# René Descartes

Born 1596.

After studying law in college,....

> *I entirely abandoned the study of letters. Resolving to seek no knowledge other than that of which could be found in myself or else in the great book of the world, I spent the rest of my youth traveling, visiting courts and armies, mixing with people of diverse temperaments and ranks, gathering various experiences, testing myself in the situations which fortune offered me, and at all times reflecting upon whatever came my way so as to derive some profit from it.*

He had a practice of lying in bed in the morning, thinking about mathematics...

# Coordinate systems

In 1618, he had an idea...
while lying in bed and watching a fly on the ceiling.

He could describe the location of the fly in terms of two numbers: its distance from the two walls.

He realized that this works even if the two walls were not perpendicular.

He realized that you could express geometry in algebra.

- The walls play role of what we now call *axes*.
- The two numbers are what we now call *coordinates*

# Coordinate systems

In terms of vectors (and generalized beyond two dimensions),

- *coordinate system* for a vector space $\mathcal{V}$ is specified by generators $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathcal{V}$
- Every vector $\mathbf{v}$ in $\mathcal{V}$ can be written as a linear combination

$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n$$

- We represent vector $\mathbf{v}$ by the vector $[\alpha_1, \ldots, \alpha_n]$ of coefficients.
  called the *coordinate representation* of $\mathbf{v}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

But assigning coordinates to points is not enough. In order to avoid confusion, we must ensure that each point is assigned coordinates in exactly one way. How?

We will discuss unique representation later.

# Coordinate representation

**Definition:** The *coordinate representation* of $\mathbf{v}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is the vector $[\alpha_1, \ldots, \alpha_n]$ such that

$$\mathbf{v} = \alpha_1\, \mathbf{a}_1 + \cdots + \alpha_n\, \mathbf{a}_n$$

In this context, the coefficients are called the *coordinates*.

**Example:** The vector $\mathbf{v} = [1, 3, 5, 3]$ is equal to

$$1\,[1, 1, 0, 0] + 2\,[0, 1, 1, 0] + 3\,[0, 0, 1, 1]$$

so the coordinate representation of $\mathbf{v}$ in terms of the vectors $[1, 1, 0, 0], [0, 1, 1, 0], [0, 0, 1, 1]$ is $[1, 2, 3]$.

**Example:** What is the coordinate representation of the vector $[6, 3, 2, 5]$ in terms of the vectors $[2, 2, 2, 3], [1, 0, -1, 0], [0, 1, 0, 1]$?

Since

$$[6, 3, 2, 5] = 2\,[2, 2, 2, 3] + 2\,[1, 0, -1, 0] - 1\,[0, 1, 0, 1],$$

the coordinate representation is $[2, 2, -1]$.

# Coordinate representation

**Definition:** The *coordinate representation* of $\mathbf{v}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is the vector $[\alpha_1, \ldots, \alpha_n]$ such that

$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n$$

In this context, the coefficients are called the *coordinates*.

Now we do an example with vectors over $GF(2)$.

**Example:** What is the coordinate representation of the vector [0,0,0,1] in terms of the vectors [1,1,0,1], [0,1,0,1], and [1,1,0,0]?

Since

$$[0,0,0,1] = 1\,[1,1,0,1] + 0\,[0,1,0,1] + 1\,[1,1,0,0]$$

the coordinate representation of $[0,0,0,1]$ is $[1,0,1]$.

# Coordinate representation

**Definition:** The *coordinate representation* of $\mathbf{v}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ is the vector $[\alpha_1, \ldots, \alpha_n]$ such that

$$\mathbf{v} = \alpha_1\, \mathbf{a}_1 + \cdots + \alpha_n\, \mathbf{a}_n$$

In this context, the coefficients are called the *coordinates*.

Why put the coordinates in a vector?

Makes sense in view of linear-combinations definitions of matrix-vector multiplication.

Let $A = \left[\begin{array}{c|c|c} & & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ & & \end{array}\right]$.

- "$\mathbf{u}$ is the coordinate representation of $\mathbf{v}$ in terms of $\mathbf{a}_1, \ldots, \mathbf{a}_n$" can be written as matrix-vector equation $A\mathbf{u} = \mathbf{v}$
- To go from a coordinate representation $\mathbf{u}$ to the vector being represented, we multiply $A$ times $\mathbf{u}$.
- To go from a vector $\mathbf{v}$ to its coordinate representation, we can solve the matrix-vector equation $A\mathbf{x} = \mathbf{v}$.