Linear functions: Which functions can be expressed as a matrix-vector product?

In each example, we *assumed* the function could be expressed as a matrix-vector product.

How can we verify that assumption?

We'll state two algebraic properties.

- If a function can be expressed as a matrix-vector product $\mathbf{x} \mapsto M * \mathbf{x}$, it has these properties.
- If the function from $\mathbb{F}^C$ to $\mathbb{F}^R$ has these properties, it can be expressed as a matrix-vector product.

Linear functions: Which functions can be expressed as a matrix-vector product?

Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over a field $\mathbb{F}$.

Suppose a function $f : \mathcal{V} \longrightarrow \mathcal{W}$ satisfies two properties:

Property L1: For every vector $\mathbf{v}$ in $\mathcal{V}$ and every scalar α in $\mathbb{F}$,

$$f(\alpha\, \mathbf{v}) = \alpha\, f(\mathbf{v})$$

Property L2: For every two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathcal{V}$,

$$f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$

We then call f a *linear function*.

Proposition: Let M be an $R \times C$ matrix, and suppose $f : \mathbb{F}^C \mapsto \mathbb{F}^R$ is defined by $f(\mathbf{x}) = M * \mathbf{x}$. Then f is a linear function.

Proof: Certainly $\mathbb{F}^C$ and $\mathbb{F}^R$ are vector spaces.

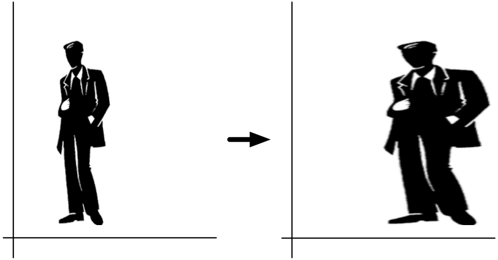
We showed that $M * (\alpha\, \mathbf{v}) = \alpha\, M * \mathbf{v}$. This proves that f satisfies Property L1.

We showed that $M * (\mathbf{u} + \mathbf{v}) = M * \mathbf{u} + M * \mathbf{v}$. This proves that f satisfies Property L2.

QED

Which functions are linear?

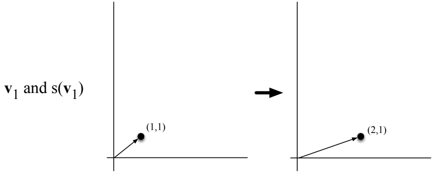
Define $s([x, y])$ = stretching by two in horizontal direction



Which functions are linear?

Define $s([x, y])$ = stretching by two in horizontal direction

Property L1: $s(\mathbf{v}_1 + \mathbf{v}_2) = s(\mathbf{v}_1) + s(\mathbf{v}_2)$



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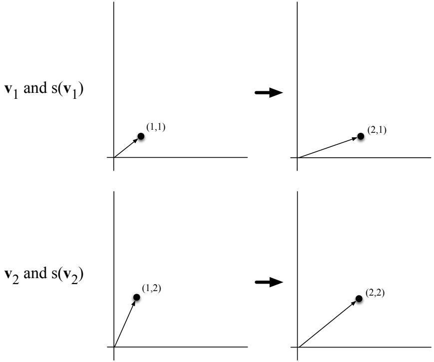

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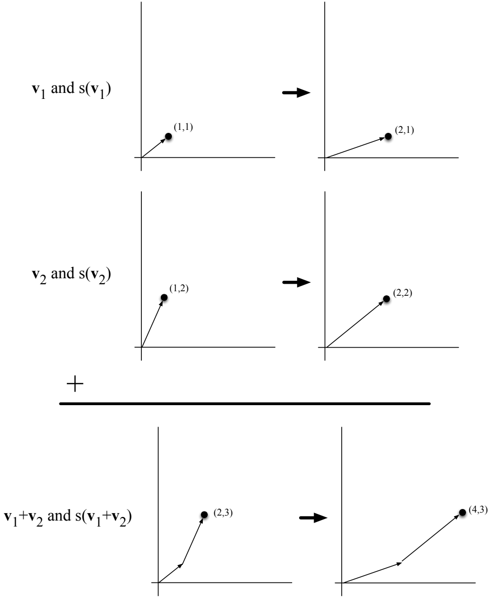
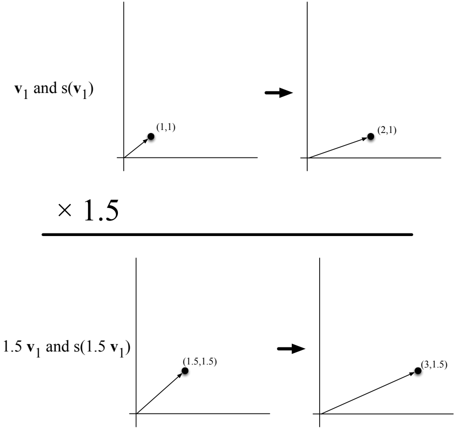


Which functions are linear?

