The size of a basis

Key fact for this week: all bases for a vector space have the same size.

We use this as the “basis” for answering many pending questions.

Morphing Lemma

Morphing Lemma: Let $\mathcal{V}$ be a vector space. Suppose S is a set of generators for $\mathcal{V}$, and B is a basis for $\mathcal{V}$. Then $|B| \leq |S|$.

Before we prove it—what good is this lemma?

Theorem: Any basis for $\mathcal{V}$ is a smallest generating set for $\mathcal{V}$.

Proof: Certainly B is a generating set for $\mathcal{V}$. Let S be a smallest generating set for $\mathcal{V}$. Then, by the Morphing Lemma, B is no bigger than S, so B is also a smallest generating set.

Theorem: All bases for a vector space $\mathcal{V}$ have the same size.

Proof: They are all smallest generating sets.

Proof of the Morphing Lemma

Morphing Lemma: Let $\mathcal{V}$ be a vector space. Suppose S is a set of generators for $\mathcal{V}$, and B is a basis for $\mathcal{V}$. Then $|B| \leq |S|$.

Proof outline: modify S step by step, introducing vectors of B one by one, without increasing the size.

How? Using the Exchange Lemma....

Review of Exchange Lemma

Exchange Lemma: Suppose S is a set of vectors and A is a subset of S. Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
Then there is a vector $\mathbf{w} \in S - A$ such that

$$\text{Span } S = \text{Span } (S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$$

Proof of the Morphing Lemma

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$. Define $S_0 = S$.
Prove by induction on $k \leq n$ that there is a generating set S_k of $\mathcal{V}$ that contains $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and has size $|S|$.

Base case: $k = 0$ is trivial.

To go from S_{k-1} to S_k: use the Exchange Lemma.

- $A_k = \{\mathbf{b}_1, \ldots, \mathbf{b}_{k-1}\}$ and $\mathbf{z} = \mathbf{b}_k$

Exchange Lemma $\Rightarrow$ there is a vector $\mathbf{w}$ in S_{k-1} such that

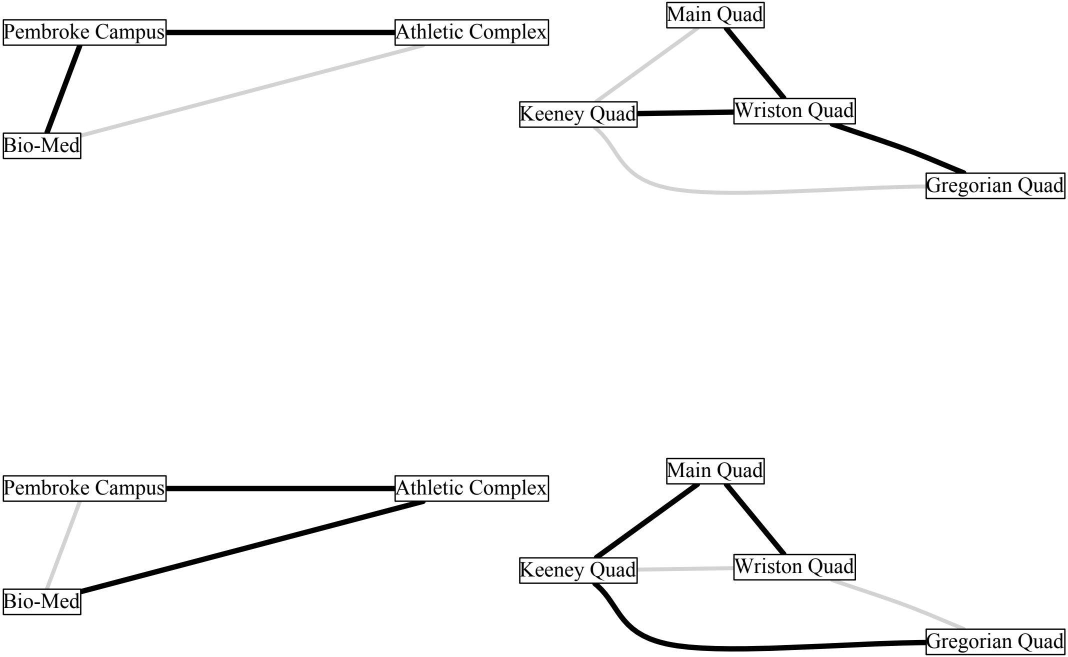
$$\text{Span } (S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}) = \text{Span } S_{k-1}$$

