Basis

If they successfully finish, the Grow algorithm and the Shrink algorithm each find a set of vectors spanning the vector space $\mathcal{V}$. In each case, the set of vectors found is linearly independent.

Definition: Let $\mathcal{V}$ be a vector space. A *basis* for $\mathcal{V}$ is a linearly independent set of generators for $\mathcal{V}$.

Thus a set S of vectors of $\mathcal{V}$ is a *basis* for $\mathcal{V}$ if S satisfies two properties:

Property B1 (*Spanning*) Span $S = \mathcal{V}$, and

Property B2 (*Independent*) S is linearly independent.

Most important definition in linear algebra.

Basis: Examples

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Example: Let $\mathcal{V} = \text{Span } \{[1,0,2,0],[0,-1,0,-2],[2,2,4,4]\}$.
Is $\{[1,0,2,0],[0,-1,0,-2],[2,2,4,4]\}$ a basis for $\mathcal{V}$?

The set *is* spanning but is *not* independent

$$1\,[1,0,2,0] - 1\,[0,-1,0,-2] - \frac{1}{2}\,[2,2,4,4] = \mathbf{0}$$

so not a basis

However, $\{[1,0,2,0],[0,-1,0,-2]\}$ *is* a basis:

- Obvious that these vectors are independent because each has a nonzero entry where the other has a zero.
- To show
  $$\text{Span } \{[1,0,2,0],[0,-1,0,-2]\} = \text{Span } \{[1,0,2,0],[0,-1,0,-2],[2,2,4,4]\},$$
  can use Superfluous-Vector Lemma:
  $$[2,2,4,4] = 2\,[1,0,2,0] - 2\,[0,-1,0,-2]$$

Basis: Examples

Example: A simple basis for $\mathbb{R}^3$: the standard generators $\mathbf{e}_1 = [1, 0, 0], \mathbf{e}_2 = [0, 1, 0], \mathbf{e}_3 = [0, 0, 1]$.

- *Spanning:* For any vector $[x, y, z] \in \mathbb{R}^3$,

$$[x, y, z] = x\,[1, 0, 0] + y\,[0, 1, 0] + z\,[0, 0, 1]$$

- *Independent:* Suppose

$$\mathbf{0} = \alpha_1\,[1, 0, 0] + \alpha_2\,[0, 1, 0] + \alpha_3\,[0, 0, 1] = [\alpha_1, \alpha_2, \alpha_3]$$

  Then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Instead of "standard generators", we call them *standard basis vectors*.
We refer to $\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ as *standard basis* for $\mathbb{R}^3$.

In general the standard generators are usually called *standard basis vectors*.

Basis: Examples

Example: Another basis for $\mathbb{R}^3$: $[1,1,1],[1,1,0],[0,1,1]$

- *Spanning:* Can write standard generators in terms of these vectors:

$$\begin{aligned}[1,0,0] &= [1,1,1]-[0,1,1]\\ [0,1,0] &= [1,1,0]+[0,1,1]-[1,1,1]\\ [0,0,1] &= [1,1,1]-[1,1,0]\end{aligned}$$

  Since $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3$ can be written in terms of these new vectors, every vector in Span $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is in span of new vectors. Thus $\mathbb{R}^3$ equals span of new vectors.

- *Linearly independent:* Write zero vector as linear combination:

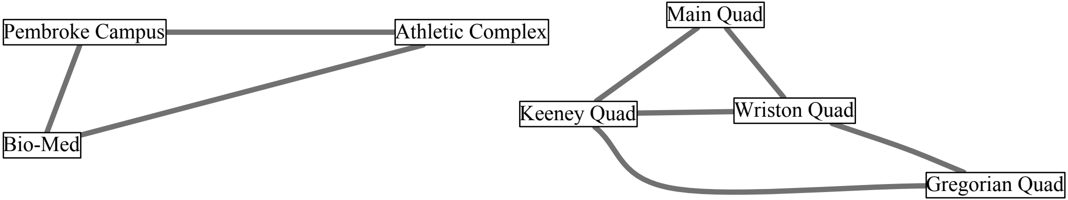
$$\mathbf{0} = x\,[1,1,1]+y\,[1,1,0]+z\,[0,1,1] = [x+y, x+y+z, x+z]$$

