CALCULUSES AND FORMAL SYSTEMS

by Haskell B. CURRY, Pennsylvania State University, U.S.A.

In his recent book [1] Lorenzen presents a notion of calculus which is in some respects more precise than similar notions which preceded it. The present paper is a study of this notion and its relation to a notion of formal system which I have advocated elsewhere. [2] The presentation is informal, except that in § 4 there is an explicit theorem about symbolic structure which includes, as a special case, the theorem that the Łukasiewicz parenthesis-free notation can be interpreted in only one way.

1. Definition of an L_1-calculus. The basic notion of Lorenzen will be called here an *L_1-calculus*. It will be described in terms which are not quite literal translations of those which he uses.

We begin with an *alphabet*, say $\mathfrak{A}$, which is simply a finite set of simple symbols, here called *letters*. A finite string (or sequence) of letters will be called a *word*. Letters and words of $\mathfrak{A}$ will sometimes be called $\mathfrak{A}$-letters or $\mathfrak{A}$-words.

Beside the $\mathfrak{A}$-letters we need a supply of additional letters which we call *word-variables*. We suppose they constitute a (possibly infinite) alphabet $\mathfrak{X}$. Words in the alphabet $\mathfrak{A} \cup \mathfrak{X}$ will be called *$\mathfrak{A}$-formulas*.

Given $\mathfrak{A}$, an L_1-calculus $\mathfrak{K}$ over $\mathfrak{A}$ is defined by a set of *rules* each of which is of the form

$$(1) \qquad A_1, A_2, \ldots, A_m \vdash A_0,$$

where the A_i are $\mathfrak{A}$-formulas, and $m > 0$. A rule with $m = 0$ will be called an *initiation rule*; those with $m > 0$ are *derivation rules*.

[1] [17], Chapter 1. (Numbers in brackets refer to the Bibliography at the end.) Similar notions appear in the work of Post (see [20], [21]).

[2] See [12]. An earlier edition is in [7]. Still earlier editions, using slightly different notation are in [10], [8].

If $\mathfrak{A}$-words are substituted for the variables in a rule, so that the A_i become themselves $\mathfrak{A}$-words, we say we have an *instance* of the rule. For a rule or rule-instance (1) the $A_1, \ldots, A_m$ are called the *premises*, A_0 the *conclusion*. The conclusion of an instance of an initiation rule will be called an *initial word* of $\mathfrak{K}$.

The rules are interpreted as an inductive definition of a class of $\mathfrak{A}$-words, called the *theses* of $\mathfrak{K}$ (or $\mathfrak{K}$-theses). Using the terminology of Kleene [1], this inductive definition is as follows: I. (basic step) every initial word of $\mathfrak{K}$ is a thesis; II. (induction step) if all the *premises* of a rule-instance are $\mathfrak{K}$-theses, the conclusion is also a $\mathfrak{K}$-thesis.

Corresponding to this inductive definition there are various techniques for exhibiting a proof that an $\mathfrak{A}$-word is a $\mathfrak{K}$-thesis. Lorenzen gives a very explicit one. However, it is not necessary to go into this here.

If $\mathfrak{A}$ is a finite alphabet (i.e. with only a finite number of letters) and the number of rules (1) is finite, the calculus will be said to have a finite *basis*; otherwise to have an *infinite basis*. Lorenzen considers, at least in the purely logical part of his book, only calculuses with a finite basis. From a foundational point of view that is the only rational procedure; indeed a calculus with an infinite basis can be understood constructively only when it is based on some more fundamental one with a finite basis (cf. Examples 2 and 5 in § 3). Nevertheless, for many purposes, the restriction to a finite basis is irrelevant; since the infinite case can be reduced to a finite one, it is admissible as a generalization.

Lorenzen gives a number of elementary examples of L_1-calculuses. Further examples will be given in § 3.

As Lorenzen points out, it is of no consequence what view we take as to the objects denoted by the $\mathfrak{A}$-letters and $\mathfrak{A}$-words. [2] The study of a calculus is, of course, an activity carried out in the U-language. [3] The $\mathfrak{A}$-letters are to be taken, to begin with, as new symbols to be adjoined to the U-language as nouns; and concatenation of these letters is a process of forming further

[1] See [15], pp. 258 ff.
[2] This is also emphasized by Carnap, e.g. in [3], p. 6.
[3] This term for the language being used was introduced in [6].

U-nouns. But it is immaterial whether these nouns are to be regarded as names of themselves (i.e. as used autonymously), as designating other symbols in some suitable object language, or as designating objects of some other sort. In the latter two cases the objects denoted must have the same degree of definiteness that letters do, and the formation of a word from a sequence of letters must designate some analogous process among those objects. Following Lorenzen (and Hilbert) I shall use these symbols autonymously; a translation into some other idiom would not cause any difficulty.

On the other hand the variables are symbols which are to be used as such in the U-language. They are therefore what I have elsewhere called U-variables.[1] As already stated they stand for unspecified A-words. Likewise the symbol « $\vdash$ » is to be regarded as belonging to the U-language, and plays the role of a verb. If we take « $\vdash$ » as a prefix expressing « is a $\mathcal{K}$-thesis, » then we can regard (1) as an abbreviation for

$$\text{(2)} \quad \begin{array}{c} \text{« If } A_1, \ldots, A_m \text{ are } \mathfrak{A}\text{-words such that} \\ \vdash A_1 \text{ and } \vdash A_2 \text{ and } \ldots \text{ and } \vdash A_m, \text{ then } \vdash A_0. \text{ »} \end{array}$$

Thus (1) is defined terms of the predicate $\vdash$ as in [12] § **2**B5.

2. Generalizations. The notion of L_1-calculus is capable of various generalizations. We shall consider some of them here.

One such generalization was considered by Lorenzen himself. Let $\mathcal{K}_1$ be an L_1-calculus in the alphabet $\mathfrak{A}$. Then we may form a second calculus $\mathcal{K}_2$ as in § 1 except that in the rules (1) we allow a second type of variables, here called $\mathcal{K}_1$-variables, for which arbitrary $\mathcal{K}_1$-theses may be substituted. Lorenzen calls these new variables object-variables.

This suggests a further generalization. Suppose we consider a set of $r + 1$ calculuses $\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_r$, each of which is a set of rules (1). To each $\mathcal{K}_j$ let there be associated a type of variables called *$\mathcal{K}_j$-variables* and let the rules contain variables of any or all

[1] [12], § **2**C.

types. If the « $\vdash$ » in $\mathcal{K}_j$ is interpreted as « is a $\mathcal{K}_j$-thesis, » and the rules are interpreted as in (2), with the understanding that arbitrary $\mathcal{K}_i$-theses can be substituted for the $\mathcal{K}_i$-variables, then the rules form an inductive definition of all $\mathcal{K}_i$-theses simultaneously. We call such a system a *composite calculus of rank r*. The $\mathcal{K}_j$ are called the *component subcalculuses*.

