The University of Texas at Austin

Optimization

Homework 1

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Submitting solutions: Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.

1 Convex Sets, Convex Functions, Preservation of Convexity

1. (a) **Answer:**
   To show that the intersection of convex sets is convex, let $C = C_1 \cap C_2$ where C_1 and C_2 are convex sets. Let $\mathbf{x}, \mathbf{y} \in C$. This means $\mathbf{x}, \mathbf{y} \in C_1$ and $\mathbf{x}, \mathbf{y} \in C_2$.
   For any $\lambda \in [0, 1]$, consider the point $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}$.
   Since C_1 is convex, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C_1$.
   Since C_2 is convex, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C_2$.
   Therefore, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in C_1 \cap C_2 = C$.
   This proves that C is convex, and thus the intersection of convex sets is convex.

   (b) **Answer:**
   An example where the union of two convex sets is not convex:
   Consider two disjoint circles in $\mathbb{R}^2$. Each circle is a convex set, but their union is not convex because a line segment connecting a point in one circle to a point in the other circle would not be entirely contained within the union.

   (c) **Answer:**
   To show that the maximum of convex functions is convex, let f_1 and f_2 be convex functions, and $f_{\max}(x) = \max\{f_1(x), f_2(x)\}$.
   For any $\mathbf{x}, \mathbf{y}$ and $\lambda \in [0, 1]$:
   $f_{\max}(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}) = \max\{f_1(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y}), f_2(\lambda\mathbf{x} + (1 - \lambda)\mathbf{y})\}$
   $\leq \max\{\lambda f_1(\mathbf{x}) + (1 - \lambda)f_1(\mathbf{y}), \lambda f_2(\mathbf{x}) + (1 - \lambda)f_2(\mathbf{y})\}$
   $\leq \lambda \max\{f_1(\mathbf{x}), f_2(\mathbf{x})\} + (1 - \lambda) \max\{f_1(\mathbf{y}), f_2(\mathbf{y})\}$
   $= \lambda f_{\max}(\mathbf{x}) + (1 - \lambda)f_{\max}(\mathbf{y})$
   This proves that $f_{\max}$ satisfies the definition of convexity, and thus the maximum of convex functions is convex.

2 More Convex Sets, Convex Functions, Preservation of Convexity

(a) i. **Answer:**
   An example where the minimum of two convex functions is not convex:
   Consider $f_1(x) = x^2$ and $f_2(x) = (x - 2)^2$. Both are convex functions. Let $f_{\min}(x) = \min\{f_1(x), f_2(x)\}$.
   $f_{\min}(x)$ is not convex because it forms a "V" shape with a non-convex kink at $x = 1$, where the two parabolas intersect.

   ii. **Answer:**
   An example of two closed convex sets that are disjoint but cannot be strictly separated:
   Consider in $\mathbb{R}^2$: $C_1 = \{(x, y) : y \geq e^x\}$ $C_2 = \{(x, y) : y \leq 0\}$

These sets are closed, convex, and disjoint. However, they cannot be strictly separated because for any separating hyperplane (in this case, a line), there will always be points from both sets arbitrarily close to the hyperplane as $x \to -\infty$.

iii. **Answer:**
To show that any sub-level set of a convex function is convex:
Let f be a convex function and $L_c = \{x : f(x) \leq c\}$ be a sub-level set. Take any $x_1, x_2 \in L_c$ and $\lambda \in [0, 1]$.
$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$ (by convexity of f) $\leq \lambda c + (1-\lambda)c = c$
Therefore, $\lambda x_1 + (1-\lambda)x_2 \in L_c$, proving L_c is convex.
Example of a non-convex function with convex sub-level sets:
Consider $f(x) = -x^2$. This function is concave (thus not convex), but all its sub-level sets are convex intervals. For any c, $L_c = \{x : -x^2 \leq c\} = [-\sqrt{-c}, \sqrt{-c}]$, which is a convex set.

3 Half-Space Representation of Points Closer to v1 than v2

1. Two dimensions:

   Answer:

   For $v_1 = (-1, 0)^T$ and $v_2 = (1, 0)^T$, the set of points closer to v_1 than v_2 forms a half-space. This half-space is the left half of the plane, separated by the vertical line x = 0. The shaded region would be all points with x ¡ 0.

2. Finding c and d for the two-dimensional case:

   Answer:

   For the given example, we need to find $c = (c_1, c_2)^T$ and d such that:

   $\{(x_1, x_2)^T : c_1x_1 + c_2x_2 \leq d\}$

   represents the shaded region (x ¡ 0).

   We can choose: $c = (1, 0)^T$ and $d = 0$

   This gives us the inequality: $x_1 \leq 0$, which correctly describes the left half-plane.

3. Generalization to n dimensions:

   Answer:

   For general points $v_1, v_2 \in \mathbb{R}^n$, we can find $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ as follows:

   $c = v_2 - v_1$ $d = \frac{1}{2}(||v_2||^2 - ||v_1||^2)$

   To prove this, we start with the condition $||x - v_1||_2 \leq ||x - v_2||_2$:

   $(x - v_1)^T(x - v_1) \leq (x - v_2)^T(x - v_2)$

   Expanding and simplifying:

   $x^Tx - 2v_1^Tx + v_1^Tv_1 \leq x^Tx - 2v_2^Tx + v_2^Tv_2$

   $-2v_1^Tx + v_1^Tv_1 \leq -2v_2^Tx + v_2^Tv_2$

   $2(v_2 - v_1)^Tx \leq v_2^Tv_2 - v_1^Tv_1$

   $(v_2 - v_1)^Tx \leq \frac{1}{2}(v_2^Tv_2 - v_1^Tv_1)$

   This is equivalent to $c^Tx \leq d$ with c and d as defined above.

   Thus, we have shown that the set of points in $\mathbb{R}^n$ that are closer to point v_1 than to point v_2 indeed form a half-space, represented by $\{x : c^Tx \leq d\}$.