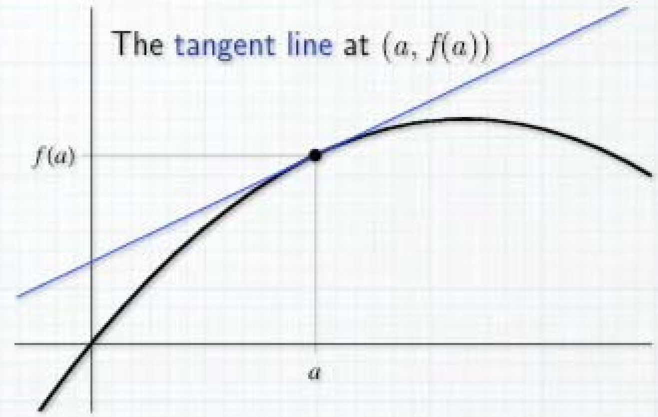
The Derivative

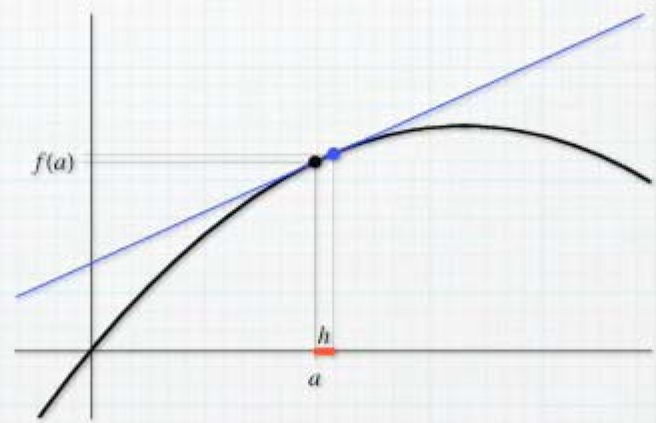
The Slope of a Curve

The key to solving a wide range of problems is the ability to compute the slope of the **tangent line** to the graph of a function f at a given point $(a, f(a))$.



The slope of the **secant line** through $(a, f(a))$ and a second point $(a+h, f(a+h))$ provides an approximation when h is small.

The *exact* slope of the tangent line is the limit of secant line slopes as $h \longrightarrow 0$.

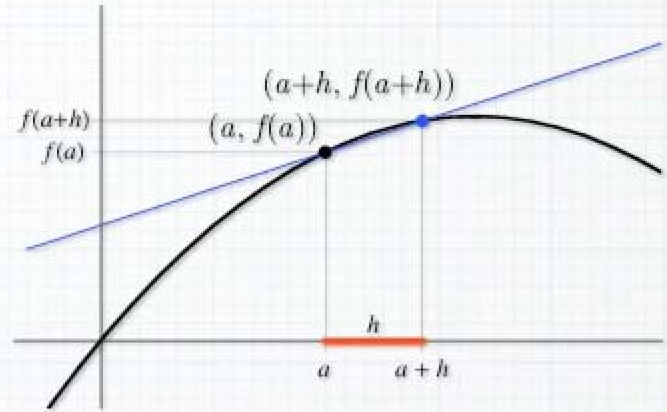


The slope of the secant line through $(a, f(a))$ and $(a+h, f(a+h))$ is

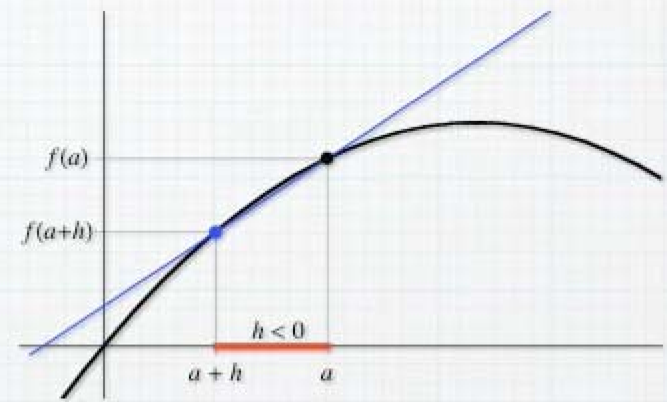
$$\frac{f(a+h) - f(a)}{h}$$

So the slope of the tangent line at $(a, f(a))$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



We should emphasize that in order for this limit to exist, the corresponding one-sided limits, as always, must exist and coincide. In other words, we are concerned with both positive and negative values of h.



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A function f is said to be **differentiable at** a if the limit

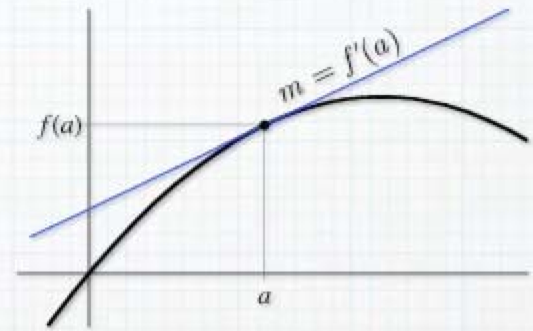
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

The value of this limit is called the **derivative of** f **at** a and is denoted by $f'(a)$. (Read f-*prime* of a.)

In other words, we define

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit exists.



Remarks:

(1) Since the slope of the tangent line at $(a, f(a))$ is $f'(a)$, the *equation* of the tangent line is

"point-slope" equation of a line

$$\boxed{y - f(a) = f'(a)(x - a).}$$

(2) $f'(a)$ is often referred to as the **rate of change** in $f(x)$ at $x = a$.

Example: Find the slope of the graph of $f(x) = \sqrt{x}$ at $(4, 2)$ and write the equation of the tangent line there.

First we form and simplify the expression that gives the slope of the secant line through $(4,\ 2)$ and $(4+h, \sqrt{4+h}\,)$:

$$\frac{f(4+h)-f(4)}{h} = \frac{\sqrt{4+h}-2}{h}\,\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$
$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)} = \frac{1}{\sqrt{4+h}+2} \quad \text{for } h \neq 0.$$

Now we take the limit as as $h \to 0$ to obtain the slope of the tangent line:

$$f'(4) = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{4}$$

Finally, the equation of the tangent line is

$$y - 2 = \frac{1}{4}(x-4) \quad \text{or} \quad y = \frac{1}{4}x + 1$$

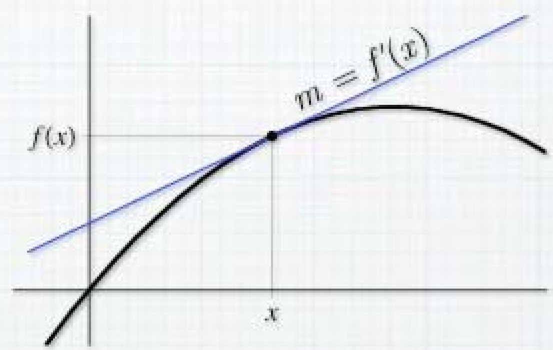
The Derivative as a Function

Given a function f we define a function f' (the **derivative** of f) by

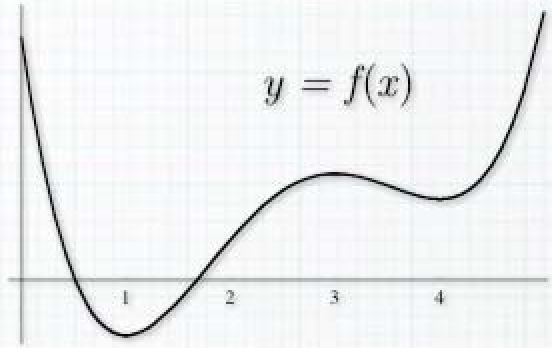
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

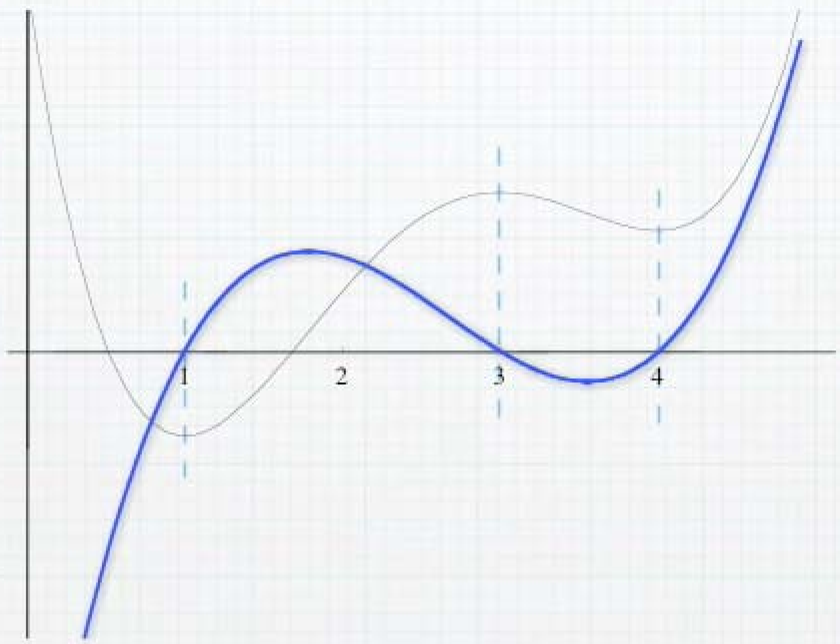
The domain of f' is understood to be the set of all x for which the defining limit exists, *i.e.*, all x at which f is *differentiable*.

Note that wherever $f'(x)$ exists, it is the slope of the graph of f at the point $(x, f(x))$ and the *rate of change* in $f(x)$.



Example: Sketch the graph of $f'(x)$, given the graph of f on the right.





