F.L.C.T.

the FUNNY LITTLE CALCULUS TEXT

VOLUME 1:
SINGLE VARIABLE

by ROBERT GHRIST

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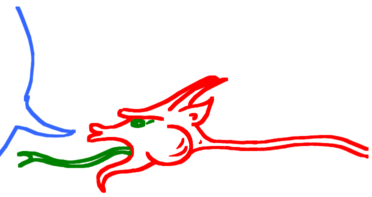
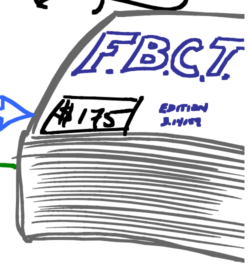
What is this book?



It's a Calculus text. It's little. It's funny.

It's not a replacement for your standard, problemfull text.



This text is meant to be read and enjoyed.

PREREQUISITES

This text assumes you've seen some Calculus before: you know what to do [differentiate/integrate] and how to do it, but you don't know what it really means -- like everything else in Life...

INSTRUCTIONS

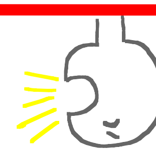
BASICALLY

```
10 READ
20 THINK
30 SOLVE PROBLEMS
40 GOTO 10
```

This is for the "big picture" (& some little pictures too...). You should find a good source for homework problems, details, & proofs.

AND THEY ALL SAID...

"WHAT is the deal with this font?"



"THE COLORS! IT'S TOO MUCH!!!"

(not to mention the pretentious literary allusions commingled with crude humor)

"Are you, in fact, Barking Mad?"

Yes, north by northwest...

THANKS!

To my students, for teaching me...

This book was written on a Fujitsu tablet PC using Microsoft Journal.

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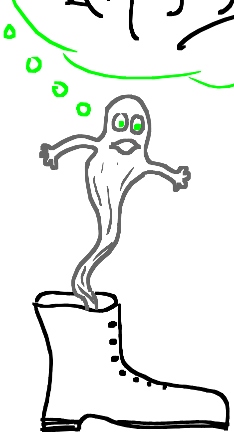
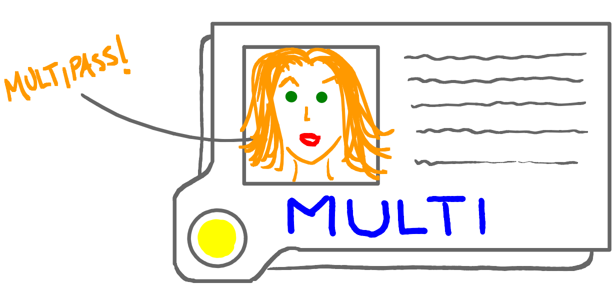
This is single-variable Calculus: stay tuned for Volume II

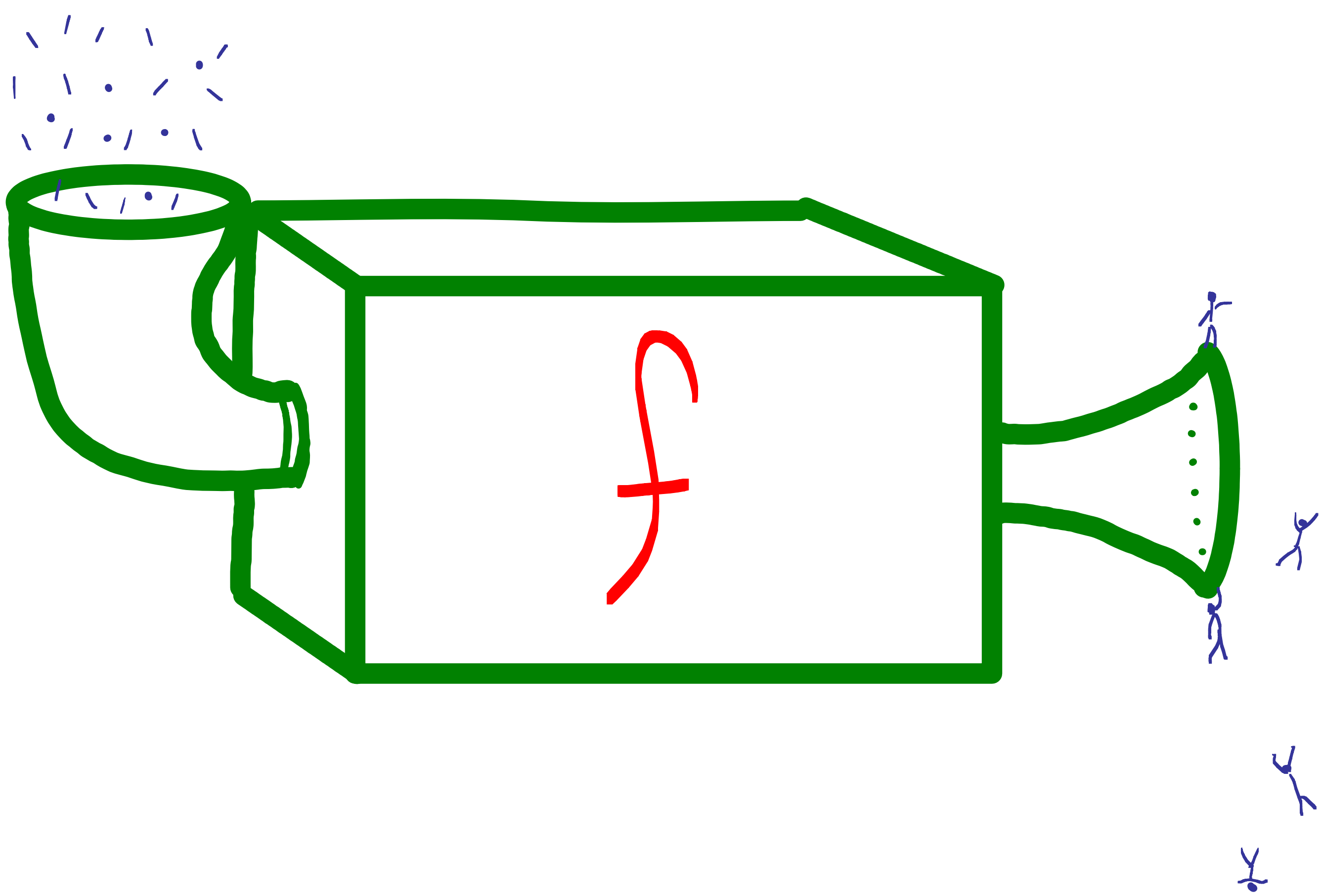




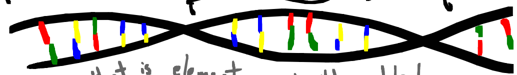
~~MULTIVARIABLE~~

F.L.C.T. Chapter 1:

Functions



You can't get far in Calculus without understanding functions: they are the DNA of Mathematics...



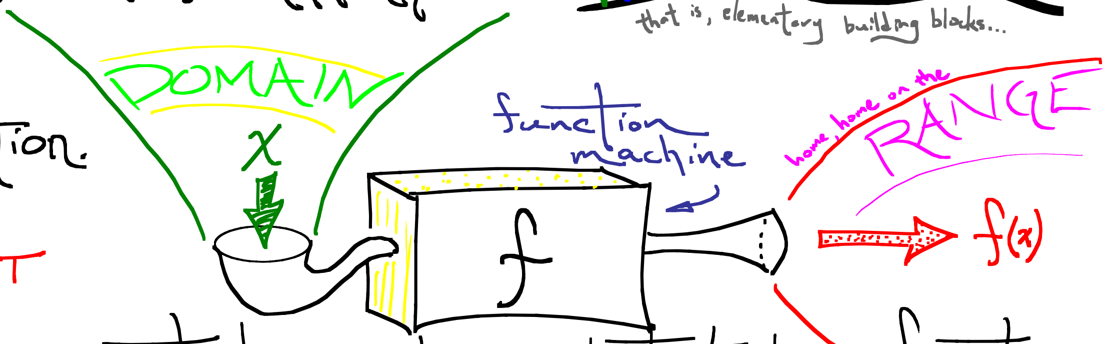
that is, elementary building blocks...

FUNCTIONS

This is a function.

x = INPUT

$f(x)$ = OUTPUT



These may or may not be numbers... but to be a function:

SAME INPUT ⟹ SAME OUTPUT

EXAMPLES

$f(x) = x^2$ is a function

$f(x) = \tan^{-1}(x)$ is not a function { many angles have $\tan = x$, so we use "ARCTAN" to fix range...

$f(\underbrace{\text{location, time}}_{\text{1 INPUT}}) = (\underbrace{\text{temperature, windspeed}}_{\text{1 OUTPUT}})$ is a MULTIVARIATE function

$f(\text{hardwork}) = \text{success}$ is not a function

(success depends on more than one variable...)

OPERATIONS

You can do the typical things to typical functions:

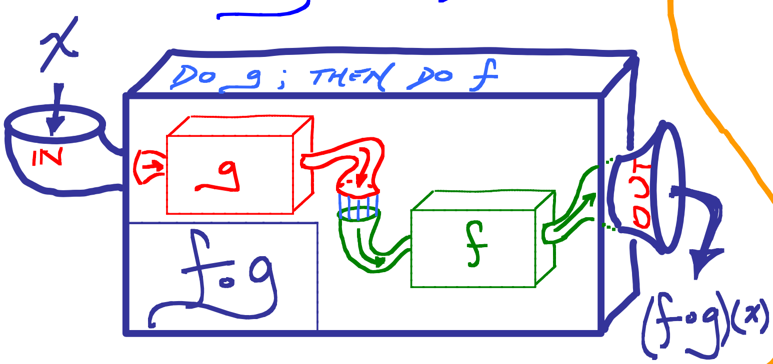
× + − ( ) ÷

MY DEAR AUNT SALLY

The most important operations are the following:

COMPOSITION

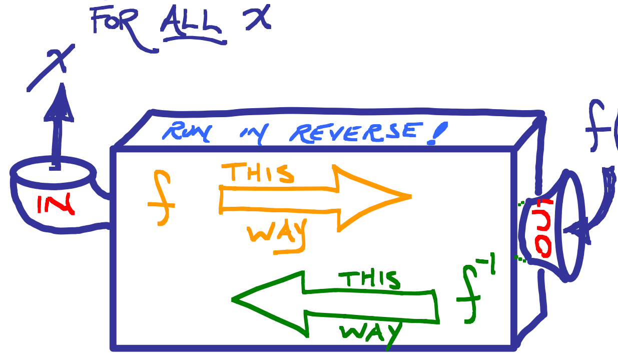
$$(f\circ g)(x) = f(g(x))$$



INVERSE

$$(f^{-1}\circ f)(x) = x = (f\circ f^{-1})(x)$$

FOR ALL x

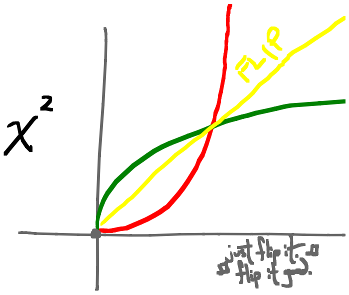


EXAMPLES

$\sqrt[3]{x}$ is the inverse of x^3

$\sqrt{x}$ [positive square root] is the inverse of x^2 $(x \geq 0)$

a^x and $\ln_a x$ are inverses



POLYNOMIALS & POWERS

A MONOMIAL is a function of the form: ax^n (let's say $n=0,1,2\ldots$)

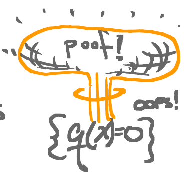
A POLYNOMIAL is $\displaystyle p(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ ELLIPSIS

"MANY" → POLY

n = DEGREE

A RATIONAL function is a quotient $r(x) = \dfrac{p(x)}{q(x)}$ POLYS

"BLOWS UP" sometimes



$\{q(x)=0\}$

Let's think... what is... CALCULATING EXISTENTIAL ANGST...

x^0 : EASY. $x^0 = 1$ for all x

x^{-1} : EASY. $x^{-1} = \dfrac{1}{x}$ for all $x \neq 0$

x^{-n} : EASY. $x^{-n} \cdot x^n = x^{-n+n} = x^0 = 1 \longrightarrow x^{-n} = \dfrac{1}{x^n}$

$x^{\frac{1}{n}}$: EASY. $(x^{\frac{1}{n}})^n = x^{\frac{n}{n}} = x \longrightarrow x^{\frac{1}{n}} = \sqrt[n]{x}$

But: $x^{\sqrt{2}} = ?$ $x^{\pi} = ?$ $x^{\sqrt{-3}} = ?$ What are these?

If you try to compute 2^{π}, you might approximate...

$$2^{\pi} \approx 2^{\frac{314}{100}} = \sqrt[100]{2^{314}} \quad \text{a } 100^{\text{th}} \text{ root...}$$

A certain limiting process seems inevitable. The principled approach to such phenomena is encapsulated in...

THE EXPONENTIAL FUNCTION

The most important function in Calculus is

$$\mathrm{EXP}(x) := e^x$$

where $e = 2.718281828459045235360287471352662497757247093\ldots$

and x is, well, x

actually, e is just e

But there is a better, subtler definition:

SERIES [$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \frac{1}{720}x^6 + \cdots$ AD INFINITUM $\rightarrow \infty$

DO NOT FEAR $\sum$ Sigma = "SUM"

$$= \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

where $k! = k(k-1)(k-2)\cdots(3)(2)(1)$ and $0! = 1$ by convention

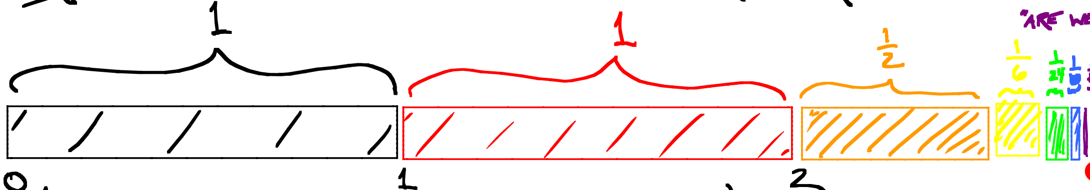
!u...

Oh yeah? Well, !u2

! ⇒ "FACTORIAL"

This is the definition of e^x: it requires an INFINITE SERIES

"ARE WE THERE YET?"

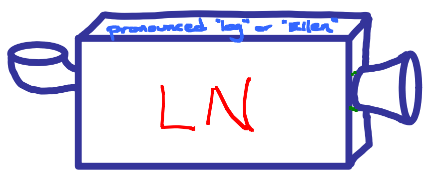


$e = 2.71828\ldots$ is the LIMIT of this sum...

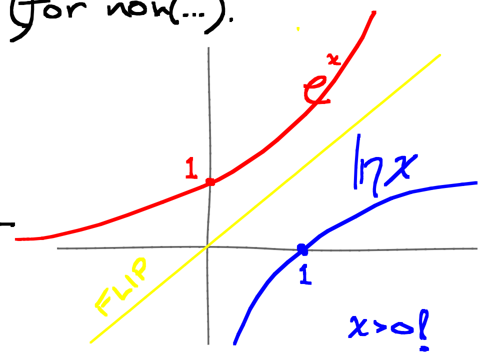
Although this looks complicated, it only takes a few terms to get a decent approximation. For the time being, pretend that such infinite sums are polynomials. I promise no harm will come to you in so doing (for now...).

SPECIAL FUNCTIONS

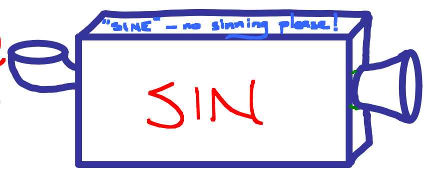
LOGS



= "natural log" = inverse of e^x



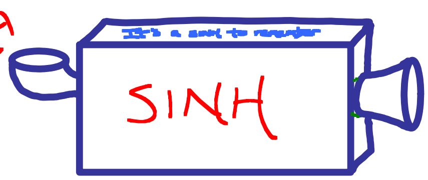
TRIGS



... & inverse other trigs, like ARCCOS, ARCTAN, ETC...

and the other trigs: e.g., ARCSIN, COS, TAN, SEC, CSC, COT...

HYPERBOLIC TRIGS



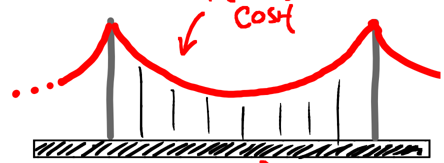
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

I must put it in my head.

This is a COSH



very useful...

& SECH, CSCH, COTH ← NOT

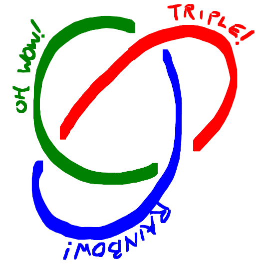
EULER'S FORMULA

NOT "yew-lurs" but "oi-lers": it's a Swiss thing

The special functions you know & love are all connected

$$\cos^2 x + \sin^2 x = 1$$

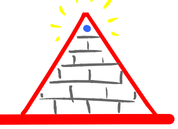
$$\cosh^2 x - \sinh^2 x = 1$$

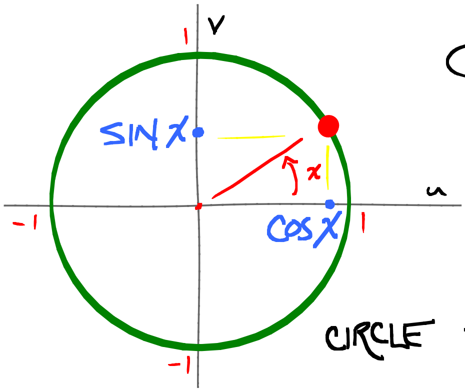


RECALL: $i = \sqrt{-1}$ so $i^2 = -1$

$$\boxed{\cos x + i \sin x = e^{ix}}$$

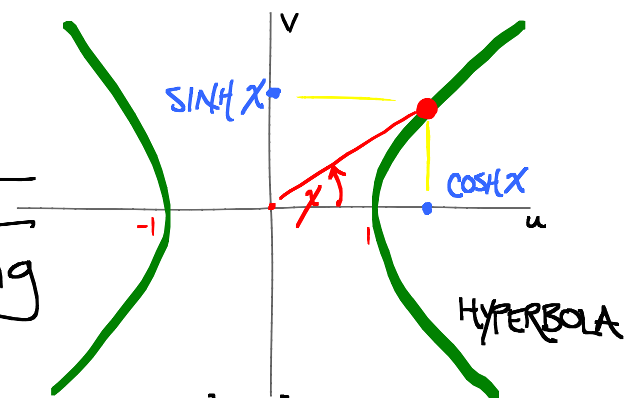
These & similar formulae are illuminative





$u^2 + v^2 = 1$

The ability to switch between algebraic and geometric thinking is crucial...



$u^2 - v^2 = 1$

SPECIAL SERIES

WE KNOW...

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots$$

chillax

Let us see what happens to other special functions:

$$\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

EXPAND e^x & e^{-x}!

can i do that?

$$= \frac{1}{2}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots\right)$$

$$- \frac{1}{2}\left(1 + (-x) + \frac{1}{2!}(-x)^2 + \frac{1}{3!}(-x)^3 + \frac{1}{4!}(-x^4) + \frac{1}{5!}(-x^5) + \dots\right)$$

$$= \cancel{0} + x + \cancel{0}x^2 + \frac{1}{3!}x^3 + \cancel{0}x^4 + \frac{1}{5!}x^5 + \dots$$

$$\sinh x = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

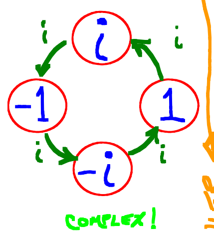
How ODD!

likewise...

$$\cosh x = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

How EVEN!