If we admit such composite calculuses, then the notion of word-variable becomes superfluous. For if $\mathfrak{A}$ is the alphabet $a_1, a_2, \ldots$, the $\mathfrak{A}$-words are precisely the theses in a calculus, $\mathcal{K}_0$ as follows.

$$(3) \qquad \begin{array}{rl} \vdash a_i & \qquad i = 1, 2, \ldots \\ x \vdash xa_i & \qquad i = 1, 2, \ldots \end{array}$$

Here the variable x ranges over the $\mathcal{K}_0$-theses. We therefore require, as part of the definition of composite calculus of rank r, that the only variables which occur are the $\mathcal{K}_j$-variables for $j = 0, 1, \ldots, r$. Thus an L_1-calculus over $\mathfrak{A}$ is a composite calculus of rank 1 containing no $\mathcal{K}_1$-variables and such that $\mathcal{K}_0$ is of the special form (2).

Special interest attaches to the case $r = 0$. Such a calculus I shall call an L_0-calculus. There is only one kind of variable in such a calculus; Lorenzen calls these variables eigenvariables; in the present terminology they are simply $\mathcal{K}_0$-variables. They may also be called *thesis-variables*. The calculus with rules (3) is an L_0-calculus; the theses are the A-words.

Most cases of any interest satisfy the following additional condition: any $\mathcal{K}_s$ for $s < r$ contains only $\mathcal{K}_j$-variables for $j < s$; in other words the $\mathcal{K}_j$ for $j < s$ form a composite calculus of rank s. Such a calculus is here called *graduated*.

It is not necessary that all the $\mathcal{K}_j$ have the same alphabet $\mathfrak{A}$; but there is no loss of generality in assuming that they do, since $\mathfrak{A}$ can be the union of the alphabets of all the $\mathcal{K}_j$.

A composite calculus does not represent the most general inductively generated system based upon words of an alphabet. It has two peculiarities. In the first place, it defines, essentially, a set of $r + 1$ unary predicates, whereas we may have systems defining also relations and predicates of arbitrary degree. In the second place, the restriction that the rules have the form (1), where the A_i are formulas, is quite stringent; one can conceive of rules which determine words or word sequences inductively and are not of that character. [1] The term « *syntactical system* » is used here for generalizations of that sort. The sole requirement is that a proposed proof that a word be a thesis of such and such type, or that some specified relation hold between given words, can be effectively checked.

A composite calculus of rank r can always be reduced to an L_1-calculus. In fact let $\mathfrak{A}$ be the alphabet of the calculus, and let t_0, t_1, ..., be letters distinct from each other and from all the letters of $\mathfrak{A}$. For each rule (1) of $\mathcal{K}_j$, form the rule

$$t_j A_1, t_j A_2, \ldots, t_j A_m \vdash t_j A_0 ; \tag{4}$$

and then, for every $\mathcal{K}_i$-variable which occurs, say x, add the premise $t_i x$. The variables can then be taken as word variables; if they are so taken the theses of the resulting calculus $\mathcal{K}_1$ are precisely words of the form $t_j B$ ($j = 0, 1, \ldots, r$) where B is a $\mathcal{K}_j$-thesis.

This reduction is analogous to the reduction of a formal system to a logistic one [2], and has somewhat the same advantages. It can be extended so as to enable us to formulate relations between words, like free occurrences of a variable, in much the same way that the t_j (and $\mathcal{K}_j$-theses) formulate classes of words. [3] Sometimes it is expedient to omit one of the t_j, say t_r. [4] However, for many purposes the formulation as a composite calculus is more natural.

The word-variables arising in the above process can of course be formulated as an L_0-calculus $\mathcal{K}_0$ of form (3). But it may happen that we can get along with a $\mathcal{K}_0$ which is less extensive than this. It will be convenient to extend the term « L_1-*calculus* » to include all calculuses of rank 1 without $\mathcal{K}_1$-variables. When the $\mathcal{K}_0$-theses include all words in the alphabet, the calculus will be called *pantactic*; when the $\mathcal{K}_0$-theses are a proper subclass of these words,

[1] For illustrations of these possibilities see Examples 6 and 7 in § 3.
[2] [12], § 1E2.
[3] See Example 7 in § 3.
[4] The examples given by Rosenbloom in [21] are actually so formulated.

they will be called well-formed words (wefs) [1] and the calculus itself *eutactic.* When a composite calculus is reduced to an L_1-calculus we use these terms relative to the original alphabet; there is evidently no necessity of including the letters $t_0, \ldots, t_r$ among the wefs.

3. Examples. The following examples illustrate some of the principles mentioned in § 2. In these we use letters « x », « y », « z », « u », « v », « w » for variables. We attach superscripts, where necessary, to indicate the component calculus; thus the superscript j indicates that the variable is a $\mathfrak{K}_j$-variable. Where no superscripts occur the variables are understood to be $\mathfrak{K}_0$-variables. We also attach superscripts to « $\vdash$ » in order to avoid certain possibilities of confusion.

EXAMPLE 1. The classical propositional algebra is formulated in Church [5] as follows:
Alphabet: (,), $\supset$, $\neg$, p_1, p_2, ...
$\mathfrak{K}_0$ (propositions)

$$\vdash^0 p_i, \qquad i = 1, 2, \ldots$$

$$x, y \vdash^0 (x \supset y),$$

$$x \vdash^0 (\neg x).$$

$\mathfrak{K}_1$ (assertions)

$$\vdash^1 (x \supset (y \supset x)),$$

$$\vdash^1 \big((x \supset (y \supset z)) \supset ((x \supset y) \supset (x \supset z))\big),$$

$$\vdash^1 \big(((\neg x) \supset (\neg y)) \supset (y \supset x)\big),$$

$$(x \supset y), x \vdash^1 y.$$

EXAMPLE 2. The formulation of Example 1 had an infinite basis. But it is easy (cf. [21]) to get a formulation with a finite basis as follows:
Alphabet: (,), $\supset$, $\neg$, p, a.
$\mathfrak{K}_0$ (variables)

$$\vdash^0 p,$$

$$x \vdash^0 xa.$$

[1] This abbreviation is preferred to « wff » because it can be pronounced.

$\mathfrak{K}_1$ (propositions)

$$\vdash^1 x^0,$$
$$x^1, y^1 \vdash^1 (x^1 \supset y^1),$$
$$x^1 \vdash^1 (\neg\, x^1),$$

$\mathfrak{K}_2$ (assertions)

$$\vdash^2 \big(x^1 \supset (y^1 \supset x^1)\big),$$
$$\vdash^2 \Big(\big(x^1 \supset (y^1 \supset z^1)\big) \supset \big((x^1 \supset y^1) \supset (x \supset z^1)\big)\Big),$$
$$\vdash^2 \Big(\big((\neg\, x^1) \supset (\neg\, y^1)\big) \supset (y^1 \supset x^1)\Big).$$

$x^1 \supset y^1, x^1 \vdash^2 y^1.$

EXAMPLE 3. The following reformulation of Example 2 as an L_1-calculus illustrates the technique at the end of § 2.
Alphabet: (,), $\supset$, $\neg$, p, a, V, P, T.
$\mathfrak{K}_0$ as in (3), for alphabet of Example 2.

