Limits and Graphs

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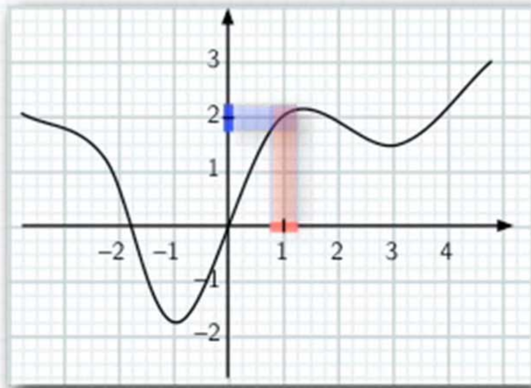
Suppose that a function f has the graph shown on the right.

Our goal is to describe the behavior of f in the vicinity of $x = 1$ in a concise manner.



Notice that $f(1) = 1$. function value at 1

Yet, if $x \approx 1$ and $x \neq 1$, then $f(x) \approx 2$.

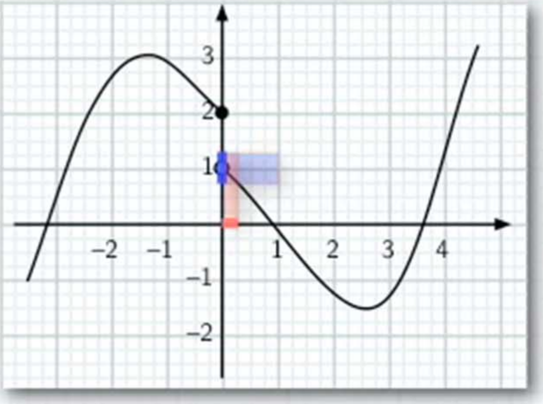
So, the number 2 is crucial in describing the behavior of f *near* 1.

We say that 2 is the *limit* of $f(x)$ as x *approaches* 1. This is written compactly as

$$\lim_{x \to 1} f(x) = 2.$$

limit as x approaches 1

Suppose that a function f has the graph shown on the right.



Here the interesting behavior of f is in the vicinity of $x = 0$.

Notice that $f(0) = 2$. function value at 0

If $x \approx 0$ and $x < 0$, then $f(x) \approx 2$.

But if $x \approx 0$ and $x > 0$, then $f(x) \approx 1$.

Therefore, the limit of $f(x)$ as x approaches 0 *does not exist*.

However, we can say that 2 is the *limit* of $f(x)$ as x *approaches* 0 *from the left* (or *from below*) and express this by writing

$$\lim_{x\to 0^-} f(x) = 2,$$

limit as x approaches 0 from the left

and we can say that 1 is the *limit* of $f(x)$ as x *approaches* 0 *from the right* (or *from above*) and express this by writing

$$\lim_{x\to 0^+} f(x) = 1.$$

limit as x approaches 0 from the right

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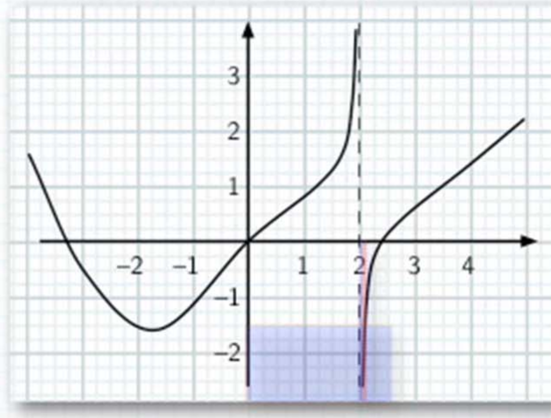
Suppose that a function f has the graph shown on the right.



Here the interesting behavior of f is in the vicinity of $x = 2$.

Notice that $f(2)$ is undefined and the line $x = 2$ is a vertical asymptote.

If $x \approx 2$ and $x < 2$, then $f(x)$ is large and positive.

But if $x \approx 2$ and $x > 2$, then $f(x)$ is large and negative.

Therefore, the limit of $f(x)$ as x approaches 2 *does not exist*.

In fact, *neither of the one-sided limits exists*.

However, we can describe the nature of the vertical asymptote by writing

$$\lim_{x \to 2^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = -\infty.$$

Given the function f whose graph is below, determine the following:

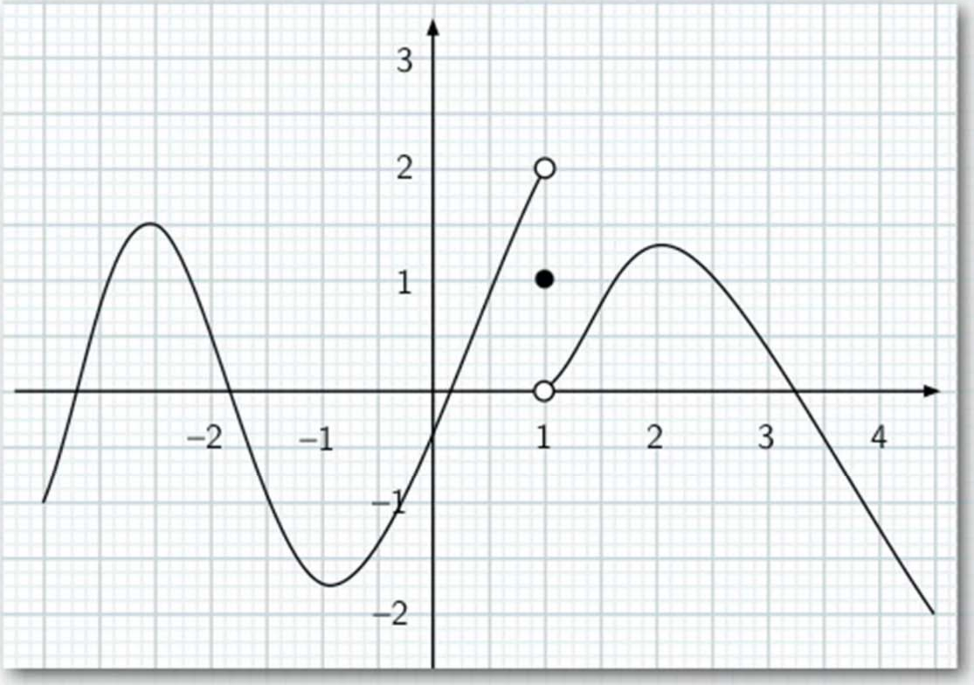
a) $f(1)$ b) $\lim_{x\to 1^-} f(x)$ c) $\lim_{x\to 1^+} f(x)$ d) $\lim_{x\to 1} f(x)$

a) $f(1) = 1$

b) $\lim_{x\to 1^-} f(x) = 2$

c) $\lim_{x\to 1^+} f(x) = 0$

d) $\lim_{x\to 1} f(x)$ does not exist



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Given the function f whose graph is below, determine the following:

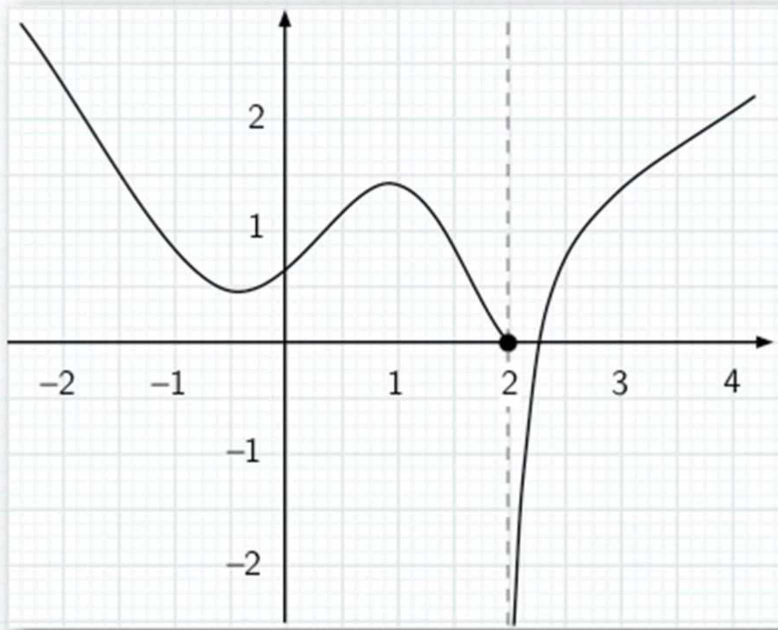
a) $f(2)$ b) $\lim_{x \to 2^-} f(x)$ c) $\lim_{x \to 2^+} f(x)$ d) $\lim_{x \to 2} f(x)$

