Critical Numbers

and

The First Derivative Test

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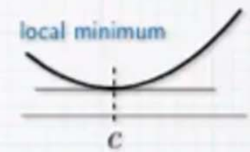
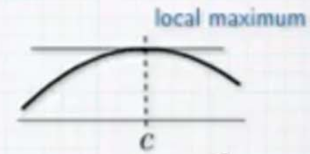
Definition

If there is an open interval I containing c in which either

$$f(c) \leq f(x) \text{ for all } x \text{ in } I \text{ or } f(c) \geq f(x) \text{ for all } x \text{ in } I,$$

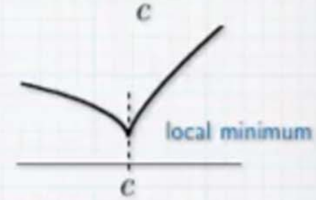
then we say that $f(c)$ is a **local extreme value** of f.





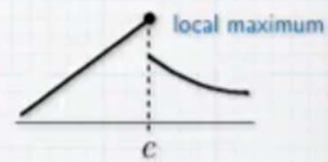
Theorem

If $f(c)$ is a local extreme value,
then either $f'(c) = 0$ or $f'(c)$ does not exist.



Definition

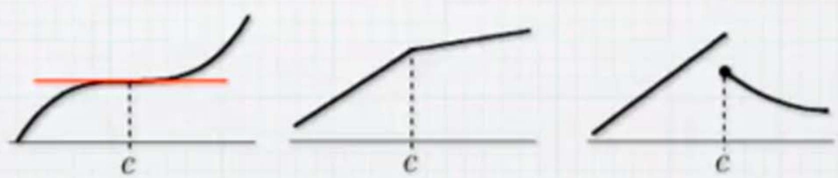
A number c in the domain of f is called a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.



Theorem

If $f(c)$ is a local extreme value, then c is a critical number of f.

Caution:
The converse is false.

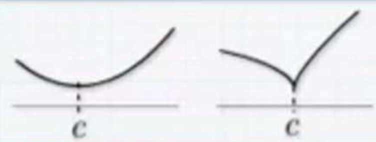
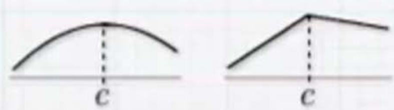
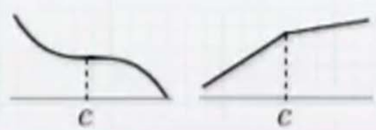


Critical Point Classification

Given a critical number c of a function f, how can we tell whether $f(c)$ is a local extreme value? And if it is, how can we tell whether it's a local minimum or a local maximum?

The First Derivative Test

Let c be an *isolated* critical number of f.

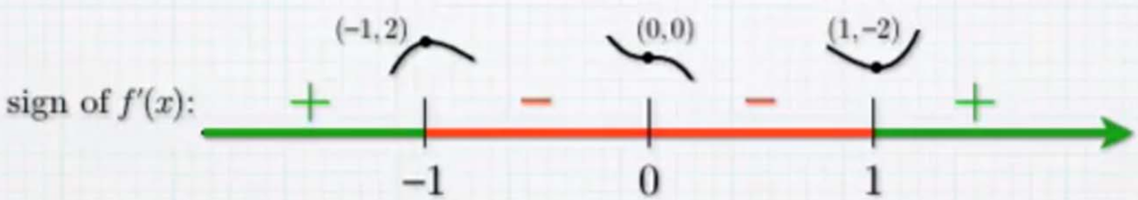
sign change in $f'(x)$ at c	**picture**	**conclusion**
$-\mid+$		local minimum
$+\mid-$		local maximum
none ($-\mid-$ or $+\mid+$)		not a local extremum

Example

Let $f(x) = 3x^5 - 5x^3$. Find the critical numbers of f, and classify the value of f at each one as a local minimum, local maximum, or neither.

$$\begin{aligned} f(x) &= 3x^5 - 5x^3 \\ f'(x) &= 15x^4 - 15x^2 \\ &= 15x^2(x^2-1) = 15x^2(x+1)(x-1) \end{aligned}$$

Critical numbers: $0, \pm 1$



$f(-1) = 2$ is a local maximum.

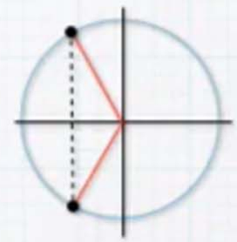
$f(0) = 0$ is neither a local minimum nor a local maximum.

$f(1) = -2$ is a local minimum.

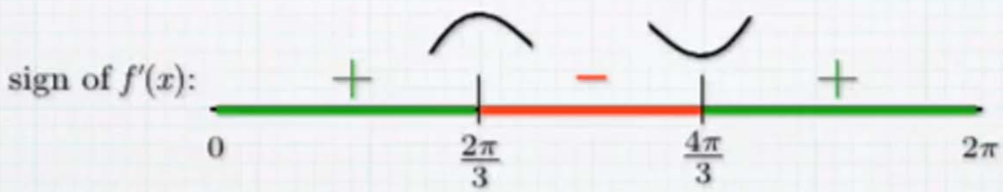
Example

Let $f(x) = x + 2\sin x$. Find the critical numbers of f in the interval $[0, 2\pi]$, and classify the value of f at each one as a local minimum, local maximum, or neither.

$$f(x) = x + 2\sin x.$$
$$f'(x) = 1 + 2\cos x$$
$$= 0 \text{ when } \cos x = -\tfrac{1}{2}$$



Critical numbers: $x = \frac{2\pi}{3}, \frac{4\pi}{3}$



$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$ is a local maximum.

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$ is a local minimum.

Example

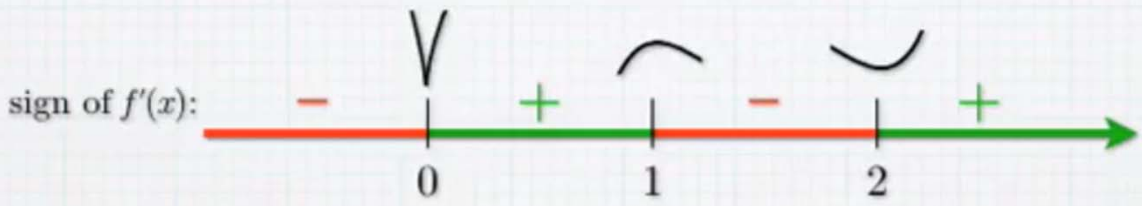
Let $f(x) = x^{2/3}(15x^2 - 72x + 120)$. Find the critical numbers of f, and classify the value of f at each one as a local minimum, local maximum, or neither.

$$f(x) = 15x^{8/3} - 72x^{5/3} + 120x^{2/3}$$

$$f'(x) = 40x^{5/3} - 120x^{2/3} + 80x^{-1/3}$$

$$= \frac{40(x^2 - 3x + 2)}{\sqrt[3]{x}} = \frac{40(x-1)(x-2)}{\sqrt[3]{x}}$$

Critical numbers: $0, 1, 2$



$f(0) = 0$ is a local minimum.

$f(1) = 63$ is a local maximum.

$f(2) = 36\sqrt[3]{4} \approx 57.15$ is a local minimum.

Example Let $f(x) = \dfrac{1-x}{x^2}$.

Find the critical numbers of f, and classify the value of f at each one as a local minimum, local maximum, or neither.

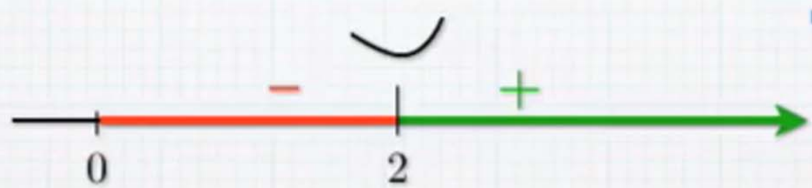
$$f(x) = \frac{1-x}{x^2}$$

$$f'(x) = \frac{-x^2 - (1-x)2x}{x^4} = \frac{x(x-2)}{x^4} = \frac{x-2}{x^3}$$

Only one critical number: 2

Note: 0 is not a critical number because it's not in the domain of f.

sign of $f'(x)$:



$f(2) = -1/4$ is a local minimum.

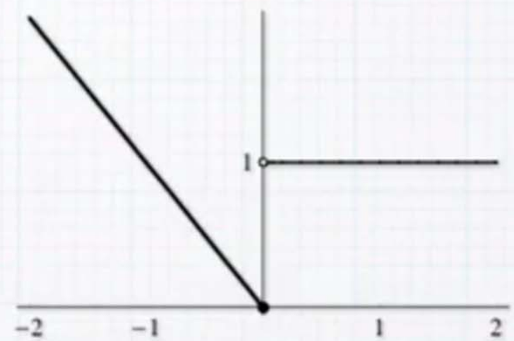
Example (non-isolated critical numbers)

Let $f(x) = \begin{cases} -x & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Then $f'(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$

Critical numbers: all $x \geq 0$

None are isolated!



$f(0) = 0$ is a local minimum.

$f(x) = 1$ is *both a local minimum and a local maximum* for all $x > 0$.

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Concavity

and

The Second Derivative Test

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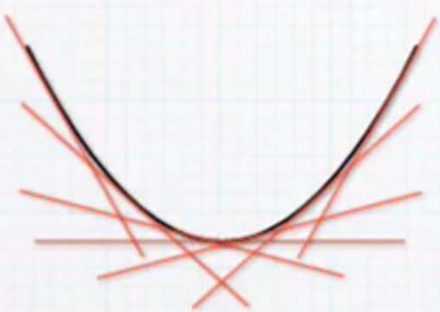
Concavity

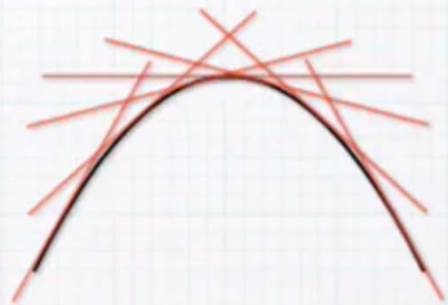
Let f be differentiable at each x in an open interval I.

