Direct Sum

Let $\mathcal{U}$ and $\mathcal{V}$ be two vector spaces consisting of D-vectors over a field $\mathbb{F}$.

Definition: If $\mathcal{U}$ and $\mathcal{V}$ share only the zero vector then we define the *direct sum* of $\mathcal{U}$ and $\mathcal{V}$ to be the set

$$\{\mathbf{u} + \mathbf{v} \ : \mathbf{u} \in \mathcal{U}, \mathbf{v} \in \mathcal{V}\}$$

written $\mathcal{U} \oplus \mathcal{V}$

That is, $\mathcal{U} \oplus \mathcal{V}$ is the set of all sums of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

In Python, `[u+v for u in U for v in V]`

Direct Sum: Example

Vectors over $GF(2)$:

Example: Let $\mathcal{U} = \text{Span }\{1000, 0100\}$ and let $\mathcal{V} = \text{Span }\{0010\}$.

- Every nonzero vector in $\mathcal{U}$ has a one in the first or second position (or both) and nowhere else.
- Every nonzero vector in $\mathcal{V}$ has a one in the third position and nowhere else.

Therefore the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Therefore $\mathcal{U} \oplus \mathcal{V}$ is defined.

$\mathcal{U} \oplus \mathcal{V} = \{0000 + 0000, 1000 + 0000, 0100 + 0000, 1100 + 0000, 0000 + 0010, 1000 + 0010, 0100 + 0010, 1100 + 0010\}$
which is equal to $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 1110\}$.

Direct Sum: Example

Vectors over $\mathbb{R}$:

Example: Let $\mathcal{U} = \text{Span } \{[1, 2, 1, 2], [3, 0, 0, 4]\}$ and let $\mathcal{V}$ be the null space of $\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$.

- The vector $[2, -2, -1, 2]$ is in $\mathcal{U}$ because it is $[3, 0, 0, 4] - [1, 2, 1, 2]$
- It is also in $\mathcal{V}$ because

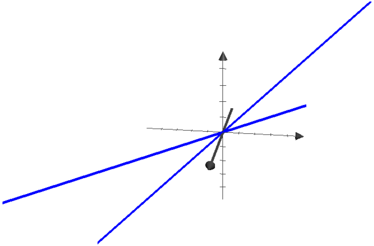
$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore we cannot form $\mathcal{U} \oplus \mathcal{V}$.

Direct Sum: Example

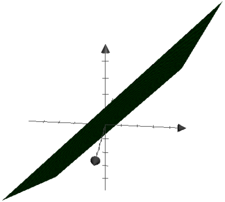
Vectors over $\mathbb{R}$:

Example:



- Let $\mathcal{U} = \text{Span } \{[4, -1, 1]\}$.
- Let $\mathcal{V} = \text{Span } \{[0, 1, 1]\}$.

The only intersection is at the origin, so $\mathcal{U} \oplus \mathcal{V}$ is defined.



- $\mathcal{U} \oplus \mathcal{V}$ is the set of vectors $\mathbf{u} + \mathbf{v}$ where $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{V}$.
- This is just $\text{Span } \{[4, -1, 1], [0, 1, 1]\}$
- Plane containing the two lines

Properties of direct sum

Lemma: $\mathcal{U} \oplus \mathcal{V}$ is a vector space.

(Prove using Properties V1, V2, V3.)

Lemma: The union of

- a set of generators of $\mathcal{U}$, and
- a set of generators of $\mathcal{V}$

is a set of generators for $\mathcal{U} \oplus \mathcal{V}$.

Proof: Suppose $\mathcal{U} = \text{Span } \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ and $\mathcal{V} = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.
Then

- every vector in $\mathcal{U}$ can be written as $\alpha_1 \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m$, and
- every vector in $\mathcal{V}$ can be written as $\beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$

so every vector in $\mathcal{U} \oplus \mathcal{V}$ can be written as

$$\alpha_1 \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m \; + \; \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$$

QED

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof: Clearly

- a basis of $\mathcal{U}$ is a set of generators for $\mathcal{U}$, and
- a basis of $\mathcal{V}$ is a set of generators for $\mathcal{V}$.

Therefore the previous lemma shows that

- the union of a basis for $\mathcal{U}$ and a basis for $\mathcal{V}$ is a generating set for $\mathcal{U} \oplus \mathcal{V}$.

We just need to show that the union is linearly independent.

