Change of Variables

(Substitution)

Differentials

Let $y = f(x)$. Then we *define* the **differential** dy by

$$\boxed{dy = f'(x)\,dx = \frac{dy}{dx}\,dx\,.}$$

$y = f(x)$	derivative	differential
$y = x^3$	$\dfrac{dy}{dx} = 3x^2$	$dy = 3x^2\,dx$
$u = \cos t$	$\dfrac{du}{dt} = -\sin t$	$du = -\sin t\,dt$
$x = \tan\theta$	$\dfrac{dx}{d\theta} = \sec^2\theta$	$dx = \sec^2\theta\,d\theta$
$w = \sqrt{x^2+1}$	$\dfrac{dw}{dx} = \dfrac{x}{\sqrt{x^2+1}}$	$dw = \dfrac{x}{\sqrt{x^2+1}}\,dx$

Old Formulas in Terms of Differentials

Linearity $d(u+v) = du + dv$ $\qquad d(cu) = c\,du$

Product rule $d(uv) = u\,dv + v\,du$

Quotient rule $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

Chain rule $d\,f(u) = f'(u)\,du$

Antidifferentiation

$$\int \underbrace{F'(x)\,dx}_{\uparrow} = F(x) + C$$

dy where $y = F(x)$

$$\int dy = y + C$$

Change of Variables ("u-Substitution")

$$\boxed{\int \underbrace{f(u(x))}_{f(u)}\,\underbrace{u'(x)\,dx}_{du} = \int f(u)\,du}$$

Reverse Chain Rule

Example

$$\int \underbrace{(x^2+1)^3}_{u^3}\,\underbrace{2x\,dx}_{du} = \int u^3 du = \frac{1}{4}u^4 + C$$

$$= \frac{1}{4}(x^2+1)^4 + C$$

Example

$$\int \underbrace{(x^2+1)^3}_{u^3}\,\underbrace{x\,dx}_{\frac{1}{2}du} = \frac{1}{2}\int u^3 du = \frac{1}{8}u^4 + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

$$u = x^2 + 1$$
$$du = 2x\,dx$$
$$x\,dx = \tfrac{1}{2}du$$

Example

$$\int \sqrt{3x+2}\,dx = \int \sqrt{u}\ \tfrac{1}{3}du = \tfrac{1}{3}\int \sqrt{u}\,du$$

$$u = 3x+2$$
$$du = 3\,dx$$
$$dx = \tfrac{1}{3}du$$

$$= \tfrac{1}{3}\,\tfrac{2}{3}\,u^{3/2} + C$$

$$= \tfrac{2}{9}\,(3x+2)^{3/2} + C$$

Example

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{x^2+1}\ \underline{x\,dx} = \int \sqrt{u}\ \tfrac{1}{2}\,du$$

$$u = x^2+1$$
$$du = 2x\,dx$$
$$x\,dx = \tfrac{1}{2}du$$

$$= \tfrac{1}{2}\int \sqrt{u}\,du = \tfrac{1}{2}\,\tfrac{2}{3}\,u^{3/2} + C$$

$$= \tfrac{1}{3}\,(x^2+1)^{3/2} + C$$

Example

$$\int \sqrt{1+\sin x}\,\underline{\cos x\,dx} = \int \sqrt{u}\,du = \tfrac{2}{3}\,u^{3/2} + C$$

$$u = 1 + \sin x$$
$$du = \cos x\,dx$$

$$= \tfrac{2}{3}\,(1+\sin x)^{3/2} + C$$

Example

$$\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \int \frac{\underline{x^2 dx}}{\sqrt{x^3+1}} = \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$$

$$u = x^3 + 1$$
$$du = 3x^2 dx$$
$$x^2 dx = \tfrac{1}{3} du$$

$$= \frac{1}{3}\,2\sqrt{u} + C$$

$$= \frac{2}{3}\sqrt{x^3+1} + C$$

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Example $\displaystyle\int x^2 \sin(x^3+1)\,dx = \int \sin(x^3+1)\,\underline{x^2dx}$

$$u = x^3 + 1$$
$$du = 3x^2dx$$
$$x^2dx = \tfrac{1}{3}du$$

$$= \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C$$
$$= -\frac{1}{3}\cos(x^3+1) + C$$

Example $\displaystyle\int \sin^3 x\,\underline{\cos x\,dx} = \int u^3 du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$

$$u = \sin x$$
$$du = \cos x\,dx$$

Example $\displaystyle\int \tan^2 x\,\underline{\sec^2 x\,dx} = \int u^2 du = \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

$$u = \tan x$$
$$du = \sec^2 x\,dx$$

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A Different Kind of Example

$$\int \frac{x}{(x-1)^3}dx = \int \frac{u+1}{u^3}\, du = \int (u^{-2}+u^{-3})du$$

$$= -u^{-1} - \frac{1}{2}u^{-2} + C$$

$$= \frac{-2u-1}{2u^2} + C$$

$$= \frac{-2\,(x-1)-1}{2\,(x-1)^2} + C$$

$$u = x - 1$$
$$du = dx$$
$$x = u + 1$$

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Definite Integrals

$$\boxed{\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du}$$

Example $$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int_{u(0)}^{u(2)} \frac{1}{\sqrt{u}}\,du = \frac{1}{3}\int_1^9 \frac{1}{\sqrt{u}}\,du$$

$$u = x^3 + 1$$
$$du = 3x^2 dx$$
$$x^2 dx = \tfrac{1}{3} du$$

$$= \frac{2}{3}\sqrt{u}\,\Big|_1^9 = \frac{2}{3}(3-1) = \frac{4}{3}$$

Example $$\int_0^{\pi/2} \sin^3 x \cos x\,dx = \int_{u(0)}^{u(\pi/2)} u^3\,du = \int_0^1 u^3\,du = \frac{1}{4}\,u^4\,\Big|_0^1$$

$$u = \sin x$$
$$du = \cos x\,dx$$

$$= \frac{1}{4}(1-0) = \frac{1}{4}$$

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Example

$$\int_0^1 (x+1)\sqrt{1-x^2}\,dx = \int_0^1 x\sqrt{1-x^2}\,dx + \int_0^1 \sqrt{1-x^2}\,dx$$

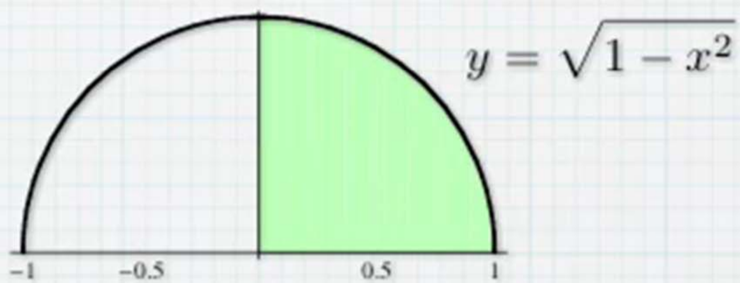
$$= \frac{1}{3} + \frac{\pi}{4} = \frac{4+3\pi}{12}$$