$\mathfrak{K}_1$

$$\vdash Vp,$$
$$Vx \vdash Vxa,$$
$$Vx \vdash Px,$$
$$Px, Py \vdash P\,(x \supset y),$$
$$Px \vdash P\,(\neg\, x),$$
$$Px, Py \vdash T\,\big(x \supset (y \supset x)\big),$$
$$Px, Py, Pz \vdash T\,\Big(\big(x \supset (y \supset z)\big) \supset \big((x \supset y) \supset (x \supset z)\big)\Big),$$
$$Px, Py \vdash T\,\Big(\big((\neg\, x) \supset (\neg\, y)\big) \supset (y \supset x)\Big),$$
$$T\,(x \supset y), Tx \vdash Ty.$$

EXAMPLE 4. (Łukasiewicz prefix notation).
Alphabet: two infinite sequences

$$e_1, e_2, e_3, \ldots$$
$$f_1, f_2, f_3, \ldots$$

$\mathfrak{K}_0$

$$\vdash e_i$$
$$x_1, \ldots, x_{n_k} \vdash f_k x_1 x_2 \ldots x_{n_k}.$$

The integer n_k associated with f_k we call the degree of f_k.

EXAMPLE 5. Suppose that in Example 4 there are infinitely many e_i, and for every positive integer n there are infinitely many f_k of degree n. Let m_k be the number of f_i for $i \leq k$ such that $n_i = n_k$. Let $\mathfrak{B}$ be an alphabet consisting of the two letters a, b, and let $\mathfrak{A}$ be the alphabet of Example 4. If r is an integer, let a^r stand for a continuous sequence of r a's. Then we can form a translation from $\mathfrak{A}$ into $\mathfrak{B}$ as follows

$$\begin{array}{ll} e_i & ba^i \\ f_k & bba^{n_k}ba^{m_k}. \end{array}$$

Then it is decidable which $\mathfrak{B}$-words are translations of $\mathfrak{A}$-words, and the original $\mathfrak{A}$-word, when it exists, is uniquely determined. The $\mathfrak{B}$-words which are translations of theses of Example 4 are precisely the $\mathfrak{K}_2$-theses of the following graduated calculus of rank 2 on the alphabet $\mathfrak{B}$:

$\mathfrak{K}_0$ (words beginning with b)

$$\vdash^0 b,$$
$$x \vdash^0 xa,$$
$$x \vdash^0 xb.$$

$\mathfrak{K}_1$ (atoms)

$$\vdash^1 ba,$$
$$x \vdash^1 xa.$$

$\mathfrak{K}_2$ (wefs)

$$\vdash^2 x^1,$$
$$x \vdash^2 bbay^1x,$$
$$x, by^1z \vdash^2 by^1azx.$$

EXAMPLE 6. (Gödel representation.) In Example 5 we have in principle translated Example 4 into an alphabet consisting of two symbols. A translation into an alphabet with one symbol may be obtained by the methods of Gödel. For, if $\mathfrak{C}$ is the alphabet consisting of one symbol c, the $\mathfrak{C}$-words can differ only in the number of occurrences of c; hence they may be identified with the positive natural numbers. Using the ordinary arithmetical notations for such numbers, we can translate from the alphabet $\mathfrak{A}$ (Example 4) into the alphabet $\mathfrak{C}$ thus:

$$\begin{array}{lll} e_i & 2i+1 & i = 1, 2, \ldots \\ f_k & 2^{n_k}\cdot 3^{m_k} & k = 1, 2, \ldots \end{array}$$

where m_k, n_k are as in Example 5. This suggests the following rules, in which $n = n_k$, $m = m_k$, p_j is the jth prime:

$$\begin{array}{lll} \mathfrak{K}_0 & \vdash^0 c & \\ & x \vdash^0 xc & \\ \mathfrak{K}_1 & \vdash^1 2i+1 & i = 1, 2, \ldots \\ & x_1, \ldots, x_n \vdash^1 2^n \cdot 3^m \cdot p_3{}^{x_1} \ldots p_{n+2}{}^{x_n}. & \end{array}$$

This system does not, however, come under our definition of a calculus, because the conclusion of the second rule of $\mathfrak{K}_1$ is not a $\mathfrak{C}$-formula. However, that rule is constructive in the sense that, given any $\mathfrak{C}$-words $x_1, \ldots, x_n$, the conclusion is a (constructively) definite $\mathfrak{C}$-word. Thus we have a syntactical system which is not a calculus. This system has the property that it is decidable whether or not a $\mathfrak{C}$-word is a $\mathfrak{K}$-thesis; and if it is a $\mathfrak{K}$-thesis, then it corresponds to a unique thesis of Example 4. It could be made into a calculus by introducing additional letters, but it would then have no advantage over Example 5.

In the general case there will be infinitely many rules in $\mathfrak{K}_1$. But if the number of letters in $\mathfrak{A}$ is finite, the number of rules is finite also. But in that case simpler representations suffice. If the number of letters in $\mathfrak{A}$ is q, then an $\mathfrak{A}$-word can be regarded simply as a number in the q-adic system. Again, if there is a finite upper bound n to all the n_k, we can replace the expansion in terms of a prime-power product into one in terms of an enumeration of all n-tuples. If such a function is J_n, we can replace the rules of $\mathfrak{K}_1$ by

$$\vdash^1 J_{n+1}(0, i, 0, \ldots, 0),$$

$$x_1, \ldots, x_{n_k} \vdash^1 J_{n+1}(k, x_1, \ldots, x_{n_k}, 0, \ldots, 0).$$

EXAMPLE 7. The following formulation of the Church theory of λ-conversion [1] is an illustration of a syntactical system which is not a calculus. The alphabet consists of the four letters e, a, $*$, λ. The predicates consist of two unary prefixes V, W; one quaternary prefix S, and seven binary infixes L, J, F, F', B, B', $=$. The interpretations of these are as follows, the conditions in parentheses being understood as presuppositions:

Vx	x is a variable,
Wx	x is a wef,
$Sxyzu$	the substitution of y for x in z yields u (Vx, Wy, Wz, Wu),
$x\,L\,y$	x precedes y in the list of variables (Vx, Vy),
$x\,J\,y$	x and y are distinct variables (Vx, Vy),
$x\,F\,y$	x is free in y (Vx, Wy),
$x\,F'\,y$	x is not free in y (Vx, Wy),
$x\,B\,y$	x is bound in y (Vx, Wy),
$x\,B'\,y$	x is not bound in y (Vx, Wy),
$x = y$	x is convertible to y (Wx, Wy).

