Orthogonal complement

Let $\mathcal{U}$ be a subspace of $\mathcal{W}$.
For each vector $\mathbf{b}$ in $\mathcal{W}$, we can write $\mathbf{b} = \mathbf{b}^{||\mathcal{U}} + \mathbf{b}^{\perp\mathcal{U}}$ where

- $\mathbf{b}^{||\mathcal{U}}$ is in $\mathcal{U}$, and
- $\mathbf{b}^{\perp\mathcal{U}}$ is orthogonal to every vector in $\mathcal{U}$.

Let $\mathcal{V}$ be the set $\{\mathbf{b}^{\perp\mathcal{U}} \;:\; \mathbf{b} \in \mathcal{W}\}$.
Definition: We call $\mathcal{V}$ the *orthogonal complement* of $\mathcal{U}$ in $\mathcal{W}$

Easy observations:

- Every vector in $\mathcal{V}$ is orthogonal to every vector in $\mathcal{U}$.
- Every vector $\mathbf{b}$ in $\mathcal{W}$ can be written as the sum of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

Maybe $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$? To show direct sum of $\mathcal{U}$ and $\mathcal{V}$ is defined, we need to show that the only in vector that is in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Any vector $\mathbf{w}$ in both $\mathcal{U}$ and $\mathcal{V}$ is orthogonal to itself.
Thus $0 = \langle \mathbf{w}, \mathbf{w} \rangle = \|\mathbf{w}\|^2$.
By Property N2 of norms, that means $\mathbf{w} = \mathbf{0}$.

Therefore $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$. **Recall:** $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$

Orthogonal complement: example

Example: Let $\mathcal{U} = \text{Span } \{[1,1,0,0],[0,0,1,1]\}$. Let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{U}$ in $\mathbb{R}^4$. What vectors form a basis for $\mathcal{V}$?

Every vector in $\mathcal{U}$ has the form $[a,a,b,b]$.

Therefore any vector of the form $[c,-c,d,-d]$ is orthogonal to every vector in $\mathcal{U}$.

Every vector in $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is orthogonal to every vector in $\mathcal{U}$....
... so $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is a subspace of $\mathcal{V}$, the orthogonal complement of $\mathcal{U}$ in $\mathbb{R}^4$.

Is it the whole thing?

$\mathcal{U} \oplus \mathcal{V} = \mathbb{R}^4$ so $\dim \mathcal{U} + \dim \mathcal{V} = 4$.

$\{[1,1,0,0],[0,0,1,1]\}$ is linearly independent so $\dim \mathcal{U} = 2$... so $\dim \mathcal{V} = 2$

$\{[1,-1,0,0],[0,0,1,-1]\}$ is linearly independent
so $\dim \text{Span } \{[1,-1,0,0],[0,0,1,-1]\}$ is also 2....
so $\text{Span } \{[1,-1,0,0],[0,0,1,-1]\} = \mathcal{V}$.

Orthogonal complement: example

Example: Find a basis for the null space of $A = \left[\begin{array}{cccc} 1 & 0 & 2 & 4 \\ \hline 0 & 5 & 1 & 2 \\ \hline 0 & 2 & 5 & 6 \end{array}\right]$

By the dot-product definition of matrix-vector multiplication, a vector $\mathbf{v}$ is in the null space of A if the dot-product of each row of A with $\mathbf{v}$ is zero.

Thus the null space of A equals the orthogonal complement of Row A in $\mathbb{R}^4$.

Since the three rows of A are linearly independent, we know $\dim \text{Row } A = 3$...

so the dimension of the orthogonal complement of Row A in $\mathbb{R}^4$ is $4 - 3 = 1$....

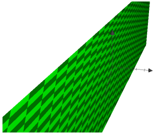
The vector $[1, \frac{1}{10}, \frac{13}{20}, \frac{-23}{40}]$ has a dot-product of zero with every row of A...

so this vector forms a basis for the orthogonal complement.

and thus a basis for the null space of A.

Using orthogonalization to find intersection of geometric objects

Example: Find the intersection of



- the plane spanned by $[1,0,0]$ and $[0,1,-1]$
- the plane spanned by $[1,2,-2]$ and $[0,1,1]$

The orthogonal complement in $\mathbb{R}^3$ of the first plane is Span $\{[4,-1,1]\}$.

Therefore first plane is $\{[x,y,z] \in \mathbb{R}^3 \ : \ [4,-1,1] \cdot [x,y,z] = 0\}$

The orthogonal complement in $\mathbb{R}^3$ of the second plane is Span $\{[0,1,1]\}$.

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The intersection of these two sets is the set

$$\{[x,y,z] \in \mathbb{R}^3 \ : \ [4,-1,1] \cdot [x,y,z] = 0 \text{ and } [0,1,1] \cdot [x,y,z] = 0\}$$

By Row-Space/Null-Space Duality, a basis for this vector space is a basis for the null space of $A = \left[\begin{array}{ccc} 4 & -1 & 1 \\ \hline 0 & 1 & 1 \end{array}\right]$

The null space of A is the orthogonal complement of Span $\{[4,-1,1],[0,1,1]\}$ in $\mathbb{R}^3$...
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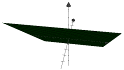

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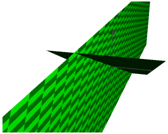

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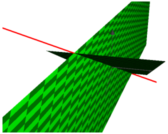

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The null space of A is the orthogonal complement of Span $\{[4, -1, 1], [0, 1, 1]\}$ in $\mathbb{R}^3$...
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Computing the orthogonal complement

Suppose we have a basis $\mathbf{u}_1, \ldots, \mathbf{u}_k$ for $\mathcal{U}$ and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_n$ for $\mathcal{W}$. How can we compute a basis for the orthogonal complement of $\mathcal{U}$ in $\mathcal{W}$?

One way: use `orthogonalize(vlist)` with

$$\text{vlist} = [\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_n]$$

Write list returned as $[\mathbf{u}_1^*, \ldots, \mathbf{u}_k^*, \mathbf{w}_1^*, \ldots, \mathbf{w}_n^*]$

These span the same space as input vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots, \mathbf{w}_n^*$, namely $\mathcal{W}$, which has dimension n.

Therefore exactly n of the output vectors $\mathbf{u}_1^*, \ldots, \mathbf{u}_k^*, \mathbf{w}_1^*, \ldots, \mathbf{w}_n^*$ are nonzero.

The vectors $\mathbf{u}_1^*, \ldots, \mathbf{u}_k^*$ have same span as $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and are all nonzero since $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are linearly independent.

Therefore exactly $n - k$ of the remaining vectors $\mathbf{w}_1^*, \ldots, \mathbf{w}_n^*$ are nonzero.

Every one of them is orthogonal to $\mathbf{u}_1, \ldots, \mathbf{u}_n$...
so they are orthogonal to every vector in $\mathcal{U}$...
so they lie in the orthogonal complement of $\mathcal{U}$.
By Direct-Sum Dimension Lemma, orthogonal complement has dimension $n - k$, so the remaining nonzero vectors are a basis for the orthogonal complement.