1st term
$$\int_0^1 x\sqrt{1-x^2}\,dx = -\frac{1}{2}\int_1^0 \sqrt{u}\,du = \frac{1}{2}\int_0^1 \sqrt{u}\,du = \frac{1}{2}\,\frac{2}{3}\,u^{3/2}\Big|_0^1$$

$$= \frac{1}{3}$$

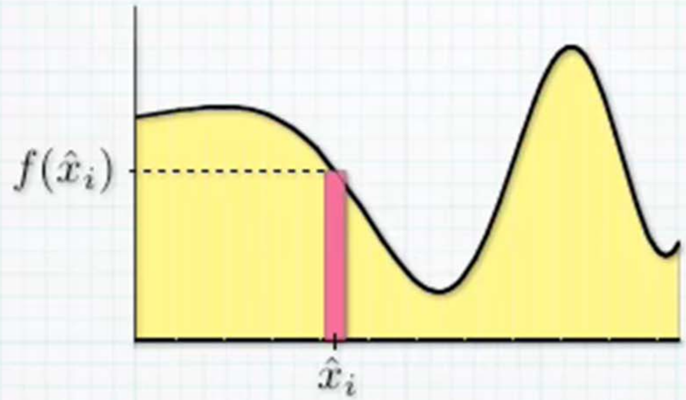
$$u = 1-x^2$$
$$du = -2x\,dx$$
$$x\,dx = -\tfrac{1}{2}\,du$$

2nd term
$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$



Areas Between Curves

Area of a Region Defined by $0 \leq y \leq f(x)$ and $a \leq x \leq b$



Area of a typical rectangle $\Delta A_i = f(\hat{x}_i)\Delta x$

Riemann sum $A \approx \sum \Delta A_i = \sum f(\hat{x}_i)\Delta x$

Area of a typical thin *slice* $dA = f(x)\,dx$ **The *area differential***

Definite integral $A = \int_{x=a}^{x=b} dA = \int_a^b f(x)\,dx$

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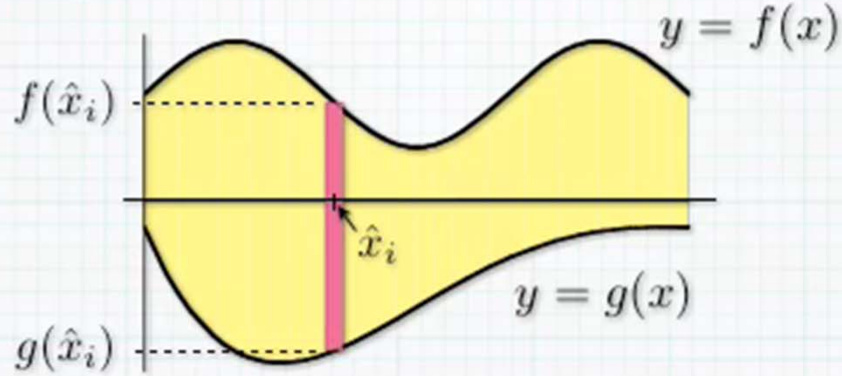
Area of a Region Defined by $g(x) \leq y \leq f(x)$ and $a \leq x \leq b$



Area of a typical rectangle $\Delta A_i = \big(f(\hat{x}_i) - g(\hat{x}_i)\big)\Delta x$

Riemann sum $A \approx \sum \Delta A_i = \sum \big(f(\hat{x}_i) - g(\hat{x}_i)\big)\Delta x$

Area of a typical thin *slice* $dA = \big(f(x) - g(x)\big)\, dx$ **The *area differential***

Definite integral $A = \displaystyle\int_{x=a}^{x=b} dA = \int_a^b \big(\underset{\substack{\uparrow \\ \text{top} \\ \text{curve}}}{f(x)} - \underset{\substack{\uparrow \\ \text{bottom} \\ \text{curve}}}{g(x)}\big)\, dx$

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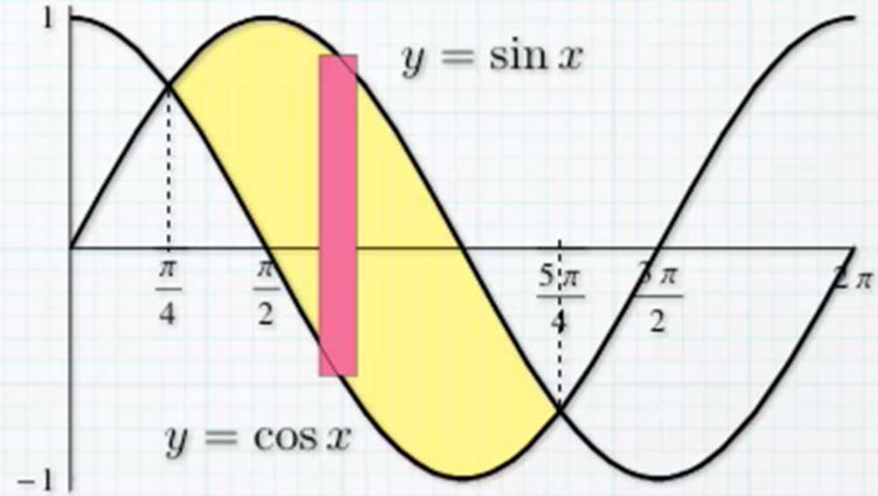
Example Find the area of the region bounded by the graphs of $y = \sin x$ and $y = \cos x$ between $x = \pi/4$ and $x = 5\pi/4$.



$$\Delta A_i = (\sin \hat{x}_i - \cos \hat{x}_i)\Delta x$$

$$dA = (\sin x - \cos x)\, dx$$

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4}$$

$$= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

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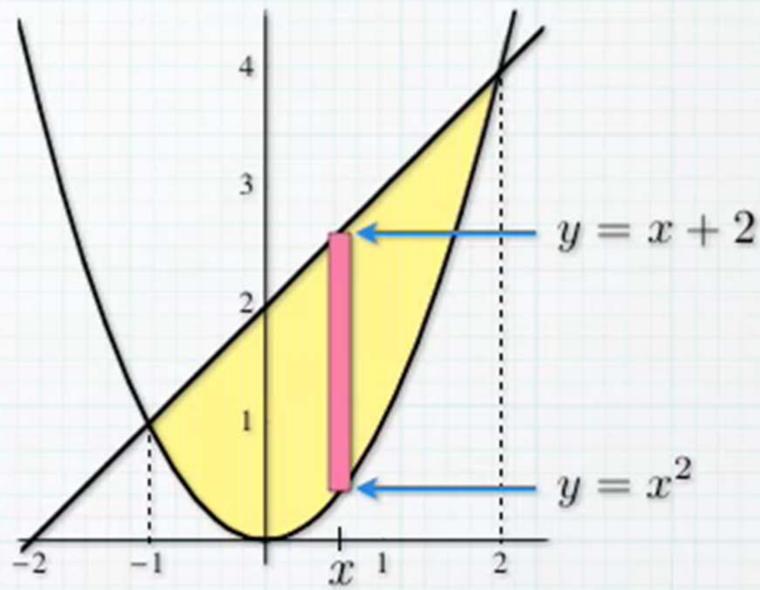
Example Find the area of the region bounded by the graphs of $y = x^2$ and $y = x + 2$.