Example: Find $f'(x)$, given that $f(x) = x + \dfrac{1}{x}$.

First we form and simplify the expression $\dfrac{f(x+h) - f(x)}{h}$.

$$\frac{\left((\cancel{x} + h) + \frac{1}{x+h}\right) - \left(\cancel{x} + \frac{1}{x}\right)}{h} = \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \frac{h}{h} + \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \frac{(x+h)x}{(x+h)x}$$

$$= 1 + \frac{-\cancel{h}}{\cancel{h}(x+h)x}$$

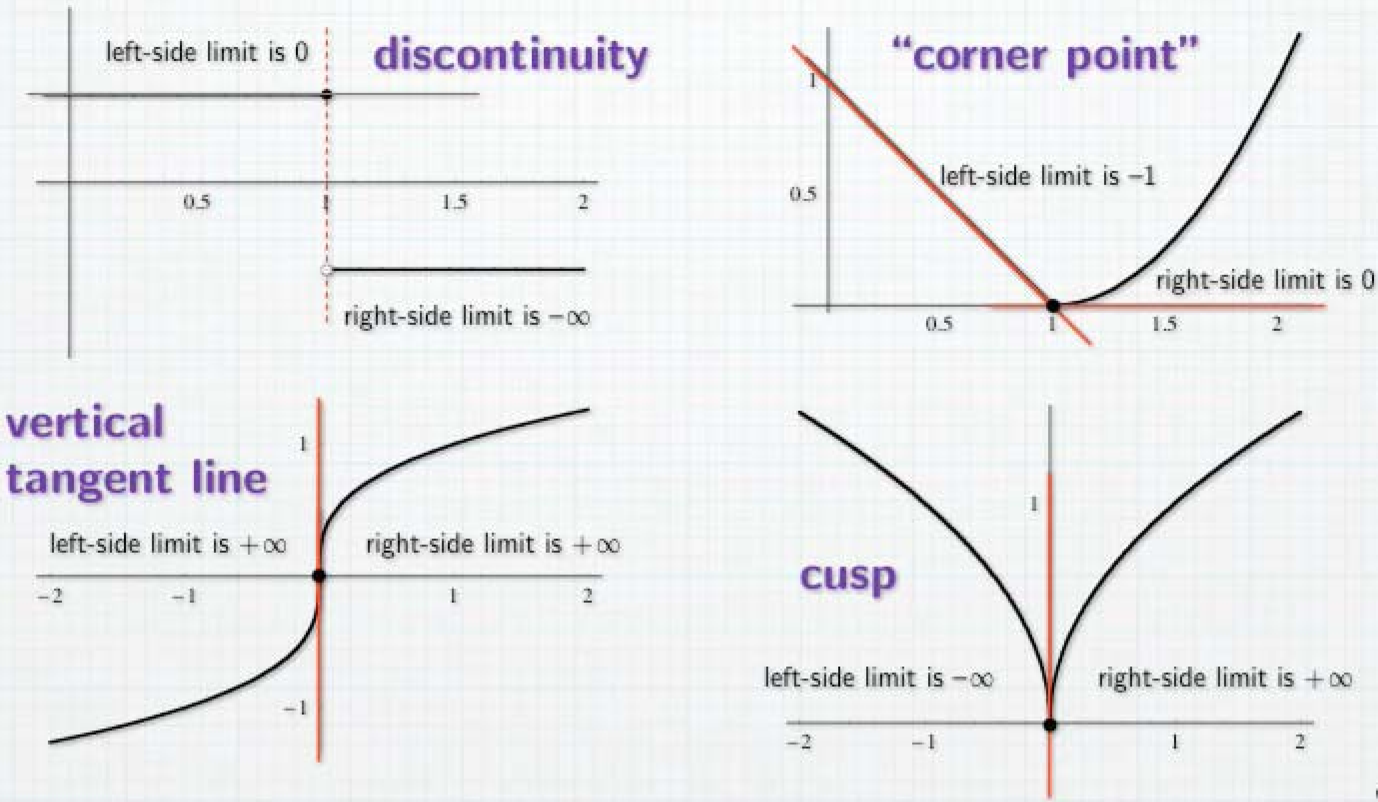
$$= 1 - \frac{1}{(x+h)x}$$

$$\therefore \quad f'(x) = \lim_{h \to 0} \left(1 - \frac{1}{(x+h)x}\right) = 1 - \frac{1}{(x+0)x} = \boxed{1 - \frac{1}{x^2}}$$

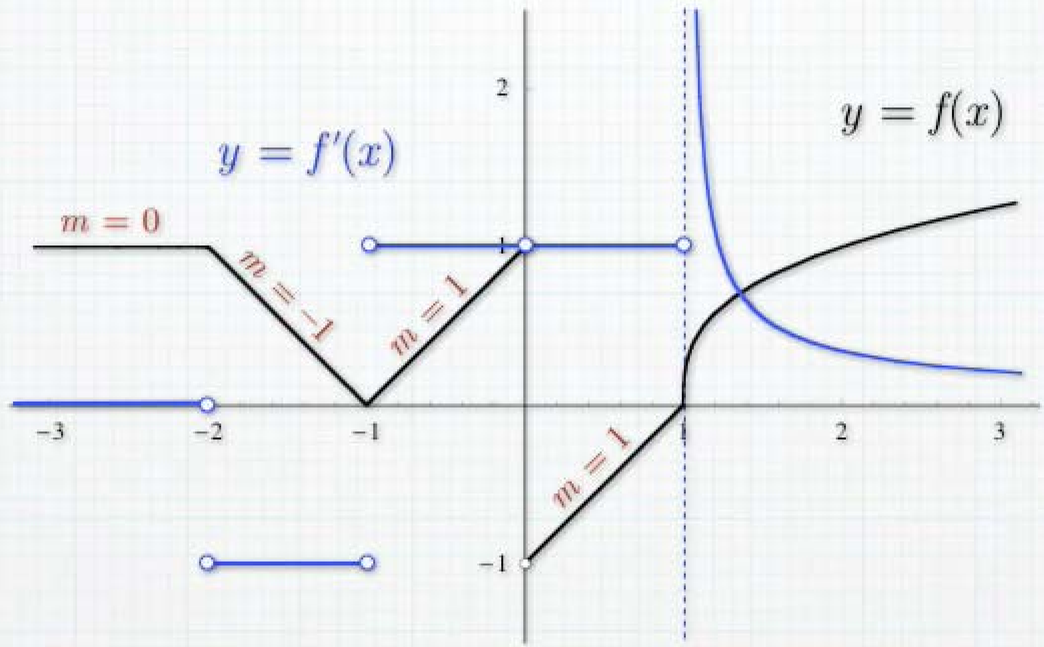
Nondifferentiablity

A function f is *not* differentiable at a if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \textit{fails to exist.}$$



Example: Sketch the graph of $f'(x)$, given the graph of f below.



Calculation of Derivatives

Power Rule
Product Rule
Reciprocal Rule
Quotient Rule

We begin by noting the derivatives of a few basic functions.

Any constant function $f(x) = c$ has derivative $f'(x) = 0$.

The identity function $f(x) = x$ has derivative $f'(x) = 1$.

Power Rule

If n is a positive integer, then $f(x) = x^n$ has derivative $f'(x) = n\,x^{n-1}$.

$f(x)$	$f'(x)$
x^2	$2x$
x^5	$5x^4$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h} \\ &= \frac{x^n + n\,x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + h^n - x^n}{h} \\ &= n\,x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + h^{n-1} \quad \longrightarrow n\,x^{n-1}\end{aligned}$$

as $h \to 0$

Linearity Properties

1. $\boxed{(f+g)'(x) = f'(x) + g'(x)}$

$$\frac{(f(x+h)+g(x+h)) - (f(x)+g(x))}{h} = \frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}$$

$$\longrightarrow f'(x) + g'(x) \text{ as } h \to 0$$

2. $\boxed{(c\,f)'(x) = c\,f'(x)}$

$$\frac{c\,f(x+h) - c\,f(x)}{h} = c\,\frac{f(x+h)-f(x)}{h} \quad \longrightarrow c\,f'(x) \text{ as } h \to 0$$

$f(x)$	$f'(x)$
$x^2 + x^3$	$2x + 3x^2$
$3x^4$	$3(4x^3) = 12x^3$
$3x^2 - x$	$3(2x) - (1) = 6x - 1$

Example: Find $p'(x)$ if $p(x) = x^5 + 2x^3 - 5x^2 + 3x + 2$.

$$\begin{aligned} p'(x) &= 5x^4 + 2(3x^2) - 5(2x) + 3(1) + 0 \\ &= 5x^4 + 6x^2 - 10x + 3 \end{aligned}$$

Example: Let $p(x) = 2x^3 - 3x^2 - 12x + 5$.

