A warning on the use of linear logit models in transport mode choice studies

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This article argues that linear logit models are inappropriate to use for transport demand studies because, they impose many rigid a priori *restrictions on the parameters of price responsiveness of demand, such as the elasticities of substitution and cross price elasticities, and the structure of technology (or preference) underlying the linear logit models has severe irregularity and inconsistency. This has a serious implication for the credibility of the findings of several recent studies which attempted to measure the welfare loss of traffic misallocation attributable to the Interstate Commerce Commission's railroad minimum rate regulation.*

1. Introduction

■ This article argues that linear logit models[1] are not appropriate to use in studying the nature of demand functions, such as measuring the price responsiveness of demand and simulating traffic allocation across modes under an alternative pricing scenario for three reasons. First, the linear logit models impose many rigid *a priori* restrictions on the parameters of price responsiveness of demand, such as the elasticities of substitution and the cross price elasticities. Second, in those types of linear logit models which include a ratio of prices as an explanatory variable, the parameters of price responsiveness of demand are not invariant with respect to choice of the "base" mode which serves as the denominator in every logit equation. Finally, the structure of technology (or preference) underlying the linear logit models is severely irregular and inconsistent. Therefore, we do not recommend using logit models beyond the prediction of choice probabilities. This is a serious matter in view of the recent widespread use of linear logit models in empirical studies of both freight and passenger demands.

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[1] Throughout this paper "linear logit models" refer to those estimated from "aggregate" data using logarithms of demand ratios (or equivalently, ratios of market shares) as the dependent variables, and thus are distinguished from those estimated from purely "disaggregate" data which use the discrete choice variables "0 or 1" directly as the dependent variables. Clearly, it is meaningless to talk about preference structure and elasticities of substitution for the disaggregate discrete choice models.

Although the discussion in this paper refers to freight transportation only, the results hold for passenger transportation in the same way.

The outline of this article follows. Section 2 postulates the two types of linear logit models which have been used most widely in transport demand studies. Section 3 examines the implications of each of the two types on the elasticities of intermodal substitution and on the cross price elasticities. Section 4 uses the logit models reported in Boyer (1977) to compute the elasticity of rail-truck substitution to demonstrate empirically inconsistencies among the results and thus the inadequacy of using a linear logit model as a demand model.

2. Linear logit models

■ Beginning with Berkson's applications to bioassay problems (1944, 1951, 1953, 1955), linear logit models have been widely used in many fields including biology, psychology, and economics. The primary reasons for this popularity are that the underlying "logistic" probability function gives a close approximation to the cumulative normal probability function (Berkson, 1944), that the resultant logit model is computationally far easier to estimate than the "probit" model (Berkson, 1951), and that the logit analysis constrains the probability estimates to lie between zero and one. (See Berkson (1953, 1955), Cox (1970), and McFadden (1974a) for additional properties of these models.)

For a two-mode discrete choice case, the logistic probability function and the resultant logit model can be written as in (1) and (2), respectively:

$$P(i) = \frac{1}{1 + \exp[-a_0 - a_1x_1 - \sum_{n=2}^{N} a_nx_n]} \tag{1}$$

$$\log\left[\frac{P(i)}{1 - P(i)}\right] = a_0 + a_1x_1 + \sum_{n=2}^{N} a_nx_n, \tag{2}$$

where

$P(i)$ = the conditional probability of choosing mode i;
x_1 = the price of mode i relative to that of the other mode;
x_n = other exogenous factors affecting mode choice, $n = 2, 3, \ldots, N$;
a_n = the parameter of the logistic probability function, $n = 0, 1, 2, \ldots, N$.

For the general case of an M-mode discrete choice, one of the modes, say mode M, is arbitrarily chosen to serve as the "base" mode. We then specify the logit model as:[2]

$$\log\left[\frac{P(i)}{P(M)}\right] = a_{i0} + a_{i1}x_{i1} + \sum_{n=2}^{N} a_{in}x_{in} \qquad \text{for all} \qquad i = 1, 2, \ldots, M-1, \tag{3}$$

where

$P(m)$ = the conditional probability of choosing mode m, $m = 1, 2, \ldots, M$;
x_{i1} = the price of mode i relative to that of the base mode M;
x_{in} = other exogenous factors affecting mode choice, $n = 2, 3, \ldots, N$;

[2] The Nerlove-Press (1972) form of logit model restricts the parameters other than the constant terms to be equal across the $M - 1$ equations; i.e., $a_{in} = a_n$ for all $i = 1, 2, \ldots, M - 1$ and $n = 1, 2, \ldots, N$. Most results of this study remain unchanged even in Nerlove-Press form.

a_{i0}, a_{i1}, and a_{in}'s = parameters of the logit equation for ith mode.

