Representations of vector spaces

Two important ways to represent a vector space:

As the solution set of homogeneous linear system

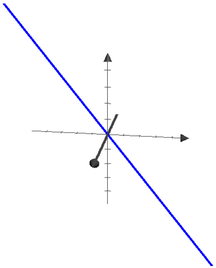
$$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$$

Equivalently, Null $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

As Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

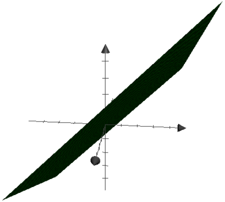
Equivalently, Row $\begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_k \end{bmatrix}$

Conversions between the two representations



$$\{[x, y, z] \ : \\ [4, -1, 1] \cdot [x, y, z] = 0, \\ [0, 1, 1] \cdot [x, y, z] = 0\}$$

$$\text{Span } \{[1, 2, -2]\}$$



$$\text{Span } \{[4, -1, 1], [0, 1, 1]\}$$

$$\{[x, y, z] \ : \ [1, 2, -2] \cdot [x, y, z] = 0\}$$

Conversions for affine spaces?

- From representation as solution set of linear system to representation as affine hull
- From representation as affine hull to representation as solution set of linear system

Conversions for affine spaces?

From representation as solution set of linear system to representation as affine hull

- *input:* linear system $A\mathbf{x} = \mathbf{b}$
- *output:* vectors whose affine hull is the solution set of the linear system.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

1 Let $\mathbf{u}$ be one solution to the linear system.

$\mathbf{u} = [-0.5, 0.75, 0]$

2 Consider the corresponding homogeneous system $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Its solution set, the null space of A, is a vector space $\mathcal{V}$.

3 Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be generators for $\mathcal{V}$.

$\mathbf{b}_1 = [2, -3, 4]$

4 Then the solution set of the original linear system is the affine hull of $\mathbf{u}, \mathbf{b}_1 + \mathbf{u}, \mathbf{b}_2 + \mathbf{u}, \ldots, \mathbf{b}_k + \mathbf{u}$.

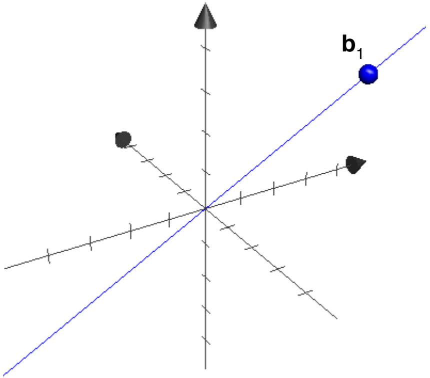
$[-0.5, -.75, 0]$ and
$[-0.5, -.75, 0] + [2, -3, 4]$

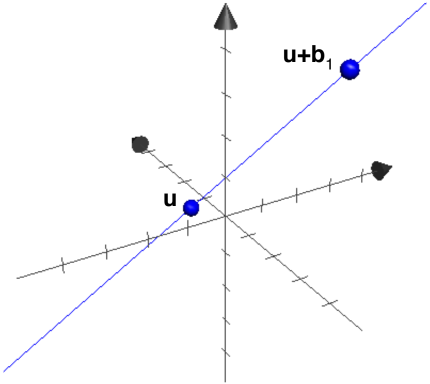
From representation as solution set to representation as affine hull

One solution to equation $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\mathbf{u} = [-0.5, 0.75, 0]$

Null space of $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is Span $\{\mathbf{b}_1\}$:

Solution set of equation is $\mathbf{u} +$ Span $\{\mathbf{b}_1\}$,
i.e. the affine hull of $\mathbf{u}$ and $\mathbf{u} + \mathbf{b}_1$





Representations of vector spaces

Two important ways to represent a vector space:

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Equivalently, Null $\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

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Equivalently, Row $\begin{bmatrix} \mathbf{b}_1 \\ \hline \vdots \\ \hline \mathbf{b}_k \end{bmatrix}$

How to transform between these two representations?