Set $S_k = S_{k-1} \cup \{\mathbf{b}_k\} - \{\mathbf{w}\}$.

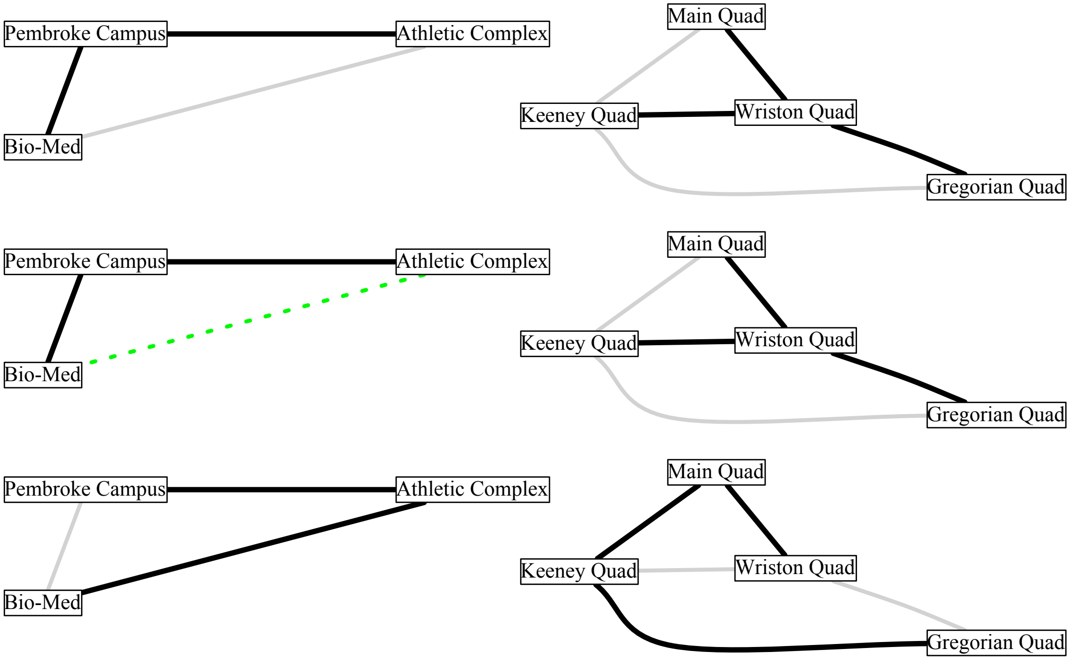
QED

This induction proof is an algorithm.