The graph of f is **concave up** on I if f' is increasing on I.

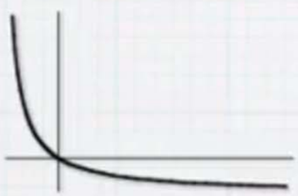
The graph of f is **concave down** on I if f' is decreasing on I.



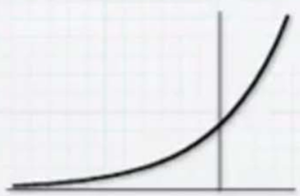
Increasing Slope
Graph is "**concave up**" (convex).



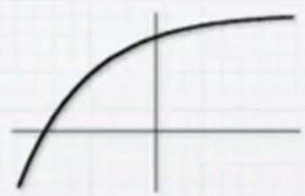
Decreasing Slope
Graph is "**concave down**" (concave).



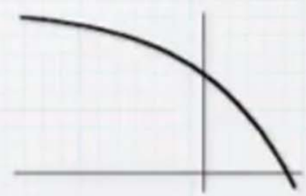
concave up
and decreasing



concave up
and increasing



concave down
and increasing



concave down
and decreasing

"Slopeless" Interpretation of Concavity

Let I be an open interval in the domain of f.

- If the graph of f is concave up on I, then

$$f(x) < \frac{f(x+h)+f(x-h)}{2} \quad \text{whenever } x \pm h \text{ are in } I.$$

- If the graph of f is concave down on I, then

$$f(x) > \frac{f(x+h)+f(x-h)}{2} \quad \text{whenever } x \pm h \text{ are in } I.$$

Proof of the first part:

Suppose that f' is increasing on I, and let $x \pm h$ be in I, where $h > 0$.

We'll show that $f(x+h) - 2f(x) + f(x-h) > 0$.

$$\begin{aligned}
\frac{f(x+h)-2f(x)+f(x-h)}{h} &= \frac{f(x+h)-f(x)}{h} - \frac{f(x)-f(x-h)}{h} \\
&= f'(t) - f'(s) \quad \text{Mean-value Theorem} \\
&\qquad \text{where } x-h < s < x < t < x+h \\
&> 0 \quad \text{since } f' \text{ is increasing on } I.
\end{aligned}$$

Therefore $f(x+h) - 2f(x) + f(x-h) > 0$.

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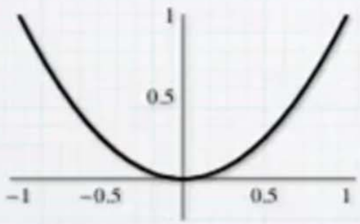
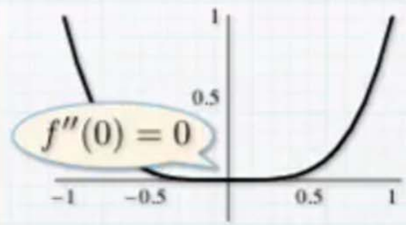
Concavity and the Second Derivative

Let I be an open interval in the domain of f, and suppose that $f''(x)$ exists for all x in I. Then:

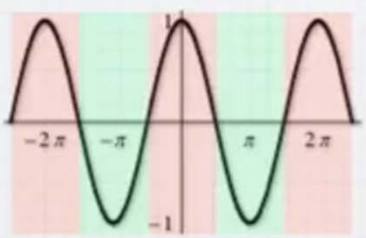
- If the graph of f is concave up on I, then $f''(x) \geq 0$ for all x in I.
- If $f''(x) > 0$ for all x in I, then the graph of f is concave up on I.
- If the graph of f is concave down on I, then $f''(x) \leq 0$ for all x in I.
- If $f''(x) < 0$ for all x in I, then the graph of f is concave down on I.



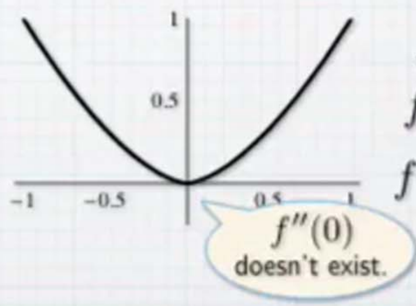
$$f(x) = x^2$$
$$f'(x) = 2x$$
$$f''(x) = 2$$



$$f(x) = x^4$$
$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$



$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$



$$f(x) = x^{8/5}$$
$$f'(x) = \frac{8}{5}\,x^{3/5}$$
$$f''(x) = \frac{24}{25}\,x^{-2/5}$$

Example Let $f(x) = 2x^6 - 9x^5 + 10x^4$.

Find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down.

$$f'(x) = 12x^5 - 45x^4 + 40x^3$$

$$f''(x) = 60x^4 - 180x^3 + 120x^2$$

$$= 60x^2(x^2 - 3x + 2) = 60x^2(x-1)(x-2)$$

$$f''(x) \begin{cases} > 0 & \text{if } x < 0 \\ > 0 & \text{if } 0 < x < 1 \\ < 0 & \text{if } 1 < x < 2 \\ > 0 & \text{if } x > 2 \end{cases}$$

The graph of f is concave up on $(-\infty, 0)$, $(0, 1)$, and $(2, \infty)$.

The graph of f is concave down on $(1, 2)$.

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Example Let $f(x) = 2x^6 - 9x^5 + 10x^4$.

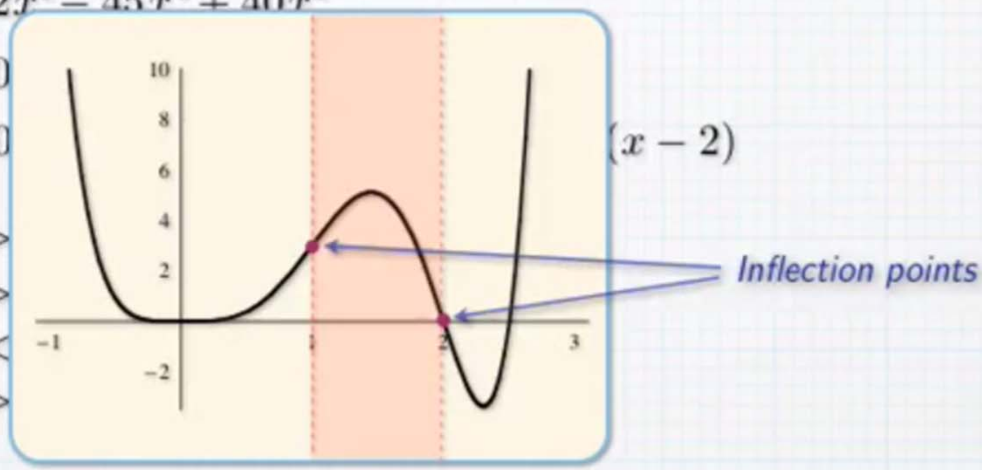
Find the intervals on which the graph of f is concave up and the intervals on which the graph of f is concave down.

$f'(x) = 12x^5 - 45x^4 + 40x^3$

$f''(x) = 60$

$= 60 \quad (x-2)$

$f''(x) \begin{cases} > \\ > \\ < \\ > \end{cases}$



The graph of f is concave up on $(-\infty, 1)$, and $(2, \infty)$.

The graph of f is concave down on $(1, 2)$.

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Critical Points and Local Extrema

Recall the following definition and theorem from the preceding lecture:

Definition

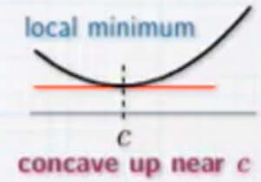
A number c in the interior of the domain of f is called a **critical number** of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Theorem

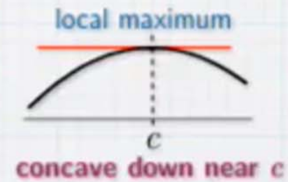
If $f(c)$ is a local extreme value, then c is a critical number of f.

Now suppose that c is an *isolated* critical number of f, where $f'(c) = 0$.

- $f(c)$ is a local minimum if and only if the graph of f is concave up on an open interval containing c.



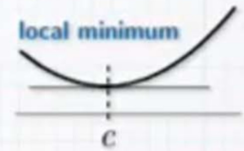
- $f(c)$ is a local maximum if and only if the graph of f is concave down an open interval containing c.



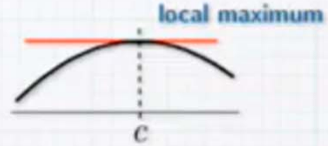
The Second Derivative Test

Let c be a critical number of f where $f'(c) = 0$ and $f''(c)$ exists.

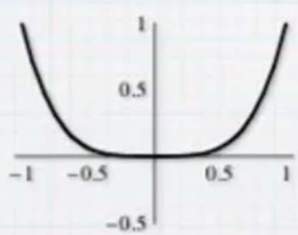
If $f''(c) > 0$, then $f(c)$ is a local minimum.

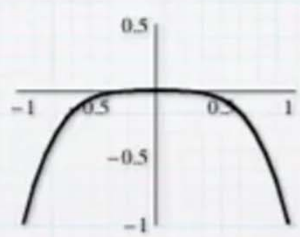


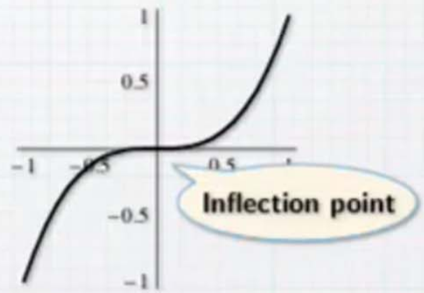
If $f''(c) < 0$, then $f(c)$ is a local maximum.