From left to right:

- *input:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
- *output:* generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

From right to left:

- *input:* generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$,
- *output:* homogeneous linear system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

Annihilator of a vector space

From left to right: • *input* system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$,
• *output:* generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Solution set is the set of vectors $\mathbf{u}$ such that $\mathbf{a}_1 \cdot \mathbf{u} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{u} = 0$

$$\underbrace{\begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}}_{A} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Equivalent:

Given rows of a matrix A, find generators for Null A

$$\text{rows of a matrix } A \;\downarrow\; \boxed{\text{Algorithm X}} \;\downarrow\; \text{generators for Null } A$$

If $\mathbf{u}$ is such a vector then

$$\mathbf{u} \cdot (\alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_m \, \mathbf{a}_m) = 0$$

for any coefficients $\alpha_1, \ldots, \alpha_m$.

Definition: The set of vectors $\mathbf{u}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$ is called the *annihilator* of $\mathcal{V}$. Written as $\mathcal{V}^o$.

Example: The annihilator of Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ is the solution set for $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$

$$\text{generators for a vector space } \mathcal{V} \;\downarrow\; \boxed{\text{Algorithm X}} \;\downarrow\; \text{generators for annihilator } \mathcal{V}^o$$

Annihilator of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *annihilator* of $\mathcal{V}$, written $\mathcal{V}^o$, is

$$\mathcal{V}^o = \{\mathbf{u} \in \mathbb{F}^n \ : \ \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^o = \text{Span } \{[1,0,-1]\}$:

- Note that $[1,0,-1] \cdot [1,0,1] = 0$ and $[1,0,-1] \cdot [0,1,0] = 0$.
  Therefore $[1,0,-1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- For any scalar β,
  $$\beta\,[1,0,-1] \cdot \mathbf{v} = \beta\,([1,0,-1] \cdot \mathbf{v}) = 0$$
  for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Which vectors $\mathbf{u}$ satisfy $\mathbf{u} \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$? Only scalar multiples of $[1,0,-1]$.

Example over $GF(2)$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^o = \text{Span } \{[1,0,1]\}$:

- Note that $[1,0,1] \cdot [1,0,1] = 0$ (remember $GF(2)$ addition) and $[1,0,1] \cdot [0,1,0] = 0$.
- Therefore $[1,0,1] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- Of course $[0,0,0] \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in Span $\{[1,0,1],[0,1,0]\}$.
- $[1,0,1]$ and $[0,0,0]$ are the only such vectors.

Annihilator of a vector space

Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1],[0,1,0]\}$. Then $\mathcal{V}^o = \text{Span } \{[1,0,-1]\}$
$\dim \mathcal{V} + \dim \mathcal{V}^o = 3$

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Example over $\mathbb{R}$: Let $\mathcal{V} = \text{Span } \{[1,0,1,0],[0,1,0,1]\}$.
Then $\mathcal{V}^o = \text{Span } \{[1,0,-1,0],[0,1,0,-1]\}$.
$\dim \mathcal{V} + \dim \mathcal{V}^o = 4$

Annihilator Dimension Theorem: $\dim \mathcal{V} + \dim \mathcal{V}^o = n$

Proof: Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be generators for $\mathcal{V}$.

Let $A = \begin{bmatrix} \mathbf{a}_1 \\ \hline \vdots \\ \hline \mathbf{a}_m \end{bmatrix}$

Rank-Nullity Theorem states that

$$\begin{array}{ccccc} \text{rank } A & + & \text{nullity } A & = & n \\ \dim \mathcal{V} & + & \dim \mathcal{V}^o & = & n \end{array}$$

Then $\mathcal{V}^o = \text{Null } A$.

QED

Annihilator of a vector space

Definition: For a subspace $\mathcal{V}$ of $\mathbb{F}^n$, the *annihilator* of $\mathcal{V}$, written $\mathcal{V}^o$, is

$$\mathcal{V}^o = \{\mathbf{u} \in \mathbb{F}^n \;:\; \mathbf{u} \cdot \mathbf{v} = 0 \text{ for every vector } \mathbf{v} \in \mathcal{V}\}$$

$$\begin{array}{ccc} \text{rows of a matrix } A & & \text{generators for a vector space } \mathcal{V} \\ \downarrow & & \downarrow \\ \boxed{\text{Algorithm X}} & = & \boxed{\text{Algorithm X}} \\ \downarrow & & \downarrow \\ \text{generators for Null } A & & \text{generators for annihilator } \mathcal{V}^o \end{array}$$

From left to right: Given system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$, find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

Algorithm X solves left-to-right problem....

what about right-to-left problem?

