Linear function invertibility

How to tell if a linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$ is invertible?

- *One-to-one?* f is one-to-one if its kernel is trivial. *Equivalent:* if its kernel has dimension zero.
- *Onto?* f is onto if its image equals its co-domain

Recall that the image of a function f with domain $\mathcal{V}$ is $\{f(\mathbf{v}) \ : \mathbf{v} \in \mathcal{V}\}$.

Lemma: The image of f is a subspace of $\mathcal{W}$.

How can we tell if the image of f equals $\mathcal{W}$?

Dimension Lemma: If $\mathcal{U}$ is a subspace of $\mathcal{W}$ then

Property D1: $\dim \mathcal{U} \leq \dim \mathcal{W}$, and

Property D2: if $\dim \mathcal{U} = \dim \mathcal{W}$ then $\mathcal{U} = \mathcal{W}$

Proof: Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be a basis for $\mathcal{U}$.
By Superset-Basis Lemma, there is a basis B for $\mathcal{W}$ that contains $\mathbf{u}_1, \ldots, \mathbf{u}_k$.

- Thus $k \leq |B|$, and
- If $k = |B|$ then $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} = B$

QED

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Use Property D2 with $\mathcal{U} = \text{Im } f$.
Shows that the function f is onto iff $\dim \text{Im } f = \dim \mathcal{W}$

We conclude:

$$\boxed{f \text{ is invertible } \dim \text{Ker } f = 0 \text{ and } \dim \text{Im } f = \dim \mathcal{W}}$$

Linear function invertibility

$$\boxed{f \text{ is one-to-one if } \dim \operatorname{Ker} f = 0 \text{ and } \dim \operatorname{Im} f = \dim \mathcal{W}}$$

How does this relate to dimension of the domain?

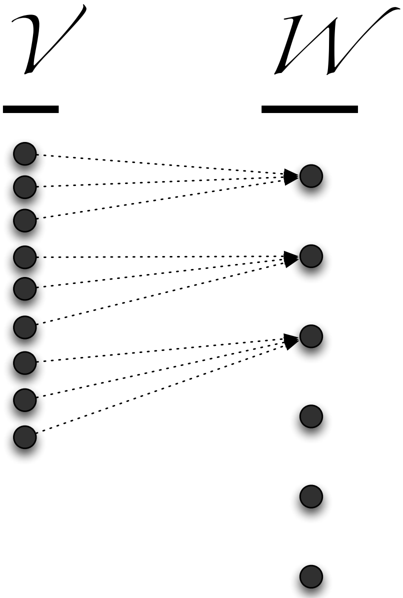
Conjecture: For f to be invertible, need $\dim \mathcal{V} = \dim \mathcal{W}$.

Extracting an invertible function

Starting with a linear function f we will extract a largest possible subfunction that is invertible.

Make it onto by setting co-domain to be image of f.

Make it one-to-one by getting rid of extra domain elements sharing same image.



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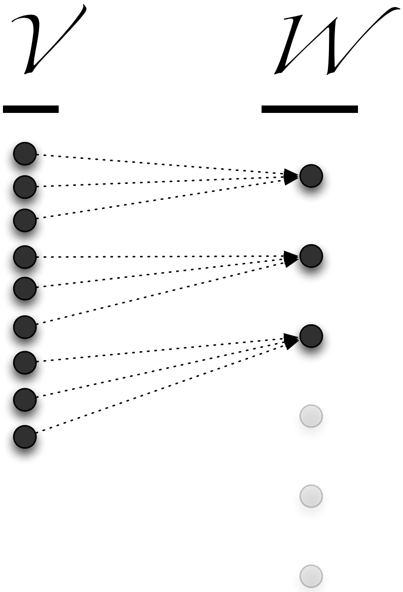

