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Exercise 1: F_1, F_2 - σ-algebras, $F = F_1 \times F_2 = \{A \times B, A \in F_1, B \in F_2\}$

F is σ-alg. $\overset{?}{\Leftrightarrow}$ At least one of F_1 and F_2 is a trivial σ-algebra $\{\emptyset, X\}$

Proof: ⊕ Suppose that F_1 is a trivial σ-algebra. (It's symmetric with F_2)

1) We have $\emptyset = \emptyset \times \emptyset \in F_1 \times F_2$ and $X_1 \times X_2 \in F_1 \times F_2$

(X_1, X_2 are sets of F_1, F_2)

2) Take $A \in F_2$, any element of F is $\emptyset \times A = \emptyset$ or $X_1 \times A$

And $\bigcup_{i=1}^{\infty} (X_1 \times A_i) = X_1 \times (\bigcup_{i=1}^{\infty} A_i) \in F$ because $\bigcup_{i=1}^{\infty} A_i \in F_2$

3) Then we also have $X_1 \times A_1 \setminus X_1 \times A_2 = X_1 \times (A_1 \setminus A_2) \in F$

because $A_1 \setminus A_2 \in F_2$

⊕ Suppose that both F_1 & F_2 are non-trivial.

$\exists A_1 \in F_1 : A_1 \neq F_1$ & $A_1 \neq \emptyset$

$\exists A_2 \in F_2 : A_2 \neq F_2$ & $A_2 \neq \emptyset$

Suppose that F is ~~alsa~~ a σ-algebra $\Rightarrow \begin{cases} A_1 \times X_2 \in F \\ X_1 \times A_2 \in F \end{cases}$

$\Rightarrow (A_1 \times X_2) \cup (X_1 \times A_2) \in F$

$\quad\quad\quad\quad\quad\quad\quad\quad \| $

$\quad\quad\quad\quad\quad\quad B_1 \times B_2$

where $B_1 \in F_1$, $B_2 \in F_2$

Take function $f(x,y) = x$ for $(x,y) \in B_1 \times B_2$

We have $f : B_1 \times B_2 \to X_1$ (because $X_1 \times A_2 \subset B_1 \times B_2$)

$\Rightarrow B_1 = X_1$

Similarly, we have $B_2 = X_2$

However, $X_1 \times X_2 \neq A_1 \times X_2 \cup X_1 \times A_2 = B_1 \times B_2$

We get a contradiction here, so F can't be σ-algebra when F_1 & F_2 are both non-trivial. $(*_1)$

⊕ From 1) 2) & 3) we can conclude that F is σ-algebra $(*_2)$

And from $(*_1)$, $(*_2)$, F is σ-alg. if & only if at least one F_1 & F_2 is a trivial σ-algebra

Excercise 2: $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) = \mathcal{B}(\mathbb{R}^{n+m})$

Proof: ⊕ The Cartesian product of $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^m)$ is a σ-algebra $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m)$ such that $\mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) = \sigma(\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^n), B \in \mathcal{B}(\mathbb{R}^m)\})$

\+ Borel σ-algebra in $\mathbb{R}^n$ is the smallest σ-algebra, containing all open rectangles, i.e. $\mathcal{B}(\mathbb{R}^n) = \sigma(\{ \times_{i=1}^{n} (a_i, b_i) \mid a_i, b_i \in \mathbb{R}\})$

So $\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^n), B \in \mathcal{B}(\mathbb{R}^m)\}$ ~~will~~ contains rectangles $\times_{i=1}^{n+m} (a_i, b_i)$

$\Rightarrow \mathcal{B}(\mathbb{R}^{n+m}) \subseteq \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m)$ $(*_1)$

⊕ We have all rectangles $\times_{i=1}^{n+m} (a_i, b_i)$ are contained by $\mathcal{B}(\mathbb{R}^{n+m})$

$\rightarrow$ All elements of $\times_{i=1}^{n} (a_i, b_i) \times F^m$ are contained by $\mathcal{B}(\mathbb{R}^{n+m})$

With $A \in \mathcal{B}(\mathbb{R}^n)$, all elements of $A \times F^m$ are contained by $\mathcal{B}(\mathbb{R}^{n+m})$ ①

Because $\begin{cases} A_1 \times F^m \cup A_2 \times F^m = (A_1 \cup A_2) \times F^m \\ A_1 \times F^m \setminus A_2 \times F^m = (A_1 \setminus A_2) \times F^m \end{cases}$

Similarly, we have $B \in \mathcal{B}(\mathbb{R}^m)$, all elements of $F^n \times B$ are contained by $\mathcal{B}(\mathbb{R}^{n+m})$ ②

From ① and ②, we have that $\mathcal{B}(\mathbb{R}^{n+m})$ contains all elements from $\{A \times B \mid A \in \mathcal{B}(\mathbb{R}^n), B \in \mathcal{B}(\mathbb{R}^m)\}$.

$\Rightarrow \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m) \subseteq \mathcal{B}(\mathbb{R}^{n+m})$ $(*_2)$

From $(*_1)$ and $(*_2)$, $\mathcal{B}(\mathbb{R}^{n+m}) = \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^m)$ $\square$

Exercise 3: Construct an example of (Ω, F) and $\mathcal{E}: \Omega \to \mathbb{R}$ such that
$\mathcal{E}^{-1}(c) = \{\omega \in \Omega : \mathcal{E}(\omega) = c\} \in F \quad \forall c \in \mathbb{R}$, $\mathcal{E}$ not r.v.

