Linear Dependence: The Superfluous-Vector Lemma

Grow and Shrink algorithms both test whether a vector is superfluous in spanning a vector space $\mathcal{V}$. Need a criterion for superfluity.

Superfluous-Vector Lemma: For any set S and any vector $\mathbf{v} \in S$, if $\mathbf{v}$ can be written as a linear combination of the other vectors in S then Span $(S - \{\mathbf{v}\}) =$ Span S

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Suppose $\mathbf{v}_n = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}$

To show: every vector in Span S is also in Span $(S - \{\mathbf{v}_n\})$.

Every vector $\mathbf{v}$ in Span S can be written as $\mathbf{v} = \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots \beta_n \mathbf{v}_n$

Substituting for $\mathbf{v}_n$, we obtain

$$\begin{aligned}
\mathbf{v} &= \beta_1 \mathbf{v}_1 + \beta_2 \mathbf{v}_2 + \cdots + \beta_n \left(\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_{n-1} \mathbf{v}_{n-1}\right) \\
&= (\beta_1 + \beta_n \alpha_1)\mathbf{v}_1 + (\beta_2 + \beta_n \alpha_2)\mathbf{v}_2 + \cdots + (\beta_{n-1} + \beta_n \alpha_{n-1})\mathbf{v}_{n-1}
\end{aligned}$$

which shows that an arbitrary vector in Span S can be written as a linear combination of vectors in $S - \{\mathbf{v}_n\}$ and is therefore in Span $(S - \{\mathbf{v}_n\})$.

QED

Defining linear dependence

Definition: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly dependent* if the zero vector can be written as a **nontrivial** linear combination of the vectors:

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

In this case, we refer to the linear combination as a *linear dependency* in $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

On the other hand, if the *only* linear combination that equals the zero vector is the trivial linear combination, we say $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly *in*dependent.

Example: The vectors $[1, 0, 0]$, $[0, 2, 0]$, and $[2, 4, 0]$ are linearly dependent, as shown by the following equation:

$$2\,[1, 0, 0] + 2\,[0, 2, 0] - 1\,[2, 4, 0] = [0, 0, 0]$$

Therefore:

$2\,[1, 0, 0] + 2\,[0, 2, 0] - 1\,[2, 4, 0]$ is a linear dependency in $[1, 0, 0]$, $[0, 2, 0]$, $[2, 4, 0]$.

Linear dependence

Example: The vectors $[1, 0, 0]$, $[0, 2, 0]$, and $[0, 0, 4]$ are linearly independent.
How do we know?
Easy since each vector has a nonzero entry where the others have zeroes.
Consider any linear combination

$$\alpha_1 [1, 0, 0] + \alpha_2 [0, 2, 0] + \alpha_3 [0, 0, 4]$$

This equals $[\alpha_1, 2\alpha_2, 4\alpha_3]$

If this is the zero vector, it must be that $\alpha_1 = \alpha_2 = \alpha_3 = 0$
That is, the linear combination is trivial.

We have shown the only linear combination that equals the zero vector is the trivial linear combination.

Linear dependence in relation to other questions

$\boxed{\text{How can we tell if vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent?}}$

Definition: Vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are *linearly dependent* if the zero vector can be written as a nontrivial linear combination $\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$

By linear-combinations definition, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent iff there is a nonzero vector $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$ such that $\begin{bmatrix} & & \\ \mathbf{v}_1 & \cdots & \mathbf{v}_n \\ & & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \mathbf{0}$

Therefore, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent iff the null space of the matrix is nontrivial.

This shows that the question

$\boxed{\text{How can we tell if vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent?}}$

is the same as a question we asked earlier:

$\boxed{\text{How can we tell if the null space of a matrix is trivial?}}$

Linear dependence in relation to other questions

The question

$$\boxed{\text{How can we tell if vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent?}}$$

is the same as a question we asked earlier:

$$\boxed{\text{How can we tell if the null space of a matrix is trivial?}}$$

Recall: *solution set of a homogeneous linear system*

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

is the null space of matrix $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$.