a) $f(2) = 0$

b) $\lim_{x \to 2^-} f(x) = 0$

c) $\lim_{x \to 2^+} f(x) = -\infty$

d) $\lim_{x \to 2} f(x)$ does not exist



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Sketch the graph of a function f*, defined for* $-1 < x < 3$*, for which the following are true:*

$$\lim_{x\to -1^+} f(x) = 2$$

$$f(0) = 1$$

$$\lim_{x\to 0} f(x) = 0$$

$$f(1) = 0$$

$$\lim_{x\to 1^-} f(x) = \infty$$

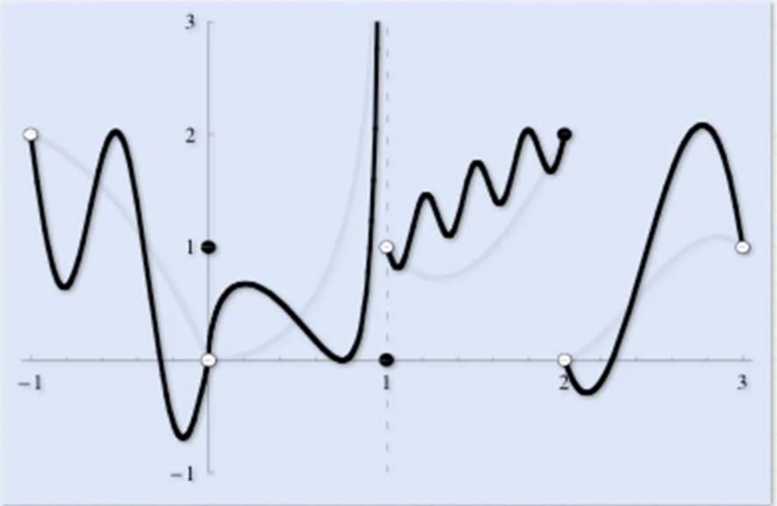
$$\lim_{x\to 1^+} f(x) = 1$$

$$f(2) = 2$$

$$\lim_{x\to 2^-} f(x) = 2$$

$$\lim_{x\to 2^+} f(x) = 0$$

$$\lim_{x\to 3^-} f(x) = 1$$



There are many other ways this graph could have been drawn. Other possibilities need only indicate the correct values at, and limiting behavior near, x = −1, 0, 1, 2, and 3.

Calculation of Limits

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Several basic, general facts about limits are helpful in the calculation of specific limits.

Assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ each exists.

Then the following are true:

1. $\lim_{x\to a} (f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} (c\, f(x)) = c \lim_{x\to a} f(x)$
3. $\lim_{x\to a} (f(x)\, g(x)) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right)$
4. If $\lim_{x\to a} g(x) \neq 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. If $\lim_{x\to a} f(x) \neq 0$ while $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist.
6. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ may or may not exist.

We have the following limits regarding two specific, simple functions :

$$\lim_{x \to a} c = c \quad \text{and} \quad \lim_{x \to a} x = a$$

Example: (a typical polynomial)

$$\begin{aligned}
\lim_{x \to 2} \left(2x^3 + 3x + 5\right) &= \lim_{x \to 2} 2x^3 + \lim_{x \to 2} 3x + \lim_{x \to 2} 5 \\
&= 2 \lim_{x \to 2} x^3 + 3 \lim_{x \to 2} x + \lim_{x \to 2} 5 \\
&= 2 \left(\lim_{x \to 2} x\right)^3 + 3 \lim_{x \to 2} x + \lim_{x \to 2} 5 \\
&= 2\,(2)^3 + 3 \cdot 2 + 5 \\
&= 27
\end{aligned}$$

Notice that the result is simply the *value* of the polynomial at $x = 2$.

In fact, this always happens with polynomials; that is,

If $p(x)$ is a polynomial and a is any real number, then

$$\lim_{x \to a} p(x) = p(a).$$

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Example:

(a rational function whose denominator does not approach zero)

$$\lim_{x\to 3}\frac{2x^2-x+2}{x^2+1} = \frac{\lim\limits_{x\to 3}(2x^2-x+2)}{\lim\limits_{x\to 3}(x^2+1)}$$

$$= \frac{2\cdot 3^2-3+2}{3^2+1}$$

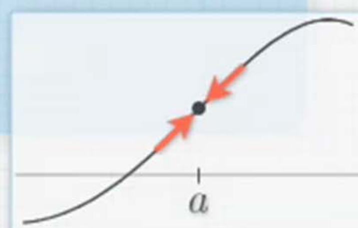
$$= \frac{17}{10}$$

Again, notice that the result is simply the *value* of the function at $x = 3$. In fact, this always happens with rational functions, provided that the denominator isn't zero.

If $p(x)$ *and* $q(x)$ *are polynomials and* $q(a) \neq 0$*, then*

$$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

Continuity Property



Example:
(a rational function whose denominator approaches zero while its numerator does not)

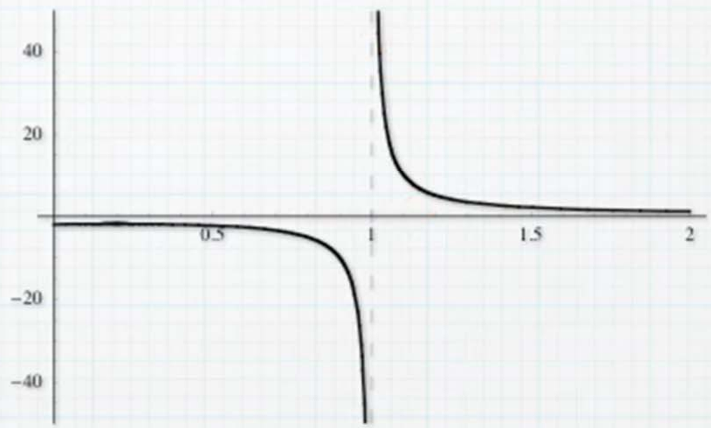
$$\lim_{x \to 1} \frac{2x^2 - x + 2}{x^3 - 1} \quad \text{does not exist,}$$

because $\lim_{x \to 1}(x^3 - 1) = 0$ while $\lim_{x \to 1}(2x^2 - x + 2) = 3$.

Can we say something about one-sided limits?

$$\lim_{x \to 1^-} \frac{2x^2 - x + 2}{x^3 - 1} = -\infty,$$

$$\lim_{x \to 1^+} \frac{2x^2 - x + 2}{x^3 - 1} = +\infty.$$



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Recall the last of the six facts with which we began:

6. If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ may or may not exist.

It is this situation that is the most interesting and, in fact, is the main reason why we're discussing limits at all!

Computation of limits in this "$\frac{0}{0}$" case often involves use of the following additional fact about limits:

7. If $f(x) = \varphi(x)$ for all x near but not equal to a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} \varphi(x).$$

Here, think of $\varphi(x)$ as a "simplified" version of $f(x)$ whose limit as $x \to a$ is easy to determine, such as a polynomial or a rational function whose numerator and denominator do not both approach 0.

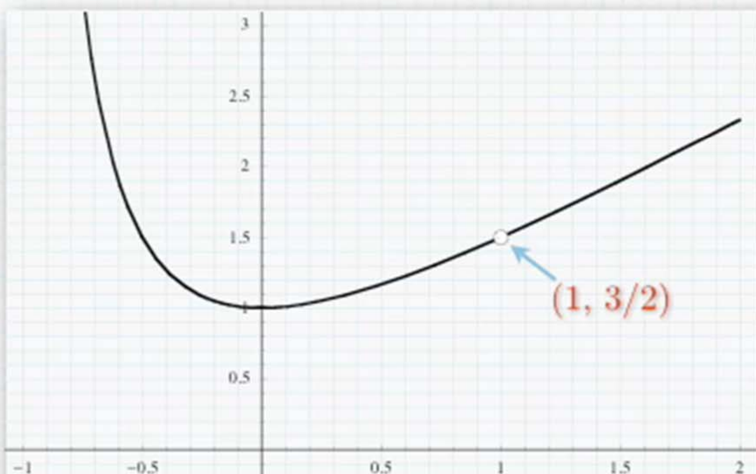
Example: Find $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$.