Points of intersection

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1,\ 2$$

$$dA = (x + 2 - x^2)\,dx$$

$$A = \int_{-1}^{2} (x + 2 - x^2)\,dx$$

$$= \left(\frac{1}{2}x^2 + 2x - \frac{1}{3}x^3\right)\Bigg|_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - \frac{9}{3} - \frac{1}{2} = \frac{9}{2}$$

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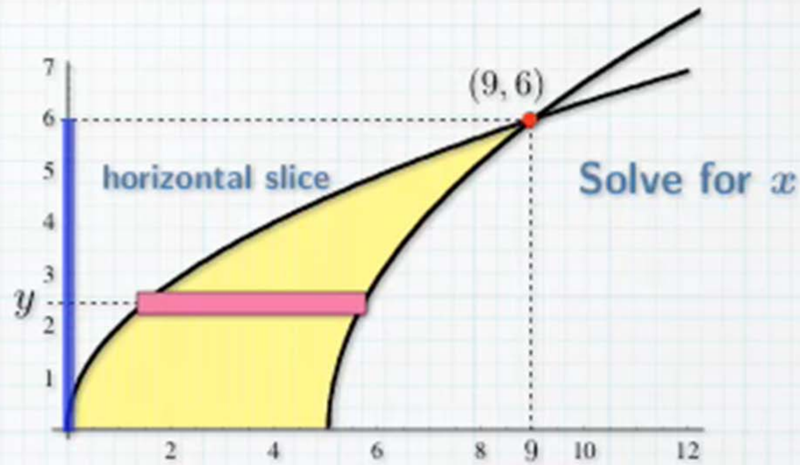
Example Find the area of the region bounded by the graphs of

$$y = 2\sqrt{x},\ \ y = 0, \text{ and } y = 3\sqrt{x-5}.$$



$$\begin{aligned} dA &= \left(\tfrac{1}{9}\,y^2 + 5 - \tfrac{1}{4}\,y^2\right) dy \\ &= \left(5 - \tfrac{5}{36}\,y^2\right) dy \\ &= \tfrac{5}{36}\left(36 - y^2\right) dy \end{aligned}$$

$$A = \frac{5}{36}\int_0^6 (36 - y^2)\,dy \ = \frac{5}{36}\left(36y - \frac{1}{3}\,y^3\right)\Bigg|_0^6$$

$$= \frac{5}{36}\left(216 - \frac{1}{3}\,216 - 0\right) = \frac{5}{36}(144)$$

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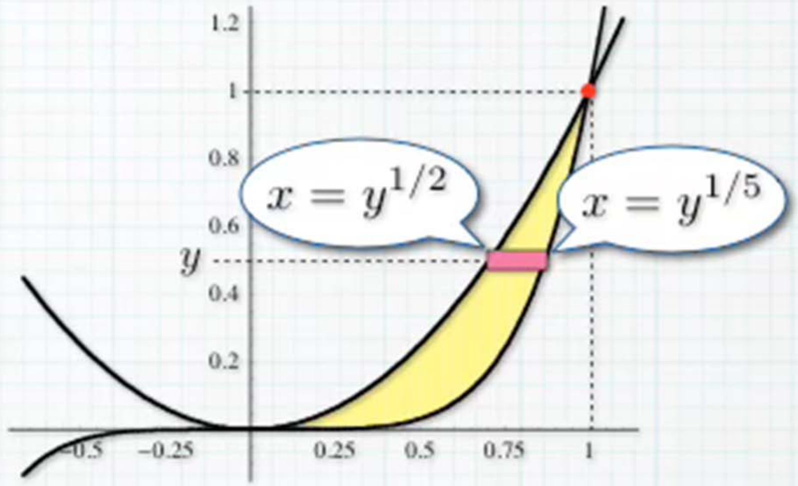
Example Find the area of the region bounded by the graphs of $y = x^2$ and $y = x^5$ two ways.



Vertical slices:
Integration with respect to x

$$dA = (x^2 - x^5)\,dx$$

$$A = \int_0^1 (x^2 - x^5)\,dx = \left(\tfrac{1}{3}\,x^3 - \tfrac{1}{6}\,x^6\right)\Big|_0^1 = \tfrac{1}{3} - \tfrac{1}{6} = \tfrac{1}{6}$$

Horizontal slices: Integration with respect to y

$$dA = (y^{1/5} - y^{1/2})\,dy$$

$$A = \int_0^1 (y^{1/5} - y^{1/2})\,dy = \left(\tfrac{5}{6}\,y^{6/5} - \tfrac{2}{3}\,y^{3/2}\right)\Big|_0^1 = \tfrac{5}{6} - \tfrac{2}{3}$$

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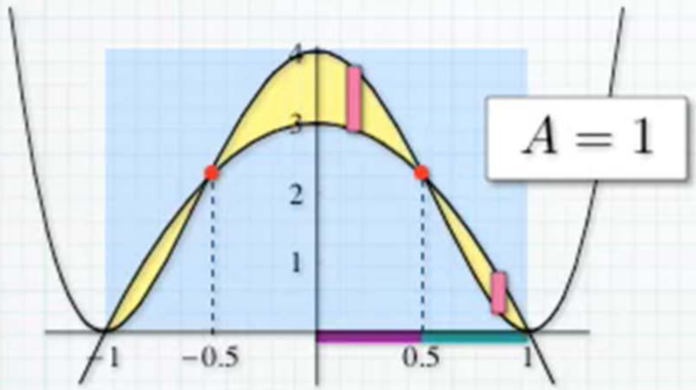
Example Find the area of the region bounded by the graphs of $y = 3\,(1 - x^2)$ and $y = 4\,(1 - x^2)^2$.



