Gaussian Elimination: Solving system of equations

Key idea: keep track of transformations performed in putting matrix in echelon form.

Given matrix A, compute matrices M and U such that $MA = U$

- U is in echelon form
- M is invertible

To solve $A\mathbf{x} = \mathbf{b}$:

- Compute M and U so that $MA = U$
- Compute the matrix-vector product $M\mathbf{b}$, and solve $U\mathbf{x} = M\mathbf{b}$.

Claim: This gives correct solution to $A\mathbf{x} = \mathbf{b}$
Proof: Suppose $\mathbf{v}$ is a solution to $U\mathbf{x} = M\mathbf{b}$, so $U\mathbf{v} = M\mathbf{b}$

- Multiply both sides by M^{-1}: $M^{-1}(U\mathbf{v}) = M^{-1}M\mathbf{b}$
- Use associativity: $(M^{-1}U)\mathbf{v} = (M^{-1}M)\mathbf{b}$
- Cancel M^{-1} and M: $(M^{-1}U)\mathbf{v} = \mathbb{1}\mathbf{b}$
- Use $M^{-1}U = A$: $A\mathbf{v} = \mathbb{1}\mathbf{b} = \mathbf{b}$

How to solve $U\mathbf{x} = M\mathbf{b}$?

- If U is triangular, can solve using *back-substitution* (`triangular_solve`)
- In general, can use similar algorithm

Gaussian Elimination: Finding basis for null space

Instead of finding basis for null space of A, find basis for $\{\mathbf{u} \ : \ \mathbf{u} * A = \mathbf{0}\} = \text{Null } A^T$

Input:

	A	B	C	D
0	1	0	1	0
1	1	1	1	0
2	0	1	0	1
3	1	1	1	1
4	0	0	0	1

Find M, U such that $MA = U$ and U is in echelon form and M is invertible

$$\underbrace{\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 1 \end{array}}_{M} * \underbrace{\begin{array}{c|cccc} & A & B & C & D \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 \end{array}}_{A} = \underbrace{\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}}_{U}$$

Last two rows of U are zero vectors

- Row 3 of U is (row 3 of M) * A
- Row 4 of U is (row 4 of M) * A

Gaussian Elimination: Finding basis for null space

Find M, U such that $MA = U$ and U is in echelon form and M is invertible

$$\underbrace{\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 & 0 & 1 \end{array}}_{M} \quad * \quad \underbrace{\begin{array}{c|cccc} & A & B & C & D \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 0 & 1 \end{array}}_{A} \quad = \quad \underbrace{\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}}_{U}$$

Last two rows of U are zero vectors

- Row 3 of U is (row 3 of M) * A
- Row 4 of U is (row 4 of M) * A

Therefore two rows in $\{\mathbf{u} : \mathbf{u} * A = \mathbf{0}\}$ are rows 3 and 4 of M
To show that these two rows form a basis for $\{\mathbf{u} : \mathbf{u} * A = \mathbf{0}\}$....
$\dim \text{Row } A = 3$
By Rank-Nullity Theorem, $\dim \text{Row } A + \dim \text{Null } A^T = \text{number of rows} = 5$
Shows that $\dim \text{Null } A^T = 2$
Since M is invertible, all its rows are linearly independent.

Gaussian elimination: recording the transformations

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & U_1 & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & M_1 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & U_2 & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & M_2 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & M_1 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & U_3 & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & M_3 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & M_2 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & M_1 & \\ & & \end{bmatrix}\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & U_4 & \\ & & \end{bmatrix}$$

Gaussian elimination: recording the transformations

$$\begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \text{-}2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 0 & -1 & 2 & -6 & -6 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \text{-}2.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \text{-}2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 0 & -1 & 2 & -6 & -6 \\ 0 & -2.5 & 0 & -10.5 & -2 \end{bmatrix}$$

Gaussian elimination: recording the transformations

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \text{-2} & 1 & 0 \\ 0 & \text{-2.5} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 0 & -1 & 2 & -6 & -6 \\ 0 & -2.5 & 0 & -10.5 & -2 \end{bmatrix}$$

Gaussian elimination: recording the transformations

$$\begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \text{-}2 & 1 & 0 \\ 0 & \text{-}2.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 4 & 1 & 2 & 4 & 2 \\ 5 & 0 & 0 & 2 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 & 8 \\ 2 & 1 & 0 & 5 & 4 \\ 0 & 0 & 4 & -5 & -2 \\ 0 & -2.5 & 0 & -10.5 & -2 \end{bmatrix}$$

Gaussian elimination: recording the transformations

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Gaussian elimination: recording the transformations

- Maintain M (initially identity) and U (initially A)
- Whatever transformations you do to U, do same transformations to M

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1
\end{array}
\quad * \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\quad = \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
$$

ColumnA:
select row 1
add it to rows 2,3

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$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array} \quad * \quad \begin{array}{c|cccc} & A & B & C & D \\ \hline 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \end{array} \quad \checkmark \quad = \begin{array}{c|cccc} & A & B & C & D \\ \hline 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 & 1 \end{array}$$

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2 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1
\end{array}
\quad * \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\begin{array}{c} \\ \\ \checkmark \\ \\ \\ \end{array}
\quad = \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
$$