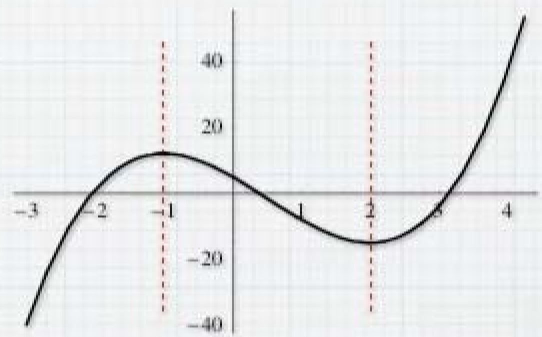
Find all x where $p'(x) > 0$ and all x where $p'(x) < 0$.

$$\begin{aligned} p'(x) &= 2(3x^2) - 3(2x) - 12(1) + 0 \\ &= 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x+1)(x-2) \end{aligned}$$

$p'(x) = 0$ at $x = -1$ and $x = 2$

$p'(x) < 0$ for $-1 < x < 2$

$p'(x) > 0$ for $x < -1$ and $x > 2$



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The square-root function $f(x) = \sqrt{x}$ has derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$
\begin{aligned}
\frac{\sqrt{x+h}-\sqrt{x}}{h}\;\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} &= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+h}+\sqrt{x}} \longrightarrow \frac{1}{2\sqrt{x}} \quad \text{as } h \to 0
\end{aligned}
$$

Example: Find and simplify $f'(x)$ if $f(x) = 3\sqrt{x} - x^2$.

$$
\begin{aligned}
f'(x) &= 3\left(\frac{1}{2\sqrt{x}}\right) - 2x \\
&= \frac{3}{2\sqrt{x}} - 2x = \frac{3}{2\sqrt{x}} - \frac{4x\sqrt{x}}{2\sqrt{x}} = \frac{3-4x\sqrt{x}}{2\sqrt{x}}
\end{aligned}
$$

Product Rule

3. $\boxed{(f\,g)'(x) = f'(x)g(x) + f(x)g'(x)}$

$$f(x)g(x) \qquad\qquad f'(x)g(x) + f(x)g'(x)$$

$$x\sqrt{x} \qquad\qquad (1)\ \sqrt{x} \ +\ x\ \frac{1}{2\sqrt{x}} \ = \sqrt{x} + \frac{1}{2}\sqrt{x} \ = \frac{3}{2}\sqrt{x}$$

$$p(x) = f(x)g(x)$$

$$\frac{p(x+h)-p(x)}{h} = \frac{\big(f(x+h)g(x+h) - f(x)g(x+h)\big) + \big(f(x)g(x+h) - f(x)g(x)\big)}{h}$$

$$= \frac{(f(x+h)-f(x))g(x+h) + f(x)(g(x+h)-g(x))}{h}$$

$$= \frac{f(x+h)-f(x)}{h}\,g(x+h) \ + f(x)\,\frac{g(x+h)-g(x)}{h}$$

$$\longrightarrow f'(x)g(x) + f(x)g'(x) \text{ as } h \to 0$$

Example: Let $f(x) = (x^2 - 1)\sqrt{x}$. Find and simplify $f'(x)$.

$$\begin{aligned} f'(x) &= (2x - 0)\sqrt{x} + (x^2 - 1)\,\frac{1}{2\sqrt{x}} \\ &= 2x\sqrt{x}\,\frac{2\sqrt{x}}{2\sqrt{x}} + \frac{x^2 - 1}{2\sqrt{x}} \\ &= \frac{4x^2}{2\sqrt{x}} + \frac{x^2 - 1}{2\sqrt{x}} \quad = \frac{5x^2 - 1}{2\sqrt{x}} \end{aligned}$$

Example: Let $f(x) = (x^2 - x)(x^3 + x^2 - x + 1)$. Find $f'(1)$.

$$f'(x) = (2x - 1)(x^3 + x^2 - x + 1) + (x^2 - x)(3x^2 + 2x - 1 + 0)$$

$$f'(1) = (2 - 1)(1 + 1 - 1 + 1) + (1 - 1)(3 + 2 - 1) \; = 2$$

Example: Let $f(x) = (x^3 + x - 3)^2$. Find and simplify $f'(x)$.

$$f(x) = (x^3 + x - 3)(x^3 + x - 3)$$

$$f'(x) = (3x^2+1)(x^3+x-3) + (x^3+x-3)(3x^2+1)$$

$$= 2(x^3+x-3)(3x^2+1)$$

Example: Let $f(x) = (g(x))^2$. Find $f'(x)$ in terms of $g(x)$ and $g'(x)$.

$$f(x) = g(x)g(x)$$

$$f'(x) = g'(x)g(x) + g(x)g'(x)$$

$$= 2\,g(x)g'(x)$$

Reciprocal Rule

4. $$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g(x)^2}$$

$$f(x) = \frac{1}{g(x)}$$

$$f(x)g(x) = 1$$

$$f'(x)g(x) + f(x)g'(x) = 0$$

$$f'(x)g(x) = -f(x)g'(x)$$

$$= -\frac{1}{g(x)}g'(x)$$

$$f'(x) = -\frac{g'(x)}{g(x)^2}$$

Example: Let $f(x) = \dfrac{1}{x^2+1}$. Find $f'(x)$.

$$f'(x) = -\frac{2x}{(x^2+1)^2}$$

Example: Let $f(x) = \dfrac{1}{\sqrt{x}}$.

Find and simplify $f'(x)$.

$$f'(x) = -\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = -\frac{1}{2x\sqrt{x}}$$

$$= -\frac{1}{2x^{3/2}}$$

$$= -\frac{\sqrt{x}}{2x^2}$$

Reciprocal Powers

If m is a positive integer, then $f(x) = \dfrac{1}{x^m}$ has derivative $f'(x) = -\dfrac{m}{x^{m+1}}$.

$$f'(x) = -\frac{m\,x^{m-1}}{x^{2m}} = -m\,x^{-m-1} = -\frac{m}{x^{m+1}}$$

If $f(x) = x^{-m}$, then $f'(x) = -m\,x^{-m-1}$.

If n is *any* integer, then $f(x) = x^n$ has derivative $f'(x) = n\,x^{n-1}$.

$$f(x) = \frac{1}{x^3} \qquad\qquad f(x) = x^{-3}$$

$$f'(x) = -\frac{3x^2}{x^6} = -\frac{3}{x^4} \qquad\qquad f'(x) = -3x^{-4} = -\frac{3}{x^4}$$

Quotient Rule

5. $$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$q(x) = \frac{f(x)}{g(x)} = f(x)\frac{1}{g(x)}$$

$$q'(x) = f'(x)\frac{1}{g(x)} + f(x)\left(-\frac{g'(x)}{g(x)^2}\right)$$

$$= \frac{f'(x)g(x)}{g(x)^2} - \frac{f(x)g'(x)}{g(x)^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Example: Let $f(x) = \dfrac{2x-1}{x+1}$. Find and simplify $f'(x)$.

$$f'(x) = \frac{(2)(x+1) - (2x-1)(1)}{(x+1)^2}$$

$$= \frac{2x+2-2x+1}{(x+1)^2} = \frac{3}{(x+1)^2}$$

Example: Let $f(x) = \dfrac{x^3}{x^2+1}$. Find and simplify $f'(x)$.

$$f'(x) = \frac{(3x^2)(x^2+1) - x^3(2x)}{(x^2+1)^2}$$

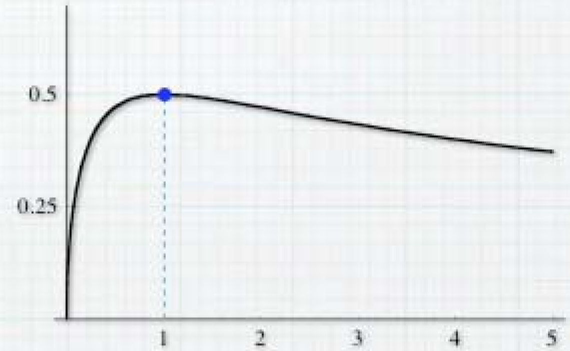
$$= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}$$

Example: Let $f(x) = \dfrac{\sqrt{x}}{x+1}$. Find and simplify $f'(x)$.

$$f'(x) = \frac{\left(\frac{1}{2\sqrt{x}}\right)(x+1) - \sqrt{x}\,(1)}{(x+1)^2} \quad \frac{2\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{x+1 - 2x}{(x+1)^2\, 2\sqrt{x}}$$

$$= \frac{1-x}{2\sqrt{x}\,(x+1)^2}$$



Summary

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(c\,f)'(x) = c\,f'(x)$
3. $(f\,g)'(x) = f'(x)g(x) + f(x)g'(x)$
4. $\left(\dfrac{1}{g}\right)'(x) = -\dfrac{g'(x)}{g(x)^2}$
5. $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

Any constant function $f(x) = c$ has derivative $f'(x) = 0$.

The identity function $f(x) = x$ has derivative $f'(x) = 1$.

If n is *any* integer, then $f(x) = x^n$ has derivative $f'(x) = n\,x^{n-1}$.

The square-root function $f(x) = \sqrt{x}$ has derivative $f'(x) = \dfrac{1}{2\sqrt{x}}$.

Derivatives of Trigonometric Functions

$$f(x) = \sin x$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x\ \cos h + \cos x\ \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1) + \cos x\ \sin h}{h} \\ &= \lim_{h\to 0} \frac{\sin x\,(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x\ \sin h}{h} \\ &= \sin x\ \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x\ \lim_{h\to 0} \frac{\sin h}{h} \\ &= \sin x\ (0) + \cos x\ (1) \\ &= \cos x \end{aligned}$$

$$f(x) = \cos x$$

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \, \cos h - \sin x \, \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\cos x \, (\cos h - 1) - \sin x \, \sin h}{h} \\
&= \lim_{h \to 0} \frac{\cos x \, (\cos h - 1)}{h} - \lim_{h \to 0} \frac{\sin x \, \sin h}{h} \\
&= \cos x \, \boxed{\lim_{h \to 0} \frac{\cos h - 1}{h}} - \sin x \, \boxed{\lim_{h \to 0} \frac{\sin h}{h}} \\
&= \cos x \, (0) - \sin x \, (1) \\
&= -\sin x
\end{aligned}
$$

The derivative of $\sin x$ is $\cos x$.

The derivative of $\cos x$ is $-\sin x$.