Euler's formula gives us a series formula for SIN/COS



$$e^{ix} = 1 + ix + \frac{1}{2!}(ix)^2 + \frac{1}{3!}(ix)^3 + \frac{1}{4!}(ix)^4 + \frac{1}{5!}(ix)^5 + \dots$$

$$= 1 + ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \dots$$

$$= \underbrace{\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right)}_{\text{REAL}} + i\underbrace{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right)}_{\text{IMAGINARY}}$$

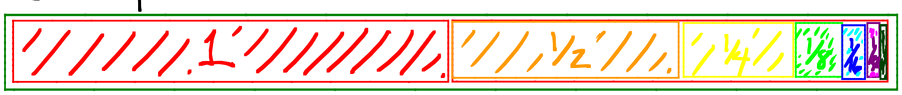
$$\overset{\text{EULER}}{=} \cos(x) + i\sin(x)$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

TRICKY...

Other special series come with RULES



$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \dots = 2 \qquad \sum_{k=0}^{\infty} \left(\tfrac{1}{2}\right)^k = 2$$

This is a special case of the GEOMETRIC SERIES:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots = \sum_{k=0}^{\infty} x^k$$

BUT! ONLY FOR $|x| < 1$

unless you are a physicist

(proof: $(1-x)(1+x+x^2+x^3+\dots) = (1+x+x^2+x^3+\dots) - (x+x^2+x^3+x^4+\dots) = 1$ VOILA!)

but does $1+2+4+8+16+\dots = \frac{1}{1-2} = -1$? No! Remember! $|x|<1$!

TAYLOR SERIES

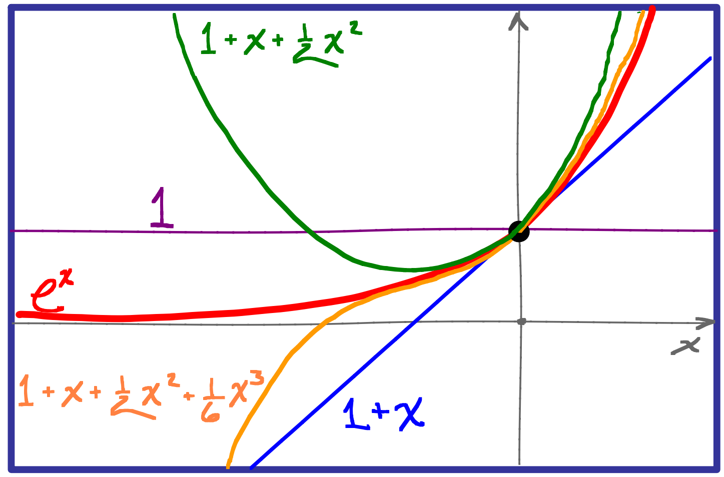
It seems as though many standard functions may have infinite series expressions. This leads to the first big idea of FLCT:

ANY "REASONABLE" $f(x)$ "EQUALS" $\sum_{k=0}^{\infty} c_k x^k$ FOR SOME c_k CONSTANTS

Such a series expansion is called a TAYLOR SERIES about $x=0$. It means that, near 0, $f(x)$ looks like a polynomial... (this also goes by the name "MACLAURIN" SERIES)

It's best to think in terms of POLYNOMIAL APPROXIMATION. You know that the tangent line to $y=e^x$ at $(0,1)$ is $1+x$. The best fit quadratic function is $1+x+\frac{1}{2}x^2$. Keep going: the TAYLOR POLYNOMIAL to e^x at $x=0$ of degree n is:

$$1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n$$



= LEVEL TO MORE =

To get a tangent line to any $f(x)$ at $x=0$, you use $f'(0)$, the derivative. It should not surprise you to learn that the other polynomial approximations to $f(x)$ are also based on derivatives.

that's so derivative

The TAYLOR SERIES of $f(x)$ about $x=0$ is:

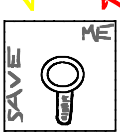
$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)\,x+\frac{1}{2!}f''(0)\,x^2+\frac{1}{3!}f'''(0)\,x^3+\cdots \text{ + going + going + going}$$

$f^{(k)}$ means "k^{th} derivative"

$$= f(x) \quad \text{for most functions, at least}$$

By changing coordinates $x \mapsto x-a$ we obtain the TAYLOR SERIES of $f(x)$ ABOUT $x=a$:

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = \underbrace{f(a)+f'(a)(x-a)+\frac{1}{2!}f''(a)(x-a)^2+\frac{1}{3!}f'''(a)(x-a)^3+\cdots}$$



For x near a, a truncation of this to a polynomial gives the best polynomial approximation to f (near a).

COMPUTING TAYLOR SERIES

Since we all know how to differentiate like a superhero, it is no big deal to compute a few examples:

EXAMPLE:

$$\frac{d}{dx}(e^x)=e^x \implies (e^x)^{(k)}(0)=e^0=1 \quad \text{Thus,}$$

$$e^x = \underbrace{1+\frac{1}{1!}x+\frac{1}{2!}x^2+\frac{1}{3!}x^3}+\cdots \quad \text{AHA!}$$

BUT... ABOUT $x=10$, THIS IS A BAD APPROXIMATION: TRY...

$$e^x = e^{10}+\frac{e^{10}}{1!}(x-10)+\frac{e^{10}}{2!}(x-10)^2+\frac{e^{10}}{3!}(x-10)^3+\cdots = e^{10}\cdot e^{x-10}$$

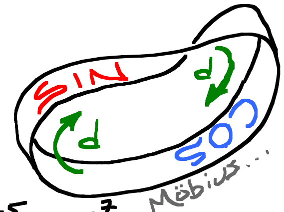
OF COURSE

EXAMPLE

$$\sin x \xrightarrow{\frac{d}{dx}} \cos x \xrightarrow{\frac{d}{dx}} -\sin x \xrightarrow{\frac{d}{dx}} -\cos x \xrightarrow{\frac{d}{dx}} \sin x$$

$$\downarrow x=0 \qquad \downarrow x=0 \qquad \downarrow x=0 \qquad \downarrow x=0$$

$$0 \qquad\quad 1 \qquad\quad 0 \qquad\quad -1$$



So...
$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

But, in general, computing lots of derivatives is NO FUN. Fortunately, pretending that Taylor series are polynomials allows one to perform computions like these:

EXAMPLE

$$e^{\sin(x^2)} = e^{\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right)} = 1 + \left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots\right) + \frac{1}{2}\left(x^2 - \frac{x^6}{3!} + \cdots\right)^2 + \frac{1}{3!}\left(x^2 + \cdots\right)^3 + \cdots$$

SIMPLIFY

$$= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{24}x^8 + \text{H.O.T.}$$

"HIGHER ORDER TERMS"
i.e., stuff I don't care about...

This requires ALGEBRA ☹, but only polynomial algebra.

EXAMPLE

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

GEOMETRIC SERIES
$|x| < 1$!

SUBS

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n x^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n$$

DIFF

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^n + \cdots = \sum_{n=0}^{\infty} n x^n$$

INT

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

SUBS

$$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \cdots + (-1)^{n-1}\frac{(x-1)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n$$

$|x| < 1$!

IS THIS LEGIT ?

MIRABILE DICTU, you can pull this off. No 'blemo, as long as you remember (in this case) $|x| < 1$.

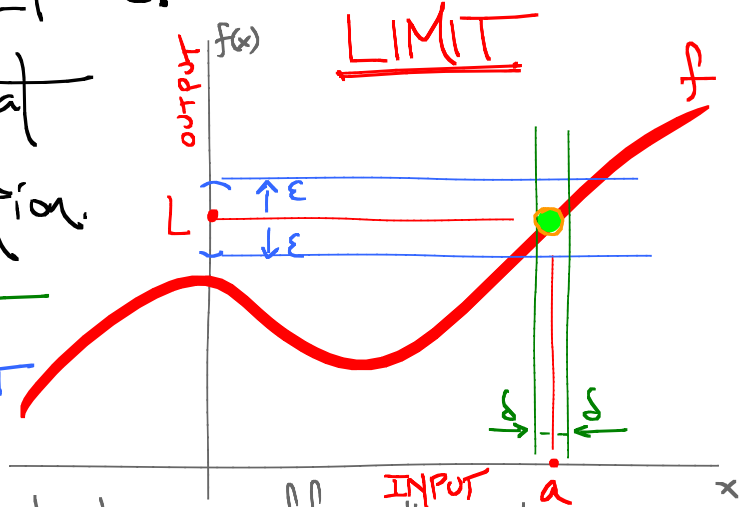
(= Latin: "coo dat. word."

"CALCVLVM VIRVMQVE CANO"

AND NOW THE BEGINNING

The time has come to do Calculus. We begin at the beginning with the notion of LIMITS. Perhaps you recall...

$\lim_{x \to a} f(x) = L \iff$ For every $\varepsilon > 0$, there exists a $\delta > 0$ so that whenever $0 < |x - a| < \delta$ one has $|f(x) - L| < \varepsilon$.



You have an intuition for what this means, but it is worth learning to parse the definition.

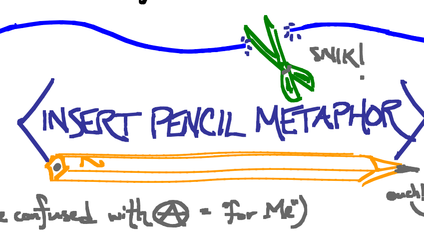
HINTS: δ = TOLERANCE on INPUT

ε = TOLERANCE on OUTPUT

This is like an adversarial game.
Player 1 picks ε; Player 2 must choose a sufficiently small $\delta > 0$.
Engineers/scientists consistently use tolerances for error: same deal.

It is a slight generalization to define $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
Note that here, as ever, existence is not automatic.
Our motivation for investigating limits is to define.....

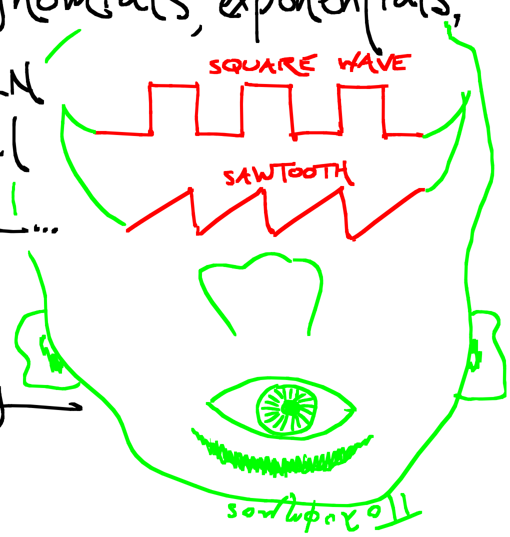
continuity



A continuous function is one for which

DEFN $\boxed{\lim_{x \to a} f(x) = f(a) \quad \forall a}$ $\forall$ = "for all" (not to be confused with Ⓐ = "for Me")

In particular, the limits (2-SIDED) must exist! Most functions you see in real life are continuous: polynomials, exponentials, sines, cosines, etc.; beware SEC, CSC, LN and rational functions. Certain digital signals are discontinuous. But no biggie...



The end of this Chapter has arrived, in which the hard work of understanding functions as Taylor series pays off...

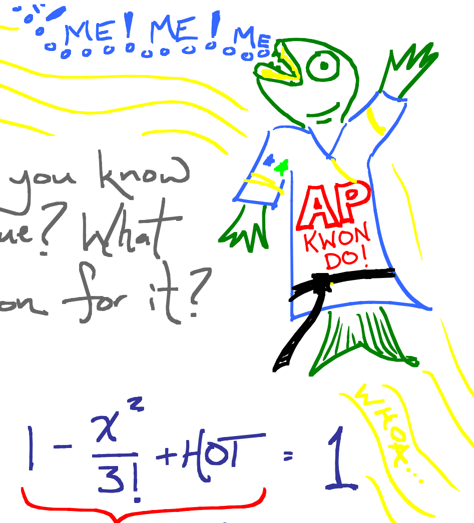
LIMITS & SERIES

EXAMPLE

who remembers this one?

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

But: do you know why it is true? What is the reason for it?



THE REASON:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \frac{x^3}{3!} + \text{HOT}}{x} = \lim_{x\to 0} \underbrace{1 - \frac{x^2}{3!} + \text{HOT}} = 1$$

LOTS OF x

CANCEL!

THE TAYLOR SERIES ABOUT $x=0$ IS: $1 + \text{HOT}$...

"AHA!", you say, "I could have used l'Hôpital's rule." Well guess what: I just used something better: l'Hôpital's WHOLEGOSHDARNEDRAISONDÊTRE!

HULK SMASH!

If all you know is THISRULE and THATRULE and TRICKSINMYHEAD, then you don't know Calculus -- you are just a PATTERNMATCHINGAUTOMAZOMBIE.

Don't. Be. That. Way. Learn the reasons why the rules/tools work!

Why is it that l'Hôpital's rule doesn't always work? Or, if it doesn't, why you keep on taking derivatives?

Assume: $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$

LOTS OF $(x-a)$'s TO FACTOR

THEN: Expand about $x=a$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\cancel{f(a)}^{\,0} + f'(a)\boxed{(x-a)} + \boxed{\tfrac{1}{2}} f''(a)\boxed{(x-a)^2} + \boxed{\text{HOT}}}{\cancel{g(a)}_{\,0} + g'(a)\boxed{(x-a)} + \boxed{\tfrac{1}{2}} g''(a)\boxed{(x-a)^2} + \boxed{\text{HOT}}}$$

OH HAI!

these both vanish...

DO YOU LIKE MY HAT?

et cetera...

This is why l'Hôpital's rule is.

EXAMPLE

$$\lim_{x\to 1} \frac{1-\cos(x-1)}{\ln^2 x} = \lim_{x\to 1} \frac{1-\left(1-\frac{(x-1)^2}{2!}+\text{HOT}\right)}{\left((x-1)-\frac{(x-1)^2}{2}+\text{HOT}\right)^2} = \lim_{x\to 1} \frac{\frac{1}{2!}\boxed{(x-1)^2} + \text{HOT}}{\boxed{(x-1)^2} + \text{HOT}}$$

$(x-1)$ FACTORS

CANCEL!

$$= \frac{1}{2}$$

You need the 2^{nd} order term to compute the limit.

ORDERS OF GROWTH

Calculus requires a familiarity with functional growth. We have used H.O.T. for terms such as $(x-a)^{\text{big}}$, which go to zero quickly as $x \to a$. How quickly? What if $x \to \infty$?

CONSIDER:

DOMINANT TERMS — COEFFICIENTS

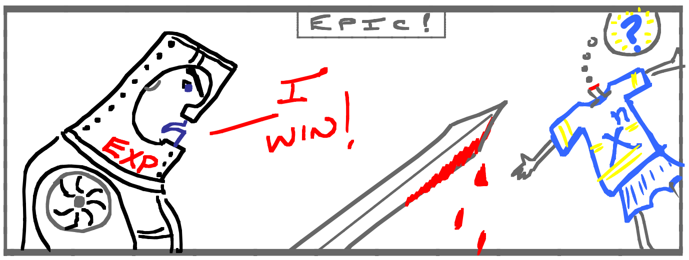
$$\lim_{x\to\infty} \frac{\boxed{2x^2} - 5x + 3}{\boxed{3x^2} - 11} = \lim_{x\to\infty} \frac{2 - \overset{\searrow 0}{5/x} + \overset{\searrow 0}{3/x^2}}{3 - \underset{\searrow 0}{11/x^2}} = \frac{2}{3}$$

or...

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots}{x^2} = \lim_{x\to\infty} \underset{\searrow 0}{\frac{1}{x^2}} + \underset{\searrow 0}{\frac{1}{x}} + \frac{1}{2!} + \overset{\nearrow +\infty}{\frac{x}{3!}} + \cdots \overset{+\infty}{\nearrow} \overset{+\infty}{\nearrow}$$

WHOA...

$$= +\infty$$



This line of reasoning shows that $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$ for all $n > 0$

EXPONENTIALS BEAT POLYNOMIALS (as $x \to \infty$)

There is a hierarchy of functions based on ASYMPTOTICS: It pays to know who is on their way to the top at what speed...

FACTORIAL GROWTH	x^x or $x!$	(Protein conformations; Ways to live your life -- assuming no fate)
BEATS ↓ EXPONENTIAL GROWTH	e^x or $\cosh x$ or $\sinh x$	(Compound investment or credit card debt or MOORE'S LAW)
BEATS ↓ POLYNOMIAL GROWTH	x^n $n>0$	(area/volume growth; network utility; falling bodies)
BEATS ↓ LOGARITHMIC GROWTH	$\ln x$	(crypto key length; Richter scale; pH)
BEATS ↓ NO GROWTH	CONST or SIN	(mess on my desk)

In the next chapter, we continue with derivatives...

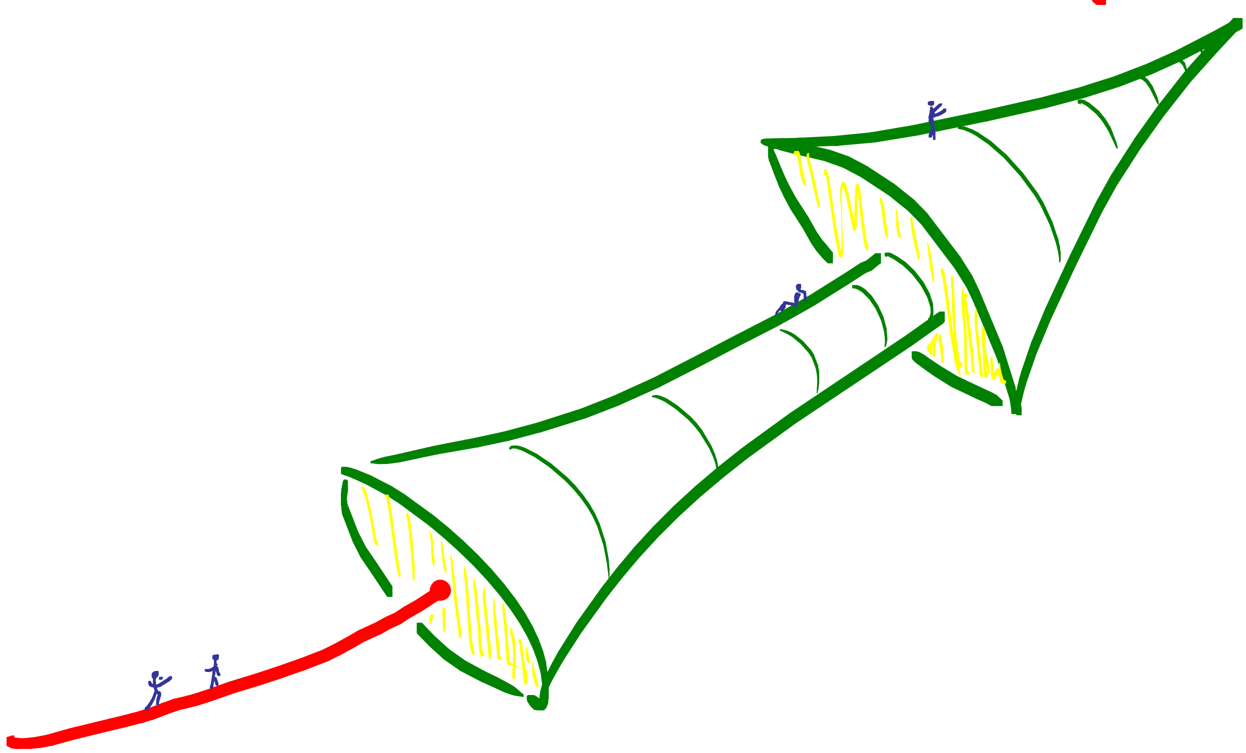
of course, $|\sin(x)| \leq 1$ & $|\cos| \leq 1$

Don't try this with sin or cos: $+ - + - + - \ldots$ is fatal as $x \to \infty$...

