Dimension

Definition: We define the *dimension* of a vector space to be the size of a basis for that vector space. The dimension of a vector space $\mathcal{V}$ is written $\dim \mathcal{V}$.

Definition: We define the *rank* of a set S of vectors as the dimension of Span S. We write rank S.

Example: The vectors $[1, 0, 0], [0, 2, 0], [2, 4, 0]$ are linearly dependent.
Therefore their rank is less than three.
First two of these vectors form a basis for the span of all three, so the rank is two.

Example: The vector space Span $\{[0, 0, 0]\}$ is spanned by an empty set of vectors.
Therefore the rank of $\{[0, 0, 0]\}$ is zero

Row rank, column rank

Definition: For a matrix M, the *row rank* of M is the rank of its rows, and the *column rank* of M is the rank of its columns.
Equivalently, the row rank of M is the dimension of Row M, and the column rank of M is the dimension of Col M.

Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 4 & 0 \end{bmatrix}$$

whose rows are the vectors we saw before: $[1, 0, 0], [0, 2, 0], [2, 4, 0]$
The set of these vectors has rank two, so the row rank of M is two.
The columns of M are $[1, 0, 2]$, $[0, 2, 4]$, and $[0, 0, 0]$.
Since the third vector is the zero vector, it is not needed for spanning the column space.
Since each of the first two vectors has a nonzero where the other has a zero, these two are linearly independent, so the column rank is two.

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Example: Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 0 & 7 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

Each of the rows has a nonzero where the others have zeroes, so the three rows are linearly independent. Thus the row rank of M is three.
The columns of M are $[1, 0, 0]$, $[0, 2, 0]$, $[0, 0, 3]$, and $[5, 7, 9]$.
The first three columns are linearly independent, and the fourth can be written as a linear combination of the first three, so the column rank is three.

Row rank, column rank

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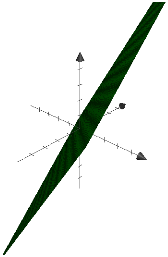
Does column rank always equal row rank? ☺

Geometry

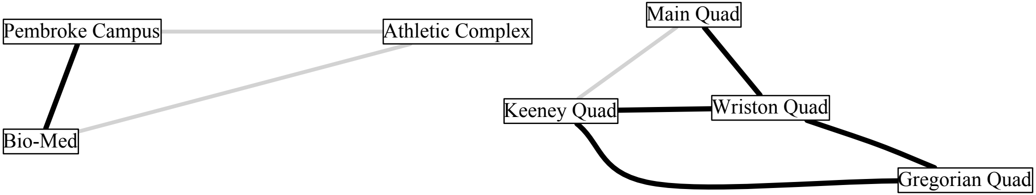


We have asked:

Fundamental Question: How can we predict the dimensionality of the span of some vectors?

Now we can answer:
Compute the rank of the set of vectors.

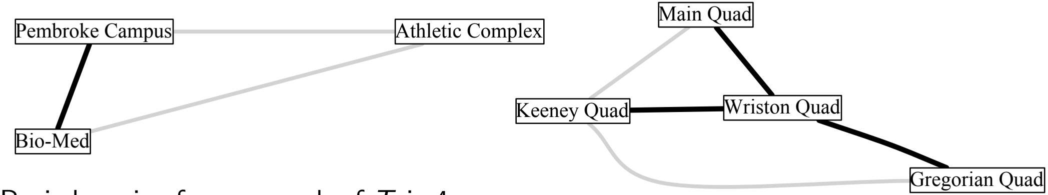
Dimension and rank in graphs



Let

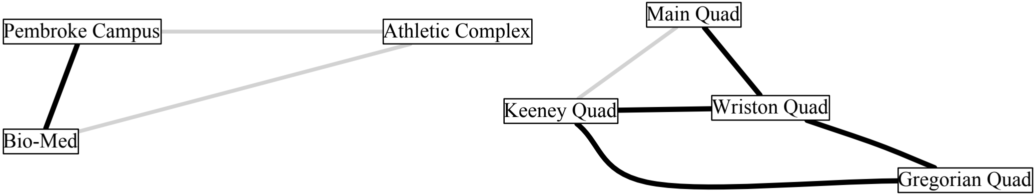
T = set of dark edges

Basis for Span T:



Basis has size four, so rank of T is 4.

Dimension and rank in graphs



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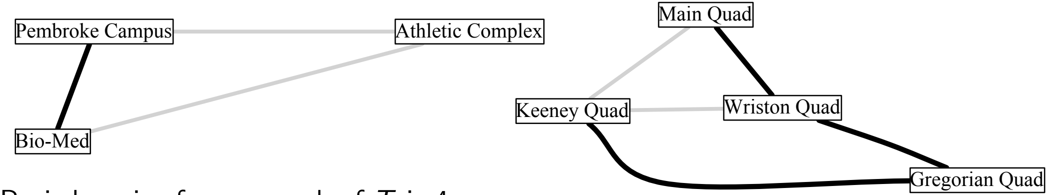

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Cardinality of a vector space over $GF(2)$

Recall *checksum problem*
Checksum function $\mathbf{x} \mapsto [\mathbf{a}_1 \cdot \mathbf{x}, \ldots, \mathbf{a}_{64} \cdot \mathbf{x}]$

Original "file" $\mathbf{p}$, transmission error $\mathbf{e}$ so corrupted file is $\mathbf{p} + \mathbf{e}$.

What is probability that corrupted file has the same checksum as original?

If error is chosen according to uniform distribution,

$$\begin{aligned}
&\text{Probability}\,(\mathbf{p} + \mathbf{e} \text{ has same checksum as } \mathbf{p}) \\
&= \text{Probability}\,(\mathbf{e} \text{ is a solution to homogeneous linear system}) \\
&= \frac{\text{number of solutions to homogeneous linear system}}{\text{number of } n\text{-vectors}} \\
&= \frac{\text{number of solutions to homogeneous linear system}}{2^n}
\end{aligned}$$

raising Question

$$\boxed{\text{How to find number of solutions to a homogeneous linear system over } GF(2)\text{?}}$$

Cardinality of a vector space over $GF(2)$

How to find number of solutions to a homogeneous linear system over $GF(2)$?

Solution set of a homogeneous linear system is a vector space.
Question becomes

How to find out cardinality of a vector space $\mathcal{V}$ over $GF(2)$?

- Suppose basis for $\mathcal{V}$ is $\mathbf{b}_1, \ldots, \mathbf{b}_n$.
- Then $\mathcal{V}$ is set of linear combinations
$$\beta_1 \, \mathbf{b}_1 + \cdots + \beta_n \, \mathbf{b}_n$$
- Number of linear combinations is 2^n
- By Unique-Representation Lemma, every linear combination gives a different vector of $\mathcal{V}$
- Thus cardinality is $2^{\dim \mathcal{V}}$

Cardinality of a vector space over $GF(2)$

Cardinality of a vector space $\mathcal{V}$ over $GF(2)$ is $2^{\dim \mathcal{V}}$

How to find dimension of solution set of a homogeneous linear system?

Write linear system as $A\mathbf{x} = \mathbf{0}$

How to find dimension of the null space of A?

Answers will come later.