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$\frac{x^3-1}{x^2-1} = \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)}$$

$$= \frac{x^2+x+1}{x+1} \text{ for } x \neq 1$$

$$\therefore \quad \lim_{x\to 1}\frac{x^3-1}{x^2-1} = \lim_{x\to 1}\frac{x^2+x+1}{x+1} = \frac{1^2+1+1}{1+1} = \frac{3}{2}$$



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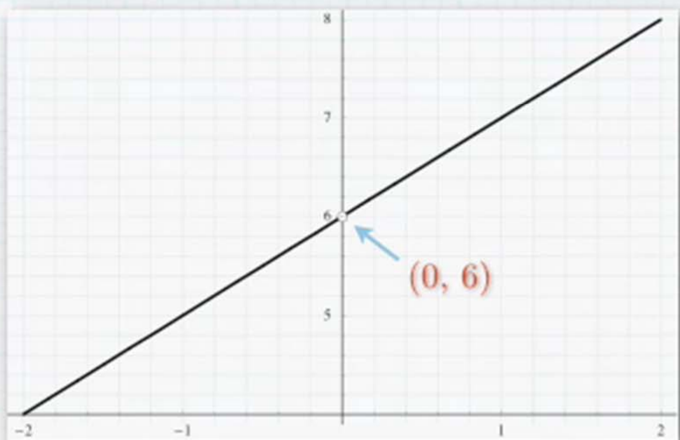
Example: Find $\lim_{x\to 0} \dfrac{(x+3)^2-9}{x}$.

$$\begin{aligned}\frac{(x+3)^2-9}{x} &= \frac{x^2+6x}{x}\\ &= \frac{\cancel{x}(x+6)}{\cancel{x}}\\ &= x+6 \text{ for } x \neq 0\end{aligned}$$

$$\therefore \quad \lim_{x\to 0} \frac{(x+3)^2-9}{x} = \lim_{x\to 0}(x+6) = 6$$



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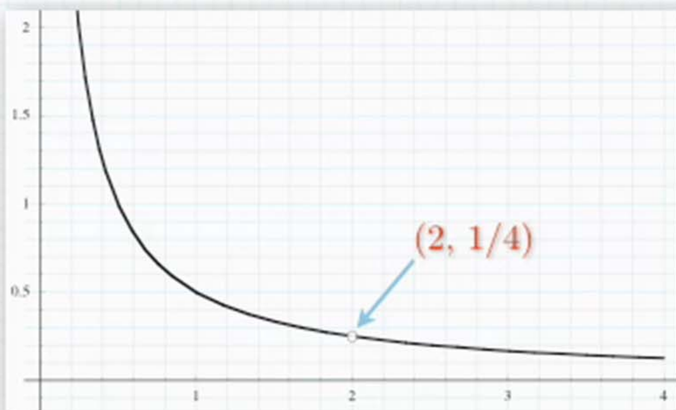
Example: Find $\lim\limits_{x \to 2} \dfrac{1/2 - 1/x}{x-2}$.

$$\frac{1/2 - 1/x}{x-2} = \frac{\cancel{x-2}}{2x\,\cancel{(x-2)}}$$
$$= \frac{1}{2x} \quad \text{for } x \neq 2$$

$$\therefore \quad \lim_{x \to 2} \frac{1/2 - 1/x}{x-2} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$$



Example: Find $\displaystyle\lim_{x\to 2}\frac{x^2-x-2}{x^2-4x+4}$.

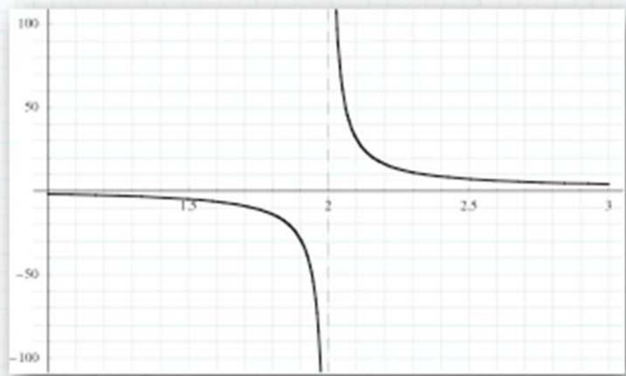
$$\frac{x^2-x-2}{x^2-4x+4} = \frac{\cancel{(x-2)}(x+1)}{(x-2)^{\cancel{2}}}$$

$$= \frac{x+1}{x-2} \quad \text{for } x \neq 2$$

$$\therefore \quad \lim_{x\to 2}\frac{x^2-x-2}{x^2-4x+4} \text{ does not exist.}$$

$$\lim_{x\to 2^-}\frac{x^2-x-2}{x^2-4x+4} = \lim_{x\to 2^-}\frac{x+1}{x-2} = -\infty$$

$$\lim_{x\to 2^+}\frac{x^2-x-2}{x^2-4x+4} = \lim_{x\to 2^+}\frac{x+1}{x-2} = +\infty$$



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For the next example, we will need an additio Continuity Property nit:

$$\lim_{x\to a}\sqrt{x} = \sqrt{a} \text{ for all } a>0, \text{ and } \lim_{x\to 0^+}\sqrt{x} = 0.$$

Example: Find $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$.

$$\begin{aligned}\frac{\sqrt{x}-1}{x-1} &= \frac{\sqrt{x}-1}{x-1}\times\frac{\sqrt{x}+1}{\sqrt{x}+1}\\ &= \frac{\cancel{x-1}}{\cancel{(x-1)}(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1} \quad \text{for } x\neq 1\end{aligned}$$

$$\therefore \quad \lim_{x\to 1}\frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1}\frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

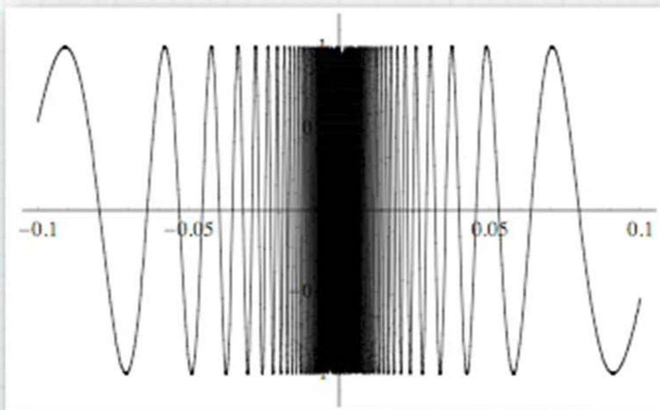
Nonexistent one-sided limits

We have seen that limits fail to exist when the corresponding one-sided limits don't coincide. But in all the examples we've seen so far, nonexistent one-sided limits have been assigned one of the symbols $\pm\infty$.

So is there ever a situation when that is not possible?

The answer is yes (but not with a rational function). A simple example is provided by the function

$$f(x) = \sin(1/x)$$



As x approaches 0 from either the left or the right, $f(x)$ oscillates between -1 and 1 infinitely often. Thus neither one-sided limit at 0 exists.

Some trigonometric limits

Continuity of sine and cosine

For all real numbers a,

$$\lim_{x\to a} \sin x = \sin a \quad \text{and} \quad \lim_{x\to a} \cos x = \cos a$$

Example: Find $\displaystyle\lim_{x\to 0} \frac{\sin x + 2\cos x + \cos^2 x}{\sin x + \cos 2x}$.

$$\lim_{x\to 0} \frac{\sin x + 2\cos x + \cos^2 x}{\sin x + \cos 2x} = \frac{\lim_{x\to 0}(\sin x + 2\cos x + \cos^2 x)}{\lim_{x\to 0}(\sin x + \cos 2x)}$$

$$= \frac{\sin 0 + 2\cos 0 + \cos^2 0}{\sin 0 + \cos 0}$$

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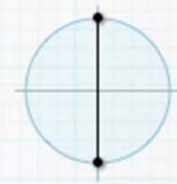
Other Trig Functions

Tangent

If a is not an odd multiple of $\frac{\pi}{2}$, then $\cos a \neq 0$ and so

$$\lim_{x\to a} \tan x = \lim_{x\to a} \frac{\sin x}{\cos x} = \frac{\sin a}{\cos a} = \tan a.$$



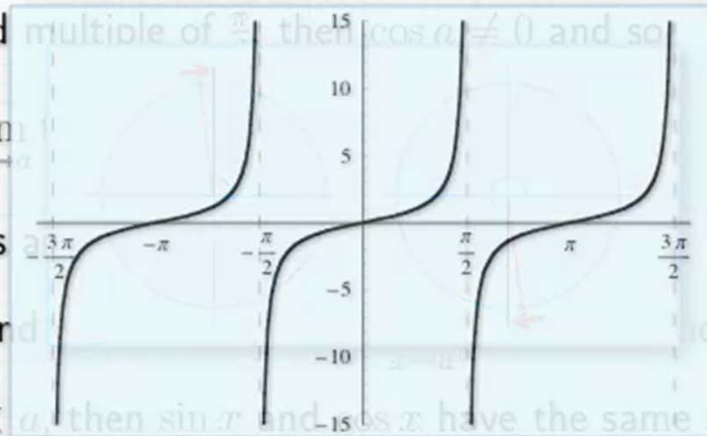
Suppose that a is an odd multiple of $\frac{\pi}{2}$.

