The University of Texas at Austin
**Optimization**

Homework 7

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**Submitting solutions:** *Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.*

1. **Strong Convexity and Smoothness**. Consider the quadratic function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top Q\boldsymbol{x} + \boldsymbol{q}^\top \boldsymbol{x} + c,$$

where $Q, \boldsymbol{q}$, and $c$ are given as in a previous problem set:

$$Q = \begin{bmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{bmatrix}, \qquad \boldsymbol{q} = \begin{pmatrix} -22 \\ -14.5 \\ 13 \end{pmatrix}, \qquad c = -1.$$

   (a) Is this function smooth? If so, give the smoothness parameter, otherwise report "$\infty$".

   (b) Is this function strongly convex? If so, report the strong convexity parameter, otherwise report "0".

2. **Gradient Descent and Line Search.** For the function $f(\boldsymbol{x})$ given in the previous problem above, implement gradient descent and plot the suboptimality vs iteration for: (a) gradient descent implemented using a fixed step size computed as suggested in the lectures (i.e., as function of the smoothness parameter you computed above), and (b) gradient descent using line search, as explained in the lectures.

3. **Condition Number**. We saw in class that a fixed step size is able to guarantee linear convergence. The choice of step size we gave in class, however, depended on the function $f$. Show that it is not possible to choose a fixed step size $t$, that gives convergence for any strongly convex function. That is, for any fixed step size $t$, show that there exists (by finding one!) a smooth (twice continuously-differentiable) strongly convex function with bounded Hessian, such that a fixed-stepsize gradient algorithm starting from some point $x_0$, does not converge to the optimal solution.

4. **Convex functions**

   (a) If $f_i$ are convex functions, show that $f(x) := \sup_i f_i(x)$ is also convex.

   (b) Show that the largest eigenvalue of a symmetric matrix is a convex function of the matrix (i.e. $\lambda_{\max}(M)$ is a convex function of $M$). Is the same true for the eigenvalue of largest magnitude ?

   (c) Consider a weighted graph with edge weight vector $w$. Fix two nodes $a$ and $b$. The *weighted shortest path* from $a$ to $b$ is the path whose sum of edge weights is the minimum, among all paths with one endpoint at $a$ and another at $b$. Let $f(w)$ be the weight of this path. Show that $f$ is a concave function of $w$.