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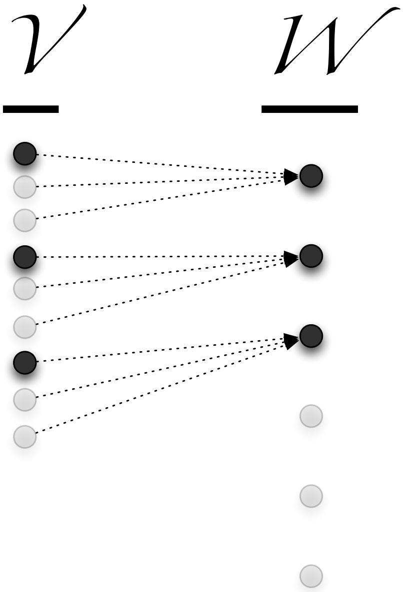


Extracting an invertible function

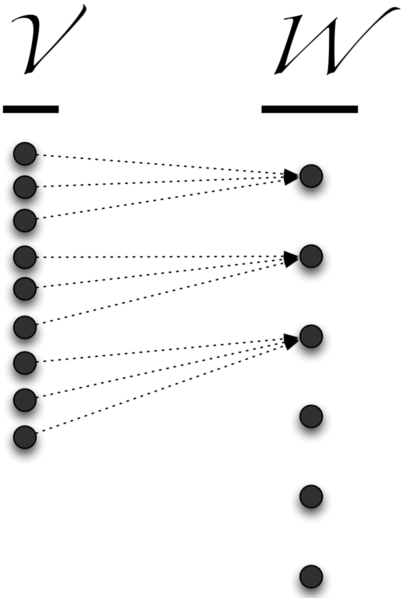
Start with linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

Step 1: Choose smaller co-domain $\mathcal{W}^*$

Step 2: Choose smaller domain $\mathcal{V}^*$

Step 3: Define function $f^* : \mathcal{V}^* \longrightarrow \mathcal{W}^*$ by $f^*(\mathbf{x}) = f(\mathbf{x})$

In fact, we will end up selecting a *basis* of $\mathcal{W}^*$ and a basis of $\mathcal{V}^*$.



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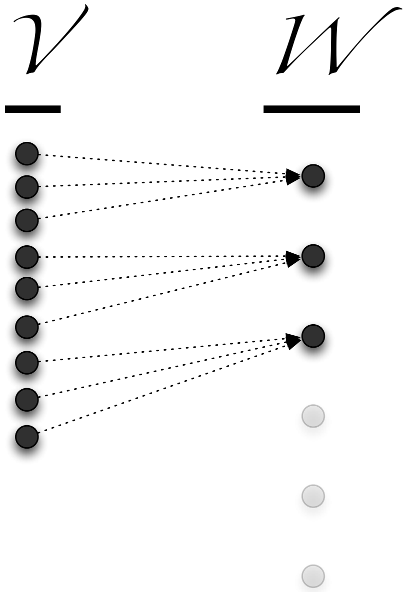

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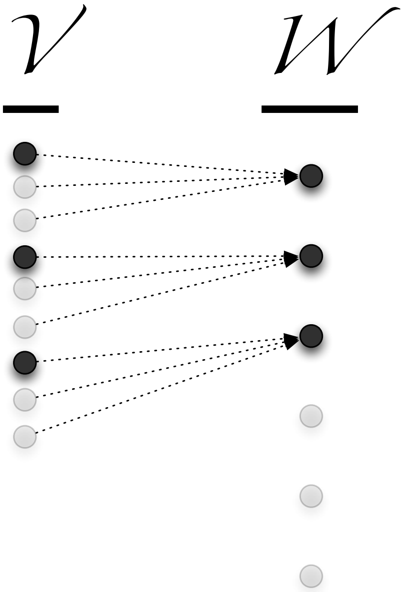

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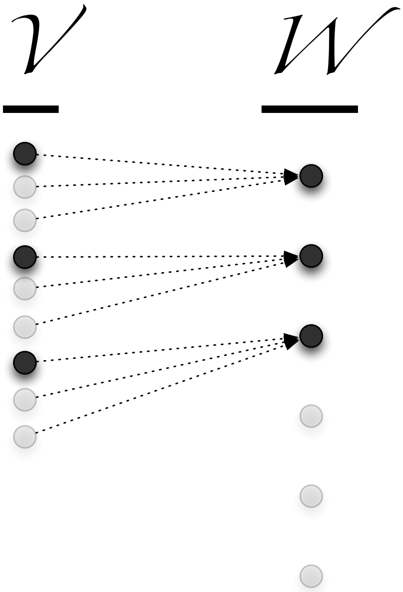

