Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \;:\; A * \mathbf{u} = \mathbf{0}\}$. Written *Null* A

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Example:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [0, 0, 0] = [0, 0]$$

so the null space includes $[0, 0, 0]$

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * [6, -1, -1] = [0, 0]$$

so the null space includes $[6, -1, -1]$

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By dot-product definition,

$$\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right] * \mathbf{u} = [\mathbf{a}_1 \cdot \mathbf{u}, \; \ldots, \; \mathbf{a}_m \cdot \mathbf{u}]$$

Thus $\mathbf{u}$ is in null space of $\left[\begin{array}{c} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{array}\right]$ if and only if $\mathbf{u}$ is a solution to the homogeneous linear system

$$\begin{aligned} \mathbf{a}_1 \cdot \mathbf{x} &= 0 \\ &\vdots \\ \mathbf{a}_m \cdot \mathbf{x} &= 0 \end{aligned}$$

Null space of a matrix

We just saw:

Null space of a matrix $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

equals the solution set of the homogeneous linear system $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}}$

This shows: $\boxed{\textit{Null space of a matrix is a vector space.}}$

Can also show it directly, using algebraic properties of matrix-vector multiplication:

Property V1: Since $A * \mathbf{0} = \mathbf{0}$,the null space of A contains $\mathbf{0}$

Property V2: if $\mathbf{u} \in$ Null A then $A * (\alpha\, \mathbf{u}) = \alpha\, (A * \mathbf{u}) = \alpha\, \mathbf{0} = \mathbf{0}$ so $\alpha \mathbf{u} \in$ Null A

Property V3: If $\mathbf{u} \in$ Null A and $\mathbf{v} \in$ Null A
then $A * (\mathbf{u} + \mathbf{v}) = A * \mathbf{u} + A * \mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$
so $\mathbf{u} + \mathbf{v} \in$ Null A

Null space of a matrix

Definition: *Null space* of a matrix A is $\{\mathbf{u} \ : \ A * \mathbf{u} = \mathbf{0}\}$. Written *Null* A

Proposition: Null space of a matrix is a vector space.

Example:

$$\text{Null} \left[\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 9 \end{array} \right] = \text{Span } \{[6, -1, -1]\}$$

Solution space of a matrix-vector equation

Earlier, we saw:

If $\mathbf{u}_1$ is a solution to the linear system $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array}}$

then the solution set is $\mathbf{u}_1 + \mathcal{V}$,

where $\mathcal{V}$ = solution set of $\boxed{\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}}$

Restated: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$

where $\mathcal{V} = \text{Null } A$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

Example:

- Null space of $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix}$ is Span $\{[6, -1, -1]\}$.
- One solution to $\boxed{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 9 \end{bmatrix} * \mathbf{x} = [1, 1]}$ is $x = [-1, 1, 0]$.
- Therefore solution set is $[-1, 1, 0] +$ Span $\{[6, -1, -1]\}$
- For example, solutions include
  - $[-1, 1, 0] + [0, 0, 0]$
  - $[-1, 1, 0] + [6, -1, -1]$
  - $[-1, 1, 0] + 2\,[6, -1, -1]$

$$\vdots$$

Solution space of a matrix-vector equation

Proposition: If $\mathbf{u}_1$ is a solution to $A * \mathbf{x} = \mathbf{b}$ then solution set is $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V} = \text{Null } A$

- If $\mathcal{V}$ is a trivial vector space then $\mathbf{u}_1$ is the only solution.
- If $\mathcal{V}$ is not trivial then $\mathbf{u}_1$ is *not* the only solution.

Corollary: $A * \mathbf{x} = \mathbf{b}$ has at most one solution iff Null A is a trivial vector space.

Question: How can we tell if the null space of a matrix is trivial?

Answer comes later...