

The π -calculus

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I want to thank my hamsters, Boris Becker, and this bottle of Merlot.

A Word About Formatting

One of the pedagogical aims of this thesis is to be as clear and reader-assistive as possible during its presentation. To that end, I have taken several liberties in the formatting of this thesis. Figures, equations, definitions, theorems and examples all share one numbering scheme in hopes that it will make them easier to locate. Where *definitions* appear, they are clad in italics as well as in a margin box to make them easier to find. Equation numbers also appear in margin boxes. When definitions or equations are referred to later in the text, an assistive link will appear in the margin to avoid index-fingering. To this end, the wired reader will note the presence within the backcover of a searchable pdf version of the text, wherein all references are hyper-textual (clickable). The included CD also contains all the code referenced in this thesis.

definitions

definitions
↪ [page iii](#)

Abstract

In this thesis I will blow your mind, and possibly (hopefully?) my own in the process.

1

Introduction

Writing distributed systems are often held up as being some of the toughest problems that software engineers face today. Managing threads, locks and synchronization, avoiding deadlocks and livelocks and dealing with failure and latency all contribute to distributed programming nightmarish reputation. In the meantime, we steadily progress towards a future where all computation is distributed, where raw computing power is nothing more than a service that can be consumed like energy, and where huge systems sprawl across machines and networks to do their bidding.

Perhaps there is a better way. Many computer scientists are now reevaluating the tools they use – namely the languages – in hopes that there may be a way to cut the gordian knot by making very low level changes. The hope is that there is a breed of languages that offer a much more natural way to perform distributed computing, that do not have all the high-level nightmares that currently plague us. The aim of this thesis is to explore what such a language might look like, and perhaps offer some concrete features with which we might judge the overall value of a candidate language.

For several decades theoretical computer scientists have been studying a branch of math known as *processes algebras*. The hope is to discover an algebraic model for computation that might eventually be the foundation for a new wave of programming languages that greatly ease the difficulty of distributed programming by considering computation via its natural distributed unit - the process. A process can be a task of any size, but the important thing about it is that it shares no state with the processes in its environment. Instead, processes interact via message passing – sending data back-and-forth via named channels. The idea is that by modeling computation in such terms, we might discover natural ways to solve distributed problems and not have to worry about the many headaches that would otherwise arise at a very high level in a language whose primitives are not built for a distributed setting.

One of the most successful and influential process algebras to arise has been the π -calculus, first given by Robin Milner, Joachim Parrow and David Walker. It improved upon the process algebras that came before it by allowing processes a kind of mobility within their environments, which in turn allows for much more dynamic and powerful systems. We will give a formal presentation of the π -calculus, as a sort of starting point for our discussion of process algebra and the various languages they have inspired. In addition, the presentation will give us some valuable tools and vocabulary for evaluating these languages.

There has been a mushroom of process algebra that have sprung up around the π -

calculus, evidence of its fertility but also of the uncertainty of what the right system is. We will look at a few of these variations at a somewhat higher level than the π -calculus, and attempt to pull out some of the trends and major debates and clarify how they relate to the needs of distributed systems.

We will also look at a number of implementation of these languages, and use some example systems as a litmus test for how naturally they lend themselves to solving the problems.

2

The π -Calculus

In the 1930's, Alonzo Church and Stephen Kleene introduced the λ -calculus as an algebraic model for describing computation. In the λ -calculus, computation is the invocation of functions. In the 1980's, Robin Milner introduced his Calculus of Communicating Systems (CCS), in which computation is modeled as communication via message passing. Because these channels can be shared and used an arbitrary number of times, there is not a concrete naming system for a chunk of computation the way there is in a function. Instead, in the CCS any bit of functionality can be referred to as a *process*, with no specification of the granularity. Where we might think of a program in the λ -calculus as a bunch of functions invoking one another, in the the CCS we say that a *system* is composed of many communicating processes. It was this CCS that later became the basis for the π -calculus.

Although it can be defined in other ways, one of the ways of giving a process's *location* is in terms of the communication channels which can be used to access it. Since processes are the units populating the space of a CCS system, it is natural to think of a process's location in terms of the processes which are 'near' it - those it can communicate with via named channels. In this case, changing the channel topology of a system changes the locations of its component processes. In the CCS, channel topology is static, whereas the π -calculus allows communication channels to be dynamically established and relinquished between processes. This gives a kind of *mobility* of processes, which vastly expands the capabilities of interaction in a system.

In this chapter, we will give the syntax of the *asynchronous* π -calculus and a discussion of its features.¹ Following this, we will introduce a notion of equivalence of terms in the language. Finally, we will give some semantic reduction rules that provide computational behavior via steps of reduction.

2.1 Syntax

The terms of the π -calculus operate on a space of *identifiers* v, w, \dots which for the time being will consist of names n, m, \dots for communication channels, and variables x, y, \dots which, as we will see below, can refer to channels or to recursive process bodies.

identifiers

¹Several versions of the π -calculus have been proposed. We will loosely follow the recent presentation given in [Hen07].

Process terms		
$R, S, B :=$	$R_1 \mid R_2$	Composition
	$n!\langle V \rangle$	Send
	$n?(X).R$	Receive
	$\text{new}(n).R$	Restriction
	$\text{if } v_1 = v_2 \text{ then } R_1 \text{ else } R_2$	Matching
	$\text{rec } x.B$	Recursion
	stop	Termination
Systems		
$\mathbb{M}, \mathbb{N} :=$	$\text{new}(c_1, \dots, c_n).R_1 \mid \dots \mid R_m \quad n, m > 0$	

Figure 2.1.1: Terms in the asynchronous π -calculus

Given two or more processes, we can compose them using the \mid operator, which means that the composed processes will be executed concurrently.

We denote the sending of a tuple V over a channel n by $n!\langle V \rangle$. Here V is a tuple of identifiers in the form $V = (v_1, \dots, v_k) : k \geq 0$. We say that V has *arity* k . In the case $n = 0$ nothing is being transmitted; the communication acts as a *handshake* or signal. We will denote this case by $n!\langle \rangle$. It is a feature of the language that sending is not a blocking operation – that is, the process will not wait or depend on the value actually being received or operated on by another process, but will simply terminate after sending the value. Non-blocking sends make our π -calculus an *asynchronous* language - processes that send values do not wait to sync operation with the processes that consume the values. However, we will show in [Example 2.1.9](#) that synchronous behavior can still be modeled in our language.