If $f''(c) = 0$, then $f(c)$ may be a local minimum, a local maximum, neither, or both. That is, **the test fails**.







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Example

Let $f(x) = 3x^5 - 5x^3$. Find the critical numbers of f, and classify the value of f at each one as a local minimum, local maximum, or neither.

$$f(x) = 3x^5 - 5x^3$$

1st derivative and critical numbers

$$f'(x) = 15x^4 - 15x^2$$
$$= 15x^2(x^2 - 1) = 15x^2(x+1)(x-1)$$

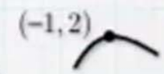
Critical numbers: $0, \pm 1$

2nd derivative

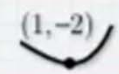
$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

2nd derivative test

$f''(-1) = -30 < 0$. So $f(-1) = 2$ is a local maximum.



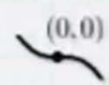
$f''(1) = 30 > 0$. So $f(1) = -2$ is a local minimum.



$f''(0) = 0$. So the 2nd derivative test fails.

1st derivative test

Since $f'(x)$ does not change sign at $x = 0$, the *1st derivative test* tells us that $f(0) = 0$ is neither a local minimum nor a local maximum.



Example

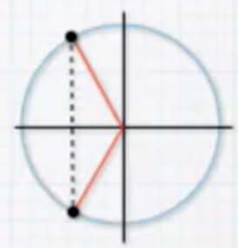
Let $f(x) = x + 2\sin x$. Find the critical numbers of f in the interval $[0, 2\pi]$, and classify the value of f at each one as a local minimum, local maximum, or neither.

1st derivative and critical numbers

$$f(x) = x + 2\sin x.$$

$$f'(x) = 1 + 2\cos x$$

$$= 0 \text{ when } \cos x = -\tfrac{1}{2}$$



Critical numbers: $x = \frac{2\pi}{3},\ \frac{4\pi}{3}$

2nd derivative test

$$f''(x) = -2\sin x$$

$$f''(\tfrac{2\pi}{3}) = -2\tfrac{\sqrt{3}}{2} = -\sqrt{3} < 0$$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} + \sqrt{3}$

is a local maximum.

$$f''(\tfrac{4\pi}{3}) = -2(-\tfrac{\sqrt{3}}{2}) = \sqrt{3} > 0$$

$f(\frac{4\pi}{3}) = \frac{4\pi}{3} - \sqrt{3}$

is a local minimum.

Example Let $f(x) = x + \dfrac{1}{x}$.

Find the critical numbers of f, and classify the value of f at each one as a local minimum, local maximum, or neither.

$$f(x) = x + \frac{1}{x}$$

1st derivative and critical numbers

$$f'(x) = 1 - x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

Critical numbers: ± 1

Note: 0 is not a critical number because it's not in the domain of f.

2nd derivative test

$$f''(x) = 2x^{-3} = \frac{2}{x^3}$$

$f''(-1) = -2 < 0$ $\frown$

$f(-1) = -3/2$ is a local maximum.

$f''(1) = 2 > 0$ $\smile$

$f(1) = 3/2$ is a local minimum.

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Curve Sketching

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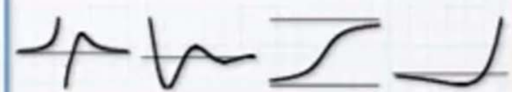
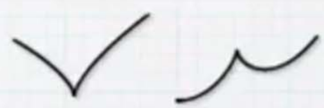
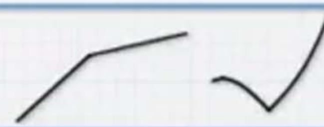
Summary of Graphical Features

$f'(x) > 0 \;\; / \;\; f'(x) < 0$	increasing / decreasing	
$f''(x) > 0 \;\; / \;\; f''(x) < 0$	concave up/down	
$\lim_{x\to-\infty} f(x) = b$ or $\lim_{x\to\infty} f(x) = b$	horizontal asymptote at $y = b$	
$\lim_{x\to a^-} f(x) = \pm\infty$ or $\lim_{x\to a^+} f(x) = \pm\infty$	vertical asymptote at $x = a$	
$\lim_{x\to a} f'(x) = \pm\infty$	vertical tangent at $x = a$	
$\lim_{x\to a^-} f'(x) = \pm\infty$, $\lim_{x\to a^+} f'(x) = \mp\infty$	cusp at $x = a$	
$\lim_{x\to a^-} f'(x) \neq \lim_{x\to a^+} f'(x)$	"corner" at $x = a$	

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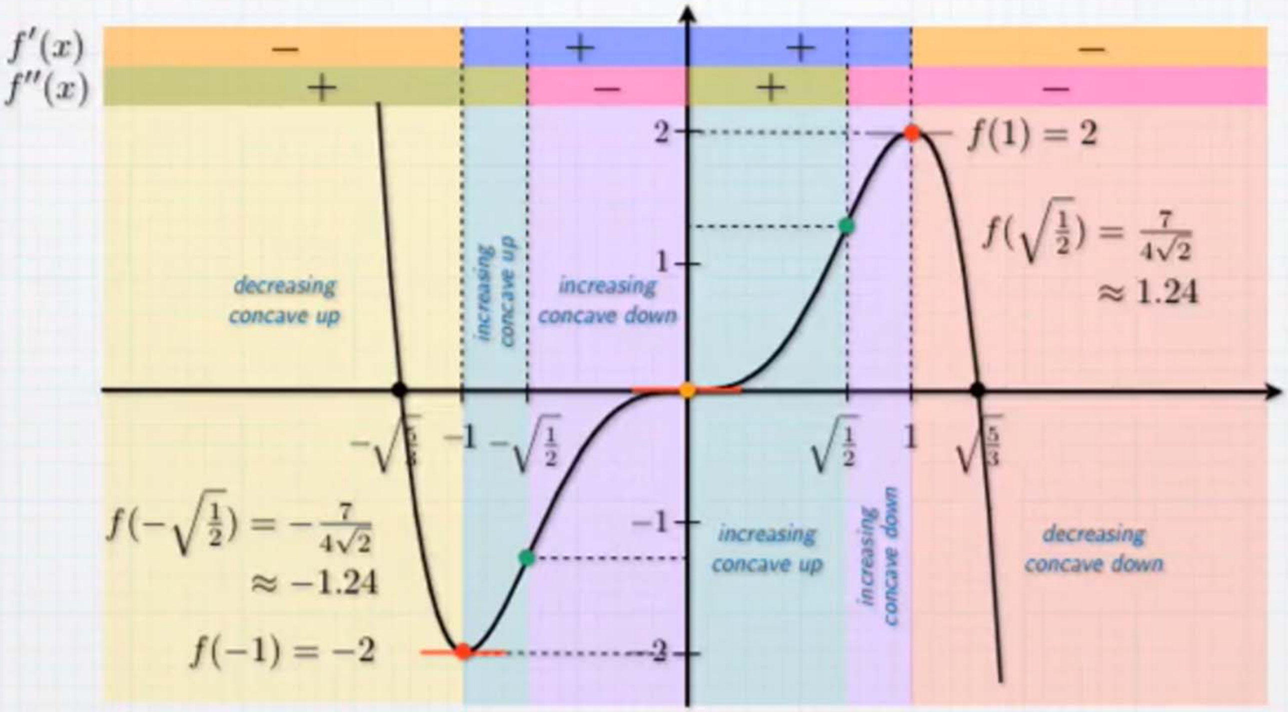
Example Sketch the graph of $f(x) = 5x^3 - 3x^5$.

$$f(x) = 5x^3 - 3x^5 = x^3(5 - 3x^2) = 0 \quad \text{at } x = 0, \pm\sqrt{5/3}$$
$$f'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 0 \quad \text{at } x = 0, \pm 1$$
$$f''(x) = 30x - 60x^3 = 30x(1 - 2x^2) = 0 \quad \text{at } x = 0, \pm\sqrt{1/2}$$



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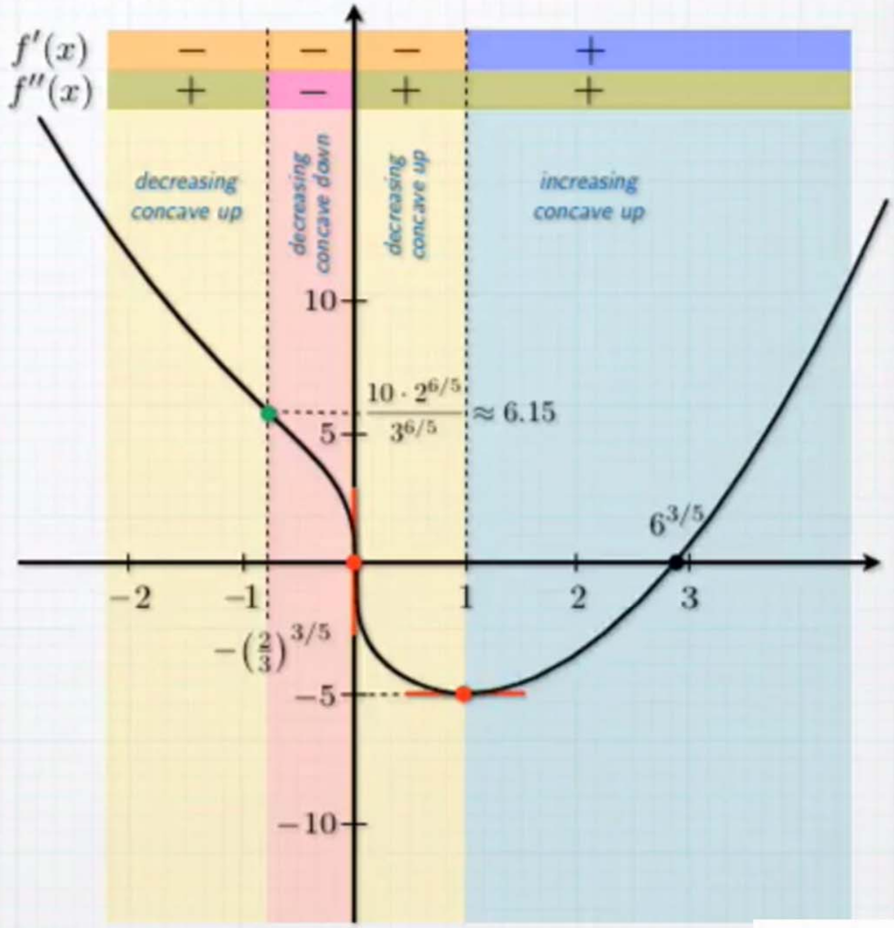
Example Sketch the graph of $f(x) = x^2 - 6x^{1/3}$.

