

Vector addition: Translation and vector addition

- ▶ With complex numbers, translation achieved by adding a complex number, e.g.
 $f(z) = z + (1 + 2i)$
- ▶ Let's do the same thing with vectors...
 - ▶ **Definition of vector addition:**

$$[u_1, u_2, \dots, u_n] + [v_1, v_2, \dots, v_n] = [u_1 + v_1, u_2 + v_2, \dots, u_n + v_n]$$

- ▶ For 2-vectors represented in Python as 2-element lists, addition procedure is
`def add2(v,w): return [v[0]+w[0], v[1]+w[1]]`

Vector addition: Translation and vector addition

Quiz: Suppose we represent n -vectors by n -element lists. Write a procedure `addn(v, w)` to compute the sum of two vectors so represented.

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```
def addn(v, w): return [v[i]+w[i] for i in range(len(v))]
```

Vector addition: The zero vector

The D -vector whose entries are all zero is the *zero vector*, written $\mathbf{0}_D$ or just $\mathbf{0}$

$$\mathbf{v} + \mathbf{0} = \mathbf{v}$$

Vector addition: Vector addition is associative and commutative

- ▶ *Associativity*

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$$

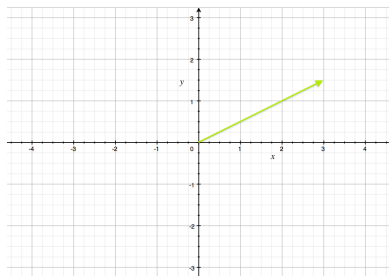
- ▶ *Commutativity*

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

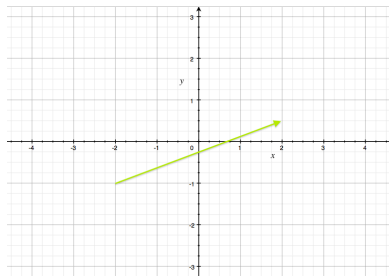
Vector addition: Vectors as arrows

Like complex numbers in the plane, n -vectors over \mathbb{R} can be visualized as *arrows* in \mathbb{R}^n .

The 2-vector $[3, 1.5]$ can be represented by an arrow with its tail at the origin and its head at $(3, 1.5)$.



or, equivalently, by an arrow whose tail is at $(-2, -1)$ and whose head is at $(1, 0.5)$.

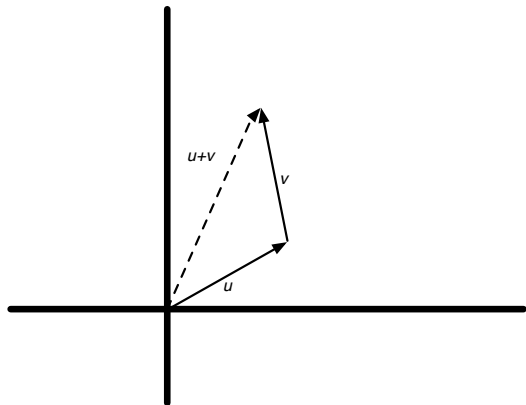


Vector addition: Vectors as arrows

Like complex numbers, addition of vectors over \mathbb{R} can be visualized using arrows.

To add \mathbf{u} and \mathbf{v} :

- ▶ place tail of \mathbf{v} 's arrow on head of \mathbf{u} 's arrow;
- ▶ draw a new arrow from tail of \mathbf{u} to head of \mathbf{v} .



Scalar-vector multiplication

With complex numbers, *scaling* was multiplication by a real number $f(z) = r z$

For vectors,

- ▶ we refer to field elements as *scalars*;
- ▶ we use them to scale vectors:

$$\alpha \mathbf{v}$$

Greek letters (e.g. α, β, γ) denote scalars.

Scalar-vector multiplication

Definition: Multiplying a vector \mathbf{v} by a scalar α is defined as multiplying each entry of \mathbf{v} by α :

$$\alpha [v_1, v_2, \dots, v_n] = [\alpha v_1, \alpha v_2, \dots, \alpha v_n]$$

Example: $2 [5, 4, 10] = [2 \cdot 5, 2 \cdot 4, 2 \cdot 10] = [10, 8, 20]$

Scalar-vector multiplication

Quiz: Suppose we represent n -vectors by n -element lists. Write a procedure `scalar_vector_mult(alpha, v)` that multiplies the vector `v` by the scalar `alpha`.

