Points for problems: 1, 1, 1, 1, 1, 1
Total: 6

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Home assignment 1

Problem 1: Let ξ be a non-negative random variable. Prove that

$$\inf_{k \in \mathbb{Z}^+} \frac{E\xi^k}{t^k} \leq \inf_{\lambda \geq 0} \frac{Ee^{\lambda \xi}}{e^{\lambda t}}, \quad \forall t > 0$$

→ Proof:

- Let $L = \inf_{k \in \mathbb{Z}^+} \frac{E\xi^k}{t^k}$

- $\forall t > 0$, we have

$$\frac{Ee^{\lambda\xi}}{e^{\lambda t}} = \frac{1}{e^{\lambda t}} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k . E\xi^k}{k!} = \frac{1}{e^{\lambda t}} \cdot \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \cdot \frac{E\xi^k}{t^k} \geq \frac{L}{e^{\lambda t}} \cdot \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = \frac{L}{e^{\lambda t}} \cdot e^{\lambda t} = L$$

$$\Rightarrow \frac{Ee^{\lambda\xi}}{e^{\lambda t}} \geq \inf_{k \in \mathbb{Z}^+} \frac{E\xi^k}{t^k}$$

Then taking the infimum over $\lambda \geq 0$

$$\Rightarrow \inf_{k \in \mathbb{Z}^+} \frac{E\xi^k}{t^k} \leq \inf_{\lambda \geq 0} \frac{Ee^{\lambda\xi}}{e^{\lambda t}} \qquad \square$$

<u>Problem 2</u>: Suppose that ε is a sub-Gaussian random variable with variance proxy σ^2.

Prove that $Var(\varepsilon) \leq \sigma^2$

$\rightarrow$ <u>Proof</u>:

ε is a sub-Gaussian random variable with variance proxy σ^2 and mean μ

$$\Rightarrow \begin{cases} E[\varepsilon - \mu] = 0 \\ E\left[e^{\lambda(\varepsilon - \mu)}\right] \leq e^{\lambda^2\sigma^2/2} \quad (*)\ \forall \lambda \in \mathbb{R} \end{cases}$$

+ The Taylor series of $(*)$-left is $1 + \lambda E[\varepsilon - \mu] + \frac{\lambda^2}{2} E[(\varepsilon - \mu)^2] + o(\lambda^2) =$

$$= 1 + \frac{\lambda^2}{2} \cancel{E}\, Var(\varepsilon) + o(\lambda^2) \qquad (1)$$

+ The Taylor series of $(*)$-right is $1 + \frac{\lambda^2\sigma^2}{2} + o(\lambda^2) \qquad (2)$

From $(*)$, (1) and (2) $\Rightarrow 1 + \frac{\lambda^2}{2} Var(\varepsilon) + o(\lambda^2) \leq 1 + \frac{\lambda^2\sigma^2}{2} + o(\lambda^2)$

$$\Rightarrow Var(\varepsilon) \leq \sigma^2 \qquad \square$$

<u>Problem 3</u>: Let ε and η be sub-Gaussian r.v. (not necessarily independent) with variance proxies σ_1^2 and σ_2^2. Prove that the sum $(\varepsilon + \eta)$ is a sub-Gaussian r.v. with variance proxy $(\sigma_1 + \sigma_2)^2$.

$\rightarrow$ <u>Proof</u>:

Assume that $E[\varepsilon] = E[\eta] = 0$

We have $E[e^{\lambda(\varepsilon+\eta)}] = E[e^{\lambda\varepsilon} . e^{\lambda\eta}] \overset{(1)}{\leq} \left(E\left[e^{\lambda\varepsilon\frac{\sigma_1+\sigma_2}{\sigma_1}}\right]\right)^{\frac{\sigma_1}{\sigma_1+\sigma_2}} . \left(E\left[e^{\lambda\eta\frac{\sigma_1+\sigma_2}{\sigma_2}}\right]\right)^{\frac{\sigma_2}{\sigma_1+\sigma_2}} \overset{(2)}{\leq}$

(1): By Holder's inequality: $E[\varepsilon\eta] \leq (E[\varepsilon^p])^{\frac{1}{p}} . (E[\eta^q])^{\frac{1}{q}}$ where $p + q = 1$

