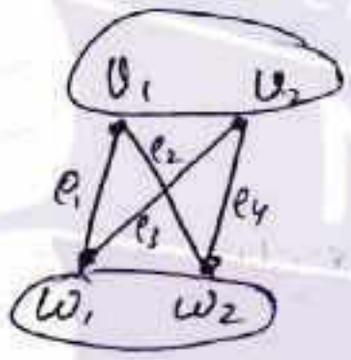
its vertex set V can be partitioned into two sets X & Y $[X \cup Y = U,\ X \cap Y = \phi]$ such that one end of each edge lies in set X where the other end lies in set Y.

*) $V = X \cup Y$ is called as BiPartition.

*) If the BiPartite Graph is simple then it is said to be <u>COMPLETE BIPARTITE GRAPH</u>

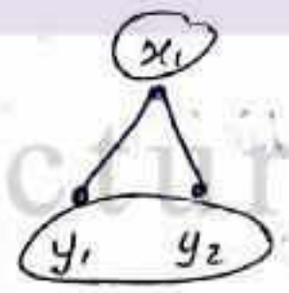
It is denoted by $\boxed{K_{m,n} \text{ where } |X|=m \ \& \ |Y|=n}$

Eg ⇒ ①

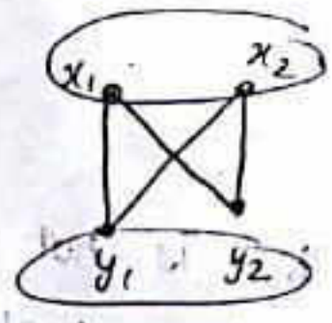


$X = \{v_1, v_2\}$
$Y = \{w_1, w_2\}$
⇒ Bipartite Graph

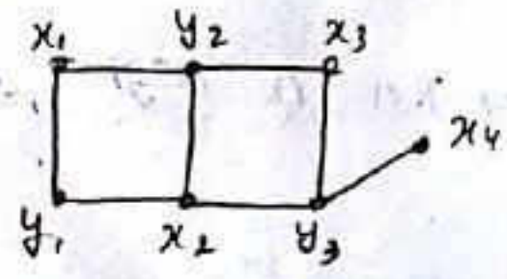
② $K_{1,2}$:



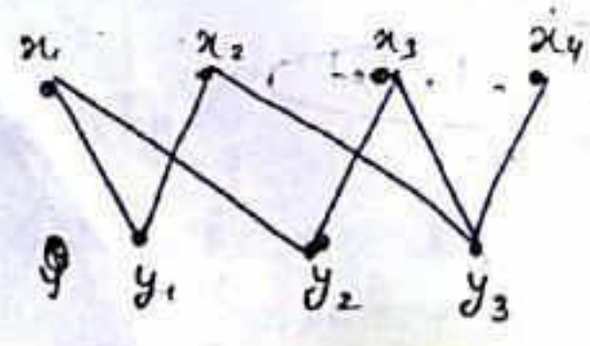
③ $K_{2,2}$:



④ Is the following graph Bipartite?



Redrawn



$X = \{x_1, x_2, x_3, x_4\}$
$Y = \{y_1, y_2, y_3\}$
Here $X \cup Y = U$
$X \cap Y = \phi$
⇒ Bipartite

LectureNotes.in

Scanned by CamScanner

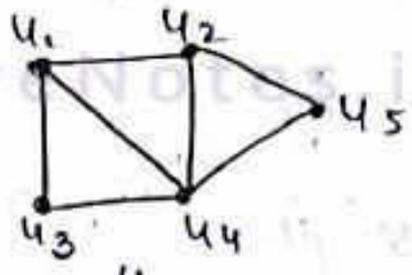
Subgraph ⇒ Let $G = (V, E)$ be a graph
$H = (V_1, E_1)$ be another graph

$\boxed{\text{H is said to be subgraph of the graph G}}$ iff

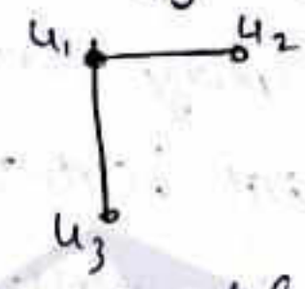
$$V(H) \subseteq V(G) \ \& \ E(H) \subseteq E(G)$$

