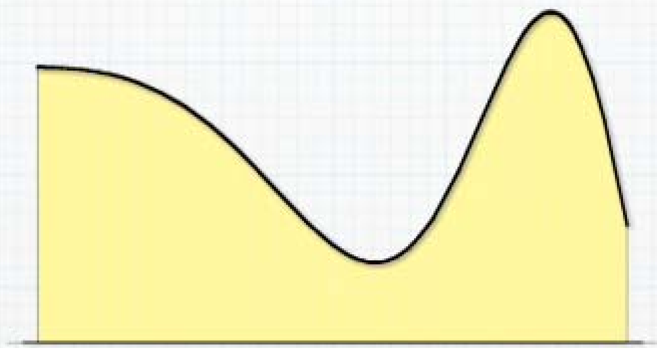
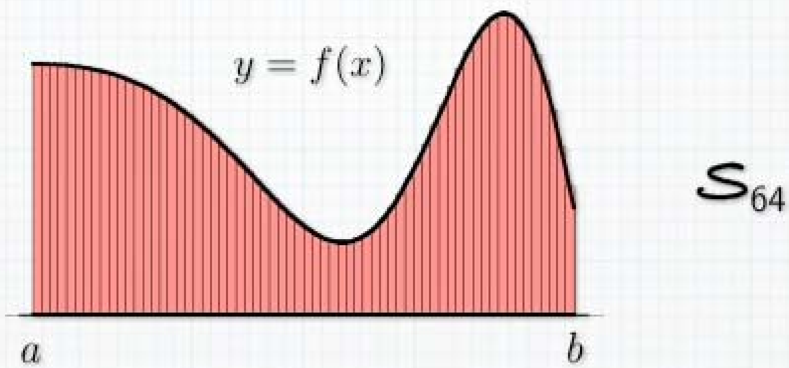
The Area Under a Curve



UNIVERSITY of HOUSTON

Approximation by Sums of Rectangle Areas



$$\text{Area} \approx \mathcal{S}_n \text{ for large } n$$

$$\text{Area} = \lim_{n\to\infty} \mathcal{S}_n$$

General Set-up of an $\mathcal{S}_n$ (uniform grid)

- n subintervals:

$$[x_{i-1}, x_i],\ i = 1, 2, \ldots, n$$

- subinterval width:

$$\Delta x = \frac{b-a}{n}$$

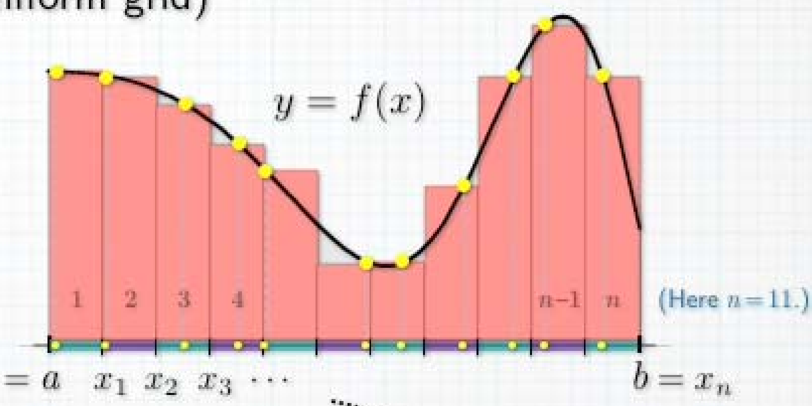
- formula for x_i:

$$x_i = a + i\,\Delta x$$

- choice of n evaluation points:

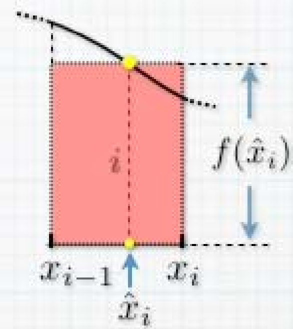
$$\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_n$$

where $x_{i-1} \le \hat{x}_i \le x_i$



area of the i^{th} rectangle

$f(\hat{x}_i)\,\Delta x$



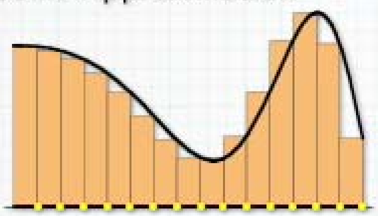
$$\text{area under the curve} \approx \mathcal{S}_n = f(\hat{x}_1)\,\Delta x + f(\hat{x}_2)\,\Delta x + \cdots + f(\hat{x}_n)\,\Delta x$$

$$= \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x$$

The sum of $f(\hat{x}_i)\,\Delta x$ as i goes from 1 to n

Convenient Choices of $\hat{x}_i$

Right-endpoint Approximation



$$\hat{x}_i = x_i = a + i\,\Delta x$$

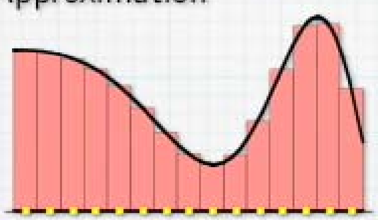
$$S_n = \sum_{i=1}^{n} f(a + i\,\Delta x)\,\Delta x$$

Left-endpoint Approximation

$$\hat{x}_i = x_{i-1} = a + (i-1)\,\Delta x$$

$$S_n = \sum_{i=1}^{n} f(a + (i-1)\Delta x)\,\Delta x$$

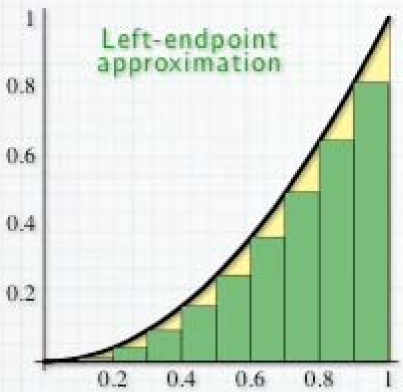
Midpoint Approximation



$$\hat{x}_i = \frac{x_{i-1} + x_i}{2} = a + (i - \tfrac{1}{2})\,\Delta x$$

$$S_n = \sum_{i=1}^{n} f(a + (i - \tfrac{1}{2})\Delta x)\,\Delta x$$

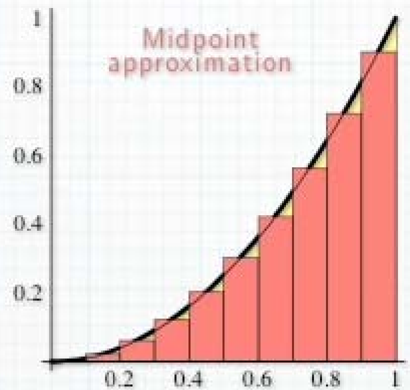
Example $f(x) = x^2$ on $[0,1]$. $n = 10$ $\Delta x = \frac{1}{10}$



$$\hat{x}_i = (i-1)\,\frac{1}{10}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left((i-1)\frac{1}{10}\right)^2 \frac{1}{10}$$

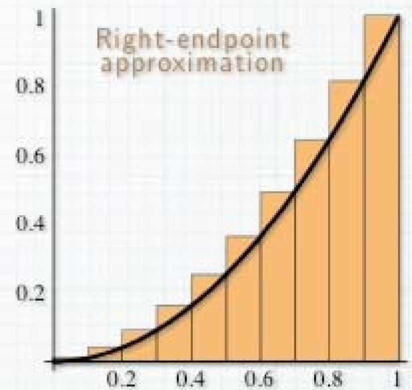
```
sum(seq((.1(I-1)
)²*.1,I,1,10)
                .285
```



$$\hat{x}_i = \left(i-\tfrac{1}{2}\right)\frac{1}{10}$$

$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left(\left(i-\tfrac{1}{2}\right)\frac{1}{10}\right)^2 \frac{1}{10}$$

```
sum(seq((.1(I-1)
)²*.1,I,1,10)
                .285
sum(seq((.1(I-.5
))²*.1,I,1,10)
               .3325
```



$$\hat{x}_i = i\,\frac{1}{10}$$

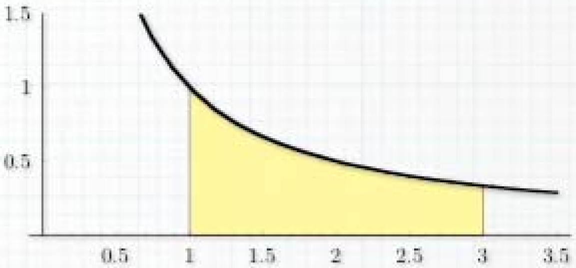
$$\mathcal{S}_{10} = \sum_{i=1}^{10} \left(i\,\frac{1}{10}\right)^2 \frac{1}{10}$$

```
                .285
sum(seq((.1(I-.5
))²*.1,I,1,10)
               .3325
sum(seq((.1*I)²*
.1,I,1,10)
                .385
```

TI-83

UNIVERSITY of HOUSTON

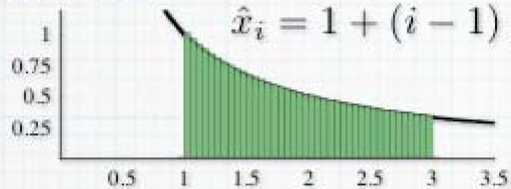
Example $f(x) = 1/x$ on $[1,3]$.



$$n = 40 \quad \Delta x = \tfrac{3-1}{40} = \tfrac{1}{20}$$

Left-endpoint approximation

$$\hat{x}_i = 1 + (i-1)\,\tfrac{1}{20}$$

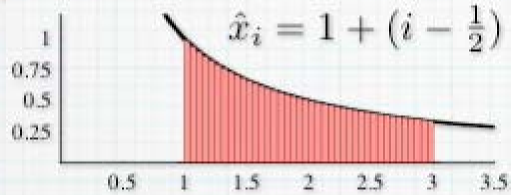


$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-1)\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-1)} \approx 1.168$$

$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+(i-1)\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+(i-1)} \approx 1.115$$

Midpoint approximation

$$\hat{x}_i = 1 + (i-\tfrac{1}{2})\,\tfrac{1}{20}$$

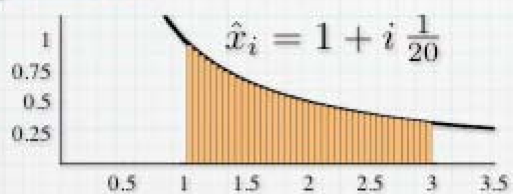


$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+(i-\frac{1}{2})\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+(i-\frac{1}{2})} \approx 1.097$$

$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+(i-\frac{1}{2})\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+(i-\frac{1}{2})} \approx 1.0985$$

Right-endpoint approximation

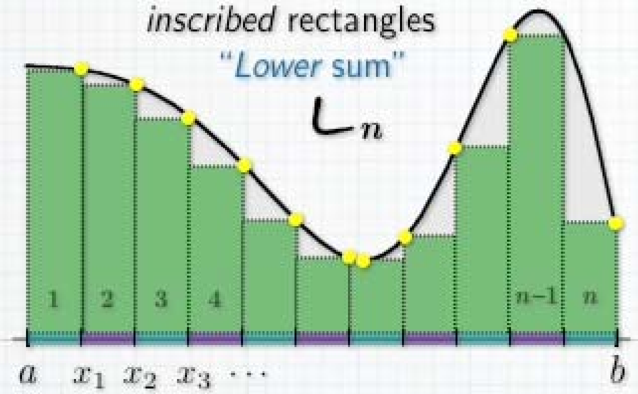
$$\hat{x}_i = 1 + i\,\tfrac{1}{20}$$

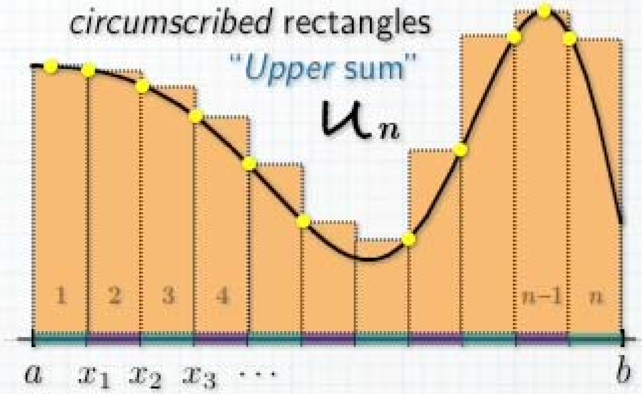


$$S_{10} = \sum_{i=1}^{10} \frac{1}{1+i\frac{1}{5}}\,\frac{1}{5} = \sum_{i=1}^{10} \frac{1}{5+i} \approx 1.035$$

$$S_{40} = \sum_{i=1}^{40} \frac{1}{1+i\frac{1}{20}}\,\frac{1}{20} = \sum_{i=1}^{40} \frac{1}{20+i} \approx 1.082$$