(2) ε and η be sub-Gaussian r.v. ~~with σ_1^2 and σ_2^2~~, so

$$E\left[e^{\lambda\varepsilon\frac{\sigma_1+\sigma_2}{\sigma_1}}\right] \leq e^{\lambda^2(\sigma_1+\sigma_2)^2/2} \quad \& \quad E\left[e^{\lambda\eta\frac{\sigma_1+\sigma_2}{\sigma_2}}\right] \leq e^{\lambda^2(\sigma_1+\sigma_2)^2/2}$$

$$\leq \left(e^{\lambda^2(\sigma_1+\sigma_2)^2/2}\right)^{\frac{\sigma_1}{\sigma_1+\sigma_2}} . \left(e^{\lambda^2(\sigma_1+\sigma_2)^2/2}\right)^{\frac{\sigma_2}{\sigma_1+\sigma_2}} = e^{\lambda^2(\sigma_1+\sigma_2)^2/2}$$

$$\Rightarrow E[e^{\lambda(\varepsilon+\eta)}] \leq e^{\lambda^2(\sigma_1+\sigma_2)^2/2}$$

$\Rightarrow (\varepsilon + \eta)$ is a sub-Gaussian r.v. with variance proxy $(\sigma_1 + \sigma_2)^2$ $\square$

Problem 4: Given any centered random variable $\mathcal{E}$, prove that the following properties are equivalent:

(i) $\mathcal{E}$ is a sub-exponential random variable with parameters (σ, b);

(ii) There are constants $c, \tau > 0$ such that $P(|\mathcal{E}| \geq t) \leq ce^{-t/\tau}$ for all $t > 0$

(iii) There exists $\theta > 0$, such that $E|\mathcal{E}|^k \leq (\theta k)^k$ for all $k \in \mathbb{N}$

→ Proof:

(i) → (ii)

- $\mathcal{E}$ is a sub-exponential r.v. with parameters $(\sigma, b) \Rightarrow E[e^{\lambda \mathcal{E}}] \leq e^{\lambda^2 \sigma^2/2}$ $\forall |\lambda| \leq \frac{1}{b}$; $b > 0$

- We have $P(\mathcal{E} \geq t) = P(\lambda\mathcal{E} \geq \lambda t) = P(e^{\lambda \mathcal{E}} \geq e^{\lambda t}) \leq \frac{E[e^{\lambda \mathcal{E}}]}{e^{\lambda t}} \leq \frac{e^{\lambda^2\sigma^2/2}}{e^{\lambda t}}$ (*)

  where $t > 0$; $0 \leq \lambda \leq \frac{1}{b}$ (Chernoff bound)

- For $0 \leq \lambda \leq \frac{1}{b}$, considerate function $f(\lambda) = \frac{\lambda^2\sigma^2}{2} - \lambda t$

  $f'(\lambda) = \lambda \frac{\cancel{2}\sigma^2}{\cancel{2}} - t = 0 \Leftrightarrow \lambda = \frac{t}{\sigma^2}$

  Case 1: $\frac{t}{\sigma^2} \geq \frac{1}{b}$ $\Rightarrow f_{min} = \cancel{f\left(\frac{t}{\sigma^2}\right)}\ f\left(\frac{1}{b}\right) = \frac{\sigma^2}{2b^2} - \frac{t}{b} \leq \frac{tb}{2b^2} - \frac{t}{b} = -\frac{t}{2b}$ $(*_1)$

  $[\sigma^2 \leq tb]$

  $(*)$ & $(*_1)$ $\Rightarrow P(\mathcal{E} \geq t) \leq e^{-\frac{t}{2b}}$

  Similarly, $\forall t > 0$: $P(\mathcal{E} \leq -t) \leq e^{-\frac{t}{2b}}$

  $\Rightarrow P(|\mathcal{E}| \geq t) \leq 2e^{-\frac{t}{2b}}$

  ~~Case 2~~: $\cancel{\frac{t}{\sigma^2} < \frac{1}{b} \Leftrightarrow \frac{b}{2\sigma^2} \cdot t < \frac{1}{2}}$

  ~~We have~~ $\cancel{P(|\mathcal{E}| \geq t) \leq 1 < \frac{2}{\sqrt{e}} = 2e^{-\frac{1}{2}} < 2e^{-\frac{b}{2\sigma^2} \cdot t}}$

  ~~Case 2~~: $\cancel{\frac{t}{\sigma^2} < \frac{1}{b} \Rightarrow f_{min} = f\left(\frac{t}{\sigma^2}\right) = \frac{t^2\sigma^2}{\sigma^4 \cdot 2} - \frac{t^2}{\sigma^2} = -\frac{t^2}{2\sigma^2}}$

  $\cancel{[\sigma^2 > tb]}$

  Case 2: $\frac{t}{\sigma^2} < \frac{1}{b} \Rightarrow \frac{b}{2\sigma^2} \cdot t < \frac{1}{2}$

  We have $P(|\mathcal{E}| \geq t) \leq 1 \leq \frac{2}{\sqrt{e}} = 2e^{-\frac{1}{2}} \leq 2e^{-\frac{b}{2\sigma^2} \cdot t}$