In this case also we can say $\boxed{\text{G is the Supergraph of H.}}$

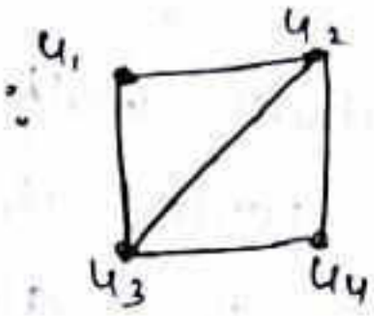
Eg ⇒ ① G :



H_1 :



H_2 :



H_1 is subgraph of G

H_2 is not a subgraph of G.

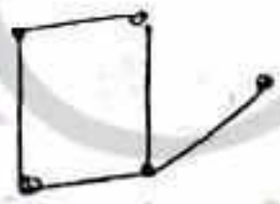
V. Imp

Spanning SubGraph ⇒ The subgraph H of the Graph G is called as spanning subgraph

iff $V(H) = V(G)$.

Eg ⇒ G :

H :

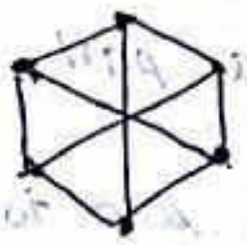


H be the spanning subgraph of G.

Complement of a Graph ⇒ Let G be a simple graph with N vertices. $\overline{G}$ is the compliment of the graph G defined as a simple Graph with the same vertex set of G where 2 vertices v_1 & v_2 are adjacent precisely when they are not adjacent in G.

Eg ⇒ ① G :

$\overline{G}$:



② G :

, $\overline{G}$:

Scanned by CamScanner

Self Complimentary

A simple graph is said to be self complimentary if it is isomorphic to its own compliment.

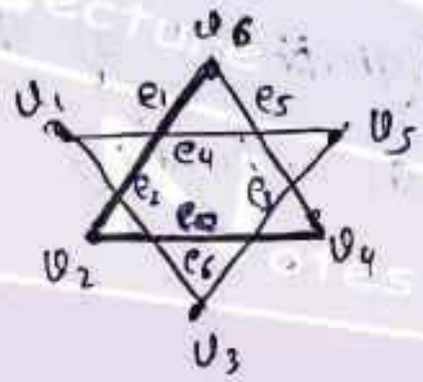
Walk ⇒ A walk in a graph is a finite sequence

$W = v_0\, e_1\, v_1\, e_2\, v_2 \ldots v_{k-1}\, e_k\, v_k$ whose terms are alternatively vertices & edges. Each edge is preceeded & succeeded by two vertices

Eg ⇒ The above walk of origin v_0 (initial vertex) & terminal vertex v_k, the remaining vertices $v_1, v_2, v_3 \ldots v_{k-1}$ are the Internal Vertices.

This walk is also called as $\boxed{v_0 - v_k \text{ walk}}$

Eg ⇒ ① G :



Walk ⇒ $v_0 - e_1 - v_2 - e_6 - v_4$

Length of the walk ⇒ No. of edges contained in the walk is length.

Closed Walk ⇒ If $v_0 = v_k$ then walk is said to be closed walk.

Open walk ⇒ If $v_0 \neq v_k$

Trail ⇒ If the edges contained in a walk are all distinct then the walk is said to be trail.

Path ⇒ Similarly, the vertices contained in a walk are all distinct then the walk is said to be a path.

Trivial Walk ⇒ The walk containing no edges is said to be a trivial walk.

Cycle ⇒ A non-trivial closed trail is said to be a cycle if its origin & all the internal vertices are distinct.

Scanned by CamScanner

28/10/11

Acyclic Graph ⇒ A Graph without cycle is said to be acyclic graph.

☆ Y Since cycles are of length 1 & parallel edges produces a cycle of length 2. So, any acyclic graph must be simple.

Tree ⇒ Any connected acyclic Graph is a Tree.

Spanning Tree ⇒ Any spanning subgraph of graph G which is a Tree is said to be spanning Tree.

Minimal ST ⇒ The spanning tree of graph G having minimum weight is said to be Minimal Spanning Tree.

How to find Minimal Spanning Tree ⇒

Kruskal's Algorithm ⇒

① Choose e_1, an edge of G, such that $w(e_1)$ is as small as possible & e_1 is not a loop.

