chapter 2:

2.1 a). exponential distn

Hazard rate $= 0.001$.

$\lambda = 0.001$.

mean life time $= \frac{1}{\lambda} = \frac{1}{0.001} = \underline{1000}$.

b) median lifetime of the bulb $\Rightarrow$.

$Pr[X > m] = \frac{1}{2}$.

$\lambda = 0.001$.

$$\int_m^\infty 0.001 e^{-0.001x} dx = \frac{1}{2}.$$

$$\int_m^\infty \frac{1}{1000} e^{-x/1000} dx = \frac{1}{2}.$$

$x/1000 = v$

$\frac{dx}{1000} = dv$

$$\int e^{-v} dv = e^{-v} + c.$$

$$e^{-m/1000} = \frac{1}{2}.$$

$$-m/1000 = -\ln 2.$$

$$\underline{m = 1000 \ln 2}.$$

c) probability of lifetime after 2000 hrs of use $\Rightarrow$

c). $P[X > 2000] = S(X = 2000)$..

$= e^{-2000/1000} = \frac{1}{e^2}$

2.3. Weibull distⁿ $\alpha = 2$, $\lambda = 0.001$.

pdf $\Rightarrow \alpha \lambda x^{\alpha - 1} e^{-\lambda x^\alpha}$

i) Rat will be ~~tumor~~ free at 30 days.

$P(X > 30) = S(X = 30)$.

survival functⁿ $= e^{-\lambda x^\alpha}$.

$= e^{-\frac{30^2}{1000}}$

$= \frac{1}{e^{\frac{900}{1000}}} = \underline{\underline{0.4}}$.

$S(X = 45) = e^{-45^2/1000}$

$= 0.132$.

$S(X = 60) = e^{-60^2/1000}$.

$= 0.027$.

ii) Mean time of ~~tumor~~. $= \frac{\Gamma(1 + 1/\alpha)}{\lambda^{1/\alpha}}$.

$= \frac{\Gamma(1 + 1/2)}{(0.001)^{1/2}} = \sqrt{1000}\,\Gamma(\tfrac{1}{2}) \cdot \tfrac{1}{2}$

$= \frac{10\sqrt{10\pi}}{2}$

$= \underline{\underline{5\sqrt{10\pi}}}$

c) Hazard rate of weibull $\Rightarrow$ $\alpha \lambda x^{\alpha-1}$

$x = 30$ $\quad = 2 (0.001)(30)^1$

$= \underline{0.06}$

$x = 45$ $\quad = 2 \times (0.001)(45)$

$= 0.09$

$x = 60$ $\quad = 2 \times (0.001) \times 60$

$= 0.12$

d) Mean time of tumour $\Rightarrow$

$Pr[X > m] = \frac{1}{2}$

$S(m) = \frac{1}{2}$

$e^{-\lambda m^\alpha} = \frac{1}{2}$

$e^{-(0.001) m^2} = \frac{1}{2}$

$\cancel{-}(0.001) m^2 = \cancel{-} \ln 2$

$m^2 = 1000 \ln 2$

$m = \sqrt{1000 \ln 2} = 17.35$

2.3) log logistic distn. pdf $\Rightarrow$ $\frac{\alpha x^{\alpha-1} \lambda}{[1 + \lambda x^\alpha]^2}$

a) survival function of loglogistic $= \frac{1}{[1 + \lambda x^\alpha]}$

$P[X > 50] = S(x = 50)$

$= \dfrac{1}{1 + 0.01 \times 50^{1.5}} = 0.220.$

$P[X > 100] = S(x = 100)$

$= \dfrac{1}{1 + 0.01 \times 100^{1.5}} = 0.09.$

$P[X > 150] = S(x = 150)$

$= \dfrac{1}{1 + 0.01 \times 150^{1.5}} = 0.05$

b) Median time.

