Simplified Exchange Lemma

We need a tool to iteratively transform one set of generators into another.

- You have a set S of vectors.
- You have a vector z you want to inject into S.
- You want to maintain same size so must eject a vector from S.
- You want the span to not change.

Exchange Lemma tells you how to choose vector to eject.

Simplified Exchange Lemma:

- Suppose S is a set of vectors.
- Suppose $\mathbf{z}$ is a nonzero vector in Span S.
- Then there is a vector $\mathbf{w}$ in S such that

$$\text{Span}\ (S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = \text{Span}\ S$$

Simplified Exchange Lemma proof

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) =$ Span S.

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Since $\mathbf{z}$ is in Span S, can write

$$\mathbf{z} = \alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n$$

By Superfluous-Vector Lemma, Span $(S \cup \{z\}) =$ Span S.
Since $\mathbf{z}$ is nonzero, at least one of the coefficients is nonzero, say α_i.
Rewrite as

$$\mathbf{z} - \alpha_1\,\mathbf{v}_1 - \cdots - \alpha_{i-1}\,\mathbf{v}_{i-1} - \alpha_{i+1}\,\mathbf{v}_{i+1} - \cdots - \alpha_n\,\mathbf{v}_n = \alpha_i\,\mathbf{v}_i$$

Divide through by α_i:

$$(1/\alpha_i)\mathbf{z} - (\alpha_1/\alpha_i)\,\mathbf{v}_1 - \cdots - (\alpha_{i-1}/\alpha_i)\,\mathbf{v}_{i-1} - (\alpha_{i+1}/\alpha_i)\,\mathbf{v}_{i+1} - \cdots - (\alpha_n/\alpha_i)\,\mathbf{v}_n = \mathbf{v}_i$$

By Superfluous-Vector Lemma, Span $(S \cup \{\mathbf{z}\}) =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$. QED

Exchange Lemma

Simplified Exchange Lemma: Suppose S is a set of vectors, and $\mathbf{z}$ is a nonzero vector in Span S. Then there is a vector $\mathbf{w}$ in S such that Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\}) = $ Span S.

Simplified Exchange Lemma helps in transforming one generating set into another...

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inject ■

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Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.

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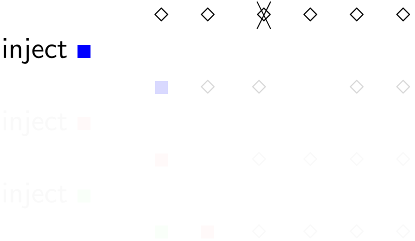

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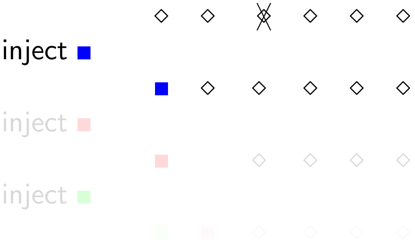

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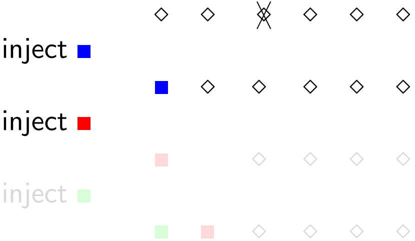

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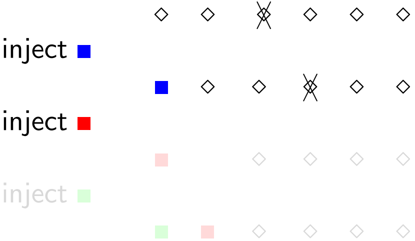

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inject $\blacksquare$

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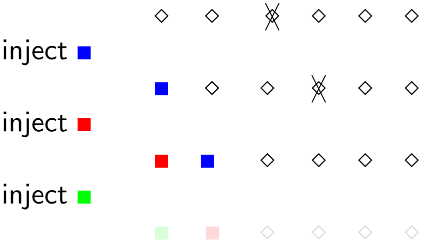

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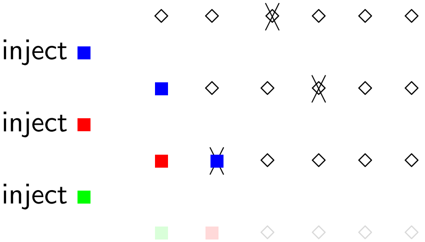

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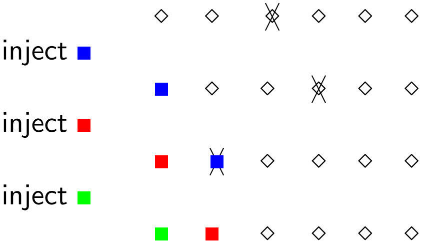

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Need to enhance this lemma. Set of *protected* elements is A:

Exchange Lemma:

- Suppose S is a set of vectors and A is a subset of S.
- Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
- Then there is a vector $\mathbf{w} \in S - A$ such that Span $S =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$

Now, not enough that $\mathbf{z}$ be nonzero—need A to be linearly independent.