$$\begin{aligned} f(x) &= x^2 - 6x^{1/3} \\ &= x^{1/3}(x^{5/3} - 6) \\ &= 0 \quad \text{at} \quad x = 0, 6^{3/5} \\ &\qquad (6^{3/5} \approx 2.93) \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x - 2x^{-2/3} \\ &= 2x^{-2/3}(x^{5/3} - 1) \\ &= 0 \quad \text{at} \quad x = 1 \end{aligned}$$

$$\lim_{x \to 0^{\pm}} f'(x) = -\infty$$

$$\begin{aligned} f''(x) &= 2 + \tfrac{4}{3}x^{-5/3} \\ &= 2x^{-5/3}\left(x^{5/3} + \tfrac{2}{3}\right) \\ &= 0 \quad \text{at} \quad x = -\left(\tfrac{2}{3}\right)^{3/5} \\ &\approx -0.784 \end{aligned}$$



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Example Sketch the graph of $f(x) = \dfrac{x^3}{x^2 - 3}$.

$$\lim_{x \to -\sqrt{3}^-} f(x) = -\infty \qquad \lim_{x \to \sqrt{3}^-} f(x) = -\infty$$

$$\lim_{x \to -\sqrt{3}^+} f(x) = \infty \qquad \lim_{x \to \sqrt{3}^+} f(x) = \infty$$

$$f'(x) = \frac{3x^2(x^2-3) - x^3 2x}{(x^2-3)^2} = \frac{x^2(x^2-9)}{(x^2-3)^2} = \frac{x^2(x+3)(x-3)}{(x^2-3)^2}$$

$$f''(x) = \frac{(4x^3 - 18x)(x^2-3)^{\cancel{2}} - (x^4 - 9x^2)\,2\cancel{(x^2-3)}\,2x}{(x^2-3)^{\cancel{4}3}}$$

$$= \frac{(4x^3 - 18x)(x^2-3) - (x^4 - 9x^2)4x}{(x^2-3)^3}$$

$$= \frac{4x^5 - 12x^3 - 18x^3 + 54x - 4x^5 + 36x^3}{(x^2-3)^3}$$

$$= \frac{6x^3 + 54x}{(x^2-3)^3} = \frac{6x(x^2+9)}{(x^2-3)^3}$$

Example Sketch the graph of $f(x) = \dfrac{x^3}{x^2-3}$.

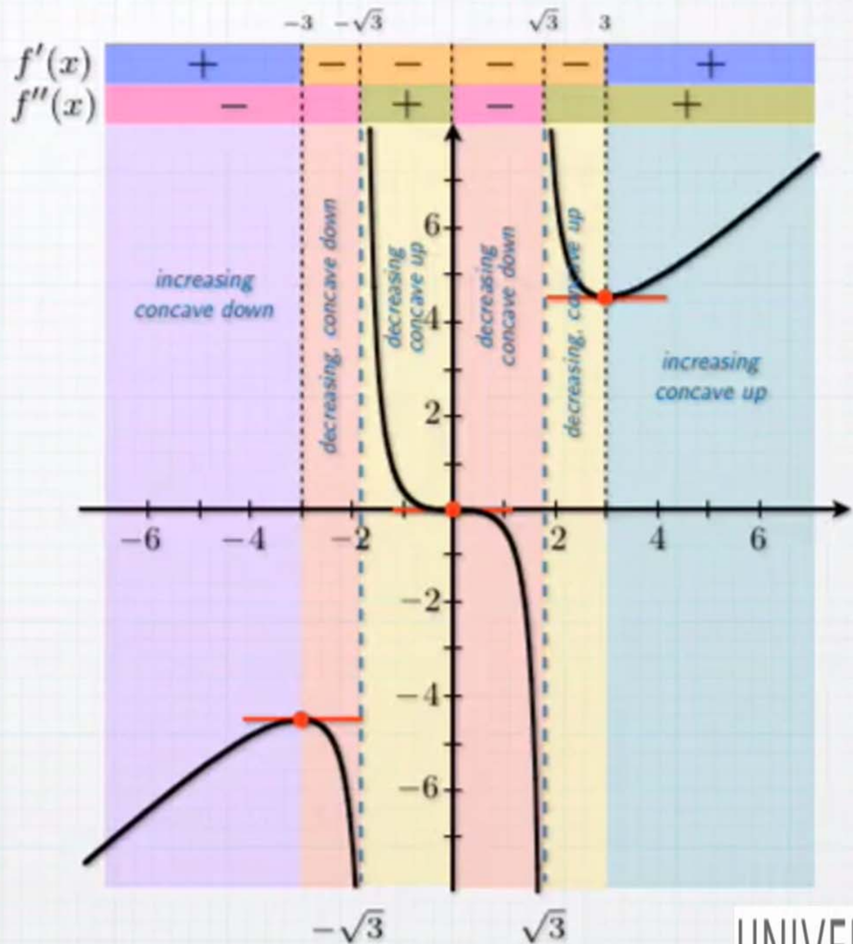
$$\lim_{x\to\pm\sqrt{3}^-} f(x) = -\infty$$

$$\lim_{x\to\pm\sqrt{3}^+} f(x) = \infty$$

$$f'(x) = \frac{x^2(x+3)(x-3)}{(x^2-3)^2}$$

$$f(\pm 3) = \pm\tfrac{9}{2}$$

$$f''(x) = \frac{6x(x^2+9)}{(x^2-3)^3}$$



Example Sketch the graph of $f(x) = \dfrac{x^5 + x^2 + 1}{x^5 + 1}$.

$$f(x) = \frac{x^2}{x^5 + 1} + 1$$

$$\lim_{x \to \pm\infty} f(x) = 1$$

$$\lim_{x \to -1^-} f(x) = -\infty \qquad \lim_{x \to -1^+} f(x) = \infty$$

$$f'(x) = \frac{x\left(2 - 3x^5\right)}{\left(x^5 + 1\right)^2}$$

$$= 0 \text{ at } x = 0, \left(\tfrac{2}{3}\right)^{1/5} \approx 0.92$$

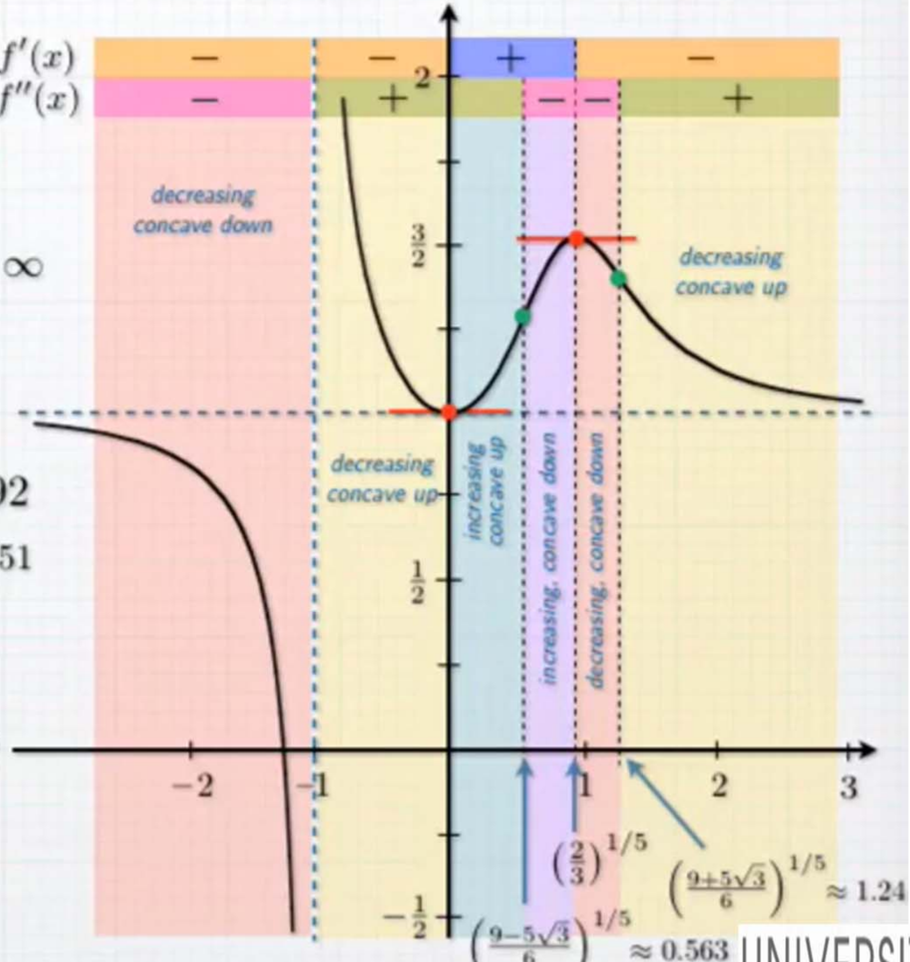
$$f\left((2/3)^{1/5}\right) = \tfrac{1}{5}\left(5 + 108^{1/5}\right) \approx 1.51$$