$P[X > m] = \frac{1}{2}.$

$$\frac{1}{1 + \lambda x^{\alpha}} = \frac{1}{2}.$$

$$\frac{1}{1 + 0.01 \times m^{1.5}} = \frac{1}{2}.$$

$$1 + 0.01 \times m^{1.5} = 2.$$

$$0.01 \times m^{1.5} = 1.$$

$$m^{1.5} = 100.$$

$$m = (100)^{2/3}.$$

$$= 21.54 \text{ days.}$$

c). $\dfrac{\alpha x^{\alpha - 1} \lambda}{(1 + \lambda x^{\alpha})}$ → hazard rate.

$\dfrac{d^2 h(x)}{dx^2} > 0.$ $\quad \dfrac{0.01 x^{0.5} \times 1.5}{1 + 0.01 x^{1.5}} = \dfrac{0.15 x^{0.5}}{1 + 0.01 x^{1.5}}$

$$h(x) = \frac{0.15\, x^{0.5}}{1+0.01\, x^{1.5}}$$

$$\frac{d}{dx} h(x) = 0.$$

$$\cancel{(1+0.01\, x^{1.5})\left(\frac{0.15}{2} x^{-1/2}\right) - (0.15\, x^{0.5})(0.15\, x^{0.5})} = 0.$$

$$(0.15)^2\, x =$$

$$0.015\, x^{0.5}\,(0.5 - 0.01\, x^{1.5}) = 0.$$

$$0.5 = 0.01\, x^{1.5}$$

$$x = \sqrt[1.5]{50}$$

$$= 13.572.$$

$x > 13.572$ decreasing.

< 13.572 increasing.

d) Mean time to death.

$$f(x) = \frac{0.015\, x^{0.5}}{(1+0.01\, x^{1.5})^2}.$$

$$E(X) = \int_0^{\infty} x \cdot \frac{0.015\, x^{0.5}}{(1+0.01\, x^{1.5})^2}\, dx.$$

$$\cancel{u = \frac{0.015\, x^{0.5}}{(1+0.01\, x^{1.5})}\, dx}.$$

$$u = \frac{1}{1+0.01\, x^{1.5}}.$$

$$du = \frac{-1}{(1+0.01\, x^{1.5})^2}\, dx.$$

$x = \sqrt[1.5]{100\left(\frac{1}{u}-1\right)}$

$E(x) = \int_0^1 \sqrt[1.5]{100\left(\frac{1}{u}-1\right)}\, du.$

$= 21.54 \int_0^1 \left(\frac{1}{u}-1\right)^{1/1.5} du.$

by Beta integral,

$$21.54 \frac{\Gamma\frac{0.5}{1.5}\,\Gamma\frac{2.5}{1.5}}{\Gamma\left(\frac{0.5}{1.5}+\frac{2.5}{1.5}\right)} = 21.54 \times 2.418.$$

$= 52.102$ days.

2.4. survival funcⁿ →

a) $S(x) = e^{(1-e^{\lambda x^{\alpha}})}$. $\alpha = 0.5$.

$\frac{d}{dx}\cancel{\left(e^{1-e^{\lambda x^{\frac{1}{2}}}}\right)} = 0$.

$\cancel{e^{1-e^{\lambda x^{\frac{1}{2}}}} \cdot \left(-e^{\lambda x^{1/2}}\right)} \left(\frac{\lambda}{2} x^{-1/2}\right) = 0.$

$\log(S(x)) = [1 - e^{(\lambda x)^{\frac{1}{2}}}]$

$-\frac{d}{dx}\log(S(x)) \cancel{= 0} =$

$-e^{(\lambda x)^{\frac{1}{2}}}\left(\frac{\lambda}{x}\right)^{\frac{1}{2}} \cdot \frac{1}{2}.$

$h''(x) = -[-\frac{1}{4}\lambda^{\frac{1}{2}} x^{-3/2} e^{(\lambda x)^{\frac{1}{2}}}$