Exchange Lemma proof

Exchange Lemma: Suppose S is a set of vectors and A is a subset of S. Suppose $\mathbf{z}$ is a vector in Span S such that $A \cup \{\mathbf{z}\}$ is linearly independent.
Then there is a vector $\mathbf{w} \in S - A$ such that Span $S =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}\})$

Proof: Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$ and $A = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$.
Since $\mathbf{z}$ is in Span S, can write

$$\mathbf{z} = \alpha_1\, \mathbf{v}_1 + \cdots + \alpha_k\, \mathbf{v}_k + \beta_1\; \mathbf{w}_1 + \cdots + \beta_\ell\; \mathbf{w}_\ell$$

By Superfluous-Vector Lemma, Span $(S \cup \{z\}) =$ Span S.
If coefficients $\beta_1, \ldots, \beta_\ell$ were all zero then we would have $\mathbf{z} = \alpha_1\, \mathbf{v}_1 + \cdots + \alpha_k\, \mathbf{v}_k$, contradicting the linear independence of $A \cup \{\mathbf{z}\}$.
Thus one of the coefficients $\beta_1, \ldots, \beta_\ell$ must be nonzero... say β_1. Rewrite as

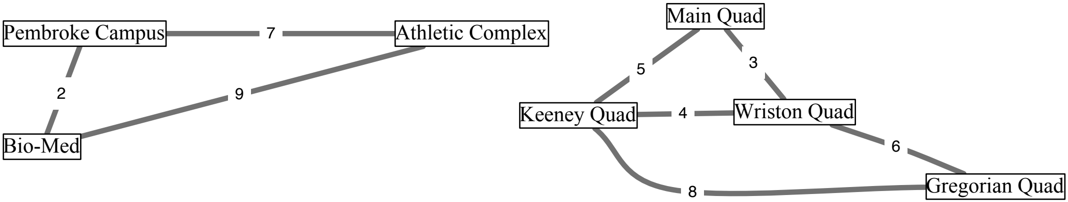
$$\mathbf{z} - \alpha_1\, \mathbf{v}_1 - \cdots - \alpha_k\, \mathbf{v}_k - \beta_2\, \mathbf{w}_2 - \cdots - \beta_\ell\, \mathbf{w}_\ell = \beta_1\, \mathbf{w}_1$$

Divide through by β_1:

$$(1/\beta_1)\mathbf{z} - (\alpha_1/\beta_1)\, \mathbf{v}_1 - \cdots - (\alpha_k/\beta_1)\, \mathbf{v}_k - (\beta_2/\beta_1)\, \mathbf{w}_2 - \cdots - (\beta_\ell/\beta_1)\, \mathbf{w}_\ell = \mathbf{w}_1$$

By Superfluous-Vector Lemma, Span $(S \cup \{\mathbf{z}\}) =$ Span $(S \cup \{\mathbf{z}\} - \{\mathbf{w}_1\})$. QED

Proof of correctness of the Grow algorithm for Minimum Spanning Forest



def $\textsc{Grow}(G)$
  $F := \emptyset$
  consider the edges in increasing order
  for each edge e:
    if e's endpoints are not yet connected
      add e to F.

We will show that this greedy algorithm chooses the minimum-weight spanning forest.

(Assume all weights are distinct.)

Let F = forest found by algorithm.

Let F^* = truly minimum-weight spanning forest.

Goal: show that $F = F^*$

Assume for a contradiction that they are different.

Proof of correctness of the Grow algorithm for Minimum Spanning Forest

Assume for a contradiction that F and F^* are different.
Let $e_1, e_2, \ldots, e_m$ be the edges of G in increasing order.
Let e_k be the minimum-weight edge on which F and F^* disagree.
Let A be the set of edges before e_k that are in both F and F^*.
Since at least one of the forests includes all of A and also e_k, we know $A \cup \{e_k\}$ has no cycles (is linearly independent).
Consider the moment when the Grow algorithm considers e_k. So far, the algorithm has chosen the edges in A, and e_k does not form a cycle with edges in A, so the algorithm must also choose e_k.
Since F and F^* differ on e_k, we infer that e_k is *not* in F^*.
Now we use the Exchange Lemma.

- A is a subset of F^*.
- $A \cup \{e_k\}$ is linearly independent.
- Therefore there is an edge e_n in $F^* - A$ such that
$$\text{Span } (F^* \cup \{e_k\} - \{e_n\}) = \text{Span } F^*$$

That is, $F^* \cup \{e_k\} - \{e_n\}$ is also spanning.
But e_k is cheaper than e_n so F^* is not minimum-weight solution. **Contradiction.**QED.