Looking at each entry, we get

$$\begin{aligned}0 &= x+y\\ 0 &= x+y+z\\ 0 &= x+z\end{aligned}$$

Plug $x+y=0$ into second equation to get $0=z$.
Plug $z=0$ into third equation to get $x=0$.
Plug $x=0$ into first equation to get $y=0$.
Thus the linear combination is trivial.

Basis: Examples in graphs



One kind of basis in a graph G: a set S of edges forming a spanning forest.

- *Spanning:* for each edge xy in G, there is an x-to-y path consisting of edges of S.
- *Independent:* no cycle consisting of edges of S

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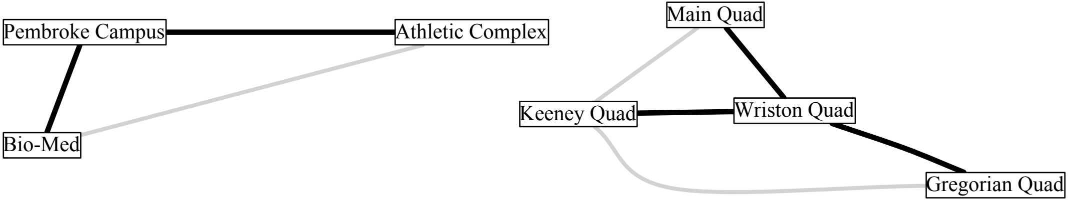

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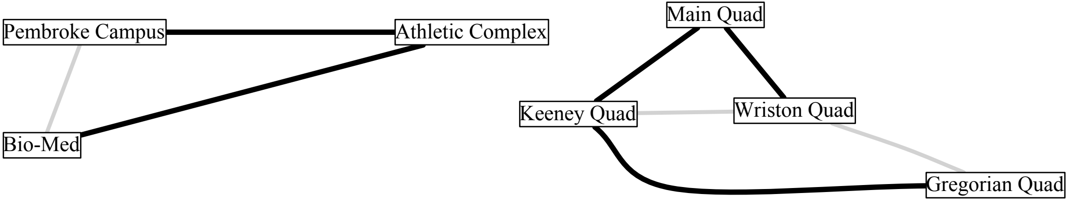

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Towards showing that every vector space has a basis

We would like to prove that every vector space $\mathcal{V}$ has a basis.

The Grow algorithm and the Shrink algorithm each provides a way to prove this, but we are not there yet:

- The Grow-Algorithm Corollary implies that, if the Grow algorithm terminates, the set of vectors it has selected is a basis for the vector space $\mathcal{V}$.
  However, *we have not yet shown that it always terminates!*

- The Shrink-Algorithm Corollary implies that, if we can run the Shrink algorithm starting with a finite set of vectors that spans $\mathcal{V}$, upon termination it will have selected a basis for $\mathcal{V}$.
  However, *we have not yet shown that every vector space $\mathcal{V}$ is spanned by some finite set of vectors!*.

Computational problems involving finding a basis

Two natural ways to specify a vector space $\mathcal{V}$:

1. Specifying generators for $\mathcal{V}$.
2. Specifying a homogeneous linear system whose solution set is $\mathcal{V}$.

Two Fundamental Computational Problems:

Computational Problem: *Finding a basis of the vector space spanned by given vectors*

- *input:* a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- *output:* a list of vectors that form a basis for Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Computational Problem: *Finding a basis of the solution set of a homogeneous linear system*

- *input:* a list $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ of vectors
- *output:* a list of vectors that form a basis for the set of solutions to the system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_n \cdot \mathbf{x} = 0$