Finding basis for null space using orthogonal complement

To find basis for null space of an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$,

find orthogonal complement of Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ in $\mathbb{R}^n$:

- Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard basis vectors $\mathbb{R}^n$.
- Let $[\mathbf{a}_1^*, \ldots, \mathbf{a}_m^*, \mathbf{e}_1^*, \ldots, \mathbf{e}_n^*] = \texttt{orthogonalize}([\mathbf{a}_1, \ldots, \mathbf{a}_m, \mathbf{e}_1, \ldots, \mathbf{e}_n])$
- Find the nonzero vectors among $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$

Algorithm for finding basis for null space

Another approach to find basis of null space of a matrix: Write matrix in terms of its columns $\mathbf{v}_0, \ldots, \mathbf{v}_n$.

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$\left[\begin{array}{c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Can transform this to express starred vectors in terms of original vectors.

$$\left[\begin{array}{c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}^{-1} = \left[\begin{array}{c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \end{array}\right]$$

Basis for null space

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are (approximately) zero vectors.

- Corresponding columns of inverse triangular matrix are nonzero vectors of the null space of the leftmost matrix.
- These columns are clearly linearly independent so they span a basis of dimension 3.
- Rank-Nullity Theorem shows that the null space has dimension 3 so these columns are a basis for null space.

Basis for null space

$$
\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\
 & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\
 & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\
 & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\
 & & & & 1 & \alpha_{45} & \alpha_{46} \\
 & & & & & 1 & \alpha_{56} \\
 & & & & & & 1
\end{bmatrix}^{-1}
$$

$$
= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array}\right]
$$

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are (approximately) zero vectors.

- Corresponding columns of inverse triangular matrix are nonzero vectors of the null space of the leftmost matrix.
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- Rank-Nullity Theorem shows that the null space has dimension 3
  so these columns are a basis for null space.

Basis for null space

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array} \right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array} \right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

def find_null_space(A):
  vstarlist = orthogonalize(columns of A)
  find upper triangular matrix T such that
    A equals (matrix with columns vstarlist) $* T$
  return list of columns of T^{-1} corresponding to zero vectors in vstarlist

How to find matrix T? How to find its inverse?

Augmenting orthogonalize(vlist)

We will write a procedure aug_orthogonalize(vlist) with the following spec:

- *input:* a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* the pair $([\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*], [\mathbf{r}_1, \ldots, \mathbf{r}_n])$ of lists of vectors such that $\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*$ are mutually orthogonal vectors whose span equals Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, and

$$\left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right] = \left[\begin{array}{c|c|c} \mathbf{v}_1^* & \cdots & \mathbf{v}_n^* \end{array}\right] \left[\begin{array}{c|c|c} \mathbf{r}_1 & \cdots & \mathbf{r}_n \end{array}\right]$$

```
def orthogonalize(vlist):
  vstarlist = []
  for v in vlist:
    vstarlist.append(
      project_orthogonal(v, vstarlist))
  return vstarlist
```

```
def aug_orthogonalize(vlist):
  vstarlist = []
  r_vecs = []
  D = set(range(len(vlist)))
  for v in vlist:
    (vstar, alphadict) =
       aug_project_orthogonal(v, vstarlist)
    vstarlist.append(vstar)
    r_vecs.append(Vec(D, alphadict))
  return vstarlist, r_vecs
```

Using aug_orthogonalize to find null space

```
def find_null_space(A):
    vstarlist = orthogonalize(columns of A)
    find upper triangular matrix T such that
                    A equals (matrix with columns vstarlist) * T
    return list of columns of T^-1 corresponding to zero vectors in vstarlist
```

$\Downarrow$

```
def find_null_space(A):
    vstarlist, r_vecs = aug_orthogonalize(columns of A)
    let T be matrix with columns given by the vectors of r_vecs
    return list of columns of T^-1 corresponding to zero vectors in vstarlist
```

How to find a column of T^{-1}?

How to find a column of T^{-1}?

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

The matrix T is square and upper triangular, with nonzero diagonal elements

$$\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} \begin{bmatrix} & & \\ & T^{-1} & \\ & & \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

To find column j of T^{-1}, solve $\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{x} \\ \\ \end{bmatrix} = \mathbf{e}_j$

Use `triangular_solve`