$+ \frac{1}{4}\left(\frac{\lambda}{x}\right) \exp((\lambda x)^{\frac{1}{2}})]$

$= 0.$

$$\cancel{\frac{1}{A}} \frac{\lambda}{x} e^{(\lambda x)^{\frac{1}{2}}} = \cancel{\frac{1}{A}} \lambda^{\frac{1}{2}} x^{-\frac{3}{2}} \cancel{e^{(\lambda x)^{\frac{1}{2}}}}$$

$$\frac{\lambda}{x} = \lambda^{1/2} x^{-3/2}.$$

$$\lambda^{\frac{1}{2}} = x^{1-3/2}.$$

$$\lambda^{\frac{1}{2}} = x^{-1/2}$$

$\underline{x = 1/\lambda}$. $\Rightarrow$ deflecn $\underline{\underline{\text{pt}}}$.

b) $h(x) = -\frac{d}{dx}(\log S(x))$

$= 1 - e^{(\lambda x)^2}.$

$\frac{d}{dx} h(x) = h'(x) = + e^{(\lambda x)^2} = -2\lambda^2 x e^{(\lambda x)^2}$

$h''(x) = + [2\lambda^2 e^{(\lambda x)^2} + 4\lambda^4 x^2 e^{(\lambda x)^2}].$

$\lambda^2 e^{(\lambda x)^2} > 0$. $\lambda^4 x^2 e^{(\lambda x)^2} > 0.$

Thus this is a strictly monotically increasing hazard $\underline{\underline{\text{funcn}}}$.

2.5). log normal distn. $\Rightarrow$

pdf. $\dfrac{\exp\left[\frac{\ln x - \mu}{\sigma}\right]}{x\sqrt{2\pi}\,\sigma}$.

Mean death time $= e^{\mu + 0.5\sigma}$

$= e^{3.177 + 2.084/2}$

Median death time $\Rightarrow P[X > m] = \frac{1}{2}$.

$$1 - \Phi\left[\frac{\ln m - \mu}{\sigma}\right] = \frac{1}{2}.$$

$$\frac{1}{2} = \Phi\left[\frac{\ln m - \mu}{\sigma}\right].$$

$$\frac{\log m - \mu}{\sigma} = \Phi^{-1}\left(\frac{1}{2}\right) = 0.$$

$$\frac{\log m - 3.177}{2.084} = 0.$$

$$\log m = 3.177$$

$$m = e^{3.177}.$$

b) $x = 100$.

$$P[X > 100] = S(100) = 1 - \Phi\left[\frac{\ln 100 - 3.177}{2.084}\right]$$

$$= 0.247.$$

similarly $S(300)$, $S(200)$.

c). $h(x) = \frac{f(x)}{S(x)}$

$$= \frac{\frac{1}{2.048\,x}\,\phi\left[\frac{\ln x - \mu}{\sigma}\right]}{1 - \Phi\left[\frac{\ln x - \mu}{\sigma}\right]}$$

2.6) gompertz distⁿ : $\Rightarrow$

$$\theta e^{\alpha x} e^{\theta/\alpha (1 - e^{\alpha x})}$$

$\theta = 0.01$

$\alpha = 0.25$.

a) $s(x) = P[X > 1] = s(1) = e^{\frac{0.01}{0.25}(1 - e^{0.25 \times 1})}$

b) will die within six months.

$P[X \leq \frac{1}{2}] = 1 - P[X > \frac{1}{2}]$.

$= 1 - s(\frac{1}{2})$.

$= 1 - e^{\frac{0.01}{0.25}(1 - e^{0.25 \times \frac{1}{2}})}$.

c) median time :

$e^{\frac{0.01}{0.25}(1 - e^{0.25 m})} = \frac{1}{2}$.

$\frac{1}{25}(1 - e^{m/4}) = -\ln 2$.

$1 - e^{m/4} = -25 \ln 2$.

$1 - 25 \ln 2 = e^{m/4}$.

$m = 4 \ln (1 - 25 \ln 2)$

2.7) gamma distⁿ $\Rightarrow$ $\frac{\lambda (\lambda x)^{\beta - 1} e^{-\lambda x}}{\Gamma \beta}$.