Lower and Upper Sums





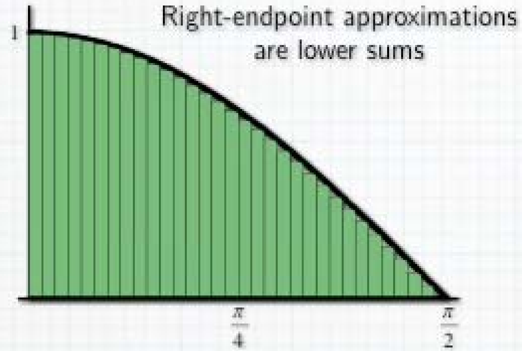
Observations: $\mathcal{L}_m \leq \text{Area} \leq \mathcal{U}_n$ for all m and n

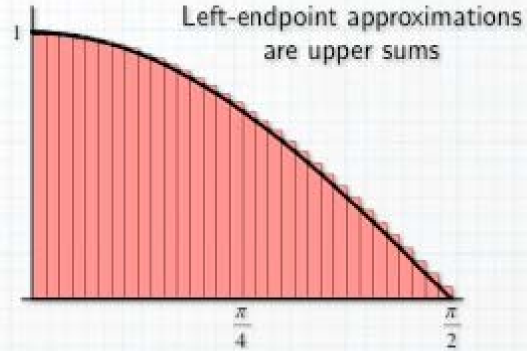
Uniform grid $\mathcal{L}_n \leq \mathcal{L}_{2n} \leq \text{Area} \leq \mathcal{U}_{2n} \leq \mathcal{U}_n$ for all n

$$\mathcal{L}_4 \leq \mathcal{L}_8 \leq \mathcal{L}_{16} \leq \cdots \leq \text{Area} \leq \cdots \leq \mathcal{U}_{16} \leq \mathcal{U}_8 \leq \mathcal{U}_4$$

For each n, $\mathcal{L}_n \leq \mathcal{S}_n \leq \mathcal{U}_n$ for every $\mathcal{S}_n$

Example $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$. $\Delta x = \frac{\pi}{2n}$





$$\mathcal{L}_4 = \sum_{i=1}^{4} \cos\left(i\tfrac{\pi}{8}\right)\tfrac{\pi}{8} \approx 0.7908 \qquad \mathcal{U}_4 = \sum_{i=1}^{4} \cos\left((i-1)\tfrac{\pi}{8}\right)\tfrac{\pi}{8} \approx 1.1835$$

$$\mathcal{L}_8 = \sum_{i=1}^{8} \cos\left(i\tfrac{\pi}{16}\right)\tfrac{\pi}{16} \approx 0.8986 \qquad \mathcal{U}_8 = \sum_{i=1}^{8} \cos\left((i-1)\tfrac{\pi}{16}\right)\tfrac{\pi}{16} \approx 1.0950$$

$$\mathcal{L}_{16} = \sum_{i=1}^{16} \cos\left(i\tfrac{\pi}{32}\right)\tfrac{\pi}{32} \approx 0.9501 \qquad \mathcal{U}_{16} = \sum_{i=1}^{16} \cos\left((i-1)\tfrac{\pi}{32}\right)\tfrac{\pi}{32} \approx 1.0483$$

$$\mathcal{L}_{32} = \sum_{i=1}^{32} \cos\left(i\tfrac{\pi}{64}\right)\tfrac{\pi}{64} \approx 0.9752 \qquad \mathcal{U}_{32} = \sum_{i=1}^{32} \cos\left((i-1)\tfrac{\pi}{64}\right)\tfrac{\pi}{64} \approx 1.0243$$

A Special Choice of $\hat{x}_i$

$$\mathcal{S}_n = \sum_{i=1}^{n} \cos(\hat{x}_i)\,\Delta x$$
$$= \sum_{i=1}^{n} \frac{\sin x_i - \sin x_{i-1}}{\pi/(2n)}\,\frac{\pi}{2n}$$
$$= \sum_{i=1}^{n} (\sin x_i - \sin x_{i-1})$$
$$= (\cancel{\sin x_1} - \sin x_0) + (\cancel{\sin x_2} - \cancel{\sin x_1}) + (\cancel{\sin x_3} - \cancel{\sin x_2}) + \cdots + (\sin x_n - \cancel{\sin x_{n-1}})$$
$$= \sin x_n - \sin x_0$$
$$= \sin\tfrac{\pi}{2} - \sin 0 \;= 1 - 0 \;= 1$$

Since $\cos$ is the derivative of $\sin$, the **mean-value theorem** says we can choose $\hat{x}_i$ so that

$$\cos(\hat{x}_i) = \frac{\sin x_i - \sin x_{i-1}}{\Delta x} = \frac{\sin x_i - \sin x_{i-1}}{\pi/(2n)}$$

$\mathcal{L}_n \le \mathcal{S}_n = 1 \le \mathcal{U}_n$ for all n!

Therefore, the exact area is 1.

What the last example illustrates (a preview of things to come)

Suppose that $f(x) \geq 0$ for all x in $[a, b]$ and we want to find the area under its graph over $[a, b]$. If $f(x) = F'(x)$ for all x in $[a, b]$, then choosing $\hat{x}_i$ so that

$$f(\hat{x}_i) = \frac{F(x_i) - F(x_{i-1})}{\Delta x}$$

mean-value theorem

results in

$$\mathcal{S}_n = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) = F(x_n) - F(x_0) = F(b) - F(a).$$

So

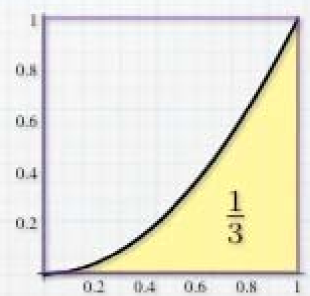
$$\mathcal{L}_n \leq F(b) - F(a) \leq \mathcal{U}_n \quad \text{for all } n.$$

Therefore, the *exact* area is $F(b) - F(a)$.

Example $f(x) = x^2$ on $[0, 1]$.

Since x^2 is the derivative of $\frac{1}{3}x^3$,

the area under the curve is $\frac{1}{3}\,1^3 - \frac{1}{3}\,0^3 = \frac{1}{3}$.



The Integral

$$\int$$

UNIVERSITY of HOUSTON

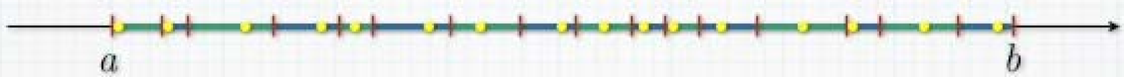
Partitions and Riemann Sums

An *augmented* **partition** $\mathcal{P}_n$ of $[a,b]$ consists of a collection of $n+1$ numbers $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$, where

$$x_0 = a < x_1 < x_2 < x_3 < \cdots < b = x_n\,,$$

and a set of n "evaluation points"

$$\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_n, \text{ where } x_{i-1} \le \hat{x}_i \le x_i\,,\ i = 1,\ldots,\, n.$$



Let $\Delta x_i = x_i - x_{i-1}$. The number $\|\mathcal{P}_n\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ is the **norm** of $\mathcal{P}_n$.

Let f be a function defined on $[a,b]$. The **Riemann sum** of f corresponding to $\mathcal{P}_n$ is

$$\mathcal{S}(f;\mathcal{P}_n) = \sum_{i=1}^{n} f(\hat{x}_i)\,\Delta x_i\,.$$

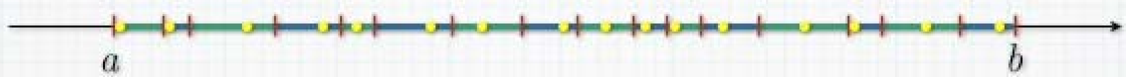
Partitions and Riemann Sums

An *augmented* **partition** $\mathcal{P}_n$ of $[a, b]$ consists of a collection of $n+1$ numbers $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$, where

$$x_0 = a < x_1 < x_2 < x_3 < \cdots < b = x_n\,,$$

and a set of n "evaluation points"

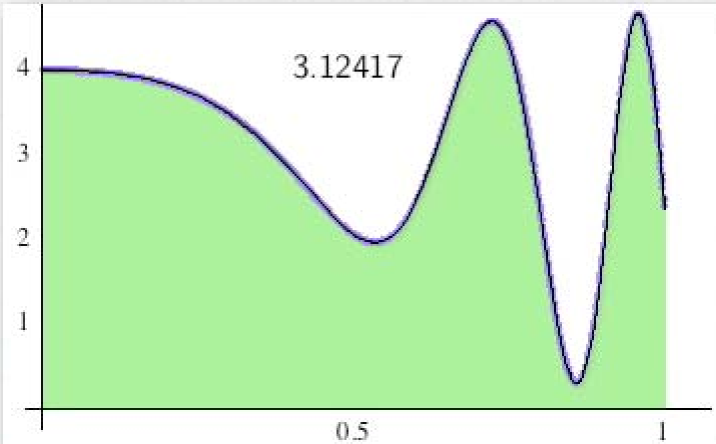
$$\hat{x}_1,\ \hat{x}_2,\ \hat{x}_3,\ \ldots,\ \hat{x}_n, \text{ where } x_{i-1} \le \hat{x}_i \le x_i\,,\ i = 1, \ldots, n.$$

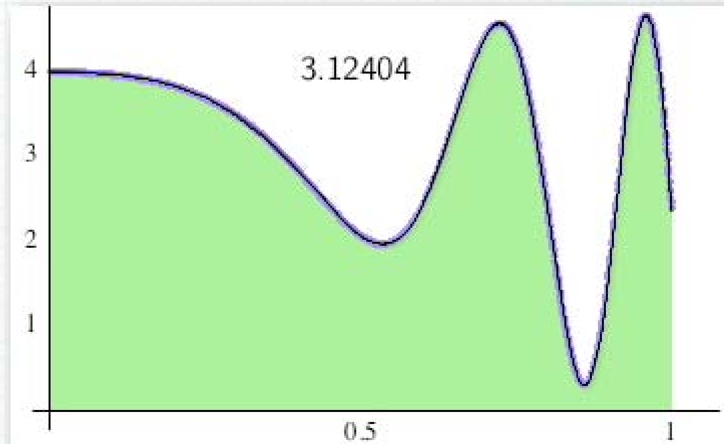


Let $\Delta x_i = x_i - x_{i-1}$. The number $\|\mathcal{P}_n\| = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_n\}$ is the **norm** of $\mathcal{P}_n$.

