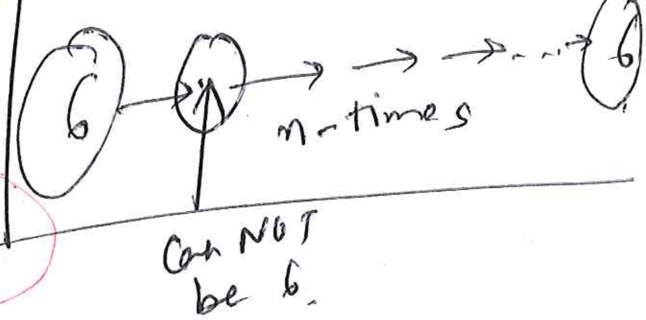
12/8

Example: A die is "fixed" so that each time it is rolled the score can not be the same as the preceding score, all other scores having probability $= \frac{1}{5}$.

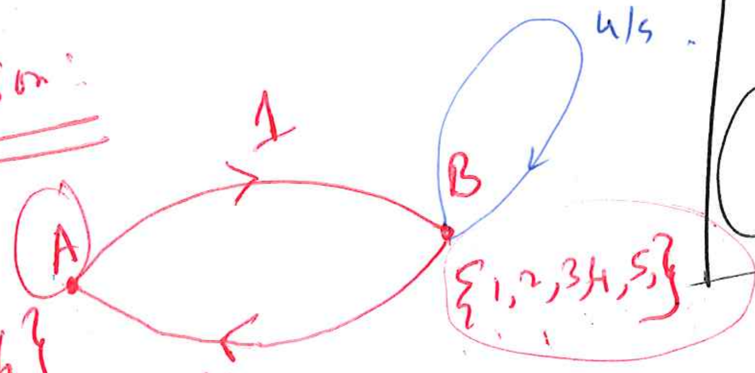
If the first score is 6, what is the probability p that the n^{th} score is 6?

What is the probability that the n^{th} score is 1?

Outcomes of a die: $\{1, 2, 3, 4, 5, 6\}$



Solution:



$$P^{(n)}_{\cancel{\{6\}\{6\}}\ 11} = \frac{1/5}{1+1/5} + \frac{1}{1+1/5}\left(1-\frac{6}{5}\right)^n$$

$$\left(\alpha = 1, \quad \beta = \frac{1}{5}\right)$$

A side (some class back)

$$P_{11}^{(n)} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n, \quad \alpha+\beta > 0$$

$P^{(n)}_{\{6\}\{6\}}$

$$\boxed{P_{11}^{(n)} = \frac{1}{6} + \frac{5}{6}\left(-\frac{1}{5}\right)^n .}$$

Probability that the n^{th} ~~stating~~ score stays at $\{6\}$ starting from $\{6\}$

HW Show that (below shown)

$$P_{12}^{(n)} = \frac{5}{6} - \frac{5}{6}\left(-\frac{1}{5}\right)^n$$

$$P = \begin{pmatrix} 0 & 1 \\ \frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

Starting at $\{6\}$ after n-steps we are at $\{1, 2, 3, 4, 5\}$

After n-steps the score $= \{1\}$ is with probability

$$\boxed{\frac{1}{5}\left(\frac{5}{6} - \frac{5}{6}\left(-\frac{1}{5}\right)^n\right)}$$

$$P^n = \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} \end{pmatrix}$$

We know (found) $\quad P_{11}^{(n)} = \frac{1}{6} + \frac{5}{6}\left(-\frac{1}{5}\right)^n$

Hence, $\quad P_{12}^{(n)} = 1 - \left(\frac{1}{6} + \frac{5}{6}\left(-\frac{1}{5}\right)^n\right)$

$$P_{12}^{(n)} = \frac{5}{6} - \frac{5}{6}\left(-\frac{1}{5}\right)^n$$

Class Structure

Markov chain $\longrightarrow$ decompose in smaller "class".

We say that "state" i leads to "state" j if

$$P_{ij}^{(n)} > 0, \quad \text{for some } n \geq 0$$

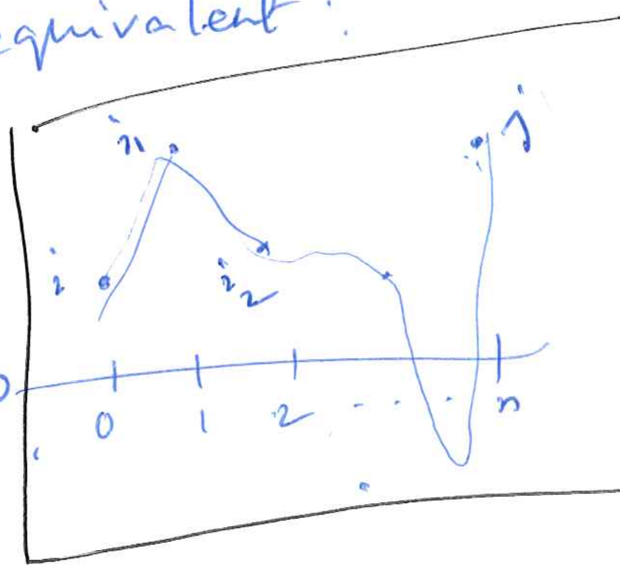
$\Rightarrow$ In this case we write

$$i \longrightarrow j$$

Theorem: for distinct states i and j the following are equivalent:

(1) $i \to j$

(2) $P_{ij}^{(n)} > 0$ for some $n \geq 0$



(3)

$P_{ii_1} P_{i_1 i_2} P_{i_2 i_3} \cdots P_{i_{n-1} j} > 0$

for some states $i_1, i_2, \ldots, i_{n-1}$.

$\cancel{P_{ij}}$

P_{ij}

Aside: Suppose

$i = 1$
$i_1 = 10$
$i_2 = \pi$
$i_3 = \sqrt{2}$
$j = 4$

$P_{ii_1} P_{i_1 i_2} P_{i_2 i_3} P_{i_3 j}$

$= P_{(1)(10)} P_{(10)(\pi)} P_{(\pi)\sqrt{2}} P_{(\sqrt{2})4}$

understood as

$\cancel{P_{12}\, P_{23}}$

$P_{01}\, P_{12}\, P_{23} \cdot P_{34}$

We say state i is **COMMUNICATING** with state j if

$i \to j$ and $j \to i$

<u>Notation:</u> $i \longleftrightarrow j$

~~Equivalent class~~

~~A class C is closed if $i \in C$, $i \to j \Rightarrow j \in C$.~~

<u>Equivalence relation</u> (R)

(1) Reflexive. $x R x$ (Example "=" $x = x$)

(2) Symmetric

if $x R y$ then $y R x$

(Example "=" $x = y \Rightarrow y = x$).

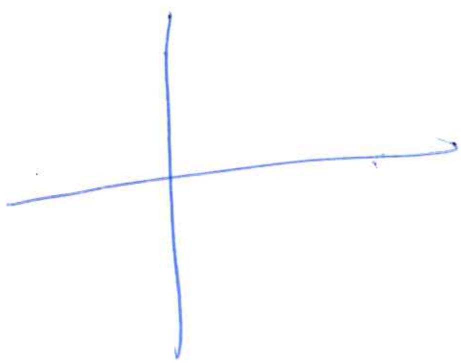
Non-example "<" $x < y \not\Rightarrow y < x$

(3) Transitive:

if xRy, yRz then xRz

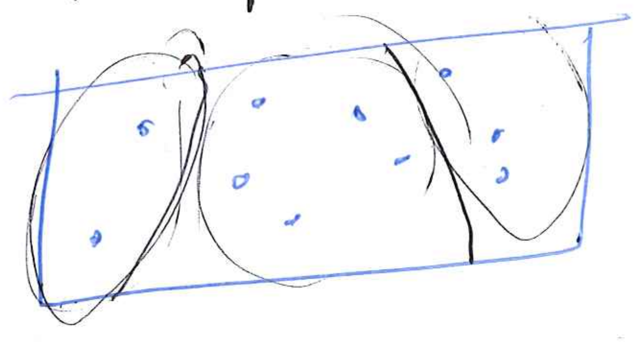
Example "=" $x=y$, $y=z$
$\Rightarrow x=z$

">" $x>y$, $y>z$
$\Rightarrow x>z$



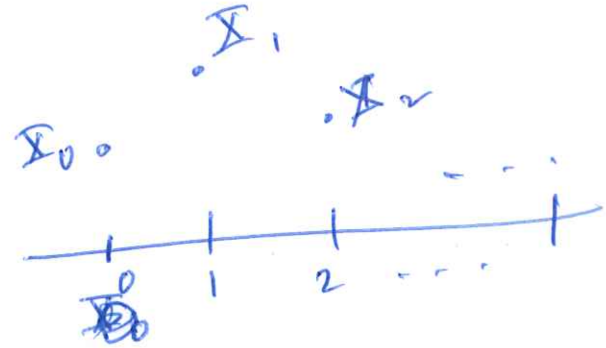
If we have a relation that is reflexive, symmetric and transitive $\Rightarrow$ equivalence relation

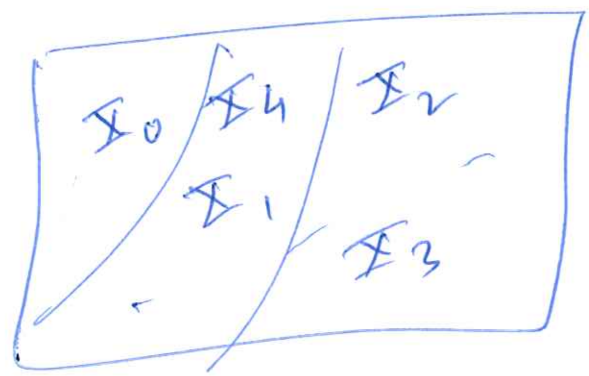
Equivalence relation partitions a given set in equivalence classes.