The system as here formulated presupposes a pantactic $\mathfrak{K}_0$ as in (3). Owing to its complexity, the rules are given names and classified, as follows:

Variables

V_1 $\quad Ve$,
V_2 $\quad Vx \rightarrow Vxa$.

Wefs

W_1 $\quad Vx \rightarrow Wx$,
W_2 $\quad Wx, Wy \rightarrow W{*}xy$,
W_3 $\quad Vx, Wy \rightarrow W\lambda xy$.

[1] The formulation is essentially that of λK-conversion in [12], but differs from it in that substitution is not always defined. The rule (η) is not accepted by Church, and he also does not accept W_3 without the hypothesis $x\,F\,y$. Application is denoted by the $*$-prefix following Chwistek (see [12], p. 31). Note that the formal variables are the words in the following list: e, ea, eaa, ...

Distinctness of variables

L_1 $\quad Vx \rightarrow x\,L\,xa,$
L_2 $\quad x\,L\,y \rightarrow x\,L\,ya,$
J_1 $\quad x\,L\,y \rightarrow x\,J\,y,$
J_2 $\quad x\,J\,y \rightarrow y\,J\,x.$

Free variables

F_1 $\quad Vx \rightarrow x\,F\,x,$
F_2 $\quad x\,F\,y,\ Wz \rightarrow x\,F\,{*}yz,$
F_3 $\quad x\,F\,y,\ Wz \rightarrow x\,F\,{*}zy,$
F_4 $\quad x\,F\,y,\ x\,J\,z \rightarrow x\,F\,\lambda zy,$
F'_1 $\quad x\,J\,y \rightarrow x\,F'\,y,$
F'_2 $\quad x\,F'\,y,\ x\,F'\,z \rightarrow x\,F'\,{*}yz,$
F'_3 $\quad Vx,\ Wy \rightarrow x\,F'\,\lambda xy,$
F'_4 $\quad xF'y,\ Vz \rightarrow x\,F'\,\lambda zy,$

Bound variables

B_1 $\quad Vx,\ Wy, \rightarrow x\,B\,\lambda xy,$
B_2 $\quad x\,B\,y,\ Wz \rightarrow x\,B\,{*}yz,$
B_3 $\quad x\,B\,y,\ Wz \rightarrow x\,B\,{*}zy,$
B_4 $\quad x\,B\,y,\ Vz \rightarrow x\,B\,\lambda zy,$
B'_1 $\quad Vx,\ Vy \rightarrow x\,B'\,y,$
B'_2 $\quad x\,B'\,y,\ x\,B'\,z \rightarrow x\,B'\,{*}yz,$
B'_3 $\quad x\,B'\,y,\ x\,J\,z \rightarrow x\,B'\,\lambda zy.$

Substitution

S_1 $\quad Vx,\ Wy \rightarrow Sxyxy,$
S_2 $\quad Vx,\ Vy,\ x\,J\,y,\ Wu \rightarrow Sxuyy,$
S_3 $\quad Sxuvw,\ Sxuyz \rightarrow Sxu{*}vy{*}wz,$
S_4 $\quad Vx,\ Wu,\ Wv \rightarrow Sxu\lambda xv\lambda xv,$
S_5 $\quad Sxuvw,\ x\,J\,y,\ y\,F'\,u \rightarrow Sxu\lambda yv\lambda yw$
S_6 $\quad Sxuvw,\ x\,J\,y,\ x\,F'\,v \rightarrow Sxu\lambda yv\lambda yw.$

Conversion

(ϱ) $\quad Wx \rightarrow x = x,$
(σ) $\quad x = y \rightarrow y = x,$

$(\tau)\quad x = y,\, y = z \rightarrow x = z,$
$(\mu)\quad x = y,\, Wz \rightarrow {}^*zx = {}^*zy,$
$(\nu)\quad x = y,\, Wz \rightarrow {}^*xz = {}^*yz,$
$(\xi)\quad Vx,\, y = z \rightarrow \lambda xy = \lambda xz,$
$(\alpha)\quad Vx,\, Vy,\, Sxyuv,\, y\,F'\,u \rightarrow \lambda xu = \lambda yv,$
$(\beta)\quad Sxuyz \rightarrow {}^*\lambda xyu = z,$
$(\eta)\quad Wy,\, x\,F'\,y \rightarrow \lambda x{}^*yx = y.$

4. Tectonics. We turn now to the study of the interrelations of the notion of calculus with the other notion mentioned in the introduction, viz. that of formal system. We consider in this section a preliminary aspect of this problem.

A characteristic feature of a formal system is that, instead of starting with words or other linguistic entities, we say that we are dealing with certain unspecified objects called *obs*. We postulate certain primitive obs or *atoms*, and certain *operations* of specified degrees; the obs are then generated from the atoms by these operations. By this it is meant that every ob is obtained from the atoms by a unique process of construction, such as can be exhibited as a genealogical tree; and different processes of construction lead to distinct obs. It is not excluded that concrete objects (or kinds of objects) be taken for the atoms, and specific modes of combinations for the operations, provided that there be one and only one construction for every ob; in particular the obs may be words in a suitable alphabet. Such a concrete realization of a formal system is called a *representation*; and the particular representation constituted by the names for the obs in the U-language is called a *presentation*.

The following question now arises: under what circumstances can the obs of a formal system $\mathfrak{S}$ be represented in the wefs (i.e. theses) of a calculus $\mathcal{K}$ in such a way that the atoms are the initial words and the operations are the applications of the rules? Evidently a necessary and sufficient condition is that it be decidable whether a word in $\mathcal{K}$ is a wef, and that the derivation of every wef in $\mathcal{K}$, when exhibited in tree form, be unique and constructively obtainable. A calculus having this property will be said to be *tectonic*; and the name *tectonics* will designate the subject whose primary concern is this property. This subject is a branch of grammatics, and includes investigations relative to dots, parentheses, brackets, etc. [1]

In this section we are concerned with the case where $\mathcal{K}$ is an L_0-calculus $\mathcal{K}_0$ with a finite or infinite basis. Sufficient conditions are derived that $\mathcal{K}_0$ be tectonic. The principal theorem (Theorem 1) includes as a special case (Corollary 1.1) the theorem that the Łukasiewicz notation (Example 4) is tectonic [2], and also (Corollaries 1.2-1.3) the more usual notations using parentheses.

Let $\mathcal{K}_0$ be subject to the following conditions:

(i) The alphabet $\mathcal{C}$ of $\mathcal{K}_0$ is the union of the two alphabets $\mathcal{A} = \{a_1, a_2, \ldots\}$ and $\mathcal{B} = \{b_1, b_2, \ldots\}$.

(ii) The rules of $\mathcal{K}_0$ are of the form

$$x_1, \ldots, x_m \vdash u_1 u_2 \ldots u_n, \tag{5}$$

where $n \geqq 2$, each u_j is either one of the x_i or a letter of $\mathcal{C}$, and every x_i occurs at least once among the u_j.