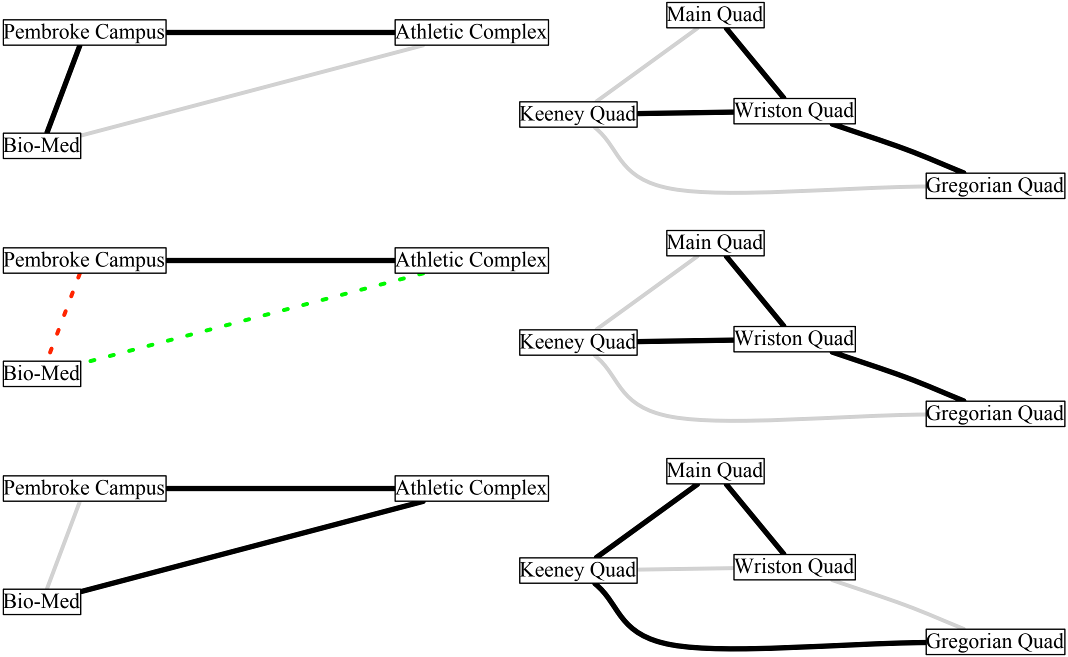
Morphing from one spanning forest to another



Morphing from one spanning forest to another



Morphing from one spanning forest to another



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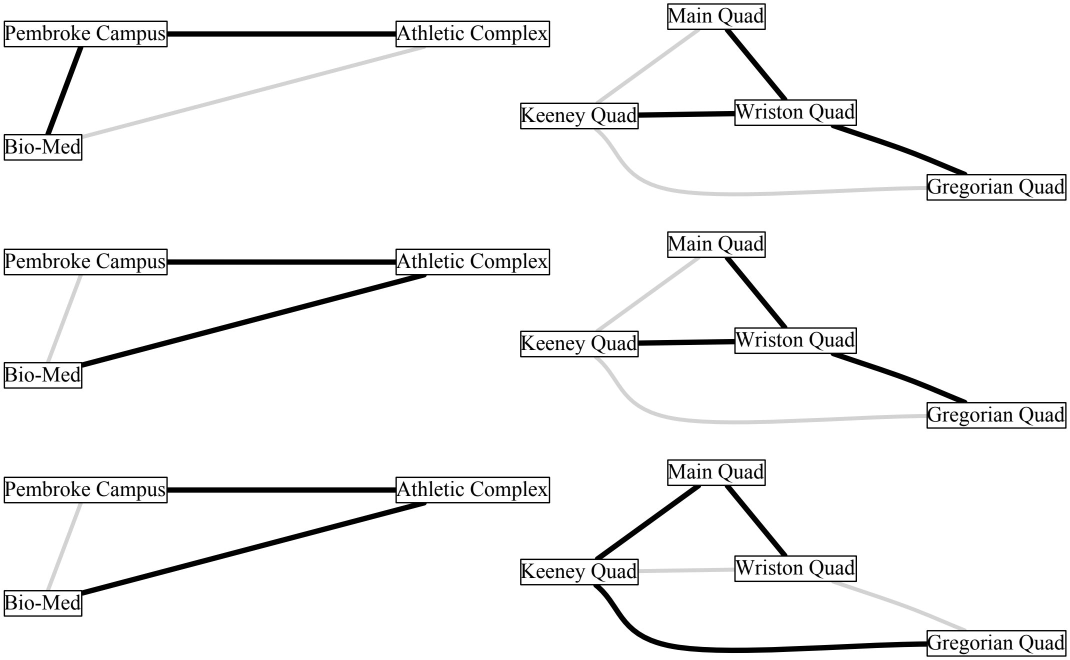

