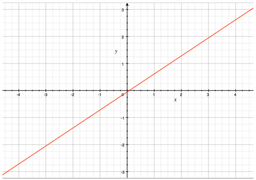
Geometric objects that exclude the origin

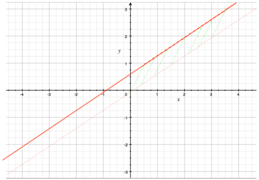
How to represent a line that does *not* contain the origin?



Start with a line that *does* contain the origin.

We know that points of such a line form a vector space $\mathcal{V}$.

Translate the line by adding a vector $\mathbf{c}$ to every vector in $\mathcal{V}$:



$$\{\mathbf{c} + \mathbf{v} \ : \ \mathbf{v} \in \mathcal{V}\}$$

(abbreviated $\mathbf{c} + \mathcal{V}$)

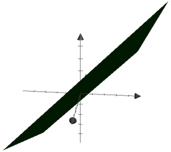
Result is line through $\mathbf{c}$ instead of through origin.

Geometric objects that exclude the origin

How to represent a plane that does *not* contain the origin?

- Start with a plane that *does* contain the origin.

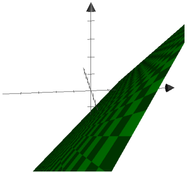
  We know that points of such a plane form a vector space $\mathcal{V}$.



- Translate it by adding a vector $\mathbf{c}$ to every vector in $\mathcal{V}$

$$\{\mathbf{c} + \mathbf{v} \ : \ \mathbf{v} \in \mathcal{V}\}$$

  (abbreviated $\mathbf{c} + \mathcal{V}$)



- Result is plane containing $\mathbf{c}$.

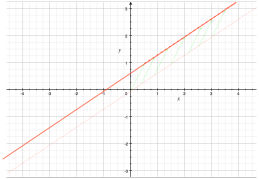
Affine space

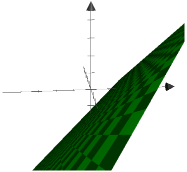
Definition: If $\mathbf{c}$ is a vector and $\mathcal{V}$ is a vector space then

$$\mathbf{c} + \mathcal{V}$$

is called an *affine space*.

Examples: A plane or a line not necessarily containing the origin.

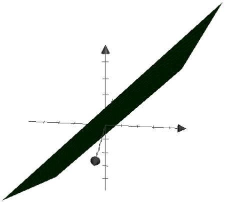




Affine space and affine combination

Example: The plane containing $\mathbf{u}_1 = [3, 0, 0]$, $\mathbf{u}_2 = [-3, 1, -1]$, and $\mathbf{u}_3 = [1, -1, 1]$.

Want to express this plane as $\mathbf{u}_1 + \mathcal{V}$
where $\mathcal{V}$ is the span of two vectors
(a plane containing the origin)

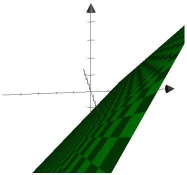


Let $\mathcal{V} = \text{Span } \{\mathbf{a}, \mathbf{b}\}$ where

$$\mathbf{a} = \mathbf{u}_2 - \mathbf{u}_1 \text{ and } \mathbf{b} = \mathbf{u}_3 - \mathbf{u}_1$$

Since $\mathbf{u}_1 + \mathcal{V}$ is a translation of a plane, it is also a plane.

- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{0}$, so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_1$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_2 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_2$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_3 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_3$.



Thus the plane $\mathbf{u}_1 + \text{Span } \{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.
Only one plane contains those three points, so this is that one.

Affine space and affine combination

Example: The plane containing $\mathbf{u}_1 = [3, 0, 0]$, $\mathbf{u}_2 = [-3, 1, -1]$, and $\mathbf{u}_1 = [1, -1, 1]$:

$$\mathbf{u}_1 + \text{Span }\{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\}$$

Cleaner way to write it?

$$\begin{aligned}
\mathbf{u}_1 + \text{Span }\{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\} &= \{\mathbf{u}_1 + \alpha\,(\mathbf{u}_2 - \mathbf{u}_1) + \beta\,(\mathbf{u}_3 - \mathbf{u}_1) : \alpha, \beta \in \mathbb{R}\} \\
&= \{\mathbf{u}_1 + \alpha\,\mathbf{u}_2 - \alpha\,\mathbf{u}_1 + \beta\,\mathbf{u}_3 - \beta\,\mathbf{u}_1 \;:\; \alpha, \beta \in \mathbb{R}\} \\
&= \{(1 - \alpha - \beta)\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3 \;:\; \alpha, \beta \in \mathbb{R}\} \\
&= \{\gamma\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3 \;:\; \gamma + \alpha + \beta = 1\}
\end{aligned}$$

Definition: A linear combination $\gamma\,\mathbf{u}_1 + \alpha\,\mathbf{u}_2 + \beta\,\mathbf{u}_3$ where $\gamma + \alpha + \beta = 1$ is an *affine combination*.