(iii) There has been assigned to all the letters of $\mathcal{A}$ and to variables the same numerical weight α, and to each letter b_j of $\mathcal{B}$ a weight β_j, where α and all β_j are rational integers, positive, negative, or zero. Then the weight of any word or formula is the algebraic sum of the weights of its constituent letters.

To state further conditions on $\mathcal{K}_0$ we need two definitions as follows:

Definition 1. A word or formula W has the *property P* just when it has weight α and every proper initial segment of W has a weight $> \alpha$.

Definition 2. A word W will be said to fit the rule (5) *in the weak sense* just when there exist words $W_1, \ldots, W_m$, each having the property P, such that W is obtained from $u_1 \ldots u_n$ by substituting W_i for x_i, $i = 1, 2, \ldots, m$. The word W will be said to fit (5) *in the strong sense* just when, in addition, the $W_1, \ldots, W_m$ are wefs.

[1] For grammatics see [12], § 1D; [10], § 15. These give references to sources.

[2] This theorem is well known; for citations see [21], [7]. The proof given in [1] is the model for that given in the text.

We now proceed to the statement of the principal theorem of this section. This will be preceded by three lemmas, of which the first two are such immediate consequences of the definitions that no proof is necessary.

LEMMA 1. *If the word or formula W, having two or more letters, has the property P, then it begins with a* $\mathfrak{B}$*-letter and ends with a letter of negative weight.*

LEMMA 2. *If the word W has property P, and W′ is formed from W by substituting word(s) having the property P for certain variable(s) in W, then W has the property P.*

LEMMA 3. *If the conclusion of* (5) *has the property P, and W is a* $\mathfrak{C}$*-word, then it is decidable whether W fits the rule in the weak sense. If it does, the* $W_1, \ldots, W_m$ *are uniquely determined.*

Proof. By Lemma 2 the decision is negative for W unless it has Property P. We therefore suppose W does have that property.

By (ii) $n > 1$. Then, by Lemma 1, u_1 must be a $\mathfrak{B}$-letter. If W does not begin with the same letter it cannot fit; if it does we have $W = U_1V_1$, where U_1 is u_1.

Suppose now that for $p \leqq n$ we have found $U_1, \ldots, U_p$, such that $U_1U_2 \ldots U_p$ is obtainable by allowable substitutions from $u_1u_2 \ldots u_p$, and W is $U_1U_2 \ldots U_pV_p$, where V_p may be void. If $p = n$ then we have a fit if and only if V_p is void. We suppose $p < n$. If u_{p+1} is a $\mathfrak{C}$-letter then W cannot fit unless V_p begins with u_{p+1}; if it does we take U_{p+1} to be u_{p+1}, V_p to be $U_{p+1}\ V_{p+1}$, and proceed to the next higher value of p. If u_{p+1} is x_i, then W cannot fit unless V_p has an initial segment, having property P, which can be taken as W_i. If there is such an initial segment, it is unique by Definition 1. If x_i has occurred among the $u_1, u_2, \ldots, u_p$, then this initial segment must agree with the W_i previously determined; if not, we take it as determining W_i. If these conditions are fulfilled we take U_{p+1} to be W_i, V_p to be $U_{p+1}\ V_{p+1}$, and proceed to the next higher value of p.

If the process continues through $p = n$ we have a fit, otherwise not, q.e.d.

THEOREM 1. Let $\mathfrak{K}_0$ *be an* L_0*-calculus satisfying the conditions* (i)-(iii) *and in addition the following:*

(iv) *Every initial word is an* $\mathfrak{A}$*-letter.*

(v) *The conclusion of every rule has property P.*

(vi) *A* $\mathfrak{C}$*-word W can fit at most one of the rules in the strong sense.*

(vii) *If the number of rules is infinite, then, given a* $\mathfrak{C}$*-word W, there is a constructive process for selecting a finite set of rules such that W cannot fit a rule not in the list, even in the weak sense.*

Then,

(a) *Every wef* ($\mathfrak{K}_0$*-thesis*) *has the property P.*

(b) $\mathfrak{K}_0$ *is tectonic.*

Proof. The property *(a)* follows by induction from (iv), (v), and Lemma 2. It remains to prove *(b)*.

If W consists of a single letter, then it must be an initial word. For if a wef is obtained by a derivation rule, whose conclusion (by (ii)) has more than one letter, the word will also have more than one letter. Hence a $\mathfrak{C}$-letter is a wef only if it is an $\mathfrak{A}$-letter, and its unique construction is then a single application of the corresponding initiation rule.

Suppose W contains $s + 1$ letters, and that the tectonic[1] problem is solvable for all words of length $< s$. Then W is a wef if and only if it is obtained by a derivation rule and hence fits that rule in the strong sense. By (vii) we can determine constructively a finite number of rules such that the rule in question must be one of them. By Lemma 3 we can determine constructively all the rules which W fits in the weak sense. For any single case the $W_1, \ldots, W_m$ are words of length $< s$, hence by the hypothesis of the induction we can determine effectively whether there is a rule which W fits in the strong sense. Thus the question of whether W is a wef is decidable.

That the construction is unique then follows by (vi), Lemma 3, and the hypothesis of the induction.

REMARK 1. The condition (vii) is superfluous for calculuses with a finite basis. In many cases we have the stronger condition

[1] That is, the problem of determining whether the word is a wef, and if so what its construction is.

that (vi) holds with respect to a fit in the weak sense, the constructive process (vii) leading to the unique case or showing that there is none.

REMARK 2. Let us examine more closely the conditions under which the construction of Lemma 3 may fail. 1) It may happen that, for some $p > 0$, U_{p+1} is a $\mathfrak{C}$-letter, but the first letter of V_p (V_0 is W) is not U_{p+1}; let us call this *disagreement with respect to constants.* 2) It may happen that a W_i is determined but does not agree with a previous determination; evidently this can happen only when there is more than one occurrence of x_i, and we call it *disagreement due to multiple variables.* 3) It may happen that U_{p+1} is x_i, but V_p begins with a letter of weight $< \alpha$; let us call this *disagreement due to negative beginning.* 4) The only remaining possibility[1] is that U_{p+1} be x_i, V_p begins with a letter of weight $\geqq \alpha$, and yet no W_i is defined. Then the weight of the initial letter must be $> \alpha$; for if it were exactly α, that letter would be the W_i. We can show that this fourth possibility cannot occur if all weights are $\geqq -1$. For let the weight of $u_1 \ldots u_p$ be $\alpha + \gamma$, $\gamma \geqq 1$ and let the first letter of V_p be v and its weight $\alpha + \delta$, $\delta > 0$. Then, since the weight of $u_1 \ldots u_p v$ is $2\alpha + \gamma + \delta$, that of W is α, there must be a decrease of $\alpha + \gamma + \delta \geqq \delta$ units beyond v. There must be then a first point at which a decrease of δ units is achieved. The segment terminating at that point will define a W_i.