Example: Let $f(x) = 3\cos x - 2\sin x$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= 3(-\sin x) - 2\cos x \\ &= -3\sin x - 2\cos x \end{aligned}$$

Example: Let $y(t) = \sin t\ \cos t$. Find $y'(t)$.

$$\begin{aligned} y'(t) &= \cos t \cos t \quad + \quad \sin t\,(-\sin t) \\ &= \cos^2 t - \sin^2 t \end{aligned}$$

Example: Let $g(\theta) = \dfrac{\cos\theta}{1+\sin\theta}$. Find and simplify $g'(\theta)$.

$$
\begin{aligned}
g'(\theta) &= \frac{-\sin\theta\,(1+\sin\theta) - \cos\theta\;\cos\theta}{(1+\sin\theta)^2} \\
&= \frac{-\sin\theta - \sin^2\theta - \cos^2\theta}{(1+\sin\theta)^2} \\
&= \frac{-\sin\theta - 1}{(1+\sin\theta)^2} \\
&= -\frac{1+\sin\theta}{(1+\sin\theta)^2} \quad = -\frac{1}{1+\sin\theta}
\end{aligned}
$$

The other trig functions

$$f(x) = \tan x = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{\cos x \, \cos x - \sin x \, (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

$$f'(x) = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x} = -\csc^2 x$$

$$f(x) = \sec x = \frac{1}{\cos x}$$

$$f'(x) = -\frac{-\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \frac{\sin x}{\cos x} = \sec x \, \tan x$$

$$f(x) = \csc x = \frac{1}{\sin x}$$

$$f'(x) = -\frac{\cos x}{\sin^2 x}$$

$$= \frac{-1}{\sin x} \frac{\cos x}{\sin x} = -\csc x \, \cot x$$

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x\ \tan x$
$\csc x$	$-\csc x\ \cot x$

The derivative of the cofunction of f
is the negative of the cofunction of f'.

Example: Let $f(x) = \sec x \, \tan x$. Find and simplify $f'(x)$.

$$f'(x) = \sec x \, \tan x \, \tan x \; + \; \sec x \, \sec^2 x$$

$$= \sec x \,(\tan^2 x \, + \sec^2 x)$$

Example: Let $f(x) = \dfrac{\sec x}{1 + \tan x}$. Find and simplify $f'(x)$.

$$f'(x) = \frac{\sec x \tan x \,(1 + \tan x) - \sec x \sec^2 x}{(1 + \tan x)^2}$$

$$= \frac{\sec x \,(\tan x + \tan^2 x - \sec^2 x)}{(1 + \tan x)^2}$$

$$\tan^2 + 1 = \sec^2 x$$
$$\tan^2 x - \sec^2 x = -1$$

$$= \frac{\sec x \,(\tan x - 1)}{(1 + \tan x)^2}$$

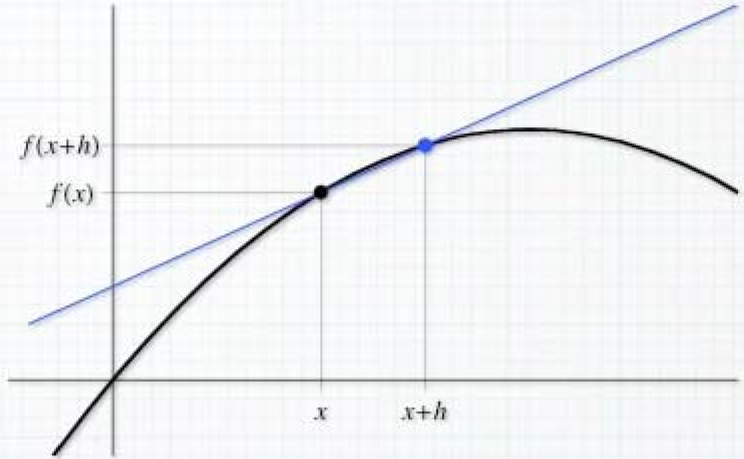
Leibniz Notation

and

The Chain Rule

Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Leibniz's $\frac{d}{dx}$ Notation

dependent variable

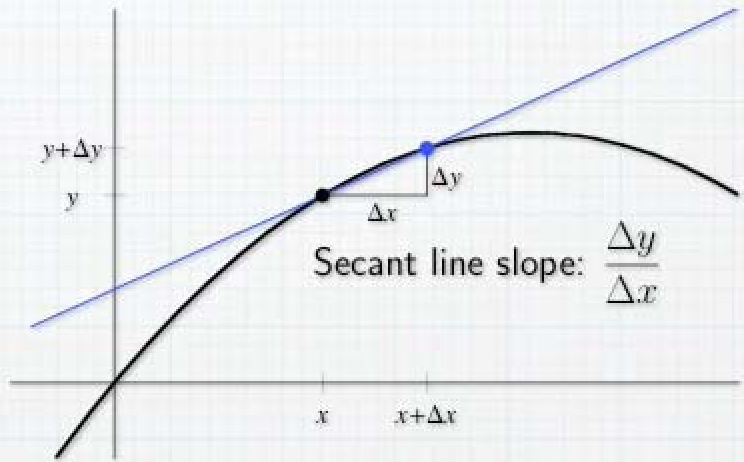
$$y = f(x)$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = f'(x)$$

slope at $x = a$

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}$$



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The $\frac{d}{dx}$ Operator

$$\frac{d}{dx}f(x) = f'(x)$$

$\frac{d}{dx}(expression) =$ the derivative of $expression$ with respect to x

$$\frac{d}{dx}x^n = n\,x^{n-1} \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

Product Rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Product Rule again

$$\frac{d}{dx}(u\,v) = \frac{du}{dx}\,v + u\,\frac{dv}{dx}$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\,v - u\,\frac{dv}{dx}}{v^2}$$

The Chain Rule

Differentiation of the composition of two functions f and u

$$\frac{d}{dx} f(u(x)) = f'(u(x))\, u'(x)$$

$$(f \circ u)' = (f' \circ u)\, u'$$

Examples

"outside" function

$$\frac{d}{dx}(x^2 + x)^3 = 3(\underbrace{x^2 + x}_{\text{"inside" function}})^2\,(2x + 1)$$

$$f(u) = (u)^3$$
$$f'(u) = 3(u)^2$$

$$\frac{d}{dx}\sqrt{x^2 + 1} = \frac{1}{2\sqrt{x^2+1}}\; 2x = \frac{x}{\sqrt{x^2+1}}$$

"inside" function

$$f(u) = \sqrt{u}$$
$$f'(u) = \frac{1}{2\sqrt{u}}$$

More Examples

"outside" function

$$\frac{d}{dx}\sin(2\pi x) = \cos(2\pi x)\,2\pi = 2\pi\cos(2\pi x)$$

$$f(u) = \sin(u)$$
$$f'(u) = \cos(u)$$

"outside" function

$$\frac{d}{dx}\cos^2 x = 2(\cos x)(-\sin x) = -2\cos x\,\sin x$$

$(\cos x)^2$

$$f(u) = u^2$$
$$f'(u) = 2u$$

"outside" function

$$\frac{d}{dx}\frac{1}{(1+\sin x)^2} = -\frac{2}{(1+\sin x)^3}\cos x$$
$$= -\frac{2\cos x}{(1+\sin x)^3}$$

$$f(u) = \frac{1}{u^2} = u^{-2}$$
$$f'(u) = -2u^{-3} = -\frac{2}{u^3}$$

Compositions of 3 functions

$$\frac{d}{dx} f(u(v(x))) = f'(u(v(x)))\, \frac{d}{dx} u(v(x))$$

$$= f'(u(v(x)))\; u'(v(x))\; v'(x)$$

Example

$$\frac{d}{dx}\left(\sin^2(3x)\right) = 2(\sin(3x))\, \frac{d}{dx} \sin(3x)$$

$$= 2(\sin(3x))\;\; \cos(3x)\; \frac{d}{dx}(3x)$$

$$= 2(\sin(3x))\; \cos(3x)\; 3$$

$$= 6 \sin(3x) \cos(3x)$$

$$(\sin(3x))^2$$

$$f(u) = u^2$$
$$f'(u) = 2u$$
$$u(v) = \sin v$$
$$u'(v) = \cos v$$
$$v(x) = 3x$$
$$v'(x) = 3$$

Example

$$\frac{d}{dx}\left(\frac{1}{1+\sqrt{x^2+1}}\right) = -\frac{1}{(1+\sqrt{x^2+1})^2}\ \frac{d}{dx}(1+\sqrt{x^2+1})$$

$$= -\frac{1}{(1+\sqrt{x^2+1})^2}\ \frac{1}{2\sqrt{x^2+1}}\ \frac{d}{dx}(x^2+1)$$

$$= -\frac{1}{(1+\sqrt{x^2+1})^2}\ \frac{1}{\cancel{2}\sqrt{x^2+1}}\ \cancel{2}x$$

$$= -\frac{x}{(1+\sqrt{x^2+1})^2\sqrt{x^2+1}}$$

$$f(u) = \frac{1}{u} = u^{-1}$$

$$f'(u) = -u^{-2} = -\frac{1}{u^2}$$

$$u(v) = 1+\sqrt{v}$$

$$u'(v) = \frac{1}{2\sqrt{v}}$$

$$v(x) = x^2+1$$

$$v'(x) = 2x$$

The Chain Rule in Leibniz Notation

Compositions of 2 functions

$$\frac{d}{dx}f(u(x)) = f'(u(x))\,u'(x)$$

$$\frac{d}{dx}f(u) = f'(u)\,\frac{du}{dx}$$

$$y = f(u) \qquad \frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

Compositions of 3 functions

$$\frac{d}{dx}f(u(v(x))) = f'(u(v(x)))\,u'(v(x))\,v'(x)$$

$$x \longmapsto v \longmapsto u \longmapsto y$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}}\,\frac{\cancel{du}}{\cancel{dv}}\,\frac{\cancel{dv}}{dx}$$

