

Rigid Configuration of Interior Zeros: Theorem, Lemma and Classification

Luis Morató de Dalmases

1. General Framework

Let:

- $L = [0, d]$ be a closed interval,
- $V \in L^\infty(0, d)$ be a symmetric potential $V(x) = V(d - x)$,
- $\mathcal{L} = -\frac{d^2}{dx^2} + V(x)$ acting on $L^2(0, d)$ with Dirichlet conditions $\psi(0) = \psi(d) = 0$.

Assume that the spectrum of \mathcal{L} is discrete and simple:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$$

We define **admissible functions** as linear combinations of the first two eigenfunctions:

$$\psi = c_1\psi_1 + c_2\psi_2, \quad (c_1, c_2) \neq (0, 0).$$

2. Main Proposition

Proposition 1 (Rigid interior zeros). *For any admissible function ψ :*

1. *There exists a central interior zero $z_C = d/2$.*
2. *There exist exactly two symmetric interior zeros $z_1 \in (0, d/2)$ and $z_2 = d - z_1$.*
3. *There are no other interior zeros.*

*This configuration is **rigid**: any admissible variation that attempts to reduce the number of interior zeros **increases the energy***

$$\Gamma[\psi] = \int_0^d (|\psi'(x)|^2 + V(x)|\psi(x)|^2) dx$$

to second order.

3. Nodal Lemma

Lemma 2 (Sturm–Liouville principle for linear combinations). *If ψ_1, ψ_2 are the first two simple eigenfunctions of \mathcal{L} :*

- ψ_1 has exactly one interior zero at $x = d/2$ (antisymmetric).
- ψ_2 has exactly two interior zeros symmetric about the midpoint (symmetric).

Any linear combination $\psi = c_1\psi_1 + c_2\psi_2$ has at most three interior zeros, located symmetrically about the midpoint.

Proof of Lemma:

By Sturm–Liouville oscillation, ψ_n has exactly n interior zeros.

The symmetry of V guarantees that ψ_1 is antisymmetric \Rightarrow zero at $d/2$.

ψ_2 is symmetric \Rightarrow two symmetric interior zeros.

A linear combination can shift interior zeros but cannot increase their number beyond the sum of the first two.

The maximum configuration is exactly three zeros: one central and two symmetric.

□

4. Classification of Possible Nodal Configurations

Let $Z = \{z_1, z_C, z_2\}$ be the interior zeros:

Configuration	Number of zeros	Position
Central only	1	$z_C = d/2$
Central + 1 lateral	2	z_C, z_1 or z_C, z_2
Central + two laterals	3	z_1, z_C, z_2 , with $z_2 = d - z_1$
No zeros	0	Impossible by Sturm–Liouville

Remarks:

- The only fixed and inevitable zero is the central one $z_C = d/2$.
- Symmetry imposes that additional zeros are always symmetric.
- No configuration has more than three interior zeros.

5. Application to Special Functions

Example 1: Completed Gamma function

$$\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$$

Consider the real embedding:

$$\psi(t) = \operatorname{Re}(\log \Gamma_R(1/2 + it))$$

The derivative $\psi'(t)$ has interior zeros satisfying the same symmetry rule.

The three positions correspond to nodes where ψ becomes extremal with respect to symmetry.

Example 2: Completed function

$$\xi(s) = \Gamma_R(s)\zeta(s)$$

The eigenfunctions of the type

$$\psi(t) = \operatorname{Re} \log \xi(1/2 + it)$$

satisfy natural Dirichlet conditions on intervals where $\zeta(1/2 + it) = 0$.

The nodal rigidity theorem guarantees that the zeros of $\xi(s)$ follow the three-node symmetric configuration in any interval where a second-order approximation can be considered.

6. Complete Proof of the Main Theorem

Proof:

By Lemma 1, ψ_1 and ψ_2 have a fixed number of interior zeros: 1 and 2 respectively.

For a linear combination $\psi = c_1\psi_1 + c_2\psi_2$, the number of interior zeros cannot exceed 3.

The symmetry of V ensures that one zero is fixed at $d/2$ (central).

The other two zeros (if they exist) are symmetric about the midpoint.

The energy functional $\Gamma[\psi]$ is coercive: any attempt to move or remove interior zeros increases Γ to second order.

Conclusion: exactly three interior zeros, with one central and two symmetric, is the only stable nodal configuration.

□

7. Conclusion

This formulation is purely spectral-topological, applicable to any symmetric self-adjoint operator on an interval.

The nodal classification is complete and exhaustive.

Special functions such as $\Gamma(s)$ or $\xi(s)$ can be treated as particular cases.

Symmetry and coerciveness guarantee nodal rigidity.