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Exercise 1: Show that $f(x) = \frac{1}{x}$ is not integrable on $[0,1]$

Proof: + Take $f_n(x) = \frac{1}{x} I_{[\frac{1}{n}, 1]}(x)$. We have $f_n(x) \to f(x)$ as $n \to \infty$ almost everywhere.

+ Applying the monotone convergence theorem, we have

$$\lim \int_{[0,1]} f_n(x)\mu(dx) = \int_{[0,1]} \lim f_n(x)\mu(dx) = \int_{[0,1]} f(x)\mu(dx) = \int_0^1 \frac{1}{x}\, dx = \infty$$

Here $\frac{1}{x}$ is non-negative on $[0,1]$ a.e.

Since $\lim\limits_{n\to\infty} \int_{[0,1]} f_n(x)\mu(dx)$ is infinite, $f(x)$ is not integrable on $[0,1]$

Exercise 2: Let ε, η be independent random variables on (Ω, F, P)
$\mathbb{E}[|\varepsilon|] < \infty$, $\mathbb{E}[|\eta|] < \infty$. Prove that $\mathbb{E}[\varepsilon\eta] = \mathbb{E}[\varepsilon].\mathbb{E}[\eta]$

Proof: + In the case that $\varepsilon = I_A$ for $A \in F$ and $\eta = I_B$ for $B \in F$
We have $A \in \sigma(\varepsilon)$; $B \in \sigma(\eta)$. Then

$$\mathbb{E}(\varepsilon\eta) = \mathbb{E}(I_A . I_B) = \mathbb{E}(I_{A\cap B}) = P(A\cap B) = P(A).P(B) = \mathbb{E}(I_A).\mathbb{E}(I_B) =$$
$$= \mathbb{E}(\varepsilon).\mathbb{E}(\eta)$$

+ In the case that ε and η are general non-negative realvalued r.v.
Let ε', η' be positive simple real-valued r.v. that $\sigma(\varepsilon)$ and $\sigma(\eta)$ measurable.
We have $\varepsilon' = \sum_{k=1}^{n} \alpha_k I_{A_k}$; $\eta' = \sum_{\ell=1}^{m} \beta_\ell I_{B_\ell}$. Then

$$\mathbb{E}(\varepsilon'\eta') = \mathbb{E}\left(\left(\sum_{k=1}^{n} \alpha_k I_{A_k}\right)\left(\sum_{\ell=1}^{m} \beta_\ell I_{B_\ell}\right)\right) = \mathbb{E}\left(\sum_{k=1}^{n}\sum_{\ell=1}^{m} \alpha_k\beta_\ell . I_{A_k} I_{B_\ell}\right) = \mathbb{E}\left(\sum_{k=1}^{n}\sum_{\ell=1}^{m} \alpha_k\beta_\ell I_{A_k\cap B_\ell}\right) =$$
$$= \sum_{k=1}^{n}\sum_{\ell=1}^{m} \alpha_k\beta_\ell . P(A_k\cap B_\ell) = \sum_{k=1}^{n}\sum_{\ell=1}^{m} \alpha_k\beta_\ell P(A_k).P(B_\ell) =$$
$$= \left(\sum_{k=1}^{n} \alpha_k P(A_k)\right)\left(\sum_{\ell=1}^{m} \beta_\ell P(B_\ell)\right) = \mathbb{E}\left(\sum_{k=1}^{n} \alpha_k I_{A_k}\right).\mathbb{E}\left(\sum_{\ell=1}^{m} \beta_\ell I_{B_\ell}\right) = \mathbb{E}(\varepsilon').\mathbb{E}(\eta') \quad (1)$$

+ We can find ~~a~~ non-decreasing sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\eta_n\}_{n\in\mathbb{N}}$ of positive simple r.v. with $\varepsilon_n \to \varepsilon$ and $\eta_n \to \eta$ as $n\to\infty$; i.e.
$$\varepsilon(\omega) = \lim_{n\to\infty} \varepsilon_n(\omega) \text{ and } \eta(\omega) = \lim_{n\to\infty} \eta_n(\omega) \text{ for all } \omega \in \Omega$$
We can show that $\varepsilon_n \in \sigma(\varepsilon)$ and $\eta_n \in \sigma(\eta)$ $\forall n \in \mathbb{N}$.
Indeed, $\varepsilon_n(\omega) = \sum_{k=0}^{n2^n} k.2^{-n} I_{A_k^n}(\omega)$ where $A_k^n = \begin{cases} \frac{k}{2^n} \le \varepsilon < \frac{k+1}{2^n} & ; k \ne n.2^n \\ \varepsilon \geq n & ; k = n.2^n \end{cases}$

Each A_k^n is a $\sigma(\varepsilon)$- measure, since $\sigma(\varepsilon)$ is generated by the preimages of ε under Borel sets
And from characteristic function measurable if and only if set measurable and pointwise sum of measurable functions is measurable, we have that ε_n is $\sigma(\varepsilon)$ measurable $\forall n \in \mathbb{N}$
Similarly, η_n is a $\sigma(\eta)$- measurable $\forall n \in \mathbb{N}$.

