The University of Texas at Austin

Optimization

Homework 3

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1 Problem 1: Computing the Dual of a Linear Program

1.1 Question

Consider the following LP. Compute its dual.

$$
\begin{aligned}
\text{min :} \quad & x_1 - x_2 \\
\text{s.t. :} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \leq 0 \\
& x_2, x_3 \geq 0.
\end{aligned}
$$

1.2 Solution

To compute the dual of the given linear program (LP), we need to follow systematic steps to transform the primal LP into its dual form.

1.2.1 Step 1: Convert Primal LP to Standard Form

First, we need to express the primal LP in a standard form where all variables are non-negative, and all constraints are equalities.

- For $x_1 \leq 0$, let $x_1' = -x_1$ so that $x_1' \geq 0$ and $x_1 = -x_1'$
- x_4 is unrestricted, so let $x_4 = x_4' - x_4''$ where $x_4', x_4'' \geq 0$
- Convert inequalities to equalities by introducing slack variables $s_1 \geq 0$ and $s_2 \geq 0$

The standard form of the primal LP becomes:

$$\begin{aligned}
\text{Minimize:} \quad & -x_1' - x_2 \\
\text{Subject to:} \quad & -2x_1' + 3x_2 - x_3 + (x_4' - x_4'') + s_1 = 0 \\
& -3x_1' + x_2 + 4x_3 - 2(x_4' - x_4'') - s_2 = 3 \\
& x_1' - x_2 + 2x_3 + (x_4' - x_4'') = 6 \\
& x_1', x_2, x_3, x_4', x_4'', s_1, s_2 \geq 0
\end{aligned}$$

1.2.2 Step 2: Set Up the Dual LP

For each constraint in the primal, we introduce a dual variable:

- y_1 for the first constraint
- y_2 for the second constraint
- y_3 for the third constraint

1.2.3 Step 3: Formulate the Dual Objective Function

The dual objective function is formed from the right-hand side of the primal constraints:

$$\text{Maximize:} \quad 0y_1 + 3y_2 + 6y_3$$

1.2.4 Step 4: Formulate the Dual Constraints

For each variable in the primal, we create a constraint in the dual:

$$\begin{aligned}
x_1' : \quad & -2y_1 - 3y_2 + y_3 \leq -1 \\
x_2 : \quad & 3y_1 + y_2 - y_3 \leq -1 \\
x_3 : \quad & -y_1 + 4y_2 + 2y_3 \leq 0 \\
x_4' : \quad & y_1 - 2y_2 + y_3 \leq 0 \\
x_4'' : \quad & -y_1 + 2y_2 - y_3 \leq 0 \\
s_1 : \quad & y_1 \leq 0 \\
s_2 : \quad & -y_2 \leq 0
\end{aligned}$$

1.2.5 Step 5: Simplify the Dual LP

Combining the constraints for x_4' and x_4'', we get:

$$y_1 - 2y_2 + y_3 = 0$$

The final dual LP is:

$$\begin{aligned} \text{Maximize:} \quad & 3y_2 + 6y_3 \\ \text{Subject to:} \quad & 2y_1 + 3y_2 - y_3 \geq 1 \\ & 3y_1 + y_2 - y_3 \leq -1 \\ & -y_1 + 4y_2 + 2y_3 \leq 0 \\ & y_1 - 2y_2 + y_3 = 0 \\ & y_1 \leq 0, \quad y_2 \geq 0, \quad y_3 \text{ unrestricted} \end{aligned}$$

This is the dual of the given linear program.

2 Problem 2: Matrix Equivalence Statements

2.1 Question

For a given matrix A, show that the following two statements are equivalent:

(i) $Ax \geq 0$ and $x \geq 0$ implies that $x_1 = 0$.

(ii) There exists some vector $p > 0$ such that $p^\top A \leq 0$, and $p^\top A_1 < 0$ (strict inequality), where A_1 denotes the first column of A.

2.2 Solution

Proof of Equivalence:

To show that statements (i) and (ii) are equivalent, we will prove two implications:

1. (i) $\Rightarrow$ (ii): Assuming (i) holds, we will show that (ii) must also hold.
2. (ii) $\Rightarrow$ (i): Assuming (ii) holds, we will demonstrate that (i) must be true.

Proof of (i) $\Rightarrow$ (ii):

Assumption (i): For all $x \geq 0$, if $Ax \geq 0$, then $x_1 = 0$.

Goal: Show that there exists a vector $p > 0$ such that $p^\top A \leq 0$ and $p^\top A_1 < 0$.

Proof:

1. **Define the Set S:** Let $S = \{x \in \mathbb{R}^n \mid x \geq 0, \, x_1 = 1, \, Ax \geq 0\}$.

   From assumption (i), there is no $x \geq 0$ with $x_1 = 1$ satisfying $Ax \geq 0$. Therefore, S is empty.