The subject of asymptotic growth keeps going: look up "big O notation" on WIKIPEDIA

F.L.C.T. Chapter 2:

Differentiation



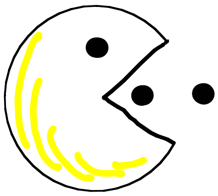
So you aced Calculus in highschool and can differentiate while unconscious. Well that's great. But do you know what it all means?

SLOPE? NOPE.

DEFINITION

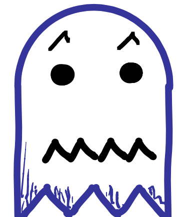
There is a distinction between definition & interpretation: this is what a derivative is

ONLY FOR SINGLE-VARIABLE FUNCTIONS!



$$\left.\frac{df}{dx}\right|_{x=a} = f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

EAT ME



Like most definitions, this one is quickly assimilated & forgotten... I assume you've been there, done that. The computation of derivatives is usually easy, based on a few simple rules: first, and most important is the overwhelming question

Oh, do not ask what is it. Let us go and make our visit.

WHAT DOES A DERIVATIVE MEAN?

(As with other things in life, interpretation matters as much as definition:-)

¡NOTATION ALERT!

f'	$\frac{df}{dx}$		OK!
$\frac{\partial f}{\partial x}$	df	$\frac{\delta f}{\delta x}$	NOT OK
$df \cdot \frac{1}{dx}$	$\frac{\cancel{d}f}{\cancel{d}x}$		DON'T GO THERE

APPROXIMATION

In keeping with the theme of **FLCT**, we begin with a Taylor series interpretation. This is, admittedly circular logic, but, in the words of the great Archimedes, "Μη μου τους κυκλους ταραττε!"

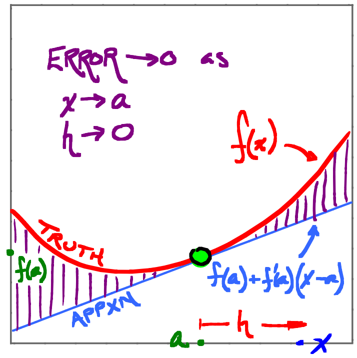
What does $f(a+h)$ look like for h SMALL?

TAYLOR EXPAND $f(x)$ ABOUT $x=a$

$$\underbrace{f(x)}_{\text{TRUTH}} = \underbrace{f(a) + f'(a)(x-a)}_{\text{LINEAR APPROXIMATION}} + \underbrace{\tfrac{1}{2}f''(a)(x-a)^2 + \text{H.O.T.}}_{\text{ERROR TERMS}}$$

Consider the substitution: $x = a + h$

$$f(a+h) = f(a) + \underbrace{f'(a)h}_{\text{LEADING TERM}} + \underbrace{\tfrac{1}{2}f''(a)h^2 + \tfrac{1}{6}f'''(a)h^3 + \text{HOT}}_{\text{VERY}^2\text{ SMALL FOR } h\approx 0}$$



from which follows a tautological computation:

$$\boxed{f'(a)} = \lim_{h\to 0} \frac{1}{h}\left(f(a+h) - f(a)\right) \quad \text{(substitute / simplify)}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(f'(a)h + \tfrac{1}{2}f''(a)h^2 + \text{HOT}\right) = \lim_{h\to 0}\left(f'(a) + \tfrac{1}{2}f''(a)h + \text{HOT}\right)$$

$$= \boxed{f'(a)}$$

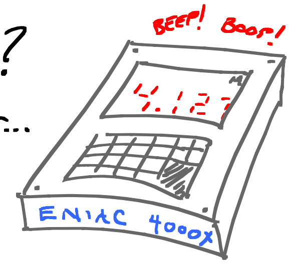
SO WHAT? If you think about it...

The derivative of f at a is the coefficient of the 1st ORDER TERM in the polynomial approximation to $f(a+h)$

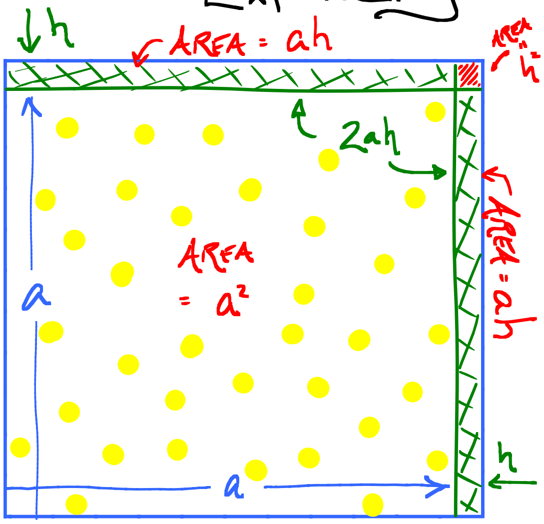
If you didn't know the definition of the derivative, you could construct it from the Taylor series...

EXAMPLE

What is the area of a square of side length $4\frac{1}{7}$?
This is easily approximated without a calculator...



Let x = SIDE LENGTH ; $f(x) = x^2$ AREA
Expanding about $a = 4$ (a nice number)...



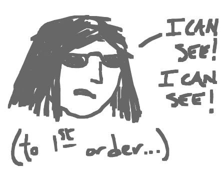
$$f(x) = f(4) + f'(4)(x-4) + \text{H.O.T.}$$

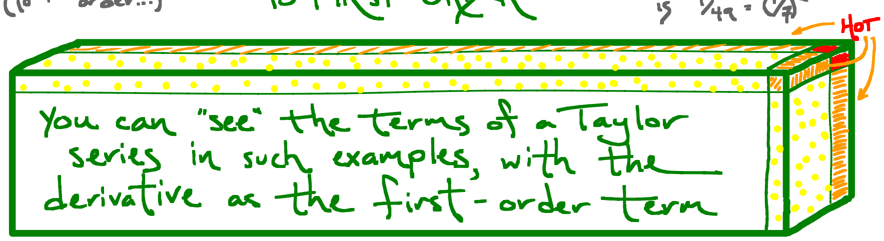
$$= 16 + 8\left(\tfrac{1}{7}\right) + \underbrace{\text{HOT}}$$

$$\approx 17\ 1/7$$

TO FIRST ORDER

The higher order term (singular) is $1/49 = (1/7)^2$

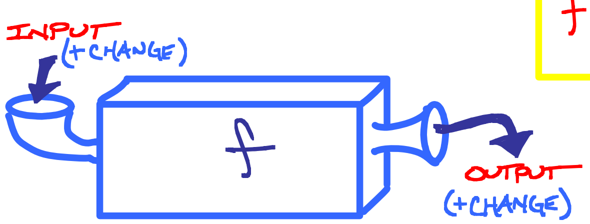




RATE OF CHANGE

Perhaps the best interpretation of $f'(a)$ is function-mechanical...

$$\boxed{f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}} \quad \begin{array}{l} \leftarrow \Delta \text{ OUTPUT} \\ \leftarrow \Delta \text{ INPUT} \end{array}$$



(PROOF: let $x = a+h$; then $\Rightarrow = \lim_{h\to 0} \frac{f(a+h) - f(a)}{(a+h) - a}$)

Thus, the derivative is a RATE of change

I said RATE of change, ribbit!

The derivative does not tell you the actual change in the output... JUST THE RATE OF CHANGE!

EXAMPLES

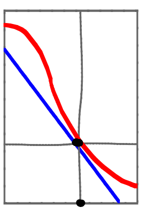
oh snap!

VELOCITY: if $x(t)$ = position of an object at time t, then the derivative $v(t) = \frac{dx}{dt}$ is the VELOCITY, where $v>0$ means ⇒ and $v<0$ means ⇐ (RECALL: VELOCITY = SPEED + DIRECTION)

ACCELERATION: in the above, $a = dv/dt = d^2x/dt^2$ = ACCELERATION. It is the rate of change of velocity.

JERK: true! JERK is the rate of change of acceleration; It's an example of a third-derivative you can feel...

CURRENT: The CURRENT running through a point in a wire is $I(t) = dQ/dt$, where $Q(t)$ gives the flow of charge along the wire.



SLOPE: as you know from your prior ~~exposure~~ indoctrination!, a derivative is a slope... FALSEFALSEFALSEFALSE !!1!! The rate of change of y with respect to x is dy/dx, which has an interpretation as the tangent line slope.

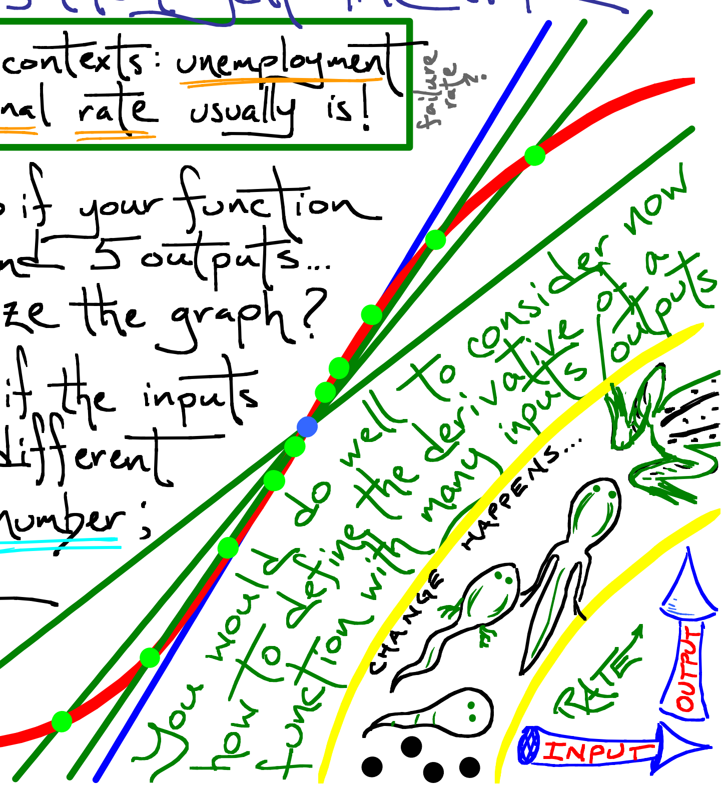
CAVE! The word "RATE" is used in other contexts: unemployment rate is NOT a derivative, while marginal rate usually is! failure rate...

Why not think of derivatives in terms of slope?

What do you do if your function has 7 inputs and 5 outputs... can you visualize the graph?

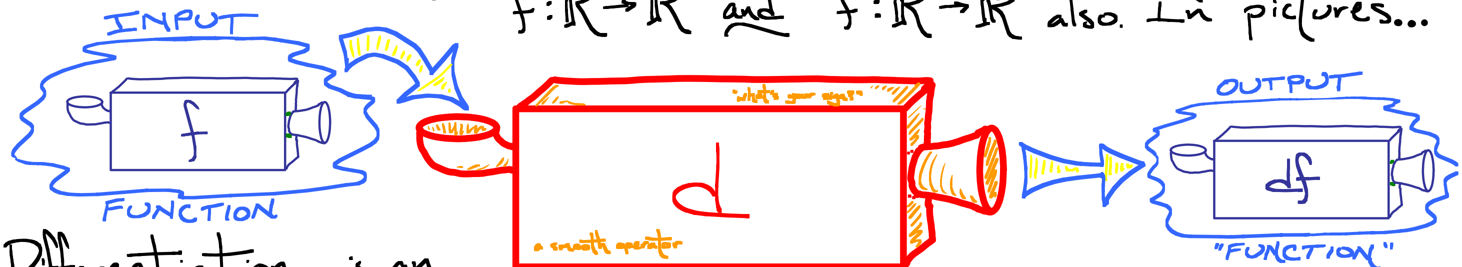
What do you do if the inputs and outputs have different units? Slope is a number; Derivatives are rates...

I SEE...



You would do well to consider now how to define the derivative of a function with many inputs/outputs

OPERATOR You should think of the derivative as a rate of change, or as part of a first-order approximation. But how should one interpret DIFFERENTIATION? The present setting is unique: $f:\mathbb{R}\to\mathbb{R}$ and $f':\mathbb{R}\to\mathbb{R}$ also. In pictures...



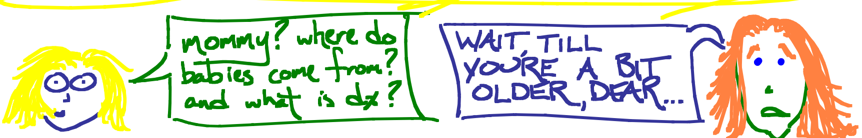
Differentiation is an OPERATOR, transforming a function f to its DIFFERENTIAL, df. The subject of differentials is full of devilish details.

Suffice to say that you should think of df as the shorthand for the function that encodes the rate(s) of change of the output(s) of f, depending on the input(s) and rate(s) of change thereof. [see FLCT II for a better explanation of differentials...]

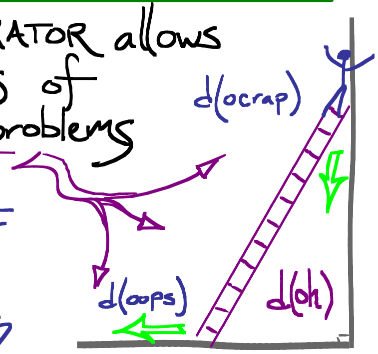
EXAMPLE You know how to work with differentials, even if only via your prior exposure to integration: $u = \sin x \implies du = \cos x\, dx$

THE RULES OF d

$d(f+g) = df + dg$	SUM
$d(fg) = f\,dg + g\,df$	PRODUCT
$d(f/g) = \dfrac{g\,df - f\,dg}{g^2}$	QUOTIENT
$d(f\circ g) = df \cdot dg$	CHAIN



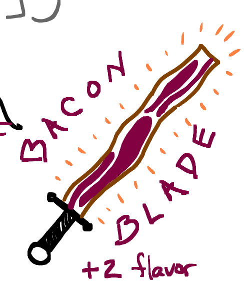
The BIG IDEA is this: using differentiation as an OPERATOR allows you to transform equations by d into equations of rates. This is the basis for various RELATED RATES problems



QUESTION A round droplet of water evaporates at a rate proportional to its surface area. At the instant when the droplet vanishes, which is changing more rapidly, its surface area? its volume? its radius?

HINT: $V = \frac{4}{3}\pi r^3$; $A = 4\pi r^2$; $dA = C\cdot A$ (C = CONST) keep going...

TRANSFORMS Differentiation, d, is not the only weapon to wield on equations: numerous other operators both respect equations and play nicely with differentiation.



One example is LOGARITHMIC DIFFERENTIATION, which expoits the LOGARITHM operator LN followed by d.

EXAMPLE

$$\boxed{y = x^x} \xrightarrow{LN} \ln y = x \ln x \xrightarrow{d} \underbrace{\frac{1}{y}dy = (\ln x)dx + x\left(\frac{1}{x}dx\right)}$$

$$\boxed{\frac{dy}{dx} = x^x(1+\ln x)} \xleftarrow{ALGEBRA} dy = y(1+\ln x)dx$$

Other examples demonstrate: OPERATORS TRANSFORM EQUATIONS

EXAMPLE

$$\boxed{y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x} \xrightarrow{LN} \ln y = \lim_{x\to\infty} x \ln\left(1+\frac{1}{x}\right) \quad \left[\begin{matrix}\text{as } x\to\infty \\ \frac{1}{x}\to 0\end{matrix}\right]$$

$$\xrightarrow{TAYLOR} \ln y = \lim_{x\to\infty} x\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} + HOT\right)$$

$$\boxed{y = e} \xleftarrow{EXP} \ln y = 1$$

This example shows how the operator LN allows for Taylor expansion. The inverse operator, EXP, recovers the solution.

---

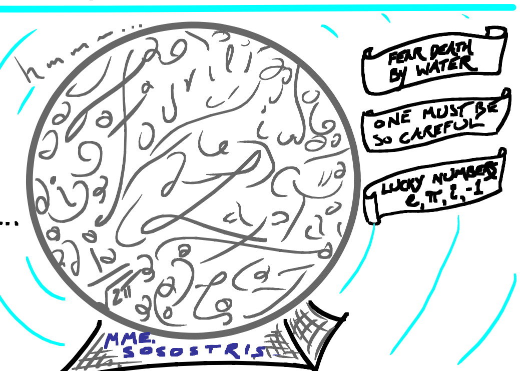
this is a...

FORESHADOWING...

Transforms are important tools in control theory, signal processing, imaging & more...

HARD-ALGEBRA PROBLEMS $\xrightarrow{\text{TRANSFORM}}$ EASY-ALGEBRA PROBLEMS

HARD-ALGEBRA PROBLEMS $\xleftarrow{\text{INVERSE}}$ EASY-ALGEBRA PROBLEMS

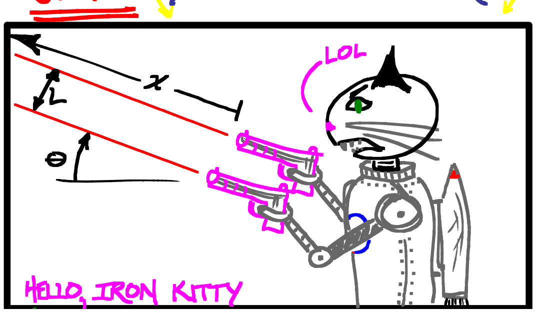


YOU MAY FIND IT HELPS YOUR INTUITION TO APPLY WHAT YOU KNOW ABOUT DERIVATIVES: HERE FOLLOW SOME USES...

OPTIMIZATION I suspect you've seen basic optimization in the guise of MAX-MIN problems. These follow a common pattern:

STEP 1: DRAW A PICTURE

STEP 2: DETERMINE ALL VARIABLES/CONSTRAINTS



STEP 3: WRITE OUT THE FUNCTION TO BE EXTREMIZED

STEP 4: DIFFERENTIATE & FIND CRITICAL POINTS

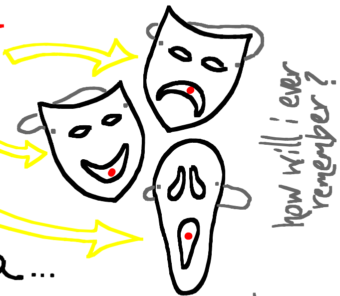
STEP 5: CLASSIFY CRITICAL POINTS (MAX/MIN/?)

STEP 6: PROFIT.

The interesting part is classification...