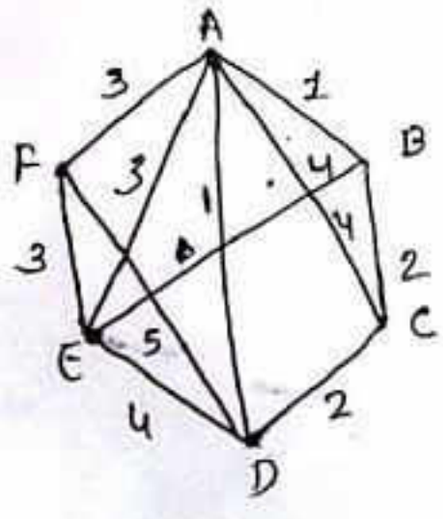
② If edges $e_1, e_2 \ldots e_i$ have been choosen, then choose the edge e_{i+1}, not already choosen, such that

i) The Induced subGraph $G[\{e_1, e_2 \ldots e_n\}]$ is acyclic

ii) $w(e_{i+1})$ is as small as possible

③ If G has n-vertices stop after (n-1) edges have been choose, otherwise repeat step 2.

Eg ⇒ Find the Minimal spanning tree by using Kruskal's Algorit



Scanned by CamScanner

Edges	Weight	
AB	1	step 1
AC	4	
AD	1	step 2
AE	3	step 4
AF	3	step 5
BC	2	step 3
BE	4	
CD	2	→ X
DE	4	
DF	5	
EF	3	

In step 1 consider that edge whose weight is smallest i.e. here AB or AD

In step 2 consider next smallest edge i.e. i.e. here AD

In step 3 ⇒ next smallest edge with weight 2 is BC, CD

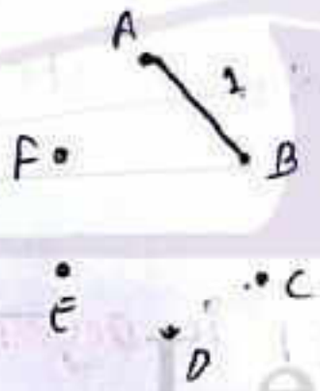
i.e. Here we consider BC

In step 4 ⇒ ∵ next smallest edge is CD but after joining CD it becomes cyclic, so we dont choose CD.

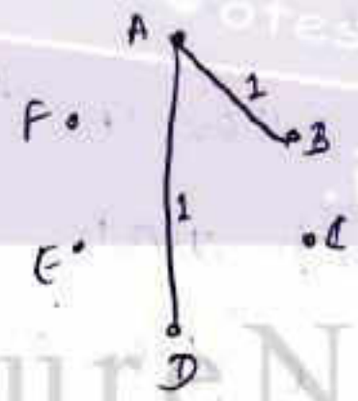
∴ the next smallest edge with weight 3 is AE, AF, EF, choose anyone i.e. here AE.

In step 5 ⇒ we choose AF.