In estimating the logit model in (2) or (3), the logarithm of market share ratio (input ratio from the shipper's viewpoint) of the two modes is used as the dependent variable. The important thing to note here is that the logit model is used to estimate the parameters of the underlying probability function rather than the ratio of demands. However, many people including Boyer (1977) and Levin (1978) have used the logit model to estimate the price sensitivities of mode-choice. This assumes that the actual allocation of traffic over the alternative modes is equal to the corresponding conditional probabilities estimated by the probability function. Although this assumption may be disputable, a more serious problem of using a logit model as a demand model is its implications on the elasticity of substitution and cross price elasticities of demands. We shall focus on this latter aspect of logit models.

Most linear logit models estimated to date can be grouped into the following two types, based upon the way to specify the relative price variable (x_1 or x_{i1}).

Type 1: The relative price variable in ith logit equation (x_{i1}), is expressed as the ratio of the price of ith mode to that of the base mode M, i.e.,

$$x_{i1} \equiv \frac{P_i}{P_M}, \qquad i = 1, 2, \ldots, M - 1,$$

where

P_i = ith mode's price;
P_M = price of the base mode M.

This type of logit model was used in Ransam *et al.* (1971), Kullman (1973), Transportational Development Agency (1975), Turner (1975), Boyer (1973), and many others.

Type 2: The relative price variable in ith logit equation (x_{i1}), is expressed as the difference between the price of ith mode and that of the base mode M, i.e.,

$$X_{i1} = P_i - P_M, \qquad i = 1, 2, \ldots, M - 1.$$

This type of logit model was used in McFadden (1974b), Transportation Development Agency (1975), Richards and Ben-Akiva (1975), Gillen (1977), Boyer (1977), Levin (1978), and many others.

In the remainder of this paper, the Type 1 and Type 2 logit models are frequently referred to as the "price-ratio" model and the "price-difference" model, respectively.

The Type 1 (price-ratio) model may be regarded as an *ad hoc* specification because it is not derived from any underlying microeconomic model of individual behavior. On the other hand, Domencich and McFadden (1975, pp. 73–78) have demonstrated that the Type 2 (price-difference) model can be derived from a random utility function which satisfies a number of specific conditions.

3. Implications on the price responsiveness of demands

■ The (partial) elasticities of substitution[3] and the price elasticities are the two most common measures of price responsiveness of demands. In what follows,

[3] The elasticity of substitution measures the proportionate change in input ratio with respect to a proportionate change in price ratio. This terminology was originally defined by Hicks (1932).

for each of the two types of logit models specified in the preceding section, the elasticities of substitution and the price elasticities will be derived in terms of the parameters of the logit model, and their implications will be examined.

□ **Type 1: price-ratio model.** The logit model actually being estimated in this case is as follows[4]:

$$\log\left(\frac{S_i}{S_M}\right) = a_{i0} + a_{i1}\left(\frac{P_i}{P_M}\right) + \sum_{n=2}^{N} a_{in}x_{in}, \tag{4}$$

$$i = 1, \quad M = 2 \quad \text{(binomial case)},$$

$$i = 1, 2, \ldots, M-1 \quad \text{(multinomial case)},$$

where X_{in} is the nth exogenous factor which is not a price variable, $n = 2, \ldots, N$, and all other variables are defined as before.

The elasticity of substitution between ith mode and the base mode M corresponding to the logit model (4) can be written as[5]:

$$\sigma_{iM} = -a_{i1}\left(\frac{P_i}{P_M}\right) \tag{5}$$

$$i = 1, \quad M = 2 \quad \text{(binomial case)},$$

$$i = 1, 2, \ldots, M-1 \quad \text{(multinomial case)}.$$

Since the value of a_{i1} is likely to be negative, the elasticity of substitution between any nonbase mode i and the base mode M is positive, as is expected. However, σ_{iM} is a strictly increasing function of the price ratio P_i/P_M. This implies that the elasticity of substitution depends on the choice of base mode.[6] This problem is illustrated in Figures 1(a) and 1(b).

Later, Allen (1938) extended the concept to the general case of more than two inputs and defined the partial elasticity of substitution as follows:

$$\sigma_{ij} \equiv \frac{d \ln\left(\frac{S_i}{S_j}\right)}{d \ln\left(\frac{P_j}{P_i}\right)} \equiv \sigma_{ji},$$

where

σ_{ij} = the (symmetric) elasticity of substitution between modes i and j,

$\frac{S_i}{S_j}$ = the ratio of the market shares of (or equivalently ratio of demands for) mode i to mode j,

$\frac{P_j}{P_i}$ = the ratio of the prices of mode j to mode i.

