Matrices and their functions

Now we study the relationship between a matrix M and the function $\mathbf{x} \mapsto M * \mathbf{x}$

- *Easy:* Going from a matrix M to the function $\mathbf{x} \mapsto M * \mathbf{x}$
- *A little harder:* Going from the function $\mathbf{x} \mapsto M * \mathbf{x}$ to the matrix M.

In studying this relationship, we come up with the fundamental notion of a *linear function*.

From matrix to function

Starting with a M, define the function $f(\mathbf{x}) = M * x$.

Domain and co-domain?
If M is an $R \times C$ matrix over $\mathbb{F}$ then

- domain of f is $\mathbb{F}^C$
- co-domain of f is $\mathbb{F}^R$

Example: Let M be the matrix

	#	@	?
a	1	2	3
b	10	20	30

and define $f(\mathbf{x}) = M * \mathbf{x}$

- Domain of f is $\mathbb{R}^{\{\#,@,?\}}$.
- Co-domain of f is $\mathbb{R}^{\{a,b\}}$.

f maps

#	@	?
2	2	-2

to

a	b
0	0

Example: Define $f(\mathbf{x}) = \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \end{bmatrix} * \mathbf{x}$.

- Domain of f is $\mathbb{R}^3$
- Co-domain of f is $\mathbb{R}^2$

f maps $[2, 2, -2]$ to $[0, 0]$

From function to matrix

We have a function $f : \mathbb{F}^A \longrightarrow \mathbb{F}^B$

We want to compute matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$.

- Since the domain is $\mathbb{F}^A$, we know that the input $\mathbf{x}$ is an A-vector.
- For the product $M * \mathbf{x}$ to be legal, we need the column-label set of M to be A.
- Since the co-domain is $\mathbb{F}^B$, we know that the output $f(\mathbf{x}) = M * \mathbf{x}$ is B-vector.
- To achieve that, we need row-label set of M to be B.

Now we know that M must be a $B \times A$ matrix....

... but what about its entries?

From function to matrix

- We have a function $f : \mathbb{F}^n \longrightarrow \mathbb{F}^m$
- We think there is an $m \times n$ matrix M such that $f(\mathbf{x}) = M * \mathbf{x}$

How to go from the function f to the entries of M?

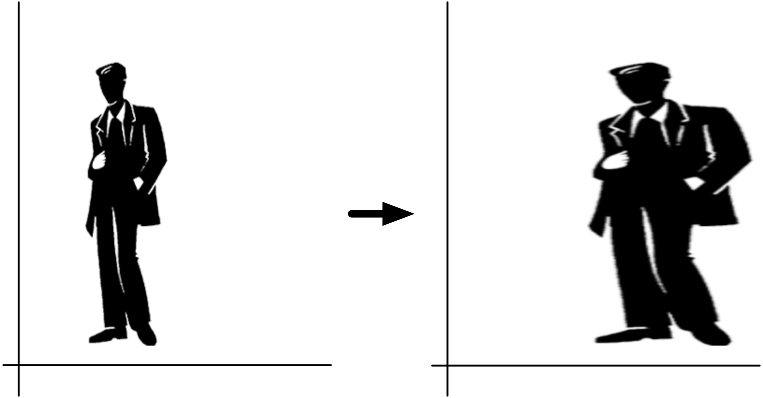
- Write mystery matrix in terms of its columns: $M = \left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right]$
- Use standard generators $\mathbf{e}_1 = [1, 0, \ldots, 0, 0], \ldots, \mathbf{e}_n = [0, \ldots, 0, 1]$ with *linear-combinations* definition of matrix-vector multiplication:

$$f(\mathbf{e}_1) = \left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right] * [1, 0, \ldots, 0, 0] = \mathbf{v}_1$$

$$\vdots$$

$$f(\mathbf{e}_n) = \left[\begin{array}{c|c|c} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{array}\right] * [0, 0, \ldots, 0, 1] = \mathbf{v}_n$$

From function to matrix: horizontal scaling



Define $s([x, y]) =$ stretching by two in horizontal direction
Assume $s([x, y]) = M * [x, y]$ for some matrix M.