The term $n?(X).R$ describes a process waiting to receive a tuple along k before continuing with R . Here X is a *pattern*, or a tuple of variables of arity k . Patterns allow us to decompose the transmitted tuple into its component values by naming them with x_1, \dots, x_k , which can be referred to in R . Thus, in the term

$$c!\langle n_1, n_2, n_3 \rangle \mid c?(x_1, x_2, x_3).R, \quad (2.1.2)$$

the names n_1, n_2, n_3 are received via c and bound via matching to x_1, x_2, x_3 (the use of x_1, x_2, x_3 is limited to being within the process R). Similarly to sending, the case of arity 0 is denoted $n?().R$ – here R will not happen until a handshake is received on n . Notice that in contrast to the case of sending, receiving is a *guarded* operation – that is, the process R will not execute until X has been received via n . For example, the term

$$c?().\text{stop} \quad (2.1.3)$$

represents a ‘listener’ process that simply consumes the value waiting on input channel c . The term *stop* describes a process that does nothing but halt.

The term $\text{new}(n).R$ describes a process in which a new channel n is created, limited to being expressed in the process R (we say the *scope* of n is *restricted* to R).

When creating multiple channel restrictions of the form $\text{new}(n).(\text{new}(m).R)$, we will abbreviate using $\text{new}(n, m).R$. It is important to note that n *can* be used outside of R if it is sent and then received by some outside process. This feature is known as *scope extrusion*, and is the underpinning for the dynamic communication topologies introduced in the π -calculus. For example, in the term

scope extrusion

$$\text{new}(n).(c!\langle n \rangle) \mid c?(a).a!\langle \rangle, \quad (2.1.4)$$

n 's scope is just the left half of the term. However, n is sent over c and is received (and bound as a) in the right hand of term. When this happens, the channel n will be able to be referred to outside of its initial scope. We will give a precise account of how this happens in our reduction rules and [Example 2.3.2](#) below.

Simple value-matching is available to provide conditional behavior. In the following term, the value received on c is checked; if it matches a then a handshake is sent along a (which is referred to by x), otherwise the process terminates.

$$c?(x).(\text{if } x = a \text{ then } x!\langle \rangle \text{ else } \text{stop}) \quad (2.1.5)$$

Recursion is built into the language using the syntax $\text{rec } x.B$. The process $\text{rec } x.B$ itself is referred to by the variable x , which is used somewhere in B to express a recursive call. For example, consider the recursive responder term below, which receives a channel x_1 via c . It then sends a handshake response on x_1 , while another responder process is run in parallel.

$$\text{rec } x.c?(x_1).(x_1!\langle \rangle \mid x) \quad (2.1.6)$$

We call a collection of parallel processes communicating over shared channels a *system*. Note that a system is itself a process. Our choice to define it is somewhat auxiliary to the π -calculus itself, but it is helpful to understanding the way that behavior can be modeled using processes as atoms of a system.

Example 2.1.7 Many presentations (including our own in the next chapter) of the π -calculus involve a choice or summation operator as in $P + Q$ for two processes terms P and Q . The meaning is essentially that either P or Q can be non-deterministically executed, and the other process will not be. However, we can model the same behavior without defining a choice operator:

$$c!\langle \rangle \mid c?().P_1 \mid c?().P_2 \quad (2.1.8)$$

In the above process, either Q or P (but not both) could be executed. This is due to the asynchronous behavior of our language – sending an empty signal along c , we cannot control which process consumes the signal (or even if it will be consumed at all). Thus, even without the choice operator our calculus is non-deterministic.

We can build systems where only one process will send a value along some channel c and only one will receive along. However, suppose we took our system and used it in building a larger system. Then we could have other processes that compete in sending or receiving along c and make our system non-deterministic. Nevertheless, we will see in the next example that we can at least learn *that* a value was consumed, although we can't say by *whom*. ■

Example 2.1.9 Perhaps we are not happy with this asynchronous transmission behavior. Surely we'd like to have blocking sends sometimes, or be able to guarantee that a value is received. However, it can be shown that our asynchronous calculus can model synchronous sending. To see why, consider the following system:

$$F_1(c) \Leftarrow \text{rec } z.(\text{new}(ack).(c!\langle ack \rangle \mid ack?().(R \mid z))) \quad (2.1.10)$$

$$F_2(c) \Leftarrow \text{rec } q.(c?(ack).(ack!\langle \rangle \mid q)) \quad (2.1.11)$$

$$Sys_1 \Leftarrow \text{new}(d).(F_1(d) \mid F_2(d)) \quad (2.1.12)$$

Here we use \Leftarrow with $F_1(c)$ to mean that the process term on the right with the channel c will be denoted by $F_1(c)$ which we will call the *interface* of a system or process. Hence, $F_1(d)$ would represent $F_1(c)$, with the occurrences of c replaced with d . Both F_1 and F_2 have (infinite) recursive behavior, and we can think of Sys_1 as a term that 'kicks off' both of them after creating a shared channel for them to communicate on.

In an iteration of F_1 , a new channel ack is created for acknowledgement. This channel is sent along c and then F_1 waits for input on ack before executing some term R and calling another iteration. F_2 receives ack along c and then uses ack to send an empty signal to F_1 that it has received input on c .

Of course, this is a silly example since the only thing being communicated is the acknowledgement channel itself – we might imagine a more complex system where F_1 sends more input along c and waits to make sure F_2 receives it. Note that the channel ack is only used *once* – we want to ensure that the acknowledgement that F_1 receives is definitely for *that* instance of communication that it just initiated. This allows us to guarantee that F_2 has executed before F_1 continues with R . ■

2.2 Structural Equivalence

A natural question at this point is, given two π -calculus terms, how can we determine if they are equivalent? Intuitively, we want them to be equivalent if they *act* the same, but actually defining this equivalence relation can be a bit subtle. First, we will look at substitution and give some rules for when we can safely interchange identifiers without creating a different term. Following this, we will develop the notion of *contexts* and then use it to build an equivalence relation among processes.

2.2.1 Identifier Substitution and α -equivalence

As a first step in our notion of equivalence, we might assert that the way things are named shouldn't change how they act. Of course, this doesn't mean we can start interchanging symbols with carefree abandon – only those identifiers whose use is *bound* in the process. There are three ways that identifiers can become bound in the π -calculus. First, a name n is bound in the restriction operator, $\text{new}(n).R$. Identifiers x_i can be bound when they are part of the pattern X matched in the receive expression $c?(X).R$. Finally, x can be bound in $\text{rec } x.B$.