[4] Although the underlying choice probability model for the binomial case is equation (1), for a multinomial case, it is in general impossible to express the choice probability of a certain mode separately from those of other modes (see Theil (1969) for detail on this point).

[5] See Appendix 1A for the derivation.

[6] In the two-mode case, it can be easily seen that this happens because the logistic probability function (1) fails to satisfy the following symmetry requirements:

$$P(i) = f(P_i, P_j)$$

$$P(j) = f(P_j, P_i) = 1 - f(P_i, P_j)$$

$$f(P_i, P_j) + f(P_j, P_i) = 1$$

for all possible assignments of the two modes to the indicators i and j, where $f(\cdot)$ is the logistic probability function which includes the price-ratio as one of the arguments.

FIGURE 1

(A) DEPENDENCE OF σ_{iM} ON CHOICE OF THE BASE MODE

(B) SHIPPER'S CHOICE BEHAVIOR IN TRANSPORTATION SECTORAL TECHNOLOGY SPACE

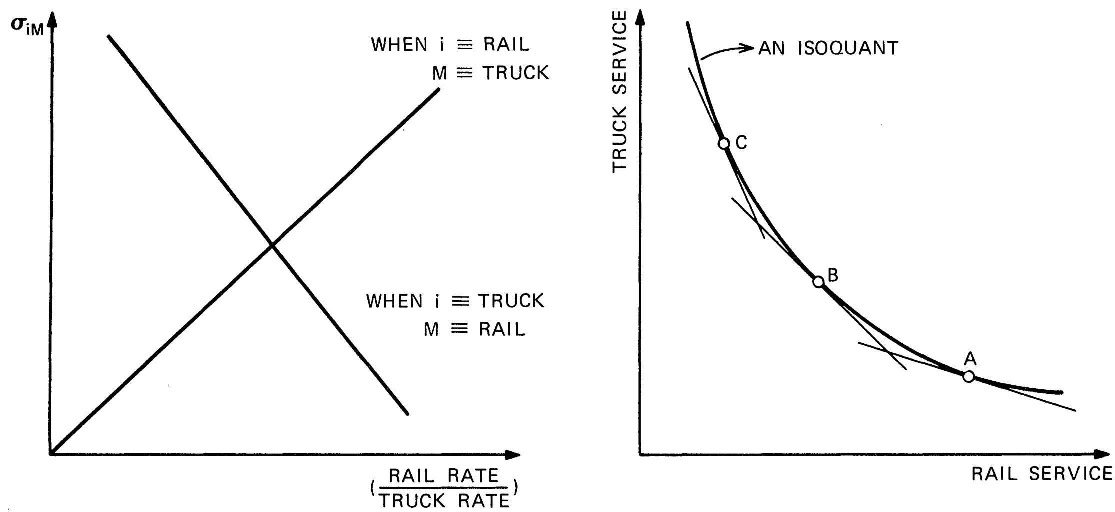

Figure (1a) exhibits the relationship between the elasticity of rail-truck substitution (σ_{rh}) and the corresponding price ratio (P_r/P_h) under the two alternative choices of "base" mode M. This shows that as the price ratio (P_r/P_h) increases, the elasticity of substitution increases if "truck" were chosen as the "base" mode M (see the schedule (a)), but decreases if instead "rail" were chosen (see the schedule (b)). This means that the choice of "base" mode M not only imposes a rigid *a priori* restriction on the relationship between the elasticity of substitution (σ_{iM}) and the corresponding price ratio (P_i/P_M), but also this restriction is contradictory to the one that would have been imposed under the alternative choice of "base" mode.

In the context of the technology (or preference) structure illustrated in Figure (1b), this can be interpreted as follows: as one moves along an isoquant (or an indifference curve) from point $A \rightarrow$ point $B \rightarrow$ point C, the elasticity of rail-truck substitution increases if "truck" were chosen as the base mode M, because the price ratio (P_r/P_h) increases, but the same elasticity of substitution decreases if instead "rail" were chosen as the base mode because, in this case, the price ratio (P_h/P_r) decreases as we move from point $A \rightarrow$ point $B \rightarrow$ point C. This means that the choice of a certain mode as the base mode M amounts to imposing a rigid *a priori* restriction on the behavior of elasticity of substitution (σ_{iM}) along an isoquant, which is contradictory to the restriction that would have been imposed under a different choice of base mode.

The elasticity of substitution between any two nonbase modes i and j corresponding to the logit model (4) can be written as[7]:

$$\sigma_{ij} = \begin{cases} -a_{i1}\left(\dfrac{P_i}{P_M}\right) & \text{if} \quad dP_j = 0 \quad \text{and} \quad dP_i \neq 0 \\ -a_{j1}\left(\dfrac{P_j}{P_M}\right) & \text{if} \quad dP_i = 0 \quad \text{and} \quad dP_j \neq 0 \end{cases} \tag{6}$$

$$\text{for all} \quad i \neq j, \quad j \neq M.$$

[7] See Appendix 1B for the derivation.