Intersections

$$3\,(1 - x^2) = 4\,(1 - x^2)^2$$
$$3 = 4\,(1 - x^2)$$
$$3 = 4 - 4x^2$$
$$4x^2 = 1$$
$$x = \pm\tfrac{1}{2}$$

$$\frac{1}{2}\,A = \int_0^{1/2} \left(4\,(1 - x^2)^2 - 3\,(1 - x^2)\right) dx + \int_{1/2}^{1} \left(3\,(1 - x^2) - 4\,(1 - x^2)^2\right) dx$$

$$= \int_0^{1/2} \left(4x^4 - 5x^2 + 1\right) dx + \int_{1/2}^{1} \left(-4x^4 + 5x^2 - 1\right) dx$$

$$= \left(\tfrac{4}{5}\,x^5 - \tfrac{5}{3}\,x^3 + x\right)\Big|_0^{1/2} + \left(-\tfrac{4}{5}\,x^5 + \tfrac{5}{3}\,x^3 - x\right)\Big|_{1/2}^{1}$$

$$= \left(\tfrac{4}{5}\,\tfrac{1}{32} - \tfrac{5}{3}\,\tfrac{1}{8} + \cancel{\tfrac{1}{2}}\right) - 0 + \left(-\tfrac{4}{5} + \tfrac{5}{3} - \cancel{1}\right) - \left(-\tfrac{4}{5}\,\tfrac{1}{32} + \tfrac{5}{3}\,\tfrac{1}{8} - \cancel{\tfrac{1}{2}}\right)$$

$$= \frac{8}{160} - \frac{10}{24} - \frac{4}{5} + \frac{5}{3} = \frac{1}{20} - \frac{5}{12} - \frac{4}{5} + \frac{5}{3} = \frac{3 - 25 - 48 + 100}{60} = \frac{30}{60} = \frac{1}{2}$$

Volume Calculations 1

Integration of Cross-sectional Area

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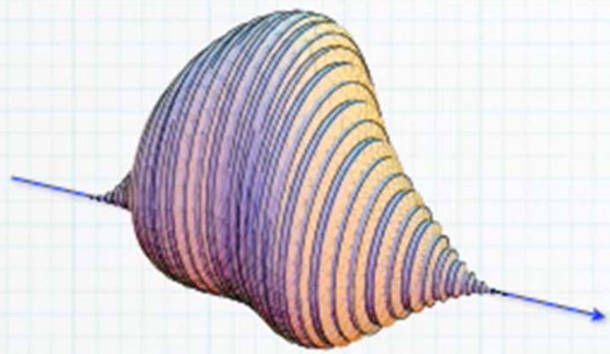
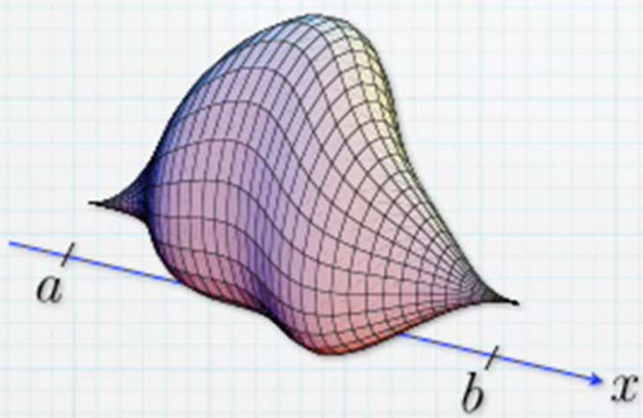
Solids with cross-sectional area $A(x)$



$$V \approx \sum A(x_i)\Delta x$$

$\begin{array}{c}\text{slice}\\\text{volume}\end{array} = A(x_i)\Delta x$ where $A(x) =$ cross-sectional area perpendicular to the x-axis



$$V = \lim_{\Delta x \to 0} \sum A(x_i)\Delta x$$

$$= \int_a^b A(x)\,dx$$

Volume is the integral of cross-sectional area.

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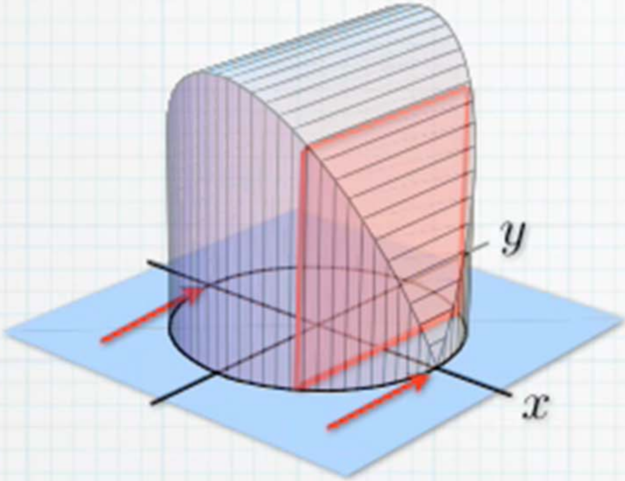
Example A solid's base is the unit disk in the xy-plane, and its vertical cross-sections parallel to the y-axis are squares. Find the volume of the solid.



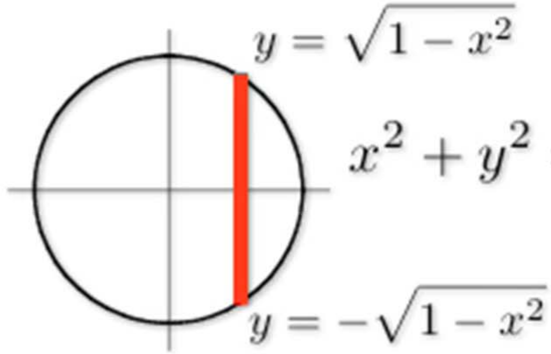
$$V \approx \sum A(x_i)\Delta x$$

$$V = \int_{-1}^{1} A(x)\, dx \; = \int_{-1}^{1} 4\,(1 - x^2)\, dx$$

$$= 2\int_{0}^{1} 4\,(1 - x^2)\, dx$$

$$= 8\,(x - \tfrac{1}{3}\,x^3)\Big|_0^1$$

$$= 8\,(\tfrac{2}{3} - 0) \; = \frac{16}{3}$$

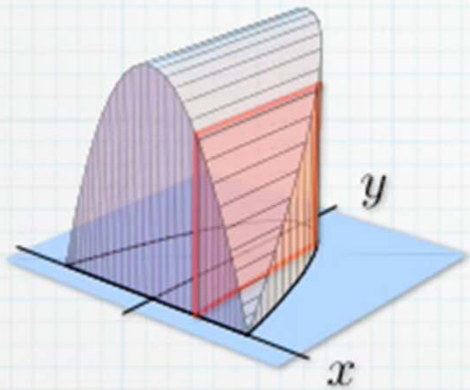


$$A(x) = \left(2\sqrt{1 - x^2}\,\right)^2 = 4\,(1 - x^2)$$

Example A solid whose base is the planar region in which

$$0 \le y \le 2\,(1-x^2)$$

has square vertical cross-sections parallel to the y-axis. Find the volume of the solid.