We denote the set of bound identifiers in a term R by $bi(R)$; all those which are not bound we call *free*, denoting them $fi(R)$. We call a term with no free variables

interface

bound identifiers

identifiers
↪ page 3

free identifiers

closed. Since such a term is self-contained in that it doesn't need any outside context to make sense, we will finally give our precise definition for *processes* as terms that are closed. *closed terms*

In term (2.1.6) above, x is a bound recursive variable, while x_1 is bound by the receive operator. Now consider the term *processes*

$$c!\langle n \rangle \mid \text{new}(n).(\text{rec } x.c?(x_1).(x_1!\langle \rangle \mid x)) \quad (2.2.1)$$

Here x_1, x are bound as before. However, the use of n being sent on c is *not* bound, since it occurs outside the scope of the scope restriction operator. This is clear in the following term, which is equivalent (see Example 2.2.8 for justification).

$$a?(x_1, x_2).x_1!\langle \rangle \mid \text{rec } x.(a!\langle n, m \rangle \mid x) \quad (2.2.2)$$

We denote the substitution of values in V for free identifiers X in R by $R[V/X]$. Of course, during the course of a substitution, we might inadvertently ‘capture’ a bound term – suppose for example we wanted to perform the substitution $P_1(c)[n/c]$ where

$$P_1(c) \Leftarrow \text{new}(n).(n!\langle x \rangle \mid c!\langle y \rangle) \quad (2.2.3)$$

This would make the free identifier c into a bound identifier n . We can avoid this by first replacing n a new variable that is ‘fresh’ – ie it does not occur elsewhere in the term:

$$P'_1(c) \Leftarrow \text{new}(n').(n'!\langle x \rangle \mid c!\langle y \rangle) \quad (2.2.4)$$

We can now safely perform the substitution $P'_1(c)[n/c]$, yielding

$$\text{new}(n').(n'!\langle x \rangle \mid n!\langle y \rangle) \quad (2.2.5)$$

Obviously we want to say $P_1(c)$ and $P'_1(c)$ are equivalent. In general when two terms are the same up to the names of bound identifiers, we say they are *α -equivalent*, and write $P_1(c) \equiv_\alpha P'_1(c)$. We will intend from now on that a term represents its entire *α -equivalency class*, and thus will not explicitly specify *α -equivalency* in the equivalence relation below. *α -equivalency*

2.2.2 Contexts and Equivalence

Perhaps the next idea we might have for building our equivalence relation is that equivalent processes should act the same when dropped into any larger system. Let us define more precisely what we mean by ‘dropping in’ a process.

Definition 2.2.6 (Context) A context \mathbb{C} is given by:

$$\mathbb{C} := \begin{cases} [] & \\ \mathcal{C} \mid Q \text{ or } Q \mid \mathcal{C} & \text{for any process } Q, \text{ context } \mathcal{C} \\ \text{new}(n).\mathcal{C} & \text{for any name } n, \text{ context } \mathcal{C}. \end{cases}$$

$\mathbb{C}[Q]$ denotes the result of replacing the placeholder $[]$ in the context \mathbb{C} with the process Q .

contextual

Notice that with contexts, we do not pay any attention to whether a name in Q is bound in \mathbb{C} . Hence, unlike with substitution, free variables in Q can become bound in $\mathbb{C}[Q]$. We say that a relation \sim between processes is *contextual* if $P \sim Q$ implies $\mathbb{C}[P] \sim \mathbb{C}[Q]$ for any context \mathbb{C} . We are now ready to define our notion of equivalency using contexts.

Definition 2.2.7 (Structural Equivalence) Structural Equivalence, denoted \equiv is the smallest contextual equivalence relation that satisfies the following axioms:

$$\begin{array}{ll}
 P \mid Q & \equiv Q \mid P & (\text{S-COMP-COMM}) \\
 (P \mid Q) \mid R & \equiv P \mid (Q \mid R) & (\text{S-COMP-ASSOC}) \\
 P \mid \text{stop} & \equiv P & (\text{S-COMP-ID}) \\
 \text{new}(c).\text{stop} & \equiv \text{stop} & (\text{S-REST-ID}) \\
 \text{new}(c).\text{new}(d).P & \equiv \text{new}(d).\text{new}(c).P & (\text{S-REST-COMM}) \\
 \text{new}(c).(P \mid Q) & \equiv P \mid \text{new}(c).Q, \text{ if } c \notin \text{fi}(P) & (\text{S-REST-COMP})
 \end{array}$$

These axioms are simply a set of properties that we expect our syntax to obey. The first and second state that composition is commutative and associate. Thus, we will omit parenthesis around compositions when our meaning is clear. (S-COMP-ID) states that a terminated process can be eliminated from a composition. (S-REST-ID) states that a scope restriction operator can be eliminated when its scope is only a terminated process. (S-REST-COMP) states that scope ordering does not matter, justifying our shorthand $\text{new}(c, d).P$. It is the last of these axioms that is most important – it is the basis for scope extrusion, upon which the language’s process mobility is based (as we shall demonstrate in [Example 2.3.2](#) below).

scope extrusion
→ [page 5](#)

Example 2.2.8 In our discussion of bound identifiers above, we asserted that the elimination of a scope restriction when none of the scoped identifiers occur in its scope. We can now show why this is permissible:

$$\begin{array}{ll}
 \text{new}(n, m).(a?(x_1, x_2).x_1!\langle \rangle) \mid \text{rec } x.(a!\langle n, m \rangle \mid x) & \\
 \equiv \text{new}(n, m).(a?(x_1, x_2).x_1!\langle \rangle \mid \text{stop}) \mid \text{rec } x.(a!\langle n, m \rangle \mid x) & (\text{S-COMP-ID}) \\
 \equiv a?(x_1, x_2).x_1!\langle \rangle \mid \text{new}(n, m).(\text{stop}) \mid \text{rec } x.(a!\langle n, m \rangle \mid x) & (\text{S-REST-COMP}) \\
 \equiv a?(x_1, x_2).x_1!\langle \rangle \mid \text{stop} \mid \text{rec } x.(a!\langle n, m \rangle \mid x) & (\text{S-REST-ID}) \\
 \equiv a?(x_1, x_2).x_1!\langle \rangle \mid \text{rec } x.(a!\langle n, m \rangle \mid x) & (\text{S-COMP-ID})
 \end{array}$$

■

2.3 Reduction Semantics

We also want to give some *semantic* properties that a process in our language should possess. Here we do not define a relation of equivalence but rather one that allows a process to internally evolve through a number of computation steps.

Definition 2.3.1 (Reduction) The *reduction relation* \longrightarrow is the smallest contextual relation that satisfies the following rules:

$$\begin{array}{lll}
c!\langle V \rangle \mid c?(X).R & \longrightarrow & R[V/X] \quad (\text{R-COMM}) \\
\text{rec } x.B & \longrightarrow & B[\text{rec } x.B/x] \quad (\text{R-REP}) \\
\text{if } v = v \text{ then } P \text{ else } Q & \longrightarrow & P \quad (\text{R-EQ}) \\
\text{if } v_1 = v_2 \text{ then } P \text{ else } Q & \longrightarrow & Q \quad (\text{where } v_1 \neq v_2) \quad (\text{R-NEQ}) \\
\frac{P \equiv P', P \longrightarrow Q, Q \equiv Q'}{P' \longrightarrow Q'} & & (\text{R-STRUCT})
\end{array}$$

We use the notation $P \cdots \longrightarrow Q$ when an arbitrary number of these rules have been applied in reducing P to Q .