Let f be a function defined on $[a, b]$. The **Riemann sum** of f corresponding to $\mathcal{P}_n$ is

$$\mathcal{S}(f; \mathcal{P}_n) = \sum_{i=1}^{n} f(\hat{x}_i)\, \Delta x_i\,.$$

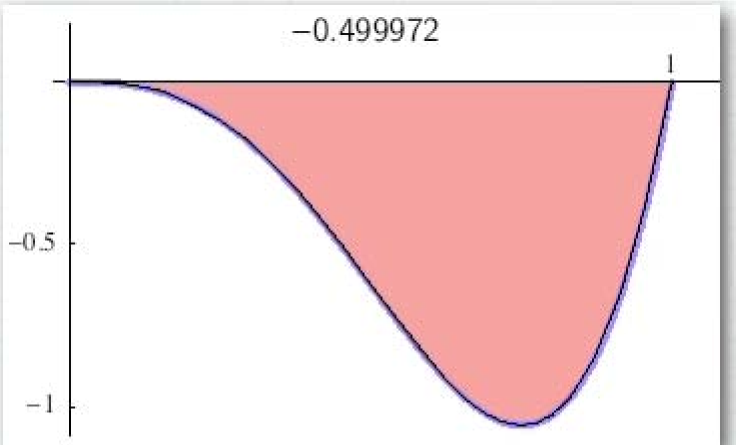


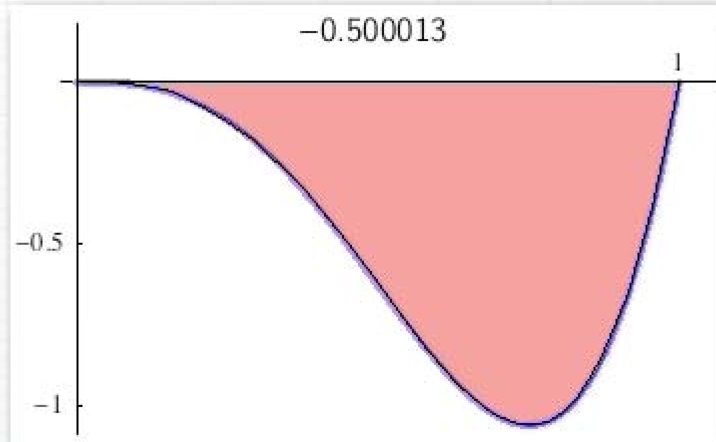


Let f be continuous on $[a, b]$. If $f(x) \geq 0$ for all x in $[a, b]$, and if $f(x) > 0$ for *some* x in $[a, b]$, then

$$\int_a^b f(x)\,dx > 0$$

and equals the area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.



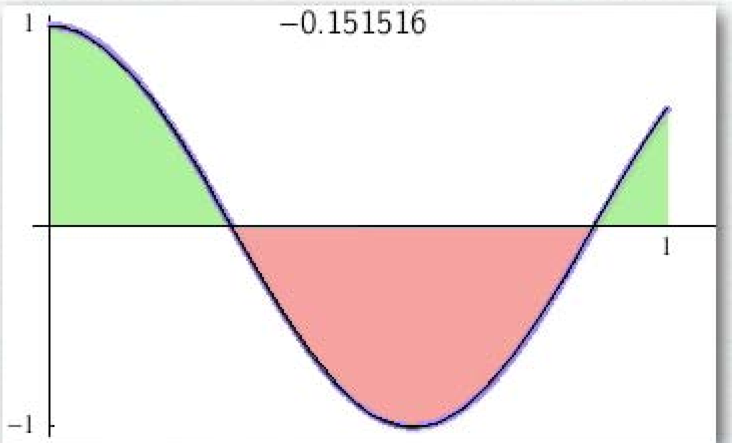


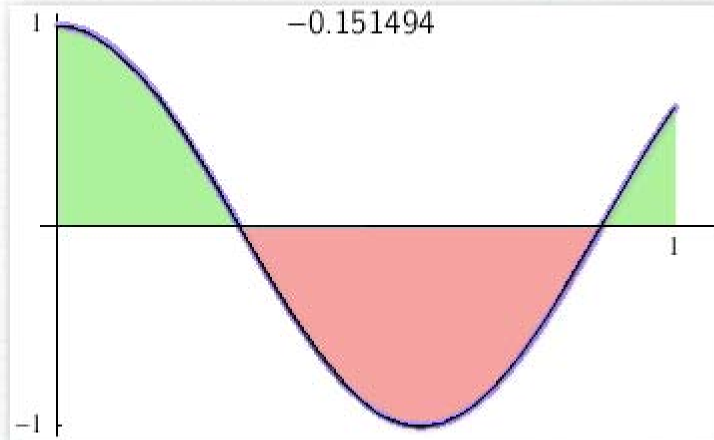
Let f be continuous on $[a, b]$. If $f(x) \leq 0$ for all x in $[a, b]$, and if $f(x) < 0$ for *some* x in $[a, b]$, then

$$\int_a^b f(x)\,dx < 0$$

and $-\int_a^b f(x)\,dx$ equals the area of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.

UNIVERSITY of **HOUSTON**



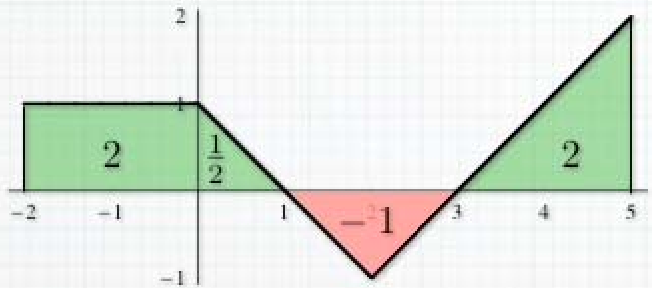


$\int_a^b f(x)\,dx$ equals the *difference* between the area under the graph of f *above* the x-axis and the area above the graph of f *below* the x-axis between $x = a$ and $x = b$.

This is the **signed** (or *net*) **area** of the region bounded by the graph of f and the x-axis between $x = a$ and $x = b$.

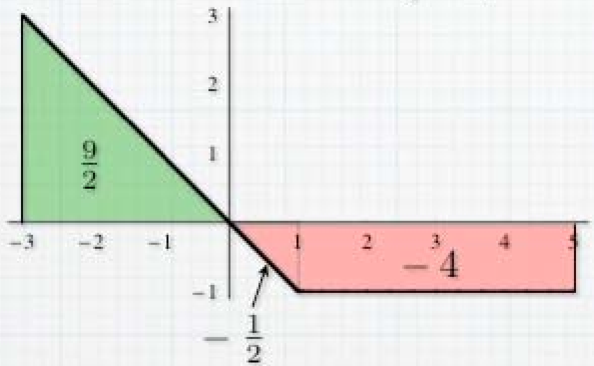
UNIVERSITY of HOUSTON

Example Let $f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 1 - x, & \text{if } 0 \le x < 2. \\ x - 3, & \text{if } 2 \le x \end{cases}$ Find $\int_{-2}^{5} f(x)\,dx$.



$$\int_{-2}^{5} f(x)\,dx = 2 + \frac{1}{2} - 1 + 2 = \frac{7}{2}$$

Example Let $f(x) = \begin{cases} -x, & \text{if } x < 1 \\ -1, & \text{if } 1 \le x \end{cases}$. Find $\int_{-3}^{5} f(x)\,dx$.



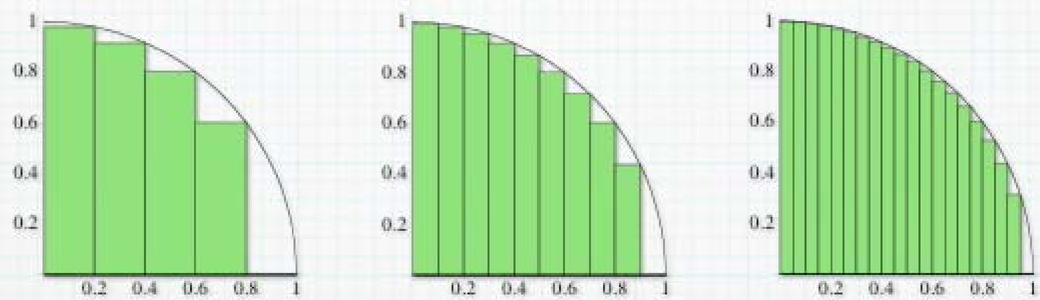
$$\int_{-3}^{5} f(x)\,dx = \frac{9}{2} - \frac{1}{2} - 4 = 0$$

UNIVERSITY of HOUSTON

Example Find $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2}$.

$$\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2}=\sum_{i=1}^{n}\sqrt{\frac{n^2-i^2}{n^2}}\,\frac{1}{n}=\sum_{i=1}^{n}\sqrt{1-(i/n)^2}\,\frac{1}{n}$$

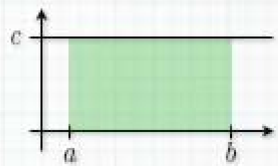
$$f(x)=\sqrt{1-x^2}$$



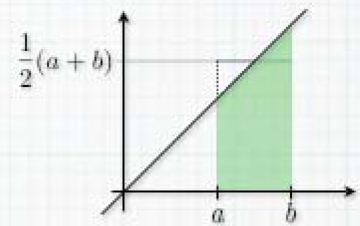
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\sqrt{n^2-i^2}}{n^2}=\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4}$$

Geometric Evaluation

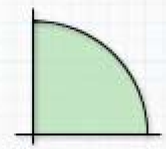
$$\int_a^b dx = b - a \qquad \int_a^b c\, dx = c\,(b-a)$$



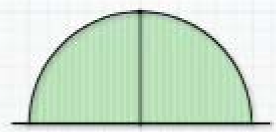
$$\int_a^b x\, dx = \frac{1}{2}(a+b)(b-a) = \frac{1}{2}\,(b^2 - a^2)$$



$$\int_0^r \sqrt{r^2 - x^2}\, dx = \frac{1}{4}\,\pi r^2$$



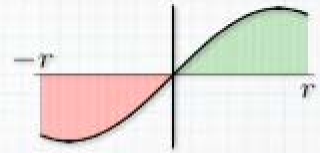
$$\int_{-r}^r \sqrt{r^2 - x^2}\, dx = \frac{1}{2}\,\pi r^2$$



Symmetries Suppose that f is integrable on $[-r, r]$.

$f(-x) = -f(x)$

If f is an *odd function*, then $\int_{-r}^{r} f(x)\,dx = 0.$

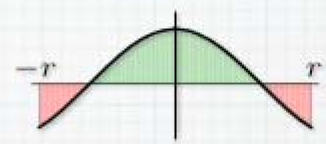


For example,

$$\int_{-2}^{2} x^3\,dx = 0 \qquad \int_{-\pi/6}^{\pi/6} \sin x\,dx = 0 \qquad \int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0$$

$f(-x) = f(x)$

If f is an *even function*, then $\int_{-r}^{r} f(x)\,dx = 2\int_{0}^{r} f(x)\,dx.$



For example,

$$\int_{-2}^{2} x^2\,dx = 2\int_{0}^{2} x^2\,dx \qquad \int_{-\pi/6}^{\pi/6} \cos x\,dx = 2\int_{0}^{\pi/6} \cos x\,dx$$

$$\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx = 2\int_{0}^{1} x^2\sqrt{1-x^2}\,dx$$

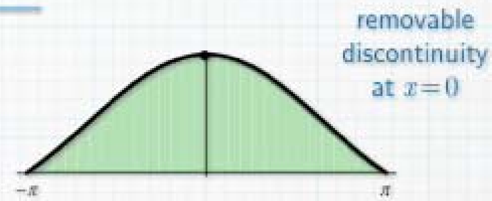
Extensions of the Definition

(1) If f is integrable on $[a, b]$, and if $g(x) = f(x)$ for all but finitely many x in $[a, b]$, then

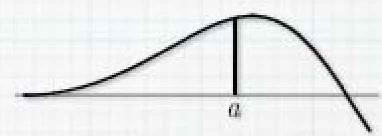
$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx\,.$$

