

Universal quantum computation with a nonlinear oscillator network

Hayato Goto

Frontier Research Laboratory, Corporate Research & Development Center,
Toshiba Corporation, 1, Komukai Toshiba-cho, Saiwai-ku, Kawasaki-shi, 212-8582, Japan

It has recently been shown that a parametrically driven oscillator with Kerr nonlinearity yields a Schrödinger cat state via quantum adiabatic evolution through its bifurcation point and a network of such nonlinear oscillators can be used for solving combinatorial optimization problems by bifurcation-based adiabatic quantum computation [H. Goto, Sci. Rep. **6**, 21686 (2016)]. Here we theoretically show that such a nonlinear oscillator network with controllable parameters can also be used for universal quantum computation. The initialization is achieved by a quantum-mechanical bifurcation based on quantum adiabatic evolution, which yields a Schrödinger cat state. All the elementary quantum gates are also achieved by quantum adiabatic evolution, in which dynamical phases accompanying the adiabatic evolutions are controlled by the system parameters. Numerical simulation results indicate that high gate fidelities can be achieved, where no dissipation is assumed.

PACS numbers: 03.67.Lx, 05.45.-a, 42.50.-p, 42.65.-k

Introduction. The standard model for quantum computation consists of quantum bits (qubits) and quantum gates [1] as present-day digital computers consist of bits and logic gates. A qubit is often represented by two discrete quantum states of various physical systems, such as electron or nuclear spins in neutral atoms, ions, molecules, or solids, polarization states or optical modes of single photons, and superconducting artificial atoms with Josephson junctions [2]. Another kind of implementation of a qubit uses a harmonic oscillator, which is described by an infinite-dimensional Hilbert space. In this case, the two computational basis states are defined as two orthogonal states of a harmonic oscillator, such as two coherent states with largely different amplitudes or two cat states with opposite parity [3]. It is known that a universal gate set for such coherent-state qubits can be achieved by gate teleportation with cat states [4, 5]. Recently, universal quantum computation with harmonic oscillators accompanied by nonlinear losses has also been proposed [6, 7]. Recent advances in circuit quantum electrodynamics with superconducting devices [8, 9] make the proposals promising.

More recently, it has been shown that a parametrically driven oscillator with Kerr nonlinearity (hereafter KPO) can yield a cat state via its quantum-mechanical bifurcation based on quantum adiabatic evolution, and a network of such nonlinear oscillators can be used for adiabatic quantum computation to find the ground states of the Ising model [10]. In the bifurcation-based adiabatic quantum computation, two coherent states, $|\alpha\rangle$ and $|\alpha^*\rangle$, corresponding to two stable branches of the KPO are regarded as up and down states of an Ising spin. It may be natural to expect that the two coherent states of a KPO can also be utilized as a qubit in the standard gate-based model of quantum computation. However, it has not been obvious so far whether a universal gate set can be achieved for such qubits with KPOs. (Note that no quantum gates are used in the bifurcation-based adiabatic quantum computation, in which the necessary operation is only to increase pump amplitudes monoton-

ically. It should also be noted that the proposals in Refs. [6, 7] cannot be directly applied to the KPOs because of the differences in their nonlinearities.)

In this Rapid Communication, we theoretically show that a universal gate set can be achieved for the above qubits with KPOs. Figure 1 shows a schematic of the proposed quantum computer. All the elementary gates are based on quantum adiabatic evolution, in which dynamical phases accompanying the adiabatic evolutions are controlled by system parameters. In the following, we first describe the definition of qubits in the present model. The physical implementation of the KPO is also mentioned. Next, we explain how to perform three kinds of elementary gates on the qubits. Numerical simulation results supporting the proposal are also provided. Finally, the conclusion is presented. Note that in the present work, we assume that there are no control errors and no decoherence sources such as losses. Such errors will be considered in a future work.

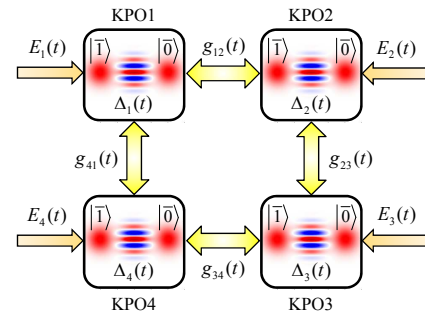


FIG. 1. Schematic of the proposed quantum computer. The R_z gate in Eq. (3) is performed by a driving field denoted by $E(t)$. The R_x gate in Eq. (4) is performed by controlling a detuning denoted by $\Delta(t)$. The two-qubit gate in Eq. (5) is performed by controlling a coupling coefficient denoted by $g(t)$. Each KPO is represented by the Wigner function of a cat state. The coherent states with positive and negative amplitudes correspond to the two computational basis states $|\bar{0}\rangle$ and $|\bar{1}\rangle$, respectively.