Example
$$f(x) = \frac{1}{1+\sqrt{x^2+1}}$$

$$x \longmapsto x^2+1 \longmapsto 1+\sqrt{x^2+1} \longmapsto y = \frac{1}{1+\sqrt{x^2+1}}$$

$$x \longmapsto v = x^2+1 \longmapsto u = 1+\sqrt{v} \longmapsto y = \frac{1}{u}$$

$$\frac{dv}{dx} = 2x \qquad \frac{du}{dv} = \frac{1}{2\sqrt{v}} \qquad \frac{dy}{du} = -\frac{1}{u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{u^2}\ \frac{1}{2\sqrt{v}}\ 2x \\ &= -\frac{1}{(1+\sqrt{v})^2}\ \frac{1}{2\sqrt{x^2+1}}\ 2x \\ &= -\frac{1}{(1+\sqrt{x^2+1})^2}\ \frac{1}{2\sqrt{x^2+1}}\ 2x\end{aligned}$$

More Examples

$$\frac{d}{dx}\left((x^2+1)^3(x^3+1)^2\right) = 3(x^2+1)^2\, 2x\,(x^3+1)^2 + (x^2+1)^3\, 2(x^3+1)^1\, 3x^2$$

$$= 6x\,(x^2+1)^2\,(x^3+1)\,\big((x^3+1)+(x^2+1)x\big)$$

$$= 6x\,(x^2+1)^2(x^3+1)(\,2x^3+x+1)$$

$$\frac{d}{dx}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{(1)\sqrt{x^2+1} - x\left(\frac{d}{dx}\sqrt{x^2+1}\right)}{x^2+1}$$

$$= \frac{\sqrt{x^2+1} - x\left(\frac{1}{2\sqrt{x^2+1}}\,2x\right)}{x^2+1}\quad \frac{\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}}$$

One More Example

$$\frac{d}{dx}\sqrt{\frac{2x}{x^2+3}} = \frac{1}{2\sqrt{\frac{2x}{x^2+3}}}\ \frac{d}{dx}\left(\frac{2x}{x^2+3}\right)$$

$$= \frac{1}{2}\sqrt{\frac{x^2+3}{2x}}\ \left(\frac{2(x^2+3)-2x\,2x}{(x^2+3)^2}\right)$$

$$= \frac{1}{2}\sqrt{\frac{x^2+3}{2x}}\ \left(\frac{6-2x^2}{(x^2+3)^2}\right) = \boxed{\frac{3-x^2}{\sqrt{2x}\ (x^2+3)^{3/2}}}$$

Higher-Order Derivatives

Repeated Differentiation

$$y = f(x)$$

(First) derivative $$\frac{dy}{dx} = f'(x)$$

Second derivative $$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2} = f''(x)$$

Third derivative $$\frac{d}{dx}\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = f'''(x)$$

Fourth derivative $$\frac{d}{dx}\frac{d^3y}{dx^3} = \frac{d^4y}{dx^4} = f^{(4)}(x)$$

nth derivative $$\frac{d^ny}{dx^n} = f^{(n)}(x)$$

Example

$$y = x^3 - x^2$$
$$\frac{dy}{dx} = 3x^2 - 2x$$
$$\frac{d^2y}{dx^2} = 6x - 2$$
$$\frac{d^3y}{dx^3} = 6$$
$$\frac{d^4y}{dx^4} = 0$$
$$\frac{d^ny}{dx^n} = 0 \text{ for all } n \geq 4$$

Example

$$f(t) = \sin 2t$$
$$f'(t) = 2\cos 2t$$
$$f''(t) = -4\sin 2t$$
$$f'''(t) = -8\cos 2t$$
$$f^{(4)}(t) = 16\sin 2t$$

Example

$$w(x) = x^{1/2} = \sqrt{x}$$
$$w'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$
$$w''(x) = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}}$$
$$w'''(x) = \frac{3}{8}x^{-5/2} = \frac{3}{8\sqrt{x^5}}$$

Rectilinear Motion

Velocity and Acceleration

Velocity

Imagine a particle moving along a straight-line path in some way.

Let t be a variable representing the time elapsed since some reference time (when $t = 0$).

Let $s(t)$ be the position of the particle at time t, measured relative to some reference point (where $s = 0$).

Average velocity over a time interval $a \leq t \leq b$ is defined to be the change in position divided by the change in time:

$$v_{\text{avg}} = \frac{\Delta s}{\Delta t} = \frac{s(b) - s(a)}{b - a}$$

(Instantaneous) **velocity** at time t is defined to be the limit of the average velocities over intervals $[t,\, t + h]$ as $h \to 0$. Thus velocity is the derivative of position:

$$v(t) = s'(t) = \lim_{h \to 0} \frac{s(t + h) - s(t)}{h}$$

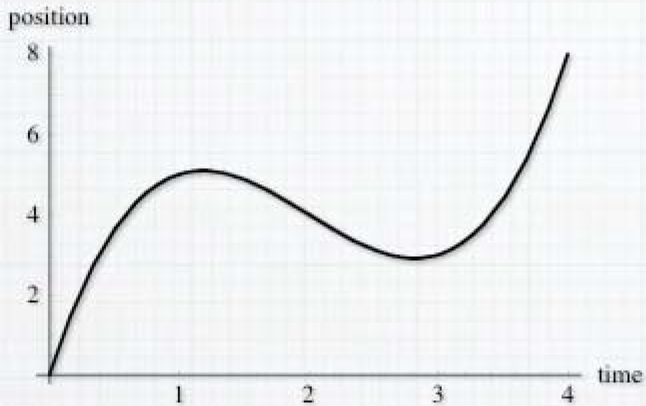
Example

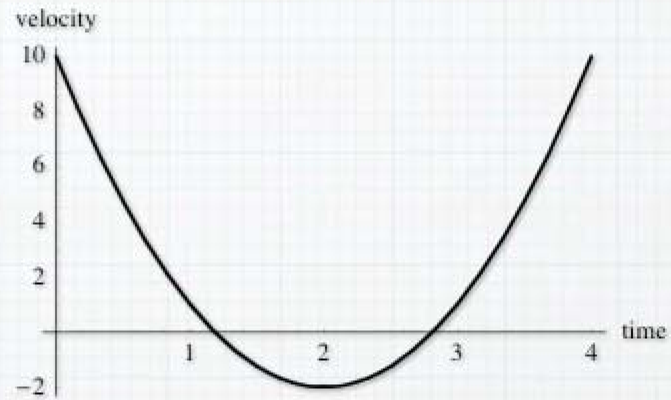
position

$$s(t) = (t-2)^3 - 2t + 8$$

velocity

$$v(t) = s'(t) = 3(t-2)^2 - 2$$





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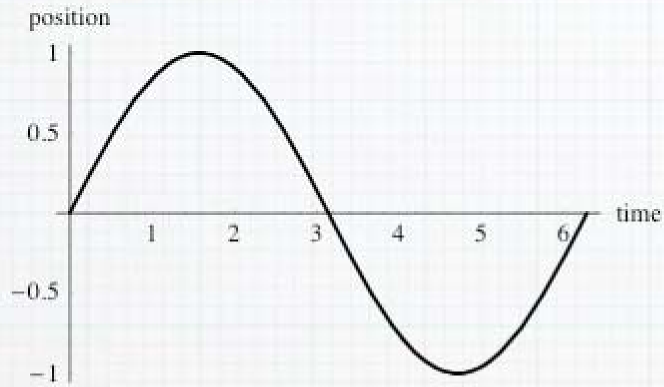
Example

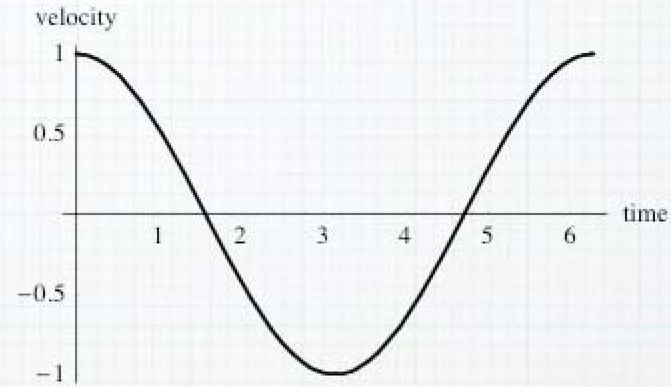
position

$$s(t) = \sin t$$



velocity

$$v(t) = s'(t) = \cos t$$



Acceleration

Just as velocity is the rate of change in position per unit time, **acceleration** is the rate of change of velocity per unit time.

Thus acceleration is the derivative of the derivative — or the *second derivative* — of position.

$$v(t) = s'(t)$$

$$a(t) = v'(t) = s''(t)$$

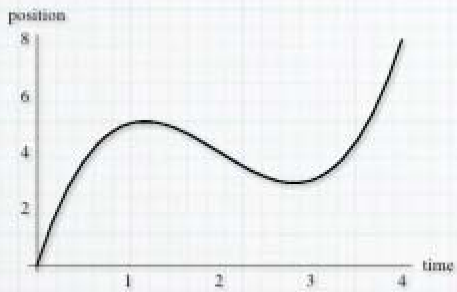
$$a(t) = \frac{dv}{dt} = \frac{d}{dt}\frac{ds}{dt} = \frac{d^2s}{dt^2}$$

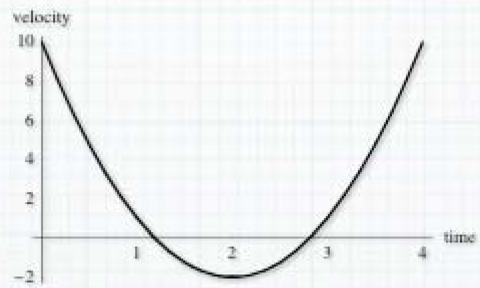
Newton's Second Law relates acceleration, mass, and force:

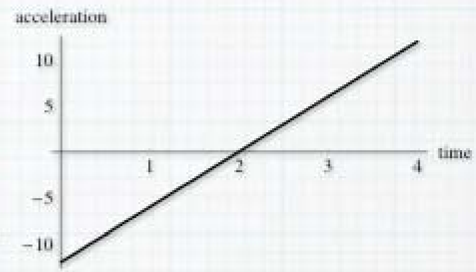
$$F = m\,a$$

Example

position	velocity	acceleration
$s(t) = (t-2)^3 - 2t + 8$	$v(t) = 3\,(t-2)^2 - 2$	$a(t) = 6\,(t-2)$







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Free fall near the Earth's surface

Let $h(t)$ be the height of an object in free fall near the surface of the Earth. The gravitational force on the object — and thus its acceleration — will be roughly constant. This acceleration due to gravity is $-g$, where the value of g is roughly 32 ft/sec^2 or 9.8 m/sec^2.

