Linear Combinations

An expression

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

is a *linear combination* of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

The scalars $\alpha_1, \ldots, \alpha_n$ are the *coefficients* of the linear combination.

Example: One linear combination of $[2, 3.5]$ and $[4, 10]$ is

$$-5\,[2, 3.5] + 2\,[4, 10]$$

which is equal to $[-5 \cdot 2, -5 \cdot 3.5] + [2 \cdot 4, 2 \cdot 10]$

Another linear combination of the same vectors is

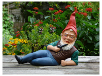
$$0\,[2, 3.5] + 0\,[4, 10]$$

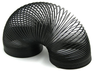
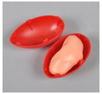
which is equal to the zero vector $[0, 0]$.

Definition: A linear combination is *trivial* if the coefficients are all zero.

Linear Combinations: JunkCo

The JunkCo factory makes five products:




using various resources.

	metal	concrete	plastic	water	electricity
garden gnome	0	1.3	.2	.8	.4
hula hoop	0	0	1.5	.4	.3
slinky	.25	0	0	.2	.7
silly putty	0	0	.3	.7	.5
salad shooter	.15	0	.5	.4	.8

For each product, there is a vector specifying how much of each resource is used per unit of product.

For making one gnome:

$\mathbf{v}_1 = \{\texttt{metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:.4}\}$

Linear Combinations: JunkCo

For making one gnome:
$\mathbf{v}_1 = \{\texttt{metal:0, concrete:1.3, plastic:0.2, water:.8, electricity:.4}\}$
For making one hula hoop:
$\mathbf{v}_2 = \{\texttt{metal:0, concrete:0, plastic:1.5, water:.4, electricity:.3}\}$
For making one slinky:
$\mathbf{v}_3 = \{\texttt{metal:.25, concrete:0, plastic:0, water:.2, electricity:.7}\}$
For making one silly putty:
$\mathbf{v}_4 = \{\texttt{metal:0, concrete:0, plastic:.3, water:.7, electricity:.5}\}$
For making one salad shooter:
$\mathbf{v}_5 = \{\texttt{metal:1.5, concrete:0, plastic:.5, water:.4, electricity:.8}\}$

Suppose the factory chooses to make α_1 gnomes, α_2 hula hoops, α_3 slinkies, α_4 silly putties, and α_5 salad shooters.

Total resource utilization is $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \alpha_4\,\mathbf{v}_4 + \alpha_5\,\mathbf{v}_5$

Linear Combinations: JunkCo: Industrial espionage

Total resource utilization is $\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2 + \alpha_3\,\mathbf{v}_3 + \alpha_4\,\mathbf{v}_4 + \alpha_5\,\mathbf{v}_5$

Suppose I am spying on JunkCo.

I find out how much metal, concrete, plastic, water, and electricity are consumed by the factory.
That is, I know the vector $\mathbf{b}$. Can I use this knowledge to figure out how many gnomes they are making?

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*

- *input:* a vector $\mathbf{b}$ and a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* a list $[\alpha_1, \ldots, \alpha_n]$ of coefficients such that

$$\mathbf{b} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

or a report that none exists.

Question: Is the solution unique?

Lights Out

Button vectors for 2×2 *Lights Out:* $\begin{bmatrix} \bullet & \bullet \\ \bullet & \end{bmatrix}$ $\begin{bmatrix} \bullet & \bullet \\ & \bullet \end{bmatrix}$ $\begin{bmatrix} \bullet & \\ \bullet & \bullet \end{bmatrix}$ $\begin{bmatrix} & \bullet \\ \bullet & \bullet \end{bmatrix}$

For a given initial state vector $\mathbf{s} = \begin{bmatrix} \bullet & \\ \bullet & \end{bmatrix}$,

$\boxed{\text{Which subset of button vectors sum to } \mathbf{s}\text{?}}$

Reformulate in terms of linear combinations.
Write

$$\begin{bmatrix} \bullet & \\ \bullet & \end{bmatrix} = \alpha_1 \begin{bmatrix} \bullet & \bullet \\ \bullet & \end{bmatrix} + \alpha_2 \begin{bmatrix} \bullet & \bullet \\ & \bullet \end{bmatrix} + \alpha_3 \begin{bmatrix} \bullet & \\ \bullet & \bullet \end{bmatrix} + \alpha_4 \begin{bmatrix} & \bullet \\ \bullet & \bullet \end{bmatrix}$$

What values for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ make this equation true?

Solution: $\alpha_1 = 0, \alpha_2 = 1, \alpha_3 = 0, \alpha_4 = 0$

$\boxed{\text{Solve an instance of } \textit{Lights Out}} \Rightarrow \boxed{\text{Which set of button vectors sum to } \mathbf{s}\text{?}}$

$\Rightarrow \boxed{\text{Find subset of } GF(2) \text{ vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ whose sum equals } \mathbf{s}} \Rightarrow \boxed{\text{Express } \mathbf{s} \text{ as a linear combination of } \mathbf{v}_1, \ldots, \mathbf{v}_n}$

Lights Out

We can solve the puzzle if we have an algorithm for

Computational Problem: *Expressing a given vector as a linear combination of other given vectors*