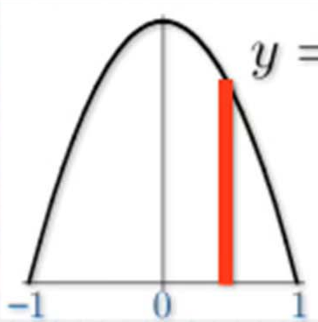
$$V = \int_{-1}^{1} A(x)\,dx \;=\; \int_{-1}^{1} 4\,(1-x^2)^2\,dx$$

$$= 8\int_{0}^{1} (1-x^2)^2\,dx$$

$$= 8\int_{0}^{1} (1-2x^2+x^4)\,dx$$

$$= 8\left(x - \frac{2}{3}\,x^3 + \frac{1}{5}\,x^5\right)\Big|_0^1$$

$$= 8\left(1-\frac{2}{3}+\frac{1}{5}\right) = \frac{64}{15}$$



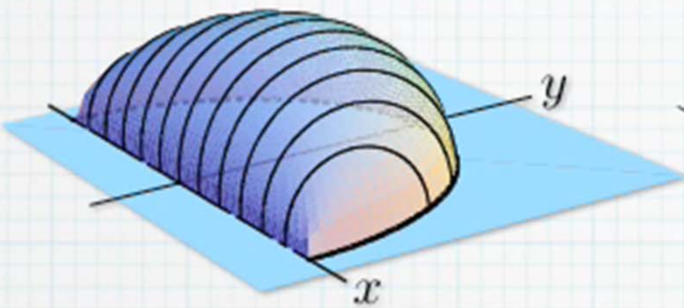
$$y = 2\,(1-x^2)$$

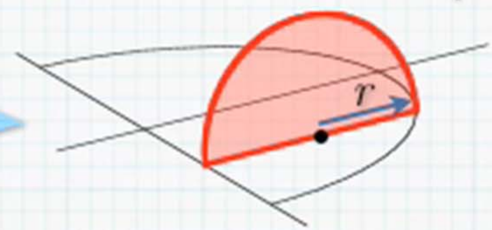
$$A(x) = \left(2\,(1-x^2)\right)^2$$
$$= 4\,(1-x^2)^2$$

Example A solid's base is the planar region in which

$$0 \le y \le \sqrt{1-x^2}$$

and its vertical cross-sections parallel to the y-axis are semi-circles. Find the volume of the solid.





$$\begin{aligned} A(x) &= \tfrac{1}{2}\,\pi\, r^2 \\ &= \tfrac{1}{2}\,\pi\left(\tfrac{1}{2}\sqrt{1-x^2}\,\right)^2 \\ &= \tfrac{\pi}{8}\,(1-x^2) \end{aligned}$$

$$\begin{aligned} V = \int_{-1}^{1} A(x)\,dx \;=\; \frac{\pi}{8}\int_{-1}^{1}(1-x^2)\,dx \;&=\; \frac{\pi}{4}\int_{0}^{1}(1-x^2)\,dx \\ &= \frac{\pi}{4}\left(x-\frac{1}{3}\,x^3\right)\Big|_0^1 \\ &= \frac{\pi}{4}\left(1-\frac{1}{3}-0\right) = \frac{\pi}{6} \end{aligned}$$

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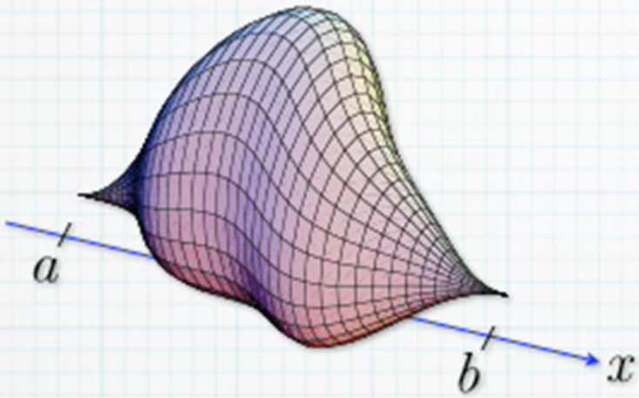
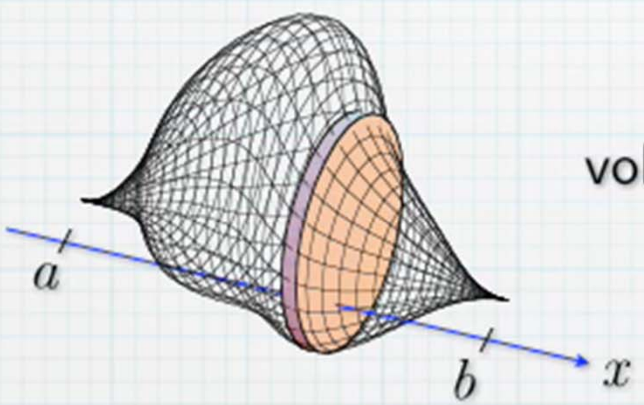
The Volume Differential



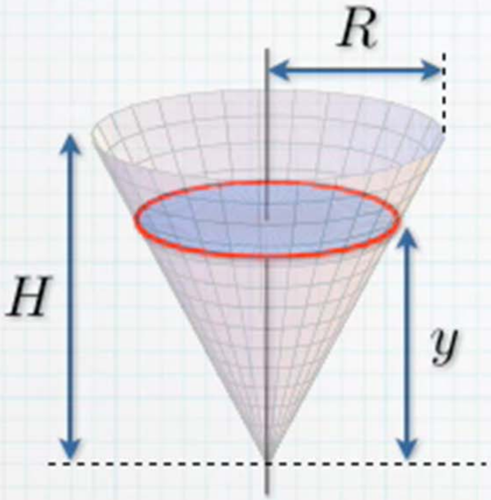
$$V = \int_a^b \underbrace{A(x)\,dx}_{dV}$$

The volume differential: $dV = A(x)\,dx$

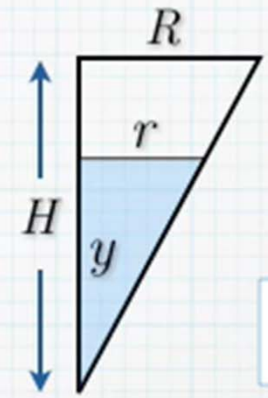


Think of this as the volume of a typical "slice" with thickness dx.