Define $s([x,y])$ = stretching by two in horizontal direction

Property L1: $s(\mathbf{v}_1 + \mathbf{v}_2) = s(\mathbf{v}_1) + s(\mathbf{v}_2)$

Property L2: $s(\alpha\, \mathbf{v}) = \alpha\, s(\mathbf{v})$



Since the function $s(\cdot)$ satisfies Properties L1 and L2, it is a linear function.

Similarly can show rotation by θ degrees is a linear function.

What about translation?

$$t([x,y]) = [x,y] + [1,2]$$

This function violates Property L1. For example:

$$t([4,5] + [2,-1]) = t([6,4]) = [7,6]$$

but

$$t([4,5]) + t([2,-1]) = [5,7] + [3,1] = [8,8]$$

Since $t(\cdot)$ violates Property L1 for at least one input, it is not a linear function.

Can similarly show that $t(\cdot)$ does not satisfy Property L2.

A linear function maps zero vector to zero vector

Lemma: If $f : \mathcal{U} \longrightarrow \mathcal{V}$ is a linear function then f maps the zero vector of $\mathcal{U}$ to the zero vector of $\mathcal{V}$.

Proof: Let $\mathbf{0}$ denote the zero vector of $\mathcal{U}$, and let $\mathbf{0}_{\mathcal{V}}$ denote the zero vector of $\mathcal{V}$.

$$f(\mathbf{0}) = f(\mathbf{0} + \mathbf{0}) = f(\mathbf{0}) + f(\mathbf{0})$$

Subtracting $f(\mathbf{0})$ from both sides, we obtain

$$\mathbf{0}_{\mathcal{V}} = f(\mathbf{0})$$

QED

Linear functions: Pushing linear combinations through the function

Defining properties of linear functions:

Property L1: $f(\alpha\,\mathbf{v}) = \alpha\, f(\mathbf{v})$

Property L2: $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$

Proposition: For a linear function f,
for any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in the domain of f and any scalars $\alpha_1, \ldots, \alpha_n$,

$$f(\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n) = \alpha_1\, f(\mathbf{v}_1) + \cdots + \alpha_n\, f(\mathbf{v}_n)$$

Proof: Consider the case of $n = 2$.

$$\begin{aligned} f(\alpha_1\,\mathbf{v}_1 + \alpha_2\,\mathbf{v}_2) &= f(\alpha_1\,\mathbf{v}_1) + f(\alpha_2\,\mathbf{v}_2) && \text{by Property L2} \\ &= \alpha_1\, f(\mathbf{v}_1) + \alpha_2\, f(\mathbf{v}_2) && \text{by Property L1} \end{aligned}$$

Proof for general n is similar. QED

Linear functions: Pushing linear combinations through the function

Proposition: For a linear function f,
$f(\alpha_1\,\mathbf{v}_1 + \cdots + \alpha_n\,\mathbf{v}_n) = \alpha_1\,f(\mathbf{v}_1) + \cdots + \alpha_n\,f(\mathbf{v}_n)$

Example: $f(\mathbf{x}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \mathbf{x}$

Verify that $f(10\,[1,-1] + 20\,[1,0]) = 10\,f([1,-1]) + 20\,f([1,0])$

$$\begin{aligned}
&\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left(10\,[1,-1] + 20\,[1,0]\right) \\
=&\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\left([10,-10] + [20,0]\right) \\
=&\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}[30,-10] \\
=&\ 30\,[1,3] - 10[2,4] \\
=&\ [30,90] - [20,40] \\
=&\ [10,50]
\end{aligned}$$

$$\begin{aligned}
&10\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * [1,-1]\right) + 20\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * [1,0]\right) \\
=&\ 10\,([1,3] - [2,4]) + 20\,(1[1,3]) \\
=&\ 10\,[-1,-1] + 20\,[1,3] \\
=&\ [-10,-10] + [20,60] \\
=&\ [10,50]
\end{aligned}$$

From function to matrix, revisited

We saw a method to derive a matrix from a function:

Given a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, we want a matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$....

- Plug in the standard generators $\mathbf{e}_1 = [1, 0, \ldots, 0, 0], \ldots, \mathbf{e}_n = [0, \ldots, 0, 1]$
- Column i of M is $f(\mathbf{e}_i)$.