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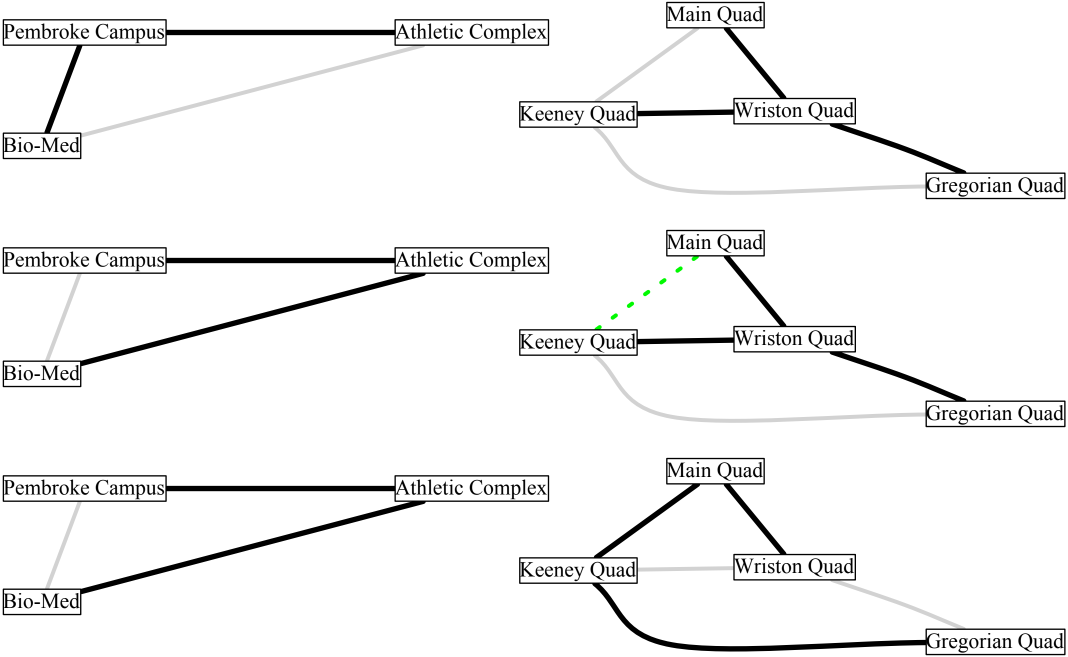

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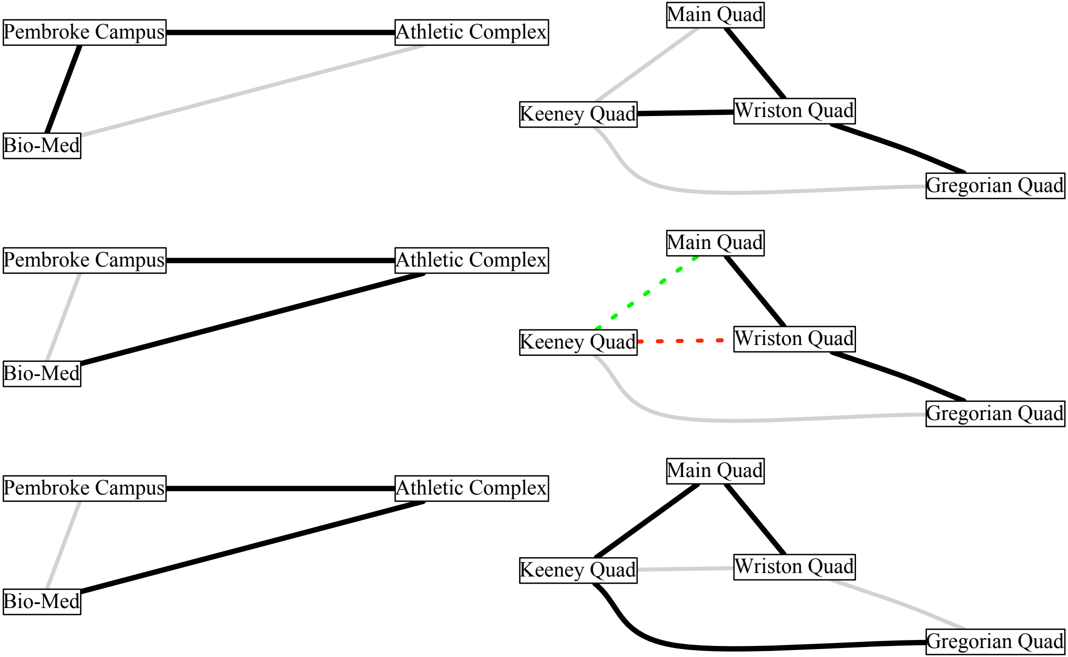

