1 Proof of Lemma 1

Proof. Let us prove first that inequalities (3) are sufficient conditions for inclusion of X into Y.

Based on Lemma 1 of [2], let us define $\gamma_{lin}(Y_+)$ as

$$\gamma_{lin}(Y_+) = \bigcap_{1\leq i\leq k} \left\{ x \in \mathbb{R}^p | \left| {u_i}^{\mathrm{T}} x \right| \leq ||Y_+ u_i||_1 \right\}$$

where each u_i is normal to the faces of $\gamma(Y)$ (or equivalently of $\gamma_{lin}(Y_+)$). Let x be any point such that $x \in \gamma(X)$. Let $x = x' + c_x$ such that $x' \in \gamma_{lin}(X_+)$ and x'' be any point such that $x'' = x' + (c_x - c_y)$. Let us assume that

$$\left| \langle u_i, c_x - c_y \rangle \right| \leq ||Y_+ u_i||_1 - ||X_+ u_i||_1, \forall i = 1, \cdots, k$$

Under this assumption and also by Lemma 2 of [2] i.e., $\sup\limits_{x' \in \gamma_{lin}(X_+)} \langle u, x' \rangle = ||X_+ u||_1$, we can say that

$$\left| \langle u_i, c_x - c_y \rangle \right| + \langle u_i, x' \rangle \leq ||Y_+ u_i||_1$$

This implies $\langle u_i, x'' \rangle \leq ||Y_+ u_i||_1$ which means $x'' \in \gamma_{lin}(Y_+)$. Thus, $x \in \gamma(Y)$ where the difference between $\gamma_{lin}(Y_+)$ and $\gamma(Y)$ is the translation c_y.

Let us prove now that inequalities (3) are necessary conditions for inclusion of X into Y.

By Lemma 4 of [2], we know that if $\gamma(X) \subseteq \gamma(Y)$ then $\forall u$, $\left| \langle u_i, c_x - c_y \rangle \right| \leq ||Y_+ u||_1 - ||X_+ u||_1$. Thus, we can say that if $\gamma(X) \subseteq \gamma(Y)$ then
$\forall i = 1, \cdots, k \left| \langle u_i, c_x - c_y \rangle \right| \leq ||Y_+ u_i||_1 - ||X_+ u_i||_1$

$\square$