$\beta = 3$ $\lambda = 0.2$.

$P[X > 18] = S(18) = 1 - I(3, 0.2 \times 18)$

$= 1 - I(3, 3.6)$.

$P[X \leq 12] = 1 - P[X > 12]$

$= 1 - S[12]$

$= 1 - 1 + I(3, 0.2 \times 12)$.

$= I(3, 2.4)$.

c) $\frac{\beta}{\lambda}$ = Mean lifetime.

$m_t = \frac{3}{0.2}$

$= \frac{\cancel{30}^{15}}{\cancel{2}} = 15$ days

2.8) Pareto dist^n $\Rightarrow \frac{\theta \lambda^\theta}{x^{\theta+1}}$.

$\theta = 4, \lambda = 5$

a) $P[X > 10] = S(10) = \frac{\lambda^\theta}{x^\theta}$.

$= \frac{5^4}{10^4} = \underline{\underline{0.0625}}$.

b) Mean lifetime of battery $= \frac{\theta\lambda}{\theta - 1}$

$= \frac{4 \times 5}{4-1} = \frac{20}{3}$

$= 6.67$.

c) $P[X > t] = 0.99$.

$\frac{5^4}{t^4} = 0.99$.

$\sqrt[4]{5^4/0.99} = t$

$t = 5.01$.

2.9). $Y = \ln(X) = 2 + 0.5z + 2W$.

a) ~~$z=1$~~

Treatment A

$z=1$

$\log(X|z) = 2 + 0.5 + 2W$

$X|z=1 = e^{2.05+2W}$.

follows log normal distn.

$x=1$

$S(x|z) = 1 - \Phi\left(\frac{\ln 1 - 2.05}{2}\right)$

$x=2$

$S(X|z) = 1 - \Phi\left(\frac{\ln 2 - 2.05}{2}\right)$

$x=5$

$= 1 - \Phi\left(\frac{\ln 5 - 2.05}{2}\right)$

Treatment B.

$z=0$.

$X|z=0 = e^{2+2W}$

$\sim$ log normal distn.

$x=1$

$= 1 - \Phi\left(\frac{\ln 1 - 2}{2}\right)$

$x=2$

$= 1 - \Phi\left(\frac{\ln 2 - 2}{2}\right)$.

$x=5$

$= 1 - \Phi\left(\frac{\ln 5 - 2}{2}\right)$.

b) $W \sim$ logistic distn.

$X|z \sim$ log logistic distn.

$e^{2.05+2W} \sim$ log logistic.

$\lambda = e^{-2.05}$

$\cancel{\frac{1}{1+(\lambda x)^\beta}}$ $\beta = \frac{1}{2}$

A

$\frac{1}{1+(e^{-2.05})^{\frac{1}{2}}}$

$\frac{1}{1+(2e^{-2.05})^{\frac{1}{2}}}$

$1/1+(5e^{-2.05})^{\frac{1}{2}}$

B

$\frac{1}{1+(e^{-2})^{\frac{1}{2}}}$

$\frac{1}{1+(2e^{-2})^{1/2}}$

$\frac{1}{1+(5e^{-2})^{1/2}}$

2.11

$S(x) = \exp[-\lambda(x-\phi)^{\alpha}]$.

a) $h(x) = -\frac{d}{dx} (\log(S(x)))$.

$\log(S(x)) = -\lambda(x-\phi)^{\alpha}$.

$-\frac{d}{dx} \log(S(x)) = \frac{d}{dx} (\lambda(x-\phi)^{\alpha})$

$= \lambda\alpha(x-\phi)^{\alpha-1}$.

So $\frac{f(x)}{S(x)} = h(x)$.