Example Derive the formula for the volume of a right circular cone with height H and radius R.



$$
\begin{aligned}
dV &= \pi r^2\, dy \\
&= \pi \left(\frac{R}{H}\, y\right)^2 dy \\
&= \frac{\pi R^2}{H^2}\, y^2\, dy
\end{aligned}
$$



similar triangles

$$\frac{r}{y} = \frac{R}{H}$$

$$\boxed{r = \frac{R}{H}\, y}$$

$$V = \int_{y=0}^{y=H} dV = \frac{\pi R^2}{H^2} \int_0^H y^2\, dy = \frac{\pi R^2}{H^2}\, \frac{1}{3}\, H^3 = \boxed{\frac{1}{3}\, \pi R^2 H}$$

Volume Calculations 2

Solids of Revolution

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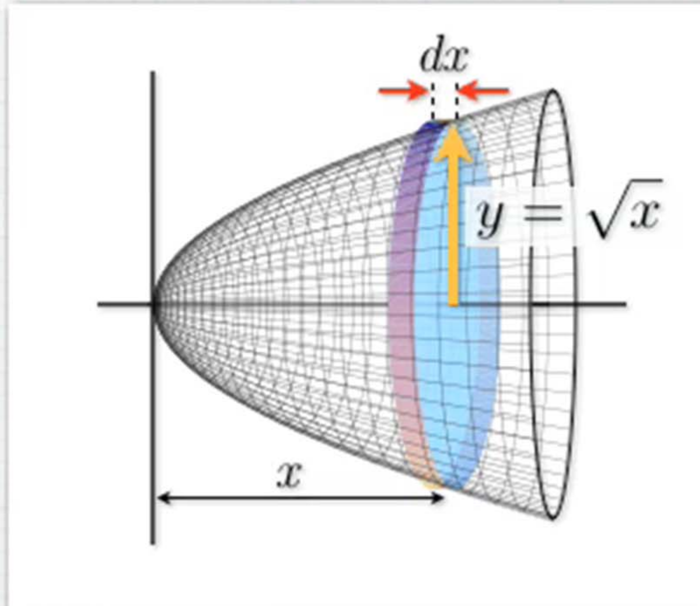
Consider, for example, the planar region in which

$$0 \le y \le \sqrt{x}, \;\; 0 \le x \le 4.$$

By revolving the region about the x-axis we obtain a **solid of revolution**.

Cross-sections perpendicular to the axis of rotation are circular; so a typical slice may be viewed as a thin *disk*.



So we compute the volume of the solid as follows:

disk radius $\quad r = y = \sqrt{x}$

disk volume $\quad dV = \pi r^2\, dx = \pi\, (\sqrt{x}\,)^2\, dx = \pi\, x\, dx$

total volume $\quad V = \displaystyle\int_0^4 \pi\, x\, dx = \frac{\pi}{2}\, x^2 \Big|_0^4 = \frac{\pi}{2}\,(4^2 - 0^2)$

$$= 8\pi$$

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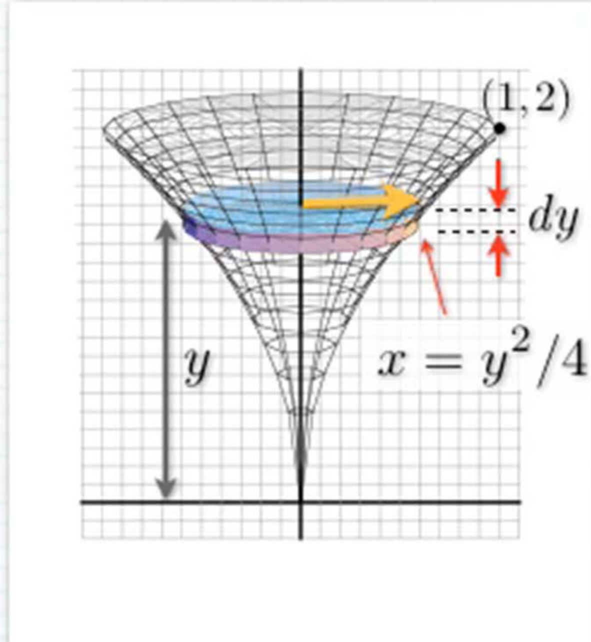
Example Let $\mathcal{R}$ be the region bounded by the graphs of

$$y = 2\sqrt{x}, \ y = 2, \ \text{and } x = 0.$$

Find the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.



disk radius $\quad r = x = y^2/4$

disk volume $\quad dV = \pi x^2 dy = \pi \dfrac{y^4}{16} dy$

total volume $\quad V = \dfrac{\pi}{16} \displaystyle\int_0^2 y^4 \, dy = \dfrac{\pi}{16} \dfrac{1}{5} y^5 \Big|_0^2$

$$= \frac{\pi}{16} \frac{1}{5} (32 - 0) = \frac{2\pi}{5}$$

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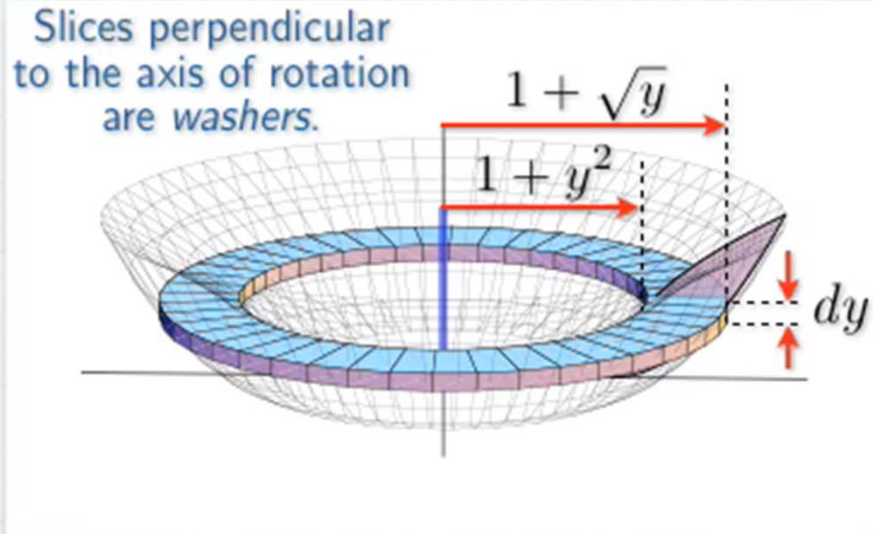
Example
Find the volume of the solid obtained by revolving the region bounded by the graphs of

$y = (x-1)^2$ and $y = \sqrt{x-1}$

about the y-axis.



washer volume: $$dV = \pi\left((1+\sqrt{y}\,)^2 - (1+y^2)^2\right)dy$$
$$= \pi\,(2\sqrt{y} + y - 2y^2 - y^4)\,dy$$

volume of the solid: $$V = \pi\int_0^1 (2\sqrt{y} + y - 2y^2 - y^4)\,dy$$
$$= \pi\left(\tfrac{4}{3}\,y^{3/2} + \tfrac{1}{2}\,y^2 - \tfrac{2}{3}\,y^3 - \tfrac{1}{5}\,y^5\right)\Big|_0^1$$
$$= \pi\left(\frac{4}{3} + \frac{1}{2} - \frac{2}{3} - \frac{1}{5}\right) = \frac{29}{30}\,\pi$$