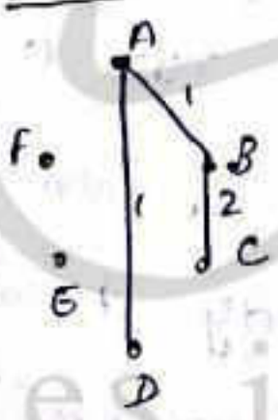
Step 1 ⇒



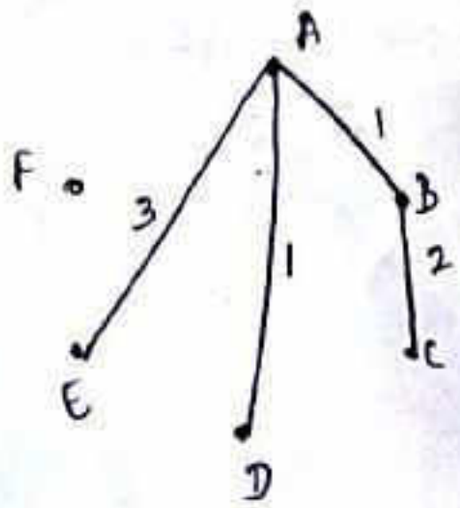
Step 2



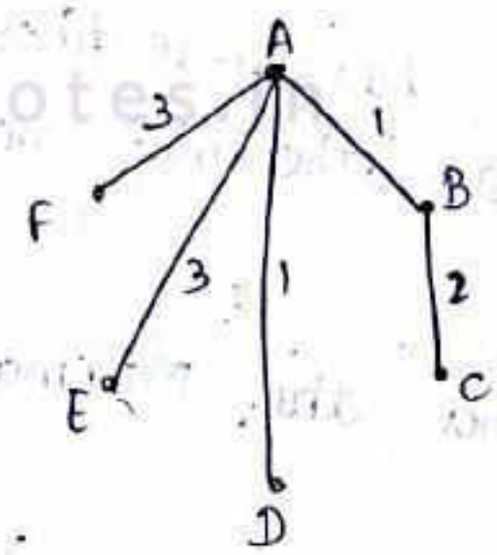
Step 3



Step 4 ⇒ The next minimum wt is 2. for the edge CD, but by considering it the subgraph will be cyclic. so it can not considered.



Step 5 ⇒



LectureNotes.in

Scanned by CamScanner

Prim's Algorithm ⇒

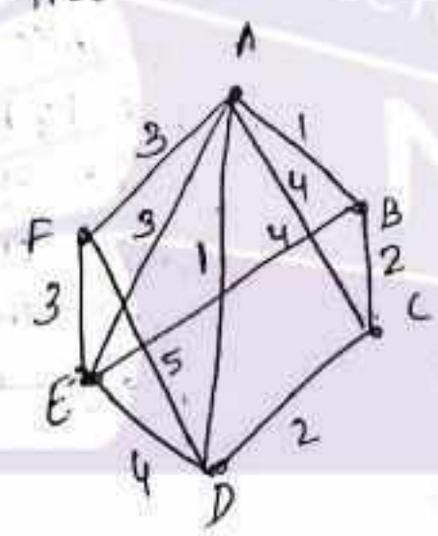
① Choose any vertex v_1 of the graph G.

② Choose an edge $e_1 = v_1 v_2$ such that $v_2 \neq v_1$ & e_1 has smallest weight among the edges of G incident with v_1.

③ If edges $e_1, e_2, e_3 - - e_i$ have been choose an edge $e_{i+1} = v_j v_k$ with $v_j \in \{v_1, v_2, - - v_{i+1}\}$ & $v_k \notin \{v_1, v_2, - - v_{i+1}\}$ such that e_{i+1} has smallest weight among the edges of G with precisely one end in $\{v_1, v_2 - - v_{i+1}\}$

④ Stop after $n-1$ edges have been choose if G has n-vertice otherwise go to step-3.

Eg ⇒ Find the minimal spanning tree by using Prim's Algo

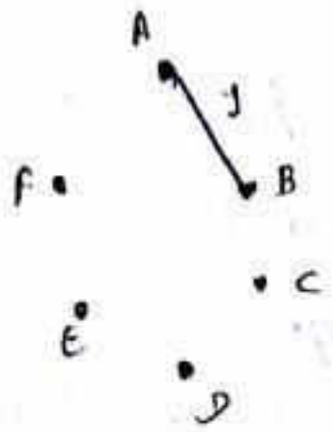


Let us choose vertex A

	Edges	weight	
I	AB	1	1
	AC	4	
	AD	1	2
	AE	3	4
	AF	3	5
II	BC	2	3
	BE	4	
III	DC	2	✗
	DE	4	
	DF	5	
IV	✗		
V	EF	3	

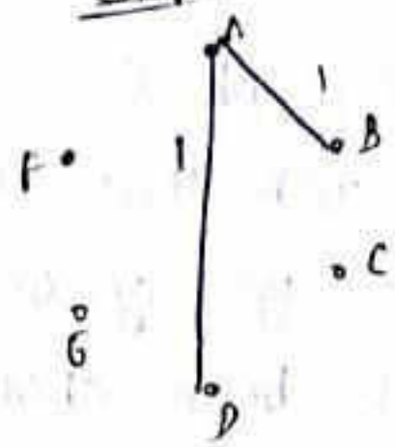
Scanned by CamScanner

Step 1 :