$\therefore f(x) = h(x) \cdot S(x)$

$= \alpha\lambda(x-\phi)^{\alpha-1} \exp[-\lambda(x-\phi)^{\alpha}]$

$\therefore h(x) = \begin{cases} 0 & x < \phi \\ \lambda\alpha(x-\phi)^{\alpha-1}. & x \geq \phi. \end{cases}$

$f(x) = \begin{cases} 0 & x<\phi \\ \lambda\alpha(x-\phi)^{\alpha-1} \exp[\lambda(x-\phi)^{\alpha}] & x \geq \phi. \end{cases}$

b). Weibull distn. $\Rightarrow \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}}$. pdf.

$\alpha = 1$

$\lambda = 0.0075$

$\phi = 100$.

Mean time $\Rightarrow$ $\frac{\Gamma(1+1/\alpha)}{\lambda^{1/\alpha}} = \frac{\Gamma 2}{\sqrt{00.75}}$

median time $P[X > m] = \frac{1}{2}$. $\quad m = (m' - 100)$

$$e^{-\lambda x^{\alpha}} = \frac{1}{2}.$$

$$\lambda x^{\alpha} = \ln 2.$$

$$x_m = \left(\frac{\ln 2}{0.0075}\right).$$

$$m' = 100 + \frac{\ln 2}{0.0075}.$$

2.12). $f(x) = \begin{cases} 1/\theta & 0 \le x \le \theta. \\ 0 & \text{o/w.} \end{cases}$

a) $S(x) = \cancel{\int_{\theta}^{\theta} \frac{1}{\theta}\, dx}$. $\quad \frac{\theta - x}{\theta - 0} = \frac{\theta - x}{\theta}$.

survival functn

$x < \theta$

b) $h(x) = -\frac{d}{dx} \ln\left(\frac{\theta - x}{\theta}\right).$

$$= (-)\frac{\cancel{\theta}}{\theta - x} \cdot \left(-\frac{1}{\cancel{\theta}}\right) = \frac{1}{\theta - x}.$$

c). mean residual life $\Rightarrow \frac{\theta + 0}{2} = \frac{\theta}{2}$

2.13) geometric distn $= p(1-p)^{x}$

a) cumulative distn $\Rightarrow$

$$P(X \le x) = 1-(1-p)^{x+1}$$

$$S(X \ge x) = (1-p)^{x} \Rightarrow \text{survival function}$$

b) hazard function $\Rightarrow$ $h(x) = \dfrac{f(x)}{S(x)}$

$$= \frac{p\cancel{(1-p)^{x}}}{\cancel{(1-p)^{x}}} = p$$

exponential distn $= \lambda e^{-\lambda x}$

hazard function $= \underline{\lambda}$

2.14) $f(x) = \dfrac{\lambda^{\beta} x^{\beta-1} e^{-\lambda x}}{\Gamma(\beta)}$

gamma distn

$$S(x) = \int_{x}^{\infty} \frac{\lambda^{\beta} x^{\beta-1} e^{-\lambda x}}{\Gamma(\beta)}\, dx$$

$=$

$$h(x) = -\frac{d}{dx} \ln(s(x)).$$

$=$.

2.15). $h(x) = \alpha + \beta x$. $\alpha > 0$ $\beta > 0$.

~~$h(x) = -\frac{d}{dx} \ln s(x)$.~~

$$e^{-\int \alpha + \beta x \, dx} = s(x).$$

$$e^{-(\alpha x + \beta/2 x^2)} = \underline{s(x)}.$$

$$\frac{f(x)}{s(x)} = h(x).$$

$$f(x) = s(x) \cdot h(x) = (\alpha + \beta x) \, e^{-(\alpha x + \beta/2 x^2)}$$

2.16) $Y = \ln X = \mu + \beta z + \sigma W$.

a). $W \sim \frac{e^w}{(1+e^w)^2}$. $-\alpha < w < \alpha$.

logistic.

$$s(x|w) = \frac{1}{1 + (\lambda x)^\beta}.$$

$\lambda = e^{-\mu}$.

$= e^{-\mu + \beta z}$.

$\beta = \frac{1}{\sigma}$.

$$s(x|\beta) = \frac{1}{1 + (e^{-\mu + \beta z} x)^{\frac{1}{\sigma}}}.$$