Definition of qubits. In the present model, we use a KPO for each qubit. Here we first describe the cat-state generation with a KPO via quantum adiabatic evolution, and next explain the definition of the qubit.

In a frame rotating at half the pump frequency of the parametric drive and in the rotating-wave approximation, the Hamiltonian for a KPO is given by

$$H_1 = \hbar\Delta a^\dagger a + \hbar\frac{K}{2}a^{\dagger 2}a^2 - \hbar\frac{p}{2}(a^{\dagger 2} + a^2), \quad (1)$$

where a and a^\dagger are the annihilation and creation operators for the KPO, Δ is the detuning of the eigenfrequency from half the pump frequency, K is the Kerr coefficient for the Kerr effect, and p is the pump amplitude for the parametric drive [10]. Hereafter, we assume that K is a positive constant and Δ is nonnegative. (When K is negative, similar discussion is straightforward.)

A cat state of the KPO is generated deterministically as follows. The KPO is initially prepared in the vacuum state $|0\rangle$. Then, p is increased sufficiently slowly from zero. Since $|0\rangle$ is the ground state for the initial Hamiltonian, the KPO adiabatically follows the instantaneous ground state of the Hamiltonian. When $\Delta = 0$, the ground state is doubly degenerate and the eigenspace is spanned by two coherent states $|\pm\sqrt{p/K}\rangle$. Since H_1 is symmetric under parity inversion $a \rightarrow -a$, the final state should have the same parity as the initial state $|0\rangle$. Thus, the final state is the even cat state [3] defined as

$$|C_+\rangle = \frac{|\sqrt{p/K}\rangle + |-\sqrt{p/K}\rangle}{\sqrt{2(1 + e^{-2p/K})}}. \quad (2)$$

When p is so large compared to K that $e^{-2p/K}$ is negligible, the two coherent states $|\pm\sqrt{p/K}\rangle$ are orthogonal to each other. Taking a value p_0 such that $e^{-2p_0/K}$ is negligible, we define the computational basis states, $|\bar{0}\rangle$ and $|\bar{1}\rangle$, of a qubit as two coherent states $|\sqrt{p_0/K}\rangle$ and $|- \sqrt{p_0/K}\rangle$, respectively, where we have used the bars to distinguish the computational basis states from the vacuum and single-photon states of the KPO. (In this Rapid Communication, the quanta of the KPO are called “photons” assuming that the KPO is implemented by an electromagnetic resonator.)

Note that the cat-state generation described above is regarded as the initialization of the qubit to $(|\bar{0}\rangle + |\bar{1}\rangle)/\sqrt{2}$, which is the standard initial state in quantum computation [1, 11, 12]. After this initialization, the pump amplitude is kept to p_0 during quantum computation. While no operation is performed, the state of each KPO is in the subspace spanned by $|\bar{0}\rangle$ and $|\bar{1}\rangle$.

Here we briefly address the physical implementation of the KPO. Since the time scale is limited by the Kerr coefficient K , a large K is desirable. (As shown below, the gate time is proportional to K^{-1} .) In particular, it is desirable that K should be larger than the loss rate of the KPO. This condition is extremely stringent for optical systems. On the other hand, superconducting

systems with Josephson junctions have already achieved this condition [13, 14]. With a superconducting circuit, parametric oscillation has also been demonstrated [15]. Thus, superconducting systems are promising for the implementation of the KPO. In this case, K is typically several tens of MHz.

Elementary quantum gates. Here we show that a universal gate set can be achieved for the qubit defined above. As a universal gate set, we choose two single-qubit gates $R_z(\phi)$ and $R_x(\theta)$, and a two-qubit gate $U(\Theta)$, where these unitary operators are defined as follows (X and Z denote Pauli operators) [1]:

$$R_z(\phi) = e^{-i\phi Z/2} = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}, \quad (3)$$

$$R_x(\theta) = e^{-i\theta X/2} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \quad (4)$$

$$U(\Theta) = e^{-i\Theta Z_1 Z_2/2} = \begin{pmatrix} e^{-i\Theta/2} & 0 & 0 & 0 \\ 0 & e^{i\Theta/2} & 0 & 0 \\ 0 & 0 & e^{i\Theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\Theta/2} \end{pmatrix}. \quad (5)$$

In the following, we explain how to perform these elementary gates in turn. [Note that $R_x(\pi/2)$ and $U(\pi/2)$ are sufficient for universality together with $R_z(\phi)$ [16].]

