Mathematical Foundations of Probability theory

Exam questions. Exam is open-book, all materials are allowed.

1. Set classes: semi-algebra, algebra, σ-algebra (semi-ring, ring, σ-ring), monotone classes. Their properties and relations between them. Definition of the measure on a semi-algebra. Extension of measure from semi-algebra to generated algebra. Measure properties on algebra.
2. Outer measure, complete measure. Hereditary classes. Class of sets, measurable w.r.t. outer measure. Theorem: restriction of outer measure to the class of measurable sets is a complete measure.
3. Outer measure corresponding to measure on algebra. Extension of measure from algebra to generated σ-algebra. Example: extension of the non σ-finite measure might be not unique. Completion of measure on σ-algebra.
4. Lebesgue measure on real line. Borel and Lebesgue sets on real line. Characterisation property of Lebesgue measure: shift invariance.
5. Measurable spaces and measurable functions. Properties of measurable functions. Almost sure convergence and convergence in measure. Almost uniform convergence. Relations between types of convergence. Egorov's theorem.
6. Approximation of measurable functions by simple functions. Construction of the Lebesgue integral. Integrable functions, their properties. Lebesgue's dominated convergence theorem.
7. L_p-spaces, Hoelder's and Minkowski's inequalities. Completeness of L_p-spaces.
8. Signed measures. The Jordan-Hahn decomposition. The Radon-Nikodym theorem.
9. Lebesgue decomposition of the measure. Integral transformation for absolutely continuous measures. Push-forward measure: measure, generated on the image of a measurable function, corresponding integral transformations.
10. Conditional mathematical expectation w.r.t. σ-algebra. Definitions and properties.
11. Conditional mathematical expectation w.r.t. a random variable $\mathbb{E}[\xi|\eta = y]$. Definitions and properties.