So question is same as:

$$\boxed{\text{How can we tell if the solution set of a homogeneous linear system is trivial?}}$$

Linear dependence in relation to other questions

The question

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is the same as a question we asked earlier:

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is the same as :

$$\boxed{\text{How can we tell if the solution set of a homogeneous linear system is trivial?}}$$

Recall:

If $\mathbf{u}_1$ is a solution to a linear system $\mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m$ then

$$\{\text{solutions to linear system}\} = \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$$

where $\mathcal{V} = \{$solutions to corresponding homogeneous linear system

$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0\}$$

Thus the question is the same as:

$$\boxed{\text{How can we tell if a solution } \mathbf{u}_1 \text{ to a linear system is the } \textit{only} \text{ solution?}}$$

Linear dependence and null space

The question

$$\boxed{\text{How can we tell if vectors } \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ are linearly dependent?}}$$

is the same as:

$$\boxed{\text{How can we tell if the null space of a matrix is trivial?}}$$

is the same as:

$$\boxed{\text{How can we tell if the solution set of a homogeneous linear system is trivial?}}$$

is the same as:

$$\boxed{\text{How can we tell if a solution } \mathbf{u}_1 \text{ to a linear system is the } \textit{only} \text{ solution?}}$$

Linear dependence

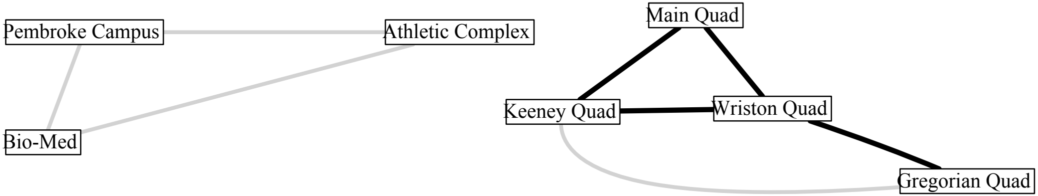
Answering these questions requires an algorithm.

Computational Problem: *Testing linear dependence*

- *input:* a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* DEPENDENDENT if the vectors are linearly dependent, and INDEPENDENT otherwise.

We'll see two algorithms later.

Linear dependence in *Minimum Spanning Forest*



We can get the zero vector by adding together vectors corresponding to edges that form a cycle: in such a sum, for each entry x, there are exactly two vectors having 1's in position x.

Example: the vectors corresponding to

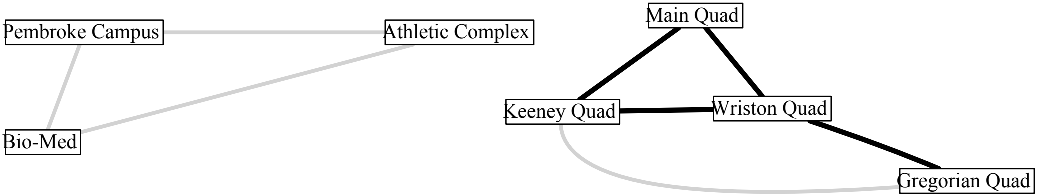
`{Main, Wriston}`,`{Main, Keeney}` `{Keeney, Wriston }`,

are as follows:

Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
			1	1		
				1	1	
			1		1	

The sum of these vectors is the zero vector.

Linear dependence in *Minimum Spanning Forest*



Sum of vectors corresponding to edges forming a cycle can make a zero vector.
Therefore if a subset of S form a cycle then S is linearly dependent.

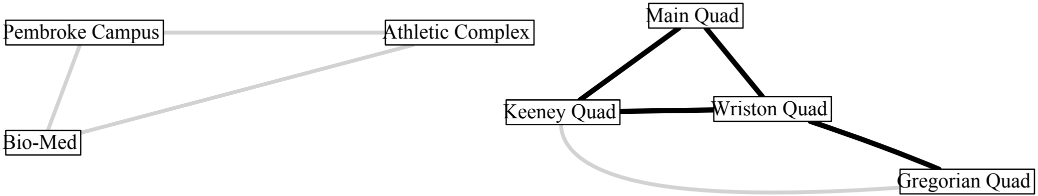
Example: The vectors corresponding to

{`Main, Keeney`}, {`Main, Wriston`}, {`Keeney, Wriston `}, {`Wriston, Gregorian`}
are linearly dependent because these edges include a cycle.