The presence of the price of base mode (P_M) in equation (6) means that even the elasticities of substitution between nonbase modes depend on the choice of base mode M. Equation (6) shows also that there are two different measures for the same elasticity of substitution (σ_{ij}): one when the jth price is held constant, and the other when the ith price is held constant. These two measures are mutually contradictory because, as the price ratio (P_j/P_i) increases, σ_{ij} decreases if P_j is held constant, but increases if, instead, P_i is held constant. This implies that there is no consistent differentiable preference (or technology) structure underlying the multinomial logit model of Type 1. This is a serious problem because it is meaningless to measure the price responsiveness of demands by using a model, the underlying preference of which has severe irregularity and inconsistency.

The price elasticities imbedded in the logit model (4) can be written as[8]:

$$E_{iM} = E_{MM} - a_{i1}\left(\frac{P_i}{P_M}\right), \qquad i = 1, 2, \ldots, M - 1. \tag{7a}$$

$$E_{ij} = E_{jj} - a_{j1}\left(\frac{P_j}{P_M}\right) \qquad \text{for all} \quad j \neq M \quad \text{and} \quad i = 1, 2, \ldots, M, \tag{7b}$$

where

E_{ij} = the elasticity of demand for ith mode with respect to the price of jth mode.

The appearance of the price of base mode (P_M) in equation (7b) means that the cross price elasticities with respect to the price of a given nonbase mode ($E_{ij}, j \neq M$) also depend on the choice of base mode. Since equation (7b) does not depend on any of the parameters of the ith logit equation or the ith price, the multinomial logit model of Type 1 restricts all cross price elasticities of demands with respect to the price of any given nonbase mode (E_{ij}, $j \neq M$, $i = 1, 2, \ldots, M$) to be identical. This is exactly the same result that Hausman (1975) discovered. This is also a severe restriction embedded in the linear logit model of Type 1.

Since equation (7a) depends on the parameter a_{i1}, in this model the cross price elasticities with respect to the price of base mode (P_M) are allowed to vary.

□ **Type 2: price-difference model.** In this case the logit model actually being estimated can be written as follows[9]:

$$\log\left(\frac{S_i}{S_M}\right) = a_{i0} + a_1(P_i - P_M) + \sum_{n=2}^{N} a_n X_{in} \tag{8}$$

$$i = 1, \qquad M = 2 \qquad \text{(binomial case)}$$

$$i = 1, 2, \ldots, M - 1 \qquad \text{(multinomial case)},$$

[8] These price elasticities are derived in Appendix 1C.

[9] Note that, except for the constant term a_{i0}, the parameters of logit model (a_1 and a_n's) are identical across all $M - 1$ equations. This is Nerlove-Press (1972) specification of the logit model of Type 2. The advantage of restricting the parameters across the $M - 1$ equations is that the under-

where

$\frac{S_i}{S_M}$ = the ratio of demand (or market share) of mode i to the base mode M;

P_m = the price of the mth mode;

$X_{in} \equiv X^*_{in} - X^*_{Mn}$, $n = 2, 3, \ldots, N$;

X^*_{mn} = nth exogenous factor of mode m which affects mode-choice;

a_{i0}, a_1, and a_n = parameters of the logit model.

The elasticities of substitution and cross price elasticities embedded in the logit model of Type 2 can be written as[10]:

$$\sigma_{ij} = \begin{cases} -a_1 P_i & \text{when} \quad dP_j = 0, \quad dP_i \neq 0 \\ -a_1 P_j & \text{when} \quad dP_i = 0, \quad dP_j \neq 0 \end{cases} \tag{9a}$$

$$\text{for all} \quad i \neq j \quad \text{and} \quad i, j = 1, 2, \ldots, M;$$

$$E_{ij} = E_{jj} - a_1 P_j \quad \text{for all} \quad i \neq j \quad \text{and} \quad i, j = 1, 2, \ldots, M, \tag{9b}$$

where

σ_{ij} = the elasticity of substitution between modes i and j,

E_{ij} = the elasticity of demand for ith mode with respect to the price of jth mode,

and other variables are defined as in equation (8).