Answer: `def scalar_vector_mult(alpha, v):`

Scalar-vector multiplication

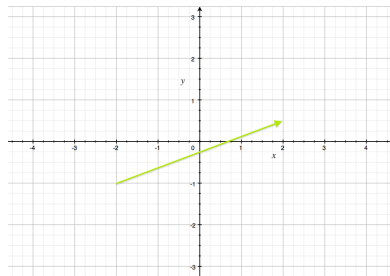
Quiz: Suppose we represent n -vectors by n -element lists. Write a procedure `scalar_vector_mult(alpha, v)` that multiplies the vector `v` by the scalar `alpha`.

Answer:

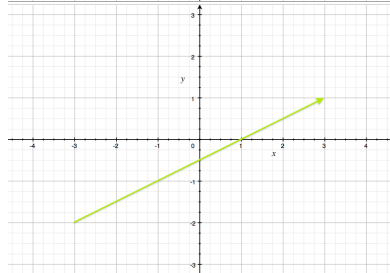
```
def scalar_vector_mult(alpha, v):    return [alpha*x for x in v]
```

Scalar-vector multiplication: Scaling arrows

An arrow representing the vector $[3, 1.5]$ is this:



and an arrow representing two times this vector is this:



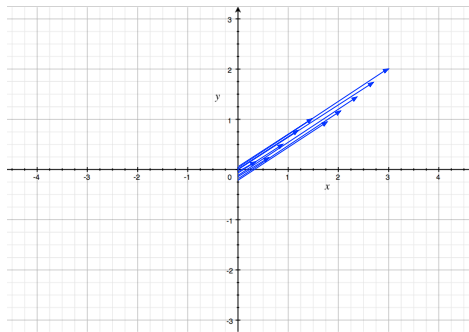
Scalar-vector multiplication: Associativity of scalar-vector multiplication

Associativity: $\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}$

Scalar-vector multiplication: Line segments through the origin

Consider scalar multiples of $\mathbf{v} = [3, 2]$:
 $\{0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$

For each value of α in this set,
 $\alpha \mathbf{v}$ is shorter than \mathbf{v} but in same direction.



Scalar-vector multiplication: Line segments through the origin

Conclusion: The set of points

$$\{\alpha \mathbf{v} : \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}$$

forms the line segment between the origin and \mathbf{v}

Scalar-vector multiplication: Lines through the origin

What if we let α range over all real numbers?

- ▶ Scalars bigger than 1 give rise to somewhat larger copies
- ▶ Negative scalars give rise to vectors pointing in the opposite direction



The set of points

$$\{\alpha \mathbf{v} : \alpha \in \mathbb{R}\}$$

forms the line through the origin and \mathbf{v}

Combining vector addition and scalar multiplication

We want to describe the set of points forming an arbitrary line segment (not necessarily through the origin).

Idea: Use the idea of translation.

Start with line segment from $[0, 0]$ to $[3, 2]$:

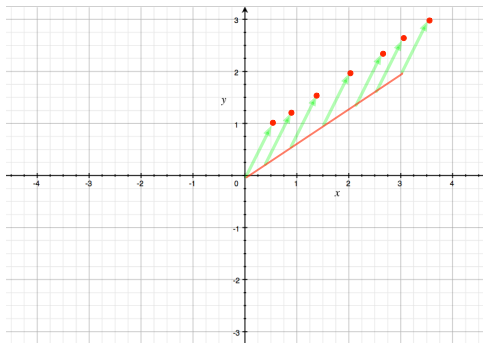
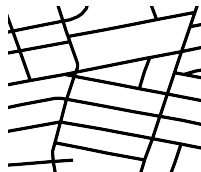
$$\{\alpha [3, 2] : 0 \leq \alpha \leq 1\}$$

Translate it by adding $[0.5, 1]$ to every point:

$$\{[0.5, 1] + \alpha [3, 2] : 0 \leq \alpha \leq 1\}$$

Get line segment from $[0, 0] + [0.5, 1]$ to

$[3, 2] + [0.5, 1]$



Combining vector addition and scalar multiplication: Distributive laws for scalar-vector multiplication and vector addition

Scalar-vector multiplication distributes over vector addition:

$$\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$$

Example:

- ▶ On the one hand,

$$2([1, 2, 3] + [3, 4, 4]) = 2[4, 6, 7] = [8, 12, 14]$$

- ▶ On the other hand,

$$2([1, 2, 3] + [3, 4, 4]) = 2[1, 2, 3] + 2[3, 4, 4] = [2, 4, 6] + [6, 8, 8] = [8, 12, 14]$$

Combining vector addition and scalar multiplication: First look at convex combinations