Example $$\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx = \int_{-\pi}^{\pi} f(x)\,dx\,,$$

where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$



(2) $$\int_a^a f(x)\,dx = 0$$



(3) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

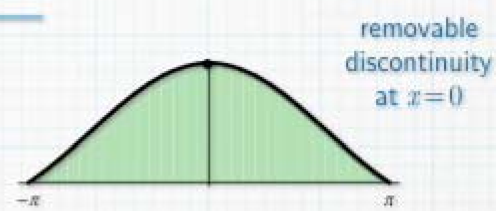
Extensions of the Definition

(1) If f is integrable on $[a,b]$, and if $g(x) = f(x)$ for all but finitely many x in $[a,b]$, then

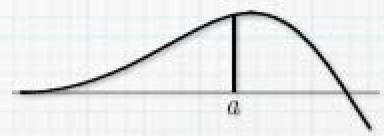
$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx\,.$$

Example $\displaystyle\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx = \int_{-\pi}^{\pi} f(x)\,dx\,,$

where $f(x) = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$



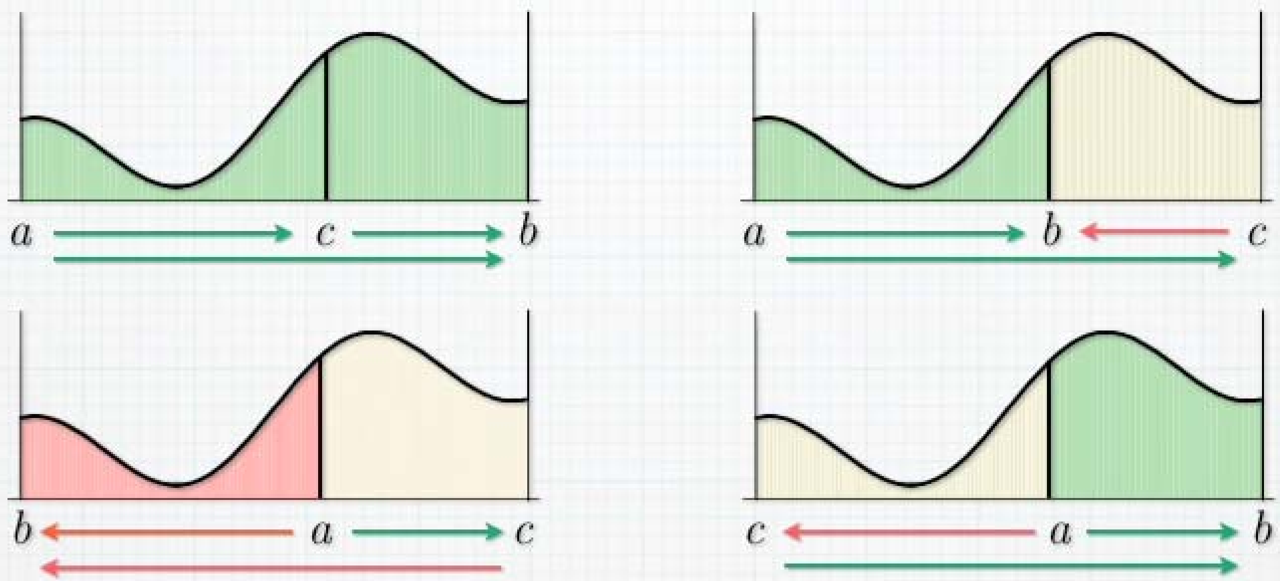
(2) $\displaystyle\int_a^a f(x)\,dx = 0$



(3) $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Interval Additivity Property

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Also: $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

Theorem

If f is bounded and has a finite number of discontinuities $t_1 < t_2 < \cdots < t_k$ in $[a, b]$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \int_a^{t_1} f(x)\,dx + \int_{t_1}^{t_2} f(x)\,dx + \cdots + \int_{t_k}^{b} f(x)\,dx\,.$$

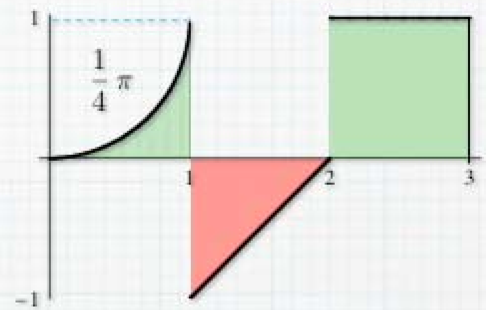
Example Let $f(x) = \begin{cases} 1-\sqrt{1-x^2}, & \text{if } 0 \le x < 1 \\ x-2, & \text{if } 1 \le x < 2 \\ 1, & \text{if } 2 < x \end{cases}$. Find $\int_0^3 f(x)\,dx$.

$$\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= \int_0^1 (1-\sqrt{1-x^2})\,dx$$

$$+ \int_1^2 (x-2)\,dx + \int_2^3 dx$$

$$= 1 - \frac{1}{4}\pi - \frac{1}{2} + 1 \quad = \frac{3}{2} - \frac{\pi}{4}$$



The Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x$$

Remarks on Terminology and Symbolism

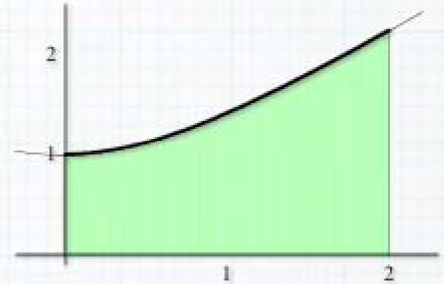
Upper limit

Integrand

Integral sign $$\int_0^2 \sqrt{1+x^2}\,dx$$ *Differential*

Lower limit

"Limits" of integration



The Dummy Variable

$$\int_a^b f(x)\,dx \qquad \int_0^2 \sqrt{1+y^2}\,dy$$

$$\int_0^1 \sqrt{k^2+r^2}\,dr \qquad \int_{-\pi}^{\pi} \cos(n\,t)\,dt \qquad \int_0^1 x^p\,dx$$

Integration with respect to r, value depends on k

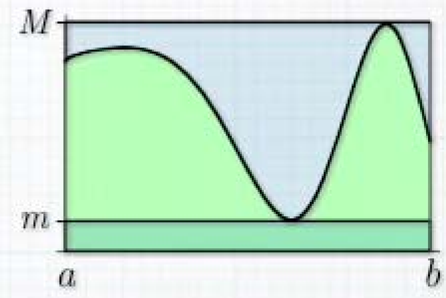
Integration with respect to t, value depends on n

Integration with respect to x, value depends on p

Average Value

Suppose that f is continuous on $[a, b]$, and let

$$m = \min_{a \le x \le b} f(x) \quad \text{and} \quad M = \max_{a \le x \le b} f(x).$$



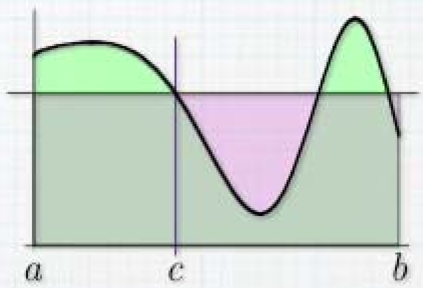
Then $m\,(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$,

i.e., $m \le \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx \le M.$

The number $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$ is the **average value** of f on $[a, b]$.

Average Value Theorem (*Mean value theorem for integrals*)

There exists a number c in (a, b) such that



$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx,$$

i.e., $\displaystyle\int_a^b f(x)\,dx = f(c)(b-a).$

UNIVERSITY of HOUSTON

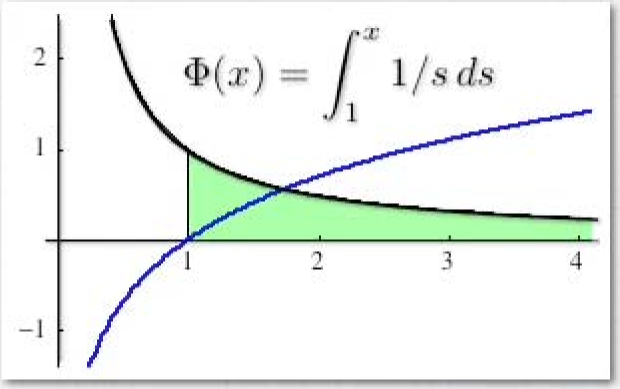
A Function Defined by Integration

Suppose that f is continuous on an interval containing a, and let

upper-case *phi*

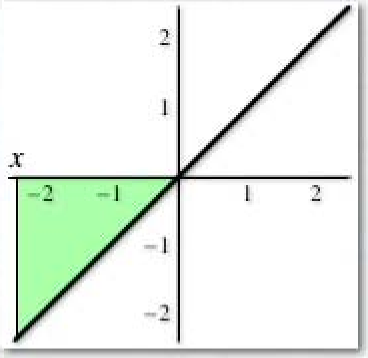
$$\Phi(x) = \int_a^x f(s)\,ds.$$

Example $f(x) = 1/x,\ a = 1$



Example Let $f(x) = x$ and $a = 0$.

Then $\Phi(x) = \displaystyle\int_0^x s\,ds = \frac{1}{2}\,x^2.$



Observations

(1) Φ is continuous.

(2) $\Phi(x)$ is increasing when $f(x) > 0$ and decreasing when $f(x) < 0$.

(3) $\Phi(a) = 0$ and $\Phi(b) = \displaystyle\int_a^b f(x)\,dx.$

UNIVERSITY of HOUSTON

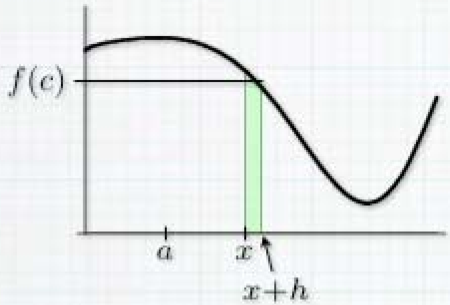
The Derivative of Φ

$$\Phi'(x) = \lim_{h\to 0} \frac{\Phi(x+h)-\Phi(x)}{h}$$

$$\frac{\Phi(x+h)-\Phi(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(s)\,ds - \int_a^x f(s)\,ds\right)$$

$$= \frac{1}{h}\int_x^{x+h} f(s)\,ds$$

$$= \frac{1}{h} f(c)\,h = f(c) \quad \text{where } x < c < x+h$$



$$\Phi'(x) = \lim_{h\to 0} f(c) = f(x)$$

Version 1 of the **Fundamental Theorem of Calculus**

$$\boxed{\frac{d}{dx}\int_a^x f(s)\,ds = f(x)}$$

for all x in any interval containing a on which f is continuous.

$$\boxed{\Phi(x) = \int_a^x f(s)\,ds \text{ is an } \textbf{antiderivative} \text{ of } f.}$$

Examples

$$\frac{d}{dx}\int_1^x \frac{1}{s}\,ds = \frac{1}{x}$$

$$\frac{d}{dx}\int_0^x \sqrt{1+s^4}\,ds = \sqrt{1+x^4}$$

$$\frac{d}{dt}\int_0^t \sin(x^2)\,dx = \sin(t^2)$$

$$\frac{d}{dr}\int_0^r \frac{t^2}{\sqrt{1+t^2}}\,dt = \frac{r^2}{\sqrt{1+r^2}}$$

Variations on $\displaystyle\frac{d}{dx}\int_a^x f(s)\,ds = f(x)$

$$\frac{d}{dx}\int_x^a f(s)\,ds = -\frac{d}{dx}\int_a^x f(s)\,ds = \underline{-f(x)}$$

$$\begin{aligned}
\frac{d}{dx}\int_{u(x)}^{v(x)} f(s)\,ds &= \frac{d}{dx}\left(\underbrace{\int_c^{v(x)} f(s)\,ds}_{\text{composition}} - \underbrace{\int_c^{u(x)} f(s)\,ds}_{\text{composition}}\right) \\
&= \frac{d}{dx}\Phi\big(v(x)\big) - \frac{d}{dx}\Phi\big(u(x)\big) \\
&= \overbrace{\Phi'\big(v(x)\big)\,v'(x)}^{\text{chain rule}} - \overbrace{\Phi'\big(u(x)\big)\,v'(x)}^{\text{chain rule}} \\
&= \underline{f\big(v(x)\big)\,v'(x) - f\big(u(x)\big)\,u'(x)}
\end{aligned}$$