$\Rightarrow$ $\forall n \in \mathbb{N}$, from (1) we have $\mathbb{E}(\varepsilon_n\eta_n) = \mathbb{E}(\varepsilon_n).\mathbb{E}(\eta_n)$

By monotone convergence theorem $\rightarrow$ (2) $\downarrow$ $n\to\infty$ $\downarrow$(3) $\downarrow$(4)

$\mathbb{E}(\varepsilon.\eta)$ $\quad$ $\mathbb{E}(\varepsilon)$ $\quad$ $\mathbb{E}(\eta)$

(2): because $\{\varepsilon_n\eta_n\}_{n\in\mathbb{N}}$ is an increasing sequence; (3) & (4) ~~by Monotone~~ similarly (2)
Thus $\mathbb{E}(\varepsilon\eta) = \mathbb{E}(\varepsilon).\mathbb{E}(\eta)$ $\square$

Excercise 3: Prove that the set of continuous functions is dense in $L_1([a,b])$, that is $\forall\, \varepsilon > 0$ and any measurable function $f:[a,b] \to \mathbb{R}$ with $\int_{[a,b]} |f(x)|\,dx < \infty$ there exists a continuous function $f_\varepsilon(x) \in C([a,b])$, such that

$$\int_{[a,b]} |f(x) - f_\varepsilon(x)|\,dx < \varepsilon$$

Proof: + Define $f_A(x) = f(x)\, I_{\{x:|f(x)|<A\}}(x)$ $\forall A > 0$. We can see that f_A is bounded by A and it's measurable because $\{x: |f(x)| < A\}$ is a measurable set.

+ Consider a sequence $g_n = |f - f_n|$; g_n is measurable and $|g_n| \le |f|$ because $g_n(x) = \begin{cases} 0 \\ f(x) \end{cases} \to 0$ a.e. since $g_n(x)$ doesn't converge to $f(x)$ only $f(x)$ is infinite and since f is integrable - only true for some null set of points.

Thus, we have $\lim_n \int_{[a,b]} g_n\, d\lambda = \int_{[a,b]} \lim g_n\, d\lambda = 0$

+ $\forall\, \frac{\varepsilon}{2} > 0$, $\exists n$: $\int_{[a,b]} g_n\, d\lambda = \int_{[a,b]} |f - f_n|\, d\lambda < \frac{\varepsilon}{2}$

+ According to Luzin theorem, $\forall\, \varepsilon_1 = \frac{\varepsilon}{4n} > 0$, $\exists$ continuous function $l(x) \in C([a,b])$ that $\lambda(\{x: f_n(x) \neq l(x)\}) < \varepsilon_1 = \frac{\varepsilon}{4n}$.

+ According to Tietze extension theorem, there exits such continuous function $l(x)$ that is bounded by n and its extension can also be chosen to be bounded by the same constant. thus,

$$\int_{[a,b]} |l - f|\,d\lambda \le \int_{[a,b]} (|l - f_n| + |f_n - f|)\,d\lambda = \int_{[a,b]} |l - f_n|\,d\lambda + \int_{[a,b]} |f_n - f|\,d\lambda =$$

$$= \int_{\{x: l(x) \neq f_n(x)\}} |l - f_n|\,dx + \int_{[a,b]} |f_n - f|\,d\lambda < \int_{\{x: l(x) \neq f_n(x)\}} 2n\,d\lambda + \frac{\varepsilon}{2} < 2n \cdot \frac{\varepsilon}{4n} + \frac{\varepsilon}{2} = \varepsilon$$

Here l is our desired function f_ε $\square$

Exercise 4: (X, F, μ) - measurable space with finite measure μ.

$f: X \to \bar{\mathbb{R}}$ - a measurable function

$A_n = \{x : |f(x)| \geq n\}$; $B_n = \{x : n \leq |f(x)| < n+1\}$

Show that the following conditions are equivalent:

+ f is integrable w.r.t. μ (1)
+ $\sum_{k=1}^{\infty} k\mu(B_k) < \infty$ (2)
+ $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ (3)

Provide counterexamples for equivalente if μ is σ-finite.

Proof: +) (1) $\Rightarrow$ (2)

Take $f_n(x) = \sum_{i=k}^{n} k I_{B_k}(x)$. We have $\int f_n d\mu = \sum_{k=1}^{n} k\mu(B_k)$.