The first of these allows a computation step for the transmission of values from one process to another. (R-REP) allows us to unravel a recursive expression into iterations of itself. (R-MATCH) enables a computational step for value-matching. Finally, (R-STRUCT) says that a reduction is defined up to structural equivalence.

Example 2.3.2 We will give a demonstration of how scope extrusion is defined using the rules and axioms of reduction and structural equivalence. Consider the expression

$$d?(X).X!\langle \rangle \mid \text{new}(c).(d!\langle c \rangle \mid c?().\text{stop}) \quad (2.3.3)$$

We can use (S-REST-COMP) to bring the restriction to the outside, giving:

$$\text{new}(c).(d?(X).X!\langle \rangle \mid d!\langle c \rangle \mid c?().\text{stop}) \quad (2.3.4)$$

Thanks to the reduction relation's contextuality, we can use (R-COMM) inside of the restriction operator and apply a substitution. Using this in conjunction with the above structural equality and (R-STRUCT), we get:

$$\text{new}(c).(c!\langle \rangle \mid c?().\text{stop}) \quad (2.3.5)$$

Finally, we can apply (R-STRUCT) again, so the process simply reduces to stop.

Now let $P(d)$ to be the right half of our original process, and \mathbb{C} to be the remaining context:

$$P(d) \Leftarrow \text{new}(c).(d!\langle c \rangle \mid c?().\text{stop}) \quad (2.3.6)$$

$$\mathbb{C} = d?(X).X!\langle \rangle \mid [] \quad (2.3.7)$$

Then we have shown that if we drop $P(d)$ into the context \mathbb{C} (forming $\mathbb{C}[P(d)]$), it will establish a new channel c and extend its scope using d . Hence it is (S-REST-COMP) which allows our system's communication topology to change dynamically via scope extrusion. ■

Example 2.3.8 Suppose we wanted to model a simple memory cell with channels for getting and setting its value. It turns out that we can simulate this just using message passing. The problem that we have to work around is that receiving a identifier from a channel removes that value from the channel – it cannot be retrieved again. Thus, we need a way to push it back in for the next time. Consider the following system:

$$\begin{aligned} \text{Cell}(\text{get}, \text{set}) \Leftarrow & \text{new}(c).(c!\langle \text{init} \rangle \\ & | \text{rec } g.(\text{get}?(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \\ & | \text{rec } s.(\text{set}?(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s))) \end{aligned} \quad (2.3.9)$$

In the cell, we first create a new channel c that will be used to ‘store’ the value. We then initialize c with the value init . The following line has a recursive listener process than, when given a channel via get , gets the value from c and sends it back via the supplied channel. In parallel, it sends the value back into c so it can be had again. The next line is similar except that two values are supplied via set - a new value and a channel. The old value is pulled from c and simply discarded (ie not used), whereas the new value x is pushed to the c this time. x is also sent via b as an acknowledgment that the cell’s new value has been set.

Also notice that the system has the free channels $\text{get}_c, \text{set}_c$ given in the interface. When some larger system binds them and uses them to interact with the components inside $\text{Cell}(\text{get}_c, \text{set}_c)$ we say that the system *instantiates* $\text{Cell}(\text{get}_c, \text{set}_c)$. In the following example we assume some larger system uses *printer* to actually print the message.

interface
↪ page 6

instantiate

$$\begin{aligned} \text{Echo}(\text{printer}) \Leftarrow & \\ & \text{new}(\text{get}_1, \text{set}_1).(\text{Cell}(\text{get}_1, \text{set}_1) \\ & | \text{new}(a).(\text{set}_1!\langle \text{“hello, world”}, a \rangle \\ & | a?(x).(\text{get}_1!\langle a \rangle \mid a?(y).\text{printer}!\langle y \rangle))) \end{aligned} \quad (2.3.10)$$

The system *Echo* creates new channels to interact with the instance of memory cell it spawns, while in parallel it puts a message in the cell, then (order is ensured using by waiting for the response via a) getting the value from the cell and sending it to the *printer* channel.

To see why this produces the behavior we expect, first observe scope restrictors for a and c can be moved out using ($S - \text{REST} - \text{COMP}$):

$$\begin{aligned} & \text{new}(\text{get}_1, \text{set}_1, a, c). \left(c!\langle \text{init} \rangle \mid \text{rec } g.(\text{get}?(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\ & \quad | \text{rec } s.(\text{set}?(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s)) \\ & \quad | \text{set}_1!\langle \text{“hello, world”}, a \rangle \\ & \quad \left. | a?(x).(\text{get}_1!\langle a \rangle \mid a?(y).\text{printer}!\langle y \rangle) \right) \end{aligned} \quad (2.3.11)$$

Two applications of (R-REP) allow us to ‘pull out’ a recursive call:

$$\begin{aligned}
& \text{new}(\text{get}_1, \text{set}_1, a, c). \\
& \left(c!\langle \text{init} \rangle \mid \text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle) \mid \text{rec } g.(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\
& \mid \text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle) \mid \text{rec } s.(\text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s)) \\
& \mid \text{set}_1!\langle \text{"hello, world"}, a \rangle \\
& \left. \mid a?(x).(\text{get}_1!\langle a \rangle \mid a?(y).\text{printer}!\langle y \rangle) \right)
\end{aligned} \tag{2.3.12}$$

Now we can apply (R-COMM) to the memory cell’s setter...

$$\begin{aligned}
& \text{new}(\text{get}_1, \text{set}_1, a, c). \\
& \left(c!\langle \text{init} \rangle \mid \text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle) \mid \text{rec } g.(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\
& \mid c?(y).(c!\langle \text{"hello, world"} \rangle \mid a!\langle \text{"hello, world"} \rangle \\
& \quad \mid \text{rec } s.(\text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s))) \\
& \left. \mid a?(x).(\text{get}_1!\langle a \rangle \mid a?(y).\text{printer}!\langle y \rangle) \right)
\end{aligned} \tag{2.3.13}$$

...which lets us strip out the initializer, and in turn the acknowledgement on a of setting “hello, world” using (R-COMM),

$$\begin{aligned}
& \text{new}(\text{get}_1, \text{set}_1, a, c). \\
& \left(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle) \mid \text{rec } g.(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\
& \mid c!\langle \text{"hello, world"} \rangle \mid \text{rec } s.(\text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s)) \\
& \left. \mid \text{get}_1!\langle a \rangle \mid a?(y).\text{printer}!\langle y \rangle \right)
\end{aligned} \tag{2.3.14}$$

We can now use (R-COMM) on the memory cell’s getter and also on its use of c to get the “hello, world” state:

$$\begin{aligned}
& \text{new}(\text{get}_1, \text{set}_1, a, c). \\
& \left(c!\langle \text{"hello, world"} \rangle \mid a!\langle \text{"hello, world"} \rangle \mid \text{rec } g.(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\
& \mid \text{rec } s.(\text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s)) \\
& \left. \mid a?(y).\text{printer}!\langle y \rangle \right)
\end{aligned} \tag{2.3.15}$$

