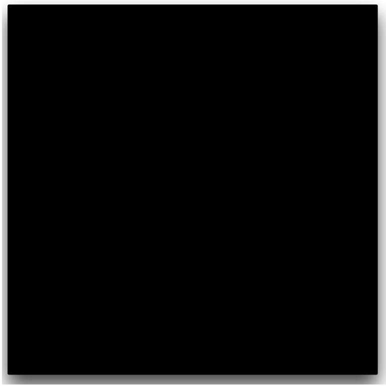
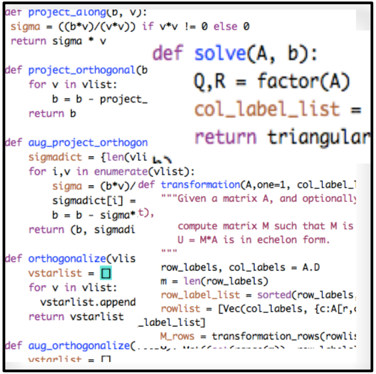
Continuing to look inside the black box



We studied Gaussian elimination, which is used in modules `solver` and `independence` when working over $GF(2)$.

We next study the methods used in these modules when working over $\mathbb{R}$.

Continuing to look inside the black box



We studied Gaussian elimination, which is used in modules `solver` and `independence` when working over $GF(2)$.

We next study the methods used in these modules when working over $\mathbb{R}$.

Fire Engine problem

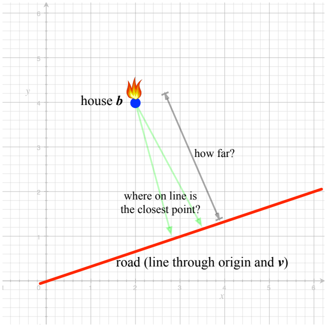
There is a burning house located at coordinates $[2, 4]$!

A street runs near the house, along the line through the origin and through $[6, 2]$—but it is near enough?

Fire engine has a hose 3.5 units long.

If we can navigate the fire engine to the point on the line nearest the house, will the distance be small enough to save the house?

We're faced with two questions: what point along the line is closest to the house, and how far is it?



What do we mean by *closest*?

Distance, length, norm, inner product

We will define the distance between two vectors $\mathbf{p}$ and $\mathbf{b}$ to be the length of the difference $\mathbf{p} - \mathbf{b}$.

This means that we must define the length of a vector.

Instead of using the term "length" for vectors, we typically use the term *norm*.

The norm of a vector $\mathbf{v}$ is written $\|\mathbf{v}\|$

Since it plays the role of length, it should satisfy the following *norm properties*:

Property N1 $\|\mathbf{v}\|$ is a nonnegative real number.

Property N2 $\|\mathbf{v}\|$ is zero if and only if $\mathbf{v}$ is a zero vector.

Property N3 for any scalar α, $\|\alpha\,\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$.

Property N4 $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality).

One way to define vector norm is to define an operation on vectors called *inner product*. Inner product of vectors $\mathbf{u}$ and $\mathbf{v}$ is written

$$\langle \mathbf{u}, \mathbf{v} \rangle$$

The inner product must satisfy certain axioms, which we outline later.

No way to define inner product for *GF*(2) so no more *GF*(2) ☹

From inner product to norm

For the real numbers and complex numbers, we have some flexibility in defining the inner product.

This flexibility is used heavily, e.g. in Machine Learning

Once we have defined an inner product, the norm of a vector **u** is defined by

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

For simplicity, we will focus on $\mathbb{R}$ and will use the most natural and convenient definition of inner product.

This definition leads to the norm of a vector being the geometric length of its arrow.

For vectors over $\mathbb{R}$, we define our inner product as the dot-product:

$$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$$

Properties of inner product of vectors over $\mathbb{R}$

- *linearity in the first argument:* $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
- *symmetry:* $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
- *homogeneity:* $\langle \alpha \mathbf{u}, \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$

For inner product = dot-product, can easily prove these properties.

From inner product to norm

We have defined $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ and $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$

Do these definitions lead to a norm that satisfies the *norm properties?*

Property N1: $||\mathbf{v}||$ is a nonnegative real number.

Property N2: $||\mathbf{v}||$ is zero if and only if $\mathbf{v}$ is a zero vector.

Property N3: for any scalar α, $||\alpha\,\mathbf{v}|| = |\alpha|||\mathbf{v}||$.

Property N4: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ (triangle inequality).

Write $\mathbf{v} = [v_1, v_2, \ldots, v_n]$.

$\|[v_1, v_2, \ldots, v_n]\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$

1. Sum of squares is a nonnegative real number so $\|\mathbf{v}\|$ is nonnegative real number.
2. If any entry v_i of $\mathbf{v}$ is nonzero then sum of squares is nonzero, so norm is nonzero.
3. Proof of third property:

$$\begin{aligned}
||\alpha\,\mathbf{v}||^2 &= \langle \alpha\,\mathbf{v}, \alpha\,\mathbf{v} \rangle && \text{by definition of norm} \\
&= \alpha\,\langle \mathbf{v}, \alpha\,\mathbf{v} \rangle && \text{by homogeneity of inner product} \\
&= \alpha\,(\alpha\,\langle \mathbf{v}, \mathbf{v} \rangle) && \text{by symmetry and homogeneity (again) of inner product} \\
&= \alpha^2||\mathbf{v}||^2 && \text{by definition of norm}
\end{aligned}$$

Thus $||\alpha\,\mathbf{v}|| = \alpha\,||\mathbf{v}||$.

We skip the proof of fourth property.

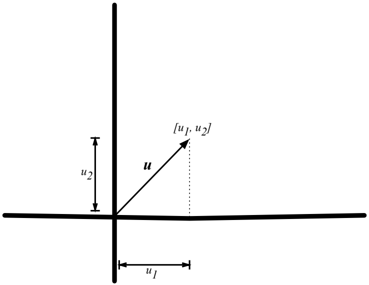
Norm is geometric length of arrow

Example: What is the length of the vector $\mathbf{u} = [u_1, u_2]$?

Remember the *Pythagorean Theorem*:

for a right triangle with side-lengths a, b, c,
where c is the length of the hypotenuse,

$$a^2 + b^2 = c^2$$



We can use this equation to calculate the length of $\mathbf{u}$:

$$(\text{length of } \mathbf{u})^2 = u_1^2 + u_2^2$$

So this notion of length agrees with the one we learned in grade school, at least for vectors in $\mathbb{R}^2$.