Matrix-matrix multiplication

If

- A is a $R \times S$ matrix, and
- B is a $S \times T$ matrix

then it is legal to multiply A times B.

- In Mathese, written AB
- In our `Mat` class, written `A*B`

AB is different from BA.

In fact, one product might be legal while the other is illegal.

Matrix-matrix multiplication

We'll see two equivalent definitions:

- one in terms of vector-matrix multiplication,
- one in terms of matrix-vector multiplication.

Matrix-matrix multiplication: vector-matrix definition

Vector-matrix definition of matrix-matrix multiplication:
For each row-label r of A,

$$\text{row } r \text{ of } AB = \underbrace{(\text{row } r \text{ of } A)}_{\text{vector}} * B$$

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \\ \quad \text{B} \quad \\ \\ \end{array}\right] = \left[\begin{array}{c} [1,0,0] * B \\ \hline [2,1,0] * B \\ \hline [0,0,1] * B \end{array}\right]$$

How to interpret $[1,0,0] * B$?

- *Linear combinations* definition of vector-matrix multiplication?
- *Dot-product* definition of vector-matrix multiplication?

Each is correct.

Matrix-matrix multiplication: vector-matrix interpretation

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\quad \text{B} \quad \right] = \left[\begin{array}{c} [1,0,0]*B \\ \hline [2,1,0]*B \\ \hline [0,0,1]*B \end{array}\right]$$

How to interpret $[1,0,0]*B$? *Linear combinations* definition:

$$[1,0,0] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \mathbf{b}_1$$

$$[0,0,1] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \mathbf{b}_3$$

$$[2,1,0] * \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = 2\,\mathbf{b}_1 + \mathbf{b}_2$$

Conclusion:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline 2\,\mathbf{b}_1 + \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right]$$

Matrix-matrix multiplication: vector-matrix interpretation

Conclusion:

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} \mathbf{b}_1 \\ \hline \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right] = \left[\begin{array}{c} \mathbf{b}_1 \\ \hline 2\,\mathbf{b}_1 + \mathbf{b}_2 \\ \hline \mathbf{b}_3 \end{array}\right]$$

We call $\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$ an *elementary row-addition matrix*.

Matrix-matrix multiplication: matrix-vector definition

Matrix-vector definition of matrix-matrix multiplication:
For each column-label s of B,

$$\text{column } s \text{ of } AB = A * (\text{column } s \text{ of } B)$$

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $B =$ matrix with columns $[4, 3]$, $[2, 1]$, and $[0, -1]$

$$B = \left[\begin{array}{c|c|c} 4 & 2 & 0 \\ 3 & 1 & -1 \end{array}\right]$$

AB is the matrix with column $i = A * ($ column i of $B)$

$A * [4, 3] = [10, -1]$ $\qquad$ $A * [2, 1] = [4, -1]$ $\qquad$ $A * [0, -1] = [-2, -1]$

$$AB = \left[\begin{array}{c|c|c} 10 & 4 & -2 \\ -1 & -1 & -1 \end{array}\right]$$

Matrix-matrix multiplication: Dot-product definition

Combine

- *matrix-vector* definition of matrix-matrix multiplication, and
- *dot-product* definition of matrix-vector multiplication

to get...

Dot-product definition of matrix-matrix multiplication:
Entry rc of AB is the dot-product of row r of A with column c of B.

Example:

$$\left[\begin{array}{ccc} 1 & 0 & 2 \\ \hline 3 & 1 & 0 \\ \hline 2 & 0 & 1 \end{array}\right] \left[\begin{array}{c|c} 2 & 1 \\ 5 & 0 \\ 1 & 3 \end{array}\right] = \left[\begin{array}{cc} [1,0,2]\cdot[2,5,1] & [1,0,2]\cdot[1,0,3] \\ [3,1,0]\cdot[2,5,1] & [3,1,0]\cdot[1,0,3] \\ [2,0,1]\cdot[2,5,1] & [2,0,1]\cdot[1,0,3] \end{array}\right] = \left[\begin{array}{cc} 4 & 7 \\ 11 & 3 \\ 5 & 5 \end{array}\right]$$

Matrix-matrix multiplication: transpose

$$(AB)^T = B^T A^T$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 19 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 19 \\ 4 & 8 \end{bmatrix}$$

You might think "$(AB)^T = A^T B^T$" but this is **false**.
In fact, doesn't even make sense!

- For AB to be legal, A's column labels $=$ B's row labels.
- For $A^T B^T$ to be legal, A's row labels $=$ B's column labels.

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$ is legal but $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ is not.

Matrix-matrix multiplication: Column vectors

Multiplying a matrix A by a one-column matrix B

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{b} \\ \\ \end{bmatrix}$$

By matrix-vector definition of matrix-matrix multiplication, result is matrix with one column: $A * \mathbf{b}$

This shows that matrix-vector multiplication is subsumed by matrix-matrix multiplication.

Convention: Interpret a vector $\mathbf{b}$ as a one-column matrix ("column vector")

- Write vector $[1, 2, 3]$ as $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Write $A * [1, 2, 3]$ as $\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ or $A\,\mathbf{b}$

Matrix-matrix multiplication: Row vectors

If we interpret vectors as one-column matrices.... what about vector-matrix multiplication?

Use transpose to turn a column vector into a row vector: Suppose $\mathbf{b} = [1, 2, 3]$.

$$[1, 2, 3] * A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{c} \\ \quad A \quad \\ \\ \end{array} \right] = \mathbf{b}^T A$$

Inner product

Let $\mathbf{u}$ and $\mathbf{v}$ be two D-vectors interpreted as matrices (column vectors).
Matrix-matrix product $\mathbf{u}^T\mathbf{v}$.

Example: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$

- First "matrix" has one row.
- Second "matrix" has one column.
- Therefore product "matrix" has one entry.

By dot-product definition of matrix-matrix multiplication,
that one entry is the dot-product of $\mathbf{u}$ and $\mathbf{v}$.

Sometimes called *inner product* of matrices.
However, that term has taken on another meaning, which we study later.

Outer product

Another way to multiply vectors as matrices.
For any $\mathbf{u}$ and $\mathbf{v}$, consider $\mathbf{u}\mathbf{v}^T$.

Example: $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 & u_1 v_4 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 & u_2 v_4 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & u_3 v_4 \end{bmatrix}$

For each element s of the domain of $\mathbf{u}$ and each element t of the domain of $\mathbf{v}$, the s, t element of $\mathbf{u}\mathbf{v}^T$ is $\mathbf{u}[s]\ \mathbf{v}[t]$.

Called *outer product* of $\mathbf{u}$ and $\mathbf{v}$.