ALGEBRA Dato Ω è F tale che

$\Omega \in F \qquad A \in F \Rightarrow A^C \in F \qquad A, B \in F \Rightarrow A \cup B \in F$

ATOMO: $A: \forall B \neq 0 \in F, B \subset A \Rightarrow B = A$

σ −ALGEBRA Dato Ω, F tale che:

$\Omega \in F \qquad A \in F \Rightarrow A^C \in F \qquad \{A_k\}_{k=1}^{\infty} \subset F \Rightarrow \cup_{k=1}^{\infty} A_k \in F$
$(\cap_{k=1}^{\infty} A_k \in F)$

MISURA

È una funzione: $m(A \cup B) = m(A) + m(B) \ \ \forall A, B \in F: A \cap B = \emptyset$

POSITIVA: $m(A) \geq 0 \ \ \forall A \in F$ **PROBABILITA':** positiva + $m(\Omega) = 1$

FORMULE

<u>Probabilità condizionata:</u> $\mathbb{P}_B(A)\mathbb{P}(B) = \mathbb{P}(A \cap B)$

<u>Densità condizionata</u> $f_{X,Y}(t, y) = f_X(t) f_{Y|X=t}(y) \qquad f_{\mathbb{P}(X \leq t)} = \left(\mathbb{P}_{X \leq t}(X \leq x)\right)'$

<u>Formula di disintegrazione</u>: $\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(A \cap B_i)$

<u>Formula delle probabilità totali</u>: $\mathbb{P}(A) = \sum_{i=1}^{\infty} \mathbb{P}(B_i)\mathbb{P}_{B_i}(A)$

<u>Formula di Bayes</u>: $\mathbb{P}_B(A_k) = \frac{\mathbb{P}(A_k)\mathbb{P}_{A_k}(B)}{\sum_{i=1}^{\infty} \mathbb{P}(A_i)\mathbb{P}_{A_i}(B)} \left(\mathbb{P}_B(A) = \frac{\mathbb{P}(A)\mathbb{P}_A(B)}{\mathbb{P}(B)}\right)$

$$\mathbb{P}(A) = \mathbb{P}(B)\mathbb{P}_B(A) + \mathbb{P}(B^c)\mathbb{P}_{B^c}(A)$$

$$\mathbb{P}(A)\mathbb{P}_A(B) = \mathbb{P}(B)\mathbb{P}_B(A)$$

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1) \cdot \mathbb{P}_{A_1}(A_2) \cdot \mathbb{P}_{A_1 \cap A_2}(A_3) \cdot \ldots \cdot \mathbb{P}_{A_1 \cap \ldots \cap A_{n-1}}(A_n)$$

INSIEMISTICA + PROPRIETA' MISURE

$m(A \cup B) = m(A) + m(B) - m(A \cap B) \qquad B \subset A \Rightarrow 0 \leq m(B) \leq m(A) \leq 1$

$B \subset A \Rightarrow m(A \setminus B) = m(A) - m(B) \qquad A = (A \cap B) \cup (A \cap B^C) = (A \cap B) \cup (A \setminus B)$

$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (B^C \cup A^C) = (A \cup B) \cap (B \cap A)^C = (A \cup B) \setminus (B \cap A)$

$A \cap B^C = A \cap (A^C \cup B^C) = A \cap (A \cap B)^C = A \setminus (A \cap B)$

INDIPENDENZA

Eventi: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ **σ-algebre**: $\mathbb{P}(F \cap G) = \mathbb{P}(F)\mathbb{P}(G) \ \ \forall F \in \mathcal{F}, G \in \mathcal{G}$

VA: $f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad X \sqcup Y \Rightarrow g(X) \sqcup f(Y) \qquad A \sqcup B \Rightarrow A \sqcup B^C, A^C \sqcup B^C$

SOMMATORIA $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \qquad \sum_{k=0}^{n} k = \frac{n(n+1)}{2} \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

BINOMIALE $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

FUNZIONI GAMMA $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \qquad \Gamma(n) = (n-1)!$

CALCOLO COMBINATORICO

ORDINE	<u>Disposizioni</u>	$\frac{n!}{(n-k)!}$
	<u>Permutazioni</u>	$n!$
	<u>Perm con ogg ripetuti</u>	$P_n^{k_1,\ldots,k_r} = \frac{n!}{k_1! \ldots k_r!}$
	<u>Permutazioni con ripetizioni</u>	n^k
NO ORDINE	<u>Combinazioni</u>	$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