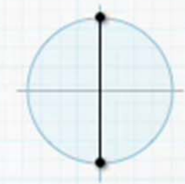
Then $\sin a \neq 0$ and $\cos a = 0$, and so $\lim_{x\to a} \tan x$ does not exist.

Other Trig Functions

Tangent

If a is not an odd multiple of $\frac{\pi}{2}$, then $\cos a \neq 0$ and so





$\lim_{x \to a}$

Suppose that a is a

Then $\sin a \neq 0$ an ... t exist.

If $x \approx a$ and $x < a$, then $\sin x$ and $\cos x$ have the same sign; so $\tan x > 0$.

Therefore

$$\lim_{x \to a^-} \tan x = \lim_{x \to a^-} \frac{\sin x}{\cos x} = \infty.$$

If $x \approx a$ and $x > a$, then $\sin x$ and $\cos x$ have opposite sign; so $\tan x < 0$.

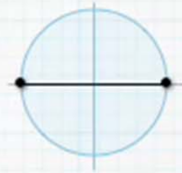
Therefore

$$\lim_{x \to a^+} \tan x = \lim_{x \to a^+} \frac{\sin x}{\cos x} = -\infty.$$

Cotangent

If a is *not* a multiple of π, then $\sin a \neq 0$, and so

$$\lim_{x\to a} \cot x = \lim_{x\to a} \frac{\cos x}{\sin x} = \frac{\cos a}{\sin a} = \cot a.$$



If a is a multiple of π, then $\lim\limits_{x\to a} \cot x$ does not exist.

Note that

$$\cot x = \tan(\pi/2 - x) = -\tan(x - \pi/2).$$

So, if a is a multiple of π, then

$$\lim_{x\to a^-} \cot x = -\lim_{x\to a^-} \tan(x-\pi/2) = -\lim_{x\to(a-\frac{\pi}{2})^-} \tan x = -\infty,$$

and similarly

$$\lim_{x\to a^+} \cot x = -\lim_{x\to a^+} \tan(x-\pi/2) = -\lim_{x\to(a-\frac{\pi}{2})^+} \tan x = \infty.$$

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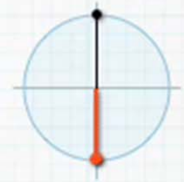
Secant

If a is not an odd multiple of $\frac{\pi}{2}$, then $\cos a \neq 0$ and so

$$\lim_{x\to a} \sec x = \lim_{x\to a} \frac{1}{\cos x} = \frac{1}{\cos a} = \sec a.$$



If a is an odd multiple of $\frac{\pi}{2}$, then $\cos a = 0$, and so

$$\lim_{x\to a} \sec x \ \text{ does not exist.}$$

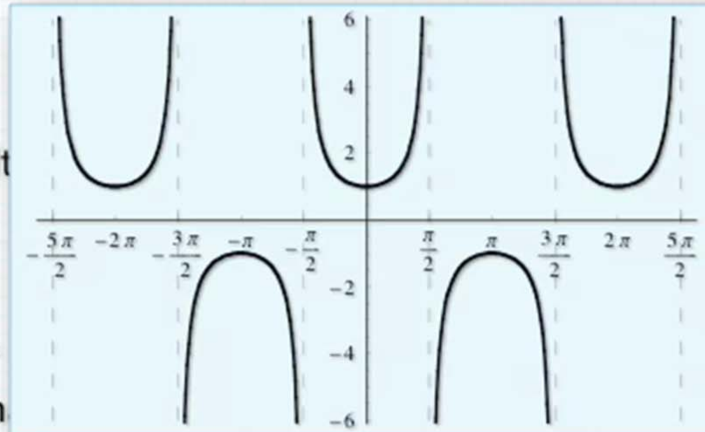
Suppose that $a = (4k+1)\pi/2$, where k is any integer. Then $\cos x$, and therefore $\sec x$, changes from positive to negative as x increases through a. Therefore

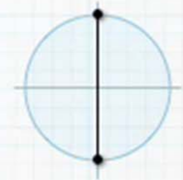
$$\lim_{x\to a^-} \sec x = \infty \quad \text{and} \quad \lim_{x\to a^+} \sec x = -\infty.$$

Now suppose that $a = (4k-1)\pi/2$, where k is any integer.

Secant

If a is not an odd multiple of $\frac{\pi}{2}$, then $\cos a \neq 0$ and so





If a is an odd mult

Suppose that $a =$ … hen $\cos x$, and therefore $\sec x$, ch … reases through a. Therefore

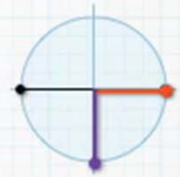
$$\lim_{x \to a^-} \sec x = \infty \quad \text{and} \quad \lim_{x \to a^+} \sec x = -\infty.$$

Now suppose that $a = (4k-1)\pi/2$, where k is any integer. Then $\cos x$, and therefore $\sec x$, changes from negative to positive as x increases through a. Therefore

$$\lim_{x \to a^-} \sec x = -\infty \quad \text{and} \quad \lim_{x \to a^+} \sec x = +\infty.$$

Cosecant

If a is *not* a multiple of π, then $\lim\limits_{x\to a} \csc x = \dfrac{1}{\sin a} = \csc a.$



If a is a multiple of π, then $\lim\limits_{x\to a} \csc x$ does not exist.

Now note that $\csc x = \sec(\pi/2 - x) = \sec(x - \pi/2).$

So if a is an *odd* multiple of π, then $a - \frac{\pi}{2}$ is of the form $\frac{(4k+1)\pi}{2}$ and so

$$\lim_{x\to a^-} \csc x = \lim_{x\to a^-} \sec(x - \pi/2) = \lim_{x\to (a-\frac{\pi}{2})^-} \sec x = \infty$$

and

$$\lim_{x\to a^+} \csc x = \lim_{x\to a^+} \sec(x - \pi/2) = \lim_{x\to (a-\frac{\pi}{2})^+} \sec x = -\infty$$

If a is an *even* multiple of π, then $a - \frac{\pi}{2}$ is of the form $\frac{(4k-1)\pi}{2}$ and so

$$\lim_{x\to a^-} \csc x = \lim_{x\to a^-} \sec(x - \pi/2) = \lim_{x\to (a-\frac{\pi}{2})^-} \sec x = -\infty$$

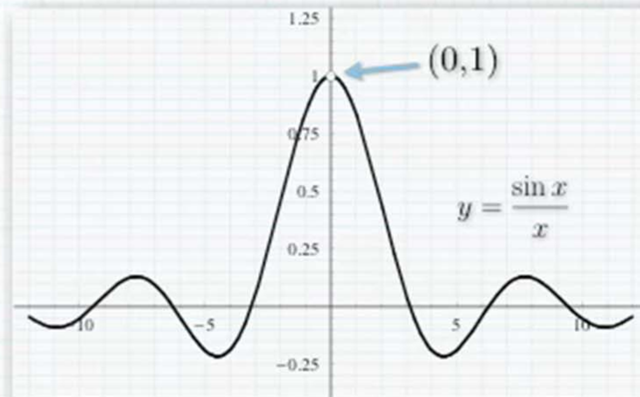
and

$$\lim_{x\to a^+} \csc x = \lim_{x\to a^+} \sec(x - \pi/2) = \lim_{x\to (a-\frac{\pi}{2})^+} \sec x = \infty.$$

Remember this limit

at will be crucial to us later:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$



Also, if θ is a function for which $\lim_{x\to a}\theta(x) = 0$, then

$$\lim_{x\to a}\frac{\sin(\theta(x))}{\theta(x)} = 1$$

For instance,

$$\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x} = 1 \quad \text{and} \quad \lim_{x\to -1/3}\frac{\sin(3x+1)}{3x+1} = 1$$

Another remark:

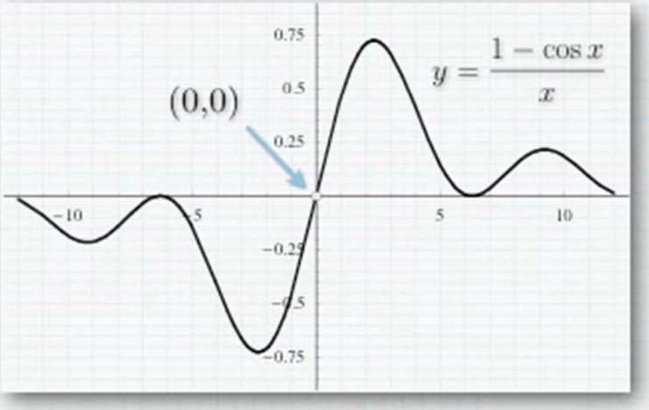
$$\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0} 1\Big/\frac{\sin x}{x} = 1\Big/\lim_{x\to 0}\frac{\sin x}{x}$$

Example: Find $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$.

$$\begin{aligned}\lim_{x\to 0} \frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} &= \lim_{x\to 0} \frac{1-\cos^2 x}{x}\cdot\frac{1}{1+\cos x}\\ &= \lim_{x\to 0} \frac{\sin^2 x}{x}\cdot\frac{1}{1+\cos x}\\ &= \lim_{x\to 0} \frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x} \quad = 1\cdot\frac{0}{2} = 0\end{aligned}$$

Another limit worth remembering:

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$$



Example: Find $\lim_{x\to 0} \dfrac{\sin^2(\pi x)}{2x^2}$.

$$\begin{aligned}\lim_{x\to 0} \frac{\sin^2(\pi x)}{2x^2} &= \lim_{x\to 0} \frac{\pi^2}{2} \frac{\sin(\pi x)}{\pi x} \frac{\sin(\pi x)}{\pi x} \\ &= \frac{\pi^2}{2} \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \lim_{x\to 0} \frac{\sin(\pi x)}{\pi x} \\ &= \frac{\pi^2}{2} \cdot 1 \cdot 1 = \frac{\pi^2}{2}\end{aligned}$$

Example: Find $\lim_{x\to 0} \dfrac{2 - 3\cos x + \cos^2 x}{\sin x}$.

$$\begin{aligned}\lim_{x\to 0} \frac{2 - 3\cos x + \cos^2 x}{\sin x} &= \lim_{x\to 0} \frac{(2-\cos x)(1-\cos x)}{\sin x} \\ &= \lim_{x\to 0} (2-\cos x) \frac{(1-\cos x)}{x} \Big/ \frac{\sin x}{x} \\ &= (2-1) \cdot 0 \,/\, 1 = 0\end{aligned}$$

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Example: Find $\lim_{x \to 0} \dfrac{1-\cos(3x)}{x^2}$.

$$
\begin{aligned}
\lim_{x \to 0} \frac{1-\cos(3x)}{x^2} &= \lim_{x \to 0} \frac{1-\cos(3x)}{x^2} \, \frac{1+\cos(3x)}{1+\cos(3x)} \\
&= \lim_{x \to 0} \frac{1-\cos^2(3x)}{x^2} \, \frac{1}{1+\cos(3x)} \\
&= \lim_{x \to 0} \frac{\sin^2(3x)}{x^2} \, \frac{1}{1+\cos(3x)} \\
&= \lim_{x \to 0} \frac{\sin(3x)}{3x} \, \frac{\sin(3x)}{3x} \, \frac{3 \cdot 3}{(1+\cos(3x))} \\
&= 1 \cdot 1 \cdot \frac{9}{2} \\
&= \frac{9}{2}
\end{aligned}
$$

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Limits at $\pm\infty$

and

Horizontal Asymptotes

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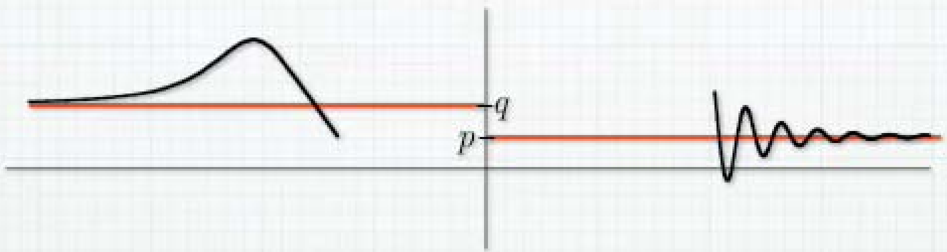
Limits at $\pm\infty$

If $f(x)$ becomes arbitrarily close to p for all sufficiently large positive x, we say that p is the limit of $f(x)$ as x approaches infinity, and we write

$$\lim_{x\to\infty} f(x) = p.$$

If $f(x)$ becomes arbitrarily close to q for all sufficiently large negative x, we say that q is the limit of $f(x)$ as x approaches negative infinity, and we write

$$\lim_{x\to-\infty} f(x) = q.$$

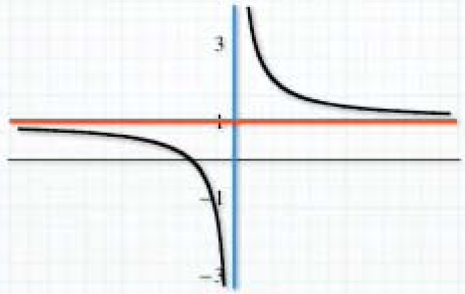


If $\lim\limits_{x\to\infty} f(x) = p$, then the line $y = p$ is a (rightward) horizontal asymptote.

If $\lim\limits_{x\to-\infty} f(x) = q$, then the line $y = q$ is a (leftward) horizontal asymptote.

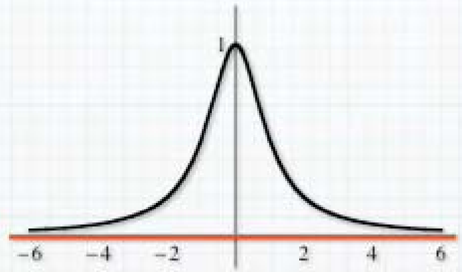
Examples

$$f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$$



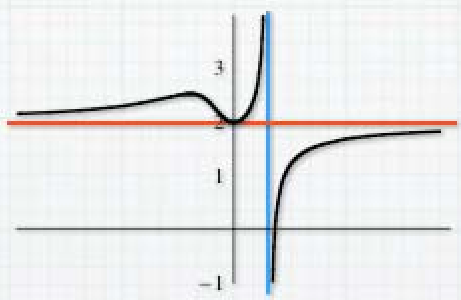
$$\lim_{x\to\infty} f(x) = 1$$
$$\lim_{x\to-\infty} f(x) = 1$$

$$f(x) = \frac{1}{x^2+1}$$



$$\lim_{x\to\infty} f(x) = 0$$
$$\lim_{x\to-\infty} f(x) = 0$$

$$f(x) = \frac{2x^3 - x^2 - 2}{x^3 - 1} = 2 - \frac{x^2}{x^3-1}$$



$$\lim_{x\to\infty} f(x) = 2$$
$$\lim_{x\to-\infty} f(x) = 2$$

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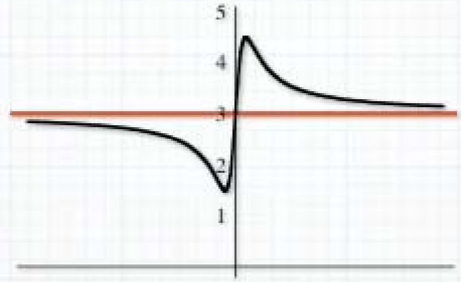
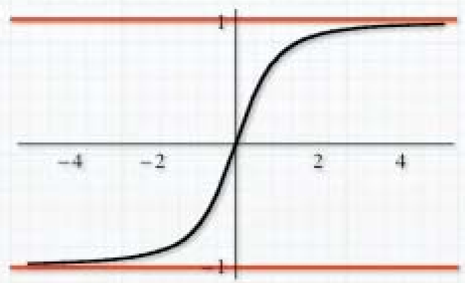
$$f(x) = \frac{3(x^2+x+1)}{x^2+1}$$
$$= 3 + \frac{3x}{x^2+1}$$