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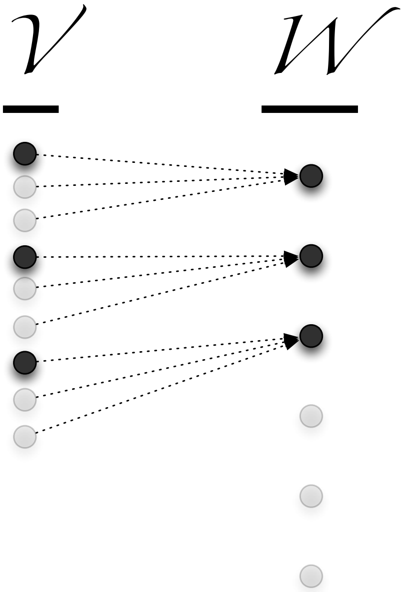

Extracting an invertible function from linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$

- Choose smaller co-domain $\mathcal{W}^*$
  Let $\mathcal{W}^*$ be image of f

  Let $\mathbf{w}_1, \ldots, \mathbf{w}_r$ be a basis of $\mathcal{W}^*$
- Choose smaller domain $\mathcal{V}^*$
  Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be pre-images of $\mathbf{w}_1, \ldots, \mathbf{w}_r$
  That is, $f(\mathbf{v}_1) = \mathbf{w}_1, \ldots, f(\mathbf{v}_r) = \mathbf{w}_r$
  Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
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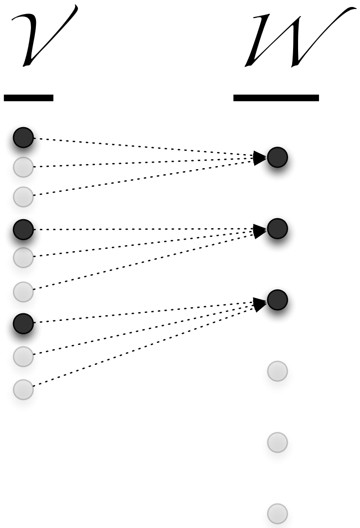

We will show:

- f^* is onto
- f^* is one-to-one (kernel is trivial)
- Bonus: $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$

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Onto:

Let $\mathbf{w}$ be any vector in co-domain $\mathcal{W}^*$.
There are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{w} = \alpha_1\,\mathbf{w}_1 + \cdots + \alpha_r\,\mathbf{w}_r$$

Because f is linear,

$$\begin{aligned} & f(\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_r\,\mathbf{v}_r) \\ &= \alpha_1\,f(\mathbf{v}_1) + \cdots + \alpha_r\,f(v_r) \\ &= \alpha_1\,\mathbf{w}_1 + \cdots + \alpha_r\,\mathbf{w}_r \end{aligned}$$

so $\mathbf{w}$ is image of $\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_r\,\mathbf{v}_r \in \mathcal{V}^*$
QED

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One-to-one:
By One-to-One Lemma, need only show kernel is trivial.

Suppose $\mathbf{v}^*$ is in $\mathcal{V}^*$ and $f(\mathbf{v}^*) = \mathbf{0}$

Because $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$, there are scalars $\alpha_1, \ldots, \alpha_r$ such that

$$\mathbf{v}^* = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r$$

Applying f to both sides,

$$\begin{aligned} \mathbf{0} &= f(\alpha_1 \mathbf{v}_1 + \cdots + \alpha_r \mathbf{v}_r) \\ &= \alpha_1 \mathbf{w}_1 + \cdots + \alpha_r \mathbf{w}_r \end{aligned}$$