A final application of (R-COMM) to a gives us

$$\begin{aligned}
& \text{new}(\text{get}_1, \text{set}_1, a, c). \\
& \left(c!\langle \text{"hello, world"} \rangle \mid \text{rec } g.(\text{get?}(b).c?(y).(c!\langle y \rangle \mid b!\langle y \rangle \mid g)) \right. \\
& \mid \text{rec } s.(\text{set?}(x, b).c?(y).(c!\langle x \rangle \mid b!\langle x \rangle \mid s)) \\
& \left. \mid \text{printer}!\langle \text{"hello, world"} \rangle \right)
\end{aligned} \tag{2.3.16}$$

which is essentially equivalent to initializing a memory cell with “hello, world” and also sending the message to *printer*. ■

2.4 Action Semantics

Our description of process behaviors so far has been limited to talking about the internal computational steps through which it might evolve. Now, we want to give a more general description of how a process might evolve in the context of a larger system. In such a system, a process can be said to either send or receive values along channels it shares with the system. To describe these abilities, we will use the notion of a *labelled transition system*, or *lts*.

Definition 2.4.1 (Labelled Transition System) A *labelled transition system* \mathcal{L} is a tuple $(\mathcal{S}, \mathcal{A})$ where \mathcal{S} is a set of processes and \mathcal{A} is a set of labels called *actions*. Furthermore, for each action a , there is a binary relation:

$$R_a \subseteq \mathcal{S} \times \mathcal{S}$$

To denote that $\langle P, Q \rangle \in R_a$, we will use the notation $P \xrightarrow{a} Q$.

Hence, the transition $P \xrightarrow{a} Q$ indicates that there is an action under which the process P becomes Q . We will refer to Q as the *residual* of P after a .

There are three types of actions that may cause a process to evolve. Note that we will use α to refer to an action of arbitrary type. First, the process might receive a value. That is, a process P can be said to be capable of the transition $P \xrightarrow{c?X} Q$, which is to say P can receive X along c resulting in the residual Q .

The second type of action available is sending. Here we need to be a bit more careful. In the case of receiving, the received name is always bound to a new name X - so we needn't worry about issues of scoping. In sending, however, we might be transmitting either a free or a bound name (or a mix of them in a tuple). In the latter case, we need to take account of the fact that the scope of the name is being extruded to whatever process receives the name. We denote the set of bound names in the send action by (B) , and we say that this set of names is *exported* by the process. Hence, the transition $P \xrightarrow{(B)c!V} Q$ represents that P can send V over c , exporting (B) and resulting in Q . For example,

$$\text{new}(a).c!\langle a \rangle \mid Q \xrightarrow{(b)c!a} Q$$

We will refer to sending and receiving as *external actions*.

Our third action we call *internal action*. This is caused by some internal evolution in P like those described by (R-COMM) in our [Section 2.3](#). We call actions like sending and receiving external since in order to occur, they need some external process (given in some system of which the process in question is a part) to do the corresponding receiving or sending. However, with internal action there is no external process needed

residual

scope extrusion
↪ page 5

exported

external action

internal action

to proceed. We use τ denote such an internal evolution step. Thus, we say $P \xrightarrow{\tau} Q$ if P is able evolve into Q by performing a reduction step without any external contributions. For example (thanks to (R-COMM)),

$$c!\langle a \rangle \mid c?(x).x!\langle \rangle \xrightarrow{\tau} c!\langle \rangle$$

We will define and further characterize τ in our discussion of (A-COMM) below.

Using these actions, we can give a set of rules describing the behavior of a π -calculus processes in an arbitrary context. Hence, we now formally define the action relation under these rules (with their preconditions listed alongside them):

Definition 2.4.2 (Action) The *action relation* \longrightarrow is the smallest relation between processes that satisfy the following rules:

$$\begin{array}{ll}
c?(X).R \xrightarrow{c?X} R[V/X] & \text{(A-IN)} \\
c!\langle V \rangle \xrightarrow{c!V} \text{stop} & \text{(A-OUT)} \\
\text{rec } x.R \xrightarrow{\tau} R[\text{rec } x.R/x] & \text{(A-REP)} \\
\text{if } v = v \text{ then } P \text{ else } Q \xrightarrow{\tau} P & \text{(A-EQ)} \\
\text{if } v_1 = v_2 \text{ then } P \text{ else } Q \xrightarrow{\tau} Q & v_1 \neq v_2 \quad \text{(A-NEQ)} \\
\frac{P \xrightarrow{\alpha} P'}{P \mid Q \xrightarrow{\alpha} P' \mid Q} & bn(\alpha) \cap fn(Q) = \emptyset \quad \text{(A-COMP)} \\
\frac{P \xrightarrow{\alpha} P'}{\text{new}(b).P \xrightarrow{\alpha} \text{new}(b).P'} & b \notin n(\alpha) \quad \text{(A-REST)} \\
\frac{P \xrightarrow{(B)c!V} P'}{\text{new}(n).P \xrightarrow{(n,B)c!V} P'} & n \neq c, n \in V \quad \text{(A-OPEN)} \\
\frac{P \xrightarrow{c?X} P', Q \xrightarrow{(B)c!V} Q'}{P \mid Q \xrightarrow{\tau} \text{new}(B).(P' \mid Q')} & (B) \cap fn(P) = \emptyset \quad \text{(A-COMM)}
\end{array}$$

The first of two of these rules simply describe the ability of processes to evolve under input or output. For example, a process $c!\langle V \rangle \mid P$ will evolve to $\text{stop} \mid P \equiv P$ when it exercises its capability for output on c . For a receiving process, evolution under (A-IN) is only possible when the substitution $[V/X]$ makes sense. That is, V and X must have the same arity and in a typed system we'd want to ensure that their types were compatible².

Meanwhile, (A-REP), (A-EQ) and (A-NEQ) serve the provide the same interval evolution capabilities as (R-REP), (R-EQ) and (R-NEQ) do in the reduction semantics.

Together, (A-COMP) and (A-REST) provide a notion contextually - with a key

Jim:
I wonder if these couldn't be simplified (ie removing A-EQ, A-REP, etc.) by allowing a transition that happens over the reduction semantics? i can't find anyone in the literature who does this but it seems like the best approach. my sense is this approach is avoided because the action relation isn't actually contextual in our above definition (since we need to be more careful about bound variable captures). but i wonder if there is a way to gracefully sidestep the issue and avoid all the redundancy...