$\mathcal{E}$ is not a measurable mapping from (Ω, F) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Solution: + Take $\Omega = \mathbb{R}$, take F contain all sets countable sets, A or A^c is countable. F is a σ-algebra because:

1) $\emptyset \in F$ & $\mathbb{R} \in F$ (since $\mathbb{R} \setminus \mathbb{R} = \emptyset$ and $\emptyset$ is countable)

2) Take $\begin{cases} A \in F \\ B \in F \end{cases}$ If A is countable, $A \setminus B \subseteq A$ is also countable

If $\mathbb{R} \setminus B$ is countable, $A \setminus B \subseteq \mathbb{R} \setminus B$ is also countable

If $\mathbb{R} \setminus A$ is countable and B is countable,

then $\mathbb{R} \setminus (A \setminus B) = (\mathbb{R} \setminus A) \cup (A \cap B)$ is also countable

$\Rightarrow A \setminus B \in F$

3). Take a sequence of sets A_n such that $A_n \in F$.

. If every A_n is countable, then $\bigcup_{n=1}^{\infty} A_n$ is countable (as a countable union of countable sets)

. For some A_i that $\mathbb{R} \setminus A_i$ is countable, then $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n \subseteq \mathbb{R} \setminus A_i$

$\Rightarrow \mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n$ is also countable. So $\bigcup_{n=1}^{\infty} A_n \in F$

+ Take $\mathcal{E}(x) = x \Rightarrow \mathcal{E}^{-1}(c) = \{c\} \in F$ because $\{c\}$ is countable.

Suppose that $\mathcal{E}$ is a measurable mapping from (Ω, F) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Because interval $(0,1) \in \mathcal{B}(\mathbb{R})$, we should have $\mathcal{E}^{-1}((0,1)) = (0,1) \in F$

However, $(0,1)$ or $\mathbb{R} \setminus (0,1)$ isn't countable.

$\Rightarrow \mathcal{E}$ is not a random variable. $\square$

Exercise 4:

- $f: \mathbb{R} \to \mathbb{R}$ - continuous function. Show that f is a Borel function (f is a measurable mapping from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$)
- $f: \mathbb{R} \to \mathbb{R}$ - monotone function. Show that f is a measurable mapping from $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$

Proof:

- Take sets $A \in \mathcal{B}(\mathbb{R})$

If $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, and if set A is open, then $f^{-1}(A)$ is also open. Thus $f^{-1}(A) \in \mathcal{B}(\mathbb{R})$ since open sets are Borel. And open sets generate Borel σ-algebra, so f is measurable.

- Let intervals $(-\infty, a)$ generate Borel σ-algebra, we have to prove that $f^{-1}((-\infty, a)) \in \mathcal{B}(\mathbb{R})$.

+) $f^{-1}((-\infty, a)) = \{x \in \Omega : f(x) < a\}$

+) f is monotone function, so if $x_1 < x_2$ and $x_2 \in f^{-1}((-\infty, a))$, then $x_1 \in f^{-1}((-\infty, a))$.

Thus $f^{-1}((-\infty, a))$ can be one of $\emptyset, \mathbb{R}, (-\infty, x), (-\infty, x]$ for some $x \in \mathbb{R}$.

All these sets are Borel, thus f is measurable. $\square$

Exercise 5 : μ- finite measure on (X,F); $\mu(X) < \infty$

$f_n : X \to \mathbb{R}$: a sequence of measurable functions

$f_n \xrightarrow{\mu} f$.

$g : \mathbb{R} \to \mathbb{R}$ a continuous function

Prove : $g(f_n) \xrightarrow{\mu} g(f)$

Proof : + $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, so we have

$$\forall \varepsilon > 0, \exists \delta > 0; \forall x, y \in \mathbb{R}\times\mathbb{R} : |x-y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$$

$$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } (|g(f_n) - g(f)| > \varepsilon) \subset (|f_n - f| > \delta) \quad (1)$$

$$+ \quad f_n \xrightarrow{\mu} f \Rightarrow \mu(|f_n - f| > \alpha) \to 0 \quad \forall \alpha > 0 \Rightarrow \lim_{n\to\infty} \mu(|f_n - f| > \delta) = 0$$

And applying μ for (1) we have $\mu(|g(f_n) - g(f)| > \varepsilon) \leq \mu(|f_n - f| > \delta)$

$$\Rightarrow \lim_{n\to\infty} \mu(|g(f_n) - g(f)| > \varepsilon) = 0 \quad \forall \varepsilon > 0$$

Thus $g(f_n) \xrightarrow{\mu} g(f)$ $\square$

Exercise 6 : μ- measure on (X,F)

$f_n : X \to \mathbb{R}$: non-negative measurable functions

$f_n \xrightarrow{\mu} f$

$f(x) \geq 0$, μ-as. ?

Proof :

+ Applying Riesz theorem, if $f_n(x) \xrightarrow{\mu} f(x)$, then exits a subsequence $\{f_{n_k}\}$ so that $f_{n_k}(x) \to f(x)$, μ-a.s. Thus $f(x) = \lim_{k\to\infty} f_{n_k}(x) \geq 0$, μ-a.s. $\square$