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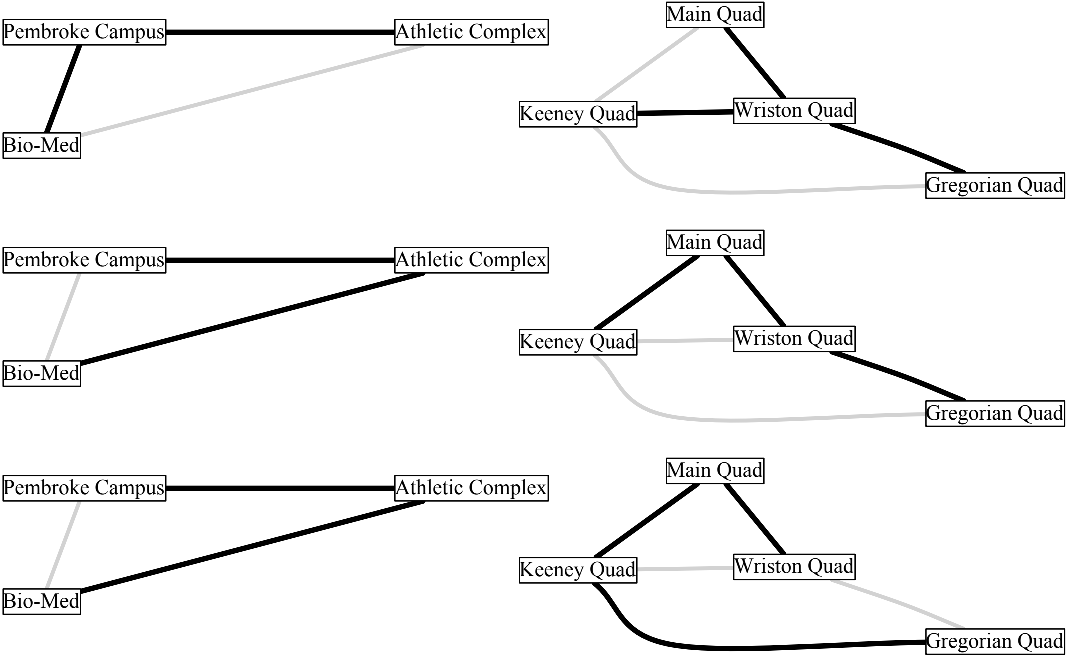