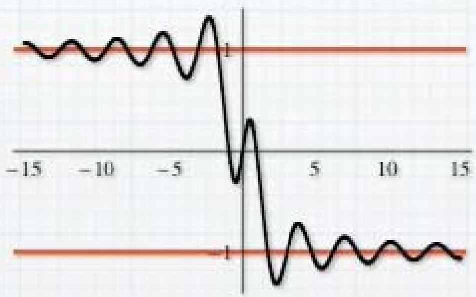
$$\lim_{x\to\infty} f(x) = 3$$
$$\lim_{x\to-\infty} f(x) = 3$$

$$f(x) = \frac{x}{\sqrt{x^2+1}}$$



$$\lim_{x\to\infty} f(x) = 1$$
$$\lim_{x\to-\infty} f(x) = -1$$

$$f(x) = \frac{\sin 2x - x}{\sqrt{x^2+1}}$$



$$\lim_{x\to\infty} f(x) = -1$$
$$\lim_{x\to-\infty} f(x) = 1$$

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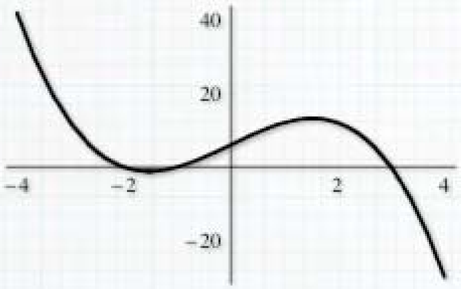
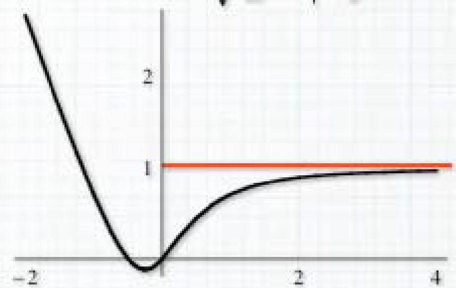
$$f(x) = 6 + 7x - x^3$$



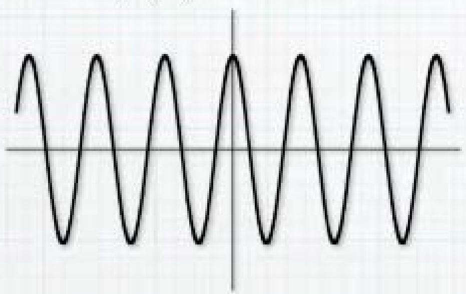
$$\lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = -\infty$$

$$f(x) = \frac{x(1 + x - |x|)}{\sqrt{x^2+1}}$$



$$\lim_{x\to-\infty} f(x) = \infty, \quad \lim_{x\to\infty} f(x) = 1$$

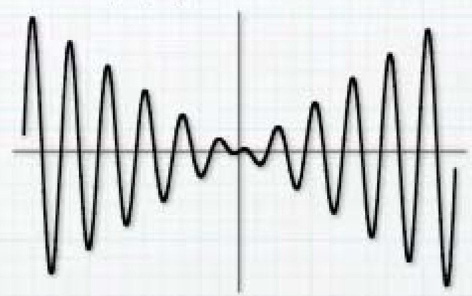
$$f(x) = \cos x$$



$\lim_{x\to\infty} f(x)$ does not exist.

$\lim_{x\to-\infty} f(x)$ does not exist.

$$f(x) = x\cos x$$



$\lim_{x\to\infty} f(x)$ does not exist.

$\lim_{x\to-\infty} f(x)$ does not exist.

Calculation of Limits at $\pm\infty$

Basic Limits

For any $p > 0$,

$$\lim_{x \to \infty} \frac{1}{x^p} = 0,$$

and, if x^p is defined when $x < 0$,

$$\lim_{x \to -\infty} \frac{1}{x^p} = 0.$$

Example Let $f(x) = \dfrac{3x^3 + x^2 - 5}{2x^3 - x + 1}$. Find $\lim\limits_{x \to \infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.

$$f(x) = \frac{3x^3 + x^2 - 5}{2x^3 - x + 1} \cdot \frac{1/x^3}{1/x^3} = \frac{3 + 1/x - 5/x^3}{2 - 1/x^2 + 1/x^3} \longrightarrow \frac{3}{2} \quad \text{as } x \to \pm\infty$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \frac{3}{2}$$

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Example Let $f(x) = \dfrac{x^2 + x + 3}{5x^3 - 1}$. Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$$f(x) = \frac{x^2 + x + 3}{5x^3 - 1} \cdot \frac{1/x^3}{1/x^3} = \frac{1/x + 1/x^2 + 3/x^3}{5 - 1/x^3} \longrightarrow \frac{0}{5} = 0 \text{ as } x \to \pm\infty$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0$$

Example Let $f(x) = \dfrac{x^3 - x^2 + 3}{3x^2 + x - 5}$. Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.

$$f(x) = \frac{x^3 - x^2 + 3}{3x^2 + x - 5} \cdot \frac{1/x^2}{1/x^2} = \frac{x + 1 + 3/x^2}{3 + 1/x - 5/x^2} \longrightarrow \begin{cases} \infty & \text{as } x \to \infty \\ -\infty & \text{as } x \to -\infty \end{cases}$$

$$\lim_{x \to \infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

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Rational Functions in General

$$f(x) = \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0} \cdot \frac{1/x^n}{1/x^n}$$

$$= \frac{a_m x^{m-n} + \cdots + \cancelto{0}{a_1 x^{1-n}} + \cancelto{0}{a_0 x^{-n}}}{b_n + \cdots + \cancelto{0}{b_1 x^{1-n}} + \cancelto{0}{b_0 x^{-n}}}$$

$$\approx \frac{a_m x^{m-n}}{b_n} \quad \text{for large } x$$

$$\longrightarrow \begin{cases} 0 & \text{if } m < n \\ a_m/b_m & \text{if } m = n \\ \pm\infty & \text{if } m > n \end{cases}$$

Note: It is not necessary that m and n be integers.

Example $\displaystyle \lim_{x\to\infty} \frac{5x^{4/3} - x^{1/2}}{3x^{4/3} - 7x + x^{2/3}} = \frac{5}{3}$

Irrational Functions

Example Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \dfrac{3x-2|x|}{x+2}$.

$$f(x) = \frac{3x-2|x|}{x+2} = \begin{cases} \dfrac{x}{x+2} & \text{if } x > 0 \\ \dfrac{5x}{x+2} & \text{if } x < 0 \end{cases} \longrightarrow \begin{cases} 1 & \text{as } x \to \infty \\ 5 & \text{as } x \to -\infty \end{cases}$$

Example Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \dfrac{3x-7}{\sqrt{5x^2+4}}$.

$$f(x) = \frac{3x-7}{\sqrt{x^2(5+4/x^2)}} = \frac{3x-7}{|x|\sqrt{5+4/x^2}} \cdot \frac{1/x}{1/x}$$

$$= \begin{cases} \dfrac{3-7/x}{\sqrt{5+4/x^2}} & \text{if } x > 0 \\ \dfrac{3-7/x}{-\sqrt{5+4/x^2}} & \text{if } x < 0 \end{cases} \longrightarrow \begin{cases} \dfrac{3}{\sqrt{5}} & \text{as } x \to \infty \\ -\dfrac{3}{\sqrt{5}} & \text{as } x \to -\infty \end{cases}$$

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Example Find $\lim_{x\to\infty} f(x)$ if $f(x) = \sqrt{x+\sqrt{x}} - \sqrt{x}$.

$$f(x) = (\sqrt{x+\sqrt{x}} - \sqrt{x})\ \frac{\sqrt{x+\sqrt{x}} + \sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}}$$

$$= \frac{x+\sqrt{x}-x}{\sqrt{x+\sqrt{x}} + \sqrt{x}}$$

$$= \frac{\sqrt{x}}{\sqrt{x+\sqrt{x}} + \sqrt{x}} \cdot \frac{1/\sqrt{x}}{1/\sqrt{x}}$$

$$= \frac{1}{\sqrt{1+1/\sqrt{x}} + 1} \longrightarrow \frac{1}{2} \text{ as } x \to \infty$$

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Continuity

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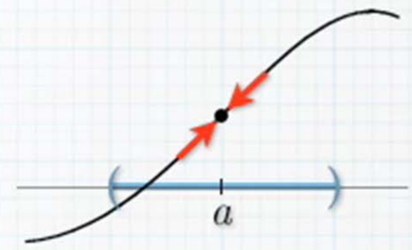
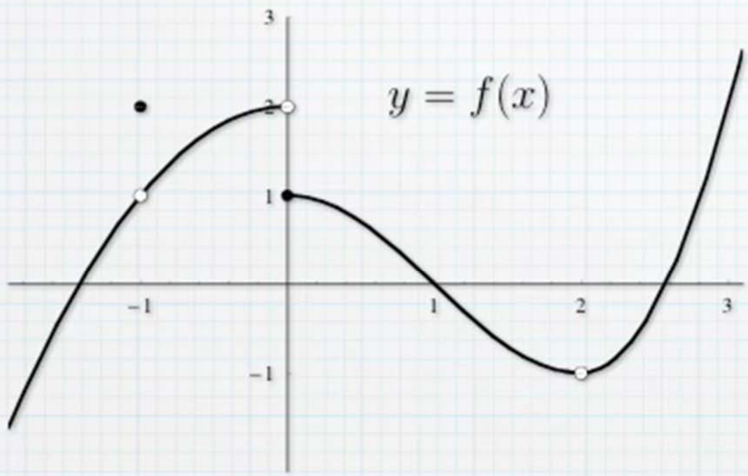
Let f be a function whose domain includes an open interval centered at $x = a$.
Then f is said to be **continuous at** a if

$$\lim_{x \to a} f(x) = f(a).$$



Example:



f is continuous everywhere *except* at $x = -1, 0,$ and 2.

f is not continuous at -1 because $\lim_{x \to -1} f(x) = 1$ while $f(-1) = 2$.

f is not continuous at 0 because $\lim_{x \to 0} f(x)$ does not exist.

f is not continuous at 2 because $f(2)$ is undefined.