To perform $R_z(\phi)$ on a qubit, we drive the KPO by a driving field with a pulse-shaped amplitude $E(t)$. Then, the additional Hamiltonian is given by

$$H_z(t) = \hbar E(t)(a + a^\dagger). \quad (6)$$

Note that the additional Hamiltonian violates the parity symmetry, and consequently induces the transition between the even and odd cat states. This driving is also interpreted as a displacement in the phase space.

When $|E(t)|$ is sufficiently small and the variation of $E(t)$ is sufficiently slow, the KPO is approximately kept in the subspace spanned by $|\bar{0}\rangle$ and $|\bar{1}\rangle$. Here we should consider the energy shifts for $|\bar{0}\rangle$ and $|\bar{1}\rangle$, which are $2\hbar E(t)\sqrt{p_0/K}$ and $-2\hbar E(t)\sqrt{p_0/K}$, respectively. These energy shifts induce dynamical phase factors, and consequently $R_z(\phi)$ is performed, where ϕ is given by (T_g is the gate time)

$$\phi = 4\sqrt{p_0/K} \int_0^{T_g} E(t)dt. \quad (7)$$

To verify the above discussion, we did numerical simulations, in which we numerically solved the Schrödinger equation with the Hamiltonian $H_1 + H_z(t)$. In the simulations, the parameters are set as $p_0 = 4K$, $\Delta = 0$, and $T_g = 2/K$, and the Hilbert space is truncated at a photon number of 20. We performed $R_z(\phi)$ on the initial state $(|\bar{0}\rangle + |\bar{1}\rangle)/\sqrt{2}$ and calculated the fidelity between the output state in the simulation and the ideal output state $(e^{-i\phi/2}|\bar{0}\rangle + e^{i\phi/2}|\bar{1}\rangle)/\sqrt{2}$. (The fidelity is defined

as the square of the absolute value of the inner product of two state vectors.) To perform $R_z(\phi)$, $E(t)$ is set as

$$E(t) = \frac{\pi\phi}{8T_g\sqrt{p_0/K}} \sin \frac{\pi t}{T_g}. \quad (8)$$

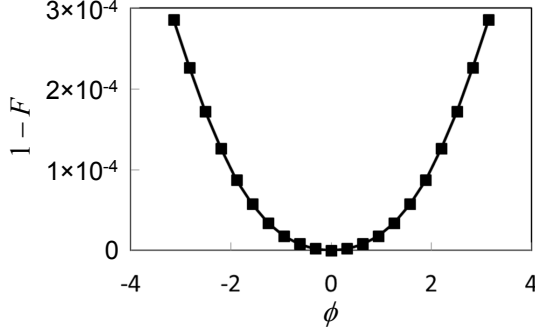


FIG. 2. Simulation result for $R_z(\phi)$. F denotes the fidelity between the output state in the simulation and the ideal output state $(e^{-i\phi/2}|\bar{0}\rangle + e^{i\phi/2}|\bar{1}\rangle)/\sqrt{2}$.

The simulation result is shown in Fig. 2. It is found that high fidelities are achieved for ϕ in the range of $-\pi$ to π , as expected. The fidelities become even higher for a longer gate time (a larger value of KT_g).

Next, we explain how to perform $R_x(\theta)$. To perform $R_x(\theta)$ on a qubit, we use the detuning Δ in Eq. (1). When Δ is slowly increased from zero to a value Δ_0 near to p_0 and then decreased to zero, the even and odd cat states, $(|\bar{0}\rangle \pm |\bar{1}\rangle)/\sqrt{2}$, obtain dynamical phase factors depending on their energy shifts due to the nonzero detuning. (Note that the even and odd states are simultaneous eigenstates of H_1 because of the parity symmetry of H_1 .) Thus, the qubit state changes as follows (θ is the relative phase between the even and odd cat states due to the dynamical phase factors):