$$f''(x) = \frac{2\left(1 - 18x^5 + 6x^{10}\right)}{\left(x^5 + 1\right)^3}$$

$$= 0 \text{ at } x = \left(\frac{9 \pm 5\sqrt{3}}{6}\right)^{1/5}$$

$$\approx 0.563,\ 1.24$$

values of f: $\approx 1.30,\ 1.39$



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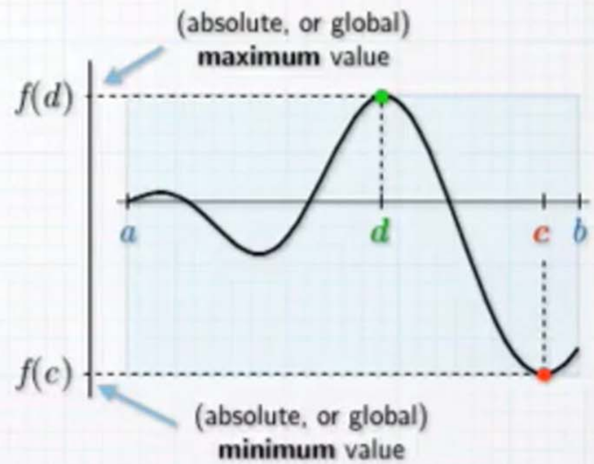
Extreme Values on Intervals

The Extreme Value Theorem

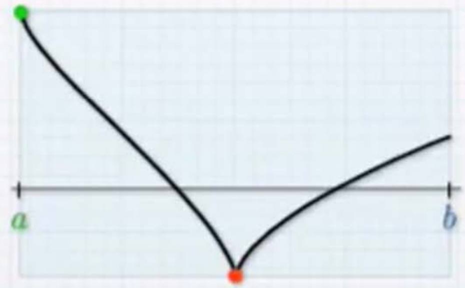
Suppose that f is continuous on a closed, bounded interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ such that

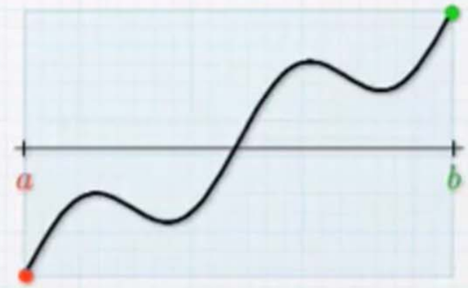
$$f(c) \le f(x) \le f(d) \text{ for all } x \text{ in } [a, b].$$

A continuous function on a closed, bounded interval attains a minimum and a maximum value.



Moreover, if either c or d is in (a, b), then it is a critical number of f.





Finding the minimum and maximum of a continuous function on a closed bounded interval.

1. Find the critical points in the interior of the interval.
2. Compute the values of f at the critical points *and at the endpoints of the interval.*
3. Select the least and greatest of the computed values.

Example Find the maximum and minimum values of $f(x) = x + 2\cos x$ on the interval $[0, 2\pi]$.

Critical numbers

$$f'(x) = 1 - 2\sin x$$
$$= 0 \text{ when } \sin x = \tfrac{1}{2}$$

Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Values at critical numbers

$$f(\tfrac{\pi}{6}) = \tfrac{\pi}{6} + 2\tfrac{\sqrt{3}}{2} = \tfrac{\pi}{6} + \sqrt{3} \approx 2.26$$
$$f(\tfrac{5\pi}{6}) = \tfrac{5\pi}{6} - 2\tfrac{\sqrt{3}}{2} = \tfrac{5\pi}{6} - \sqrt{3} \approx 0.89$$

Values at endpoints

$$f(0) = 2 \qquad f(2\pi) = 2\pi + 2 \approx 8.28$$

Therefore, the minimum value of f on $[0, 2\pi]$ is $f(\frac{5\pi}{6}) = \frac{5\pi}{6} - \sqrt{3}$, and the maximum value is $f(2\pi) = 2\pi + 2$.

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Example Find the maximum and minimum values of
$f(x) = 4x^3 - 8x^2 + 5x$ on the interval $[0, 1]$.

Critical numbers

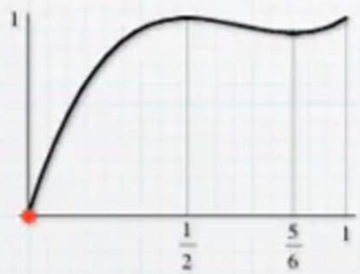
$$f'(x) = 12x^2 - 16x + 5$$
$$= (2x-1)(6x-5)$$
$$x = \tfrac{1}{2},\ \tfrac{5}{6}$$

Values at critical numbers

$$f(\tfrac{1}{2}) = \tfrac{4}{8} - \tfrac{8}{4} + \tfrac{5}{2} = 1$$
$$f(\tfrac{5}{6}) = \tfrac{4\cdot 5^3}{6^3} - \tfrac{8\cdot 5^2}{6^2} + \tfrac{5^2}{6}$$
$$= \tfrac{20\cdot 5^2 - 48\cdot 5^2 + 36\cdot 5^2}{6^3} = \tfrac{8\cdot 5^2}{6^3} = \tfrac{25}{27}$$

Values at endpoints

$$f(0) = 0 \qquad f(1) = 1$$



Therefore, the minimum value of f on $[0,1]$ is $f(0) = 0$, and the maximum value is $f(\tfrac{1}{2}) = f(1) = 1$.

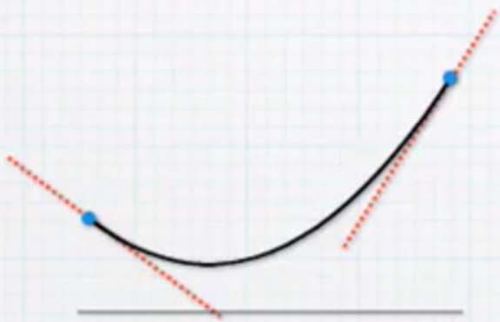
(One-sided) Derivatives at Endpoints

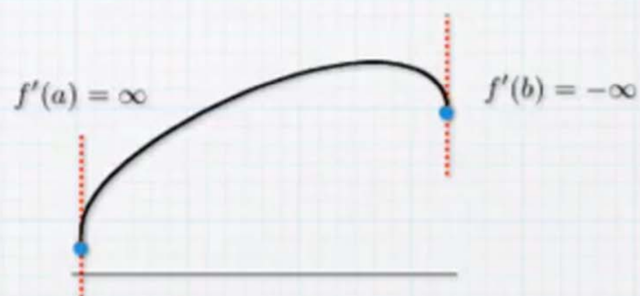
Let f be defined on a closed bounded interval $[a, b]$. The derivative of f at $x = a$ is understood to be

$$f'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}.$$

Similarly, the derivative of f at $x = b$ is understood to be

$$f'(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h} = \lim_{h \to 0^+} \frac{f(b) - f(b-h)}{h}.$$

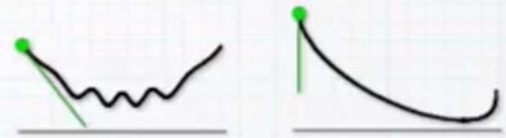




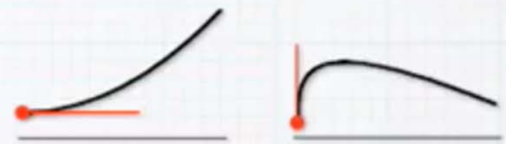
Remarks on Endpoint Extrema

Suppose that f is continuous on $[a, b]$ and that $f'(a)$ exists as a real number or as $\pm\infty$.

If $f(a)$ is the maximum value of f on $[a, b]$, then $f'(a) \le 0$ or $f'(a) = -\infty$.

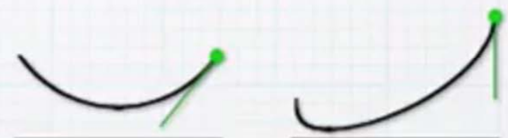


If $f(a)$ is the minimum value of f on $[a, b]$, then $f'(a) \ge 0$ or $f'(a) = \infty$.

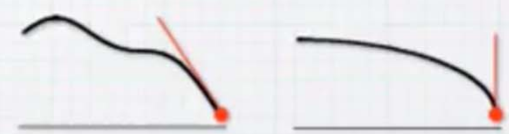


Suppose that f is continuous on $[a, b]$ and that $f'(b)$ exists as a real number or as $\pm\infty$.

If $f(b)$ is the maximum value of f on $[a, b]$, then $f'(b) \ge 0$ or $f'(b) = \infty$.



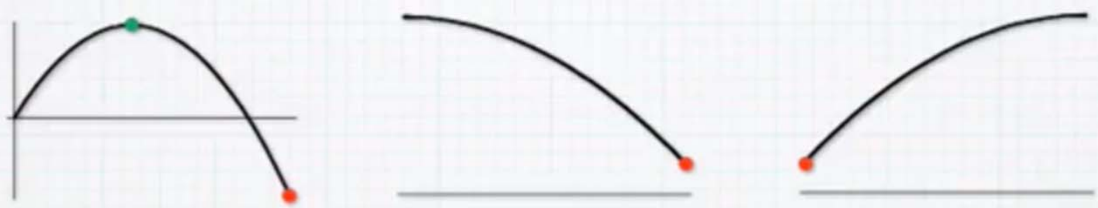
If $f(b)$ is the minimum value of f on $[a, b]$, then $f'(b) \le 0$ or $f'(b) = -\infty$.



The Second Derivative

Suppose that f is continuous on $[a, b]$ and $f''(x)$ exists for all x in (a, b).