To determine whether you have a max or a min, you will likely use the "SECOND-DERIVATIVE TEST," which is how high school students pronounce "TAYLOR SERIES". For $x=a$, a critical point of a function $f(x)$, one infers...

a IS A MAXIMUM / MINIMUM / IDUNNO OF f IF $f''(a)$ IS NEGATIVE / POSITIVE / ZERO



To grasp WHY this is true, Taylor expand @ $x=a$...

$$f(a+h) = \underbrace{f(a)}_{\text{VALUE}} + \underbrace{\cancel{f'(a)}}_{\text{0 FOR A CRITICAL}} h + \underbrace{\tfrac{1}{2}f''(a)}_{\text{SIGN}} \underbrace{h^2}_{\text{QUADRATIC}} + \text{H.O.T.}$$

(If $f''(a)$ is not defined, you must use other tools...)

This implies that near $x=a$, f is quadratic in $h=x-a$. If the second derivative also vanishes, you simply take the next term in the Taylor series (3^{rd}, 4^{th}, etc.)...

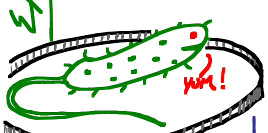
DIFFERENTIAL EQUATIONS → ODEs

An ORDINARY DIFFERENTIAL EQUATION on $x(t)$ is simply an algebraic equation implicating x and its derivative(s). These are singularly indispensible...

B=T : T=B

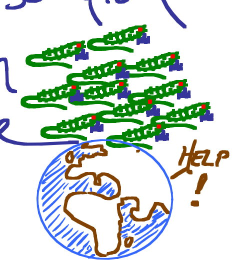
EXAMPLE

For $N(t)$ = number of bacteria in a nutrient-filled dish, one simple model for population dynamics is that N increases at a rate proportional to N. That is,



$$\frac{dN}{dt} = rN \qquad (r>0 \text{ RATE CONSTANT})$$

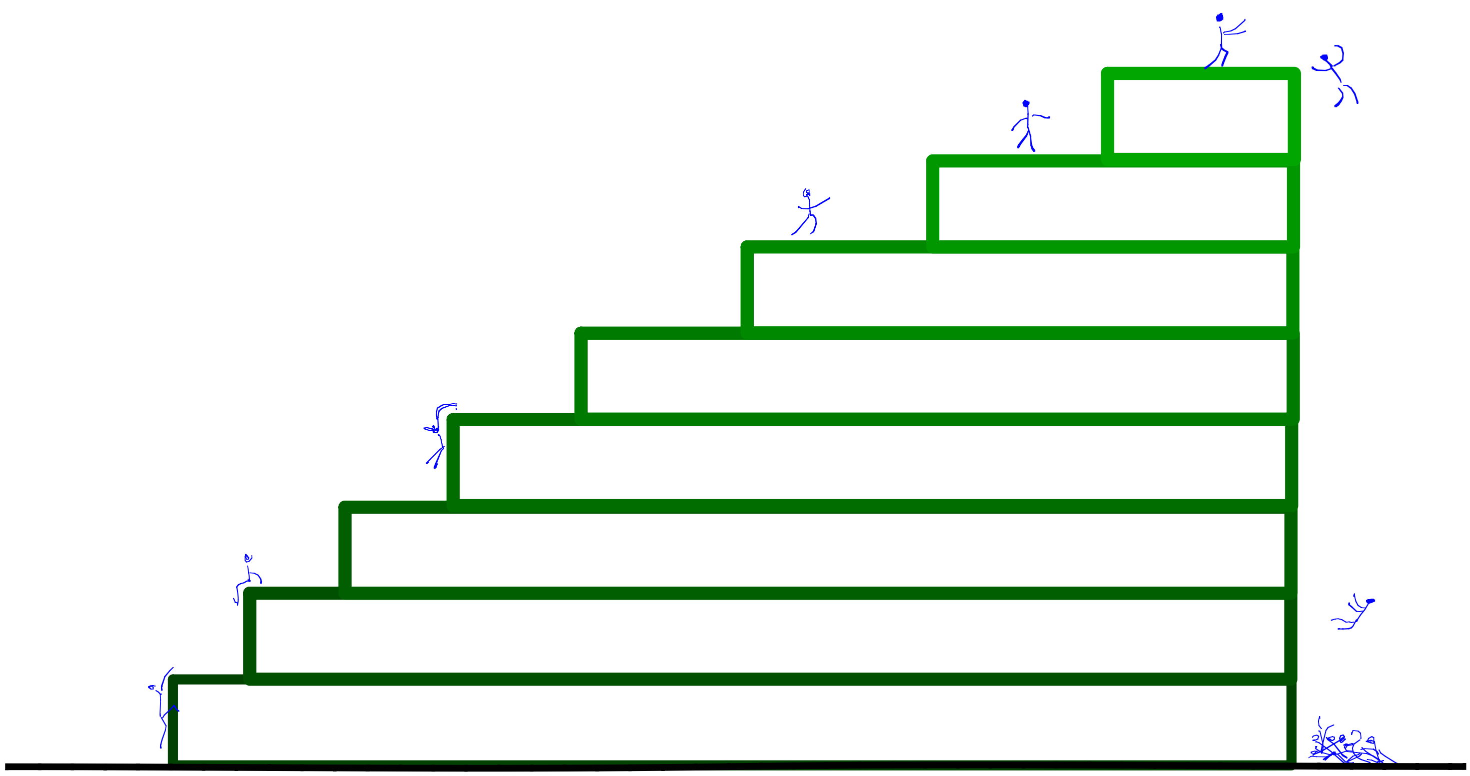
Let the reader note that $N(t)=Ce^{rt}$ is a solution for any C (constant), where $C=N(0)$. BUT... (1) This is an obviously flawed Malthusian model; (2) how do you solve this or more accurate ODE models?



STAY TUNED...

F.L.C.T. Chapter 3:

Integration



This chapter defines & computes integrals...

ANTI-DERIVATIVES The INDEFINITE INTEGRAL is easy to define: it is the inverse of the derivative.

$$\int f(x)\,dx \;:=\; \{ F(x) \text{ such that } F'(x) = f(x) \}$$

$+\,C$

Note: this is a class of functions, unique up to a constant.

EXAMPLE

$$\int \frac{1}{(1-x)^2}\,dx = \frac{x}{1-x} + C$$

BUT

$$\int \frac{1}{(1-x)^2}\,dx = \frac{1}{1-x} + C$$

HINT: these C's are not equal!

PROOF: If $F'(x) = G'(x)$ for all x, then
$0 = F'(x) - G'(x) = (F-G)'(x)$
So that $F - G = \text{CONSTANT}$ (since $\frac{d}{dx} \equiv 0$) ✓

It is a remarkable fact that anti-derivatives exist: all (smooth) functions have an indefinite integral. However, it may not be in terms of simple functions!

EH? CERF? WTF?

SEPARATION Simple differential equations motivate integrals best:

$$\frac{dx}{dt} = f(t) \quad \Longrightarrow \quad x = \int f(t)\,dt$$

E X A M P L E S . . .

Given gravitational acceleration $a(t) = -g$ integrate for VELOCITY $v(t)$ and POSITION $x(t)$ via:

$$\frac{dv}{dt} = a \;\Rightarrow\; v(t) = -gt + v_0 \qquad \frac{dx}{dt} = v \;\Rightarrow\; x = -\frac{gt^2}{2} + v_0 t + x_0$$

Slightly more complex models lead to these forms of ODE's:

$$\frac{dx}{dt} = f(x) \qquad\qquad \frac{dx}{dt} = f(x)\,g(t)$$

[AUTONOMOUS 1ST ORDER ODE] [SEPARABLE 1ST ORDER ODE]

These are ubiquitous...

EXAMPLES: WORDS — ODE — CARTOON

The amount of GREENGOO decays at a rate proportional to the amount present. $\frac{dg}{dt} = -\kappa g$

The number of bunnies increases at a rate proportional to the population, and decreases at a rate proportional to its square. $\frac{dN}{dt} = rN - kN^2$

Temperature changes at a rate proportional to the difference to the ambient temperature. $\frac{dT}{dt} = c(T_a - T)$

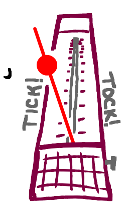
Such equations are solved by separation & integration:

EXAMPLE: $\frac{dg}{dt} = -\kappa g \xrightarrow[g\,\&\,t]{\text{SEPARATE}} \frac{dg}{g} = -\kappa dt \xrightarrow{\text{INTEGRATE}} \int \frac{dg}{g} = \int -\kappa dt$

$\xrightarrow{\text{VOILÀ}} \ln g = -\kappa t + C \xrightarrow{\text{EXP}} g = e^{-\kappa t + C}$

$\xrightarrow{e^C = g(0)} \boxed{g(t) = g_0 e^{-\kappa t}}$ THIS IS WHAT WE FOUND EXPERIMENTALLY IN CH. 2!

$\boxed{\text{COUPLED OSCILLATORS}}$ Consider the case of two simple oscillators, modeled as an angle, say $\theta_1(t)$ and $\theta_2(t)$. Each represents a repetitive behavior: a metronome, or a vibrating string, or a beating heart. The simplest model of oscillation is:



$$\frac{d\theta_1}{dt} = \omega \qquad \frac{d\theta_2}{dt} = \omega \qquad [\omega = \text{FREQUENCY CONST}]$$

Solution: $\theta_1(t) = \omega t + C_1$; $\theta_2(t) = \omega t + C_2$

But: what if we couple them... put some small influence of θ_2 on θ_1 and vice versa. SINUSOIDAL coupling is simplest:

$$\begin{aligned} \frac{d\theta_1}{dt} &= \omega + \overset{\text{COUPLING}}{\boxed{\varepsilon}} \sin(\theta_2 - \theta_1) \\ \frac{d\theta_2}{dt} &= \omega + \underset{\text{STRENGTH}}{\boxed{\varepsilon}} \sin(\theta_1 - \theta_2) \end{aligned}$$

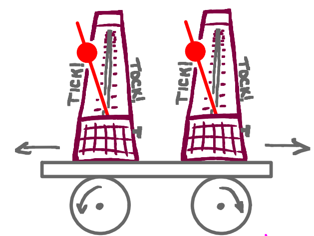
$\Big\}$ if θ_2 is a bit ahead of θ_1, it slows down θ_2 and speeds up θ_1 ...

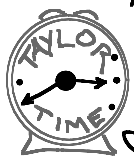
This COUPLED SYSTEM cannot be solved explicitly as above. But, we can analyze the PHASE $\phi(t) = \theta_2(t) - \theta_1(t)$ between the two:

$$\boxed{\frac{d\phi}{dt}} = \frac{d}{dt}(\theta_2 - \theta_1) = \frac{d\theta_2}{dt} - \frac{d\theta_1}{dt} = \left(\omega + \varepsilon \sin(-\phi)\right) - \left(\omega + \varepsilon \sin\phi\right) = \boxed{-2\varepsilon \sin\phi}$$

LETS TRY! $\frac{d\phi}{\sin\phi} = -2\varepsilon\, dt \Rightarrow \int \frac{d\phi}{\sin\phi} = -2\varepsilon t + C$ OH NOES! $\int \csc\phi\, d\phi = ?$

This is not an easy integral. So... let's consider ϕ not too large... For ϕ near zero,





$$\frac{d\phi}{dt} \approx -2\varepsilon\phi \xrightarrow[\text{ODE}]{\text{LINEAR}} \phi \approx \phi_0 e^{-2\varepsilon t}$$ $\boxed{\text{THUS... } \phi \to 0}$

This predicts that two coupled oscillators tend to SYNCHRONIZE! This is a remarkably observable result...

CRICK-IT! CRICK-IT! "I'm like so... grumpy..." "Like, me too! OMG!" CLAP! CLAP!

INTEGRATION TECHNIQUES

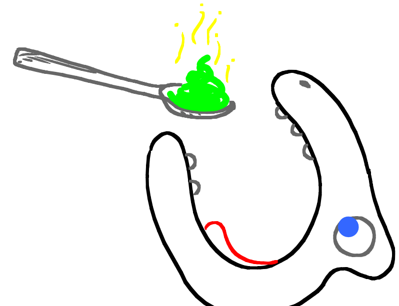
If you need to compute $\int f(x)\,dx$ and you don't recognize it, what do you do? There is no general algorithm, and even the 'puter might not help. There are some techniques worth knowing. They are:

MAGNA INTEGRA

- U-SUBSTITUTION
- INTEGRATION BY PARTS
- TRIG. SUBSTITUTIONS
- PARTIAL FRACTIONS
- COMPLETING SQUARES
- + a bunch of other stuff...

} differentiation rules reversed

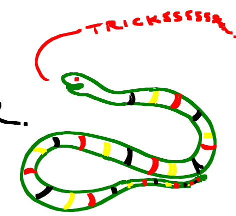
} fun with algebra



You should know that these exist, how to use them, and when each is useful. But this topic is not at the heart of Calculus.

U-SUBSTITUTION

This is the Chain Rule run in reverse.



If you substitute $u = u(x)$ then $du = \frac{du}{dx}dx$ and

$$\int f(u)\,du = \int f(u(x))\frac{du}{dx}\,dx$$

change to GIVEN

The difficulty is in choosing u well...

To be honest, your hand remembers this better than your head...

EXAMPLE:
$$\int \frac{(1+\ln x)^3}{x}\,dx = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(1+\ln x)^4 + C$$

$u = (1 + \ln x) \Rightarrow du = \frac{1}{x}\,dx$

INTEGRATION BY PARTS

This is a double-substitution method based on running the Product Rule in reverse:

PRODUCT RULE $\quad d(uv) = u\,dv + v\,du$

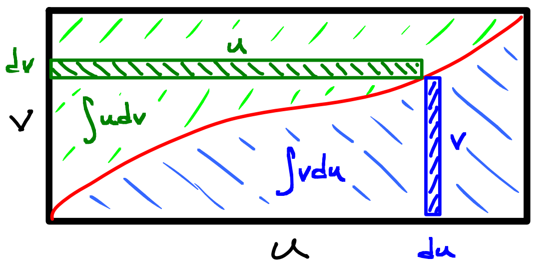
INTEGRATE $\quad \int d(uv) = \int u\,dv + \int v\,du$

$\int d☺ = ☺ \quad uv = \int u\,dv + \int v\,du$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

INTEGRATION BY PARTS FORMULA

LOVE



As with all substitution methods, the only difficulty lies in choosing u, v correctly...

UVUVUVUVUVUVUVUVUV
VUVUVUVUVUVUVUVUVU

EXAMPLE:

$u = x$
$du = dx$
$dv = e^x dx$
$v = e^x$

$$\int \underbrace{x}_{u}\ \underbrace{e^x dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\ \underbrace{dx}_{du} = e^x(x-1) + C$$

EASY!

EXAMPLE:

Let $I = \int e^x \sin x\, dx$. Computing I requires a double...

$u = \sin x$
$du = \cos x\, dx$
$dv = e^x dx$
$v = e^x$

$$I = e^x \sin x - \boxed{\int e^x \cos x\, dx} \quad J = \int e^x \cos x\, dx$$

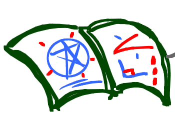
$u = \cos x$
$du = -\sin x\, dx$
$dv = e^x dx$
$v = e^x$

$$J = e^x \cos x + \boxed{\int e^x \sin x\, dx} \quad I, \text{ again! DRAT! wait...}$$

But combining these gives $I = e^x \sin x - (e^x \cos x + I)$ or

$$2I = e^x(\sin x - \cos x)$$

$$\Rightarrow I = \tfrac{1}{2} e^x (\sin x - \cos x) + C$$



can't hurt to recite this...

I LOVE IT!

J DON'T.

TRIGONOMICON

There is a body of methods based on various trigonometric substitutions. You should be familiar with the use of these methods, but do not love them, for they will not love you back. They are merely tricks based on

sigh!

$1 - \sin^2 = \cos^2 \longrightarrow u = \sin x$ } good for $\sqrt{1-x^2}$ terms

$1 + \tan^2 = \sec^2 \longrightarrow u = \tan x$ } good for $\sqrt{1+x^2}$ terms

$1 + \sinh^2 = \cosh^2 \longrightarrow u = \sinh x$

i told you he was tricksy...

EXAMPLE: $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$

Let $x = \tan u$, $dx = \sec^2 u\, du$: $\displaystyle = \int \frac{\sec^2 u\, du}{\sqrt{1+\tan^2 u}} = \int \sec u\, du$ STUCK! AAH! STUCK...

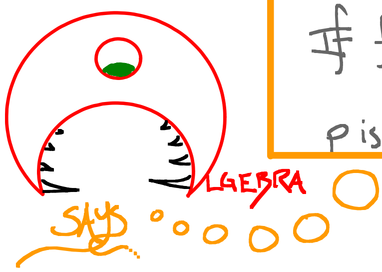
Let $x = \sinh u$, $dx = \cosh u$: $\displaystyle = \int \frac{\cosh u\, du}{\sqrt{1+\sinh^2 u}} = \int du = u + C$

$= \operatorname{arcsinh} x + C$

did I mention how much I love hyperbolic trigs?

PARTIAL FRACTIONS

This, too, is an algebraic trick worth learning. If you need to integrate a RATIONAL function (quotient of polynomials), and if the denominator factors nicely, you may split it up...



If $\displaystyle f(x) = \frac{p(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)}$ where the "roots" r_i are distinct and p is a polynomial of degree $< n$, then $f(x)$ splits up into a sum

$$f(x) = \frac{A_1}{x-r_1} + \frac{A_2}{x-r_2} + \cdots + \frac{A_n}{x-r_n}$$ for some constants A_i

EXAMPLE: $\displaystyle f(x) = \frac{4x^2-3x-4}{x^3+x^2-2x} = \frac{4x^2-3x-4}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$

put over a common denominator... $\displaystyle = \frac{A(x-1)(x+2) + Bx(x+2) + Cx(x-1)}{x(x-1)(x+2)} = \frac{4x^2-3x-4}{x(x-1)(x+2)}$

NUMERATORS $\Rightarrow$ $4x^2 - 3x - 4 = \underbrace{(A+B+C)}_{4}x^2 + \underbrace{(A+2B-C)}_{-3}x + \underbrace{(-2A)}_{-4}$ 3 EQUATIONS, 3 UNKNOWNS: #WINNING

$$\left.\begin{array}{l} A = 2 \\ B = -1 \\ C = 3 \end{array}\right\} \Rightarrow \int \frac{4x^2-3x-4}{x^3+x^2-2x}\,dx = \int \frac{2}{x} - \frac{1}{x-1} + \frac{3}{x+2}\,dx = 2\ln|x| - \ln|x-1| + 3\ln|x+2| + C$$

NOTE: the algebra in solving for A, B, C is simpler if you substitute in specific values of $x = 0, 1, -2$. Try it!

STRATEGY & INTEGRATION

Students want to know the secret algorithm or strategy for computing an integral in general. I will tell you.

BWAHAHAHAHAHAHA...

It does not exist.

You're on your own, kids. Might want to try google/wiki/alpha.

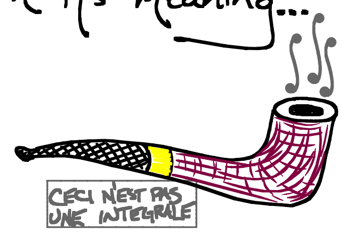
DEFINITE INTEGRALS

The indefinite integral is a (collection of) function(-s, up to a constant), given by antidifferentiation. This was motivated by finding solutions to differential equations. In contrast, the DEFINITE INTEGRAL is numerical and motivated by problems of geometry, physics, & statistics. You have no doubt computed lots of definite integrals in the past: it is now time to learn its meaning...