  Hence $\exists$ $c = 2$; $\tau = 2b$ or $\tau = \frac{2\sigma^2}{b}$ that satisfies (ii)

(ii) $\to$ (iii)

We have $E[|\varepsilon|^k] = \int_0^\infty P(|\varepsilon|^k \geq u)\,du = \begin{bmatrix} u = t^k \\ du = k.t^{k-1}\,dt \\ |\varepsilon| \to |\varepsilon|^k \end{bmatrix} = \int_0^\infty k\,t^{k-1}\,P(|\varepsilon| \geq t)\,dt \overset{(ii)}{\leq}$

$[P(|\varepsilon|\geq t) \leq ce^{-t/\tau}]$

$$\leq \int_0^\infty k.t^{k-1}.c.e^{-t/\tau}\,dt = \begin{bmatrix} r = \frac{t}{\tau} \\ dr = \frac{dt}{\tau} \end{bmatrix} = \int_0^\infty k.c.\tau^k r^{k-1}.e^{-r}\,dr =$$

$$= \left[\Gamma(k) = \int_0^\infty r^{k-1} e^{-r}\,dr\right] = k.c.\tau^k.\Gamma(k) \leq [\Gamma(k) \leq k^k] \leq k.c.(\tau.k)^k$$

$$\Rightarrow E|\varepsilon|^k \leq (\tau \sqrt[k]{c.k}.\,k)^k = (\theta k)^k \quad \forall \theta > 0,\ k \in \mathbb{N};\ c, \tau > 0$$

Thus, $\exists\, \theta = \tau \sqrt[k]{ck} > 0$ satisfies (iii)

(iii) $\to$ (i)

We have $e^{\lambda\varepsilon} = \sum_{k=0}^\infty \frac{\lambda^k \varepsilon^k}{k!}$ (definition of exponential)

$$E[e^{\lambda\varepsilon}] \leq E[e^{\lambda|\varepsilon|}] = \sum_{k=0}^\infty \lambda^k \frac{E|\varepsilon|^k}{k!} \leq \left[E|\varepsilon|^k \overset{(iii)}{\leq} (\theta k)^k\right] \leq \sum_{k=0}^\infty \lambda^k \frac{(\theta k)^k}{k!} =$$

$$= 1 + \sum_{k=2}^\infty \lambda^k \frac{(\theta k)^k}{k!} \leq \begin{bmatrix} k! \geq \left(\frac{k}{e}\right)^k \\ \text{Stirling's} \\ \text{approximation} \end{bmatrix} \leq 1 + \sum_{k=2}^\infty \lambda^k \frac{(\theta k)^k}{\left(\frac{k}{e}\right)^k} = 1 + \sum_{k=2}^\infty (\lambda\theta e)^k$$

$$= 1 + (\lambda\theta e)^2 \sum_{k=0}^\infty (\lambda\theta e)^k = \begin{bmatrix} \sum_{n=0}^\infty x^n = \frac{1}{1-x} \\ x = \lambda\theta e \\ n = k \end{bmatrix} = 1 + (\lambda\theta e)^2 \frac{1}{1-\lambda\theta e} \leq$$

$$\leq \left[\lambda\theta e < \frac{1}{2}\ (\text{restrict } \lambda < \frac{1}{2\theta e} \text{ to } Ee^{\lambda|\varepsilon|} < \infty)\right] \leq 1 + 2(\lambda\theta e)^2 \leq$$

$$\leq \begin{bmatrix} 1 + x \leq e^x \quad \forall x \geq 0 \\ x = 2(\lambda\theta e)^2 \end{bmatrix} \leq e^{2(\lambda\theta e)^2}$$

$$\Rightarrow E[e^{\lambda\varepsilon}] \leq e^{\lambda^2(2\theta e)^2/2} = e^{\lambda^2\sigma^2/2} \quad ; |\lambda| \leq \cancel{\tfrac{1}{2\theta e}}\ \# \frac{1}{b} < \frac{1}{2\theta e}$$

So that $\exists\, \sigma = 2\theta e$; $b > 2\theta e$ satisfy (i)

Problem 5: Let ε_1 and ε_2 be centered sub-Gaussian random variables with variance proxies σ_1^2 and σ_2^2, respectively ($\varepsilon_1, \varepsilon_2$ are not necessarily independent). Prove that $\varepsilon_1 \varepsilon_2$ is a sub-exponential random variable.

→ Proof:

Let ε is centered sub-Gaussian r.v., we ~~have~~ define $\|\varepsilon\|_{\psi_2}$ by

$$\|\varepsilon\|_{\psi_2} := \inf\{t>0 : E[e^{\varepsilon^2/t^2}] \le 2\}$$

Applying for ε_1 & ε_2, we have $E[e^{\varepsilon_1^2}] \le 2$ & $E[e^{\varepsilon_2^2}] \le 2$.