If we assume air resistance is negligible, then $h(t)$ is this quadratic function:

$$\boxed{h(t) = -\frac{1}{2}\, g\, t^2 + v_0 t + h_0}$$

Initial height
$h_0 = h(0)$

Velocity is linear: $v(t) = -g\,t + v_0$

Initial velocity
$v_0 = v(0)$

Acceleration is $-g$: $a(t) = -g$

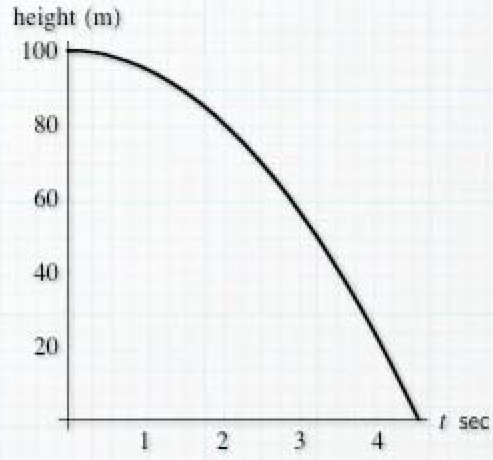
Example

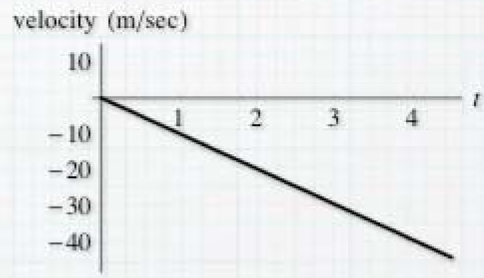
A ball dropped from a height of 100 m.

$$h_0 = 100$$
$$v_0 = 0$$
$$g = 9.8$$

$$h(t) = -4.9\,t^2 + 100 \quad \text{for} \quad 0 \le t \le 4.5$$
$$v(t) = -9.8\,t$$

$$h(t) = 0 \ \text{ when } \ 4.9\,t^2 = 100$$
$$t = \sqrt{100/4.9} \approx 4.5$$





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Example

A ball is thrown straight up from a height of 6 ft with initial velocity 64 ft/sec. How high does the ball go?

Write the height function.

$$h(t) = -\frac{1}{2}\, g\, t^2 + v_0 t + h_0 \qquad g = 32 \text{ ft/sec}^2$$

$$= -16\, t^2 + 64\, t + 6$$

Find the velocity function.

$$v(t) = -32\, t + 64 = -32(t-2)$$

Find out *when* the velocity is zero.

$$v(t) = 0 \text{ when } t = 2$$

Compute the height at the time when the velocity is zero.

$$\therefore \text{ the maximum height is } h(2) = -16 \cdot 2^2 + 64 \cdot 2 + 6$$

$$= 70 \text{ ft.}$$

Example

Show that, for all t while an object is in free fall, its height and velocity satisfy

$$\boxed{2gh(t) + v(t)^2 = 2gh_0 + v_0^2}$$

$$h(t) = -\frac{1}{2}g\,t^2 + v_0 t + h_0 \qquad\qquad v(t) = -g\,t + v_0$$

$$\begin{aligned} 2\,g\,h(t) + v(t)^2 &= -g^2\,t^2 + 2\,g\,v_0\,t + 2\,g\,h_0 + (-g\,t + v_0)^2 \\ &= \cancel{-g^2\,t^2} + \cancel{2\,g\,v_0}\,t + 2\,g\,h_0 + \cancel{g^2\,t^2} - \cancel{2\,g\,v_0}\,t + v_0^2 \\ &= 2\,g\,h_0 + v_0^2 \end{aligned}$$

Remark

Conservation of Energy

$$m\,g\,h(t) + \frac{1}{2}m\,v(t)^2 = m\,g\,h_0 + \frac{1}{2}m\,v_0^2$$

From the formula

$$2gh(t) + v(t)^2 = 2gh_0 + v_0^2$$

we can easily derive formulas for maximum height and impact velocity.

Since $h(t) = h_{\max}$ when $v(t) = 0$,

$$2gh_{\max} + 0 = 2gh_0 + v_0^2$$

$$h_{\max} = h_0 + \frac{1}{2g} v_0^2$$

Let v_{fin} be the velocity $v(t)$ at the instant when $h(t) = 0$. Then

$$0 + v_{\text{fin}}^2 = 2gh_0 + v_0^2$$

$$v_{\text{fin}} = -\sqrt{2gh_0 + v_0^2}$$

Implicit Differentiation

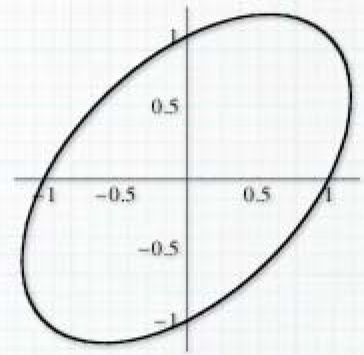
&

Derivatives of Rational Powers

Example

Suppose that x and y satisfy the equation

$$\boxed{x^2 - x\,y + y^2 = 1,}$$



whose graph is an ellipse. Our goal is to describe the slope of the curve at each point (x, y).
Just as in the case where y is a function of x,
we can define $\frac{dy}{dx}$ to be the slope of the graph at a point (x, y).

To find $\frac{dy}{dx}$, we proceed as follows:

(1) Differentiate each side of the equation with respect to x.

$$\frac{d}{dx}\left(x^2 - x\,y + y^2\right) = \frac{d}{dx}(1)$$

$$\frac{d}{dx}x^2 - \frac{d}{dx}(x\,y) + \frac{d}{dx}(y^2) = 0$$

$$2x - \left(\overset{\text{product rule}}{(1)y + x\frac{dy}{dx}}\right) + \overset{\text{chain rule}}{2y\frac{dy}{dx}} = 0$$

$$2x - y + (2y - x)\frac{dy}{dx} = 0$$

(2) Solve for $\frac{dy}{dx}$.

$$(2y - x)\frac{dy}{dx} = y - 2x$$

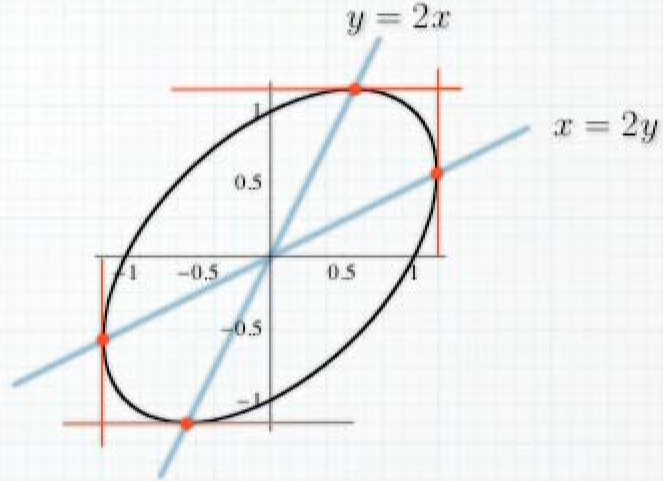
$$\boxed{\frac{dy}{dx} = \frac{y - 2x}{2y - x}}$$

This is the process known as **implicit differentiation**.

$$x^2 - x\,y + y^2 = 1$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Find the points on the ellipse where the tangent line is horizontal or vertical.



$$\frac{dy}{dx} = 0 \quad \text{if} \quad y = 2x$$

$$x^2 - x\,(2x) + (2x)^2 = 1$$

$$3x^2 = 1$$

$$x = \pm\,1/\sqrt{3}$$

$$\therefore \quad \frac{dy}{dx} = 0 \ \text{ at } \left(\pm\tfrac{1}{\sqrt{3}}, \pm\tfrac{2}{\sqrt{3}}\right).$$

$$\frac{dy}{dx} \ \text{ is undefined if } x = 2y$$

$$(2y)^2 - (2y)\,y + y^2 = 1$$

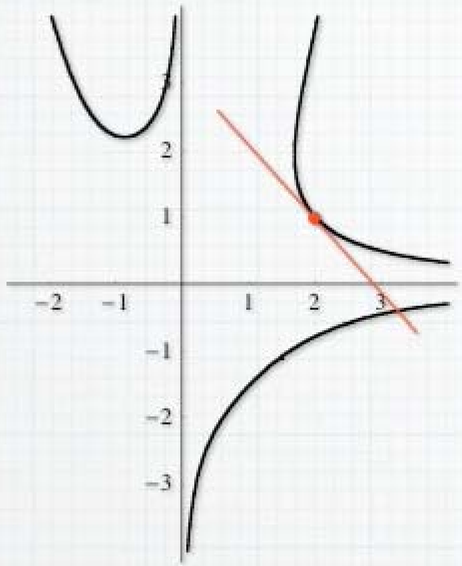
$$3y^2 = 1$$

$$y = \pm\,1/\sqrt{3}$$

$$\therefore \quad \frac{dy}{dx} \ \text{ is undefined at } \left(\pm\tfrac{2}{\sqrt{3}}, \pm\tfrac{1}{\sqrt{3}}\right).$$