$$\begin{aligned} \alpha_0|\bar{0}\rangle + \alpha_1|\bar{1}\rangle &= \frac{\alpha_0 + \alpha_1}{2}(|\bar{0}\rangle + |\bar{1}\rangle) + \frac{\alpha_0 - \alpha_1}{2}(|\bar{0}\rangle - |\bar{1}\rangle) \\ &\rightarrow \frac{\alpha_0 + \alpha_1}{2}(|\bar{0}\rangle + |\bar{1}\rangle) + \frac{\alpha_0 - \alpha_1}{2}e^{i\theta}(|\bar{0}\rangle - |\bar{1}\rangle) \\ &= e^{-i\theta/2} \left[\left(\alpha_0 \cos \frac{\theta}{2} - i\alpha_1 \sin \frac{\theta}{2} \right) |\bar{0}\rangle + \left(\alpha_1 \cos \frac{\theta}{2} - i\alpha_0 \sin \frac{\theta}{2} \right) |\bar{1}\rangle \right] \\ &= e^{-i\theta/2} R_x(\theta) (\alpha_0|\bar{0}\rangle + \alpha_1|\bar{1}\rangle). \end{aligned} \quad (9)$$

Thus, $R_x(\theta)$ is achieved by the detuning control. (The overall phase factor $e^{-i\theta/2}$ has no physical meaning and therefore can be ignored.)

Here we present numerical simulation results supporting the above discussion. We numerically solved the Schrödinger equation with H_1 in Eq. (1), where the detuning Δ is controlled as follows (T_g is the gate time):

$$\Delta(t) = \Delta_0 \sin^2 \frac{\pi t}{T_g}. \quad (10)$$

In the simulations, the parameters are set as $p_0 = 4K$ and $T_g = 10/K$, the initial state $|\psi_i\rangle$ is set to $(|\bar{0}\rangle + i|\bar{1}\rangle)/\sqrt{2}$, and the Hilbert space is truncated at a photon number of 20.

To estimate the rotation angle θ corresponding to Δ_0 , we calculated the fidelity between the output state in the simulation and $R_x(\theta)|\psi_i\rangle$, and found θ maximizing the fidelity. The results are summarized in Fig. 3, where F denotes the maximized fidelity. While Δ_0 changes from 0 to $2.5K$, the rotation angle θ changes from 0 to $-\pi$. The fidelities are also high. Thus, it has been shown that

$R_x(\theta)$ is achieved by controlling the detuning.

Finally, we explain how to perform the two-qubit gate $U(\Theta)$. To perform $U(\Theta)$ on two qubits, we control the coupling coefficient $g(t)$ between the two KPOs. (In contrast, in the bifurcation-based adiabatic quantum computation for the Ising problem [10], the coupling coefficients are set to constants depending on given problems.) The additional Hamiltonian is given by

$$H_U = \hbar g(t)(a_1 a_2^\dagger + a_1^\dagger a_2). \quad (11)$$

Note that this is the standard *linear* coupling, which describes photon exchange between two KPOs.

When $|g(t)|$ is sufficiently small and the variation of $g(t)$ is sufficiently slow, the KPOs are approximately kept in the subspace spanned by $|\bar{0}\rangle|\bar{0}\rangle$, $|\bar{0}\rangle|\bar{1}\rangle$, $|\bar{1}\rangle|\bar{0}\rangle$, and $|\bar{1}\rangle|\bar{1}\rangle$. Then, the energy shifts for $|\bar{0}\rangle|\bar{0}\rangle$ and $|\bar{1}\rangle|\bar{1}\rangle$ are $2\hbar g(t)p_0/K$ and those for $|\bar{0}\rangle|\bar{1}\rangle$ and $|\bar{1}\rangle|\bar{0}\rangle$ are $-2\hbar g(t)p_0/K$. These energy shifts induce dynamical phase factors, and consequently $U(\Theta)$ is performed,