Annihilator of a vector space

From left to right: Given system
$\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0,$
find generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$ for solution set

$$\begin{array}{c} \text{generators for a vector space } \mathcal{V} \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for annihilator } \mathcal{V}^o \end{array}$$

What happens if we apply Algorithm X to generators for annihilator $\mathcal{V}^o$?

$$\begin{array}{c} \text{generators for annihilator } \mathcal{V}^o \\ \downarrow \\ \boxed{\text{Algorithm X}} \\ \downarrow \\ \text{generators for annihilator of annihilator } (\mathcal{V}^o)^o \end{array}$$

From right to left: Given generators $\mathbf{b}_1, \ldots, \mathbf{b}_k$, find system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_m \cdot \mathbf{x} = 0$ whose solution set equals Span $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

$$\begin{array}{c} \text{generators for annihilator } \mathcal{V}^o \\ \downarrow \\ \boxed{\text{Algorithm Y}} \\ \downarrow \\ \text{generators for original space } \mathcal{V} \end{array}$$

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Theorem shows:

$$\text{Algorithm X} = \text{Algorithm Y}$$

We still must prove the Theorem...

Annihilator

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Proof:

Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be a basis for $\mathcal{V}$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_k$ be a basis for $\mathcal{V}^o$.
Since $\mathbf{b}_1 \cdot \mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathcal{V}$,

$$\mathbf{b}_1 \cdot \mathbf{a}_1 = 0, \mathbf{b}_1 \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_1 \cdot \mathbf{a}_m = 0$$

Similarly $\mathbf{b}_i \cdot \mathbf{a}_1 = 0, \mathbf{b}_i \cdot \mathbf{a}_2 = 0, \ldots, \mathbf{b}_i \cdot \mathbf{a}_m = 0$ for $i = 1, 2, \ldots, k$.

Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span } \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^o}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^o)^o$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^o)^o$, $\mathbf{a}_3$ is in $(\mathcal{V}^o)^o$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^o)^o$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^o)^o$.

Thus $\underbrace{\text{Span } \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^o)^o$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$.

Annihilator

Theorem: $(\mathcal{V}^o)^o = \mathcal{V}$ (The annihilator of the annihilator is the original space.)

Proof:
Reorganizing,

$$\mathbf{a}_1 \cdot \mathbf{b}_1 = 0, \mathbf{a}_1 \cdot \mathbf{b}_2 = 0, \ldots, \mathbf{a}_1 \cdot \mathbf{b}_k = 0$$

which implies that $\mathbf{a}_1 \cdot \mathbf{u} = 0$ for every vector $\mathbf{u}$ in $\underbrace{\text{Span }\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}}_{\mathcal{V}^o}$

This shows $\mathbf{a}_1$ is in $(\mathcal{V}^o)^o$. Similarly $\mathbf{a}_2$ is in $(\mathcal{V}^o)^o$, $\mathbf{a}_3$ is in $(\mathcal{V}^o)^o$, ..., $\mathbf{a}_m$ is in $(\mathcal{V}^o)^o$.

Therefore every vector in Span $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ is in $(V^o)^o$.

Thus $\underbrace{\text{Span }\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}}_{\mathcal{V}}$ is a subspace of $(\mathcal{V}^o)^o$.

To show that these are equal, we must show that $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$.

By Annihilator Dimension Theorem, $\dim \mathcal{V} + \dim \mathcal{V}^o = n$.

By Annihilator Dimension Theorem applied to $\mathcal{V}^o$, $\dim \mathcal{V}^o + \dim(\mathcal{V}^o)^o = n$.

Together these equations show $\dim \mathcal{V} = \dim(\mathcal{V}^o)^o$. QED