Affine combination

Definition: A linear combination

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n$$

where

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

is an *affine combination*.

Definition: The set of all affine combinations of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called the *affine hull* of those vectors.

$$\text{Affine hull of } \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n = \mathbf{u}_1 + \text{Span } \{\mathbf{u}_2 - \mathbf{u}_1, \ldots, \mathbf{u}_n - \mathbf{u}_1\}$$

This shows that the affine hull of some vectors is an affine space..

Geometric objects not containing the origin: equations

Can express a plane as $\mathbf{u}_1 + \mathcal{V}$ or affine hull of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.
More familiar way to express a plane:
The solution set of an equation $ax + by + cz = d$

In vector terms,

$$\{[x, y, z] \;:\; [a, b, c] \cdot [x, y, z] = d\}$$

In general, a geometric object (point, line, plane, ...) can be expressed as the solution set of a system of linear equations.

$$\{\mathbf{x} \;:\; \mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m\}$$

Conversely, is the solution set an affine space?

Consider solution set of a contradictory system of equations, e.g. $1\,x = 1, 2\,x = 1$:

- Solution set is empty....
- ...but a vector space $\mathcal{V}$ always contains the zero vector,
- ...so an affine space $\mathbf{u}_1 + \mathcal{V}$ always contains at least one vector.

Turns out this the only exception:

Theorem: The solution set of a linear system is either empty or an affine space.

Affine spaces and linear systems

Theorem: The solution set of a linear system is either empty or an affine space.

Each linear system corresponds to a linear system with zero right-hand sides:

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

Definition:
A linear equation $\mathbf{a} \cdot \mathbf{x} = 0$ with zero right-hand side is a *homogeneous* linear equation.
A system of homogeneous linear equations is called a *homogeneous* linear system.

We already know: The solution set of a homogeneous linear system is a vector space.

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,

$\mathbf{u}_2$ is also a solution

if and only if

$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

Affine spaces and linear systems

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & \beta_m \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{x} & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{x} & = & 0 \end{array}$$

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,

$\mathbf{u}_2$ is also a solution

if and only if

$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

Proof: We assume $\mathbf{a}_1 \cdot \mathbf{u}_1 = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{u}_1 = \beta_m$, so

$$\begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{u}_2 & = & \beta_1 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{u}_2 & = & \beta_m \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot \mathbf{u}_2 - \mathbf{a}_1 \cdot \mathbf{u}_1 & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot \mathbf{u}_2 - \mathbf{a}_m \cdot \mathbf{u}_1 & = & 0 \end{array} \quad \text{iff} \quad \begin{array}{rcl} \mathbf{a}_1 \cdot (\mathbf{u}_2 - \mathbf{u}_1) & = & 0 \\ & \vdots & \\ \mathbf{a}_m \cdot (\mathbf{u}_2 - \mathbf{u}_2) & = & 0 \end{array}$$

QED

Lemma: Let $\mathbf{u}_1$ be a solution to a linear system. Then, for any other vector $\mathbf{u}_2$,
$\mathbf{u}_2$ is also a solution
if and only if
$\mathbf{u}_2 - \mathbf{u}_1$ is a solution to the corresponding homogeneous linear system.

We use this lemma to prove the theorem:

Theorem: The solution set of a linear system is either empty or an affine space.

- Let $\mathcal{V}$ = set of solutions to corresponding homogeneous linear system.
- If the linear system has no solution, its solution set is empty.
- If it does has a solution $\mathbf{u}_1$ then

$$\begin{aligned}\{\text{solutions to linear system}\} \quad &= \quad \{\mathbf{u}_2 \ : \ \mathbf{u}_2 - \mathbf{u}_1 \in \mathcal{V}\} \\ \text{(substitute } \mathbf{v} = \mathbf{u}_2 - \mathbf{u}_1\text{)} \quad & \\ &= \quad \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}\end{aligned}$$

QED

Number of solutions to a linear system

We just proved:

If $\mathbf{u}_1$ is a solution to a linear system then

$$\{\text{solutions to linear system}\} = \{\mathbf{u}_1 + \mathbf{v} : \mathbf{v} \in \mathcal{V}\}$$

where $\mathcal{V} = \{\text{solutions to corresponding homogeneous linear system}\}$

Implications:

Long ago we asked: *How can we tell if a linear system has only one solution?*

Now we know: If a linear system has a solution $\mathbf{u}_1$ then that solution is unique if the only solution to the corresponding homogeneous linear system is $\mathbf{0}$.

Long ago we asked: How can we find the number of solutions to a linear system over $GF(2)$?

Now we know: Number of solutions either is zero or is equal to the number of solutions to the corresponding *homogeneous* linear system.