Unique representation

Recall idea of *coordinate system* for a vector space $\mathcal{V}$:

- Generators $\mathbf{a}_1, \ldots, \mathbf{a}_n$ of $\mathcal{V}$
- Every vector $\mathbf{v}$ in $\mathcal{V}$ can be written as a linear combination
$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n$$
- We represent vector $\mathbf{v}$ by its *coordinate representation* $[\alpha_1, \ldots, \alpha_n]$

Question: How can we ensure that each point has only one coordinate representation?

Answer: The generators $\mathbf{a}_1, \ldots, \mathbf{a}_n$ should form a basis.

Unique-Representation Lemma Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.

Uniqueness of representation in terms of a basis

Unique-Representation Lemma: Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.
Proof: Let $\mathbf{v}$ be any vector in $\mathcal{V}$.
The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ span $\mathcal{V}$, so there is at least one representation of $\mathbf{v}$ in terms of the basis vectors.

Suppose there are two such representations:

$$\mathbf{v} = \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n = \beta_1 \, \mathbf{a}_1 + \cdots + \beta_n \, \mathbf{a}_n$$

We get the zero vector by subtracting one from the other:

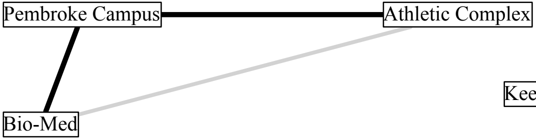
$$\begin{aligned} \mathbf{0} &= \alpha_1 \, \mathbf{a}_1 + \cdots + \alpha_n \, \mathbf{a}_n - (\beta_1 \, \mathbf{a}_1 + \cdots + \beta_n \, \mathbf{a}_n) \\ &= (\alpha_1 - \beta_1) \, \mathbf{a}_1 + \cdots + (\alpha_n - \beta_n) \, \mathbf{a}_n \end{aligned}$$

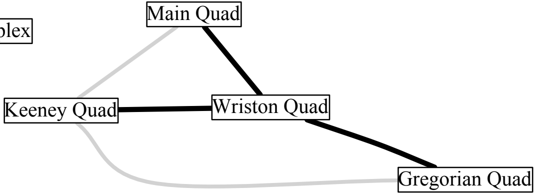
Since the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent, the coefficients $\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n$ must all be zero, so the two representations are really the same.

QED

Uniqueness of representation in terms of a basis: The case of graphs

Unique-Representation Lemma Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis for $\mathcal{V}$. For any vector $\mathbf{v} \in \mathcal{V}$, there is exactly one representation of $\mathbf{v}$ in terms of the basis vectors.





A basis for a graph is a spanning forest.

Unique Representation shows that, for each edge xy in the graph,

- there is an x-to-y path in the spanning forest, and
- there is only one such path.