TODO:
make the arrow for evolution size according to the text above it

arity
 \hookrightarrow page 4

²See [Hen07] for a discussion off type systems in the π -calculus

difference being that in action semantics we need to be careful about inadvertently capturing bound variables. Hence, in (A-REST) we require that the newly bound variable does not occur in the names of the action. Suppose we ignored this precondition and tried the following:

$$\frac{c?().stop \xrightarrow{c?} stop}{new(c).c?().stop \xrightarrow{c?} new(c).stop} \quad (\text{A-REST})$$

But it is actually impossible for $new(c).c?().stop$ to evolve since the scope of c cannot be extruded and thus no other process could ever send along $c!$ Similarly, there are problems in capturing the one of the transmitted variables.

For (A-COMP), we just need to ensure that none of the bound identifiers of α conflict with the free variables in Q . Actually, the only variables that will be bound in an action is the set (b) exported by a send action. Thus, the precondition on (A-COMP) simply requires that bound variables that are transmitted will be ‘fresh’ in the receiving process. For example, consider the following system:

$$\frac{new(b).c!\langle b \rangle \xrightarrow{(b)c!x} stop}{new(b).c!\langle b \rangle \mid b!\langle a \rangle \xrightarrow{(b)c!x} stop \mid b!\langle a \rangle} \quad (\text{A-COMP})$$

Above, we have a process sending on b but the scope of b has not actually been extruded (ie by being received on c). Hence, we need to be careful not to ‘capture’ a channel like b by allowing exported channels to have the same name as free variables outside the original scope of the process.

(A-OPEN) expresses scope extrusion of a name n . If we have that a process P can evolve to P' under the action $(B)c!V$, then we can infer that some name n , whose scope is now bound to P , can be exported in the the action $(n, B)c!V$. Note that n must actually occur in V if it is to be exported; note also that n cannot be c since restricting n (ie c) to P would cause a scoping issue that interfered with an outside processes ability to receive on c .

Finally we have (A-COMM), which expresses scope extrusion as well but this time in the context of the internal action τ . According to (A-COMM), τ is defined as the simultaneous (compositionally) occurrence of an input action and a matching output action on the same channel. In the inferred process $P \mid Q$, no external contributions are needed for evolution, which results in the process $P' \mid Q'$, with the exported names (B) now being scopes to both.

Example 2.4.3 In [Example 2.1.9](#), we defined a simple process for modeling synchronous sends:

$$F_1(c) \Leftarrow \text{rec } z. (new(ack).(c!\langle ack \rangle \mid ack?().(R \mid z)))$$

We then went on to show how this process might work as a component in a example system. But suppose we wanted to characterize this process’s behavior in general. We could use action semantics to described how it it can evolve internally (in this

case there isn't much interesting we can do except expand the replication), but we'd also want to include a description of how that process behaves externally. If possible, we'd like to do this without having to come up with a system to place the process. We don't want our characterization of the process to rely at all on some auxiliary system. We can use action semantics to provide a characterization of how $F_1(c)$ behaves externally, all without defining a particular system for it to work in. Consider the following inferences:

$$\begin{array}{lll}
 F_1(c) & \xrightarrow{\tau} & \text{new}(ack).(c!\langle ack \rangle \mid ack?().(R \mid F_1(c))) & \text{(A-REP)} \\
 & \xrightarrow{(ack)c!ack} & \text{new}(ack).(stop \mid ack?().(R \mid F_1(c))) & \text{(A-OPEN, A-OUT)} \\
 & \xrightarrow{(ack?)} & \text{new}(ack).(stop \mid (R \mid F_1(c))) & \text{(A-IN)}
 \end{array}$$

Using the [structural equivalence](#) rules (S-COMP-ID) and (S-REST-COMP). We know the final step of this inference to be equivalent to:

$$R \mid \text{new}(ack).(F_1(c))$$

contextual
↪ page 8

Above, we have shown presence of the important behaviors we expect in $F_1(c)$. That is, we have shown that $F(c)$ can:

- Recursively spawn a process that can...
- Send ack over c , extruding its scope and resulting in a residual process that can...
- Receive the acknowledgement handshake on ack , resulting in a residual that can...
- Run R and wait for the next recursive iteration to occur (internally).

Thus, we have completely characterized $F_1(c)$'s capabilities to act as a recursive synchronous output process in a system – but we have done so without having to say anything about what that system actually looks like. ■

3

Synchronicity and Expressiveness

In [Section 2.1](#), we introduced a version of the π -calculus which we said was made *asynchronous* by its use of non-blocking send operations. Rather than allowing an operation to happen after a sent value has been received by some other process, a sending process simply ends, hoping someone will eventually consume the sent message. In [Example 2.1.9](#) we gave a straightforward method of simulating synchronous sending in the asynchronous π -calculus. Also absent in our calculus was the non-deterministic choice operator, also known as the summation operator. We gave a method of simulating this in [Example 2.1.7](#).

The original calculus given by Milner, Parrow and Walker modeled synchronous message passing, and included the two operators – summation and synchronous sending – that we omitted in our asynchronous π -calculus. When combined, these two operators yield a unique behavior. The power of this behavior will become clear in our presentation of the synchronous π -calculus in [Section 3.1](#).

Our original calculus features synchronous receiving, but now sending is synchronous as well, allowing a process to be performed after the output has been consumed by some other receiver. Hence, in the synchronous π -calculus, processes can be guarded by both receiving *and* sending.

If we have a group of guarded processes joined by the summation operator, only one of those will be processes will be ‘chosen’ (non-deterministically) to be executed. The rest will simply terminate without having any effect. The power of the synchronous π -calculus comes when we have another group of summed processes, running in parallel, something like:

$$c!\langle \rangle.P_1 + d_1!\langle \rangle.P_2 + d_2?().P_3 \mid c?().R_1 + d_3!\langle \rangle.R_2$$

In the above isolated system, we can actually know which of the processes will be chosen to run, despite the non-deterministic nature of the choice operator. Because both sends and receives are guarded, both the P_k ’s and R_K ’s can only run when their respective transmission guards complete. Hence, only P_1 and R_1 will be allowed to run since sending on c is the only transmission with a matching reception on c . Now imagine a slightly more complicated system:

$$c!\langle \rangle.P_1 + d!\langle \rangle.P_2 \mid c?().R_1 + d?().R_2$$

Here we have *two* transmissions with matches, so we cannot make a guarantee about which processes are chosen. However, we *can* say that if P_1 executes on the left side,

R_1 will be chosen on the right side. We can say the same for P_2 and R_2 . There is a silent, implicit sort of communication that can happen between groups of parallel processes when a non-deterministic choice is made among them. This is a powerful feature which comes with the special operators of the synchronous π -calculus.

A natural question arises at this point: can we represent this expressive communication power using only our asynchronous π -calculus? That is, using the simulations given in Examples 2.1.9 and 2.1.7 (or perhaps some similar but more complicated approach) can we fully capture the ‘communication’ between non-deterministically chosen processes discussed above? We will explore the surprising complexity of this question and some of its implications in Section 3.2. First though, let us give a more formal discussion of the features of the synchronous π -calculus.

Jim:
On second
thought, I think
I might want
to leave the
reader with this
question, for
now - to peak
their curiosity,
perhaps.