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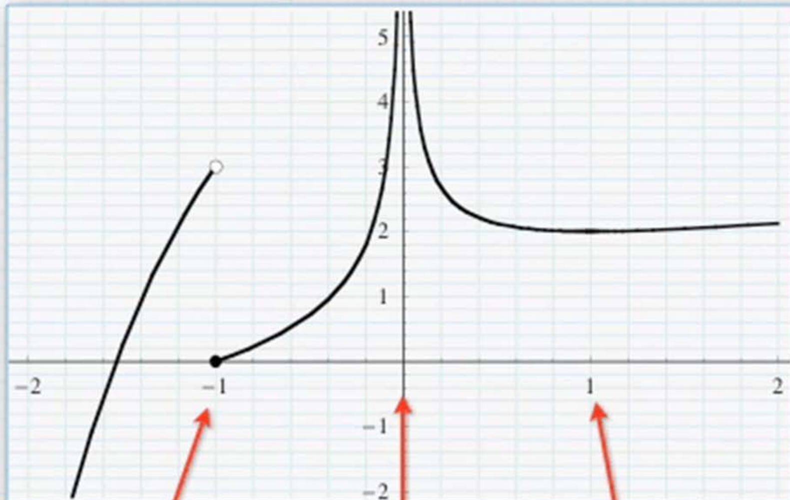
Three types of "simple" discontinuities



"Jump" discontinuity
One-sided limits exist
but are different.

"Removable" discontinuity
The limit exists but does not equal
the value of function.

"Infinite" discontinuity
(vertical asymptote)
An infinite one-sided limit.

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Polynomials are continuous everywhere.

Rational functions are continuous wherever they are defined.

sin and cos are continuous everywhere.

tan, cot, sec, and csc are continuous wherever they are defined.

Several properties of continuous functions follow directly from properties of limits.

Here are corresponding statements about continuity.

If f and g are both continuous at a, then:

1. $f+g$ and $f-g$ are continuous at a
2. cf is continuous at a
3. fg is continuous at a
4. If $g(a) \neq 0$, then f/g is continuous at a

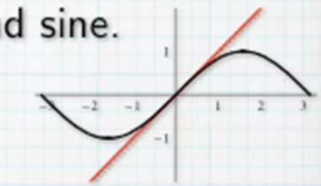
Compositions

If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Example: Let $f(x) = 3x^2 + \dfrac{x^2 - 1}{(x-1)(x - \sin x)}$.

At what values of x is f *not* continuous?
Of what type is each of its discontinuities?

First observe that $3x^2$, $x^2 - 1$, and $(x-1)(x - \sin x)$ are each continuous everywhere, since they involve only polynomials and sine.



Therefore, f is continuous wherever $(x-1)(x - \sin x) \neq 0$, that is, everywhere except at $x = 1$ and $x = 0$.

To determine the type of discontinuity at $x = 1$, we'll examine $\lim_{x \to 1} f(x)$.

$$f(x) \;=\; 3x^2 + \frac{(x-1)(x+1)}{(x-1)(x - \sin x)} \;=\; 3x^2 + \frac{x+1}{x - \sin x} \quad \text{for } x \neq 1$$

So the limit exists: $\displaystyle \lim_{x \to 1} f(x) = 3 + \frac{2}{1 - \sin 1} \approx 15.616$

Therefore, the discontinuity at $x = 1$ is **removable**, corresponding to a "hole" in the graph of f.

Example: Find numbers a and b so that the following function is continuous everywhere.

$$f(x) = \begin{cases} a\,x & \text{if } x < -1 \\ x^2 + a - b & \text{if } -1 \le x < 1 \\ b\,x & \text{if } 1 \le x \end{cases}$$

Since the "parts" f are polynomials, we only need to choose a and b so that f is continuous at $x = -1$ and 1.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$$

$$\lim_{x \to -1^-} a\,x = \lim_{x \to -1^+} x^2 + a - b$$

$$-a = 1 + a - b$$

$$-2a + b = 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\lim_{x \to 1^-} x^2 + a - b = \lim_{x \to 1^+} b\,x$$

$$1 + a - b = b$$

$$a - 2b = -1$$

$$\begin{array}{r} -2a + b = 1 \\ 2a - 4b = -2 \\ \hline -3b = -1 \\ b = \frac{1}{3} \end{array}$$

$$a - \tfrac{2}{3} = -1$$

$$a = -\tfrac{1}{3}$$

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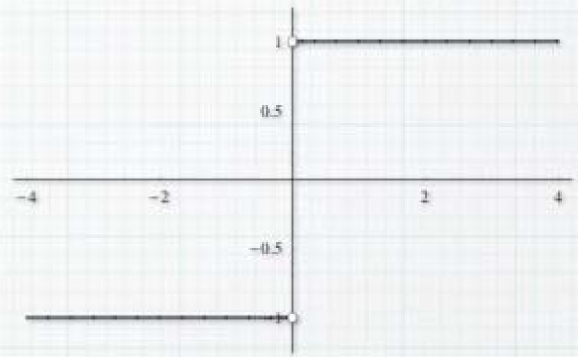
Compositions

If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.

Example

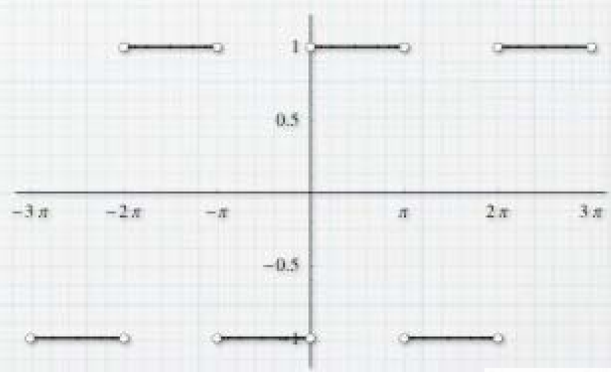
$$f(x) = \frac{x}{|x|} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } 0 < x \end{cases}$$



f is continuous except at $x = 0$, where it is undefined and has a jump discontinuity.

If we compose f with sine, the result,

$$f(\sin x) = \frac{\sin x}{|\sin x|}$$



will be continuous wherever $\sin x \neq 0$, i.e., except at multiples of π.

At multiples of π, it is undefined and has a jump discontinuity.

One-sided Continuity

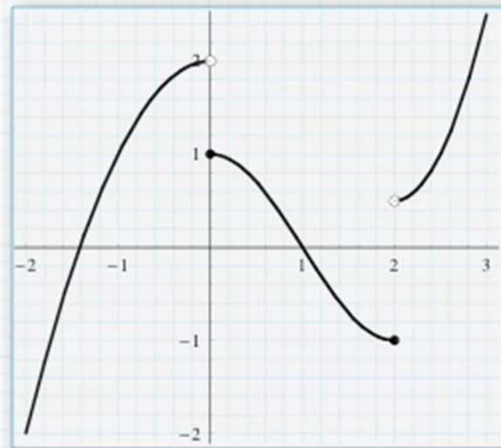
If the domain of f includes an interval whose right endpoint is a, then f is **left continuous at** a if

$$\lim_{x \to a^-} f(x) = f(a).$$

If the domain of f includes an interval whose left endpoint is a, then f is **right continuous at** a if

$$\lim_{x \to a^+} f(x) = f(a).$$

f is continuous at a if and only if f is both left continuous and right continuous at a.