- We know $s([1, 0]) = [2, 0]$ because we are stretching by two in horizontal direction
- We know $s([0, 1]) = [0, 1]$ because no change in vertical direction.

Therefore $M = \left[\begin{array}{c|c} 2 & 0 \\ 0 & 1 \end{array}\right]$

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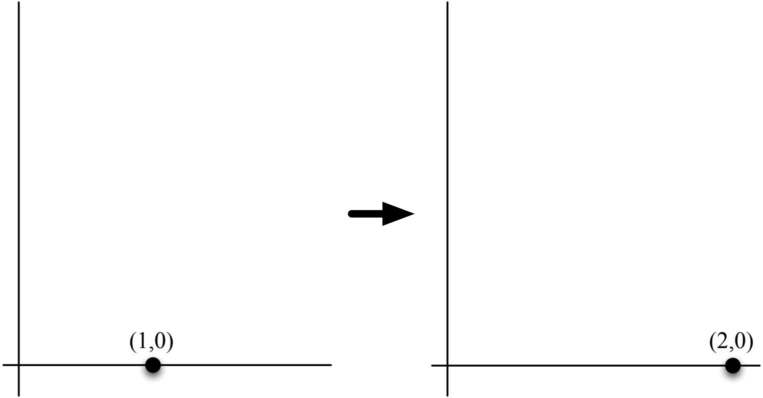

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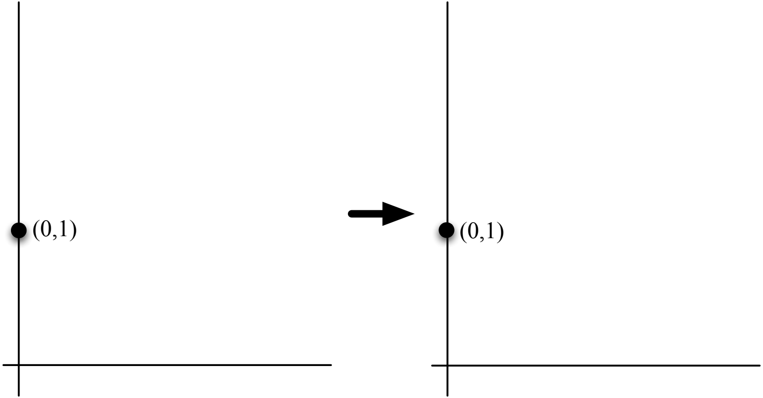

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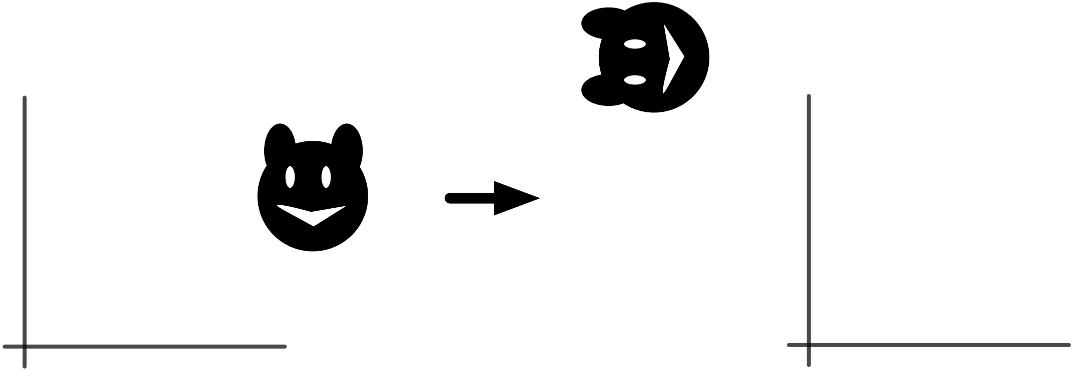
From function to matrix: rotation by 90 degrees

Define $r([x, y])$ = rotation by 90 degrees
Assume $r([x, y]) = M * [x, y]$ for some matrix M.