If $f''(x) \leq 0$ for all x in (a, b), then the minimum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical number c in (a, b), then $f(c)$ is the maximum value of f on $[a, b]$.



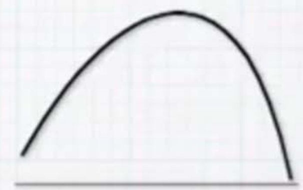
If $f''(x) \geq 0$ for all x in (a, b), then the maximum value of f on $[a, b]$ is either $f(a)$ or $f(b)$. If, in addition, f has a critical number c in (a, b), then $f(c)$ is the minimum value of f on $[a, b]$.

A Useful Theorem for Open Intervals

Let I be an open interval, and suppose that $f''(x)$ exists for all x in I.

If $f''(x) \geq 0$ for all x in I, and if there is a number c in I where $f'(c) = 0$, then $f(c)$ is the (global) minimum value of f on I.

If $f''(x) \leq 0$ for all x in I, and if there is a number c in I where $f'(c) = 0$,



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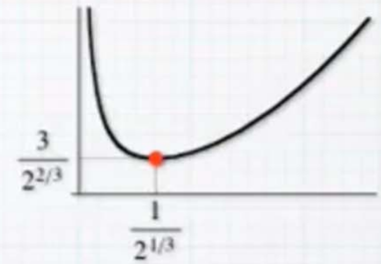
Example Find the minimum value of $f(x) = \dfrac{x^3+1}{x}$ on $(0, \infty)$.

$$f(x) = x^2 + x^{-1}$$

$$f'(x) = 2x - x^{-2} = \frac{2x^3 - 1}{x^2} = 0 \text{ at } x = \sqrt[3]{1/2}$$

$$f''(x) = 2 + 2x^{-3} > 0 \text{ for all } x \text{ in } (0, \infty).$$

Minimum value: $\quad f(\sqrt[3]{1/2}) = \dfrac{1/2 + 1}{\sqrt[3]{1/2}} = \dfrac{3\sqrt[3]{2}}{2}$



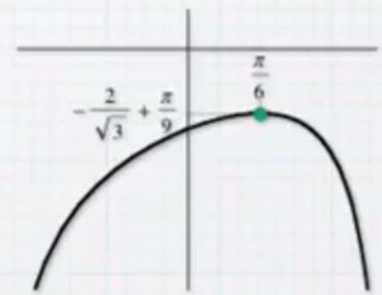
Example Find the maximum value of $f(x) = \frac{2}{3}x - \sec x$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$f'(x) = \frac{2}{3} - \sec x \tan x \qquad \frac{\sin x}{\cos^2 x} = \frac{2}{3} = \frac{1/2}{3/4} \text{ at } x = \frac{\pi}{6}$$

$$f''(x) = -\sec x \tan x \tan x - \sec x \sec^2 x$$

$$= -\sec x (\tan^2 x + \sec^2 x) < 0 \text{ on } (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

Maximum value: $f(\frac{\pi}{6}) = \frac{\pi}{9} - \frac{2}{\sqrt{3}}$



Applied Optimization Problems

Five Examples

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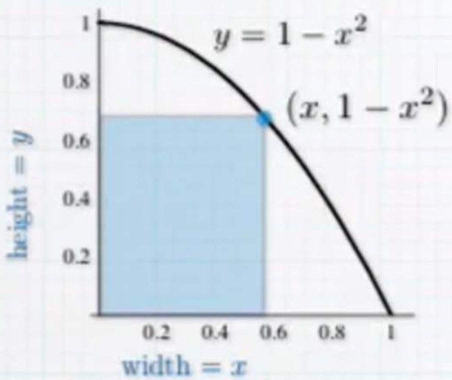
Example A rectangle is inscribed in the region in the first quadrant bounded by the coordinate axes and the parabola $y = 1 - x^2$. Find the dimensions of the rectangle that maximize its area.



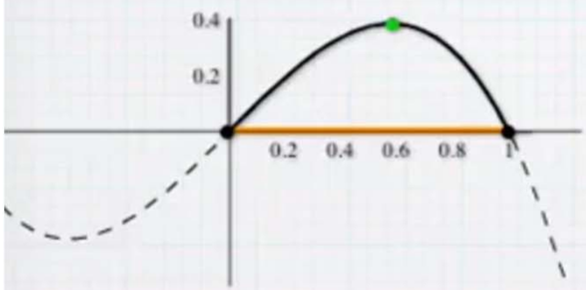
$$\overset{\text{width} \times \text{height}}{A = x\,y} \quad \text{where } y = 1 - x^2$$

$$A(x) = x\,(1 - x^2) = x - x^3 \quad \text{for } \overset{\textit{domain}}{0 \le x \le 1}$$

$$A'(x) = 1 - 3x^2 = 0 \quad \text{at } \overset{\textit{critical number}}{x = \sqrt{1/3}} \approx 0.577$$

$$A(\sqrt{1/3}) = \sqrt{\tfrac{1}{3}}\,\left(1 - \tfrac{1}{3}\right) = \frac{2}{3\sqrt{3}} \approx 0.385$$

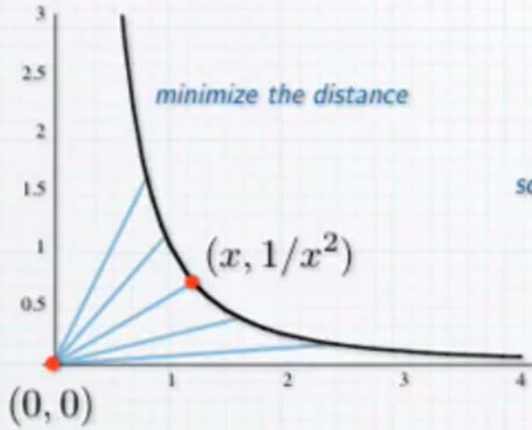
$$A(0) = 0 \qquad A(1) = 0$$



Maximum value: $A(\sqrt{1/3}) = \dfrac{2}{3\sqrt{3}}$

Optimal dimensions: width $= \sqrt{\tfrac{1}{3}}$, height $= \tfrac{2}{3}$

Example Find the point on the graph of $y = \dfrac{1}{x^2}$, $x > 0$, that is closest to the origin.



objective function

$$\sqrt{x^2 + y^2}$$

constraint

$$y = \frac{1}{x^2}$$

$$\sqrt{x^2 + (1/x^2)^2} = \sqrt{x^2 + 1/x^4}$$

squared distance $\quad f(x) = x^2 + 1/x^4 = \dfrac{x^6 + 1}{x^4}, \quad$ domain $x > 0$

$$f'(x) = 2x - 4/x^5 = \frac{2x^6 - 4}{x^5}$$

$$= 0 \text{ at } x = \sqrt[6]{2} \approx 1.12$$ critical number

$$f''(x) = 2 + 20/x^6 > 0$$

Minimum value of f: $\quad f(\sqrt[6]{2}) = \dfrac{2+1}{2^{4/6}} = \dfrac{3\sqrt[3]{2}}{2}$

Minimum distance: $\quad \sqrt{f(\sqrt[6]{2})} = \sqrt{3\sqrt[3]{2}/2} \approx 1.37$

Closest point on the curve: $\quad \left(\sqrt[6]{2}, \dfrac{1}{\sqrt[3]{2}}\right)$

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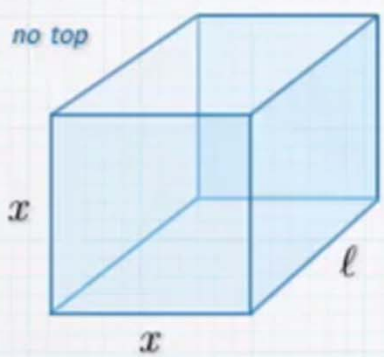
Example An aquarium is to be built to hold 20 cubic feet of water. If two ends of the aquarium are square and there is no top, find the dimensions that minimize the surface area — and thus the amount of glass used in its construction.



objective function

$$A = 2x^2 + 3x\ell$$

constraint (volume)

$$x^2\ell = 20$$

$$\ell = \frac{20}{x^2}$$

domain

$$A(x) = 2x^2 + 3x\,\frac{20}{x^2} = 2x^2 + \frac{60}{x}, \quad 0 < x < \infty$$

$$A'(x) = 4x - \frac{60}{x^2} = \frac{4\,(x^3 - 15)}{x^2}$$

critical number

$$A'(x) = 0 \ \text{ at } x = \sqrt[3]{15} \approx 2.47$$

$$A''(x) = 4 + \frac{120}{x^3} > 0$$

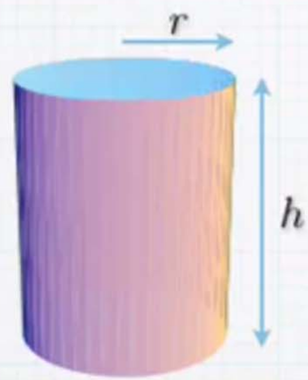
Minimum area:

$$A(\sqrt[3]{15}) = 2\,\sqrt[3]{15^2} + \frac{60}{\sqrt[3]{15}} = 6\sqrt[3]{15^2} \approx 36.49 \text{ ft}^2$$

Optimal dimensions: $x = \sqrt[3]{15} \approx 2.47$ ft,

$$\ell = \frac{20}{15^{2/3}} = \frac{4}{3}\sqrt[3]{15} \approx 3.23 \text{ ft}$$

Example A juice can (in the shape of a right circular cylinder) is to have a volume of 1 liter (1000 cm^3). Find the height and radius that minimize the surface area of the can — and thus the amount of material used in its construction.