As in the "price-ratio" model, equation (9a) shows that there are, in general, two different estimates of the elasticity of substitution between any two modes i and j at a given data point; one when the jth price is held constant, and the other when the ith price is held constant. These two measures are again mutually contradictory because, as the price ratio (P_j/P_i) increases, σ_{ij} decreases if P_j is held constant, but increases if P_i is held constant. This implies that there is no consistent differentiable preference (or technology) structure underlying the binomial or multinomial logit model of Type 2. Unlike the Type 1 model, however, the elasticities of substitution (σ_{ij}) underlying the Type 2 model do not depend on the choice of base mode M. This is probably a major reason why, recently, more and more people have started to use the Type 2 (price-difference) model.

Turning our attention to equation (9b), we notice that all cross price elastic-

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lying probability of choosing a specific mode, say mode i, can be expressed separately from those of other modes:

$$P(i) = \frac{\exp\{a_{i0} + a_1 P_i + \sum_{n=2}^{N} a_n X^*_{in}\}}{\sum_{m=1}^{M} \exp\{a_{m0} + a_1 P_m + \sum_{n=2}^{N} a_n X^*_{mn}\}}, \qquad i = 1, 2, \ldots, M,$$

where

$a_{M0} = 0$

$P(i)$ = the probability of choosing mode i, $i = 1, 2, \ldots, M$.

Owing to this advantage, most multinomial logit models estimated in the past imposed restrictions on the cross equation equalities of the parameters, except the constant term, which is allowed to vary from equation to equation.

[10] See Appendix 2 for the derivation of these formulas.

ities both with respect to the price of any nonbase mode and with respect to the price of the base mode P_M are identical.[11] This implies that, in the Type 2 (price-difference) model, Hausman's 1975 finding has an extended application in the sense that the equality of all cross price elasticities holds even with respect to the price of base mode P_M.

☐ **Summary of results.** In the preceding subsections we investigated the implications of the two types of linear logit models on the elasticities of substitution and the price elasticities. The results are summarized here.

The linear logit models of Type 1 ("price-ratio" model) have the following weaknesses as demand models:

(1) The elasticities of substitution and the price elasticities are not invariant to the choice of base mode M.
(2) A certain choice of base mode amounts to imposing rigid *a priori* restrictions on the relationships between the elasticities of substitution and the corresponding price ratios, and these restrictions are contradictory to the ones that would have been imposed under a different choice of base mode.
(3) The preference (or technology) structure underlying the multinomial logit model of Type 1 is inconsistent and irregular because there are two different measures for the elasticity of substitution between any two nonbase modes i and j: one when ith price is held constant and the other when jth price is held constant. Therefore, it is meaningless to measure the price responsiveness of demands using the multinomial logit model of Type 1.
(4) All cross price elasticities with respect to the price of any given "nonbase" mode are restricted to be equal.

Similarly, the linear logit model of Type 2 ("price-difference" model) has the following weaknesses as a demand model:

(1) The technology (or preference) structure underlying both binomial and multinomial logit models of this type is inconsistent and irregular because of the existence of two different measures for the same elasticity of substitution. Therefore, it is meaningless to try to measure the price responsiveness of demands using the logit models of Type 2.
(2) All cross price elasticities with respect to the price of any given mode, including the base mode M, are restricted to be equal. This is also an extremely unrealistic restriction.

One advantage of the Type 2 (price-difference) model over the Type 1 (price-ratio) model is that neither the elasticities of substitution nor the price elasticities depend on the choice of "base" mode M in Type 2 models.

Notice that the results derived in this section hold exactly only when aggregate data (i.e., aggregation over choices made by a particular individual and/or over choices made by various individuals during a given period of time) are used to estimate the model. When disaggregate data (i.e., discrete choice variable "0 or 1" as the dependent variable) are used to estimate the model separately for each individual, and then the predicted values are aggregated, the results hold at the individual level but not at the aggregate level owing to the aggregation of individuals with different characteristics.

[11] In general all cross price elasticities with respect to the price of base mode (P_M) would be different if the parameter a_1 of logit model (8) is allowed to vary from equation to equation.

TABLE 1

ESTIMATES OF ELASTICITY OF SUBSTITUTION USING BOYER'S LOGIT MODELS

MILEAGE BLOCK	AVERAGE FREIGHT RATE PER TON-MILE		ELASTICITY OF SUBSTITUTION		
	RAIL (P_r)	TRUCK (P_h)	LOGIT WITH PRICE-RATIO	LOGIT WITH PRICE-DIFFERENCE	
			(a_1 = −6.77) (1)	(a_1 = −1.04) (2) WHEN $dP_r = 0$	(3) WHEN $dP_h = 0$
50–100	3.21	10.41	2.1	10.83	3.34
400–500	2.05	3.98	3.5	4.14	2.13
900–1000	1.71	3.24	3.6	3.37	1.78
1500–2000	1.68	2.47	4.6	2.57	1.73

SOURCE: REGRESSIONS 3 AND 4 REPORTED IN TABLE 2, BOYER (1977).