This works correctly whenever such a matrix M really exists:

Proof: If there is such a matrix then f is linear:

- (Property L1) $f(\alpha\,\mathbf{v}) = \alpha\, f(\mathbf{v})$ and
- (Property L2) $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$

Let $\mathbf{v} = [\alpha_1, \ldots, \alpha_n]$ be any vector in $\mathbb{R}^n$.

We can write $\mathbf{v}$ in terms of the standard generators.

$$
\begin{aligned}
\mathbf{v} &= \alpha_1\,\mathbf{e}_1 + \cdots + \alpha_n\,\mathbf{e}_n \\
\text{so} & \\
f(\mathbf{v}) &= f(\alpha_1\,\mathbf{e}_1 + \cdots + \alpha_n\,\mathbf{e}_n) \\
&= \alpha_1\, f(\mathbf{e}_1) + \cdots + \alpha_n\, f(\mathbf{e}_n) \\
&= \alpha_1\,(\text{column 1 of } M) + \cdots + \alpha_n\,(\text{column} n \text{ of } M) \\
&= M * \mathbf{v}\,\mathit{QED}
\end{aligned}
$$

Linear functions and zero vectors: Kernel

Definition: *Kernel* of a linear function f is $\{\mathbf{v} \ : \ f(\mathbf{v}) = \mathbf{0}\}$

Written Ker f

For a function $f(\mathbf{x}) = M * \mathbf{x}$,

$$\text{Ker } f = \text{Null } M$$

Kernel and one-to-one

One-to-One Lemma: A linear function is one-to-one if and only if its kernel is a trivial vector space.

Proof: Let $f : \mathcal{U} \longrightarrow \mathcal{V}$ be a linear function. We prove two directions.

- Suppose Ker f contains some nonzero vector $\mathbf{u}$, so $f(\mathbf{u}) = \mathbf{0}_{\mathcal{V}}$.
  Because a linear function maps zero to zero, $f(\mathbf{0}) = \mathbf{0}_{\mathcal{V}}$ as well,
  so f is not one-to-one.

- Suppose Ker $f = \{\mathbf{0}\}$.
  Let $\mathbf{v}_1, \mathbf{v}_2$ be any vectors such that $f(\mathbf{v}_1) = f(\mathbf{v}_2)$.
  Then $f(\mathbf{v}_1) - f(\mathbf{v}_2) = \mathbf{0}_{\mathcal{V}}$
  so, by linearity, $f(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}_{\mathcal{V}}$,
  so $\mathbf{v}_1 - \mathbf{v}_2 \in$ Ker f.
  Since Ker f consists solely of $\mathbf{0}$,
  it follows that $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1 = \mathbf{v}_2$.

QED

Kernel and one-to-one

One-to-One Lemma A linear function is one-to-one if and only if its kernel is a trivial vector space.

Define the function $f(\mathbf{x}) = A * \mathbf{x}$.

If Ker f is trivial (i.e. if Null A is trivial)

then a vector $\mathbf{b}$ is the image under f of at most one vector.

That is, at most one vector $\mathbf{u}$ such that $A * \mathbf{u} = \mathbf{b}$
That is, the solution set of $A * \mathbf{x} = \mathbf{b}$ has at most one vector.

Linear functions that are onto?

Question: How can we tell if a linear function is onto?

Recall: for a function $f : \mathcal{V} \longrightarrow \mathcal{W}$, the *image* of f is the set of all images of elements of the domain:

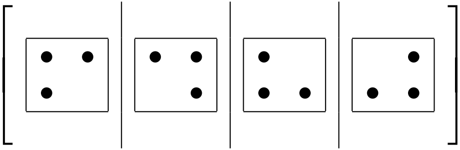
$$\{f(\mathbf{v}) \ : \mathbf{v} \in \mathcal{V}\}$$

(You might know it as the "range" but we avoid that word here.)

The image of function f is written Im f

"Is function f is onto?" same as "is Im f = co-domain of f?"

Example: *Lights Out*

Define $f([\alpha_1, \alpha_2, \alpha_3, \alpha_4]) =$  $* [\alpha_1, \alpha_2, \alpha_3, \alpha_4]$

Im f is set of configurations for which 2×2 *Lights Out* can be solved,
so "f is onto" means "2×2 *Lights Out* can be solved for every configuration"

Can 2×2 *Lights Out* be solved for every configuration? What about 5×5?
Each of these questions amounts to asking whether a certain function is onto.

Linear functions that are onto?

$\boxed{\text{"Is function } f \text{ is onto?" same as "is Im } f = \text{co-domain of } f\text{?"}}$

First step in understanding how to tell if a linear function f is onto:

- study the image of f

Proposition: The image of a linear function $f : \mathcal{V} \longrightarrow \mathcal{W}$ is a vector space