Matrix-matrix multiplication and function composition

Corresponding to an $R \times C$ matrix A over a field $\mathbb{F}$, there is a function

$$f : \mathbb{F}^C \longrightarrow \mathbb{F}^R$$

namely the function defined by $f(\mathbf{y}) = A * \mathbf{y}$

Matrix-matrix multiplication and function composition

Matrices A and B $\Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

$\boxed{\textbf{Matrix-Multiplication Lemma } f \circ g = h}$

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}$$

product $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

corresponds to function $h\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2, x_1 + x_2 \end{bmatrix}$

$f \circ g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = f\left(\begin{bmatrix} x_1 \\ x_1 + x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix}$ so $f \circ g = h$

Matrix-matrix multiplication and function composition

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

Matrix-Multiplication Lemma $f \circ g = h$

Proof: Let columns of B be $\mathbf{b}_1, \ldots, \mathbf{b}_n$. By the matrix-vector definition of matrix-matrix multiplication, column j of AB is $A *$ (column j of B).
For any n-vector $\mathbf{x} = [x_1, \ldots, x_n]$,

$$\begin{aligned} g(\mathbf{x}) &= B * \mathbf{x} && \text{by definition of } g \\ &= x_1 \mathbf{b}_1 + \cdots + x_n \mathbf{b}_n && \text{by linear combinations definition} \end{aligned}$$

Therefore

$$\begin{aligned} f(g(\mathbf{x})) &= f(x_1 \mathbf{b}_1 + \cdots x_n \mathbf{b}_n) \\ &= x_1(f(\mathbf{b}_1)) + \cdots + x_n(f(\mathbf{b}_n)) && \text{by linearity of } f \\ &= x_1(A * \mathbf{b}_1) + \cdots + x_n(A * \mathbf{b}_n) && \text{by definition of } f \\ &= x_1(\text{column 1 of } AB) + \cdots + x_n(\text{column } n \text{ of } AB) && \text{by matrix-vector def.} \\ &= (AB) * \mathbf{x} && \text{by linear-combinations def.} \\ &= h(\mathbf{x}) && \text{by definition of } h \end{aligned}$$

QED

Associativity of matrix-matrix multiplication

Matrices A and $B \Rightarrow$ functions $f(\mathbf{y}) = A * \mathbf{y}$ and $g(\mathbf{x}) = B * \mathbf{x}$ and $h(\mathbf{x}) = (AB) * \mathbf{x}$

$\boxed{\textbf{Matrix-Multiplication Lemma } f \circ g = h}$

Matrix-matrix multiplication corresponds to function composition.

Corollary: Matrix-matrix multiplication is associative:

$$(AB)C = A(BC)$$

Proof: Function composition is associative. QED

Example:

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 1 & 7 \end{bmatrix}$$