objective function

$$A = 2\pi r^2 + 2\pi r h$$

constraint (volume)

$$\pi r^2 h = 1000$$

$$h = \frac{1000}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r \frac{1000}{\pi r^2}$$

domain

$$A(r) = 2\left(\pi r^2 + \frac{1000}{r}\right), \quad 0 < r < \infty$$

$$A'(r) = 2\left(2\pi r - \frac{1000}{r^2}\right) = 4\left(\frac{\pi r^3 - 500}{r^2}\right)$$

$$= 0 \text{ at } r = \sqrt[3]{500/\pi} = 5\sqrt[3]{4/\pi}$$

critical number

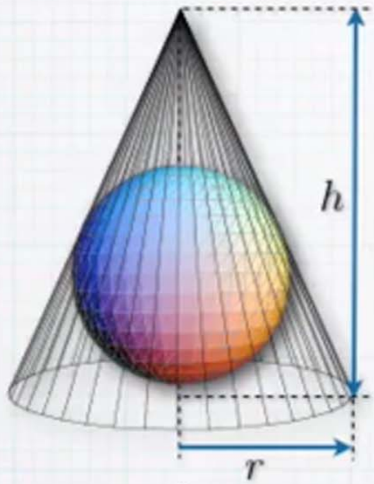
$$A''(r) = 2\left(2\pi + 2000 r^{-3}\right) > 0$$

Minimum area: $A(5\sqrt[3]{4/\pi}) = 300\sqrt[3]{2\pi} \approx 554$ cm^2

Optimal dimensions: $r = 5\sqrt[3]{4/\pi} \approx 5.42$ cm

$$h = \frac{1000}{25\pi(4/\pi)^{2/3}} = 10\sqrt[3]{4/\pi} = 2r \text{ cm}$$

Example Find the dimensions of the cone with minimum volume that can contain a sphere with radius R.



objective function

$$V = \frac{1}{3}\pi r^2 h$$

constraint: Similar Tri

$$\frac{h-R}{R} = \frac{\sqrt{h^2}}{r}$$

$$r^2 = \frac{R^2 h}{h-2R}$$

domain

$$V(h) = \frac{1}{3}\pi \frac{R^2 h}{h-2R} h = \frac{\pi R^2}{3} \frac{h^2}{h-2R}, \quad 2R < h < \infty$$

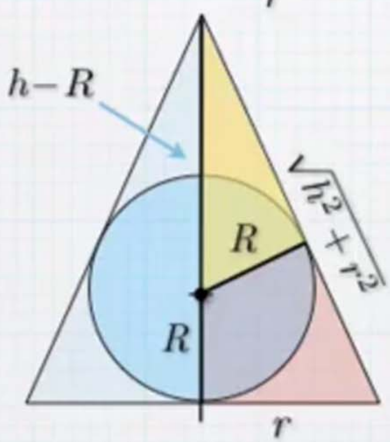
$$V'(h) = \frac{\pi R^2}{3} \cdot \frac{2h(h-2R) - h^2(1)}{(h-2R)^2}$$

critical number

$$= \frac{\pi R^2}{3} \cdot \frac{h(h-4R)}{(h-2R)^2} = 0 \text{ at } h = 4R$$

$$\lim_{h\to 2R^+} V(h) = \lim_{h\to 2R^+} \frac{\pi R^2 h^2}{3(h-2R)} = \infty$$

$$\lim_{h\to\infty} V(h) = \lim_{h\to\infty} \frac{\pi R^2 h^2}{3(h-2R)} = \infty$$



Minimum volume: $V(4R) = \frac{\pi}{3} 2R^2 \cdot 4R = \frac{8\pi R^3}{3}$

Optimal dimensions: $h = 4R \quad r^2 = \frac{R^2 \cdot 4R}{4R - 2R} = 2R^2$

The Mean-Value Theorem

and Related Results

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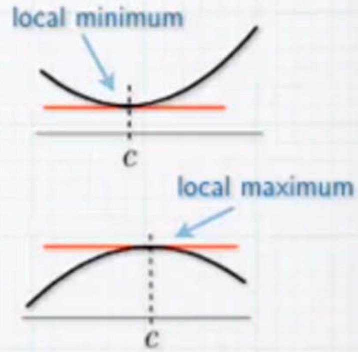
Definition

If there is an open interval I containing c in which either

$$f(c) \le f(x) \text{ for all } x \text{ in } I \text{ or } f(c) \ge f(x) \text{ for all } x \text{ in } I,$$

then we say that $f(c)$ is a **local extreme value** of f.



Theorem

If $f(c)$ is a local extreme value and if f is differentiable at $x = c$, then $f'(c) = 0$.

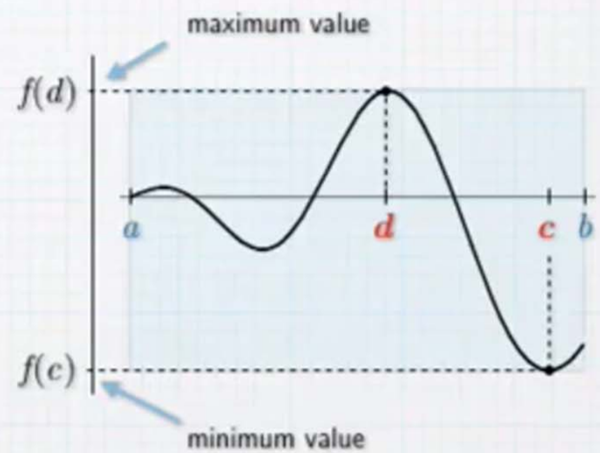
The Extreme Value Theorem

Suppose that f is continuous on a closed bounded interval $[a, b]$. Then there exist numbers c and d in $[a, b]$ such that

$$f(c) \le f(x) \le f(d) \text{ for all } x \text{ in } [a, b].$$

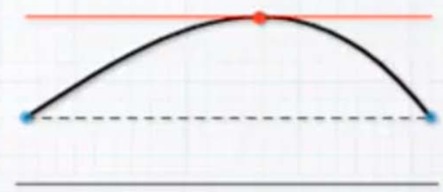
If $a < c < b$ and $f'(c)$ exists, then $f'(c) = 0$.

If $a < d < b$ and $f'(d)$ exists, then $f'(d) = 0$.



Rolle's Theorem

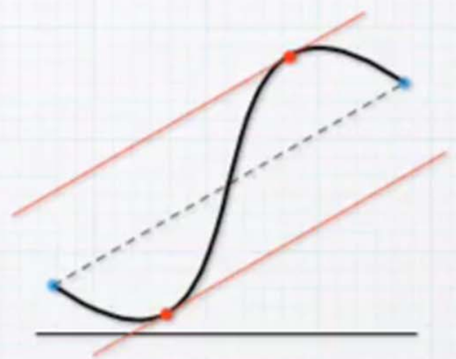
Suppose that f is continuous on $[a,b]$ and differentiable at all x in (a,b). If $f(a)=f(b)$, then $f'(c)=0$ for some c in (a,b).



The Mean-Value Theorem

Suppose that f is continuous on $[a,b]$ and differentiable at all x in (a,b). Then there is some c in (a,b) where

$$\boxed{f'(c)=\frac{f(b)-f(a)}{b-a}.}$$



Proof: Apply Rolle's theorem to $g(x)=f(x)-\dfrac{f(b)-f(a)}{b-a}(x-a)$.

$$g(a)=g(b)=f(a) \text{ and } g'(x)=f'(x)-\frac{f(b)-f(a)}{b-a}.$$

So $g'(c)=0$ for some c in (a,b).

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Rolle's Theorem

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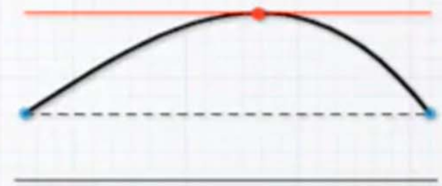


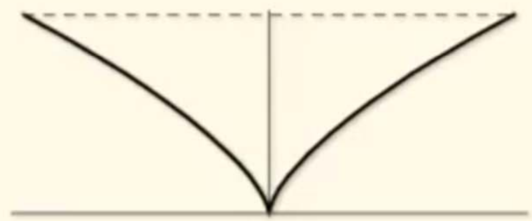
The Mean-Value Theorem

Suppose t
differential
some c in

If f fails to be differentiable at even one number in the interval, then the conclusion of the theorem may be false.

$$f(x) = x^{2/3}$$

$$f'(x) = \frac{2}{3\sqrt[3]{x}}$$



Proof: Appl

$g(a)$

So $g'(c) = 0$ for some c in (a, b).

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Example

Show that the equation $x^3 - 3x^2 + 6x = 5$ has exactly one real solution.

Let $f(x) = x^3 - 3x^2 + 6x - 5$.

f is continuous and differentiable everywhere, because it's a polynomial.

Since $f(0) = -5$ and $f(2) = 3$, the **Intermediate-value theorem** tells us that f has a root between 0 and 2.

Rolle's theorem tells us that between any *two* roots of f, there must be a root of f'.

$$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2)$$

$$\frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

So $f'(x)$ is never 0. Therefore f cannot have two roots.

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Example

Let f be continuous on $[a, b]$ and differentiable on (a, b) with $m \le f'(x) \le M$ for all x in (a, b). Show that

$$m\,(x-t) \le f(x) - f(t) \le M(x-t) \text{ if } a \le t \le x \le b.$$

Let $a \le t < x \le b$. Then, for some c between t and x,

$$f'(c) = \frac{f(x) - f(t)}{x - t}\ .$$

So

$$m \le f'(c) = \frac{f(x) - f(t)}{x - t} \le M,$$

and therefore

$$m\,(x - t) \le f(x) - f(t) \le M\,(x - t).$$

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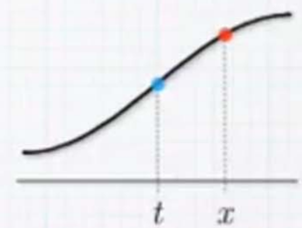
Increasing and Decreasing Values on an Interval

Let f be a function whose domain includes an interval I, and let t and x denote arbitrary numbers in I.