¡WARNING! DON'T BE FOOLED...

$$\int_a^b f(x)\,dx$$

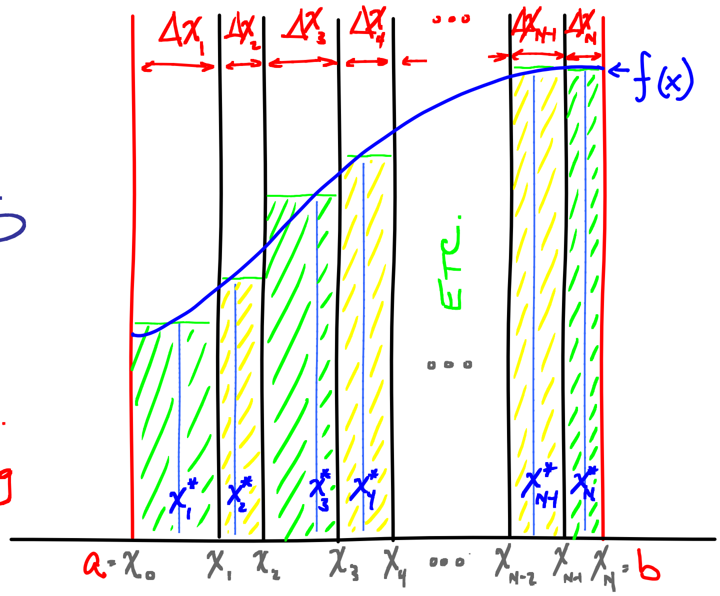
looks like the indefinite integral; however, it has an entirely different meaning!



DEFINITION: For an interval $[a,b]$, let $\mathbb{P}$ denote a PARTITION into intervals $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$. Let $\Delta x_i = x_i - x_{i-1}$, and choose, for each $i = 1 \ldots N$ a SAMPLE POINT x_i^* in $[x_{i-1}, x_i]$. The RIEMANN SUM of f on $\mathbb{P}$ is defined as:

$$S_{\mathbb{P}}(f) = \sum_{i=1}^{N} f(x_i^*)\,\Delta x_i$$

The DEFINITE INTEGRAL of f on $[a,b]$ is the limit of the Riemann sums over partitions $\mathbb{P}$ on which $|\mathbb{P}| = \max_i(\Delta x_i) \to 0$.



$$\int_a^b f(x)\,dx = \lim_{|\mathbb{P}| \to 0} \sum_i f(x_i^*)\,\Delta x_i$$

unusual... ...interesting

A definite integral is therefore a very specific sort of limit-sum. You can see why it is often associated with an area; this interpretation is perhaps not the best (it will not survive multivariable calculus!). I prefer a MASS ANALOGY...

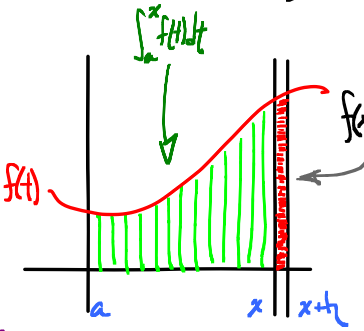
$f(x)$ = density of a rod at position x

$\int_a^b f(x)\,dx$ = mass of rod, $a \leq x \leq b$

THE FUNDAMENTAL THEOREM

Although Riemann sums and limits thereof can be computed, it is a big pain, not unlike that of the computation of the derivative from the definition. It is MUCH better to use the FTIC, which entwines the definite & indefinite:

THEOREM: Let $f(x)$ be continuous. Then...



① $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$; and

② $\underbrace{\int_a^b f(x)\,dx}_{\text{DEFINITE INTEGRAL}} = \Big[\underbrace{\int f(x)\,dx}_{\text{INDEFINITE INTEGRAL}}\Big]_{x=a}^{b}$ evaluated from a to b...

PROOF: ① $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)dt = \lim_{h\to 0}\frac{1}{h} f(x^*)\Delta x = f(x)$

$x \le x^* \le x+h$
$\Delta x = h$

② Since $\int_a^x f(t)dt$ is an antiderivative of $f(x)$, and since all antiderivatives differ by a constant, then, for $G(x)$ any antiderivative of f,

$$\int_a^x f(t)dt - G(x) = C \quad \text{CONSTANT}$$

evaluate @ $x=a$ $\Rightarrow$ $\underbrace{\cancel{\int_a^a f(t)dt}}_{0 \text{ by definition}} - G(a) = C$ $\xrightarrow{\text{THUS}}$ $C = -G(a)$ and

$$\int_a^b f(t)dt = G(b) + C \quad \text{for any antiderivative } G$$
$$= G(b) - G(a) \quad \text{"} \quad \text{"}$$
$$= \Big[\int f\,dx\Big]_{x=a}^{b}$$

QED

Quod Erat Demonstrandum = "for shizzle" in Latin

This sets the stage for using this chapter's tools for definite integrals...

EXAMPLE

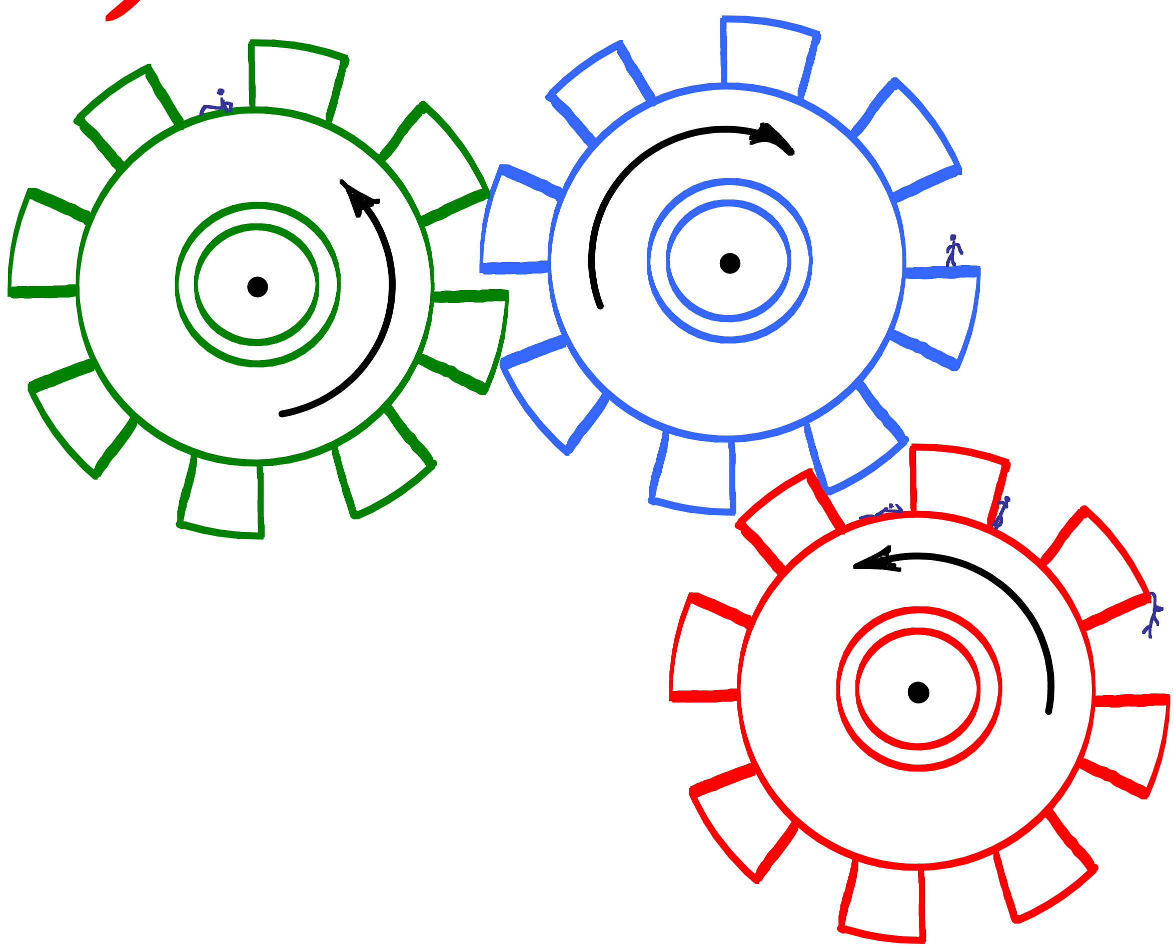
$$\int_{x=a}^{b} f(u(x))\frac{du}{dx}\,dx = \int_{u=u(a)}^{u(b)} f(u)\,du \qquad \text{U-SUBS}$$

$$\int_{x=a}^{b} u(x)\,dv(x) = u(x)\,v(x)\Big|_{x=a}^{b} - \int_{x=a}^{b} v(x)\,du(x) \qquad \text{PARTS}$$

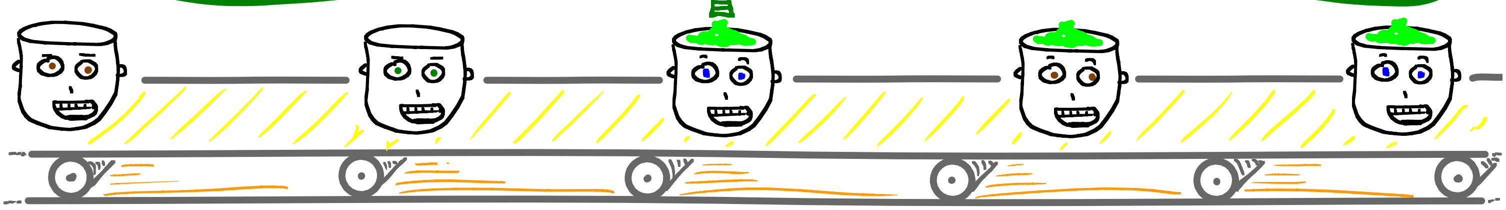
Next up...

WHAT'S IT ALL GOOD FOR?

F.L.C.T. Chapter 4: Applications



...and all the Calculus learned thus far is good for... ?



This Chapter surveys some typical applications...

"WORDLE" MAP of CHAPTER

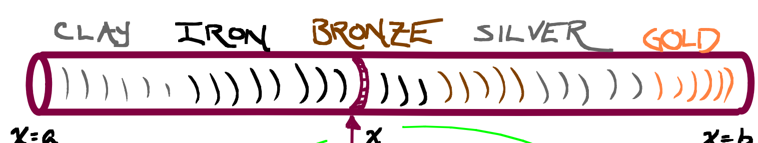
THE BIG IDEA   it's very simple...

$$\boxed{☺ = \int d☺}$$

☺ = DESIDERATUM   the "☺-element"

If you want something, determine its ELEMENT -- its rate of change -- and integrate.

MASS   Consider a rod of varying density $\rho(x)$



WHAT IS ITS MASS, m?

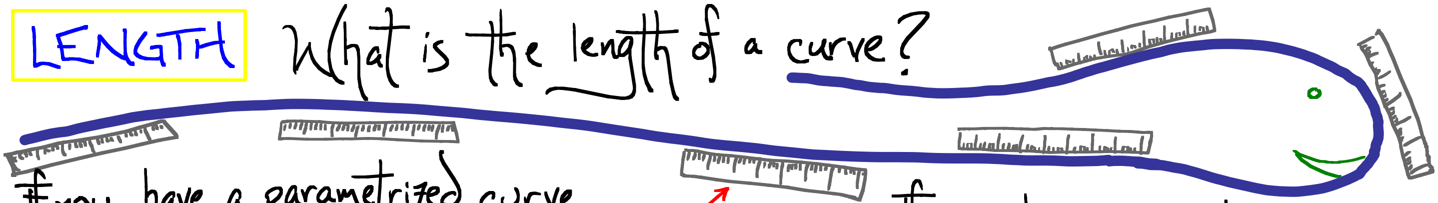
MASS ELEMENT $dm = \rho(x)\,dx$

MASS $\boxed{m = \int dm} = \int_{x=a}^{b} \rho(x)\,dx$

"So, i can think of dm as an ittybitty mass, right?"

WELL... NO. IT'S A DIFFERENTIAL. BUT, WHATEV.

LENGTH
What is the length of a curve?

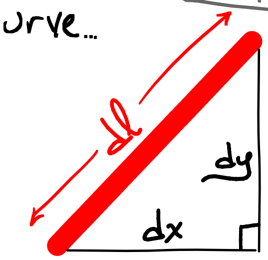


If you have a parametrized curve...

$\gamma(t) = (x(t),\, y(t))$

$d\ell = \sqrt{dx^2 + dy^2}$

$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$



If you have a graph...

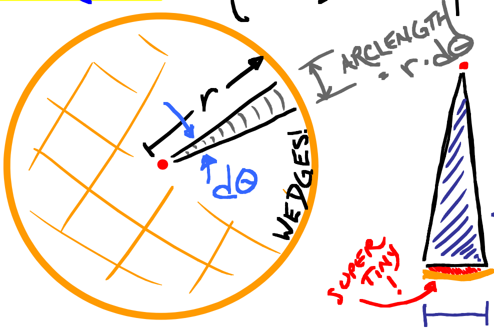
$y = f(x)$

$d\ell = \sqrt{dx^2 + dy^2}$

$= \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$

$\boxed{\ell = \int d\ell}$

AREA
Here is a simple example of the principle: AREA OF A DISC (radius r)



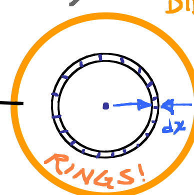
$dA = \frac{1}{2}(r\, d\theta)\, r$

$= \frac{1}{2} r^2 d\theta$

$\boxed{A = \int dA} = \int_{\theta=0}^{2\pi} \frac{1}{2} r^2 d\theta = \pi r^2$

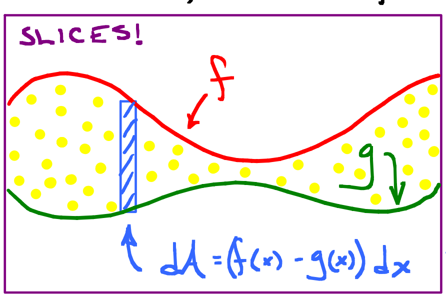
THIS IS HOW ARCHIMEDES DID IT. KINDA.

...or, use a different area element



$dA = 2\pi x\, dx$

$A = \int dA = \int_{x=0}^{r} 2\pi x\, dx = \pi r^2$



HEY! WHAT ABOUT ME? YOU FORGOT!

Yes, your favorite example, the area between two curves, works:

$A = \int dA = \int_{x=a}^{b} (f(x) - g(x))\, dx$

DON'T FORGET! Sometimes using horizontal instead of vertical is better!

SURFACE AREA
The area of a non-flat surface takes some effort to compute. For simplicity, consider a surface obtained by revolving a curve about an axis: a SURFACE OF REVOLUTION...

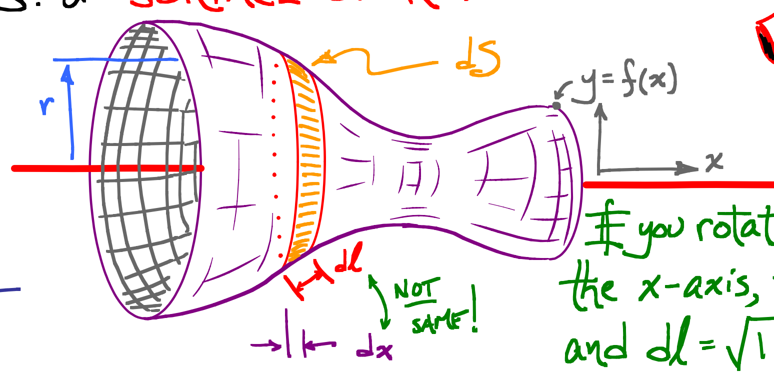
$dS = 2\pi\, \underline{r}\, \underline{d\ell}$

SURFACE AREA ELEMENT

DISTANCE TO AXIS

ARCLENGTH ELEMENT

$\boxed{S = \int dS}$



If you rotate $y = f(x)$ about the x-axis, then $r = f(x)$ and $d\ell = \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$

VOLUME

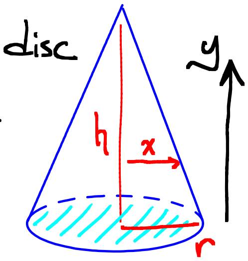
As you will have guessed by now... $\boxed{V=\int dV}$

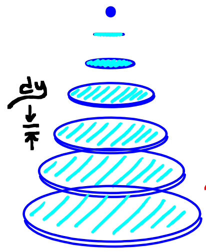
what is that squiggly sign you keep using? question marks?

and the challenge is to find a good VOLUME ELEMENT

You may recall the volume of a CONE over a disc of radius r and height h is $V=\frac{1}{3}\pi r^2 h$.

WHY THE $\frac{1}{3}$? The slicing method gives





RADIUS OF DISCS CHANGES LINEARLY $x=r(1-y/h)$

$$dV = A(y)\,dy = \pi x^2\,dy = \pi r^2(1-y/h)^2\,dy$$

so...

$$V=\int_{y=0}^{h}\pi r^2\left(1-\frac{y}{h}\right)^2 dy = \pi r^2\int_{u=0}^{1} h u^2\,du = \pi r^2 h\,\frac{u^3}{3}\Big|_0^1 = \frac{1}{3}\pi r^2 h$$

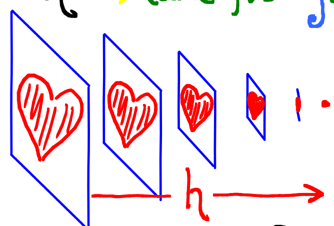
$(u = y/h - 1)$

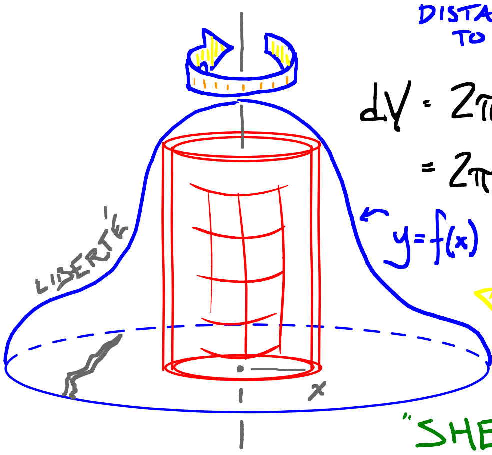
AHA! The $\frac{1}{3}$ came from $\int u^2$

But... this generalizes greatly to the volume of a cone of height h over an area B

$\boxed{V=\frac{1}{3}Bh}$

CAN YOU DO THIS?





DISTANCE TO AXIS

HEIGHT OF CYLINDER

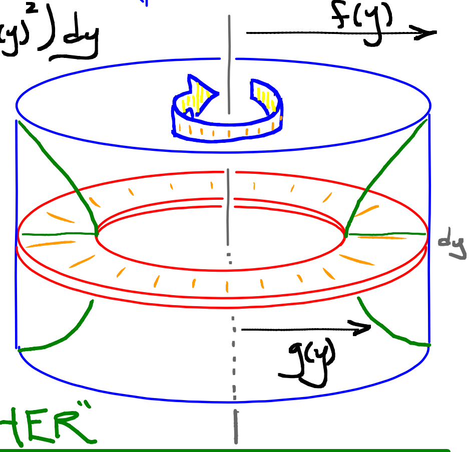
$$dV = 2\pi r\,h\,dr = 2\pi x f(x)\,dx$$

$y=f(x)$

$$dV=\pi\left(f(y)^2-g(y)^2\right)dy$$

OUTER BOUNDARY

INNER BOUNDARY



VOLUMES OF REVOLUTION can be decomposed like so

"SHELL" "WASHER"

Now for an "improper" example...