- We know rotating $[1, 0]$ should give $[0, 1]$ so $r([1, 0]) = [0, 1]$
- We know rotating $[0, 1]$ should give $[-1, 0]$ so $r([0, 1]) = [-1, 0]$

Therefore $M = \left[\begin{array}{c|c} 0 & -1 \\ 1 & 0 \end{array}\right]$



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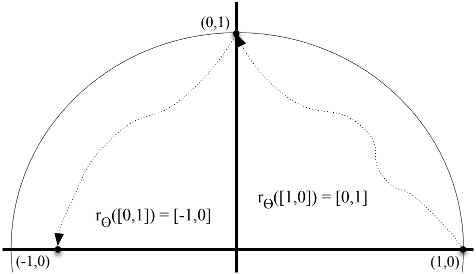
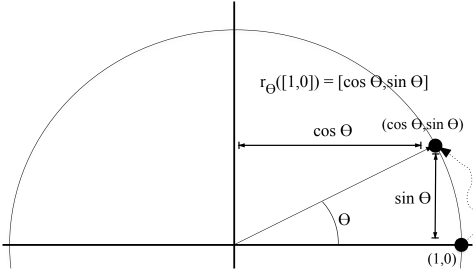


From function to matrix: rotation by θ degrees

Define $r([x, y])$ = rotation by θ.
Assume $r([x, y]) = M * [x, y]$ for some matrix M.

- We know $r([1, 0]) = [\cos\theta, \sin\theta]$ so column 1 is $[\cos\theta, \sin\theta]$
- We know $r([0, 1]) = [-\sin\theta, \cos\theta]$ so column 2 is $[-\sin\theta, \cos\theta]$

Therefore $M = \left[\begin{array}{c|c} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{array}\right]$



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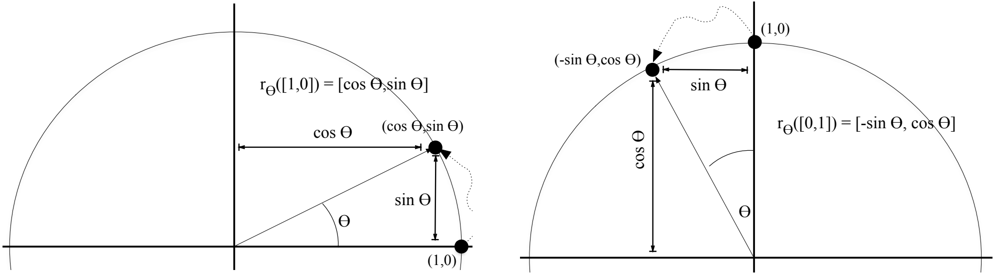

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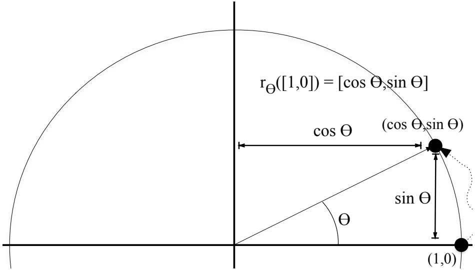

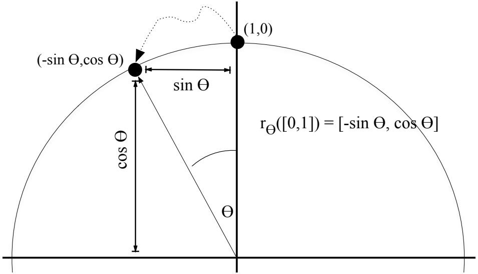

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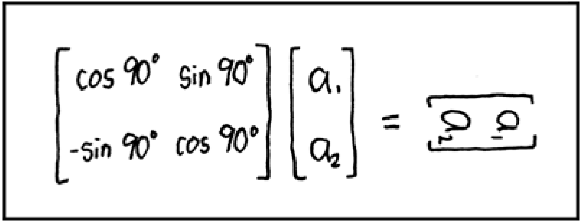
For clockwise rotation by 90 degrees, plug in $\theta =$ -90 degrees...