2. **Apply the Separating Hyperplane Theorem:** Since S is an empty convex set, there exists a non-zero vector $p \in \mathbb{R}^n$ such that:

$$p^\top x \leq 0 \quad \text{for all } x \geq 0, \, x_1 = 1, \, Ax \geq 0.$$

3. **Characterize p:**
   - **Positivity of p:** Since $x \geq 0$ and $x_1 = 1$, p must satisfy $p_i \geq 0$ for $i \geq 2$, and $p_1 \leq 0$.
   - **Non-Zero p:** p cannot be the zero vector because it must separate S from the origin.
   - **Strict Inequality for p_1:** p_1 must be strictly negative; otherwise, $p^\top x$ could be positive for some $x \in S$.

4. **Derive $p^\top A \leq 0$:** For any $x \geq 0$ with $Ax \geq 0$, we have $p^\top x \leq 0$. This implies $p^\top A \leq 0$.

5. **Show $p^\top A_1 < 0$:** Since $p_1 < 0$ and all other components of p are non-negative, $p^\top A_1 < 0$.

Proof of (ii) $\Rightarrow$ (i):

Assumption (ii): There exists $p > 0$ such that $p^\top A \leq 0$ and $p^\top A_1 < 0$.

Goal: Show that for all $x \geq 0$, if $Ax \geq 0$, then $x_1 = 0$.

Proof:

Suppose there exists $x \geq 0$ with $x_1 > 0$ and $Ax \geq 0$. Then:

$$p^\top Ax \geq 0 \quad \text{(since both } p \text{ and } Ax \text{ are non-negative)}$$
$$p^\top Ax = (p^\top A)x \leq 0 \quad \text{(since } p^\top A \leq 0\text{)}$$

This implies $p^\top Ax = 0$. However, we can write:

$$p^\top Ax = p^\top A_1 x_1 + p^\top A_{2:n} x_{2:n}$$

Since $p^\top A_1 < 0$, $x_1 > 0$, and $p^\top A_{2:n} x_{2:n} \geq 0$, we have:

$$p^\top Ax < 0$$

This contradicts our earlier conclusion that $p^\top Ax = 0$. Therefore, our assumption must be false, and x_1 must be zero whenever $x \geq 0$ and $Ax \geq 0$.

Conclusion: We have shown that (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i). Therefore, statements (i) and (ii) are equivalent.

3 Problem 3: Infeasibility in Linear Systems

3.1 Question

Consider a set of 500 equations in 100 variables:

$$Ax \leq b,$$

given by

$$a_i^\top x \leq b_i.$$

3.2 Solution

To prove this, we will use linear programming duality and exploit the properties of the dual problem to demonstrate the existence of such a subset.

Step 1: Formulate an Auxiliary Linear Program

Since the original system $Ax \leq b$ is infeasible, let's consider an auxiliary linear program that seeks the minimal amount by which we need to relax the constraints to make the system feasible.

Primal Problem:

$$\begin{aligned} \text{Minimize} \quad & t \\ \text{Subject to} \quad & Ax \leq b + t\mathbf{1} \\ & t \geq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

Here, t represents the minimal slack we need to add to the right-hand side to achieve feasibility. Since $Ax \leq b$ is infeasible, the optimal value $t^* > 0$.

Step 2: Derive the Dual Problem

To utilize duality, we formulate the dual of the above primal problem.

Dual Problem:

Let $y \in \mathbb{R}^m$ be the dual variables corresponding to the inequalities $Ax \leq b + t\mathbf{1}$.

$$\begin{aligned} \text{Maximize} \quad & b^\top y \\ \text{Subject to} \quad & A^\top y = 0 \\ & \mathbf{1}^\top y = 1 \\ & y \geq 0 \end{aligned}$$

Explanation:

- The constraint $A^\top y = 0$ comes from the fact that the variables x in the primal problem are unrestricted in sign.
- The constraint $\mathbf{1}^\top y = 1$ corresponds to the slack variable t in the primal problem.
- The dual objective function $b^\top y$ reflects the primal objective t.

Step 3: Analyze the Dual Problem

Since the primal problem has an optimal value $t^* > 0$, by strong duality, the dual problem also has an optimal value $b^\top y^* = t^* > 0$.

The constraints $A^\top y^* = 0$ and $y^* \geq 0$ imply that y^* lies in the null space of $A^\top$ and is a non-negative vector. Additionally, $\mathbf{1}^\top y^* = 1$ means that the sum of the components of y^* is 1.

Since $A^\top y^* = 0$, the positive components of y^* correspond to a set of at most $n + 1 = 101$ inequalities (due to the rank-nullity theorem and the fact that the null space can be spanned by $m - \text{rank}(A^\top)$ vectors). Let's denote this set as I.

Assume, for contradiction, that the inequalities corresponding to I are feasible. Then there exists an x such that $a_i^\top x \leq b_i$ for all $i \in I$. Multiplying each inequality by $y_i^* > 0$ and summing, we get:

$$\sum_{i \in I} y_i^* a_i^\top x \leq \sum_{i \in I} y_i^* b_i$$

But since $A^\top y^* = 0$, the left side equals zero, leading to:

$$0 \leq b^\top y^* = t^* > 0$$

This is a contradiction. Therefore, the subset I of at most 101 inequalities is itself infeasible.

Conclusion: By using duality and complementary slackness, we have shown that there exists a subset of at most 101 inequalities from the original set that is infeasible.