1. "Communicating class" is an equivalence relation. ✓

2. We say a class C is <u>closed</u> if $i \in C,\ i \to j \Rightarrow j \in C$.





<u>Example:</u> Find the <u>communicating classes</u> associated to the transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

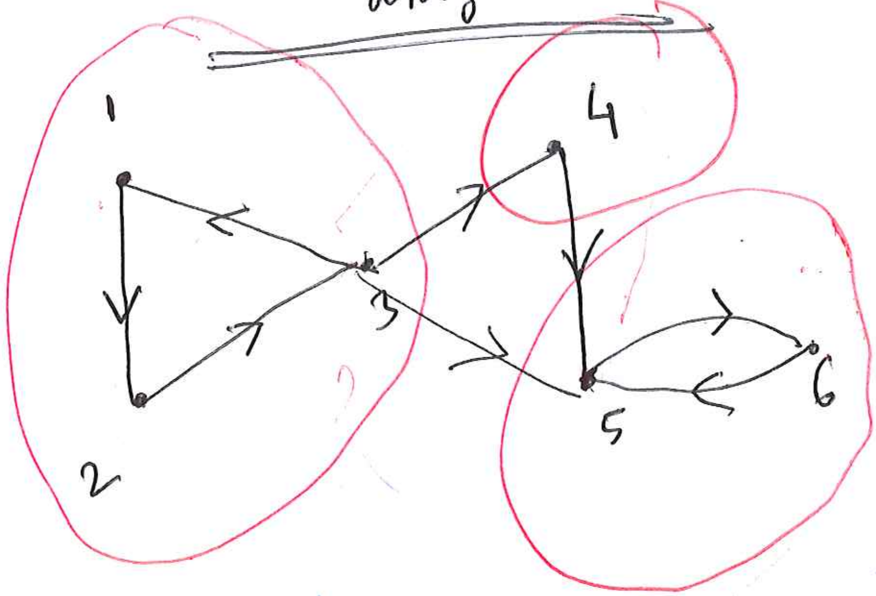
Solution:

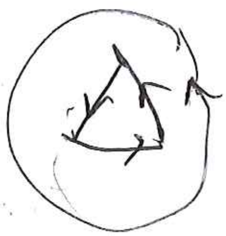
1 2 3 4 5 6

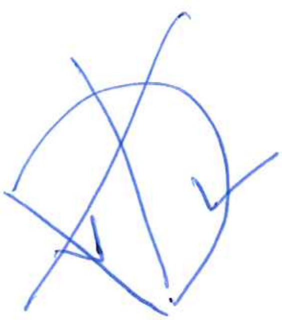
My list:

- 1 → 1 ✓
- 1 → 2 ✓
- 2 → 3 ✓
- 3 → 1 ✓
- 3 → 4 ✓
- 3 → 5 ✓
- 4 → 4 ✓
- 4 → 5 ✓
- 5 → 6 ✓
- 6 → 5 ✓

Communicating diagram







Equivalent / Communicating class:

$\{1, 2, 3\}$, $\{4\}$, $\{5, 6\}$

The ONLY closed class

Example (1.2.1)

Identify the communicating classes of the following transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

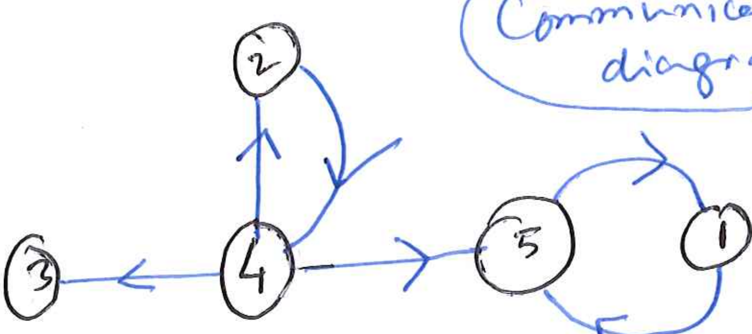
Which classes are closed?

Solution:

1 2 3 4 5

Communicating diagram

$1 \to 5$ ✓
$2 \to 4$ ✓
$4 \to 2$ ✓
$4 \to 3$ ✓
$4 \to 5$ ✓
$5 \to 1$ ✓



Equivalence / communicating classes

✓ $\{1, 5\}$, $\{2, 4\}$, $\{3\}$

closed class:

$\checkmark$ $\{1, 5\}$, $\{3\}$