ROTATE $y=x^{-p}$ ABOUT x-AXIS $1\le x<\infty$

VOLUME

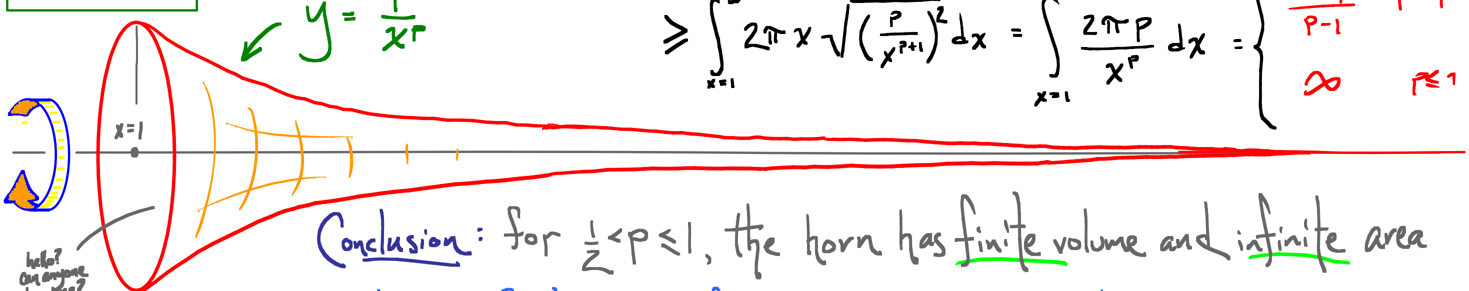
$$V=\int_{x=1}^{\infty}\pi\left(\frac{1}{x^p}\right)^2 dx = \begin{cases}\frac{\pi}{1-2p}\,x^{1-2p}\Big|_1^{\infty} & (p\neq\frac{1}{2}) & = \frac{\pi}{2p-1} & (p>\frac{1}{2}) \\ & & = \infty & (0<p<\frac{1}{2}) \\ \pi\ln x\Big|_1^{\infty} & (p=\frac{1}{2}) & = \infty & (p=\frac{1}{2})\end{cases}$$

SURFACE AREA

$$S=\int_{x=1}^{\infty}2\pi x\sqrt{1+\left(\frac{p}{x^{p+1}}\right)^2}\,dx = ? \quad \text{ESTIMATE!}$$

$$\geq \int_{x=1}^{\infty}2\pi x\sqrt{\left(\frac{p}{x^{p+1}}\right)^2}\,dx = \int_{x=1}^{\infty}\frac{2\pi p}{x^p}\,dx = \begin{cases}\frac{2\pi p}{p-1} & p>1 \\ \infty & p\leq 1\end{cases}$$

$y=\frac{1}{x^p}$



Conclusion: for $\frac{1}{2}<p\leq 1$, the horn has finite volume and infinite area

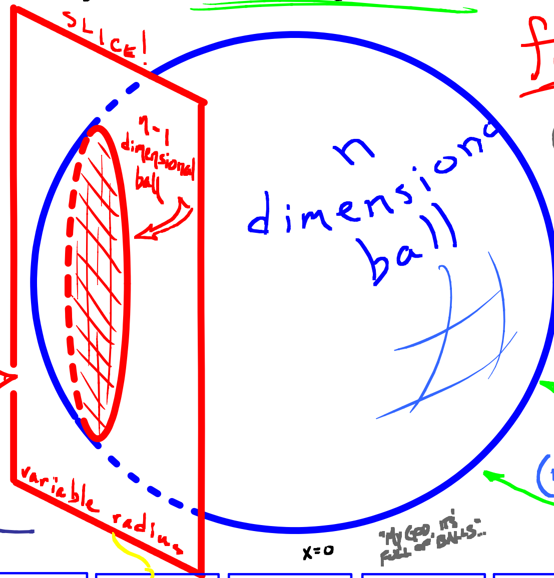
"You mean it holds a finite amount of paint but cannot be finitely painted?" KINDA.

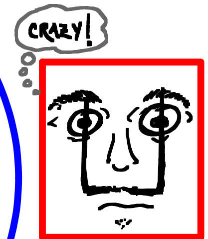
BALLS Let's compute the "volume" of an n-dimensional ball, radius r

for any n

WHAT IS... THE n^{th} DIMENSION?

$\mathbb{R}^n = \{(x_1, \dots, x_n) \text{ with each } x_i \in \mathbb{R}\}$
ordered n-tuples of reals...
The n-dimensional ball is
$B^n(r) = \{(x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 \le r^2\}$



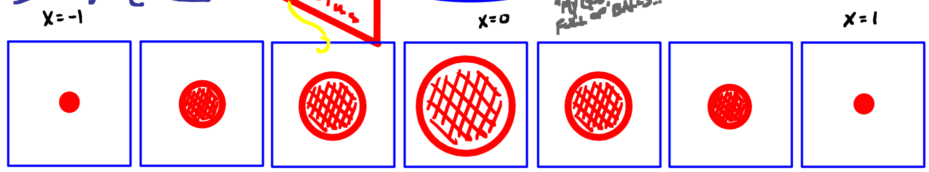


SOLID (not a sphere)

IDEA: Use the "slicing" method
A "hyperplane" cuts the ball into a ball of one dimension less. As you sweep across, the radius of the $(n-1)$ dimensional ball equals

$$r = \sqrt{1-x^2} \quad -1 \le x \le 1$$

(assuming the original is a unit n-ball)



Let $V_n(r)$ = "volume" of n-dimensional ball of radius r, with $\boxed{V_n := V_n(1)}$

LEMMA: $V_n(r) = V_n r^n$ (think: units & scaling)

LEMMA: $V_n = \int_{x=-1}^{1} V_{n-1}\left(\sqrt{1-x^2}\right)^{n-1} dx = V_{n-1} \underbrace{\int_{x=-1}^{1} (1-x^2)^{\frac{n-1}{2}} dx}_{=: C_n \text{ CONSTANT}}$

$$\boxed{V_n = C_n V_{n-1}}$$

Using the substitution $x = \sin\theta$ and integration by parts, one obtains... = NOT ME...

$$C_n \underset{\text{SUBS!}}{=} \int_{\theta=-\pi}^{\pi} (1-\sin^2\theta)^{\frac{n-1}{2}} \cos\theta \, d\theta = \int_{\theta=-\pi}^{\pi} \cos^n\theta \, d\theta \underset{\text{PARTS!}}{=} \frac{n-1}{n} \int_{\theta=-\pi}^{\pi} \cos^{n-2}\theta \, d\theta$$

i.e., $\boxed{C_n = \frac{n-1}{n} C_{n-2}}$ By hand, one computes... EASY

n	C_n	V_n
0	π	1
1	2	2
2	$\frac{\pi}{2}$	π
3	$\frac{4}{3}$	$\frac{4}{3}\pi$
4	$\frac{3\pi}{8}$	$\frac{\pi^2}{2}$
5	$\frac{16}{15}$	$\frac{8}{15}\pi^2$
6	$\frac{5\pi}{16}$	$\frac{\pi^3}{6}$
7	$\frac{32}{35}$	$\frac{16}{105}\pi^3$
8	$\frac{35\pi}{128}$	$\frac{\pi^4}{24}$

This gives an inductive approach for computing C_n
With some effort, an induction argument yields...

The volume of the unit ball in dimension n equals...

LENGTH... AREA... VOLUME... HYPERVOLUME...

$$V_n = \begin{cases} \dfrac{\pi^k}{k!} & : \quad n = 2k \quad \text{EVEN} \\[2ex] \dfrac{2^n \pi^k k!}{n!} & : \quad n = 2k+1 \quad \text{ODD} \end{cases}$$

QUESTION What is $\lim_{n\to\infty} V_n$? Are you frightened?

"NOT FRIGHTENED ENOUGH, I KNOW WHAT SCARES YOU."

AVERAGES

One of the best applications of integrals is in defining the average, or MEAN, of a function. The mean of $f(x)$ over $a \le x \le b$ is defined to be

$$\boxed{\bar{f} = \frac{\int_a^b f(x)\,dx}{\int_a^b dx}}$$

Why write the denominator thus? It makes more sense in a little while...

PLEASE STAND BY

BIG DUH $\int_a^b dx = b-a$

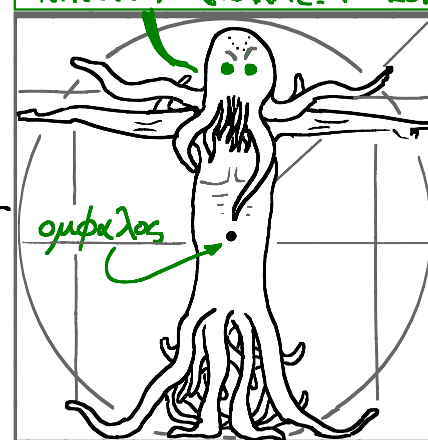
EXAMPLE: The ROOT MEAN SQUARE of f is $RMS(f) = \left(\overline{(f^2)}\right)^{1/2}$

WAVE POWER: $RMS(A\sin\omega t) = \left(\frac{\int_0^{2\pi/\omega}(A\sin\omega t)^2\,dt}{2\pi/\omega}\right)^{1/2} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}}$

CENTERS

What is the exact center of a given body? It is the CENTROID, the point whose coordinates are averages. Consider the region $R = \{(x,y) : a \le x \le b,\ g(x) \le y \le f(x)\}$ between two curves. Its centroid is the point $(\bar{x}, \bar{y})$ given by the average x and y values over R.

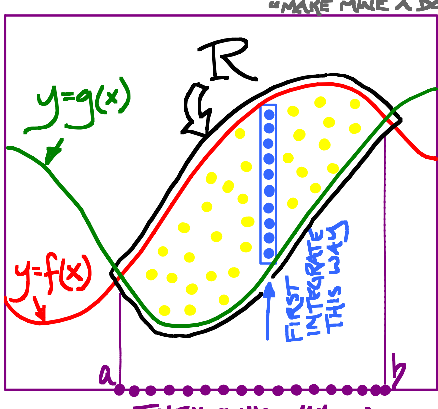
PHTHAAGT! BRYNMAWR! PO-LU.



TICKLE ME CTHULHU

THIS CALLS FOR TWO INTEGRALS, IN SUCCESSION:

"MAKE MINE A DOUBLE"



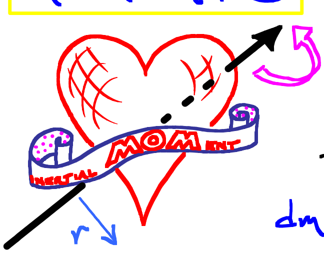
$$\bar{x} = \frac{\int_R x\,dA}{\int_R dA} = \frac{\int_{x=a}^b\left(\int_{y=g(x)}^{f(x)} x\,dy\right)dx}{\int_{x=a}^b f(x)-g(x)\,dx} = \frac{\int_{x=a}^b x\,(f(x)-g(x))\,dx}{\int_{x=a}^b f(x)-g(x)\,dx}$$

$$\bar{y} = \frac{\int_R y\,dA}{\int_R dA} = \frac{\int_{x=a}^b\left(\int_{y=g(x)}^{f(x)} y\,dy\right)dx}{\int_{x=a}^b f(x)-g(x)\,dx} = \frac{\int_{x=a}^b \frac{1}{2}\left[(f(x))^2-(g(x))^2\right]dx}{\int_{x=a}^b f(x)-g(x)\,dx}$$

These formulae give CENTROIDS: to get CENTER-OF-MASS, use a DENSITY $\rho(x,y)$ and do all integrals with respect to the MASS ELEMENT $dm = \rho\,dA$

MOMENTS

For an object rotated about an axis, the RMS of distance to the axis is called RADIUS OF GYRATION. The numerator is called the MOMENT OF INERTIA.



$$I = \int r^2\,dm$$

dm = MASS ELEMENT; r = DISTANCE

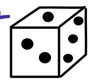
This moment represents the resistance to rotation about the axis...

PROBABILITY

A simple approach to probability is via MEASURES

COUNTING

Odds that a fair die lands on an odd number?

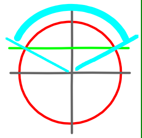


$$\frac{\text{ODDS}}{\text{TOTAL}} = \frac{3}{6} = \frac{1}{2}$$

LENGTH

Odds that a random angle has $\sin \geq 1/2$?

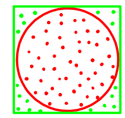
$$\frac{\pi/3}{2\pi} = \frac{1}{3}$$



AREA

Odds that a random point in a square lies in the inscribed circle?

$$\frac{\pi r^2}{(2r)^2} = \frac{\pi}{4}$$



VOLUME

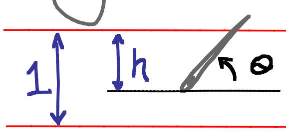
HYPER-VOLUME

This is the setting of a uniform random variable: probability is a volume fraction...

Let's try it out...

THE BUFFON NEEDLE PROBLEM

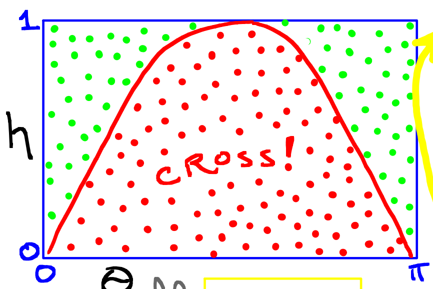
Given a needle of length 1, drop it on notebook paper with unit-distance lines. With what probability does the needle cross one of the lines?



The natural variables are: h, the distance from the low tip to the next line up; and θ, the angle.

NOTE: $0 \leq h < 1$ & $0 \leq \theta < \pi$

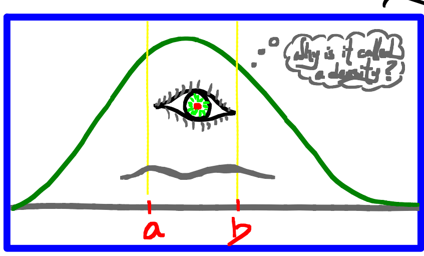
These variables are UNIFORM and INDEPENDENT: one has a crossing iff $\boxed{h \leq \sin\theta}$



What are the odds of a random (h, θ) satisfying this?

$$\frac{\int_0^\pi \sin\theta \, d\theta}{\pi \cdot 1} = \frac{2}{\pi}$$

THE AREA FRACTION



Typical random variables are not uniform, but vary according to a PDF: PROBABILITY DENSITY FUNCTION, $p(x)$, satisfying:

PROBABILITY THAT A "RANDOM" x LIES BETWEEN a AND b

$$\boxed{\mathbb{P}\{a \leq x \leq b\} = \int_{x=a}^{b} p(x)\,dx}$$

Of course, a PDF must satisfy

$$\underbrace{\int_{-\infty}^{\infty} p(x)\,dx}_{\text{PROBABILITY THAT } -\infty < x < \infty} = 1$$

LIKE, TOTALLY FOR SURE

Many definitions in probability & statistics are expressed in terms of integrals:

MEAN

$$\mu = \int x \, p(x)\,dx$$

The "average" value of x

HEY! THAT LOOKS LIKE CENTER-OF-MASS!

VARIANCE = VAR

$$\mathbb{V} = \int (x-\mu)^2 p(x)\,dx$$

How "spread out" the distribution is

HEY! THAT IS THE MOMENT OF INERTIA ABOUT THE μ-AXIS!

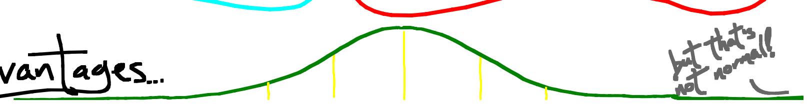
STANDARD DEVIATION = STDDEV

$$\sigma = \sqrt{\mathbb{V}}$$

Like $\mathbb{V}$, but with better units

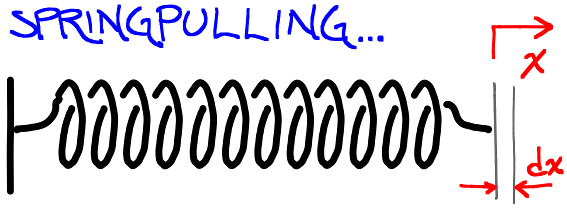
HEY! THAT'S THE RADIUS OF GYRATION ABOUT $x = \mu$!

Thinking of $p(x)dx$ as a MASS ELEMENT dm has some advantages...



WORK As you know, work equals FORCE · DISTANCE; but this is only true locally; one needs to integrate the WORK ELEMENT in general.

SPRINGPULLING...



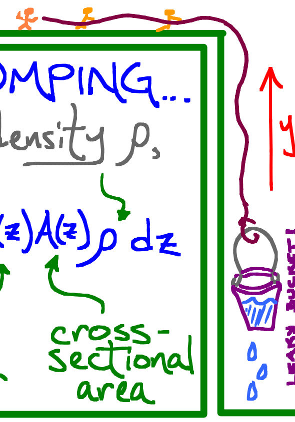
If the force required to displace a spring from equilibrium ($x=0$) is $F(x)$, then the work element is...

$$dW = F(x)\,dx$$

ROPELIFTING...

To pull an object whose weight $\omega(y)$ varies with height y...

$$dW = \omega(y)\,dy$$



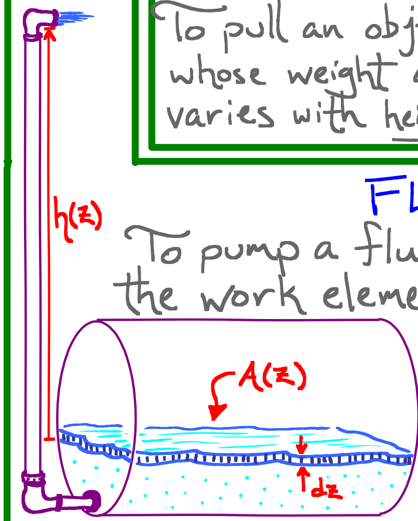
FLUIDPUMPING...

To pump a fluid of density ρ, the work element is...

$$dW = h(z)A(z)\rho\,dz$$

net height pumped

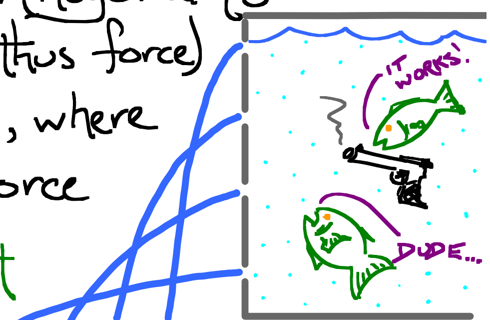
cross-sectional area



FORCE A fluid exerts a FORCE on a submerged surface equal to PRESSURE times area, orthogonal to the surface. The pressure (and thus force) varies with depth and equals ρz, where z is depth and ρ is density. The force element is...

$$dF = \rho z\,dA$$

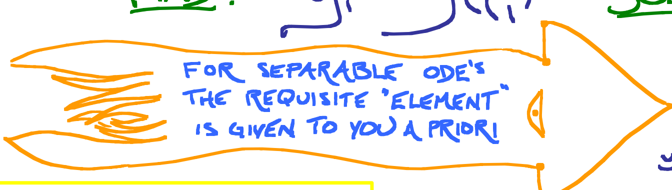
area element



DIFFERENTIAL EQUATIONS We initially motivated indefinite integrals via differential equations, but definite integrals are useful, too, for INITIAL VALUE PROBLEMS: in the separable setting, one gets...