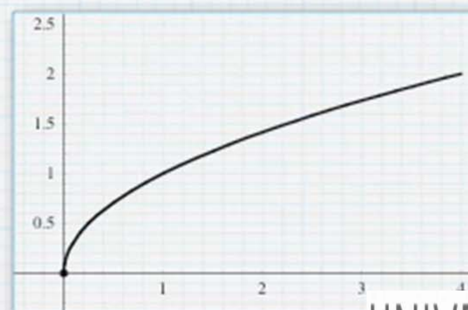


right continuous at 0, left continuous at 2

Example

$\lim_{x \to a} \sqrt{x} = \sqrt{a}$ for all $a > 0$, and $\lim_{x \to 0^+} \sqrt{x} = 0$.

So the square root function is continuous at every positive number and right continuous at 0.



Continuity on an interval

Let I be an interval, i.e., a set of one of the following forms
(a,b), $[a,b]$, $[a,b)$, $(a,b]$, (a,∞), $[a,\infty)$, $(-\infty,b)$, $(-\infty,b]$, or $(-\infty,\infty)$.

A function f whose domain includes I is said to be **continuous on** I,
if for every number c in I,

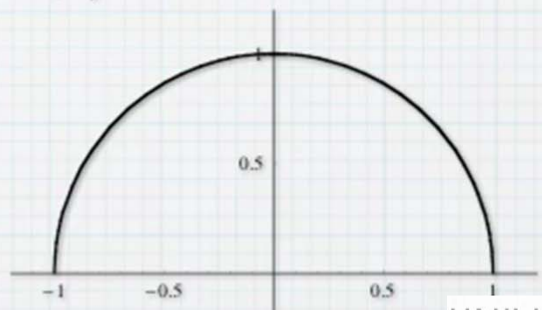
- f is continuous at c if c is not an endpoint of I,
- f is right continuous at c if c is a left endpoint of I,
- f is left continuous at c if c is a right endpoint of I.

If the domain of f is an interval, and if f is continuous on that interval,
then we may simply say that f is a **continuous function**.

$f(x) = \dfrac{1}{x}$ is continuous on $(0,\infty)$ and on $(-\infty,0)$.

$f(x) = \sqrt{x}$ is continuous on $[0,\infty)$.

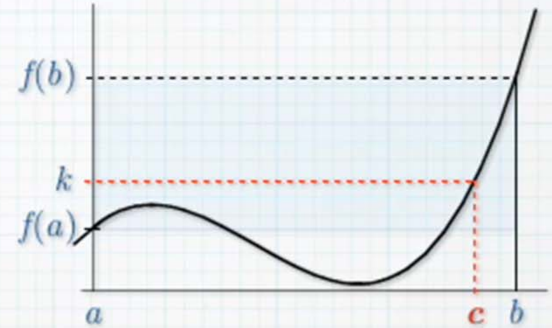
$f(x) = \sqrt{1-x^2}$ is continuous on $[-1,1]$.



Consequences of Continuity

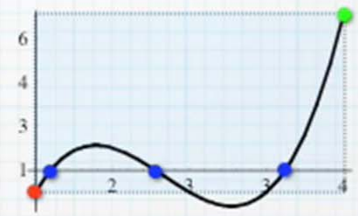
The Intermediate Value Theorem

Suppose that f is continuous on a closed bounded interval $[a,b]$. Then for each number k between $f(a)$ and $f(b)$, there exists a number c in $[a,b]$ where $f(c)=k$.



Corollary

If f is continuous on $[a,b]$, and if $f(a)$ and $f(b)$ have opposite signs, then there exists a number c in (a,b) where $f(c)=0$.

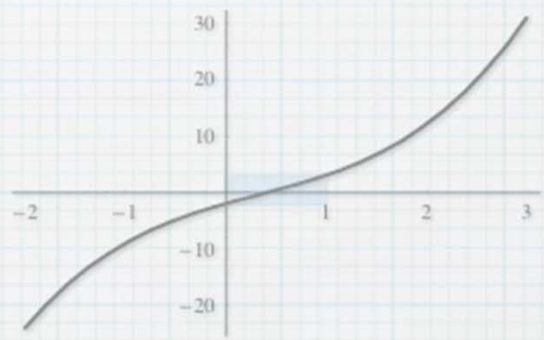


Example

Consider the polynomial

$$f(x) = x^3 - x^2 + 5x - 2$$

Since $f(0) = -2$ and $f(1) = 3$, it follows that $f(x) = 0$ for some x between 0 and 1.



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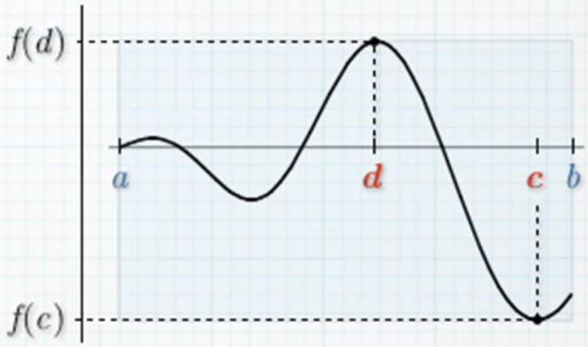
The Extreme Value Theorem

Suppose that f is continuous on a closed bounded interval $[a,b]$. Then there exist numbers c and d in $[a,b]$ such that

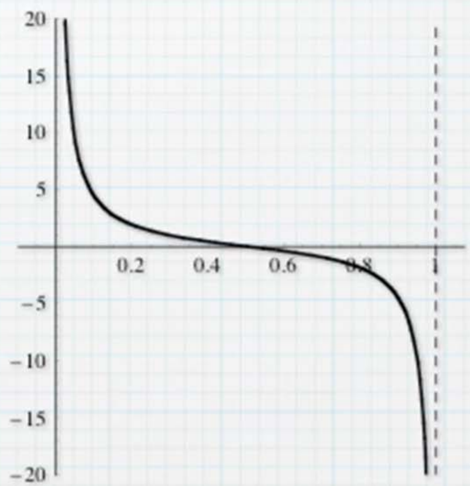
$$f(c) \leq f(x) \leq f(d) \text{ for all } x \text{ in } [a,b].$$

minimum value maximum value



Remarks

Here f is continuous on the open interval $(0,1)$. It has no minimum value, and it has no maximum value.



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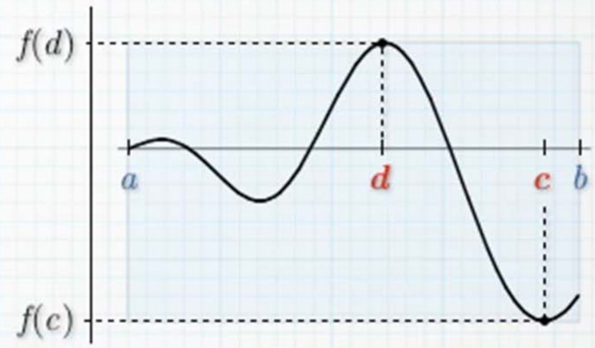
The Extreme Value Theorem

Suppose that f is continuous on a closed bounded interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ such that

$$\underbrace{f(c)}_{\text{minimum value}} \leq f(x) \leq \underbrace{f(d)}_{\text{maximum value}} \text{ for all } x \text{ in } [a, b].$$



Remarks

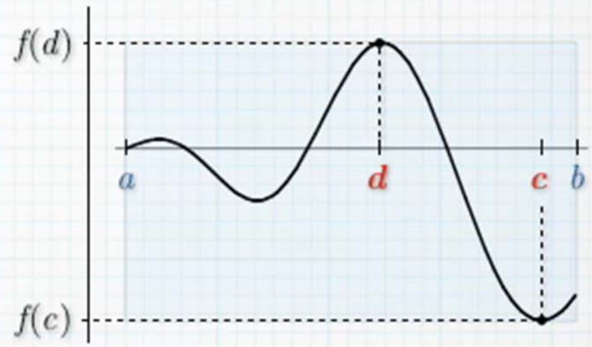
Here f is defined, but not continuous, on the closed interval $[0, 1]$. Its maximum value is 1, but it has no minimum value.

The Extreme Value Theorem

Suppose that f is continuous on a closed bounded interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d) \text{ for all } x \text{ in } [a, b].$$

minimum value maximum value



Remarks

Finally, we make note of a trivial case. Here f is a constant function. Its maximum value and its minimum value are the same, and that value is attained at every point in the interval.