The University of Texas at Austin
Optimization

HOMEWORK 4

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Submitting solutions: *Please submit your solutions as a single pdf file. If you have code or figures, please include these in the pdf.*

1. **(Unit eigenvectors of stochastic matrices)** One of the most powerful modeling ideas in stochastics and probability, used in Machine Learning, Reinforcement Learning, and many many other fields of mathematics, engineering and applied science, is the concept of the Markov Chain: this is a stochastic process whose state makes past and future independent. In other words, if you want to know what will happen tomorrow (and what you should do today), you only need to look at where you are, and not how you got there.

   A foundational fact that is extremely useful in the study of Markov Chains, is the fact that every finite state Markov chain has an invariant probability distribution. It is beyond the scope of this class (and certainly this homework!) to discuss this any further; however, as you will see, we have already developed enough tools to actually prove this, using optimization and duality.

   We say that an $n \times n$ matrix P, with entries p_{ij}, is stochastic if all of its entries are nonnegative and
   $$\sum_{j=1}^{n} p_{ij} = 1, \qquad \forall i,$$
   that is, the sum of the entries of each row is equal to 1.

   Use duality to show that if P is a stochastic matrix, then the system of equations,
   $$P^T \lambda = \lambda, \qquad \lambda \geq 0,$$
   has a nonzero solution. (Note that the vector λ can be normalized so that its components sum to one – this normalized vector is the invariant probability distribution.)

2. **(Duality in piecewise linear convex optimization)** Consider the problem of minimizing $\max_{i=1,\dots,m}(a_i^T x - b_i)$ over all $x \in \mathbb{R}^n$. Let v be the value of the optimal cost, assumed finite. Let A be the matrix with rows $a_1, \dots, a_m$, and let b be the vector with components $b_1, \dots, b_m$.

   (a) Consider any vector $p \in \mathbb{R}^m$ that satisfies $A^T p = 0$, $p \geq 0$, and $\sum_{i=1}^{m} p_i = 1$. Show that $-p^T b \leq v$.

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{array}{ll} \text{maximize} & -p^T b \\ \text{subject to} & A^T p = 0 \\ & p^T e = 1 \\ & p \geq 0, \end{array}$$

where e is the vector with all components equal to 1. Show that the optimal cost in this problem is equal to v.

3. Let A be a symmetric square matrix. Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq c \\ & x \geq 0. \end{array}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.

4. (**Robust Linear Programming**). Consider the following problem:

$$\begin{array}{ll} \min: & c_1 x_1 + c_2 x_2 + c_3 x_3 \\ \text{s.t.}: & A\boldsymbol{x} \geq \boldsymbol{b}, \quad \forall \boldsymbol{b} \in \mathcal{U}, \end{array}$$

where the uncertainty set $\mathcal{U}$ is the unit box in m dimensions, i.e.,

$$\mathcal{U} = \{\boldsymbol{b} \,:\, 0 \leq b_i \leq 1, i = 1, ..., m, \sum_{i=1}^{m} b_i \leq 1\}.$$

Show that this is equivalent to the LP

$$\begin{array}{ll} \min: & c_1 x_1 + c_2 x_2 + c_3 x_3 \\ \text{s.t.}: & A\boldsymbol{x} \geq \hat{\boldsymbol{b}}, \end{array}$$

where

$$\hat{\boldsymbol{b}} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

5. (**Robust Linear Programming – Optional**). Consider the following problem:

$$\begin{array}{ll} \min: & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.}: & A^\top \boldsymbol{x} \leq \boldsymbol{b}, \quad \forall \boldsymbol{b} \in \mathcal{U}_1 \\ & \boldsymbol{a}_2^\top \boldsymbol{x} \leq \boldsymbol{b}_2, \quad \forall \boldsymbol{a}_2 \in \mathcal{U}_2 \\ & \boldsymbol{a}_3^\top \boldsymbol{x} \leq \boldsymbol{b}_3, \quad \forall \boldsymbol{a}_3 \in \mathcal{U}_3 \end{array}$$

where A is a $m \times n$ matrix, and hence the first robust constraint is in fact m robust constraints, and the uncertainty sets are given by: $\mathcal{U}_1$ is a general polytope, and

$$\mathcal{U}_2 = \{\boldsymbol{a} \,:\, D_2\boldsymbol{a} = \boldsymbol{d}_2,\, \boldsymbol{a} \geq 0\},$$

and

$$\mathcal{U}_2 = \{\boldsymbol{a} \,:\, D_3\boldsymbol{a} \geq \boldsymbol{d}_3,\, \boldsymbol{a} \leq 0\}.$$

Your solution needs to be defined in terms of finite constraints and finite variables. Some of the parameters in your problem may themselves be a solution to another optimization problem.