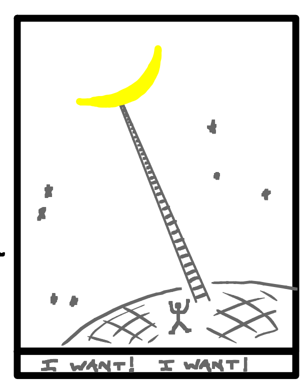
GIVEN: $\frac{dy}{dt} = f(y)g(t)$ THE DIFFERENTIAL EQUATION $\quad y(t_0) = y_0$ THE INITIAL CONDITION $\quad t_1$ THE FINAL TIME

FIND: $y_1 := y(t_1)$ SOLUTION: Integrate and solve...

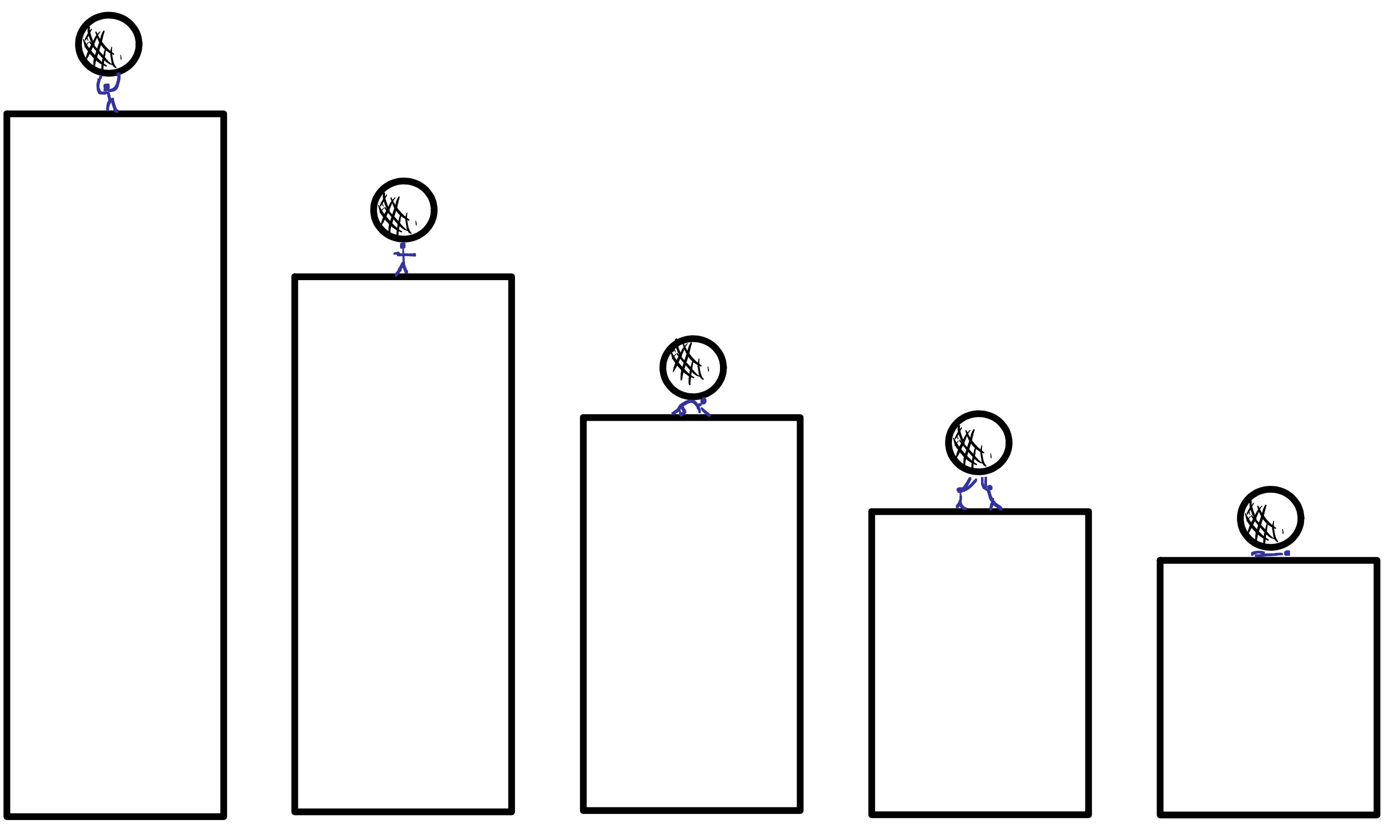


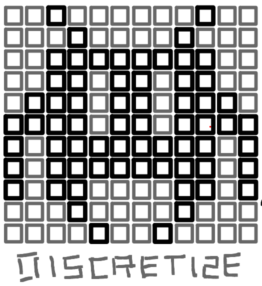
$$\int_{y=y_0}^{y_1} \frac{dy}{f(y)} = \int_{t=t_0}^{t_1} g(t)\,dt$$

THE MORAL Everybody wants to have a black box formula for solving stuff: do not do that. Construct an "element" and integrate. Carefully.



F.L.C.T. Chapter 5: Discretization





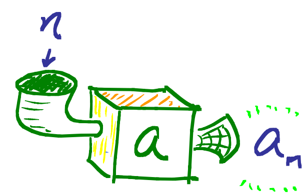
Calculus is built for continuous objects; the world is full of discrete, quantized things. The synthesis of smooth & discrete Calculus is the focus of this final chapter of 7LCT...

SEQUENCES

Calculus is done with functions.

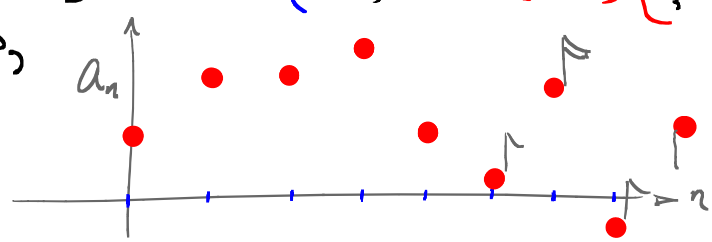
In single-variable calculus, the typical function has one real input and one real output: $f : \mathbb{R} \to \mathbb{R}$

The most common discretization is from $\mathbb{R}$ to $\mathbb{N}$, the "natural" numbers, $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$. The discrete analogue of a function is a SEQUENCE, $a : \mathbb{N} \to \mathbb{R}$, whose input is discrete. Thus,



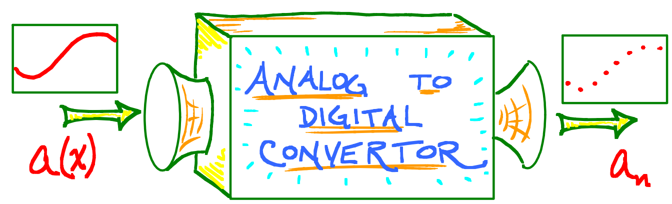
$$a = (a_0, a_1, a_2, a_3, \ldots)$$

a(0) a(1) etc.



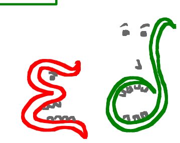
EXAMPLES

Unemployment rate as a function of *month*
Exam scores as a function of *student*
Dow Jones index as a function of *day*

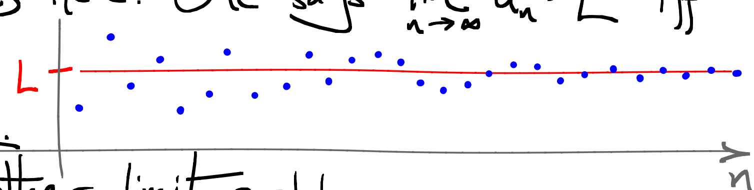


LIMITS

The first step in Calculus is limits. Relax...

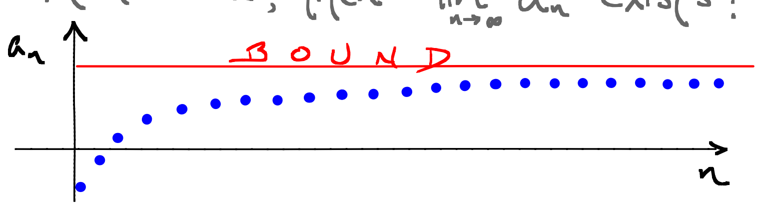


There are no tricky definitions here. One says $\lim_{n\to\infty} a_n = L$ iff a_n gets as close as you like to L for n sufficiently large.

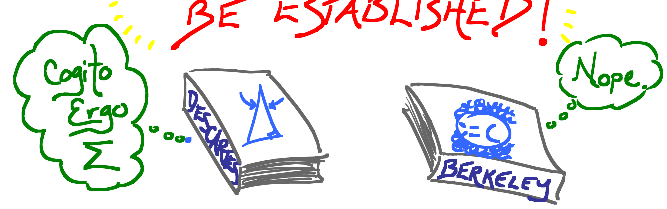


The same deal holds as in other limit problems:

LEMMA: If $a = (a_n)_0^\infty$ is BOUNDED and MONOTONE, then $\lim_{n\to\infty} a_n$ exists!



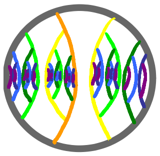
EXISTENCE MUST BE ESTABLISHED!



EXAMPLE

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}} = \boxed{\frac{1+\sqrt{5}}{2}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

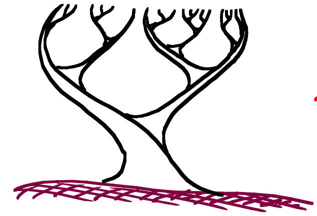
These sequences a_n are bounded and increasing... L EXISTS!



$$a_{n+1} = \sqrt{1+a_n} \xleftarrow{\text{RECURSION RELATION}} a_{n+1} = 1 + \frac{1}{a_n}$$

$$L = \sqrt{1+L} \xleftarrow{\text{LET } L = \lim_{n\to\infty} a_n} L = 1 + \frac{1}{L}$$

$$L^2 = 1 + L \xleftarrow{\text{SOLVE FOR } L>0} L^2 = L + 1$$



Fortunately, one may use CONTINUOUS techniques to establish limits in discrete settings...

EXAMPLE

$$\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin 1/n}{1/n} \overset{\text{let } x = 1/n}{=} \lim_{x\to 0^+} \frac{\sin x}{x} \overset{\text{STANDARD STUFF}}{\underset{\text{TAYLOR / L'HÔPITAL}}{=}} 1$$

i'm so lonely...

NOTE: $x = \frac{1}{n}$ are in $\mathbb{R}$, not $\mathbb{N}$...

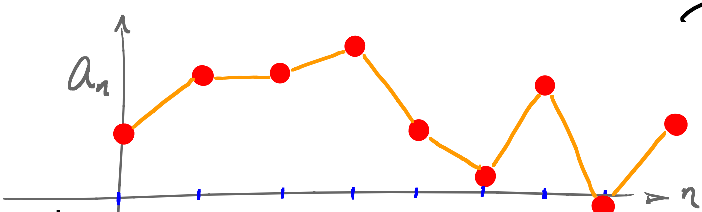
Your knowledge of Taylor series & l'Hôpital's rule is fair game...

(Isn't it strange that one can use derivatives to compute limits of sequences? Even though you cannot differentiate with respect to n? Well...)

DERIVATIVES

What is the analogue of the derivative for sequences?

These are called FORWARD/BACKWARD DIFFERENCES

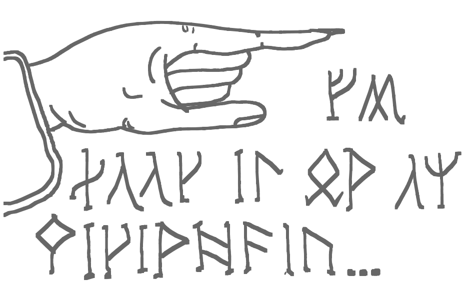


$$\Delta a_n = a_{n+1} - a_n \quad \text{FORWARD}$$

$$\nabla a_n = a_n - a_{n-1} \quad \text{BACKWARD}$$

Notice how Δa and ∇a are sequences that express rates of change as you increase (Δ) or decrease (∇) the input by 1.

(I wonder what happens if you average the two? Hmmm...)



There is an entire calculus for difference operators; Δ, ∇

E.g.) If (a_n) and (b_n) are sequences, then there is a "product rule" for differences of $(ab)_n := (a_n b_n)$

$$\Delta(ab)_n = a_n \Delta b_n + b_n \Delta a_n + \Delta a_n \Delta b_n$$

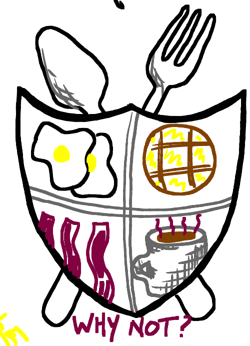
$$\Delta(ab)_n = a_n \Delta b_n + b_n \Delta a_n - \Delta a_n \Delta b_n \quad \text{H.O.T.}$$

This, as you might guess, is just the beginning...

IN PRINCIPIO

DIFFERENTIAL EQUATIONS

If you can define "derivatives" of a sequence, why not differential equations?



$$\frac{dx}{dt} = f(x) \quad \xrightarrow{\text{cf.}} \quad \Delta x_n = f(x_n)$$

DIFFERENTIAL vs. DIFFERENCE

which, since $\Delta x_n = x_{n+1} - x_n$,

$$\cong \quad \boxed{x_{n+1} = x_n + f(x_n)}$$

UPDATE RULE

Any "initial condition" x_0 yields a discrete-time "solution" sequence (x_n) via iteration

These "difference equations" are very useful in continuous settings:

such as...

NEWTON'S METHOD

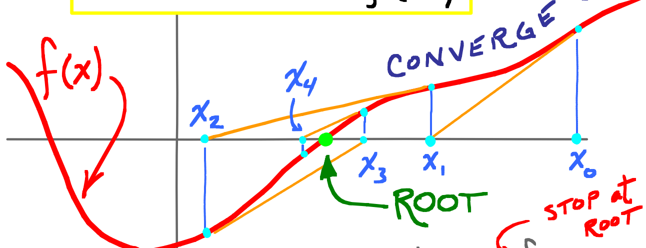
GIVEN: $f: \mathbb{R} \to \mathbb{R}$ continuous

PROBLEM: find a root of f

CHOOSE: starting guess x_0

SOLUTION: iterate the equation

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



INTUITION: cf. $\frac{dx}{dt} = -\frac{f(x)}{f'(x)}$

STOP at ROOT

ALWAYS MOVE TOWARD ROOT

CAVEAT: You may not converge to the root you wanted, nor to anything!

EXAMPLE: $f(x) = 1 - x^2$

$$\text{If } x_0 \text{ is} \begin{cases} <0 : & x_n \to -1 \\ =0 : & x_1 = \#\text{FAIL!} \\ >0 : & x_n \to +1 \end{cases}$$

or perhaps

EULER'S METHOD

GIVEN: ODE $\frac{dx}{dt} = f(x,t)$

initial condition $x_0 = x(t_0)$

PROBLEM: find $x(T)$ for $T > t_0$

CHOOSE: step size δt $(= {T-t_0}/{N})$

SOLUTION: iterate the equation

$$\boxed{\begin{aligned} x_{n+1} &= x_n + f(x_n, t_n) \\ t_{n+1} &= t_n + \delta t \end{aligned}}$$

INTUITION: cf. $\frac{dx}{dt} = f(x,t)$

TIME KEEPS ON (SLIPPIN')[3] INTO THE FUTURE $\rightarrow \frac{dt}{dt} = 1$

CAVEAT: When you get to T, you have only an approximation to $x(T)$, the accuracy of which depends upon δt.

EXAMPLE: $\frac{dx}{dt} = x \qquad x_0 = 1 \quad t_0 = 0$

We know $x(t) = e^t$ so $x(1) = e$.

CHOOSE: $\delta t = 1/N$; then

$$x_{n+1} = x_n + x_n/N = x_n(1 + 1/N)$$
$$t_{n+1} = t_n + 1/N = n/N$$

$x_N = (1 + 1/N)^N$ approximates e

As $\delta t \to 0$, $N \to \infty$ and

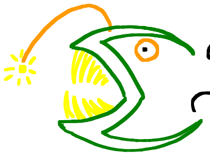
$$\lim_{N\to\infty} x_N = \lim_{N\to\infty} \left(1 + \tfrac{1}{N}\right)^N = e$$

...while Adam took no thought nor Eve to iterate...

"IS THAT ALL THERE IS?" NOPE: TAKE A NUMERICAL ANALYSIS COURSE.

INTEGRALS

The analogue of the definite integral for a (finite) sequence $\{a_n\}$ is simply the sum $\sum_{n=0}^{K} a_n$.

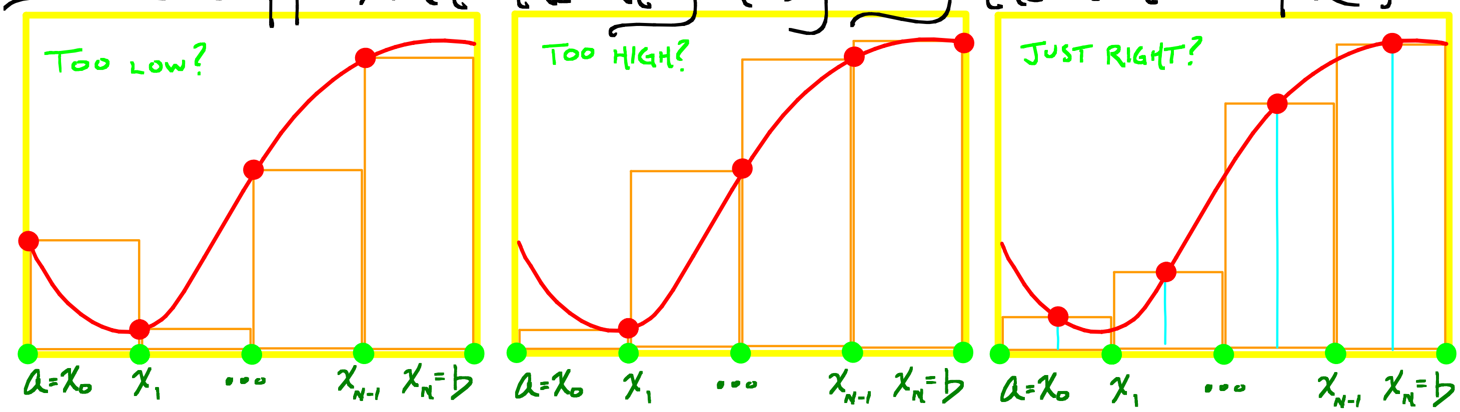


There's nothing fishy here, unless your goal is to do **NUMERICAL INTEGRATION** -- the approximation of $\int_a^b f(x)\,dx$ from a discrete sampling of f... This requires some choices!

