The University of Texas at Austin
Optimization

HOMEWORK 5

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Submitting solutions: *Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.*

1. **Linear Algebra**

   (a) Compute the eigenvalues and eigenvectors of the matrix

   $$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

   and show that the eigenvectors are orthogonal to each other.

   (b) Compute the eigenvalues and eigenvectors of the matrix

   $$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

   and show that the eigenvectors are not orthogonal to each other.

   (c) Compute the eigenvalues and eigenvectors of the matrix

   $$A_3 = \begin{bmatrix} 4 & 11 & 11 & 12 \\ 11 & 13 & 14 & 12 \\ 11 & 14 & 16 & 16 \\ 12 & 12 & 16 & 18 \end{bmatrix}.$$

   Is this matrix positive semidefinite? Are the eigenvectors orthogonal to each other?

   (d) Repeat this with random matrices of your choosing. Note that symmetric matrices, whether positive definite or note (i.e., whether or not they have nonnegative eigenvalues) will always have eigenvectors that are perpendicular (orthogonal) to each other.

2. **(Optional)** Given any functions $f_1(x), f_2(x), ..., f_n(x)$, prove that the following are equivalent.

   (a)

   $$\min_x \sum_{i=1}^{k} f_{[i]}(x),$$

   where $f_{[i]}(x)$ is the i^{th} largest of the set of values $f_1(x), f_2(x), ..., f_n(x)$. (Note that which f_i are the largest may change as x changes.)

(b)

$$\begin{array}{lll}\min\limits_{x,y,t} & kt + \sum\limits_{i=1}^{n} y_i & \\ \text{subject to} & f_j(x) \le t + y_j & \forall j = 1, ..., n \\ & y \ge 0. & \end{array}$$

3. **(Optional)** Consider the LP from Homework 3:

$$\begin{array}{rl}\min: & x_1 - x_2 \\ \text{s.t.}: & 2x_1 + 3x_2 - x_3 + x_4 \le 0 \\ & 3x_1 + x_2 + 4x_3 - 2x_4 \ge 3 \\ & -x_1 - x_2 + 2x_3 + x_4 = 6 \\ & x_1 \le 0 \\ & x_2, x_3 \ge 0.\end{array}$$

We stated in the lecture that SDP is a generalization of LP, i.e., that any LP can be written as an LP. Do this here: write the given LP explicitly as an SDP. Thus, your objective must be linear in the decision variables you choose, and your constraints can have equality constraints and semidefinite constraints, but not coordinate-wise inequality constraints as we have in LP.

4. Consider the following non-linear optimization problem:

$$\begin{array}{rl}\min: & \dfrac{(\boldsymbol{c}^\top \boldsymbol{x})^2}{\boldsymbol{d}^\top \boldsymbol{x}} \\ \text{s.t.}: & A\boldsymbol{x} \ge \boldsymbol{b},\end{array}$$

where we assume that $A\boldsymbol{x} \ge \boldsymbol{b} \Rightarrow \boldsymbol{d}^\top \boldsymbol{x} > 0$.

(a) Show that this problem is convex.

(b) Reformulate this problem as an SDP. *(Hint: it may be useful for you to add an additional dummy variable that will allow you to move the objective function into the constraints).*

5. **Using KKT**.

Consider the problem we saw in the lecture:

$$\begin{array}{rl}\min: & x_1^2 - 6x_1 + x_2^2 - 4x_2 \\ \text{s.t.}: & x_1^2 - 2x_1 + x_2^2 \le 0 \\ & \frac{1}{2}x_1^2 - 2x_1 - x_2 + 2 \le 0 \\ & x_1 - x_2 \le 0\end{array}$$

(a) Plot the constraints and shade the feasible region.

(b) Verify that the point $\boldsymbol{x} = (1,0)^\top$ is an optimal solution, using the KKT equations. We did part of this in class. We did not show that complementary slackness holds.

(c) Compute the other two corner points (the intersection of constraints 1 and 2, and then the intersection of constraints 1 and 3). Verify, using KKT, that neither of these is an optimal solution.

(d) We have thus checked that of the three corner points, one is an optimal solution, and the other two are not. There are still points in the interior, that could potentially be optimal solutions. Use the KKT conditions to show that this cannot be the case, i.e., for any interior point $\tilde{\boldsymbol{x}}$, that is, any point for which the three constraints are *strictly* satisfied, $\tilde{\boldsymbol{x}}$ is not an optimal solution.

6. Consider the optimization problem:

$$\begin{aligned} \min: \quad & \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x} + \boldsymbol{q}^\top \boldsymbol{x} + c_0 \\ \text{s.t.}: \quad & \boldsymbol{x} \in [-1,1]^3, \end{aligned}$$

where the data of the problem are given as:

$$Q = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} -22 \\ -14.5 \\ 13 \end{pmatrix}, \qquad c_0 = -1.$$

(a) Is this a convex optimization problem? Explain your answer precisely.

(b) Show that $\hat{\boldsymbol{x}}^\top = (1, 0.5, -1)$ is an optimal solution.

(c) Are there other optimal solutions?