ColumnA:
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\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 \\
3 & 0 & 1 & 0 & 1
\end{array}
\quad * \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\begin{array}{c} \\ \\ \checkmark \\ \\ \checkmark \end{array}
\quad = \quad
\begin{array}{c|cccc}
 & A & B & C & D \\
\hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 0 & 0 & 1 & 0 \\
3 & 0 & 1 & 0 & 0
\end{array}
$$

ColumnB:
select row 3
add it to no rows

ColumnC:
select row 0
add it to row 2

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$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1
\end{array}
\quad * \quad
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\begin{array}{c}
 \\ \\ \checkmark \\ \\ \\
\end{array}
\quad = \quad
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
$$

ColumnA:
select row 1
add it to rows 2,3

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & \color{red}{1} & 1 & 0 \\
3 & 0 & \color{red}{1} & 0 & 1
\end{array}
\quad * \quad
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
\begin{array}{c}
 \\ \checkmark \\ \checkmark \\ \\ \checkmark
\end{array}
\quad = \quad
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & \color{red}{0} & \color{red}{0} & \color{red}{1} & \color{red}{0} \\
3 & \color{red}{0} & \color{red}{1} & \color{red}{0} & \color{red}{0}
\end{array}
$$

ColumnB:
select row 3
add it to no rows

ColumnC:
select row 0
add it to row 2

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- Maintain M (initially identity) and U (initially A)
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$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1
\end{array}
\ast
\begin{array}{c|ccccc}
 & A & B & C & D & \\ \hline
0 & 0 & 0 & 1 & 1 & \\
1 & 1 & 0 & 1 & 1 & \checkmark \\
2 & 1 & 0 & 0 & 1 & \\
3 & 1 & 1 & 1 & 1 &
\end{array}
=
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & 1 & 0 & 0 & 1 \\
3 & 1 & 1 & 1 & 1
\end{array}
$$

ColumnA:
select row 1
add it to rows 2,3

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & 0 & \color{red}{1} & 1 & 0 \\
3 & 0 & \color{red}{1} & 0 & 1
\end{array}
\ast
\begin{array}{c|ccccc}
 & A & B & C & D & \\ \hline
0 & 0 & 0 & 1 & 1 & \checkmark \\
1 & 1 & 0 & 1 & 1 & \checkmark \\
2 & 1 & 0 & 0 & 1 & \\
3 & 1 & 1 & 1 & 1 & \checkmark
\end{array}
=
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & \color{red}{0} & \color{red}{0} & \color{red}{1} & \color{red}{0} \\
3 & \color{red}{0} & \color{red}{1} & \color{red}{0} & \color{red}{0}
\end{array}
$$

ColumnB:
select row 3
add it to no rows
ColumnC:
select row 0
add it to row 2

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 \\
2 & \color{red}{1} & 1 & 1 & 0 \\
3 & 0 & 1 & 0 & 1
\end{array}
\ast
\begin{array}{c|ccccc}
 & A & B & C & D & \\ \hline
0 & 0 & 0 & 1 & 1 & \checkmark \\
1 & 1 & 0 & 1 & 1 & \checkmark \\
2 & 1 & 0 & 0 & 1 & \checkmark \\
3 & 1 & 1 & 1 & 1 & \checkmark
\end{array}
=
\begin{array}{c|cccc}
 & A & B & C & D \\ \hline
0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 \\
2 & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{1} \\
3 & 0 & 1 & 0 & 0
\end{array}
$$

ColumnD:
select row 2
done

Code for finding transformation to echelon form

- Initialize `rowlist` to be list of rows of A
- Initialize `M_rowlist` to be list of rows of identity matrix

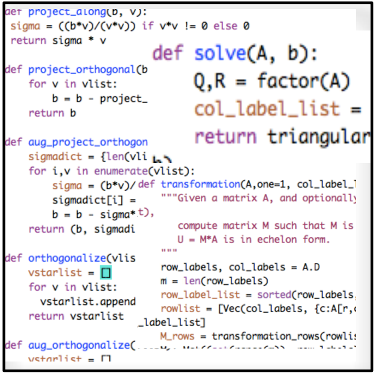
```
for c in sorted(col_labels, key=hash):
    rows_with_nonzero = [r for r in rows_left if rowlist[r][c] != 0]
    if rows_with_nonzero != []:
        pivot = rows_with_nonzero[0]
        rows_left.remove(pivot)
        new_M_rowlist.append(M_rowlist[pivot])
        for r in rows_with_nonzero[1:]:
            multiplier = rowlist[r][c]/rowlist[pivot][c]
            rowlist[r] -= multiplier*rowlist[pivot]
            M_rowlist[r] -= multiplier*M_rowlist[pivot]

    for r in rows_left: new_M_rowlist.append(M_rowlist[r])
```

Finally, return matrix M formed from `M_rowlist`

Code provided in module `echelon`

The black box starts to become less opaque



The modules `independence` and `solver` both Gaussian elimination when working over $GF(2)$:

- The procedure `solve(A, b)` computes a matrix M such that MA is in echelon form, and uses M to try to find a solution.
- The procedure `rank(L)` converts to echelon form and counts the nonzero rows to find the rank of L.

We saw that Gaussian elimination can be used to find a nonzero vector in the null space of a matrix.... You will use this in an algorithm for factoring integers.