4. An empirical illustration

■ Recently Boyer (1977) reported binomial logit models of both Type 1 and Type 2 which were estimated from an identical data of intercity rail-truck freight transportation. To see how sensitive the estimate of the elasticity of rail-truck substitution is to the type of logit model used, equations (5) and (9a) were applied to the Boyer's logit models. Table 1 reports the results by each selected mileage block.

A comparison across columns (1), (2), and (3) shows that the estimates of the elasticity of substitution are enormously different between the two types of models and between the two different measures of the Type 2 model. Column (1) shows that the elasticity of substitution measured from the Type 1 logit model increases with distance, whereas columns (2) and (3) show that those measured from the Type 2 model decrease with distance. These results are mutually contradictory. Since both types of logit models have the weaknesses indicated in Section 3, both are unlikely to give a close approximation to the true demand functions. This clearly jeopardizes the credibility of the Boyer's estimate of the welfare loss attributable to the ICC's railroad minimum rate regulation.

In conclusion, if one is interested in studying the nature of demand functions, such as measuring the price responsiveness of aggregate demands or predicting modal demands under alternative pricing scenarios, it is inappropriate to use a logit model. "Flexible" functions such as translog functions[12] (Christensen *et al.*, 1971, 1973), generalized Leontief functions (Diewert, 1971) and generalized quadratic functions (Denny, 1972, 1974) appear to be the right substitutes for the logit models because these functions allow free variations of the elasticities of substitution and the price elasticities.

Appendix 1

Derivation of elasticities of substitution and price elasticities

■ **For type 1 (price-ratio) logit model.**

$$\log\left(\frac{S_i}{S_M}\right) = a_{i0} + a_{i1}\left(\frac{P_i}{P_M}\right) + \sum_{n=2}^{N} a_{in}X_{in}, \qquad i = 1, 2, \ldots, M-1. \quad \text{(A1)}$$

[12] Applications of translog functions in freight demand study can be found in Friedlaender and Spady (1977) and Oum (1977).

□ **A: elasticity of substitution between the base mode M and a nonbase mode i.**

$$\sigma_{iM} \equiv \frac{d \log\left(\frac{S_i}{S_M}\right)}{d \log\left(\frac{P_M}{P_i}\right)} = \frac{d\left[a_{i1}\left(\frac{P_i}{P_M}\right)\right]}{d \log\left(\frac{P_M}{P_i}\right)}$$

$$= \frac{d[a_{i1}t^{-1}]}{dt}\frac{dt}{dy}, \qquad \text{where} \qquad t \equiv \frac{P_M}{P_i}, \qquad y \equiv \log t$$

$$= -a_{i1}t^{-2} \cdot t, \qquad \text{since} \qquad \frac{dt}{dy} = t.$$

Therefore,

$$\sigma_{iM} = -a_{i1}\left(\frac{P_i}{P_M}\right), \qquad \text{since} \qquad t^{-1} = \frac{P_i}{P_M}$$

$$\text{for all} \qquad i = 1, 2, \ldots, M-1.$$

□ **B: elasticity of substitution between any two nonbase modes i and j.**

$$\sigma_{ij} \equiv \frac{d \log\left(\frac{S_i}{S_j}\right)}{d \log\left(\frac{P_j}{P_i}\right)} \qquad \text{for} \qquad i \neq j \qquad \text{and} \qquad i, j \neq M.$$

From the model it is possible to write the following:

$$\log\left(\frac{S_i}{S_j}\right) = \log\left(\frac{S_i}{S_M}\right) - \log\left(\frac{S_j}{S_M}\right)$$

$$= (a_{i0} - a_{j0}) + \left[a_{i1}\left(\frac{P_i}{P_M}\right) - a_{j1}\left(\frac{P_j}{P_M}\right)\right] + \sum_{n=2}^{N} (a_{in}X_{in} - a_{jn}X_{jn}). \quad \text{(A2)}$$

Therefore,

$$\sigma_{ij} = \frac{1}{P_M} \cdot \frac{d[a_{i1}P_i - a_{j1}P_j]}{d \log\left(\frac{P_j}{P_i}\right)}$$

$$= \frac{1}{P_M}\frac{d[a_{i1}P_i - a_{j1}P_j]}{dt}\frac{dt}{dy}, \qquad \text{where} \qquad t \equiv \frac{P_j}{P_i}, \qquad y \equiv \log t.$$