Set of points making up the the $[0.5, 1]$ -to- $[3.5, 3]$ segment:

$$\{\alpha [3, 2] + [0.5, 1] : \alpha \in \mathbb{R}, 0 \leq \alpha \leq 1\}$$

Not symmetric with respect to endpoints ☹️

Use distributivity:

$$\begin{aligned}\alpha [3, 2] + [0.5, 1] &= \alpha ([3.5, 3] - [0.5, 1]) + [0.5, 1] \\ &= \alpha [3.5, 3] - \alpha [0.5, 1] + [0.5, 1] \\ &= \alpha [3.5, 3] + (1 - \alpha) [0.5, 1] \\ &= \alpha [3.5, 3] + \beta [0.5, 1]\end{aligned}$$

where $\beta = 1 - \alpha$

New formulation:

$$\{\alpha [3.5, 3] + \beta [0.5, 1] : \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Symmetric with respect to endpoints 😊

Combining vector addition and scalar multiplication: First look at convex combinations

New formulation:

$$\{\alpha [3.5, 3] + \beta [0.5, 1] : \alpha, \beta \in \mathbb{R}, \alpha, \beta \geq 0, \alpha + \beta = 1\}$$

Symmetric with respect to endpoints 😊

An expression of the form

$$\alpha \mathbf{u} + \beta \mathbf{v}$$

where $0 \leq \alpha \leq 1, 0 \leq \beta \leq 1$, and $\alpha + \beta = 1$ is called a *convex combination* of \mathbf{u} and \mathbf{v}

The \mathbf{u} -to- \mathbf{v} line segment consists of the set of convex combinations of \mathbf{u} and \mathbf{v} .










Combining vector addition and scalar multiplication: First look at convex combinations

$$\mathbf{u} = \text{img}_1 \quad \text{and} \quad \mathbf{v} = \text{img}_2$$

Use scalars $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$:

$$\frac{1}{2} \text{img}_1 + \frac{1}{2} \text{img}_2 = \text{img}_3$$

“Line segment” between two faces:

								
$1\mathbf{u} + 0\mathbf{v}$	$\frac{7}{8}\mathbf{u} + \frac{1}{8}\mathbf{v}$	$\frac{6}{8}\mathbf{u} + \frac{2}{8}\mathbf{v}$	$\frac{5}{8}\mathbf{u} + \frac{3}{8}\mathbf{v}$	$\frac{4}{8}\mathbf{u} + \frac{4}{8}\mathbf{v}$	$\frac{3}{8}\mathbf{u} + \frac{5}{8}\mathbf{v}$	$\frac{2}{8}\mathbf{u} + \frac{6}{8}\mathbf{v}$	$\frac{1}{8}\mathbf{u} + \frac{7}{8}\mathbf{v}$	$0\mathbf{u} + 1\mathbf{v}$

Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at convex combinations



Combining vector addition and scalar multiplication: First look at affine combinations

Infinite line through $[0.5, 1]$ and $[3.5, 3]$?

Our formulation so far ☹️

$$\{[0.5, 1] + \alpha [3, 2] : \alpha \in \mathbb{R}\}$$

Nicer formulation 😊:

$$\{\alpha [3.5, 3] + \beta [0.5, 1] : \alpha \in \mathbb{R}, \beta \in \mathbb{R}, \alpha + \beta = 1\}$$

An expression of the form $\alpha \mathbf{u} + \beta \mathbf{v}$ where $\alpha + \beta = 1$ is called an *affine* combination of \mathbf{u} and \mathbf{v} .

The line through \mathbf{u} and \mathbf{v} consists of the set of affine combinations of \mathbf{u} and \mathbf{v} .