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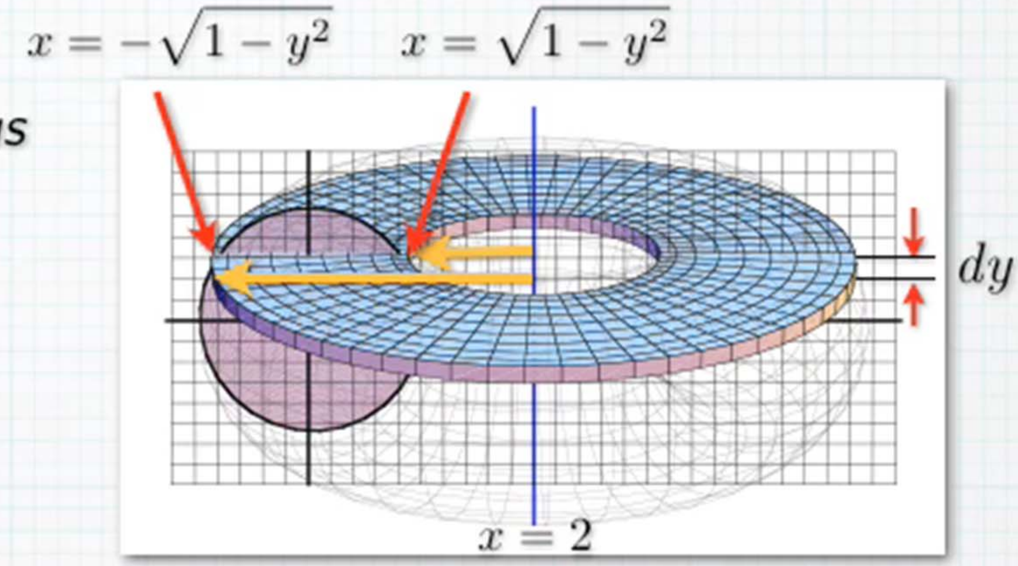
Example
Find the volume of the *torus* obtained by revolving the unit disk

$$x^2 + y^2 \leq 1$$

about the line $x = 2$.



inner radius $= \ 2 - \sqrt{1-y^2}$

outer radius $= \ 2 + \sqrt{1-y^2}$

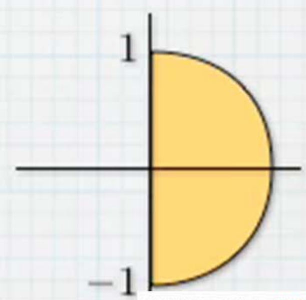
washer volume:

$$dV = \pi \left(\left(2 + \sqrt{1-y^2}\right)^2 - \left(2 - \sqrt{1-y^2}\right)^2 \right) dy$$

$$= 8\pi \sqrt{1-y^2}\, dy$$

volume of the torus:

$$V = 8\pi \int_{-1}^{1} \sqrt{1-y^2}\, dy = 8\pi \cdot \frac{\pi}{2} = 4\pi^2$$



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Volume Calculations 3

The Cylindrical Shell Method

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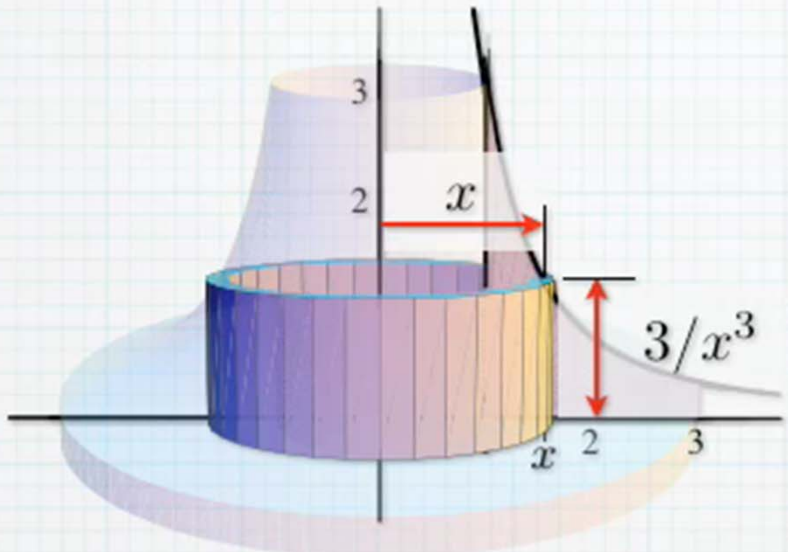
Example Find the volume of the solid generated by revolving about the y-axis the region defined by

$$0 \le y \le 3/x^3 \text{ and } 1 \le x \le 3.$$



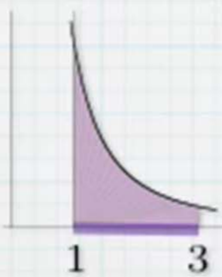
typical shell volume

$$\Delta V \approx \overbrace{2\pi x}^{\text{circumference}} \underbrace{(3/x^3)}_{\text{height}} \overbrace{\Delta x}^{\text{thickness}}$$