Therefore,

$$\sigma_{ij} = \begin{cases} -a_{i1}\left(\frac{P_i}{P_M}\right), & \text{when} \quad dP_j = 0 \quad \text{and} \quad dP_i \neq 0, \\ -a_{j1}\left(\frac{P_j}{P_M}\right), & \text{when} \quad dP_i = 0 \quad \text{and} \quad dP_j \neq 0, \end{cases}$$

because

$$dt = -\frac{P_j}{P_i^2}\,dP_i + \frac{1}{P_i}\,dP_j,$$

and

$$\frac{dP_i}{dt} = -\frac{P_i^2}{P_j}, \qquad \text{when} \qquad dP_j = 0,$$

$$\frac{dP_j}{dt} = P_i, \qquad \text{when} \qquad dP_i = 0,$$

$$\frac{dt}{dy} = t = \frac{P_j}{P_i}.$$

Therefore,

$$\sigma_{ij} = \frac{1}{P_M}\left(-a_{i1}\frac{P_i^2}{P_j}\right)\left(\frac{P_j}{P_i}\right) = -a_{i1}\left(\frac{P_i}{P_M}\right), \qquad \text{when} \qquad dP_j = 0,$$

$$= \frac{1}{P_M}(-a_{j1}P_i)\left(\frac{P_j}{P_i}\right) = -a_{j1}\left(\frac{P_j}{P_M}\right), \qquad \text{when} \qquad dP_i = 0.$$

☐ **C: Cross price elasticities.**

With respect to the price of "base" mode (P_M).

It is possible to interpret (S_i/S_M) as the ratio of demands between modes i and M. Then the demand for the ith mode can be written as:

$$S_i = S_M\cdot\exp\left\{a_{i0} + a_{i1}\left(\frac{P_i}{P_M}\right) + \sum_{n=2}^{N} a_{in}X_{in}\right\}$$

$$E_{iM} \equiv \frac{\partial S_i}{\partial P_M}\frac{P_M}{S_i}$$

$$= \frac{P_M}{S_i}\left[\frac{\partial S_M}{\partial P_M}\cdot\left(\frac{S_i}{S_M}\right) + S_M\cdot\left(\frac{S_i}{S_M}\right)\cdot\left(\frac{-a_{i1}P_i}{P_M^2}\right)\right]$$

$$= \frac{\partial S_M}{\partial P_M}\frac{P_M}{S_M} - a_{i1}\left(\frac{P_i}{P_M}\right).$$

Therefore,

$$E_{iM} = E_{MM} - a_{i1}\left(\frac{P_i}{P_M}\right) \qquad \text{for all} \qquad i = 1, 2, \ldots, M-1.$$

With respect to the price of a nonbase mode (P_j).

Using equation (A2) of this Appendix, it is possible to write the demand for the ith mode (S_i) as:

$$S_i = S_j\cdot\exp\left\{(a_{i0} - a_{j0}) + \left[a_{i1}\left(\frac{P_i}{P_M}\right) - a_{j1}\left(\frac{P_j}{P_M}\right)\right] + \sum_{n=2}^{N}[a_{in}X_{in} - a_{jn}X_{jn}]\right\}.$$

$$E_{ij} \equiv \frac{\partial S_i}{\partial P_j}\frac{P_j}{S_i},$$

$$E_{ij} = \frac{P_j}{S_i}\left[\frac{\partial S_j}{\partial P_j}\left(\frac{S_i}{S_j}\right) + S_j \cdot \left(\frac{S_i}{S_j}\right) \cdot \left(-\frac{a_{j1}}{P_M}\right)\right]$$

$$= \frac{\partial S_j}{\partial P_j}\frac{P_j}{S_j} - a_{j1}\left(\frac{P_j}{P_M}\right) .$$

Therefore,

$$E_{ij} = E_{jj} - a_{j1}\left(\frac{P_j}{P_M}\right) \qquad \text{for all} \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j \neq M \\ j \neq i. \end{array}$$

Appendix 2

Derivation of elasticities of substitution and price elasticities

■ **For type 2 (price-difference) logit model.**

$$\log\left(\frac{S_i}{S_M}\right) = a_{i0} + a_1(P_i - P_M) + \sum_{n=2}^{N} a_{in}X_{in} \qquad i = 1, 2, \ldots, M - 1. \qquad \text{(A3)}$$

□ **A: elasticity of substitution between the base mode and any other mode.**

$$\sigma_{iM} \equiv \frac{da_1(P_i - P_M)}{d \log\left(\frac{P_M}{P_i}\right)}$$

$$= \frac{da_1(P_i - P_M)}{dt}\frac{dt}{dy}, \qquad \text{where} \qquad t \equiv \frac{P_M}{P_i}, \qquad y \equiv \log t.$$