Properties of direct sum

Direct Sum Basis Lemma:

Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Proof, cont'd: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a basis for $\mathcal{U}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for $\mathcal{V}$.
We need to show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is independent.
Suppose

$$\mathbf{0} = \alpha_1\ \mathbf{u}_1 + \cdots + \alpha_m \mathbf{u}_m + \beta_1\ \mathbf{v}_1 + \cdots + \beta_n\ \mathbf{v}_n.$$

Then

$$\underbrace{\alpha_1\ \mathbf{u}_1 + \cdots + \alpha_m\ \mathbf{u}_m}_{\text{in } \mathcal{U}} = \underbrace{(-\beta_1)\ \mathbf{v}_1 + \cdots + (-\beta_n)\ \mathbf{v}_n}_{\text{in } \mathcal{V}}$$

Left-hand side is a vector in $\mathcal{U}$, and right-hand side is a vector in $\mathcal{V}$.

By definition of $\mathcal{U} \oplus \mathcal{V}$, the only vector in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

This shows:

$$\mathbf{0} = \alpha_1\ \mathbf{u}_1 + \cdots + \alpha_m\ \mathbf{u}_m$$

and

$$\mathbf{0} = (-\beta_1)\ \mathbf{v}_1 + \cdots + (-\beta_n)\ \mathbf{v}_n$$

By linear independence, the linear combinations must be trivial. QED

Direct Sum

Direct Sum Basis Lemma:
Union of a basis of $\mathcal{U}$ and a basis of $\mathcal{V}$ is a basis of $\mathcal{U} \oplus \mathcal{V}$.

Direct Sum Dimension Corollary: $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$

Proof: A basis for $\mathcal{U}$ together with a basis for $\mathcal{V}$ forms a basis for $\mathcal{U} \oplus \mathcal{V}$. QED

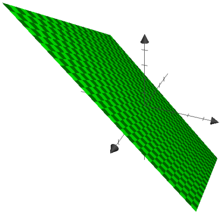
Complementary subspace

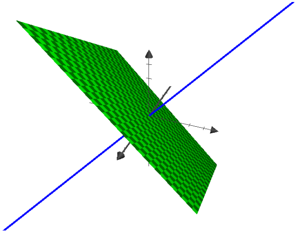
If $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$, we say $\mathcal{U}$ and $\mathcal{V}$ are *complementary* subspaces of $\mathcal{W}$.

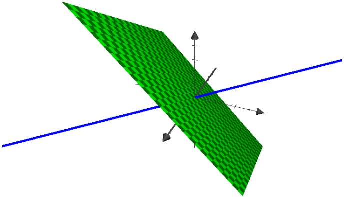
Example: Suppose $\mathcal{U}$ is a plane in $\mathbb{R}^3$.

Then any line through the origin that does not lie in $\mathcal{U}$ is complementary subspace with respect to $\mathbb{R}^3$

Example illustrates that, for a given subspace $\mathcal{U}$ of $\mathcal{W}$, there can be many different subspaces $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are complementary.







Complementary subspace

Proposition: For any finite-dimensional vector space $\mathcal{W}$ and any subspace $\mathcal{U}$, there is a subspace $\mathcal{V}$ such that $\mathcal{U}$ and $\mathcal{V}$ are complementary.

Proof: Let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be a basis for $\mathcal{U}$. By Superset-Basis Lemma, there is a basis for $\mathcal{W}$ that includes $\mathbf{u}_1, \ldots, \mathbf{u}_k$:

$$B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_r\}$$

Let $\mathcal{V} = \text{Span } \{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$.

Any vector in $\mathcal{W}$ can be written in terms of its basis:

$$\mathbf{w} = \underbrace{\alpha_1\, \mathbf{u}_1 + \cdots + \alpha_k\, \mathbf{u}_k}_{\text{in } \mathcal{U}} + \underbrace{\beta_1\, \mathbf{v}_1 + \cdots + \beta_r\, \mathbf{v}_r}_{\text{in } \mathcal{V}}$$

If some vector $\mathbf{v}$ is in Span $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ and in Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$
then $\mathbf{v} = \alpha_1\, \mathbf{u}_1 + \cdots + \alpha_k\, \mathbf{u}_k$ and $\mathbf{v} = \beta_1\, \mathbf{v}_1 + \cdots + \beta_r\, \mathbf{v}_r$
so

$$\alpha_1\, \mathbf{u}_1 + \cdots + \alpha_k\, \mathbf{u}_k = \beta_1\, \mathbf{v}_1 + \cdots + \beta_r\, \mathbf{v}_r$$

$$\mathbf{0} = \alpha_1\, \mathbf{u}_1 + \cdots + \alpha_k\, \mathbf{u}_k - \beta_1\, \mathbf{v}_1 - \cdots - \beta_r\, \mathbf{v}_r$$

so $\alpha_1 = \cdots = \alpha_k = \beta_1 = \cdots = \beta_r = 0$ so $\mathbf{v} = \mathbf{0}$. QED