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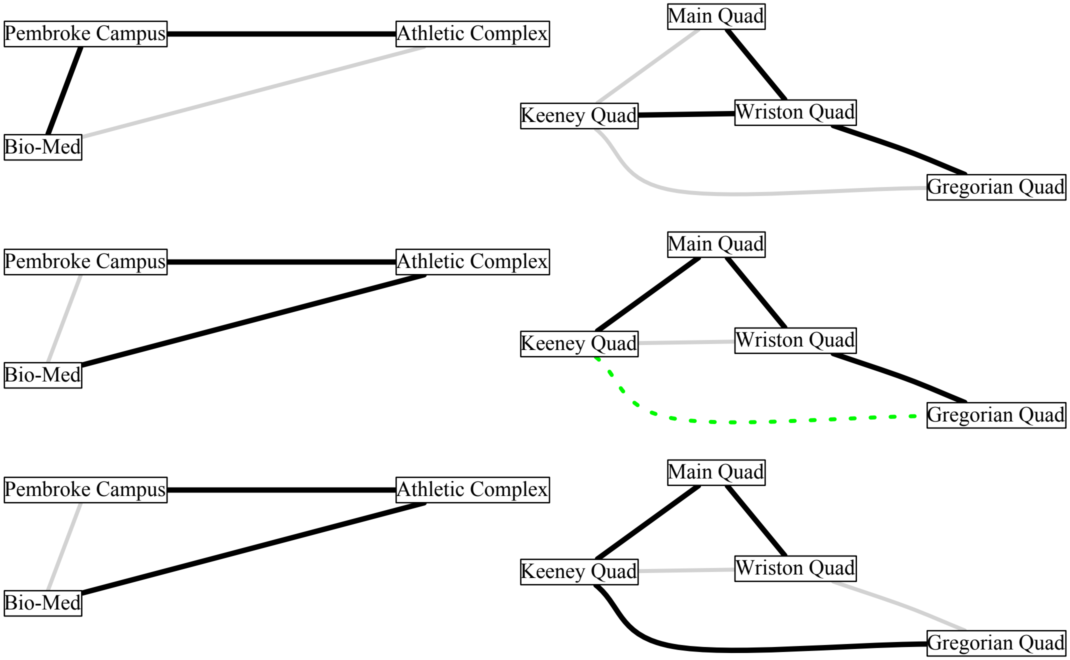

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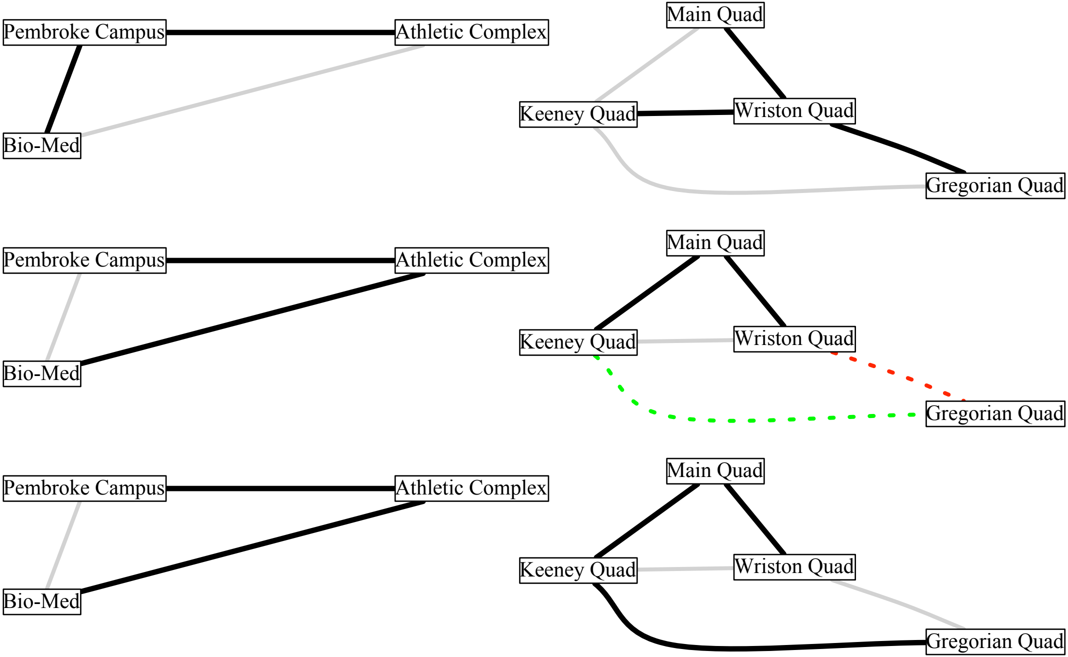

Morphing from one spanning forest to another