3.1 The Synchronous π -Calculus

<i>Action Prefixes</i>	
$\pi :=$	$n!\langle V \rangle$ Send
	$n?(X)$ Receive
	τ Internal Evolution
<i>Process terms</i>	
$R, S, B :=$	$\sum_{i \in \{1, \dots, n\}} \pi_i.R_i$ Summation ($n \in \mathbb{Z}$)
	$R_1 \mid R_2$ Composition
	$\text{new}(n).R$ Restriction
	$\text{if } v_1 = v_2 \text{ then } R_1 \text{ else } R_2$ Matching
	$\text{rec } x.B$ Recursion
	stop Termination

Figure 3.1.1: Terms in the synchronous π -calculus

Note the important difference between Figure 2.1.1 and Figure 3.1.1. First, we have grouped sending and receiving together as *action prefixes* (along with a new prefix for internal evolution). These prefixes are made available to the language via the summation operator.

internal evolution
 \hookrightarrow page ??

Consider the term:

$$\sum_{i \in \{1, \dots, n\}} \pi_i.R_i$$

The notation $\pi_i.R_i$ requires that the action π_i happen before the guarded process R_i will be executed. If R_a is executed in this way, then for all $j \neq a$, both the capabilities π_j and the execution of R_j are lost. In other words, the summation ensures that only

one of n guarded processes will be executed. Which of these processes is picked depends on which action prefix capability is exercised first.

For the cases that $n = 1$ and $n = 2$, we will use the notation $\pi.R$ and $\pi_1.R_1 + \pi_2.R_2$, respectively. We add to our congruence relation given in [Definition 2.2.7](#) that summation is commutative and associative.

The summation case of $n = 0$ is what is meant by our *stop* termination process. Thus the following fact, which we will call (S-SUM-ID), is trivially true:

$$\pi.R + 0 \equiv \pi.R$$

Note also that in our action prefixes, we have made sending a guarded operation, which means that R will not execute until some other process receives V along n . Receiving is also guarded, as in the asynchronous version. In the case where π is some internal evolution τ , we simply mean that R can proceed after the process does something which we cannot observe. This internal behavior is captured in the new reduction rule (R-TAU) below, while (R-SYNC) expresses the external evolution of the summation operator.

$\tau.R + P \longrightarrow R$	(R-TAU)
$c!\langle V \rangle.P + Q \mid c?(X).R + B \longrightarrow (P \mid R)[[V/X]]$	(R-SYNC)
$\text{rec } x.B \longrightarrow B[[\text{rec } x.B/x]]$	(R-REP)
$\text{if } v = v \text{ then } P \text{ else } Q \longrightarrow P$	(R-EQ)
$\text{if } v_1 = v_2 \text{ then } P \text{ else } Q \longrightarrow Q \quad (\text{where } v_1 \neq v_2)$	(R-NEQ)
$\frac{P \equiv P', P \longrightarrow Q, Q \equiv Q'}{P' \longrightarrow Q'}$	(R-STRUC)

Figure 3.1.2: Reduction rules for the synchronous π -calculus

Example 3.1.3 It is not hard to show that the synchronous π -calculus can model the asynchronous version. To see why, first note that asynchronous sending can be encoded simply by $n!\langle V \rangle.\text{stop}$. This we will abbreviate with the familiar notation $n!\langle V \rangle$. As we noted above, the summation notation allows for a single guarded process. If we limit ourselves to these single summations, disallow use of the internal action τ , and limit all sending to be of the form $n!\langle V \rangle.\text{stop}$, then we have the asynchronous π -calculus. To see why the reduction semantics are compatible, note that (R-TAU) is not needed in the asynchronous calculus since we have excluded its operator. The following shows that (R-SYNC) is a more general form of (R-COMM) of [Definition 2.3.1](#):

$c!\langle V \rangle.\text{stop} + \text{stop} \mid c?(X).R + \text{stop} \equiv c!\langle V \rangle.\text{stop} \mid c?(X).R,$	(S-SUM-ID)
$c!\langle V \rangle.\text{stop} \mid c?(X).R \longrightarrow (\text{stop} \mid R)[[V/X]],$	(R-SYNC)
$(\text{stop} \mid R)[[V/X]] \equiv R[[V/X]]$	(S-COMP-ID)
$c!\langle V \rangle \mid c?(X).R \longrightarrow R[[V/X]]$	(R-STRUC)

It will be obvious from our presentation of synchronous action rules below that they needn't be shown to be a general version of the asynchronous rules – they are

equivalent simply by ignoring the extra rules and using our $n!\langle V \rangle.stop$ encoding of sending. ■

Neither do the action rules for the synchronous π -calculus, given below, differ significantly from [those](#) in the asynchronous version. As we would expect, (A-OUT) has been modified to express synchronous sending: it now evolves over output to a process R and not simple the termination process $stop$. We have also added two new rules. The first, (A-TAU) describes the behavior of a process guarded by an internal action. The second rule, (A-SUM), characterizes the summation operator. It says, for example, that if some process P is able to evolve to P' over some action α , then α will also cause $P + Q$ to evolve via choice to P' .

$c?(X).R$	$\xrightarrow{c?X}$	$R[V/X]$	(A-IN)
$c!\langle V \rangle.R$	$\xrightarrow{c!V}$	R	(A-OUT)
$\text{rec } x.R$	$\xrightarrow{\tau}$	$R[\text{rec } x.R/x]$	(A-REP)
$\text{if } v = v \text{ then } P \text{ else } Q$	$\xrightarrow{\tau}$	P	(A-EQ)
$\text{if } v_1 = v_2 \text{ then } P \text{ else } Q$	$\xrightarrow{\tau}$	Q	$v_1 \neq v_2$ (A-NEQ)
$\tau.R$	$\xrightarrow{\tau}$	R	(A-TAU)
$\sum_{i \in \{1, \dots, n\}} \frac{P_i \xrightarrow{\alpha} P'_i}{\pi_i.P_i \xrightarrow{\alpha} P'_i}$			(A-SUM)
$\frac{P \xrightarrow{\alpha} P'}{P \mid Q \xrightarrow{\alpha} P' \mid Q}$		$bn(\alpha) \cap fn(Q) = \emptyset$	(A-COMP)
$\frac{P \xrightarrow{\alpha} P'}{\text{new}(b).P \xrightarrow{\alpha} \text{new}(b).P'}$		$b \notin n(\alpha)$	(A-REST)
$\frac{P \xrightarrow{(B)c!V} P'}{\text{new}(n).P \xrightarrow{(n,B)c!V} P'}$		$n \neq c, n \in V$	(A-OPEN)
$\frac{P \xrightarrow{c?X} P', Q \xrightarrow{(B)c!V} Q'}{P \mid Q \xrightarrow{\tau} \text{new}(B).(P' \mid Q')}$		$(B) \cap fn(P) = \emptyset$	(A-COMM)

Figure 3.1.4: Action rules for the synchronous π -calculus

Example 3.1.5 Stuff ■

3.2 Separation Results

When trying to compare the expressive power of different calculi, one good approach is to, as above, provide explicit encodings from one language to another¹. We say that we are trying to encode from *source language* into the terms of a *target language*, and if we succeed, we have shown that the target language is at least as expressive as the source. Hence, in example [Example 3.1.3](#) we showed that the synchronous calculus is at least as expressive as the asynchronous calculus by giving an encoding from the asynchronous to the synchronous. We will also use the notation

$$\llbracket P \rrbracket \stackrel{def}{=} Q$$

to mean that P in the source language is encoded by Q in the target language.