CHOOSE: step size $\delta x = (b-a)/N$ for some N

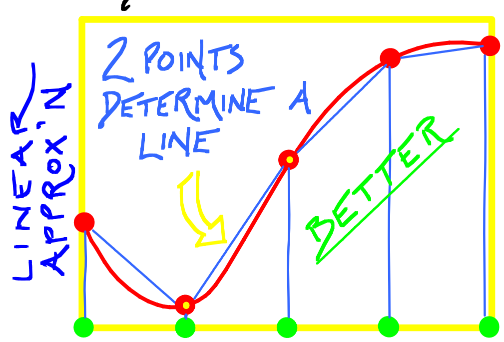
DEFINE: sample points $x_n = a + n\cdot\delta x$ for $n = 0 \dots N$

You can approximate the integral by using the x_i's to sample f:



LEFT ENDPOINTS	RIGHT ENDPOINTS	MIDPOINTS
$\int_a^b f(x)dx \simeq \sum_{n=0}^{N-1} f(x_n)\cdot\delta x$	$\int_a^b f(x)dx \simeq \sum_{n=1}^{N} f(x_n)\cdot\delta x$	$\int_a^b f(x)dx \simeq \sum_{n=0}^{N-1} f\left(\frac{x_n+x_{n+1}}{2}\right)\cdot\delta x$

These give simple-to-compute, but coarse, approximations. To better approximate, use a Taylor series approach, approximating f locally with polynomials

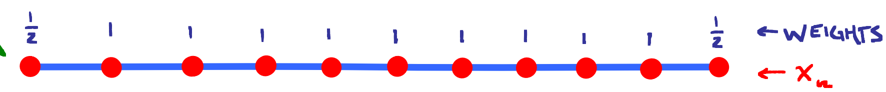


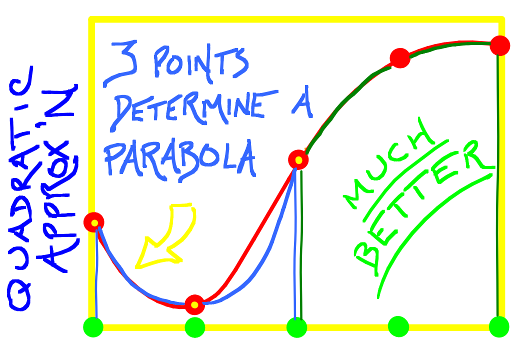
TRAPEZOIDAL RULE

Linear interpolation on pairs of points yields

$$\int_a^b f(x)dx \simeq \frac{\delta x}{2}\left(f(x_0)+\sum_{n=1}^{N-1} 2f(x_n) + f(x_N)\right)$$

CARTOON OF "WEIGHTS"



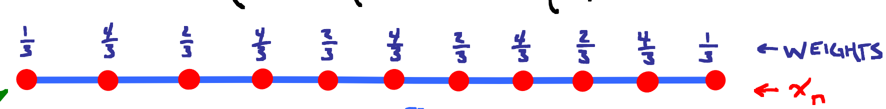


SIMPSON'S RULE

is a little trickier... doh!

$$\int_a^b f(x)dx \simeq \frac{\delta x}{3}\cdot\left(f(x_0)+\sum_{n=1}^{\frac{N}{2}-1} 2f(x_{2n}) + \sum_{n=1}^{N/2} 4f(x_{2n-1}) + f(x_N)\right)$$

This estimate is more accurate.



"IS THAT ALL THERE IS?" NOPE: TAKE A NUMERICAL ANALYSIS COURSE.

SERIES

The analogue of an improper integral $\left(\int_0^\infty f(x)\,dx\right)$ is:

$$\boxed{\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \dots}$$

THIS IS

BEATIFIC! $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \dots = 2$

ENIGMATIC! $1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots = ?$

is it zero? one? or $\frac{1}{1-(-1)} = \frac{1}{2}$?

What to do? Use limits, as with the rest of Calculus.

DEFINE: $S_i = \sum_{k=0}^{n} a_k = a_0 + a_1 + \dots + a_n$ "PARTIAL SUMS"

FULL FATHOM FIVE THY SEQUENCE LIES! THOSE ARE PEARLS THAT WERE S_i"

$$\sum_{k=0}^{\infty} a_k := \lim_{n\to\infty} S_n$$ "CONVERGES" or "DIVERGES"

$$\boxed{\int_{x=0}^{\infty} f(x)\,dx = \lim_{T\to\infty}\int_0^T f(x)\,dx \quad \text{cf.} \quad \sum_{k=0}^{\infty} a_k = \lim_{L\to\infty}\sum_{k=0}^{L} a_k}$$

This is the proper definition, and fits with our notion of the SERIES as a discrete integral...

QUESTIONS:

1: Does a given series $\sum_{k=0}^{\infty} a_k$ converge?] NOT SO HARD

2: If so, to which value does it converge?] NOT SO EASY...

3: Why is an infinite sum called a "series"?] I'LL TELL YOU LATER...

CONVERGENCE TESTS

There are numerous tests for series to ascertain convergence or divergence. Be careful of hypotheses!

DIVERGENCE TEST

HYPOTHESES: none

TEST: if $\lim_{n\to\infty} a_n \neq 0$, then $\sum_n a_n$ DIVERGES

EXAMPLE: $\sum_n \cos(n\pi)$ DIVERGES! $= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \dots$

COMPARISON TEST

HYPOTHESES: $a_n \geq 0$; $b_n \geq 0$

TEST:

if $a_n \leq b_n$ / $b_n \leq a_n$ and $\sum_n b_n$ CONVERGES / DIVERGES then $\sum_n a_n$ also CONVERGES / DIVERGES since $\sum a_n \leq \sum b_n$ / $\sum_n b_n \leq \sum_n a_n$

EXAMPLE:

$$\sum_n \frac{\ln(n)}{n} \geq \sum \frac{1}{n} = \infty \quad \text{DIV}$$

$$\sum_n \frac{\sqrt{n-1}}{n^2} \leq \sum_n \frac{1}{n^{3/2}} < \infty \quad \text{CONV}$$

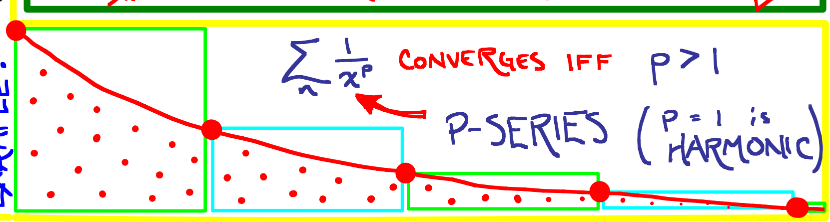
INTEGRAL TEST

HYPOTHESES: $a_n \geq 0$, decreasing

TEST: choose $a(x)$ continuous, decreasing, with $a(n) = a_n$. Then

$$\sum_n a_n \quad \text{CONVERGE or DIVERGE TOGETHER!} \quad \int_1^\infty a(x)\,dx$$

EXAMPLE:



$\sum_n \frac{1}{x^p}$ CONVERGES IFF $p > 1$

P-SERIES ($p = 1$ is HARMONIC)

LIMIT TEST

HYPOTHESES: $a_n \geq 0$, $b_n \geq 0$

TEST: if $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ STRICT!

then $\sum_n a_n$ CONVERGE OR DIVERGE TOGETHER! $\sum_n b_n$

EXAMPLE: $\sum_n \frac{n^2-2n+3^n}{5^{n-1}+n^3}$ CONVERGES since $\sum_n \left(\frac{3}{5}\right)^n$ does...

RATIO TEST

my fav-o-rite!

HYPOTHESES: none! yay!

TEST: Compute $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

then $\sum a_n$ CONVERGES if $\rho < 1$; DIVERGES if $\rho > 1$

EXAMPLE: $\sum_n \frac{n!}{1\cdot 3\cdot 5 \cdots (2n-1)}$ CONVERGES since $\rho = 1/2$

ROOT TEST

HYPOTHESES: none! yay!

TEST: Compute $\rho = \lim_{n\to\infty} \sqrt[n]{a_n}$

then $\sum a_n$ CONVERGES if $\rho < 1$; DIVERGES if $\rho > 1$

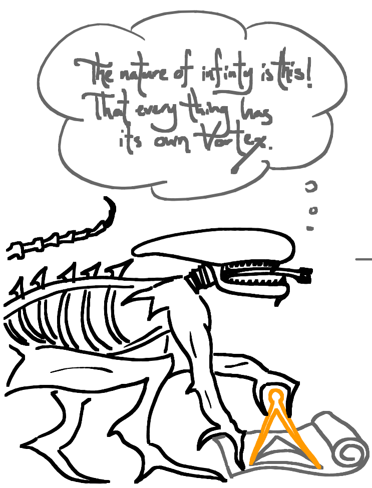
EXAMPLE: $\sum_n \frac{\pi^n}{n e^n}$ DIVERGES since $\rho = \pi/e$

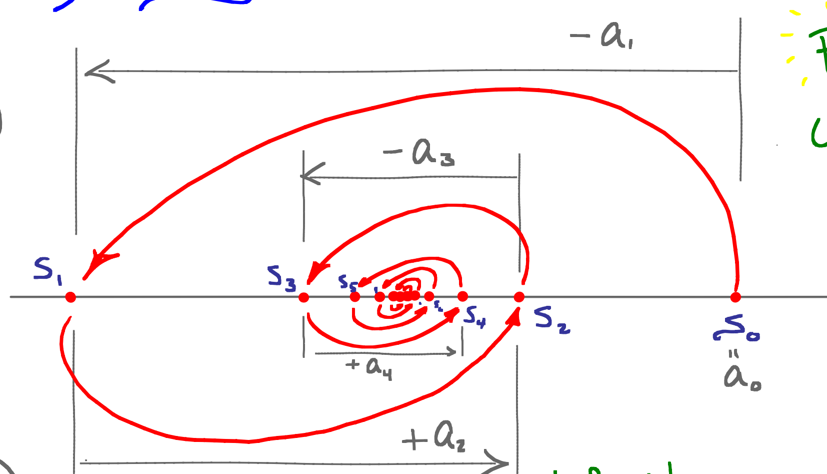
It takes a lot of practice to know which test to use when... Some types of series are not so bad, however...

ALTERNATING SERIES

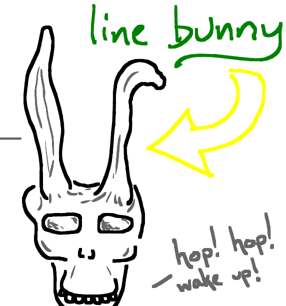
An alternating series alternates $\sum_{n=0}^{\infty} (-1)^n a_n$ (with $a_n \geq 0$) positive and negative terms. For such, it is easy to determine convergence: An alternating series (with decreasing terms a_n) converges IF & ONLY IF the limit $\lim_{n\to\infty} a_n = 0$

REALLY? GEE WHIZ!





PROOF: Use the number line bunny



Partial sums hop back & forth by length a_n

CONVERGENCE

There are degrees of convergence; one says $\sum_n a_n$ CONVERGES ABSOLUTELY if $\sum_n |a_n|$ CONVERGES. A series $\sum_n a_n$ CONVERGES CONDITIONALLY if it converges but not absolutely.

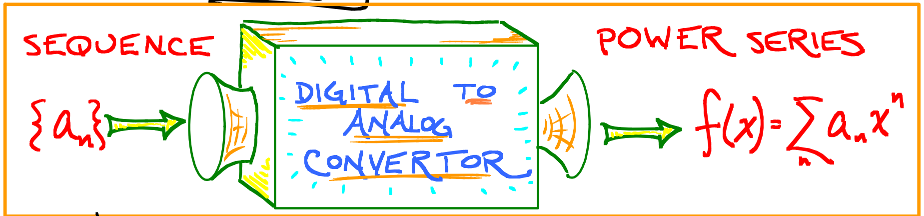
EXAMPLES: $\sum_n \frac{(-1)^n}{n^p}$

DIVERGES	$p \leq 0$	divergence test
CONVERGES CONDITIONALLY	$0 < p \leq 1$	alternating test
CONVERGES ABSOLUTELY	$1 < p$	p-series test

POWER SERIES

The theme of this chapter is the discretization of functions to sequences and the resulting Calculus. It is time to reverse the process. A POWER SERIES in x is

$$\sum_{n=0}^{\infty} a_n x^n$$



for some coefficients a_n. This converts a series into a function of x (at least when the series converges!). When does it converge?

IT'S TIME FOR A TEST!

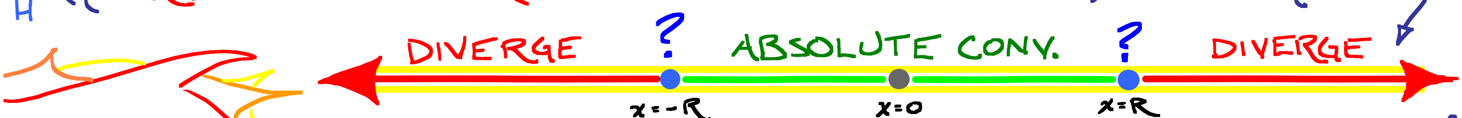
my fav-o-rite!

Let $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

Then $f(x) = \sum_n a_n x^n$ CONVERGES ABSOLUTELY if $\rho < 1$ (and DIVERGES if $\rho > 1$) by the ratio test.

THE TEST...

Let $\boxed{R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|}$. Solving $\rho < 1$ yields $|x| < R$; hence, R is called the RADIUS OF CONVERGENCE. So, a power series behaves like so:



MORE TESTS REQUIRED AT $x = R$ & $x = -R$!

R can be any number, such as:

- ∞ for $a_n = \frac{1}{n!}$
- 1 for $a_n = 1$
- 0 for $a_n = n!$

The Taylor series of Chapter 1 return. For $f(x) = \sum_n a_n x^n$ and $|x| < R$, the Taylor series of f is, precisely, the power series...

Power series of the form $\sum_n a_n (x-c)^n$ have an INTERVAL OF CONVERGENCE centered at c

BUT: you must be careful at $x = \pm R$!

EXAMPLE: By taking $\lim_{x\to 1^-} \ln(1+x)$, we get...

RECALL: for $|x| < 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$

DIVERGES AT $x=-1$

CONVERGES AT $x=1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$$

CONVERGES!

$$\frac{1}{2}\ln 2 = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$$

OK!

ADD

$$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

Noooooooo.....! These are the same terms, just in a different order...

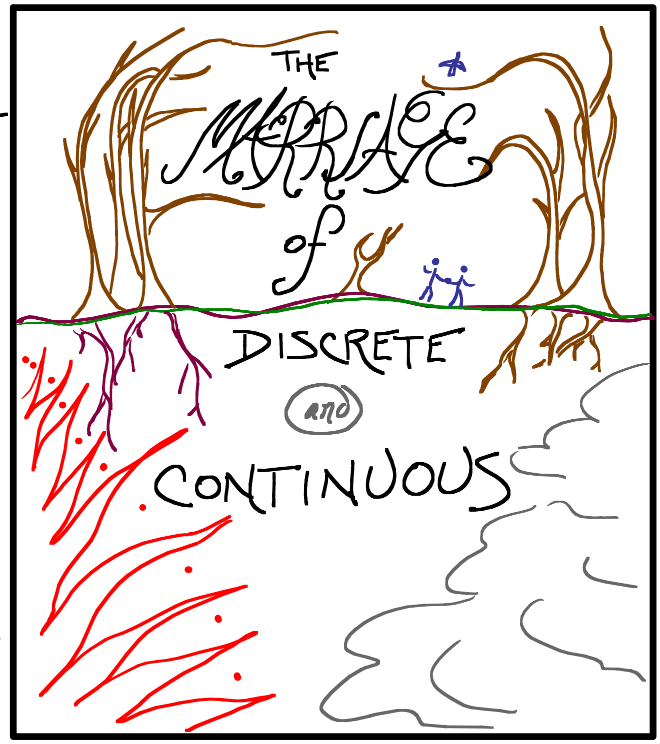
Yes, dear reader, a conditionally convergent series can be rearranged to sum up to ANY NUMBER YOU WANT. Beware! Fortunately, absolutely convergent series behave as nicely as can be...

TAYLOR, AGAIN

All this work, just to come back to Taylor series? Yep. But notice the language we have developed; ex gratia –

THEOREM: Within the radius of convergence of a power series for $f(x)$, one can differentiate and integrate f or the series equivalently.

This, then, is a synthesis of discrete and continuous Calculus: one can evaluate a discrete "integral" by taking a Taylor series and using all the tools & tricks of Calculus:



EXAMPLE: Evaluate the geometric series at $-x^2$ to obtain

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots \qquad |x|<1$$

Integration yields:

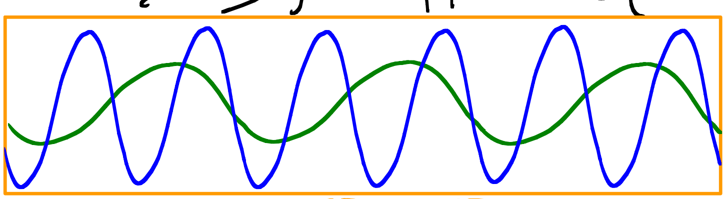
$$\text{ARCTAN}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad |x|<1$$

Taking the limit as $x \to 1^-$ yields...

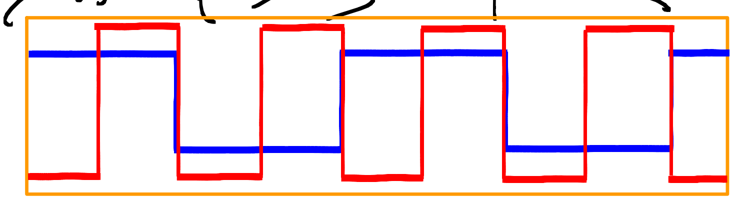
$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \qquad \text{SURPRISE!}$$

THE END?

The use of power series relies on polynomials as a basis for approximation. Different bases are possible:

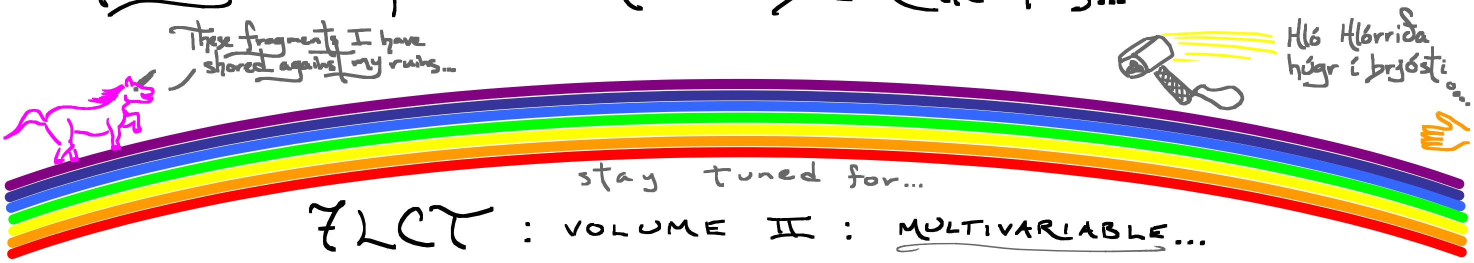


FOURIER SERIES



WAVELETS

The most exciting thing in Calculus is how the same basic ideas recur, no matter how deep one treads. This will be evident when you explore multivariable Calculus...



stay tuned for...

7LCT : VOLUME II : MULTIVARIABLE...