MOMENTI

$$\mathbb{E}[XY] = \sum_i \sum_j x_k y_j \mathbb{P}(X = x_k, Y = y_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f_{X,Y}(x, y) dx dy$$

$$\mathbb{E}[X] = \sum_n \mathbb{E}_{\{N=n\}}[X]\mathbb{P}(N = n)$$

$$\boxed{A = \cos(\theta)\sin(\theta) \Rightarrow \mathbb{E}[A] = \cos \int (\theta)\sin(\theta) f_\theta d\theta}$$

$X \sqcup Y \Rightarrow \mathbb{E}[XY] = E[X]\mathbb{E}[Y]$ e $\mathbb{E}[X^2Y^2] = \mathbb{E}[X^2]\mathbb{E}[Y^2] \qquad \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$V(X + Y) = V(X) + V(Y) + 2Cov(X, Y) \qquad X \sqcup Y \Rightarrow Cov(X, Y) = 0$

$$Cov(X, Y) = E[(X - E[X])(Y - E[Y])] = \mathbb{E}[XY] - \mathbb{E}[Y]\mathbb{E}[X]$$

$$V(X) = \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X|Y = y] = \sum_x x\mathbb{P}(X = x|Y = y)$$

$$\mathbb{E}[g(X)|Y = y] = \int_{\mathbb{R}} g(x) f_{X|Y=y} dx$$

LEMMI

$|\mathbb{E}[X]| \leq \mathbb{E}[|X|] \qquad \exists n + 1^{\circ} \Rightarrow \exists n^{\circ}$

$$\mathbb{E}[g(X)] = \sum g(x_k)\mathbb{P}(X = x_k) = \int_{-\infty}^{\infty} g(x) f(x) dx$$

COEFFICIENTE DI CORRELAZIONE $p(X, Y) = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}}$

VA CORRELATE: $\mathbb{E}[XY] \neq 0$

DISUGUAGLIANZE

<u>MARKOV</u>: $\mathbb{P}(X \geq t) \leq \frac{\mu_X}{t}$

<u>CHEBYSHEV</u>: $\mathbb{P}((X - \mu)^2 \geq t^2) \leq \frac{\sigma^2}{t^2} \Leftrightarrow \mathbb{P}(|X - \mu| > t) \leq \frac{V^2(X)}{t^2}$

<u>CAUCHY-SCHWARTZ</u>: $X, Y \in L^2$: $|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\mathbb{E}[Y^2]}$, $0 \Leftrightarrow Y = cX$