Examples

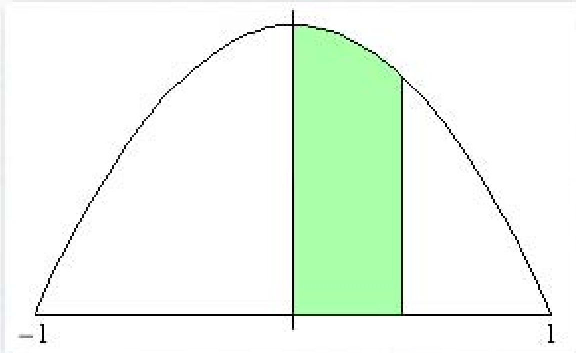
$$q(x) = \int_0^{\sin x} (1 - s^2)\, ds$$

$$q'(x) = \frac{d}{dx} \int_0^{\sin x} (1 - s^2)\, ds$$

chain rule

$$= (1 - \sin^2 x) \cos x$$

$$= \cos^3 x$$

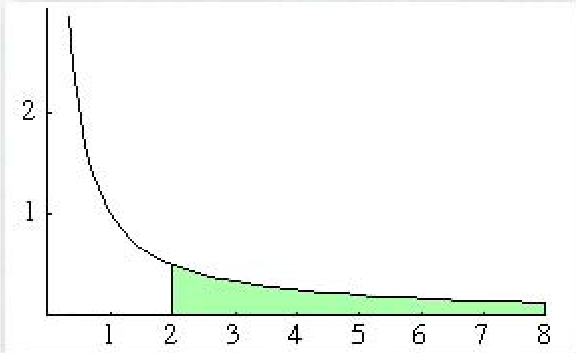


$$g(x) = \int_x^{x^3} \frac{1}{s}\, ds$$

$$g'(x) = \frac{d}{dx} \int_x^{x^3} \frac{1}{s}\, ds$$

$$= \frac{1}{x^3}\, 3x^2 - \frac{1}{x}$$

chain rule



UNIVERSITY of HOUSTON

Computing the Integral

Let f be continuous on $[a, b]$, and let $\Phi(x) = \int_a^x f(s)\,ds$ for $a \leq x \leq b$.

Now let F be any function such that

$$F'(x) = f(x) \text{ for } a \leq x \leq b.$$

F is an antiderivative of f.

Then, since F and Φ have the same derivative on $[a, b]$, they differ by a constant, i.e.,

$$\Phi(x) - F(x) = C.$$

Since $\Phi(a) = 0$, we can find the constant by evaluating at $x = a$:

$$0 - F(a) = C.$$

So

$$\Phi(x) - F(x) = -F(a).$$

i.e.,

$$\Phi(x) = F(x) - F(a).$$

Therefore, we can evaluate at $x = b$ and find that

$$\Phi(b) = \int_a^b f(s)\,ds = F(b) - F(a).$$

UNIVERSITY of HOUSTON

Version 2 of the
Fundamental Theorem of Calculus

$$\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$$

So the heart of the problem becomes that of finding $F(x)$, given $F'(x)$. This is called **antidifferentiation**.

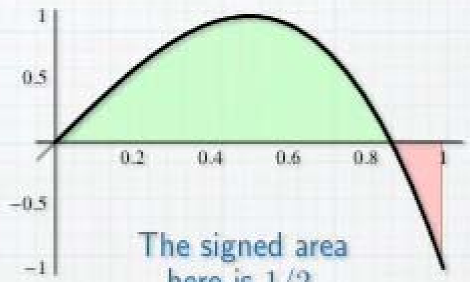
Notation $\quad F(x)\Big|_a^b = F(b) - F(a)$ $\quad$ *F evaluated between a and b*

So the above formula can be written $\displaystyle\int_a^b F'(x)\,dx = F(x)\Big|_a^b$.

Example Compute $\displaystyle\int_0^1 (3x - 4x^3)dx$, and interpret it as a signed area.

Since $3x - 4x^3$ is the derivative of $\frac{3}{2}x^2 - x^4$,

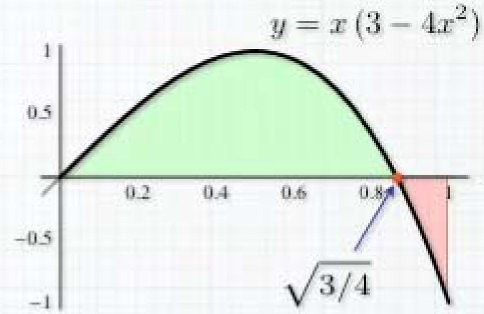
$$\int_0^1 (3x - 4x^3)dx = \frac{3}{2}x^2 - x^4\Big|_0^1$$
$$= \left(\frac{3}{2} - 1\right) - 0 = \frac{1}{2}$$



The signed area here is $1/2$.

Example Find the area bounded by the graph of $y = 3x - 4x^3$ between $x = 0$ and $x = 1$.

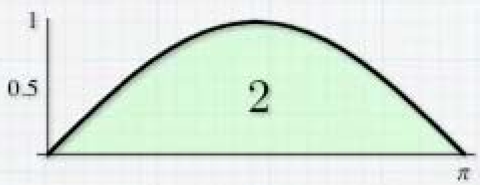
$3x - 4x^3$ is the derivative of $\frac{3}{2}x^2 - x^4$.



$$\int_0^{\sqrt{3/4}} (3x - 4x^3)dx = \frac{3}{2}x^2 - x^4 \Big|_0^{\sqrt{3/4}}$$
$$= \left(\frac{3}{2}\,\frac{3}{4} - \frac{3^2}{4^2}\right) - 0 = \frac{9}{16}$$
$$\int_{\sqrt{3/4}}^{1} (3x - 4x^3)dx = \frac{3}{2}x^2 - x^4 \Big|_{\sqrt{3/4}}^{1}$$
$$= \left(\frac{3}{2} - 1\right) - \frac{9}{16} = -\frac{1}{16}$$

Area $= \frac{9}{16} + \frac{1}{16} = \frac{5}{8}$

Example Find the area under one arch of the graph of $y = \sin x$.



Since $\sin x$ is the derivative of $-\cos x$,

$$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = 1 - (-1) = 2\,.$$

The Fundamental Theorem of Calculus

(1) If f is continuous on an interval containing a, then

$$\frac{d}{dx}\int_a^x f(s)\,ds = f(x) \quad \text{for all } x \text{ in that interval.}$$

(2) If f is continuous and $F'(x) = f(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Coming next: Antidifferentiation

Antidifferentiation
and
Indefinite Integrals

UNIVERSITY of HOUSTON

The Fundamental Theorem of Calculus

Given a continuous function f, the function Φ defined by

$$\Phi(x) = \int_a^x f(s)\,ds$$

is an **antiderivative** of f; that is, $\Phi' = f$.

If F is *any* antiderivative of f; that is, if $F' = f$, then

$$\Phi(x) = F(x) - F(a),$$

and therefore

$$\Phi(b) = \int_a^b f(s)\,ds = F(b) - F(a).$$

Method for computing $\int_a^b f(s)\,ds$

(1) Find an antiderivative F of f.

(2) Compute $F(b) - F(a)$.

UNIVERSITY of HOUSTON

Important Facts About Antiderivatives

(1) If F is an antiderivative of f, then so is $F+C$ for any constant C, simply because the derivative of a constant function is 0.

Example Let $f(x) = 3x^2$. One antiderivative is $F(x) = x^3$, and so is $x^3 + 1$, $x^3 - 5$, and $x^3 + C$ for any number C.

Now recall that one consequence of the mean-value theorem is that if $f'(x) = 0$ for all x in some interval, then $f(x)$ must be constant on that interval.

From that it follows that if two functions have the same derivative on an interval, then those functions must *differ by a constant* on that interval.

(2) In other words, **any two antiderivatives** of f on an interval **differ by a constant** on that interval. So, if F is an antiderivative of f, then so is $F+C$ for any constant C, *and every antiderivative has that form.*

Example Let $f(x) = 3x^2$. One antiderivative is $F(x) = x^3$. Therefore, *every* antiderivative has the form $x^3 + C$ for some number C.

The Indefinite Integral

Given one antiderivative F of f, the expression $F(x) + C$, where C represents an arbitrary constant, provides a description of *all* antiderivatives of f.

This description of all antiderivatives of f is the **indefinite integral** of f, which is commonly denoted by

$$\int f(x)\,dx.$$

So, given any one antiderivative F of f, we write

$$\boxed{\int f(x)\,dx = F(x) + C.}$$

Examples

$$\int 3\,dx = 3x + C \qquad \int 2x\,dx = x^2 + C$$

$$\int \cos x\,dx = \sin x + C \qquad \int \frac{1}{2\sqrt{x}}\,dx$$

UNIVERSITY of HOUSTON

Differentiation

$$\frac{d}{dx}x^3 = 3x^2$$

$$\frac{d}{dx}2\sqrt{x} = \frac{1}{\sqrt{x}}$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\begin{aligned}\frac{d}{dx}(x\sin x + \cos x)\\ &= \sin x + x\cos x\\ &\quad - \sin x\\ &= x\cos x\end{aligned}$$

Indefinite Integration

$$\int 3x^2\,dx = x^3 + C$$

$$\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\begin{aligned}\int x\cos x\,dx\\ &= x\sin x + \cos x + C\end{aligned}$$

Definite Integration

$$\int_0^2 3x^2\,dx = x^3\Big|_0^2 = 8$$

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{x}}\,dx &= 2\sqrt{x}\Big|_1^2\\ &= 2(\sqrt{2}-1)\end{aligned}$$

$$\begin{aligned}\int_0^\pi \sin x\,dx &= -\cos x\Big|_0^\pi\\ &= 1-(-1) = 2\end{aligned}$$

$$\begin{aligned}\int_0^{\pi/2} x\cos x\,dx\\ &= (x\sin x + \cos x)\Big|_0^{\pi/2}\\ &= \frac{\pi}{2} - 1\end{aligned}$$

UNIVERSITY of HOUSTON

	Differentiation	Antidifferentiation
Power Rule	$\frac{d}{dx}x^p = p\,x^{p-1}$	$\boxed{\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C \quad \text{if } p \neq -1}$
	$\frac{d}{dx}x^3 = 3x^2$	$\int x^2\,dx = \frac{1}{3}x^3 + C$
	$\frac{d}{dx}x^{2/3} = \frac{2}{3}x^{-1/3}$	$\int x^{-1/3}dx = \frac{3}{2}x^{2/3} + C$

Linearity Properties

$$\frac{d}{dx}\big(F(x) \pm G(x)\big) = F'(x) \pm G'(x) = f(x) \pm g(x)$$

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C$$

$$\frac{d}{dx}\big(c\,F(x)\big) = c\,F'(x) = c\,f(x)$$

$$\int c\,f(x)\,dx = c\int f(x)\,dx = c\,F(x) + C$$

Examples

(1)
$$\begin{aligned}\int (5x^2 - 4x + 1)\,dx &= \int 5x^2\,dx + \int (-4x)\,dx + \int 1\,dx \\ &= 5\int x^2\,dx - 4\int x\,dx + \int dx \\ &= 5\cdot\tfrac{1}{3}\,x^3 - 4\cdot\tfrac{1}{2}\,x^2 + x + C \\ &= \tfrac{5}{3}\,x^3 - 2x^2 + x + C\end{aligned}$$

(2)
$$\int (8x^3 - \sqrt{3x})\,dx = \int (8x^3 - \sqrt{3}\,x^{1/2})\,dx = 2x^4 - \tfrac{2\sqrt{3}}{3}\,x^{3/2} + C$$