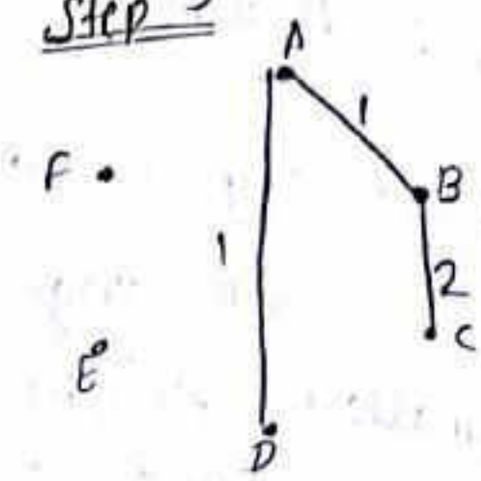
B is ending point so we start to write the weight of that edges which starts from B as then compare all the weight starting from Edge A to the last i.e. from AB - BE

Step 2



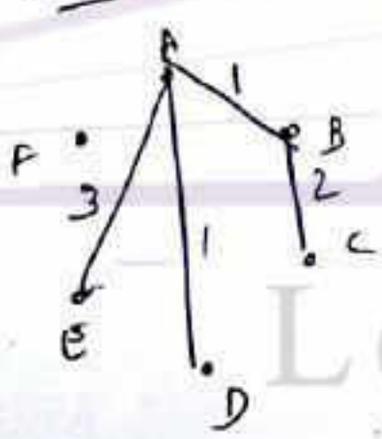
By comparing we find that AD is the next smallest weight so we consider this. Now end point is D. so we start writing the weight of that edges which starts from D & compare from AB to DF

Step 3



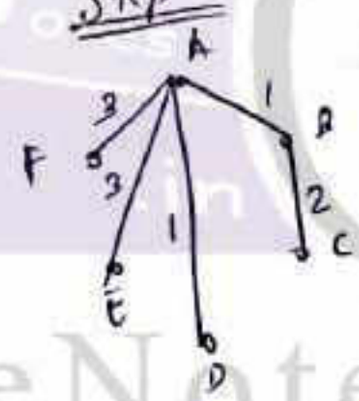
On comparing from AB to DF we found that BC is smallest one so we draw this. Now C is end point so we write the weight of that edges which start from C but we see that all that edges started from C are already written so we choose next smallest weighted edge i.e. DE but by making DC it becomes cyclic. so we don't consider DC & chose next smallest weighted edge i.e. AE.

Step 4 :



Now E is the ending point. Now write the weight of that edges which starts from E & then compare from AB - EF. & found the smallest one

Step 5



from AB-EF the smallest one is AF, EF. so we took AF. draw the AF. & hence we got MST.

Scanned by CamScanner

Related to trail & cycle there are two graphs ⇒

Euler Trail ⇒ A trail is said to be euler trail if it includes every edge of the graph G.

Tour ⇒ A tour is a closed walk which includes every edge atleast ones

So,

Euler Tour ⇒ It is a tour which includes each edge exactly ones.

Eulerian Graph ⇒ ~~if a graph~~ If a graph has an Euler ~~Tool~~ Tour then the graph is said to be Eulerian Graph.

Hamiltonian Path ⇒ A path is said to be Hemiltanion path if it contains every vertex of the graph.

Hemiltanion Cycle ⇒ It is a cycle which contains every vertex of the Graph.

Hemiltanion Graph ⇒ If a graph contain Hemiltanion Cycle then the graph is said to be Hemiltanion Graph.

Matrix Representation ⇒

Non directed Graph ⇒

Adjacency Matrix ⇒ Let $G = (V, E)$ be a simple graph with

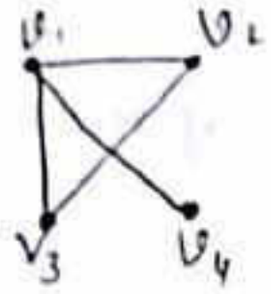
$$V = \{v_1, v_2, \ldots, v_n\}$$

Then the adjacency matrix is denoted by A_G where $A_G = (a_{ij})_{n \times n}$

$$a_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Scanned by CamScanner

Eg = ①



$$\Rightarrow A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

*) The adjacency matrix of a simple non directed graph is to be symmetric.