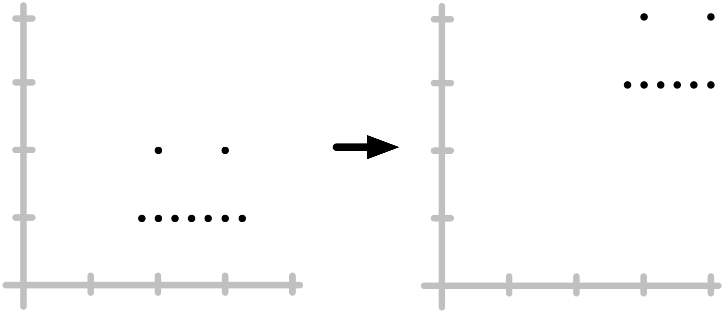
Matrix Transform (`http://xkcd.com/824`)

From function to matrix: translation

$t([x, y])$ = translation by $[1, 2]$. Assume $t([x, y]) = M * [x, y]$ for some matrix M.

- We know $t([1, 0]) = [2, 2]$ so column 1 is $[2, 2]$.
- We know $t([0, 1]) = [1, 3]$ so column 2 is $[1, 3]$.

Therefore $M = \left[\begin{array}{c|c} 2 & 1 \\ 2 & 3 \end{array}\right]$



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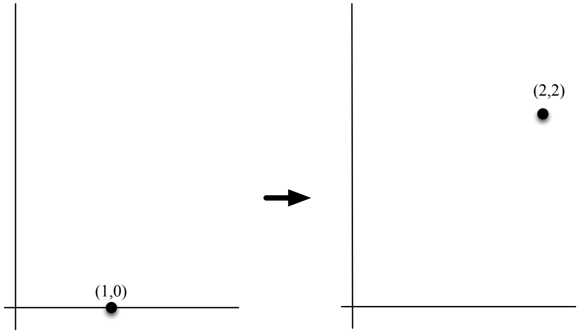

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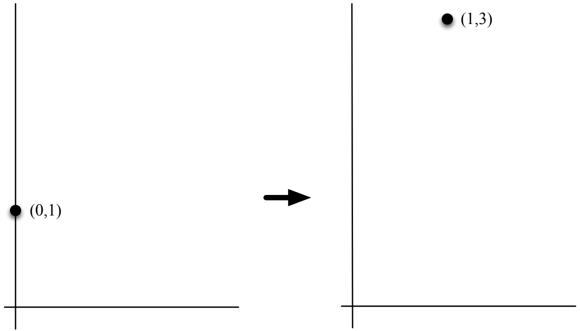

From function to matrix: identity function

Consider the function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined by $f(\mathbf{x}) = \mathbf{x}$
This is the identity function on $\mathbb{R}^4$.

Assume $f(\mathbf{x}) = M * \mathbf{x}$ for some matrix M.

Plug in the standard generators
$\mathbf{e}_1 = [1, 0, 0, 0], \mathbf{e}_2 = [0, 1, 0, 0], \mathbf{e}_3 = [0, 0, 1, 0], \mathbf{e}_4 = [0, 0, 0, 1]$

- $f(\mathbf{e}_1) = \mathbf{e}_1$ so first column is $\mathbf{e}_1$
- $f(\mathbf{e}_2) = \mathbf{e}_2$ so second column is $\mathbf{e}_2$
- $f(\mathbf{e}_3) = \mathbf{e}_3$ so third column is $\mathbf{e}_3$
- $f(\mathbf{e}_4) = \mathbf{e}_4$ so fourth column is $\mathbf{e}_4$

So $M = \left[\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

Identity function $f(\mathbf{x})$ corresponds to identity matrix $\mathbb{1}$

Diagonal matrices

Let $d_1, \ldots, d_n$ be real numbers. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the function such that $f([x_1, \ldots, x_n]) = [d_1 x_1, \ldots, d_n x_n]$. The matrix corresponding to this function is

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Such a matrix is called a *diagonal* matrix because the only entries allowed to be nonzero form a diagonal.

> **Definition:** For a domain D, a $D \times D$ matrix M is a *diagonal* matrix if $M[r, c] = 0$ for every pair $r, c \in D$ such that $r \neq c$.

Special case: $d_1 = \cdots = d_n = 1$. In this case, $f(\mathbf{x}) = \mathbf{x}$ (*identity function*)

The matrix $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ is an identity matrix.