Because $\mathbf{w}_1, \ldots, \mathbf{w}_r$ are linearly independent, $\alpha_1 = \cdots = \alpha_r = 0$

so $\mathbf{v}^* = \mathbf{0}$ QED

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Need only show linear independence

Suppose $\mathbf{0} = \alpha_1 \, \mathbf{v}_1 + \cdots + \alpha_r \, \mathbf{v}_r$

Applying f to both sides,

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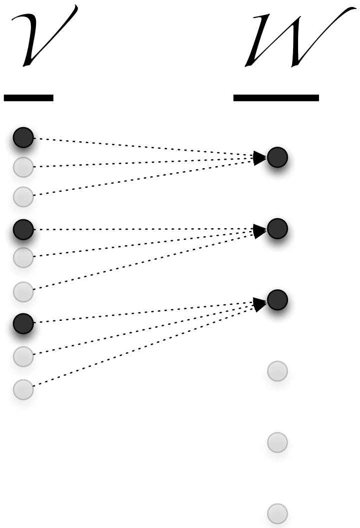

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Example:

Let $A = \left[\begin{array}{c|c|c} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right]$, and define $\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

Define $\mathcal{W}^* = \text{Im } f = \text{Col } A = \text{Span } \{[1,2,1],[2,1,2],[1,1,1]\}$.

One basis for $\mathcal{W}^*$ is $\mathbf{w}_1 = [0,1,0]$, $\mathbf{w}_2 = [1,0,1]$

Pre-images for $\mathbf{w}_1$ and $\mathbf{w}_2$:
$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$,
for then $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = \mathbf{w}_2$.

Let $\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Then $f^* : \mathcal{V}^* \longrightarrow \text{Im } f$ is onto and one-to-one

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To show about original function f:
original domain $\mathcal{V} = \text{Ker } f \oplus \mathcal{V}^*$
Must prove two things:

1. Ker f and $\mathcal{V}^*$ share only zero vector
2. every vector in $\mathcal{V}$ is the sum of a vector in Ker f and a vector in $\mathcal{V}^*$

We already showed kernel of f^* is trivial.
This shows only vector of Ker f in $\mathcal{V}^*$ is zero vector. —thing 1 is proved

Let $\mathbf{v}$ be any vector in $\mathcal{V}$, and let $\mathbf{w} = f(\mathbf{v})$.
Since f^* is onto, its domain $\mathcal{V}^*$ contains a vector $\mathbf{v}^*$ such that $f(\mathbf{v}^*) = \mathbf{w}$
Therefore $f(\mathbf{v}) = f(\mathbf{v}^*)$ so
$f(\mathbf{v}) - f(\mathbf{v}^*) = \mathbf{0}$ so $f(\mathbf{v} - \mathbf{v}^*) = \mathbf{0}$
Thus $\mathbf{u} = \mathbf{v} - \mathbf{v}^*$ is in Ker f
and $\mathbf{v} = \mathbf{u} + \mathbf{v}^*$

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original domain $\mathcal{V}$ = Ker $f \oplus \mathcal{V}^*$

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define $\mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $f(\mathbf{x}) = A\mathbf{x}$.

$\mathbf{v}_1 = [\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}]$ and $\mathbf{v}_2 = [-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}]$

$\mathcal{V}^* = \text{Span } \{\mathbf{v}_1, \mathbf{v}_2\}$

Ker $f = \text{Span } \{[1, 1, -3]\}$

Therefore
$\mathcal{V} = (\text{Span } \{[1, 1, -3]\}) \oplus (\text{Span } \{\mathbf{v}_1, \mathbf{v}_2\})$

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original domain $\mathcal{V} = \text{Ker } f \oplus \mathcal{V}^*$

By Direct-Sum Dimension Corollary,

$$\dim \mathcal{V} = \dim \text{Ker } f + \dim \mathcal{V}^*$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathcal{V}^*$,

$$\dim \mathcal{V}^* = r = \dim \text{Im } f$$

We have proved...

Kernel-Image Theorem:
For any linear function $f : \mathcal{V} \to W$,

$$\dim \text{Ker } f + \dim \text{Im } f = \dim \mathcal{V}$$