The University of Texas at Austin
Optimization

HOMEWORK 3

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Submitting solutions: *Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.*

This problem set gives us practice with computing the dual, and with theorems of the alternative. Duality theory is one of the most powerful aspects of convex optimization and it is useful for many applied and theoretical purposes. I consider problem 3 one of the more conceptually challenging problems you will see in this course. It illustrates (depending on how you solve it) the power of duality and complementary slackness, as the hint suggests. It's a deep statement about geometry of linear inequalities, and I encourage you to imagine applications of this (or a similar) result.

1. Consider the following LP. Compute its dual.

$$\begin{aligned}
\min: \quad & x_1 - x_2 \\
\text{s.t.}: \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \leq 0 \\
& x_2, x_3 \geq 0.
\end{aligned}$$

2. For a given matrix A, show that the following two statements are equivalent:

   (i) $A\boldsymbol{x} \geq 0$ and $\boldsymbol{x} \geq 0$ implies that $x_1 = 0$.

   (ii) There exists some vector $\boldsymbol{p} \geq 0$ such that $\boldsymbol{p}^\top A \leq 0$, and $\boldsymbol{p}^\top A_1 < 0$ (strict inequality), where A_1 denotes the first column of A.

3. Consider a set of 500 equations in 100 variables:

$$A\boldsymbol{x} \leq \boldsymbol{b},$$

   given by

$$\boldsymbol{a}_i^\top \boldsymbol{x} \leq b_i.$$

   Suppose that these are not feasible, i.e., there is no solution that satisfies all of them. Show that in fact, there must be a subset of at most 101 which are not feasible. (Hint: Use duality and complementary slackness).