Computing a basis

Proposition: Mutually orthogonal nonzero vectors are linearly independent.

What happens if we call the `orthogonalize` procedure on a list `vlist`$=[\mathbf{v}_0, \ldots, \mathbf{v}_n]$ of vectors that are linearly dependent?

$\dim \text{Span } \{\mathbf{v}_0, \ldots, \mathbf{v}_n\} < n + 1$.

`orthogonalize(`$[\mathbf{v}_0, \ldots, \mathbf{v}_n]$`)` returns $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*]$

The vectors $\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*$ are mutually orthogonal.

They can't be linearly independent since they span a space of dimension less than $n + 1$.

Therefore some of them must be zero vectors.

Leaving out the zero vectors does not change the space spanned...

Let S be the subset of $\{\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*\}$ consisting of nonzero vectors.

Span $S =$ Span $\{\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*\} =$ Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$

Proposition implies that S is linearly independent.

Thus S is a basis for Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$.

Computing a basis

Therefore in principle the following algorithm computes a basis for Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$:

def find_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
    "Return the list of nonzero starred vectors."
    $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*] = \texttt{orthogonalize}([\mathbf{v}_0, \ldots, \mathbf{v}_n])$
    return $[\mathbf{v}^*$ for $\mathbf{v}^*$ in $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*]$ if $\mathbf{v}^*$ is not the zero vector$]$

Example:

Suppose $\texttt{orthogonalize}([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6])$ returns $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*, \mathbf{v}_4^*, \mathbf{v}_5^*, \mathbf{v}_6^*]$ and the vectors $\mathbf{v}_2^*$,$\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero.
Then the remaining output vectors $\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*$ form a basis for Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Recall

Lemma: Every finite set T of vectors contains a subset S that is a basis for Span T.

What about finding a subset of $\mathbf{v}_0, \ldots, \mathbf{v}_n$ that is a basis?

Proposed algorithm:

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):

Computing a basis

Therefore in principle the following algorithm computes a basis for Span $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$:

def find_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
  "Return the list of nonzero starred vectors."
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Recall

Lemma: Every finite set T of vectors contains a subset S that is a basis for Span T.

What about finding a subset of $\mathbf{v}_0, \ldots, \mathbf{v}_n$ that is a basis?

Proposed algorithm:

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
  "Return the list of original vectors that correspond to nonzero starred vectors."
  $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*] = \texttt{orthogonalize}([\mathbf{v}_0, \ldots, \mathbf{v}_n])$
  Return $[\mathbf{v}_i$ for i in $\{0, \ldots, n\}$ if $\mathbf{v}_i^*$ is not the zero vector$]$

Is this correct?

Correctness of find_subset_basis

def find_subset_basis($[\mathbf{v}_0, \ldots, \mathbf{v}_n]$):
  $[\mathbf{v}_0^*, \ldots, \mathbf{v}_n^*] =$ `orthogonalize`$([\mathbf{v}_0, \ldots, \mathbf{v}_n])$
  Return $[\mathbf{v}_i$ for i in $\{0, \ldots, n\}$ if $\mathbf{v}_i^*$ is not the zero vector$]$

```
def orthogonalize(vlist):
 vstarlist = []
 for v in vlist:
   vstarlist.append(
    project_orthogonal(v, vstarlist))
 return vstarlist
```

Example: `orthogonalize`$([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6])$ returns $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_3^*, \mathbf{v}_4^*, \mathbf{v}_5^*, \mathbf{v}_6^*]$

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

In third iteration of `orthogonalize`, project_orthogonal$(\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*])$ computes $\mathbf{v}_3^*$:

- subtract projection of $\mathbf{v}_3$ along $\mathbf{v}_0^*$,
- subtract projection along $\mathbf{v}_1^*$,
- subtract projection along $\mathbf{v}_2^*$—but since $\mathbf{v}_2^* = \mathbf{0}$, the projection is the zero vector

Result is the same as project_orthogonal$(\mathbf{v}_3, [\mathbf{v}_0^*, \mathbf{v}_1^*])$. Zero starred vectors are ignored.
Thus `orthogonalize`$([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6])$ would return $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$.
Since $[\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*]$ is a basis for $\mathcal{V} = \text{Span}\,\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$
and $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$ spans the same space, and has the same cardinality

Correctness of find_subset_basis

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and $[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$ spans the same space, and has the same cardinality
$[\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6]$ is also a basis for $\mathcal{V}$.

Correctness of find_subset_basis

Another way to justify `find_subset_basis`...

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$\left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c|c} \\ \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^* \\ \\ \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & & \alpha_{0n} \\ & 1 & \alpha_{12} & & \alpha_{1n} \\ & & 1 & & \alpha_{2n} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

Correctness of find_subset_basis

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \mathbf{v}_3^* & \mathbf{v}_4^* & \mathbf{v}_5^* & \mathbf{v}_6^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\ & 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\ & & 1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\ & & & 1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\ & & & & 1 & \alpha_{45} & \alpha_{46} \\ & & & & & 1 & \alpha_{56} \\ & & & & & & 1 \end{bmatrix}$$

Delete zero columns and the corresponding rows of the triangular matrix. Shows Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq$ Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$
so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$
Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.
Resulting triangular matrix is invertible. Move it to other side.
Shows Span $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq$ Span $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$.

Correctness of find_subset_basis

$$\left[\begin{array}{c|c|c|c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \end{array}\right]$$

Let $\mathcal{V} = \text{Span}\ \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

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Correctness of find_subset_basis

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Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$$\left[\begin{array}{c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{array}\right] = \left[\begin{array}{c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\ & 1 & \alpha_{13} & \alpha_{16} \\ & & 1 & \alpha_{36} \\ & & & 1 \end{bmatrix}$$

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Correctness of find_subset_basis

$$\left[\begin{array}{c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\ & 1 & \alpha_{13} & \alpha_{16} \\ & & 1 & \alpha_{36} \\ & & & 1 \end{bmatrix}^{-1}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

Suppose $\mathbf{v}_2^*$, $\mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$$= \left[\begin{array}{c|c|c|c} \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \end{array}\right]$$

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Correctness of find_subset_basis

$$\left[\begin{array}{c|c|c|c} \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \end{array}\right] \begin{bmatrix} 1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\ & 1 & \alpha_{13} & \alpha_{16} \\ & & 1 & \alpha_{36} \\ & & & 1 \end{bmatrix}^{-1}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$.

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Roundoff error in computing a basis

In principle the following algorithm computes a basis for Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$:

def find_basis($[\mathbf{v}_1, \ldots, \mathbf{v}_n]$)
  Use `orthogonalize` to compute $[\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]$
  Return the list consisting of the nonzero vectors in this list.

However: the computer uses floating-point calculations.

Due to round-off error, the vectors that are supposed to be zero won't be exactly zero.

Instead, consider a vector $\mathbf{v}$ to be zero if $\mathbf{v} * \mathbf{v}$ is very small (e.g. smaller than 10^{-20}):

def find_basis($[\mathbf{v}_1, \ldots, \mathbf{v}_n]$)
  Use `orthogonalize` to compute $[\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]$
  Return the list consisting of vectors in this list
  whose squared norms are greater than 10^{-20}

Can use this procedure in turn to define `rank(vlist)` and `is_independent(vlist)`.
Use same idea in other procedures such as `find_subset_basis`