This leads to the following corollaries.

COROLLARY 1.1. *The representation of Łukasiewicz* (§ 4, Example 4) *is tectonic; moreover Property P is both necessary and sufficient that a word be a wef.*

Proof. Take $\mathfrak{E}$ to be $e_1, e_2, \ldots$, $\mathfrak{B}$ to be $f_1, f_2, \ldots$ with b_k as f_k. Let $\alpha = -1$, $\beta_k = n_k - 1$. Then conditions (i)-(v) are obviously fulfilled. Disagreement as to constants (cf. Remark 2) can only occur at the beginning; since (by Lemma 1) every word with property P is an e_i or begins with some f_k, there will be agreement at the beginning with one and only one rule. Thus (vi) and (vii) hold

[1] V_p cannot be void unless $p = n$, since W has property P and $U_1 U_2 \ldots U_p$ has weight $> \alpha$. Likewise if $p = n$, V_p must be void if W has property P.

in the strong sense (cf. Remark 1). The tectonicness of $\mathcal{K}_0$ then follows by the theorem. Since there are no multiple occurrences of variables, no letters of weight $< a$, and none of weight < -1, disagreements of types 2)-4) (Remark 2) are impossible. It follows that a W having property P will always fit, in the weak sense, the unique rule which begins with the same constant. The reduction used in the proof of Theorem 1 will show that this weak fit will also be a strong one. Thus P, which is necessary by *(a)* of Theorem 1, is also sufficient, q.e.d.

COROLLARY 1.2. *Let $\mathfrak{B}$ contain the letters (,), let them be assigned the weights $+1$, and -1 respectively, and let all other $\mathfrak{C}$-letters be assigned the weight* 0. *Let the conditions (i), (ii), (iv)-(vii) hold. Then $\mathcal{K}_0$ is tectonic.*

This hardly requires proof. It includes however, the $\mathcal{K}_0$ of Example 1 (§ 3) and, indeed, practically all cases where we present a formal system using parentheses to indicate grouping. This is shown by the following corollary, in which there is a slight extension, which is self-explanatory, of the terms « wef » and « tectonic ».

COROLLARY 1.3. *Let $\mathcal{K}_0$ satisfy* (i), (iv)*; and let $\mathfrak{B}$ and the assignment of weights be as in Corollary 1.2. Let the rules of $\mathcal{K}_0$ consist of any number of rules of the forms*

(a) $$x_1, x_2, \ldots, x_m \vdash (b_j x_1 x_2 \ldots x_m),$$

(b) $$x_1, x_2, \ldots, x_m \vdash (x_1 x_2 \ldots x_m b_j),$$

(c) $$x_1, x_2, \ldots, x_m \vdash (b_j x_1 \ldots x_n b_j),$$

(d) $$x, y \vdash (x b_j y),$$

where the b_j are all distinct from (,), and from each other in distinct rules; and let there be at most one rule

(e) $$x, y \vdash (xy).$$

Let wefs be formed from $\mathcal{K}_0$-theses by omitting certain external parentheses and certain parentheses in cases (d) *and* (e) *according to the principle of association to the left. Then the wefs are tectonic relative to the operations of $\mathcal{K}_0$.*

Proof. Let $\mathcal{K}'_0$ be derived from $\mathcal{K}_0$ by adjoining all rules of the form

$$\text{(f)} \qquad x_0, x_1, \ldots, x_m \vdash (x_0 b_{j1} x_1 b_{j2} \ldots b_{jm} x_m),$$

where $m > 1$ and all b_{jk} are letters appearing in (d) or are omitted entirely. Then all wefs are $\mathcal{K}'_0$-theses or are derived from them by omitting external parentheses. Since $\mathcal{K}'_0$ has obviously the property (v), every wef has the property that it has weight 0, and every initial segment of it has weight $\geqq 0$. Let W be any word having that property. Then W can be expressed in a unique manner on the form $V_1 V_2 \ldots V_q$, where each V_j has property P; in fact, we simply take V_1 as the minimal segment of W, V_2 as that of the rest of W, etc. If $q = 1$, let W' be W; if $q > 1$, let W' be (W). Then W is a wef if and only if W' is a $\mathcal{K}_0'$-thesis.

Now $\mathcal{K}'_0$ satisfies the conditions (i), (ii), (iii), (iv), (v). It also satisfies (vi) in the strong sense of Remark 1. In fact W' can fit one of the rules (*a*)-(*f*), even in the weak sense, only if W has the proper q[1] and has the proper b_j in the proper places. This is possible in only one way. Hence $\mathcal{K}_0$ is tectonic by the same reasoning as in Corollary 1.2. But an application of *(f)* can be analysed in one and only one way into a series of applications of *(d)*, *(e)*. This completes the proof.

Various generalizations would be possible allowing more than one kind of parentheses, superfluous parentheses, certain omissions in cases *(a)*, *(b)*, additional cases similar to *(a)*, *(b)*, *(c)*, etc.

Although Theorem 1 covers several interesting cases, yet it does not exhaust the possibilities of a tectonic notation. It may happen, e.g., that we assign weights which depend on presence of other letters. Various notations involving dots have also a tectonic character; yet they hardly come under Theorem 1. Furthermore, in Example 6 we have a representation of a formal system in a system of notation which is evidently tectonic, but is not a calculus.

[1] In the case where W' is W, q refers to the word obtained by removing outside parentheses from W.

5. Interrelations of formal systems and calculuses. We shall now discuss the question mentioned at the beginning of § 4 from a more general standpoint.

By Corollary 1.1 the calculus of Example 4 is tectonic; if the e_i are taken as the atoms and the f_k as the operations of a formal system $\mathfrak{S}$, then $\mathfrak{S}$ is represented in Example 4. [1] So represented, $\mathfrak{S}$ is by definition a syntactical system. Two conditions are necessary in order that this system be an L_1-calculus: *(a)* that the formal system be logistic, and *(b)* that the rules be such that the conditions of § 1 are satisfied in the representation.

As for condition *(a)* it is known (see [12] § 1E2) that any formal system can be reduced to a logistic one with one auxiliary category. The auxiliary category is a new unary predicate which can itself be eliminated by a technique analogous to that at the close of § 2 (cf. Example 3). There is therefore no difficulty about condition *(a)*.

Let us call a formal system $\mathfrak{S}$ *elementary* just when its deductive rules are of the form

$$S_1 \;\&\; S_2 \;\&\; \ldots \;\&\; S_n \rightarrow S_0, \tag{6}$$

where each S_i is an elementary statement of the extension of $\mathfrak{S}$ formed by adjoining certain indeterminates $x_1, \ldots, x_m$ to the atoms, the rule instances being obtained by specifying the x_i to be specific obs. The above transformations will carry such rules over into rules satisfying the conditions of § 1. This takes care of condition *(b)*.