(3)
$$\int \frac{t^3+1}{t^2}\,dt = \int (t + t^{-2})\,dt = \tfrac{1}{2}\,t^2 - t^{-1} + C = \frac{t^3-2}{2t} + C$$

(4)
$$\begin{aligned}\int (y-3)\sqrt{y}\,dy = \int (y^{3/2} - 3y^{1/2})\,dy &= \tfrac{2}{5}\,y^{5/2} - 2y^{3/2} + C \\ &= \tfrac{2}{5}\,y^{3/2}(y-5) + C\end{aligned}$$

Differentiation | **Antidifferentiation**

$$F'(x) = f(x) \qquad \int f(x)\,dx = F(x) + C$$

Trigonometric Functions

$$\frac{d}{dx}\sin x = \cos x \qquad \int \cos x\,dx = \sin x + C$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \int \sin x\,dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\frac{d}{dx}\sec x = \sec x\,\tan x \qquad \int \sec x\,\tan x\,dx = \sec x + C$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\frac{d}{dx}\csc x = -\csc x\,\cot x \qquad \int \csc x\cot x\,dx = -\csc x + C$$

Guess and Check

Example $\displaystyle\int \cos 2x\, dx \overset{\checkmark}{=} \tfrac{1}{2}\sin 2x + C$

$\frac{d}{dx}\sin 2x = 2\cos 2x$

$\frac{d}{dx}\frac{1}{2}\sin 2x = \cos 2x$

$$\int \cos 2x\, dx = \frac{1}{2}\int 2\cos 2x\, dx = \frac{1}{2}\sin 2x + C$$

Example $\displaystyle\int \sqrt{3x+1}\, dx \overset{\checkmark}{=} \tfrac{1}{3}\tfrac{2}{3}(3x+1)^{3/2} + C$

$\displaystyle = \tfrac{2}{9}(3x+1)^{3/2} + C$

$\frac{d}{dx}\frac{2}{3}(3x+1)^{3/2}$

$= (3x+1)^{1/2}\cdot 3$

$+1)^{3/2} = \sqrt{3x+1}$

$$\int \sqrt{3x+1}\, dx = \frac{2}{9}\int 3\,\frac{3}{2}\sqrt{3x+1}\, dx = \frac{2}{9}(3x+1)^{3/2} + C$$

Example $\displaystyle\int \sqrt{x^2+1}\, dx$ $\quad \cancel{\tfrac{1}{2x}\tfrac{2}{3}(x^2+1)^{3/2}}$

$\frac{d}{dx}\frac{2}{3}(x^2+1)^{3/2} = 2x\sqrt{x^2+1}$

$$\int \sqrt{x^2+1}\, dx \not= \frac{1}{3x}\int \frac{3}{2}\,2x\sqrt{x^2+1}\, dx \overset{\checkmark}{=} \frac{1}{3x}(x^2+1)^{3/2} + C$$

$)^{3/2} \neq \sqrt{x^2+1}$!

Example Find the function f, given that

$$f'(x) = \sin 2x + 2\cos 3x \text{ and } f(0) = 1.$$

indefinite integral

$$\int (\sin 2x + 2\cos 3x)\, dx = -\frac{1}{2}\cos 2x + \frac{2}{3}\sin 3x + C$$

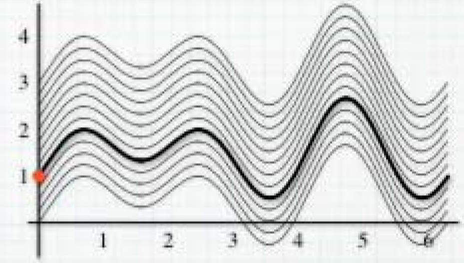
So $f(x) = -\frac{1}{2}\cos 2x + \frac{2}{3}\sin 3x + C$ for some C.

Find C

$$1 = -\frac{1}{2}\cos 0 + \frac{2}{3}\sin 0 + C$$

$$1 = -\frac{1}{2} + 0 + C$$

So $C = \frac{3}{2}$.



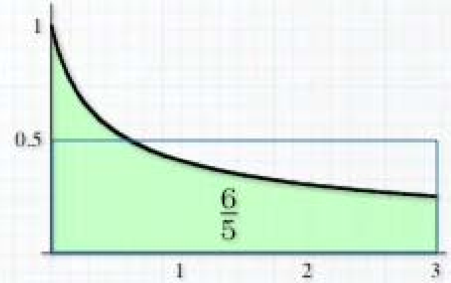
Therefore $f(x) = -\frac{1}{2}\cos 2x + \frac{2}{3}\sin 3x + \frac{3}{2} = \frac{1}{6}(4\sin 3x - 3\cos 2x + 9)$

Example

Find the area under the graph of

$$f(x) = \frac{1}{\sqrt{5x+1}}$$

between $x = 0$ and $x = 3$.



indefinite integral

$$\int (5x+1)^{-1/2}\,dx = \frac{1}{5}\,2\int 5\,\frac{1}{2}(5x+1)^{-1/2}\,dx$$

$$= \frac{2}{5}\,(5x+1)^{1/2} + C = \frac{2}{5}\sqrt{5x+1} + C$$

definite integral

$$\int_0^3 (5x+1)^{-1/2}\,dx = \frac{2}{5}\sqrt{5x+1}\,\Big|_0^3 = \frac{2}{5}\left(\sqrt{5x+1}\,\Big|_0^3\right)$$

$$= \frac{2}{5}\left(\sqrt{16} - \sqrt{1}\right) = \frac{6}{5}$$

UNIVERSITY of HOUSTON

The Natural Logarithm

UNIVERSITY of HOUSTON

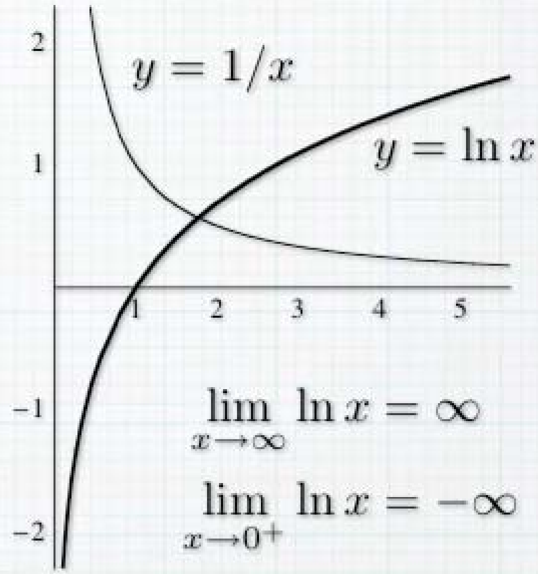
Definition of the Natural Log Function

$$\ln x = \int_1^x \frac{1}{t}\,dt \quad \text{for all } x > 0$$

This is equivalent to:

$$\ln 1 = 0, \text{ and}$$

$$\frac{d}{dx}\ln x = \frac{1}{x} \quad \text{for all } x > 0$$



Examples

$$\frac{d}{dx}(x\ln x) = (1)\ln x + x\,\frac{1}{x} = \ln x + 1$$

$$\frac{d}{dx}\ln(\cos x) = \frac{1}{\cos x}\,\frac{d}{dx}\cos x = \frac{-\sin x}{\cos x} = -\tan x$$

Algebraic Properties

$$\boxed{\ln(xy) = \ln x + \ln y}$$

$$\ln(1/x) = -\ln x$$

$$\ln(x/y) = \ln x - \ln y$$

$$\ln(x^p) = p \ln x$$

proof

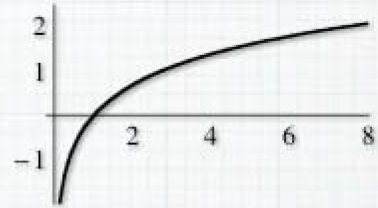
	derivative w.r.t. x	value at $x = 1$
left side:	$\frac{d}{dx}\ln(xy) = \frac{1}{xy}\,y = \frac{1}{x}$	$\ln(1y) = \ln y$
right side:	$\frac{d}{dx}(\ln x + \ln y) = \frac{1}{x} + 0$	$\ln 1 + \ln y = \ln y$

Limits

$$\lim_{x\to\infty} \ln x = \infty \text{ because } \ln(x^p) = p\ln x \implies \ln(2^x) = x\ln 2$$

Now, $\displaystyle\lim_{x\to 0^+} \ln x = \lim_{x\to\infty} \ln(1/x)$

$$= \lim_{x\to\infty} (-\ln x) = -\infty$$



Theorem

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^r} = 0 \text{ for any } r > 0}$$

$\ln x$ grows *slower* than any positive power as $x \to \infty$.

Proof Choose p such that $0 < p < 1$ and $p > 1 - r$.

Then, for $x > 1$,

$$\ln x = \int_1^x \frac{1}{t}\,dt < \int_1^x \frac{1}{t^p}\,dt = \frac{1}{1-p}\,t^{1-p}\Big|_1^x = \frac{1}{1-p}\,(x^{1-p} - 1)$$

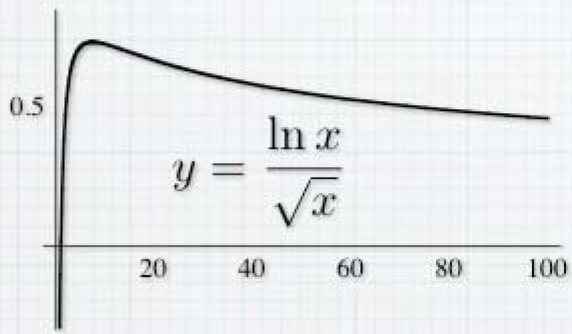
and so $\displaystyle 0 < \frac{\ln x}{x^r} < \frac{1}{1-p}\,(x^{1-p-r} - x^{-r}) \longrightarrow 0$ as $x \to \infty$.

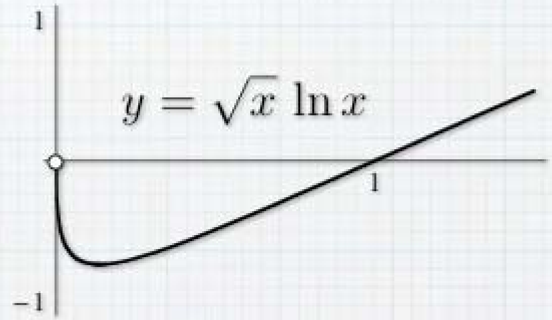
UNIVERSITY of HOUSTON

$$\lim_{x \to \infty} \frac{\ln x}{x^r} = 0 \quad \text{for any } r > 0$$

Corollary $$\lim_{x \to 0^+} x^r \ln x = 0 \quad \text{for any } r > 0$$

Proof $$\lim_{x \to 0^+} x^r \ln x = \lim_{x \to \infty} (1/x)^r \ln(1/x) = -\lim_{x \to \infty} \frac{\ln x}{x^r} = 0$$

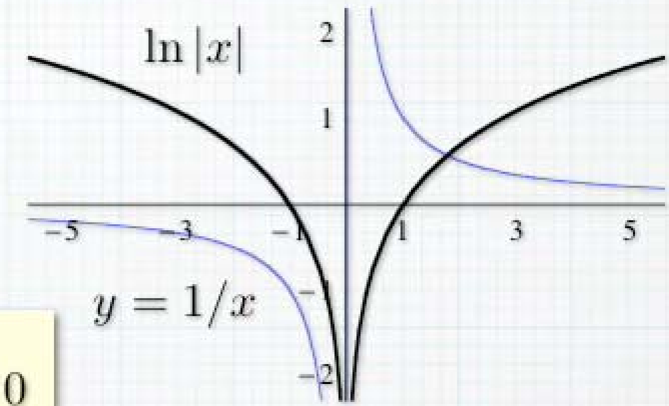




The function $\ln|x|$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \text{for all } x \neq 0$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{for all } x \neq 0$$



Power rule: $$\int x^n\,dx = \begin{cases} \frac{1}{n+1}x^{n+1} + C, & \text{if } n \neq -1 \\ \ln|x| + C, & \text{if } n = -1 \end{cases}$$