DISTRIBUZIONI

DISCRETA	$\mathbb{P}(\boldsymbol{X=0})=\boldsymbol{1-p}$ $\quad \mathbb{P}(\boldsymbol{X=1})=\boldsymbol{p}$ $\quad \mathbb{E}[\boldsymbol{X}]=\sum \boldsymbol{k}\,\mathbb{P}(\boldsymbol{X=k})$ $\quad \boldsymbol{X\geq 0}\Rightarrow \mathbb{E}[\boldsymbol{X}]=\sum_{\boldsymbol{k=0}}^{\infty}\boldsymbol{1-F_X(k)}$
UNIFORME	$X\sim Unif(\{1,\dots,N\})\Rightarrow \mathbb{E}[X]=\frac{N+1}{2}$ $\quad V[X]=\frac{N^2-1}{12}$ $\quad Y\sim Unif(a,b)\Rightarrow \mathbb{E}[Y]=\frac{a+b}{2}$ $\quad V(Y)=\frac{(b-a)^2}{12}$ $\quad f(x)=\frac{1}{b-a}\mathbb{I}_{[a,b]}(x)$ $Y\sim Unif(0,a)\Rightarrow M(t)={}^1\!/_{at}\,(e^{at}-1)$
GEOMETRICA	$k\in[1,\infty)\to \mathbb{P}(X=k)=(1-p)^{k-1}p$ $\quad \mathbb{P}(X\leq k)=1-(1-p)^k$ $\quad \mathbb{P}(X>n)=(1-p)^n$ $\mathbb{P}_{X>k}(X>k+j)=\mathbb{P}(N>j)$ $\quad \mathbb{E}[X]=\frac{1}{p}$ $\quad \mathbb{E}[X^2]=\frac{2(1-p)}{p^2}+\frac{1}{p}$ $\quad Var(X)=\frac{1-p}{p^2}$ $\quad M_X(t)=\frac{p}{1-p}\frac{(1-p)e^t}{1-(1-p)e^t}$
CONTINUA	$\left(\mathbb{P}(X\leq t)\right)'\mathbb{I}_{(\dots)}=f(x)=\frac{1}{b-a}\mathbb{I}_{(a,b)}(x)\Rightarrow \mathbb{P}\left(X\in(t,t+\epsilon)\right)=\frac{\epsilon}{b-a}$ $\quad \mathbb{P}(X\in A)=\int_A f(x)dx$ $\quad \mathbb{E}[X]=\int_{-\infty}^{\infty}x\,f(x)dx$ $X\geq 0\Rightarrow \mathbb{E}[X]=\int_0^{\infty}1-F_X(y)dy$
POISSON	$k\in\{0,1,\dots\}\to \mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$ $\quad \lambda>0$ $\quad \mathbb{E}[X]=\lambda$ $\quad \mathbb{E}[X^2]=\lambda^2+\lambda$ $\quad \mathbb{E}[X^3]=\lambda^3+3\lambda^2+\lambda$ $\mathbb{E}[X^4]=\lambda^4+6\lambda^3+7\lambda^2+\lambda$ $\quad V(X)=\lambda$ $\quad M_X(t)=e^{-\lambda}e^{\lambda e^t}$
ESPONENZIALE	$f(x)=\lambda e^{-\lambda x}\,\mathbb{I}_{[0,\infty)}(x)$ $\quad \mathbb{P}(X>t)=e^{-\lambda t}$ $\quad \mathbb{P}_{X>k}(X>k+j)=\mathbb{P}(N>j)$ $\quad \mathbb{E}[X]=\frac{1}{\lambda}$ $\quad Var(X)=\frac{1}{\lambda^2}$ $\quad M_X(t)=\frac{\lambda}{\lambda-t}$
BINOMIALE (**LIMITE CENTRALE**)	$n\geq 1,k=0,\dots,n\to X\sim B(n,p)\Rightarrow \mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$ $\quad \mathbb{E}[X]=np$ $\quad V(X)=npq$ $\quad M_X(t)=(1-p+pe^t)^n$
GAMMA	$X\sim\Gamma(n,\lambda)\Rightarrow f(x)=\frac{\lambda^n}{\Gamma(n)}x^{n-1}e^{-\lambda x}\mathbb{I}_{(0,\infty)}(x)$ $\quad \mathbb{E}[X^p]=\frac{\Gamma(n+p)}{\Gamma(n)\lambda^p}\Rightarrow \mathbb{E}[X]=\frac{n}{\lambda},\mathbb{E}[X^2]=\frac{n(n+1)}{\lambda^2}$ $\quad V(X)=\frac{n}{\lambda^2}$ $\quad M_X(t)=\left(\frac{\lambda}{\lambda-t}\right)^n$
BERNOULLI	$B(1,p)=B(p)$ $\quad \mathbb{E}[X]=p$ $\quad V(X)=p(1-p)$ $\quad M_X(t)=1-p+pe^t$ $\quad \boxed{\Gamma(n+1)=n!\text{ se intero positivo}}$
BETA	$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\mathbb{I}_{(0,1)}(x)$ $\quad \mathbb{E}[X]=\frac{a}{a+b}$ $\quad Var(X)=\frac{ab}{(a+b)^2(a+b+1)}$
GAUSSIANA	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$ $\quad X\sim N(\mu,\sigma^2)\Rightarrow M_X(t)=e^{\mu t+\frac{\sigma^2t^2}{2}}$ $\quad \mathbb{E}[X]=\mu$ $\quad Var(X)=\sigma^2$
GAUSSIANA STANDARD	$\mu=0,\ \ \sigma^2=1$ $\quad \mathbb{E}[X]=0$ $\quad \mathbb{E}[X^2]=1=Var(Z)$ $\quad \mathbb{P}(Z\leq x)=\phi(x)$