Suppose that $\|\varepsilon_1\|_{\psi_2} \neq 0$

Then $\tilde{\varepsilon}_1 = \dfrac{\varepsilon_1}{\|\varepsilon_1\|_{\psi_2}}$ is sub-Gaussian with $\|\tilde{\varepsilon}_1\|_{\psi_2} = \dfrac{\|\varepsilon_1\|_{\psi_2}}{\|\varepsilon_1\|_{\psi_2}} = 1$.

Thus we assume that $\|\varepsilon_1\|_{\psi_2} = \|\varepsilon_2\|_{\psi_2} = 1$.

Using formula $xy \le \dfrac{x^2}{2} + \dfrac{y^2}{2}$ $\forall x,y \in \mathbb{R}$ (due to $(x-y)^2 \ge 0$)

~~for $x = E\varepsilon_1$, $y = E\varepsilon_2$~~ ~~$x = Ee^{\varepsilon_1}$, $y = Ee^{\varepsilon_2}$~~ $\Rightarrow \varepsilon_1 . \varepsilon_2 \le \dfrac{\varepsilon_1^2}{2} + \dfrac{\varepsilon_2^2}{2}$

$$\Rightarrow E[e^{\varepsilon_1\varepsilon_2}] \le E[e^{(\varepsilon_1^2+\varepsilon_2^2)/2}] = E[e^{\frac{\varepsilon_1^2}{2}} . e^{\frac{\varepsilon_2^2}{2}}] \le \frac{1}{2} E[e^{\varepsilon_1^2} + e^{\varepsilon_2^2}] = \frac{1}{2}(2+2) = 2$$

This implies that $\|\varepsilon_1 \varepsilon_2\|_{\psi_1} \le 1$

By Markov's inequality $(\varepsilon = \varepsilon_1 \varepsilon_2) \Rightarrow \|\varepsilon\|_{\psi_1} \le 1$

$$P(|\varepsilon| > t) = P\left(e^{\frac{|\varepsilon|}{\|\varepsilon\|_{\psi_1}}} > e^{\frac{t}{\|\varepsilon\|_{\psi_1}}}\right) \le E\left[e^{\frac{|\varepsilon|}{\|\varepsilon\|_{\psi_1}}}\right] e^{\frac{-t}{\|\varepsilon\|_{\psi_1}}} \le 2e^{-\frac{t}{\|\varepsilon\|_{\psi_1}}} \le 2e^{-t}$$

From problem 4(i+ii), there are constants $c = 2$, $\tau = 1$ such that $P(|\varepsilon| > t) \le 2e^{-t}$ $\forall t > 0$, that $\varepsilon = \varepsilon_1 . \varepsilon_2$ is a sub-exponential r.v. □

Problem 6: Prove that the Orlicz norm $\|\cdot\|_{\psi_p}$, $p \geq 1$, satisfies the triangle inequality that is for any random variables ξ and η it holds that

$$\|\xi + \eta\|_{\psi_p} \leq \|\xi\|_{\psi_p} + \|\eta\|_{\psi_p}$$

→ Proof:

We define the Orlicz norm as $\|\xi\|_{\psi} := \inf\left\{ t > 0 : \mathbb{E}\left[\psi\left(\frac{|\xi|}{t}\right)\right] \leq 1 \right\}$

Let $x = \|\xi\|_{\psi_p}$; $y = \|\eta\|_{\psi_p}$. Then
$(x, y > 0)$

$$\psi\left(\frac{|\xi + \eta|}{x+y}\right) \leq \psi\left(\frac{|\xi| + |\eta|}{x+y}\right) = \psi\left(\frac{x}{x+y} \cdot \frac{|\xi|}{x} + \frac{y}{x+y} \cdot \frac{|\eta|}{y}\right) \leq$$

$$\leq \frac{x}{x+y}\,\psi\left(\frac{|\xi|}{x}\right) + \frac{y}{x+y} \cdot \psi\left(\frac{|\eta|}{y}\right)$$

$$\Rightarrow \mathbb{E}\left[\psi\left(\frac{|\xi + \eta|}{x+y}\right)\right] \leq \frac{x}{x+y}\,\mathbb{E}\left[\psi\left(\frac{|\xi|}{x}\right)\right] + \frac{y}{x+y} \cdot \mathbb{E}\left[\psi\left(\frac{|\eta|}{y}\right)\right] \leq$$

$$\leq \frac{x}{x+y} \cdot 1 + \frac{y}{x+y} \cdot 1 = 1$$

$$\Rightarrow \cancel{\psi}\ \|\xi + \eta\|_{\psi_p} \leq x + y = \|\xi\|_{\psi_p} + \|\eta\|_{\psi_p} \qquad \square$$