Example

Find the slope of the graph of $x^3y^2 = x\,y^3 + 6$ at the point $(2,1)$.



$$x^3y^2 = x\,y^3 + 6$$

$$\frac{d}{dx}(x^3y^2) = \frac{d}{dx}(x\,y^3) + 0$$

$$3x^2y^2 + x^3 \cdot 2y\,\frac{dy}{dx} = (1)\,y^3 + x \cdot 3y^2\,\frac{dy}{dx}$$

$$2x^3y\,\frac{dy}{dx} - 3x\,y^2\,\frac{dy}{dx} = y^3 - 3x^2y^2$$

$$x\,y\,(2x^2 - 3y)\,\frac{dy}{dx} = y^2(y - 3x^2)$$

$$\frac{dy}{dx} = \frac{y\,(y - 3x^2)}{x\,(2x^2 - 3y)}$$

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{1\,(1 - 3 \cdot 2^2)}{2\,(2 \cdot 2^2 - 3 \cdot 1)} = -\frac{11}{10}$$

Radicals and Fractional Powers

Let n be a nonzero integer. We want first to find $\frac{dy}{dx}$ if $y = x^{1/n}$.

$$y^n = x$$

$$n\,y^{n-1}\,\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{n}\,y^{1-n} = \frac{1}{n}\left(x^{1/n}\right)^{1-n} = \frac{1}{n}\,x^{\frac{1-n}{n}}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{n}\,x^{\frac{1}{n}-1}}$$

Same operation as the familiar power rule.

Examples

$$\frac{d}{dx}\,x^{1/2} = \frac{1}{2}\,x^{-1/2} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\,x^{1/3} = \frac{1}{3}\,x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$\frac{d}{dx}\,x^{-1/2} = -\frac{1}{2}\,x^{-3/2} = -\frac{1}{2x^{3/2}}$$

Now let m and n be nonzero integers. We want to find $\frac{dy}{dx}$ if $y = x^{m/n}$.

$$y = x^{m/n} = \left(x^{1/n}\right)^m$$

$$\frac{dy}{dx} = m\left(x^{1/n}\right)^{m-1} \frac{1}{n} x^{\frac{1}{n}-1}$$ chain rule

$$= \frac{m}{n} x^{\frac{m-1}{n}+\frac{1}{n}-1}$$

$$\frac{dy}{dx} = \frac{m}{n} x^{\frac{m}{n}-1}$$ Again, the same operation as the familiar power rule.

For *any* rational exponent p, $\frac{d}{dx} x^p = p\, x^{p-1}$.

Examples

$$\frac{d}{dx} x^{2/3} = \frac{2}{3} x^{-1/3} = \frac{2}{3\, x^{1/3}}$$

$$\frac{d}{dx} x^{-2/3} = -\frac{2}{3} x^{-5/3} = -\frac{2}{3\, x^{5/3}}$$

Example: Find $\dfrac{dy}{dx}$ if $2x^{5/2}+7y^{2/7}=9xy$, and compute the slope of the graph at $(1,1)$.

$$2x^{5/2}+7y^{2/7}=9xy$$

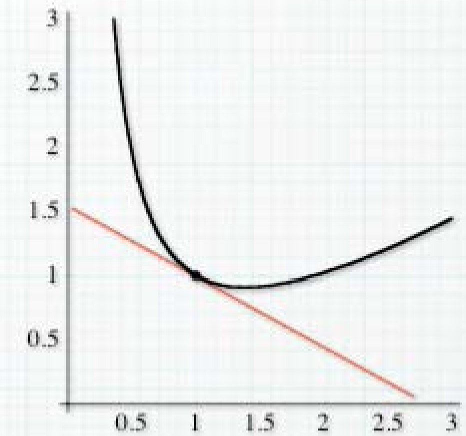
$$5x^{3/2}+2y^{-5/7}\frac{dy}{dx}=9\left((1)y+x\frac{dy}{dx}\right)$$

$$2y^{-5/7}\frac{dy}{dx}-9x\frac{dy}{dx}=9y-5x^{3/2}$$

$$\left(2y^{-5/7}-9x\right)\frac{dy}{dx}=9y-5x^{3/2}$$

$$\frac{dy}{dx}=\frac{9y-5x^{3/2}}{2y^{-5/7}-9x}\;\frac{y^{5/7}}{y^{5/7}}$$

$$\frac{dy}{dx}=\frac{\left(9y-5x^{3/2}\right)y^{5/7}}{2-9xy^{5/7}}$$



$$\therefore\quad \left.\frac{dy}{dx}\right|_{(1,1)}=\frac{9-5}{2-9}=-\frac{4}{7}$$

Example: Find and simplify $f'(t)$ if $f(t) = \dfrac{t}{(t^2+1)^{1/3}}$.

$$f'(t) = \frac{(1)(t^2+1)^{1/3} - t\,\frac{1}{3}(t^2+1)^{-2/3}\,2t}{(t^2+1)^{2/3}} \quad \frac{3\,(t^2+1)^{2/3}}{3\,(t^2+1)^{2/3}}$$

$$= \frac{3\,(t^2+1) - 2t^2}{3\,(t^2+1)^{4/3}} = \frac{t^2+3}{3\,(t^2+1)^{4/3}}$$

Example: Find and simplify $g'(r)$ if $g(r) = r^{2/3}(1-r)^{1/3}$.

$$g'(r) = \frac{2}{3} r^{-1/3}\,(1-r)^{1/3} + r^{2/3}\,\frac{1}{3}(1-r)^{-2/3}(-1)$$

$$= \frac{1}{3}\left(2\,r^{-1/3}(1-r)^{1/3} - r^{2/3}\,(1-r)^{-2/3}\right)\,\frac{r^{1/3}(1-r)^{2/3}}{r^{1/3}(1-r)^{2/3}}$$

$$= \frac{2\,(1-r) - r}{3r^{1/3}(1-r)^{2/3}} = \frac{2-3r}{3r^{1/3}(1-r)^{2/3}}$$

Rates of Change

&

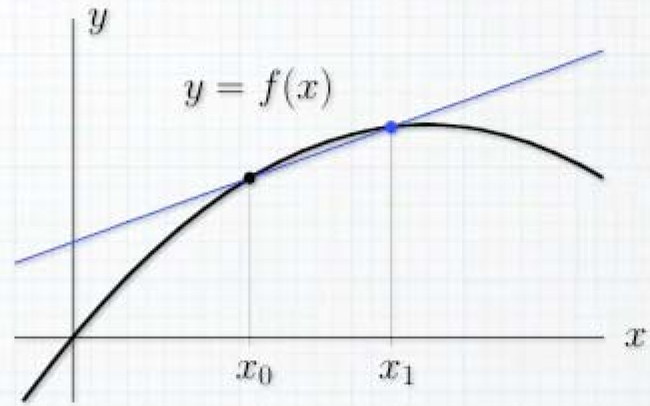
Related Rates

The Derivative as a Rate of Change

Given two quantities represented by variables x and y, with $y = f(x)$, the secant line slope

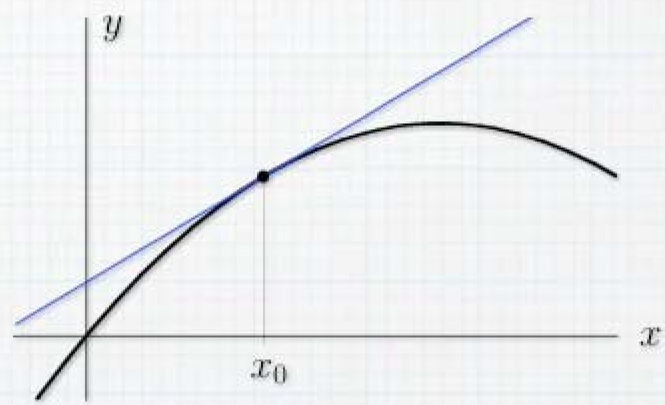
$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

is the **average rate of change** in y with respect to x over the interval $[x_0, x_1]$.

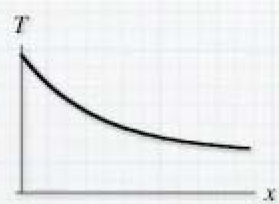
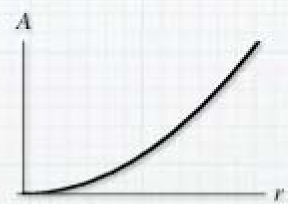
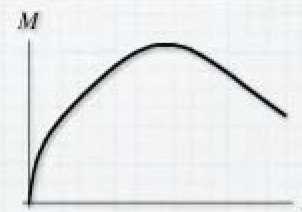


The derivative $f'(x_0)$ is the (instantaneous) **rate of change** in y with respect to x at x_0.

So the instantaneous rate of change is the limit of average rates of change as interval width approaches zero.



Examples of Rates of Change and Their Units

	water temperature	area inside a circle	automobile gas mileage
dependent variable	T in °C	A in cm^2	M in mi/gal
independent variable	depth x in ft	radius r in cm	speed s in mi/hr
rate of change	$\frac{dT}{dx}$ in C°/ft	$\frac{dA}{dr}$ in $\frac{\text{cm}^2}{\text{cm}}$ or cm	$\frac{dM}{ds}$ in $\frac{\text{mi/gal}}{\text{mi/hr}}$ or $\frac{\text{hr}}{\text{gal}}$
			

The idea of rate of change is perhaps most familiar when the quantity of interest is a function of time.