DISTRIBUZIONI PROPRIETA

$$P_0(\lambda)+P_0(\mu)=P_0(\lambda+\mu)$$

$$X\sim P_0(\lambda),Y\sim P_0(\mu)\Rightarrow \mathbb{P}_{X+Y=b}(X=a)=\binom{b}{a}\left(\frac{\lambda}{\lambda+\mu}\right)^a\left(1-\frac{\lambda}{\lambda+\mu}\right)^{b-a}$$

$$X_n\sim B(n,p_n),p_n\leq 1,np_n\to\lambda,p_n\to 0\Rightarrow X_n\overset{D}{\rightarrow}P_0(\lambda)$$

$$\Gamma(n,\lambda)+\Gamma(m,\lambda)=\Gamma(n+m,\lambda)$$

$$B(n,p)+B(m,p)=B(n+m,p)$$

$$X_i\sim\exp(\lambda)\Rightarrow\sum_k x_k\sim\Gamma(n,\lambda)$$

PROPRIETA' GAUSSIANA

$$\boxed{\begin{gathered}\mathbb{P}(|\boldsymbol{Z}|\leq\boldsymbol{x})=\mathbb{P}(-\boldsymbol{x}\leq\boldsymbol{Z}\leq\boldsymbol{x})=\mathbb{P}(\boldsymbol{Z}\leq\boldsymbol{x})-\mathbb{P}(\boldsymbol{Z}\leq-\boldsymbol{x})\\ \mathbb{P}(|\boldsymbol{Z}|>\boldsymbol{x})=\boldsymbol{1}-\mathbb{P}(|\boldsymbol{Z}|\leq\boldsymbol{x})\end{gathered}}\qquad \boxed{\boldsymbol{\phi(-t)=1-\phi(t)}}$$

$$Z\sim N(0,1)\Rightarrow -Z\sim N(0,1)$$

$$Z\sim N(0,1),X=\mu+\sigma Z\Rightarrow X\sim N(\mu,\sigma^2)$$

$$X\sim N(\mu,\sigma^2)\Rightarrow Z=\frac{X-\mu}{\sigma}\sim N(0,1)$$

COSTANTI $\mu\equiv N(\mu,0)$

$$\mathbb{E}[X]=\mu,Var(X)=\sigma^2\Rightarrow X\sim N(\mu,\sigma^2)$$

$$X\sim N(\mu,\sigma^2),Y\sim N(m,s^2),X\sqcup Y\Rightarrow X+Y\sim N(\mu+m,\sigma^2+s^2)$$

$$\int_{-\infty}^{\mu}f(x)dx=\frac{1}{2}$$

VETTORI GAUSSIANI

$$\underline{X}\,|\,Y=\sum_i a_iX_i\sim N(\mu_Y,\sigma_Y^2)\ \forall i \qquad M_{\underline{X}}(\underline{u})=\mathbb{E}[e^{<\underline{u},\underline{x}>}]$$

$\underline{X}\sim N\left(\underline{\mu},C\right)\Leftrightarrow M_{\underline{X}}(u)=e^{<\underline{u},\underline{\mu}>+{}^1\!/_2<Cu,u>}$, $\ \det(C)>0\Rightarrow f_{\underline{X}}(x)=\frac{1}{\sqrt{(2\pi)^2\det|C|}}\exp\left(-\frac{1}{2}\left\langle C^{-1}\left(\underline{x}-\underline{\mu}\right),\underline{x}-\underline{\mu}\right\rangle\right)$ con C matrice di covarianza

DEGENERE: $\underline{X}\sim(\mu,A^2)$ $\quad \det(A^2)=0\Rightarrow$non continua

VETTORE GAUSSIANO: (X,Y) vettore gaussiano: $\quad Cov(X,Y)=0\Leftrightarrow X\sqcup Y\Leftrightarrow\mathbb{E}[XY]=0$ $\quad M_{(X,Y)}(u,v)=M_X(u)M_Y(v)\Rightarrow X\sqcup Y$

CREARE VETTORI GAUSSIANI:

- $X_1,\dots,X_n$ gauss ind$\Rightarrow\underline{X}=(X_1,\dots,X_n)\sim N\left(\underline{\mu},C\right)$ con $\underline{\mu}=(\mu_1,\dots,\mu_n)$ $\quad C=diag(\sigma_1^2,\dots,\sigma_n^2)$
- $\underline{X}\in N(\mu,C),A$ matr $k\times n,k\leq n,b\in R^k\Rightarrow\underline{Y}=A\underline{X}+b$ $\quad \underline{Y}\sim N(\overline{\mu},\overline{C})$ $\quad \overline{\mu}=A\mu+b,\overline{C}=ACA^T$ matr cov
- $\underline{X}=(X_1,\dots,X_n)$ gauss multivariato $\Rightarrow X_i=(0,\dots,1,\dots,0)(X_1,\dots,X_n)^T$ $\quad X_1+X_2=(1,1,\dots,0)(X_1,\dots,X_n)^T$