Example $$\int \frac{x+1}{x^2}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2}\right)dx = \ln|x| - \frac{1}{x} + C$$

Example $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$$= \frac{1}{2}\ln(x^2+1) + C$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\,dx \end{aligned}}$$

Example $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{1}{u}(-du) = -\ln|u| + C$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x\,dx \end{aligned}}$$

Example $\displaystyle\int \frac{1}{\sqrt{x}\,(1+\sqrt{x}\,)}\,dx = \int \frac{1}{u}(2\,du) = 2\ln|u| + C$

$$= 2\ln(1+\sqrt{x}\,) + C$$

$$\boxed{\begin{aligned} u &= 1+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \end{aligned}}$$

$$\boxed{\frac{d}{dx}\ln|u(x)| = \frac{u'(x)}{u(x)}}$$

$$\boxed{\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C}$$

The natural log arises (only) when integrating a quotient *whose numerator is the derivative of its denominator* (or a constant multiple of it).

$$\int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\sin x} + C$$

UNIVERSITY of HOUSTON

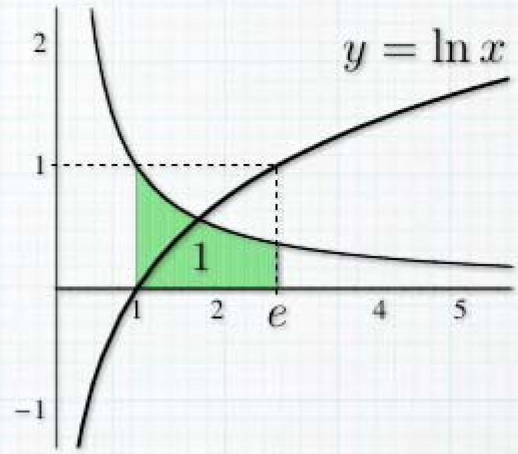
The Exponential Function

$$e^x$$

Definition of the Number e

Since $\ln 2 < 1 < \ln 4$, there is a number x between 2 and 4 where $\ln x = 1$. Name that number e.

$$\ln e = 1$$



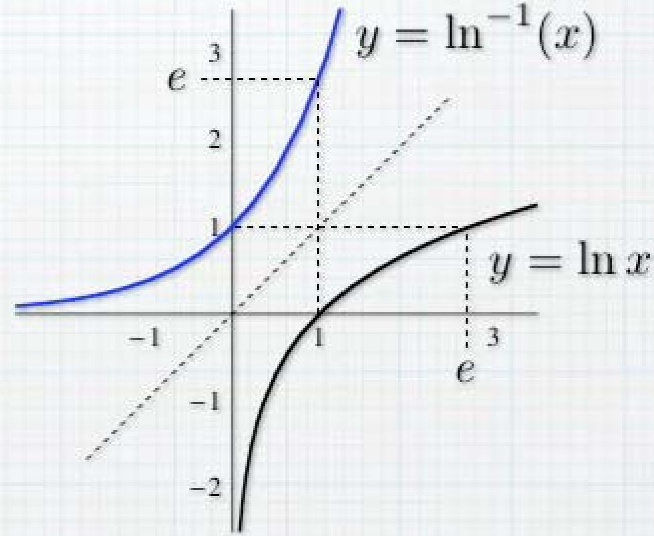
The Inverse of $\ln x$

$$\ln^{-1}(x) = y \iff \ln y = x$$

$$\ln^{-1}(1) = e \iff \ln e = 1$$

$$\ln^{-1}(0) = 1 \iff \ln 1 = 0$$

$$\ln^{-1}(x) = e^x \iff \ln e^x = x$$



Result: $\boxed{\ln^{-1}(x) = e^x \text{ for all } x}$

i.e., $e^{\ln x} = x$ for all $x > 0$ and $\ln(e^x) = x$ for all x

i.e., $\boxed{\ln x = \log_e x}$

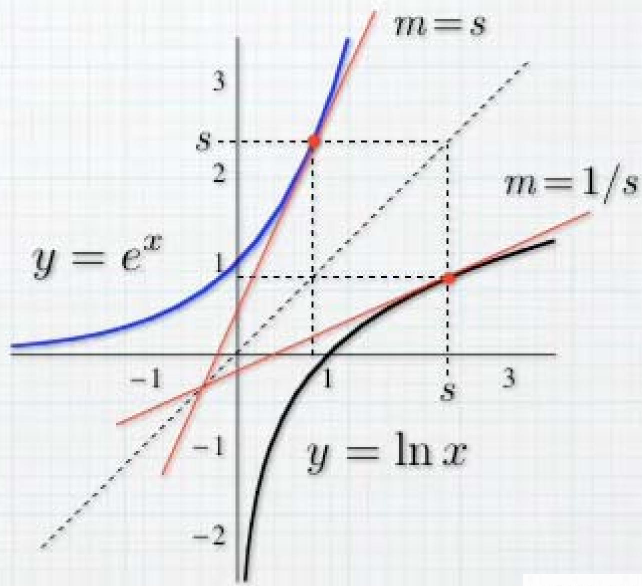
The Derivative of e^x

$$\ln(e^x) = x$$

$$\frac{d}{dx}\ln(e^x) = \frac{d}{dx}x$$

chain rule $$\frac{1}{e^x}\frac{d}{dx}e^x = 1$$

$$\boxed{\frac{d}{dx}e^x = e^x}$$



chain rule $$\boxed{\frac{d}{dx}e^u = e^u\,\frac{du}{dx}}$$ The exponent is the "inside" function.

Example

$$\frac{d}{dx}e^{-x^2} = e^{-x^2}(-2x) = -2x\,e^{-x^2}$$

Example

$$\frac{d}{dx}\left(x\,e^{-2x}\right) = (1)\,e^{-2x} + x\,e^{-2x}(-2) = (1-2x)\,e^{-2x}$$

Example

$$\frac{d}{dx}\left(\frac{1}{3-2e^{-x/2}}\right) = -\left(3-2e^{-x/2}\right)^{-2}\left(-2e^{-x/2}(-1/2)\right)$$
$$= -\frac{e^{-x/2}}{\left(3-2e^{-x/2}\right)^2}$$

UNIVERSITY of HOUSTON

Integrals

$$\boxed{\int e^u \, du = e^u + C}$$

Example

$$\int e^{-2x}\, dx = -1/2 \int e^u \, du \ = -\frac{1}{2}\, e^u + C \ = -\frac{1}{2}\, e^{-2x} + C$$

Example

$$\int \frac{1}{\sqrt{x}}\, e^{-\sqrt{x}}\, dx = \ -2 \int e^u \, du \ = -2\, e^u + C \ = -2\, e^{-\sqrt{x}} + C$$

Example

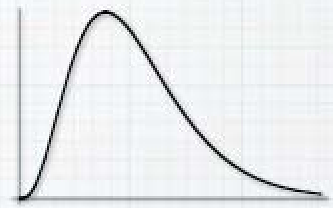
$$\int \frac{e^{-x}}{2+3e^{-x}}\, dx = -\frac{1}{3} \int \frac{1}{u}\, du \ = -\frac{1}{3} \ln(2+3e^{-x}) + C$$

UNIVERSITY of HOUSTON

Limits

$$\lim_{x\to\infty} e^{kx} = \infty \quad \text{and} \quad \lim_{x\to -\infty} e^{kx} = 0 \quad \text{for any } k > 0$$

$$\lim_{x\to\infty} e^{-kx} = 0 \quad \text{and} \quad \lim_{x\to -\infty} e^{-kx} = \infty \quad \text{for any } k > 0$$



$$\boxed{\lim_{x\to\infty} x^p e^{-kx} = 0 \quad \text{for any } k, p > 0}$$

proof

$$x^p e^{-kx} = (\ln t)^p e^{-k\ln t} = (\ln t)^p e^{\ln(t^{-k})} = \frac{(\ln t)^p}{t^k}$$

$$\boxed{x = \ln t}$$

$$= \left(\frac{\ln t}{t^{k/p}}\right)^p \longrightarrow 0$$

as $t \to \infty$

reciprocal limit

$$\boxed{\lim_{x\to\infty} \frac{e^{kx}}{x^p} = \infty \quad \text{for any } k, p > 0}$$

Example Sketch the graph of $f(x) = x^2 e^{-x}$.

$$f'(x) = 2x\,e^{-x} + x^2 e^{-x}(-1) = x\,(2-x)\,e^{-x} = (2x - x^2)\,e^{-x}$$

$$f''(x) = (2-2x)\,e^{-x} + (2x - x^2)\,(-e^{-x}) = (2 - 4x + x^2)\,e^{-x}$$

critical numbers: $x = 0,\ 2$

local minimum: $f(0) = 0$

local maximum: $f(2) = 4/e^2$

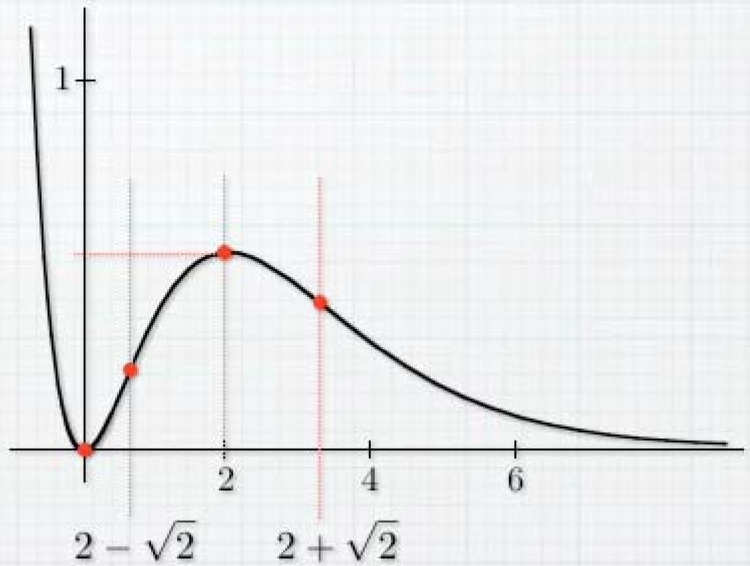
Inflections occur where

$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

horizontal asymptote:

$$\lim_{x\to\infty} x^2 e^{-x} = 0$$



Other Bases

conversion to base e $\quad b^x = e^{\ln b^x} = e^{x \ln b}$

$$\boxed{\frac{d}{dx}\, b^x = b^x \ln b \quad \text{and} \quad \int b^x\, dx = \frac{1}{\ln b}\, b^x + C}$$

Examples

$$\frac{d}{dx}\, 2^x = 2^x \ln 2$$

$$\frac{d}{dx}\, 10^{-x/3} = 10^{-x/3}(\ln 10)\,(-1/3) = -\frac{\ln 10}{3}\, 10^{-x/3}$$

$$\int 2^{-x}\, dx = -\int 2^u\, du = -\frac{1}{\ln 2}\, 2^{-x} + C$$

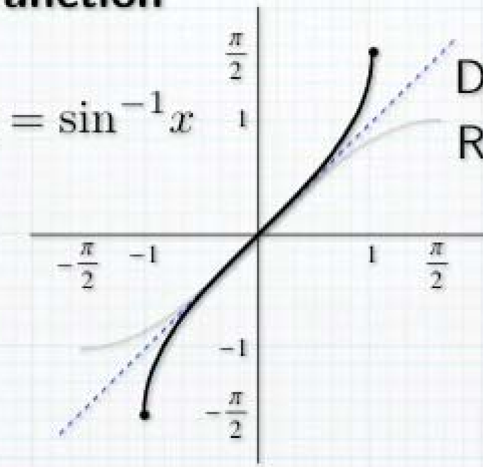
Logarithms

conversion to base e $\quad \log_b x = \frac{1}{\ln b} \ln x \implies \boxed{\frac{d}{dx} \log_b x = \frac{1}{x \ln b}}$