The zero vector is equal to the nontrivial linear combination

			Pembroke	Athletic	Bio-Med	Main	Keeney	Wriston	Gregorian
	1	*				1	1		
+	1	*				1		1	
+	1	*					1	1	
+	0	*						1	1

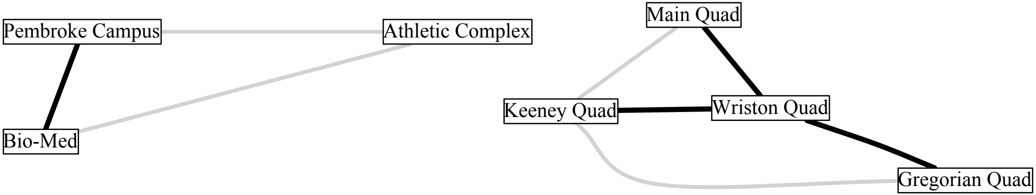
Linear dependence in *Minimum Spanning Forest*



If a subset of S form a cycle then S is linearly dependent.

On the other hand, if a set of edges contains no cycle (i.e. is a forest) then the corresponding set of vectors is linearly independent.

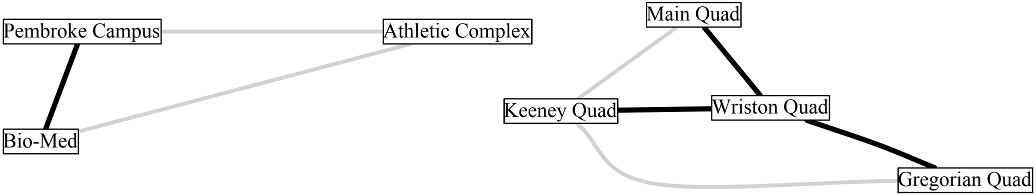
Linear dependence in *Minimum Spanning Forest*



If a subset of S form a cycle then S is linearly dependent.

On the other hand, if a set of edges contains no cycle (i.e. is a forest) then the corresponding set of vectors is linearly independent.

"Quiz"



Which edges are spanned?
Which sets are linearly dependent?

Properties of linear independence: hereditary

Lemma: A subset of a linearly independent set is linearly independent.

In graphs, if a set of edges forms no cycle then any subset of these edges forms no cycle.

Properties of linear independence: hereditary

Lemma: A subset of a linearly independent set is linearly independent.
Proof: Let S and T be subsets of vectors, and suppose S is a subset of T.
Goal" prove that if T is linearly independent then S is linearly independent. Equivalent to the contrapositive: if S is linearly dependent then T is linearly dependent. *Idea:* If

you can write $\mathbf{0}$ as a nontrivial linear combination of some set S vectors,
you can write it so even if we allow some additional vectors to be in the linear combination.

Formally:
Write $T = \{\mathbf{s}_1, \ldots, \mathbf{s}_n, \boldsymbol{t}_1, \ldots, \boldsymbol{t}_k\}$ where $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$.
Supppose that S is linearly dependent.
Then there are coefficients $\alpha_1, \ldots, \alpha_n$, not all zero, such that

$$\mathbf{0} = \alpha_1 \mathbf{s}_1 + \cdots + \alpha_n \mathbf{s}_n$$

Therefore

$$\mathbf{0} = \alpha_1 \mathbf{s}_1 + \cdots + \alpha_n \mathbf{s}_n + 0\, \boldsymbol{t}_1 + \cdots 0\, \boldsymbol{t}_k$$

Thus $\mathbf{0}$ can be written as a nontrivial linear combination of the vectors of T,
i.e. T is linearly dependent. QED

Properties of linear (in)dependence

Linear-Dependence Lemma: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors

if and only if

the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$

in which the coefficient of $\mathbf{v}_i$ is nonzero.

In graphs, the Linear-Dependence Lemma states that an edge e is in the span of other edges if there is a cycle consisting of e and a subset of the other edges.

Properties of linear (in)dependence

Linear-Dependence Lemma: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors
if and only if
the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$
in which the coefficient of $\mathbf{v}_i$ is nonzero.

Proof: *First direction:* Suppose $\mathbf{v}_i$ is in the span of the other vectors. That is, there exist coefficients $\alpha_1, \ldots, \alpha_{n-1}$ such that

$$\mathbf{v}_i = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{i-1}\mathbf{v}_{i-1} + \alpha_{i+1}\mathbf{v}_{i+1} + \cdots \alpha_n \mathbf{v}_n$$

Moving $\mathbf{v}_i$ to the other side, we can write

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + (-1)\mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

which shows that the all-zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in which the coefficient of $\mathbf{v}_i$ is nonzero.