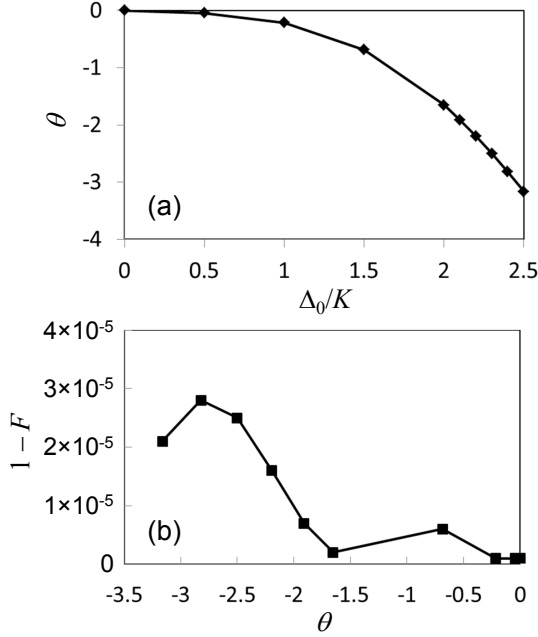


FIG. 3. Simulation result for $R_x(\theta)$. (a) θ maximizing the fidelity between the output state in the simulation and $R_x(\theta)|\psi_i\rangle$. (b) F denotes the fidelity between the output state in the simulation and $R_x(\theta)|\psi_i\rangle$.

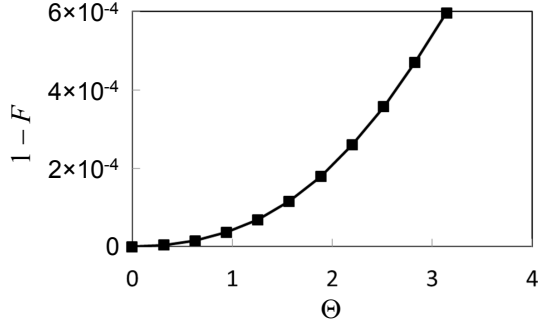


FIG. 4. Simulation result for the two-qubit gate $U(\Theta)$. F denotes the fidelity between the output state in the simulation and the ideal output state $U(\Theta)|\psi_i\rangle = (e^{-i\Theta/2}|\bar{0}\rangle|\bar{0}\rangle + e^{i\Theta/2}|\bar{0}\rangle|\bar{1}\rangle + e^{i\Theta/2}|\bar{1}\rangle|\bar{0}\rangle + e^{-i\Theta/2}|\bar{1}\rangle|\bar{1}\rangle)/2$.

where Θ is given by (T_g is the gate time)

$$\Theta = \frac{4p_0}{K} \int_0^{T_g} g(t) dt. \quad (12)$$

We did numerical simulations for the two-qubit gate, in which we numerically solved the Schrödinger equation. In the simulations, the parameters are set as $p_0 = 4K$, $\Delta = 0$, and $T_g = 2/K$, the initial state is set to $|\psi_i\rangle = (|\bar{0}\rangle + |\bar{1}\rangle)(|\bar{0}\rangle + |\bar{1}\rangle)/2$, and the Hilbert space is truncated at a photon number of 20 for each KPO. We performed $U(\Theta)$ on $|\psi_i\rangle$ and calculated the fidelity between the output state in the simulation and the ideal output state $U(\Theta)|\psi_i\rangle$. To perform $U(\Theta)$, $g(t)$ is set as

$$g(t) = \frac{\pi\Theta}{8T_g p_0/K} \sin \frac{\pi t}{T_g}. \quad (13)$$

The simulation result is shown in Fig. 4. It is found that high fidelities are achieved for Θ in the range of 0 to π , as expected. The fidelities become even higher for a longer gate time (a larger value of KT_g).

In conclusion, we have shown that the network of nonlinear oscillators called KPOs, which has been used for bifurcation-based adiabatic quantum computation, can also be used for universal quantum computation. The qubit is defined with two coherent states of a KPO. The initialization of the qubit is achieved by a quantum-mechanical bifurcation based on quantum adiabatic evolution. All the elementary gates are also performed by quantum adiabatic evolution, in which dynamical phases accompanying the adiabatic evolutions are controlled by driving field amplitudes, detunings, and coupling coefficients. The present scheme will open a new possibility for quantum nonlinear systems in quantum information science.

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plitudes are approximately orthogonal. And even and odd cat states are defined as $|\alpha\rangle + |-\alpha\rangle$ and $|\alpha\rangle - |-\alpha\rangle$, respectively, where the normalization factors have been omitted.

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- [16] $e^{i\pi/2}R_z(\pi/2)R_x(\pi/2)R_z(\pi/2)$ is equivalent to a Hadamard gate. $e^{-i\pi/4}U(\pi/2)R_z^{(1)}(-\pi/2)R_z^{(2)}(-\pi/2)$ is equivalent to a controlled- Z gate, where $R_z^{(1)}(\phi) = e^{-i\phi Z_1/2}$ and $R_z^{(2)}(\phi) = e^{-i\phi Z_2/2}$.