The Inverse Trig Functions

The Inverse Sine Function

$y = \sin^{-1} x$



Domain: $[-1, 1]$

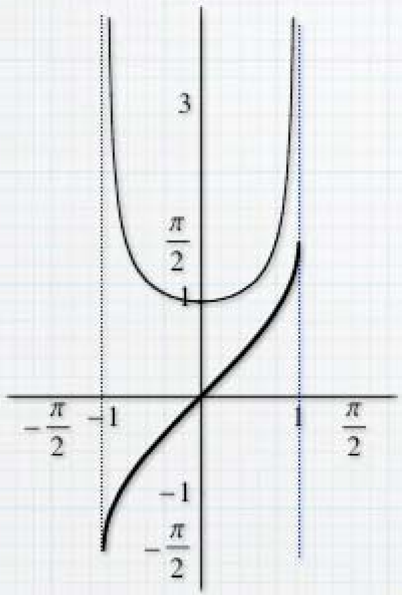
Range: $[-\pi/2, \pi/2]$

$$\sin^{-1} x = y \iff \sin y = x \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\sin^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

odd function $\quad \sin^{-1}(-x) = -\sin^{-1} x$

The Derivative of $\sin^{-1} x$

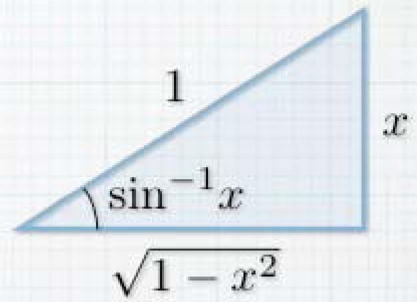


$$\sin(\sin^{-1} x) = x$$

$$\frac{d}{dx}\sin(\sin^{-1} x) = 1$$

$$\cos(\sin^{-1} x)\,\frac{d}{dx}\sin^{-1} x = 1$$

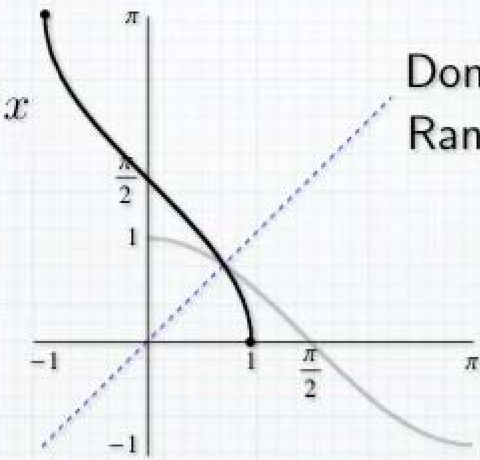
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\cos(\sin^{-1} x)}$$



$$\boxed{\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}}$$

$$\boxed{\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1} u + C}$$

The Inverse Cosine Function



Domain: $[-1, 1]$
Range: $[0, \pi]$

$$\cos^{-1} x = y \iff \cos y = x \text{ and } 0 \le y \le \pi$$

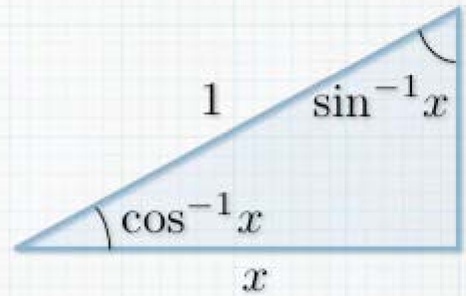
x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
$\cos^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π

The Derivative of $\cos^{-1}x$

$$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2}$$

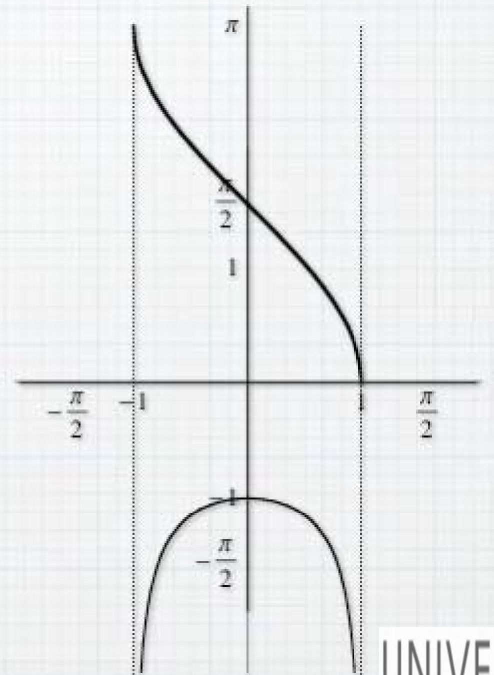
$$\cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x$$

$$\frac{d}{dx}\cos^{-1}x = -\frac{d}{dx}\sin^{-1}x$$



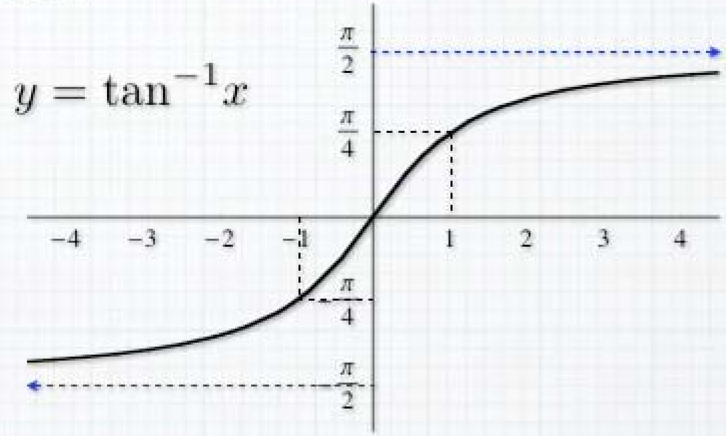
$$\boxed{\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}}$$

$$\boxed{\int \frac{1}{\sqrt{1-u^2}}\,du = -\cos^{-1}u + C}$$



The Inverse Tangent Function

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$



$$\tan^{-1} x = y \iff \tan y = x \text{ and } -\frac{\pi}{2} < y < \frac{\pi}{2}$$

x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\tan^{-1} x$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

odd function $\quad \tan^{-1}(-x) = -\tan^{-1} x$

UNIVERSITY of HOUSTON

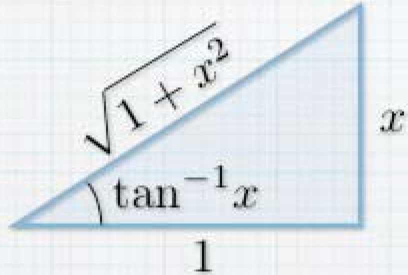
The Derivative of $\tan^{-1} x$

$$\tan(\tan^{-1} x) = x$$

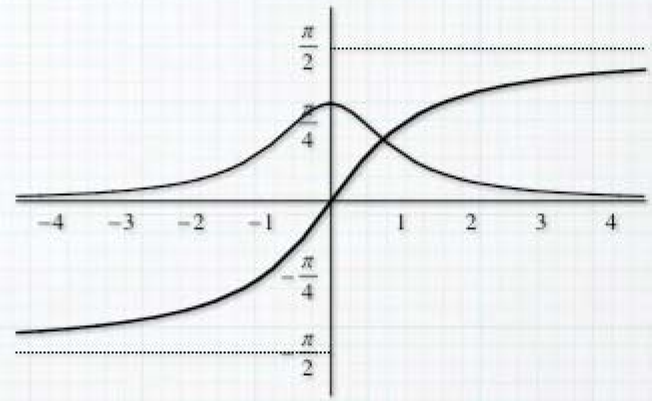
$$\frac{d}{dx}\tan(\tan^{-1} x) = 1$$

$$\sec^2(\tan^{-1} x)\,\frac{d}{dx}\tan^{-1} x = 1$$

$$\frac{d}{dx}\tan^{-1} x = \cos^2(\tan^{-1} x)$$



$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$



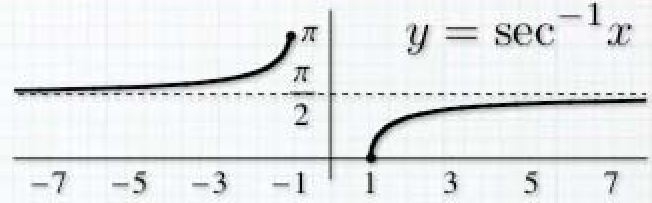
$$\boxed{\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}}$$

$$\boxed{\int \frac{1}{1+u^2}\,du = \tan^{-1} u + C}$$

The Inverse Secant Function

Domain: $(-\infty, -1] \cup [1, \infty)$

Range: $[0, \pi/2) \cup (\pi/2, \pi]$



$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\int \frac{1}{|u|\sqrt{u^2-1}}\,du = \sec^{-1}u + C$$

Inverse Cotangent and Inverse Cosecant

$$\cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}$$

$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\csc^{-1}x + \sec^{-1}x = \frac{\pi}{2}$$

$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Derivatives

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$$

Examples

$$\frac{d}{dx}\left((1-x^2)\sin^{-1}x\right) = -2x\sin^{-1}x + \sqrt{1-x^2}$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{2}\right) = \frac{1/2}{1+(x/2)^2} = \frac{2}{4+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}\sqrt{x}\right) = \frac{1}{\sqrt{x}\sqrt{(\sqrt{x})^2-1}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{x-1}}$$

Indefinite Integrals

$$\int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}u + C$$

$$\int \frac{1}{1+u^2}\,du = \tan^{-1}u + C$$

$$\int \frac{1}{|u|\sqrt{u^2-1}}\,du = \sec^{-1}u + C$$

Example

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{4\,(1-u^2)}}\;2du = \int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \sin^{-1}u + C$$

$$= \sin^{-1}(x/2) + C$$

$x = 2u \quad dx = 2du$

$4 - x^2 = 4\,(1-u^2)$

Example

$$\int \frac{1}{4+9x^2}\,dx = \int \frac{2/3}{4\,(1+u^2)}\,du = \frac{1}{6}\tan^{-1}u + C$$

$$= \frac{1}{6}\tan^{-1}(3x/2) + C$$

$$\boxed{\begin{aligned} x = \tfrac{2}{3}\,u \quad dx &= \tfrac{2}{3}\,du \\ 4+9x^2 &= 4+9(4/9)u^2 \\ &= 4\,(1+u^2) \end{aligned}}$$

Example

complete the square

$$\int \frac{x}{x^2+2x+2}\,dx = \int \frac{x}{(x+1)^2+1}\,dx = \int \frac{u-1}{u^2+1}\,du$$

$$\boxed{\begin{aligned} u &= x+1 \\ x &= u-1 \\ dx &= du \end{aligned}}$$

$$= \int \left(\frac{1}{2}\cdot\frac{2u}{u^2+1} - \frac{1}{u^2+1}\right) du$$

$$= \frac{1}{2}\ln(u^2+1) - \tan^{-1}u + C$$

$$= \frac{1}{2}\ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

Example

$$\int \frac{1}{x\sqrt{x-1}}\,dx = \int \frac{1}{u^2\sqrt{u^2-1}}\,2u\,du = 2\int \frac{1}{u\sqrt{u^2-1}}\,du$$

$$\boxed{\begin{aligned} x &= u^2 \\ dx &= 2u\,du \\ u &= \sqrt{x} \end{aligned}}$$

$$= 2\sec^{-1}u + C$$

$$= 2\sec^{-1}\sqrt{x} + C$$

Example

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = \int \frac{1}{1+u^2}\,(-du) = -\int \frac{1}{1+u^2}\,du$$

$$\boxed{\begin{aligned} u &= e^{-x} \\ du &= -e^{-x}\,dx \end{aligned}}$$

$$= -\tan^{-1}u + C$$

$$= -\tan^{-1}(e^{-x}) + C$$

UNIVERSITY of HOUSTON