Properties of linear (in)dependence

Linear-Dependence Lemma: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors.

A vector $\mathbf{v}_i$ is in the span of the other vectors
if and only if
the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$
in which the coefficient of $\mathbf{v}_i$ is nonzero.

Proof: *Now for the other direction.* Suppose there are coefficients $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_i \mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

and such that $\alpha_i \neq 0$.

Dividing both sides by α_i yields

$$\mathbf{0} = (\alpha_1/\alpha_i)\, \mathbf{v}_1 + (\alpha_2/\alpha_i)\, \mathbf{v}_2 + \cdots + \mathbf{v}_i + \cdots + (\alpha_n/\alpha_i)\, \mathbf{v}_n$$

Moving every term from right to left except $\mathbf{v}_i$ yields

$$-(\alpha_1/\alpha_i)\, \mathbf{v}_1 - (\alpha_2/\alpha_i)\, \mathbf{v}_2 - \cdots - (\alpha_n/\alpha_i)\, \mathbf{v}_n = \mathbf{v}_i$$

QED

Properties of linear (in)dependence

Linear-Dependence Lemma: Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors. A vector $\mathbf{v}_i$ is in the span of the other vectors if and only if the zero vector can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in which the coefficient of $\mathbf{v}_i$ is nonzero.

Contrapositive:

$\mathbf{v}_i$ is *not* in the span of the other vectors
if and only if
for every linear combination equaling the zero vector

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_i \mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n$$

the coefficient α_i *is* zero.

Analyzing the Grow algorithm

def $\textsc{Grow}(\mathcal{V})$
  $S = \emptyset$
  repeat while possible:
    find a vector $\mathbf{v}$ in $\mathcal{V}$ that is not in Span S, and put it in S.

Grow-Algorithm Corollary: The vectors obtained by the Grow algorithm are linearly independent.

In graphs, this means that the solution obtained by the Grow algorithm has no cycles (is a forest).

Analyzing the Grow algorithm

Grow-Algorithm Corollary: The vectors obtained by the Grow algorithm are linearly independent.

Proof: For $n = 1, 2, \ldots$, let $\mathbf{v}_n$ be the vector added to S in the n^{th} iteration of the Grow algorithm. We show by induction that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.

For $n = 0$, there are no vectors, so the claim is trivially true.
Assume the claim is true for $n = k - 1$. We prove it for $n = k$.
The vector $\mathbf{v}_k$ added to S in the k^{th} iteration is not in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.
Therefore, by the Linear-Dependence Lemma, for any coefficients $\alpha_1, \ldots, \alpha_k$ such that

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_k \mathbf{v}_k$$

it must be that α_k equals zero. We may therefore write

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1}$$

By claim for $n = k - 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$ are linearly independent, so $\alpha_1 = \cdots = \alpha_{k-1} = 0$

The linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is *trivial*. We have proved that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. This proves the claim for $n = k$. QED

Analyzing the Shrink algorithm

def $\textsc{Shrink}(\mathcal{V})$
$S =$ some finite set of vectors that spans $\mathcal{V}$
repeat while possible:
find a vector $\mathbf{v}$ in S such that Span $(S - \{v\}) = \mathcal{V}$, and remove $\mathbf{v}$ from S.

Shrink-Algorithm Corollary: The vectors obtained by the Shrink algorithm are linearly independent.

In graphs, this means that the Shrink algorithm outputs a solution that is a forest.

Recall:

Superfluous-Vector Lemma For any set S and any vector $\mathbf{v} \in S$, if $\mathbf{v}$ can be written as a linear combination of the other vectors in S then
Span $(S - \{\mathbf{v}\}) =$ Span S

Analyzing the Shrink algorithm

Shrink-Algorithm Corollary: The vectors obtained by the Shrink algorithm are linearly independent.

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be the set of vectors obtained by the Shrink algorithm. Assume for a contradiction that the vectors are linearly dependent.

Then $\mathbf{0}$ can be written as a nontrivial linear combination

$$\mathbf{0} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

where at least one of the coefficients is nonzero.

Let α_i be one of the nonzero coefficients.

By the Linear-Dependence Lemma, $\mathbf{v}_i$ can be written as a linear combination of the other vectors.

Hence by the Superfluous-Vector Lemma, Span $(S - \{\mathbf{v}_i\}) =$ Span S,

so the Shrink algorithm should have removed $\mathbf{v}_i$. QED