Therefore,

$$\sigma_{iM} = \begin{cases} -a_1P_i, & \text{when} \quad dP_M = 0 \quad \text{and} \quad dP_i \neq 0 \\ -a_1P_M, & \text{when} \quad dP_j = 0 \quad \text{and} \quad dP_M \neq 0 \end{cases}$$

$$\text{for all} \qquad i = 1, 2, \ldots, M - 1,$$

because

$$\frac{dt}{dy} = t = \left(\frac{P_M}{P_i}\right),$$

and

$$dt = -\frac{P_M}{P_i^2}dP_i + \frac{1}{P_i}dP_M,$$

$$\frac{dP_i}{dt} = -\frac{P_i^2}{P_M}, \qquad \text{when} \qquad dP_M = 0,$$

$$\frac{dP_j}{dt} = P_i, \qquad \text{when} \qquad dP_i = 0.$$

Therefore,

$$\sigma_{iM} = \begin{cases} a_1\left(\frac{-P_i^2}{P_M}\right)\left(\frac{P_M}{P_i}\right) = -a_1P_i, & \text{when} \quad dP_M = 0, \\ -a_1(P_i)\left(\frac{P_M}{P_i}\right) = -a_1P_M, & \text{when} \quad dP_i = 0. \end{cases}$$

☐ **B: elasticity of substitution between any two nonbase modes *i* and *j*.** Using equation (A3) of this Appendix, the ratio of demands for the *i*th mode to *j*th mode can be written as:

$$\log\left(\frac{S_i}{S_j}\right) = \log\left(\frac{S_i}{S_M}\right) - \log\left(\frac{S_j}{S_M}\right)$$

$$= (a_{i0} - a_{j0}) + \{a_1(P_i - P_M) - a_1(P_j - P_M)\} + \sum_{n=2}^{N}(a_{in}X_{in} - a_{jn}X_{jn})$$

$$= (a_{i0} - a_{j0}) + a_1(P_i - P_j) + \sum_{n=2}^{N}(a_{in}X_{in} - a_{jn}X_{jn}). \tag{A4}$$

Therefore,

$$\sigma_{ij} \equiv \frac{d[a_1(P_i - P_j)]}{d\log\left(\frac{P_j}{P_i}\right)}.$$

Therefore,

$$\sigma_{ij} = \begin{cases} -a_1P_i, & \text{when} \quad dP_j = 0 \quad \text{and} \quad dP_i \neq 0, \\ -a_1P_j, & \text{when} \quad dP_i = 0 \quad \text{and} \quad dP_j \neq 0 \end{cases}$$

$$\text{for all} \quad i \neq j \quad \text{and} \quad i, j = 1, 2, \ldots, M.$$

Note that this result is inclusive of the previous result on σ_{iM}.

☐ **C: cross price elasticities.**

With respect to the price of base mode P_M.

Using equation (A3), it is possible to write the demand for the *i*th mode (S_i) as:

$$S_i = S_M \cdot \exp\{a_{i0} + a_1(P_i - P_M) + \sum_{n=2}^{N} a_{in}X_{in}\}, \quad i = 1, 2, \ldots, M-1.$$

Then the cross price elasticities with respect to the price of base mode P_M become:

$$E_{iM} \equiv \frac{P_M}{S_i}\frac{\partial S_i}{\partial P_M}$$

$$= \frac{P_M}{S_i}\left[\frac{\partial S_M}{\partial P_M}\left(\frac{S_i}{S_M}\right) + S_M \cdot \left(\frac{S_i}{S_M}\right)(-a_1)\right]$$

$$= \frac{\partial S_M}{\partial P_M}\frac{P_M}{S_M} - a_iP_M.$$

Therefore,

$$E_{iM} = E_{MM} - a_iP_M \quad \text{for all} \quad i = 1, 2, \ldots, M-1.$$

With respect to the price of a given nonbase mode j.

Using equation (A4), it is possible to write the demand for *i*th mode as:

$$S_i = S_j \cdot \exp\{(a_{i0} - a_{j0}) + a_1(P_i - P_j) + \sum_{n=2}^{N}(a_{in}X_{in} - a_{jn}X_{jn})\}.$$

Then the cross price elasticity with respect to jth price can be written as:

$$E_{ij} \equiv \frac{\partial S_i}{\partial P_j} \frac{P_j}{S_i}$$

$$= \frac{P_j}{S_i}\left[\frac{\partial S_j}{\partial P_j}\left(\frac{S_i}{S_j}\right) + S_j \cdot \left(\frac{S_i}{S_j}\right)(-a_1)\right]$$

$$= \frac{\partial S_j}{\partial P_j} \frac{P_j}{S_j} - a_1 P_j.$$

Therefore,

$$E_{ij} = E_{jj} - a_1 P_j \qquad \text{for all} \qquad i \neq j, \qquad i, j = 1, 2, \ldots, M.$$

Note that this result is inclusive of the previous result on E_{iM}.

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