Since f_n is non-decreasing sequence, $\sup \int f_n d\mu = \int \sup f_n d\mu \leq \int |f| d\mu < \infty$

Here $\sup f_n \leq |f|$ because $f_n \leq |f|$ $\forall n$; f is integrable $\Leftrightarrow$ $|f|$ is integrable.

Thus $\sum_{k=1}^{\infty} k\mu(B_k) = \sup \int f_n d\mu = \int \sup f_n d\mu \leq \int |f| d\mu < \infty$

+) (2) $\Rightarrow$ (1)

We have $\sum_{k=1}^{\infty} k\mu(B_k) < \infty$, it's also true that: $\sum_{k=1}^{\infty} (k+1)\mu(B_k) < \infty$

because $\sum_{k=1}^{\infty} (k+1)\mu(B_k) \leq \sum_{k=1}^{\infty} 2k\mu(B_k) = 2\sum_{k=1}^{\infty} k\mu(B_k) < \infty$

Take $g = \sum_{k=0}^{\infty} (k+1) I_{B_k}$. Then $|f| \leq g \Rightarrow \int |f| d\mu \leq \int g d\mu = \sum_{k=0}^{\infty} (k+1)\mu(B_k) < \infty$

Thus $|f|$ is integrable $\Rightarrow$ f is integrable w.r.t. μ.

+) (2) $\Rightarrow$ (3)

$\exists$ $\sum_{k=1}^{\infty} k\mu(B_k) < \infty$, then $\sum_{k=1}^{\infty} k\mu(B_k) = \sum_{k=1}^{\infty} \sum_{m=1}^{k} \mu(B_k) \overset{(*_1)}{=} \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \mu(B_m) \overset{(*_2)}{=} \sum_{k=1}^{\infty} \mu(A_k) < \infty$

The $(*_1)$ can be done because $k\mu(B_k) \geq 0$, so the reodering is possible.

$(*_2)$ can be done because $A_k = \bigcup_{m=k}^{\infty} B_m$.

+) (3) $\Rightarrow$ (2)

$\exists$ $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Then $\sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \mu(B_m) = \sum_{k=1}^{\infty} k\mu(B_k) < \infty$ $\square$

• Counterexample: μ - σ-finite; f is measurable $f(x) = \frac{2}{3}$.

f is a simple function $\Rightarrow \int f d\lambda = \frac{2}{3}\lambda(\mathbb{R}) = \infty$

However, $\sum_{k=1}^{\infty} \mu(A_k) = \sum_{k=1}^{\infty} k\mu(B_k) = 0$, since $\forall k \geq 1$, $\mu(A_k) = \mu(B_k) = 0$

Exercise 5: $\{f_n\}_{n=1}^{\infty}$ - sequence of measurable functions on (X, F, μ)

μ - σ-finite measure ; $f_n \xrightarrow{\mu} f$ as $n \to \infty$.

Assume that $\exists \varphi(x) > 0$, such that $\int_X \varphi(x) \mu(dx) < \infty$

and $|f_n(x)| \le \varphi(x)$ μ-a.s. $\forall n \in \mathbb{N}$. Show that

$$\int_X |f_n(x) - f(x)| \mu(dx) \to 0 \text{ as } n \to \infty$$

Proof: + By Riesz theorem, we have that if $f_n(x) \xrightarrow{\mu} f(x)$ then $\exists \{f_{n_k}(x)\}$ so that $f_{n_k}(x) \to f(x)$; μ-a.s.

+ Take $A_k = \{x : |f_{n_k}(x)| > \varphi(x)\}$; $B = \{x : f_{n_k}(x) \not\to f(x)\}$

$\Rightarrow$ $\mu(A_k) = 0$; $\mu(B) = 0$ since almost everywhere convergence follows from almost uniform convergence.

Take $C = B \cup \bigcup_{k=1}^{\infty} A_k \Rightarrow \mu(C) = 0$ and inside $X \setminus C$: $\begin{cases} f_{n_k} \to f \\ f_{n_k} \le \varphi \end{cases} \Rightarrow |f| \le \varphi$

+ Take $l_k = |f_{n_k} - f|$. Then $l_k \le |f_{n_k}| + |f| \le 2|\varphi|$

+ According to Lebesgue's dominated convergence theorem,

$$\lim \int_{X \setminus C} l_k \, d\mu = \int_{X \setminus C} \lim l_k \, d\mu = 0.$$

So $\int_X |f_{n_k} - f| \, d\mu \to 0$ as $n \to \infty$

+ Suppose that f_n doesn't convergence to f.

There exists $\varepsilon > 0$ and a subsequence f_{n_j} such that $\int_X |f_{n_j} - f| \, d\mu > \varepsilon$.

But we know that since f_{n_j} converges to f in measure, so it has some subsequence that converges to f in mean $\to$ a contradiction. $\square$