To prove a separation result between languages, it is enough to show that there are problems that are not solvable in the source language that are not solvable in the target language. In [\[Pal03\]](#), Palamidessi uses the solvability of the leader election problem on symmetric networks to show that the synchronous π -calculus is strictly more expressive than the asynchronous version. Loosely, the leader election problem is the problem of having group of identifier (via integers, perhaps) processes agree on a ‘leader’ process identification in a finite amount of time. We say that these processes are a symmetric network if any two processes P_i, P_j are equivalent under structural equivalence and renaming of their identifiers. For example, consider the following symmetric network:

$$P_0 \mid P_1 \Leftarrow c_0!\langle \rangle.o!\langle 0 \rangle + c_1?().o!\langle 1 \rangle \mid c_1!\langle \rangle.o!\langle 1 \rangle + c_0?().o!\langle 0 \rangle$$

It should not be hard to see that this solves the leader election problem in the synchronous π -calculus by agreeing on a leader via the output channel o . It may be less obvious, but it is not possible to solve the leader election problem in a symmetric network of asynchronous π -calculus processes. This is a direct result of the lack of the choice operator: without it, the symmetric processes have no way to pick a leader non-deterministically without potentially disagreeing with one another. It is only through the implicit communication underlying the choice operator that synchronous processes are able to break out of their symmetry and agree on a leader.

Using these results, Palamidessi a set of useful requirements that formally separate the two calculi.

The first of those requirement is on *uniformity*, which means that:

$$\llbracket \alpha(P) \rrbracket = \alpha(\llbracket Q \rrbracket) \tag{3.2.1}$$

$$\llbracket P \mid Q \rrbracket = \llbracket P \rrbracket \mid \llbracket Q \rrbracket \tag{3.2.2}$$

(3.2.1) simply states α -renaming is not violated in the process of the encoding. (3.2.2)

¹Note that [Example 3.1.3](#) is a trivial example of such an encoding since your encodings are straight-forward enough to be equivalent under the structural equivalence. Most encodings require the more advanced structure of *bisimulation* equivalences, which relies on setting up a notion of simulation and then proving that two processes simulate one another.

source language

target language

uniformity

Jim:
When I try
to refer to
the label
 α -equivalency
here, things
break. any
idea how to fix
this without
re-engineering
my definitions?

is related to the requirements of a distributed system. That is, parallel processes really should just map to parallel processes, with not top level manager process or the like to aide the encoding. This is how Palamidessi builds the requirement of a symmetric network into a language requirement.

The other requirement is on *reasonability*. Reasonability to Palamidessi means that the language can distinguish between two processes when their actions are different on a certain given channel. This essentially encapsulates the requirements of the leader election problem. That is, a electoral system would be one were actions on the output channel are the same and we want our target language to be capable of semantically distinguishing this from a non-electoral system (where actions on the output channel differ).

TODO:
where/how?

reasonability

4

Synchronicity and Distributed Systems

The second question we might ask is about the implementation of a synchronous π -calculus. On distributed systems, which seem like the natural setting for such an implementation, we have only asynchronous available to us. Hence, depending on the answer to the first question, we may have to

However, whether the asynchronous calculus actually has the expressive power of its synchronous forebear is a matter of more complexity. In fact, we will see by straightforward counterexample (and by a rather less straightforward argument due to Palamidessi [Pal03]) that two are not truly equal. The implementation of synchronous calculi on distributed systems is a thorny issue. On the one hand, it is a useful construct in that it allows us to model many problems more naturally and easily. On the other, all communication in a distributed system is in fact asynchronous in nature, and so synchronous communication must be layered on top at some level of the implementation. It is notoriously difficult construct to implement synchronous communication efficiently without violating the requirements of a truly distributed system where there is shared memory or central control process. The issue becomes more difficult still with the Palamidessi's related requirement of symmetry, which means that a process's behavior does not depend on its location in the channel topology.

The creators of Pict, the Join-calculus, and other implementations based on the π -calculus all decided to have their primitives support only asynchronous communication, while synchronous communication is made available overtop of this via a library or higher-level language. This these greatly simplifies implementation, resulting in a cleaner, more efficient core language. The summation operator in particular is difficult and expensive to fully simulate. In the implementation of Pict, for example, David Turner notes [Tur96] that “the additional costs imposed by summation are unacceptable.”. Turner goes on to say that essential uses of summation are infrequent in practice.

How can we reconcile this with Palamidessi's result, which indicates that there are important problems that cannot be solved without the full generality of the synchronous calculus? Is such a calculus even implementable at any cost on distributed systems? We will see that relaxing any of Palamidessi's assumptions enables a full encoding. Some of such encodings derive from the classic distributed solutions to the problem of synchronous communication but strain important assumptions about symmetry in ways that we may not be comfortable allowing. Palamidessi herself gives as important probabilistic encoding [PH01] which does not break symmetry.

TODO:
make a note
about why
the original
example only
covered input
guarded choice

4.1 Symmetry in Practice

Speaking in an interview on developing the π -calculus, Robin Milner notes [Ber03]:

That was to me the challenge: picking communication primitives which could be understood at a reasonably high level as well as in the way these systems are implemented at a very low level...There's a subtle change from the Turing-like question of what are the fundamental, smallest sets of primitives that you can find to understand computation...as we move towards mobility... we are in a terrific tension between (a) finding a small set of primitives and (b) modeling the real world accurately.

This tension is quite evident in the efforts of process algebraists to find the 'right' calculus for modeling distributed systems. While the synchronous π -calculus more elegantly and (given certain assumptions) completely expresses distributed systems, actual implementation must commit to asynchronous communication as their primitives. Which we choose as a model depends in part on our goals. In any case, it is evident that by limiting ourselves to smaller calculi, many useful new concepts and structures arise in order to solve the problems posed by asynchronous communication. While these structures might not belong in the 'smallest set of primitives', they are useful for bringing the power of the π -calculus to a model that more closely resembles the implementation of distributed systems.

This section will go on to talk about on Nestmann and other implementation-driven encodings, how much they violate uniformity/reasonableness and whether that has any means for the API as a good model for implementable distributed languages.

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