This is the correspondence for vertex set to vertex set.

Adjacency Matrix of any non-directed graph ⇒

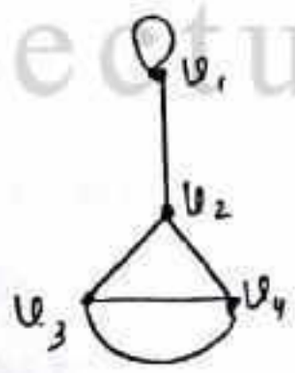
Let G be a graph with n-vertices listed as $v_1, v_2, v_3 \ldots v_n$

The adjacency matrix of G, with respect to this particular listing of the n-vertices of G is a $n \times n$ matrix.

$$A_G = (a_{ij})$$

where the (i,j)th entry is the number of edges joining the vertex v_i to v_j

Dic Eg ⇒

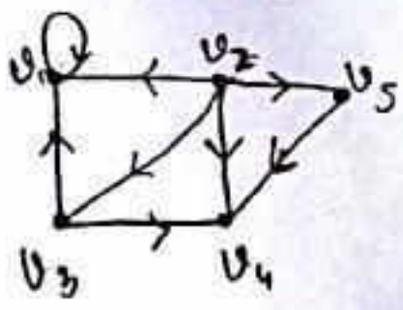


$$\Rightarrow A_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

Diagraph ⇒ Let (V,E) be the directed graph. $A_G = (a_{ij})$

where
$$a_{ij} = \begin{cases} 1, & (v_i, v_j) \in E \\ 0, & \text{otherwise.} \end{cases}$$

Eg ⇒



$$\Rightarrow A_4 = \begin{array}{c} \\ u_1 \\ v_2 \\ v_3 \\ v_4 \\ u_5 \end{array} \begin{array}{c} \begin{array}{ccccc} u_1 & u_2 & u_3 & u_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Scanned by CamScanner

Incidence Matrix ⇒ It is the matrix from vertex to edge.

for non directed graph

Let $G = (V, E)$ be a non directed graph.

with $V = \{v_1, v_2, v_3 \ldots v_m\}$

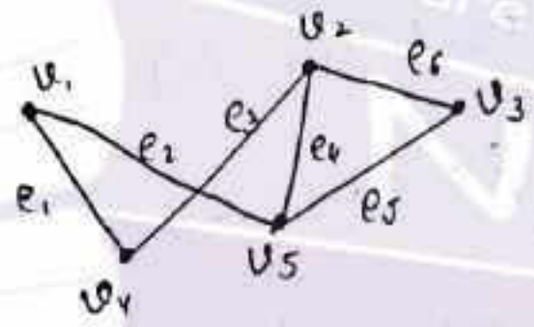
$E = \{e_1, e_2, e_3 \ldots e_n\}$

The Incidence Matrix is to be denoted by I_G.

where $I_G = (a_{ij})_{m \times n}$

$$a_{ij} = \begin{cases} 1, & \text{when } v_i \text{ is incident with } e_j \\ 0, & \text{otherwise} \end{cases}$$

Eg ⇒ G :



⇒

$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

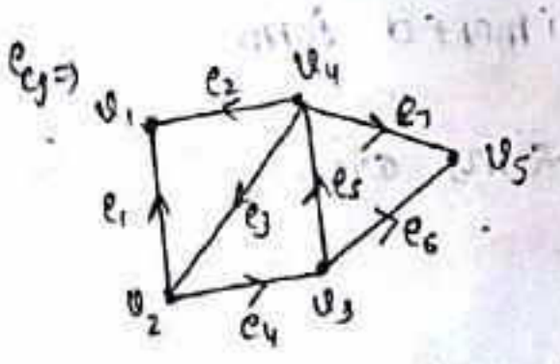
for Diagraph ⇒ Let $G = (V, E)$ be the graph with $V = \{v_1, v_2 \ldots v_m\}$

$E = \{e_1, e_2 \ldots e_n\}$

So $I_G = (a_{ij})_{m \times n}$

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the initial vertex of } e_j \\ -1, & \text{if } v_i \text{ is the terminal vertex of } e_j \\ 0, & \text{otherwise} \end{cases}$$

Eg ⇒



⇒

$$I_G = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix} \end{array}$$

Scanned by CamScanner