It follows that an arbitrary formal system can be represented as a syntactical system; it can further be reduced to an L_1-calculus if it is elementary.

We now consider the converse relation. Suppose we have given an L_1-calculus $\mathcal{K}_0$, $\mathcal{K}_1$. We have to find a formal system $\mathfrak{S}$ which is, in a sense to be explained, equivalent to it.

[1] This is on the supposition that $\mathfrak{S}$ is completely formal in the sense that its morphology has the character stated in [12], p. 32. Only such formal systems are considered in this paper. However we do not, without explicit mention, impose restrictions on the deductive rules beyond the requirement of constructiveness. Any vagueness on that point in [12] is irrelevant. Cf. the notion of elementary formal system to be presently defined.

If $\mathcal{K}_0$ is tectonic, and if each of the A_i in the rules of $\mathcal{K}_0$ can be constructed from the initial words and the variables by means of the operations of $\mathcal{K}_0$—i.e., if the A_i become $\mathcal{K}_0$-theses when the x_j are added to the initial words—, then the translation of the calculus into a formal system is immediate. In fact the calculus is by definition an elementary formal system represented in the $\mathcal{K}_0$-theses. This is the case in practically all cases actually used in ordinary logic and mathematics, provided one takes for $\mathcal{K}_0$ the eutactic calculus defining the « formulas », « wefs », or what not.

In general, however, this will not be the case. The A_i are, in fact, arbitrary strings of $\mathfrak{A}$-letters and variables; and it is not always possible to form them from the $\mathfrak{A}$-letters and variables by the operations of $\mathcal{K}_0$. Thus, if $\mathcal{K}_0$ is the pantactic calculus (3) formed on a, b, then the operations of $\mathcal{K}_0$ are postfixing a and postfixing b; and it is not possible to form the $\mathfrak{A}$-formula ax from a, b, x, by these operations. It is therefore necessary to take account of the fact that the $\mathfrak{A}$-formulas are formed from $\mathfrak{A}$-words (or $\mathfrak{A}$-letters) and variables by the general operation of word formation, which we call *concatenation*.

If $\mathcal{K}_0$ is a pantactic system (3), then the concatenation operation can be defined by a recursive process. In order to avoid confusion with the operations of $\mathcal{K}_0$, we denote concatenation by the infix «$\wedge$». Let $\mathcal{K}'_0$ be the calculus formed by adjoining to $\mathcal{K}_0$ the rule

$$(7) \qquad x, y \vdash (x \wedge y).$$

With any $\mathcal{K}_0'$-thesis A' we associate the $\mathcal{K}_0$-thesis A, called the *definiens* of A', which is obtained by omitting the letters (,), $\wedge$. This can be obtained by a series of replacements of occurrences of a left side of one of the schemes

$$(8) \qquad \begin{aligned} (x \wedge a_i) &= xa_i, \\ (x \wedge (y \wedge a_i)) &= (x \wedge y)\, a_i, \end{aligned}$$

by an occurrence of the corresponding right side. Now let us interpret a rule (1) of $\mathcal{K}_1$ in the following fashion. We regard A_1, $\ldots$, A_m as formed from $\mathcal{K}_0$-theses and variables by concatenation;

if arbitrary $\mathfrak{K}_0$-theses are substituted for the variables these become certain theses of $\mathfrak{K}'_0$; then (1) says that if the definientia of A_1, ..., A_m are $\mathfrak{K}_1$-theses, then the definiens of A_0 is a $\mathfrak{K}_1$-thesis. The so modified rules have the same effect as the original $\mathfrak{K}_1$-rules no matter how the concatenation operations are associated. Choosing, for definiteness, the convention that the maximal constant segments in the A_i be taken as $\mathfrak{K}_0$-theses, and that concatenation between these elements and variables is associative to the left, the rules can be interpreted as deductive rules in a logistic formal system $\mathfrak{S}$ in which the $\mathfrak{K}_0$-theses are obs and «is a $\mathfrak{K}_1$-thesis» is the unary predicate. The rules however, are not elementary. The system is related to an elementary system much as Example 6 is related to a calculus.

To get an elementary system we must take concatenation as one of the operations of the system and formalize its associativity. Still assuming that $\mathfrak{K}_0$ is (3), we proceed as follows.[1] Since the operations of $\mathfrak{K}_0$ can be taken as special cases of concatenation, we omit them; we can then omit the symbol for concatenation. We take in the place of $\mathfrak{K}_0$ the calculus $\mathfrak{K}_0'$, on the alphabet formed by adding parentheses to $\mathfrak{A}$, with the rules:

(9)
$$\vdash a_i,$$
$$x, y \vdash (xy).$$

Inasmuch as $\mathfrak{K}''_0$ is tectonic (Corollary 1.3), we can take its theses as obs. Moreover we can adopt the usual conventions for omission of superfluous parentheses; viz. the rule of association to the left, and that allowing the omission of the outermost parentheses. With this understanding the $\mathfrak{K}_0$-theses are included among the $\mathfrak{K}''_0$-theses. Let the $\mathfrak{K}_0$-thesis A which is obtained from a $\mathfrak{K}''_0$-thesis A'' by omission of parentheses be called the definiens of A''. We take as primitive predicates a unary one, designated by prefixed «$\vdash$», and a binary one, designated by infixed «$=$»; we suppose that neither of these symbols occurs in the alphabet of $\mathfrak{K}_0$ or $\mathfrak{K}_1$. The interpretations of these predicates are «has a $\mathfrak{K}_1$-thesis as

[1] Cf. [14], [22], [24], [25].

definiens » and « has the same definiens as ». According to the convention for omitting parentheses, the rules of $\mathcal{K}_1$ can be interpreted as deductive rules for $\vdash$. We adopt them as such; and also the following (where the variables range over the obs $\mathcal{K}_0$-theses);

$$\begin{gathered} x = x, \\ x(yz) = xyz, \\ x = y \;\&\; y = z \rightarrow x = z, \\ x = y \rightarrow zx = zy, \\ x = y \rightarrow xz = yz, \\ x = y \rightarrow y = x, \\ x = y \;\&\; x \rightarrow y. \end{gathered}$$

We then have an elementary formal system for which the interpretation above given is valid.

The restriction to a pantactic $\mathcal{K}_0$, made in the last two reductions, is no loss of generality; because the reduction of § 2 leads to that eventuality, the case of a eutactic K_0 being merely an alternative possibility. Thus, given any L_1-calculus ($\mathcal{K}_0$, $\mathcal{K}_1$), we can find a logistic formal system, in general non-elementary, whose obs are the $\mathcal{K}_0$-theses and whose assertions are the $\mathcal{K}_1$-theses; and an elementary formal system, with assertions and formalized equality, whose obs are the constructions (by concatenation) of the $\mathcal{K}_0$-theses, whose assertions are constructions of the $\mathcal{K}_1$-theses, and such that two constructions of the same $\mathcal{K}_0$-thesis are equal. The first of these transformations goes through even for a syntactical system which is not a calculus.