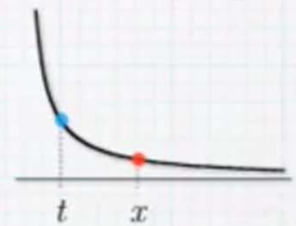
Then we say that f is **increasing** on I if

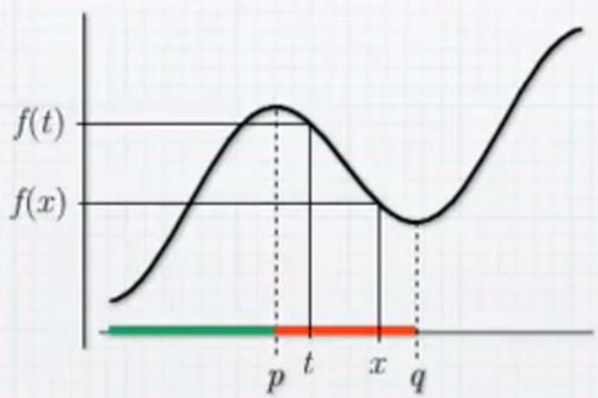
Inequality preserved by f ⟶ $$f(t) < f(x) \text{ whenever } t < x$$



and that f is **decreasing** on I if

Inequality reversed by f ⟶ $$f(t) > f(x) \text{ whenever } t < x.$$





Let I be an interval. The open interval obtained by excluding endpoints from I is called the **interior** of I.

So the interior of $[0,1]$ is $(0,1)$, the interior of $[0,\infty)$ is $(0,\infty)$, and so on.

Also, the interior of any open interval is itself.

Theorem

Let I be an interval, and suppose that f is continuous on I and differentiable on its interior. Then:

- if $f'(x) > 0$ for all x in the interior of I, then f is increasing on I;
- if $f'(x) < 0$ for all x in the interior of I, then f is decreasing on I;
- $f'(x) = 0$ for all x in the interior of I if and only if f is constant on I.

Proof

Let t and x be in I with $t < x$.
Then the **mean-value theorem** tells us that there is a number c in (t, x) where

$$f'(c) = \frac{f(x) - f(t)}{x - t}.$$

If $f'(x) > 0$ for all x in the interior of I, we have $f'(c) > 0$, which implies that $f(x) - f(t) > 0$, and we conclude that f is increasing on I.

If $f'(x) < 0$ for all x in the interior of I, we have $f'(c) < 0$, which implies that $f(x) - f(t) < 0$, and we conclude that f is decreasing on I.

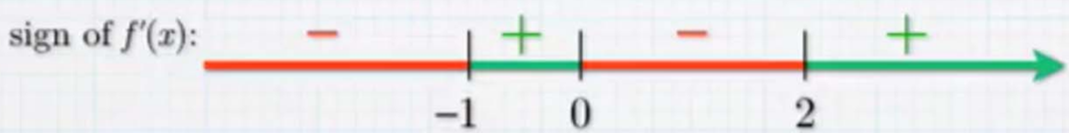
If $f'(x) = 0$ for all x in the interior of I, we have $f'(c) = 0$, which implies that $f(x) - f(t) = 0$, and we conclude that f is constant on I.

Example

Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 10$.
Find the intervals on which f is increasing and the intervals on which f is decreasing.

First we compute $f'(x)$ and simplify, factoring if possible.

$$\begin{aligned} f'(x) &= 12x^3 - 12x^2 - 24x \\ &= 12x\,(x^2 - x - 2) \\ &= 12x\,(x+1)(x-2) \\ &= 0 \text{ at } x = 0, -1, 2. \end{aligned}$$



f is decreasing on $(-\infty, -1]$ and $[0, 2]$.

f is increasing on $[-1, 0]$ and $[2, \infty)$.

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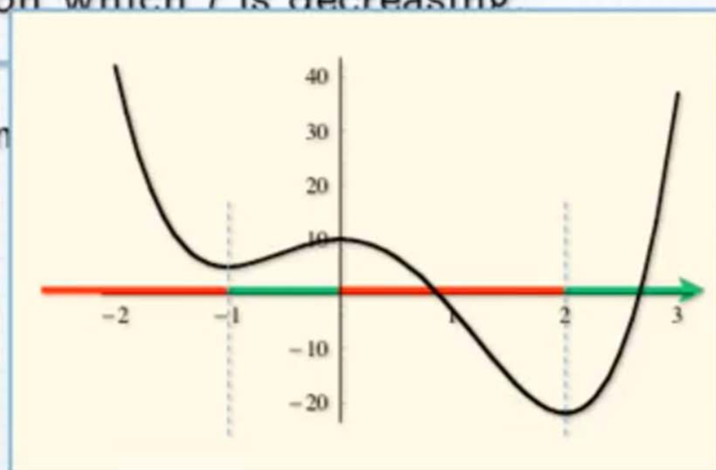

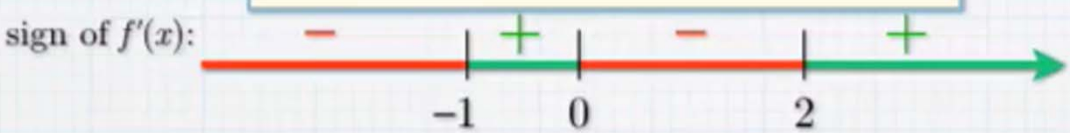

f is decreasing on $(-\infty, -1]$ and $[0, 2]$.

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Example

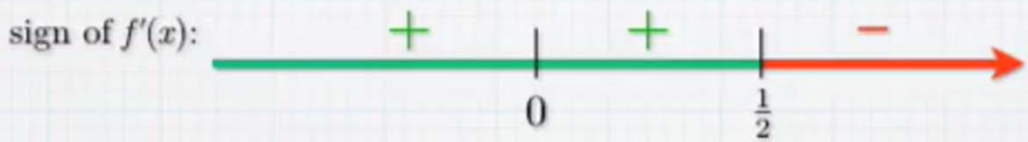
Let $f(x) = x^{1/3}(2-x)$.
Find the intervals on which f is increasing and the intervals on which f is decreasing.

First we compute $f'(x)$ and simplify.

$$f(x) = 2x^{1/3} - x^{4/3}$$

$$f'(x) = \frac{2}{3}x^{-2/3} - \frac{4}{3}x^{1/3} = \frac{2}{3x^{2/3}} - \frac{4x}{3x^{2/3}}$$

$$= \frac{2\,(1-2x)}{3x^{2/3}} \quad = 0 \text{ at } x = \tfrac{1}{2} \text{ and is undefined at } x = 0.$$



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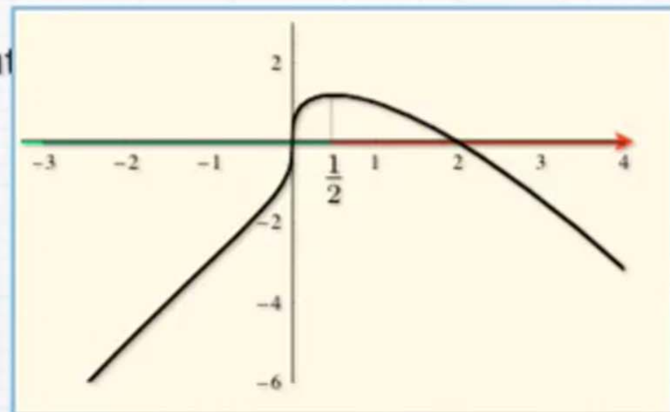
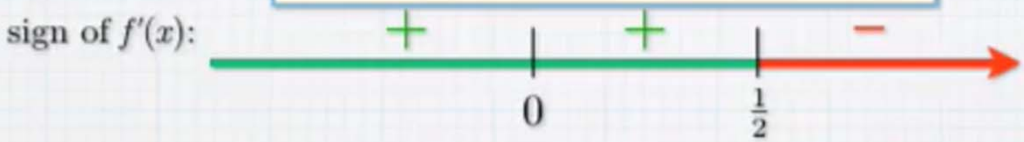


$\frac{4x}{3x^{2/3}}$

is undefined at $x = 0$.



f is decreasing on $[\frac{1}{2}, \infty)$.

f is increasing on $(-\infty, \frac{1}{2}]$.

Question: If f is increasing on I, can we conclude that $f'(x) > 0$ for all x in the interior of I? *No.*

For example, $f(x) = x^3$ is increasing on $(-\infty,\infty)$, but $f'(0) = 0$, so we only have $f'(x) \geq 0$ on $(-\infty,\infty)$.

Theorem

Let I be an interval, and suppose that f is continuous on I and differentiable on its interior. Then:

- if f is increasing on I, then $f'(x) \geq 0$ for all x in the interior of I;
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Suppose that f is increasing on I, and let x be in the interior of I. Then,

$$\frac{f(x+h) - f(x)}{h} > 0$$

for all h small enough that $x + h$ is also in the interior of I.

This implies that $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$.

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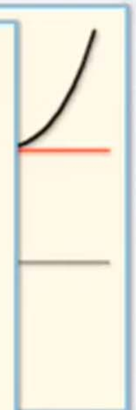
Example

Let $f(x) = 6x^5 - 15x^4 + 10x^3$.
Find the intervals on which f is increasing and the intervals on which f is decreasing.



Theorem

Let I be an interval, and suppose that f is continuous on I and differentiable on its interior. Then:

- if $f'(x) \geq 0$ for all x in the interior of I, and if $f'(x) = 0$ at only finitely many x in I, then f is increasing on I;
- if $f'(x) \leq 0$ for all x in the interior of I, and if $f'(x) = 0$ at only finitely many x in I, then f is decreasing on I.