The University of Texas at Austin
Optimization

Homework 1

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Submitting solutions: *Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.*

1. Convex Sets, Convex Functions, Preservation of Convexity

   (a) Show that the intersection of convex sets is convex. Thus, show that for any convex sets C_1 and C_2, the intersection $C = C_1 \cap C_2$ is also convex. Do this using the definition of convexity, i.e., show that if $\boldsymbol{x}, \boldsymbol{y} \in C$, then for any $\lambda \in [0, 1]$, the point $\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \in C$.

   (b) Give an example where the union of two convex sets is not convex.

   (c) Show that the maximum of convex functions is convex. Note that given any two function, $f_1 : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$, their max is defined pointwise:

$$f_{\max}(\boldsymbol{x}) = \max\{f_1(\boldsymbol{x}), f_2(\boldsymbol{x})\}.$$

2. More Convex Sets, Convex Functions, Preservation of Convexity

   (a) Give an example where the minimum of two convex functions (defined analogously to the max of two functions, as explained above) is not convex.

   (b) (Boyd and Vandenberghe, Exercise 2.23). Give an example of two closed convex sets that are disjoint but cannot be strictly separated (read the chapter for the relevant definitions).

   (c) The sub-level sets of a function $f(\boldsymbol{x})$ are the sets of the form:

$$L_c = \{\boldsymbol{x} \,:\, f(\boldsymbol{x}) \leq c\},$$

   for different constants $c \in \mathbb{R}$ (as defined in Lecture 2). Show that any sub-level set of a convex function is also convex. Also, give an example to show the converse is NOT true, i.e., give an example of a function that is not convex, but all of its sub-level sets are convex. Hint: you don't have to get too complicated: think of a simple 1-variable (univariate) function, hence, one whose graph you can draw, so that you can visually assess convexity of the sub-level sets, and lack of convexity of the function.

3. Consider two points, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$. In this exercise you will show that there exists a vector $\boldsymbol{c} \in \mathbb{R}^n$ and a scalar $d \in \mathbb{R}$ such that

$$\{\boldsymbol{x} \,:\, ||\boldsymbol{x} - \boldsymbol{v}_1||_2 \leq ||\boldsymbol{x} - \boldsymbol{v}_2||_2\} = \{\boldsymbol{x} \,:\, \boldsymbol{c}^\top \boldsymbol{x} \leq d\}.$$

- Two dimensions: consider the above problem in two dimensions, where $v_1 = (-1, 0)^\top$, and $v_2 = (1, 0)^\top$. Draw the set of points that are closer to v_1 than v_2, by shading the plane. This should be a half-space!
- For the above example, find the two-dimensional vector $c = (c_1, c_2)^\top$ and the scalar d that describes the shaded region from the above. So in other words, the shaded region you drew must correspond do:

$$\{(x_1, x_2)^\top \,:\, c_1 x_1 + c_2 x_2 \leq d\}.$$

- Finally, generalize the above. For general points $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^n$, find a vector $\boldsymbol{c} \in \mathbb{R}^n$ and a scalar $d \in \mathbb{R}$ such that

$$\{\boldsymbol{x} \,:\, ||\boldsymbol{x} - \boldsymbol{v}_1||_2 \leq ||\boldsymbol{x} - \boldsymbol{v}_2||_2\} = \{\boldsymbol{x} \,:\, \boldsymbol{c}^\top \boldsymbol{x} \leq d\}.$$

  Thus, you are showing that the set of points in $\mathbb{R}^n$ that are closer to point v_1 than to point v_2, form a half-space.

4. Prove that the set $\{\boldsymbol{x} : ||A\boldsymbol{x} + \boldsymbol{b}||_2 \leq \boldsymbol{c}^T \boldsymbol{x} + d\}$ is a convex set – **Optional**

5. (Boyd and Vandenberghe, Exercise 2.6) – **Optional**

   (a) When does one halfspace contain another? Give conditions under which

$$\{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} \leq b\} \subseteq \{\boldsymbol{x} | \tilde{\boldsymbol{a}}^T \boldsymbol{x} \leq \tilde{b}\}$$

   (where $\boldsymbol{a} \neq 0, \tilde{\boldsymbol{a}} \neq 0$). Also find the conditions under which the two halfspaces are equal.

   (b) What is the distance between the two parallel hyperplanes $\{\boldsymbol{x} \in R^n | \boldsymbol{a}^T \boldsymbol{x} = b_1\}$ and $\{\boldsymbol{x} \in R^n | \boldsymbol{a}^T \boldsymbol{x} = b_2\}$?