5. **Concluding remarks.** The following are suggested by the foregoing discussion.

(a) Calculuses and formal systems are alike in that they are inductively generated systems concerned with unspecified objects. Neither notion commits one to the view that logic and mathematics have symbolism as their main subject matter. On the other hand both notions are equally compatible with that view.

(b) The two notions differ in the way their objects are generated. In a syntactical system the objects are generated by an

associative operation of concatenation, and this associativity has to be observed by inspection. In a formal system the obs are generated by operations whose properties are postulated explicitly. In this respect the notion of formal system is the more rigorous concept of the two; but, since the notions are intertranslatable, this difference in rigor is of little consequence.

(c) The notion of formal system puts less emphasis on linguistic accidents. Thus, if we were to translate Example 1 into the notation of Łukasiewicz, the two systems, as calculuses, would be quite different; from the standpoint of formal systems, however, one would regard these as merely different presentations of the same formal system. Thus the notion of formal system agrees with the tendency in mathematics to seek intrinsic, invariant formulations, such as vectors, projective geometries, topological spaces, etc.

(d) The difficulties in the translation from a calculus to a formal system in § 5 do not arise in the systems of ordinary logic and mathematics. These systems accommodate themselves rather naturally to the notion of formal system; they are brought under the concept of calculus only with some forcing.

Bibliography

(Abbreviations conform to practice of Mathematical Reviews)

[1] Burks A. W., Warren D. W. and Wright J. B., *An analysis of a logical machine using parenthesis-free notation.* Math. Tables Aids Comput. 8 : 53-57, Washington, D. C., 1954.

[2] Carnap R., *Introduction to semantics.* Cambridge, Mass., 1942.

[3] Carnap R., *The logical syntax of language.* (Translation by Amethe Smeaton.) London and New York, 1937.

[4] Church A., *The calculi of lambda-conversion.* Ann. of Math. Studies 6. Princeton, N. J., 1941 ; 2nd ed. 1951.

[5] Church A., *Introduction to mathematical logic.* Part 1. Second edition. Princeton, N. J., 1956.

[6] Curry H. B., *Languages and formal systems.* Proc. Xth International Congress of Philosophy 1949 : 770-772.

[7] CURRY H. B., *Leçons de logique algébrique.* Paris and Louvain, 1952.

[8] CURRY H. B., *Outlines of a formalist philosophy of mathematics.* Amsterdam, 1951.

[9] CURRY H. B., *Les systèmes formels et les langues.* Colloques internationaux du Centre national de la recherche scientifique, 36, Paris, 1950, pp. 1-10 (1953).

[10] CURRY H. B., *A theory of formal deducibility.* Notre Dame Mathematical Lectures, 6, Notre Dame, Indiana, (1950); 2nd ed. 1952.

[11] CURRY H. B., *On the use of dots as brackets in logical expressions.* Journal of Symbolic Logic 2 : 26-28 (1937).

[12] CURRY H. B. and FEYS R., *Combinatory logic.* Amsterdam, 1958.

[13] GÖDEL K., *Über formal unentscheidbare Sätze der principia mathematica und verwandter Systeme 1.* Monatsh. Math. Phys. 38 : 173-198 (1931).

[14] HERMES H., *Semiotik.* Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, n. F., Heft 5 : 5-22 (1938).

[15] KLEENE S. C., *Introduction to metamathematics.* Amsterdam and Groningen, 1952.

[16] KLEENE S. C., *Proof by cases in formal logic.* Ann. of Math. (2) 35 : 529-544 (1934).

[17] LORENZEN P., *Einführung in die operative Logik und Mathematik.* Berlin-Göttingen-Heidelberg, 1955.

[18] ŁUKASIEWICZ Jan., *Sur la formalisation des théories mathématiques.* Colloques internationaux du Centre national de la recherche scientifique, 36, Paris, 1950, pp. 11-21.

[19] MARKOV A. A., *Teoriya algorifmov. (Theory of algorithms.)* Trudy Mat. Inst. Steklov, No. 42. Moscow, 1954.

[20] POST E. L., *Formal reductions of the general combinatorial decision problem.* Amer. J. Math. 65 : 197-215 (1943).

[21] ROSENBLOOM P. C., *The elements of mathematical logic.* New York, 1950.

[22] SCHRÖTER K., *Ein allgemeiner Kalkülbegriff.* Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, n. F., 6 (1941).

[23] ŠANIN N. A., *O nekotoryh logičeskih problemah aritmetiki. (On some logical problems of arithmetic.)* Trudy Mat. Inst. Steklov, No. 43. Moscow, 1955.

[24] TARSKI A., *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und der ω-Vollständigkeit.* Monatsh. Math. Physik 40 : 97-112 (1933).

[25] TARSKI A., *Der Wahrheitsbegriff in den formalisierten Sprachen.* Studia Philos. 1 : 261-405 (1936).

Abstract

Lorenzen, in his book *Einführung in die operative Logik und Mathematik* has given a relatively precise form of syntactical system which he calls a calculus. The present paper deals with the relationship of Lorenzen's notion of calculus with the notion of formal system (as explained, for example, in *Outlines of a formalist philosophy of Mathematics*, Amsterdam, 1951). It is shown that the obs of a formal system can be represented as the theses of a calculus of a certain type just when the calculus has a property called the tectonic property, and conditions are given under which one form of system can be transformed into the other.

Zusammenfassung

Lorenzen hat in seinem Buch, *Einführung in die operative Logik und Mathematik*, eine ziemlich genaue Auffassung eines syntaktischen Systems angegeben, die er Kalkül nennt. Die vorliegende Abhandlung betrifft die Verwandtschaft dieses Lorenzenschen Begriffs mit dem eines formalen Systems (wie z. B. in *Outlines of a formalist philosophy of mathematics*, Amsterdam, 1951, angegeben). Es wird bewiesen, dass die « Obs » eines formalen Systems sich als die Thesen eines passenden Kalküls genau dann repräsentieren lassen, wenn dieser Kalkül eine gewisse Eigenschaft, die «tectonic property» heisst, besitzt; die Bedingungen für die Überführung einer Art von System in die andere werden aufgezeigt.

Résumé

Lorenzen, dans son livre *Einführung in die operative Logik und Mathematik*, a exposé une forme assez précise de système syntaxique qu'il a appelée un calcul. L'article actuel traite les relations entre cette idée et celle d'un système formel (comme elle a été exposée, par exemple, dans *Outlines of a formalist philosophy of mathematics*, Amsterdam, 1951). L'auteur démontre que les « obs » d'un système formel peuvent se représenter comme les thèses d'un calcul d'une certaine espèce justement quand le calcul a une propriété que l'on appelle « the tectonic property »; et il énonce des conditions pour qu'un type de système soit transformable en l'autre.