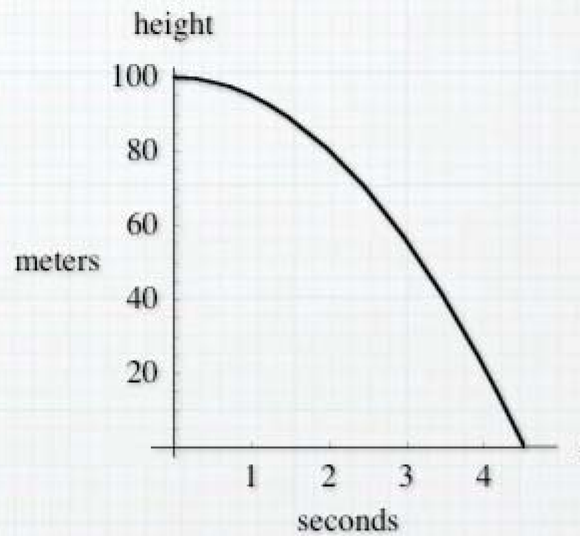
Example: A ball is dropped from the top of a building.
Let $h(t)$ be its height in meters after t seconds.
Then the derivative $h'(t)$ is the **velocity** of the ball in units of *meters per second*.

Secant line slope = *average* velocity

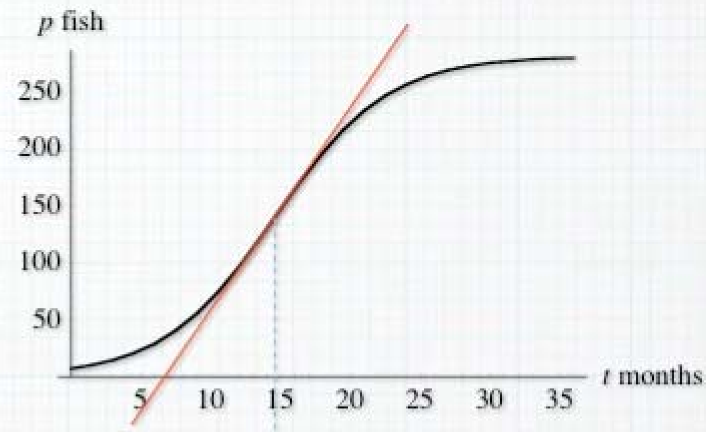
$$\frac{\Delta h}{\Delta t} \approx \frac{-100}{4.5} \approx -22 \text{ m/sec}$$

Tangent line slope = *instantaneous* velocity

For an object moving along a vertical path, **velocity** is the *rate of change* in height per unit time.

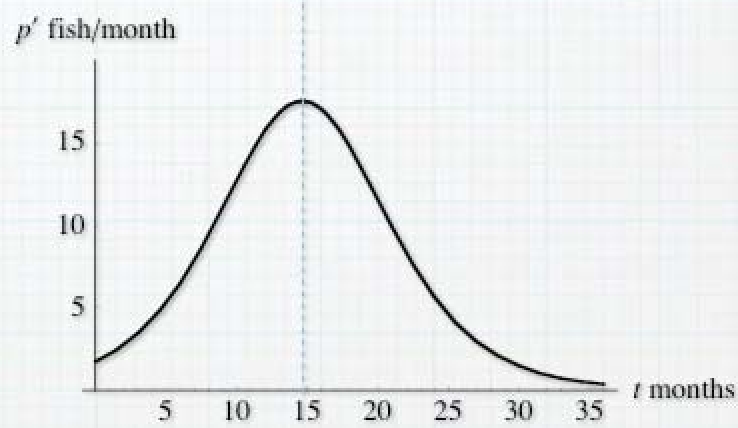


Suppose that a pond is stocked with a small number of fish. Let $p(t)$ be the number of fish in the pond t months later.



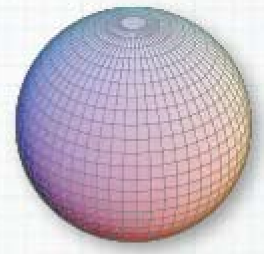
avg rate of change over 36 months $\approx 270/36$
$= 7.5$ fish/month

$p'(t)$ is the rate of change in the number of fish at time t; i.e., $p'(t)$ is the population's net growth rate.



Related Rates

Example



Let V be the volume of a sphere with radius r,

$$V = \frac{4}{3}\pi\, r^3,$$

and suppose that r and V are changing with time, with rates

$$\frac{dV}{dt} \quad \text{and} \quad \frac{dr}{dt}.$$

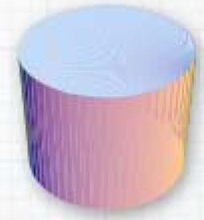
We can relate these rates by differentiating each side of the volume formula with respect to t (*implicitly*). The key is the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi\, r^3\right)\frac{dr}{dt}$$

$$\boxed{\frac{dV}{dt} = 4\,\pi\, r^2\,\frac{dr}{dt}}$$

Now, if either rate is known, we can find the other at any given r.

Example



Let V be the volume of a cylinder with radius r and height h:

$$V = \pi r^2 h,$$

and suppose that r, h, and V are changing with time, with rates

$$\frac{dV}{dt}, \ \frac{dr}{dt}, \text{ and } \frac{dh}{dt}.$$

The relationship among these rates is found by differentiating with respect to t as follows. Here, both the product rule and the chain rule are key.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\pi r^2 h\right) = \overbrace{\underbrace{\frac{d}{dt}(\pi r^2)}_{\text{chain rule}}\, h + \pi r^2 \frac{dh}{dt}}^{\text{product rule}}$$

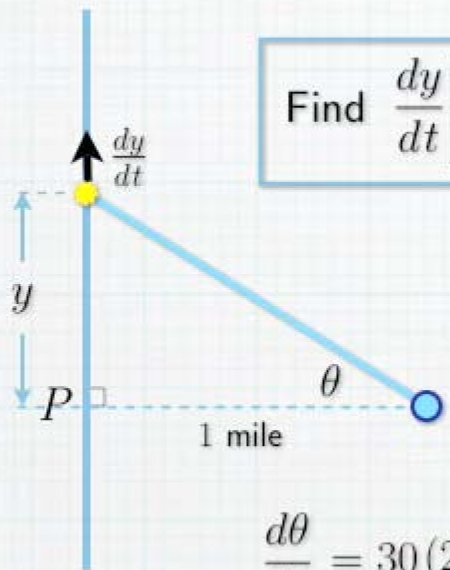
$$= 2\pi r \frac{dr}{dt}\, h + \pi r^2 \frac{dh}{dt}$$

$$\boxed{\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}}$$

Now, if two rates are known, we can find the other at any given r and h.

Example

A rotating spotlight beacon is located 1 mile from the closest point P on a straight shoreline. The spotlight makes 30 full rotations per hour. How fast does the light move past a point on the shoreline that is 1 mile from P?



Find $\left.\dfrac{dy}{dt}\right|_{\theta=\pi/4}$

$$\frac{d\theta}{dt} = 30\,(2\pi) = 60\pi \ \frac{\text{radians}}{\text{hour}}$$

$$y = \tan\theta$$

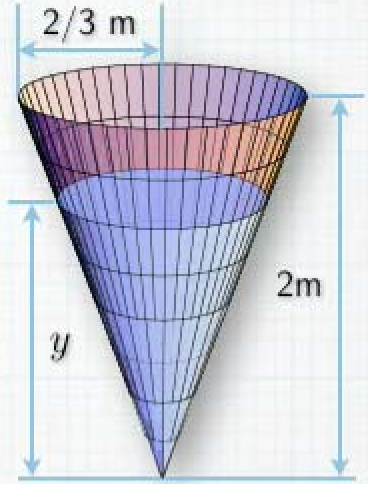
$$\frac{dy}{dt} = \frac{dy}{d\theta}\frac{d\theta}{dt}$$

$$\frac{dy}{dt} = \sec^2\theta\ \frac{d\theta}{dt}$$

$$\left.\frac{dy}{dt}\right|_{\theta=\pi/4} = \sec^2(\pi/4)\,60\pi = \frac{1}{\left(\sqrt{2}/2\right)^2}\,(60\pi) = 2\,(60\pi) = 120\,\pi \text{ mph} \approx 377 \text{ mph}$$

Example

A water tank has the shape of a cone with height 2 m and radius 2/3 m. If water is pumped into the tank at a rate of $3\,\text{m}^3/\text{hr}$, how fast is the water level rising at the instant when tank begins to overflow?



Let y = depth at time t

Given: $\dfrac{dV}{dt} = 3\ \text{m}^3/\text{hr}$

$$\boxed{\text{Find } \left.\frac{dy}{dt}\right|_{y=2}}$$

$$V = \frac{\pi}{3} r^2 y \quad \text{cone volume}$$

$$V = \frac{\pi}{3}\left(\frac{y}{3}\right)^2 y = \frac{\pi}{27} y^3$$

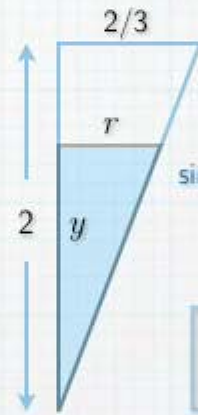
$$\frac{dV}{dt} = \frac{\pi}{9} y^2 \frac{dy}{dt} \quad \text{chain rule}$$

$$3 = \frac{\pi}{9} y^2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{27}{\pi y^2}$$

$$\left.\frac{dy}{dt}\right|_{y=2} = \frac{27}{4\pi}\ \text{m/hr} \approx 2.15\ \text{m/hr}$$

$$\approx 0.6\ \text{mm/sec}$$



similar triangles

$$\frac{r}{y} = \frac{2/3}{2}$$

$$\boxed{r = \frac{1}{3} y}$$