volume differential

$$dV = 2\pi x\,(3/x^3)\,dx = 6\pi/x^2\,dx$$

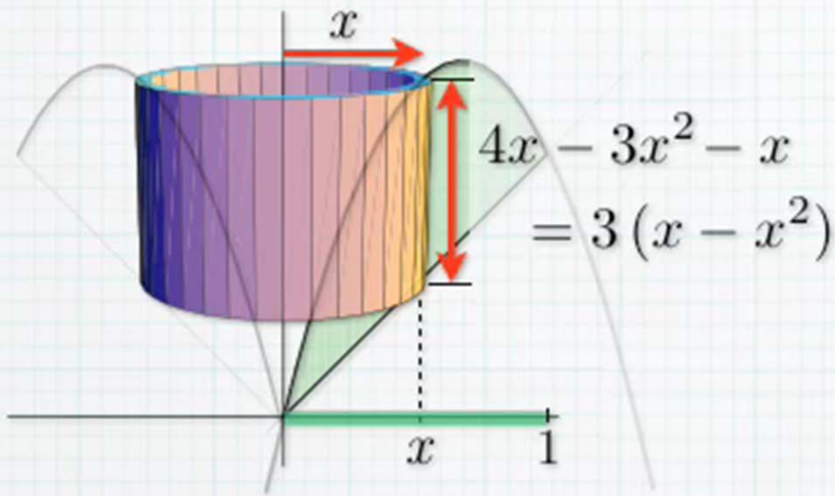
volume of the solid

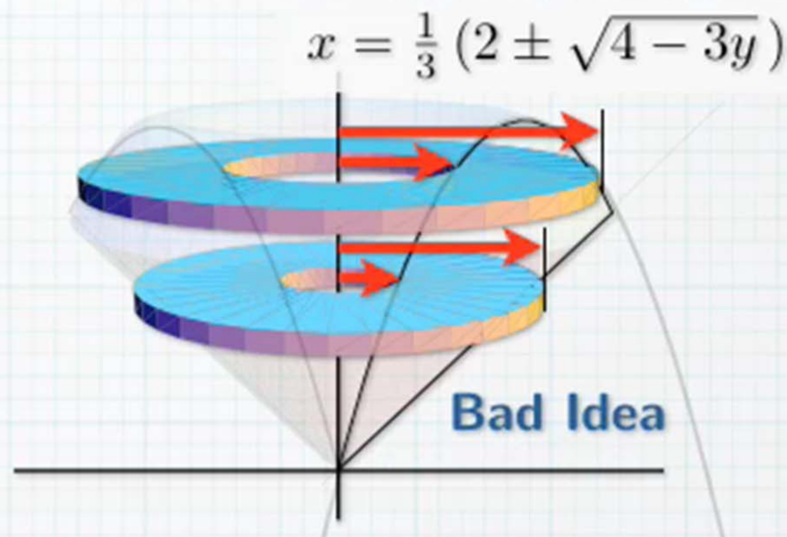


$$V = \int_1^3 6\pi x^{-2}\,dx = -6\pi x^{-1}\Big|_1^3 = -6\pi(1/3 - 1) = 4\pi$$

Example Let $\mathcal{R}$ be the region in which $x \le y \le 4x - 3x^2$.

Find the volume of the solid obtained by revolving $\mathcal{R}$ about the y-axis.





$$dV = \underbrace{2\pi x}_{\text{circumference}} \cdot \underbrace{3\,(x - x^2)}_{\text{height}} \underbrace{dx}_{\text{thickness}} = 6\pi\,(x^2 - x^3)\,dx$$

$$V = 6\pi \int_0^1 (x^2 - x^3)\,dx = 6\pi \left(\tfrac{1}{3}\,x^3 - \tfrac{1}{4}\,x^4\right)\Big|_0^1 = 6\pi\left(\tfrac{1}{3} - \tfrac{1}{4} - 0\right)$$

$$= 6\pi \cdot \tfrac{1}{12} = \frac{\pi}{2}$$

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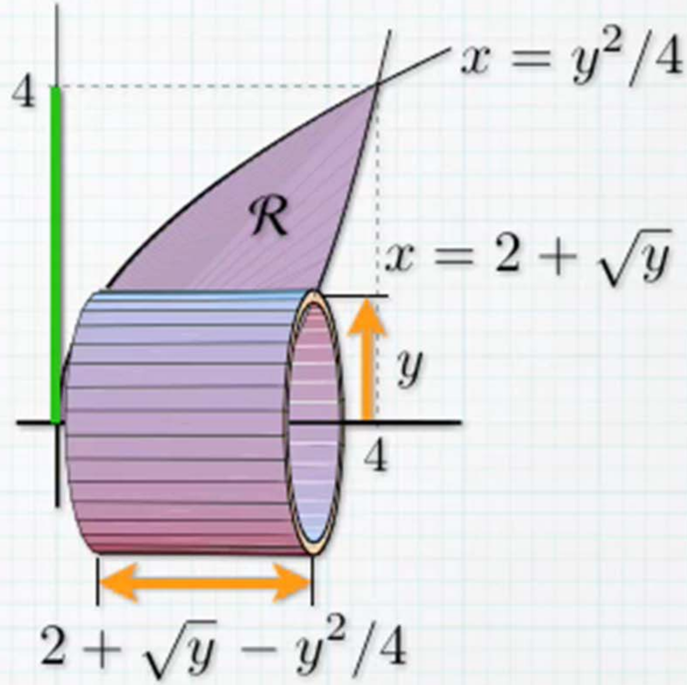
Example Let $\mathcal{R}$ be the region in which

$$y^2/4 \le x \le 2 + \sqrt{y}.$$

Find the volume of the solid obtained by revolving $\mathcal{R}$ about the x-axis.



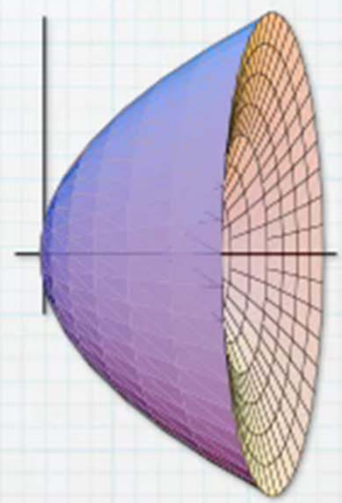
$$dV = \underbrace{2\pi y}_{\text{circumference}} \cdot \underbrace{\left(2 + \sqrt{y} - y^2/4\right)}_{\text{height}} \cdot \underbrace{dy}_{\text{thickness}}$$

$$= 2\pi \left(2y + y^{3/2} - y^3/4\right) dy$$

$$V = 2\pi \int_0^4 \left(2y + y^{3/2} - y^3/4\right) dy$$

$$= 2\pi \left(y^2 + \tfrac{2}{5}\, y^{5/2} - \tfrac{1}{16}\, y^4\right) \Big|_0^4$$

$$= 2\pi \left(16 + \tfrac{2}{5}\, 32 - \tfrac{1}{16}\, 256 - 0\right) = \frac{128}{5}\,\pi$$



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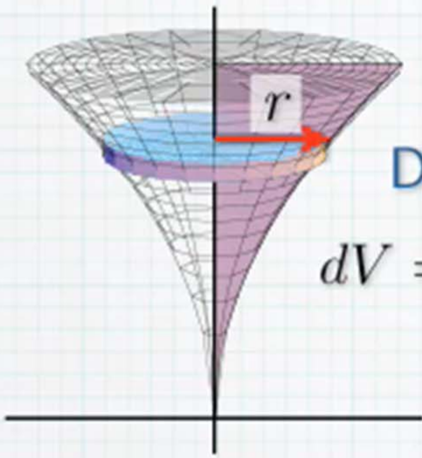
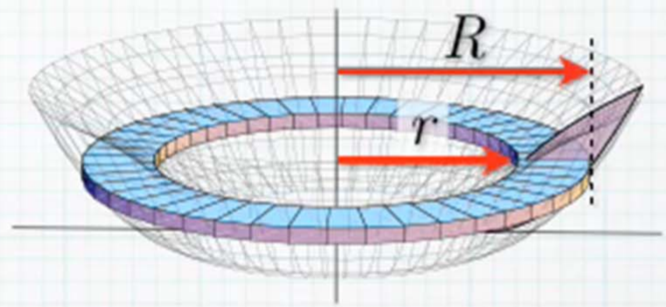
Summary Volumes of solids of revolution



Disks

$$dV = \pi\, r^2\, dy$$

Washers



$$dV = \pi\,(R^2 - r^2)\,dy$$

Cylindrical shells

$$dV = 2\pi\, r\, h\, dx$$