Roulette Game:

$1, 2, 3, \dots, 36, 0, 00$

Even	Odd
2	1
4	3
⋮	⋮
36	35

Game I: Betting on Even/Odd

$$\text{Payoff} = \begin{cases} +1 & \text{w.p. } \frac{18}{38} \\ -1 & \text{w.p. } \frac{20}{38} \end{cases}$$

$$E(\downarrow) = (+1)\frac{18}{38} + (-1)\frac{20}{38} = -\frac{2}{38} = -1/19$$

Game II:

Col I	Col II	Col III
1	13	25
2	14	26
⋮	⋮	⋮
12	24	36

Betting on Cols, $\text{Payoff} = \begin{cases} +2 & \text{w.p. } \frac{12}{38} \\ -1 & \text{w.p. } \frac{26}{38} \end{cases}$

$$E(\leftarrow) = (+2)\frac{12}{38} + (-1)\frac{26}{38} = -\frac{1}{19}$$

$S_n = X_1 + X_2 + \dots + X_n$, $S_0 =$ the money one has initially.

① $P(S_n > b_1 S_0 + b_0)$, ② $E(S_n \mid S_n > S_0)$ $\longrightarrow$ more the better

③ Find n s.t. $S_n = bS_0$ before $S_n = 0$, lower $E(n)$ the better it is!

$S \sim B(n, p)$

$S_n :=$ ① Sum of successes out of n independent trials, say $\{x_i\}$'s
② Each X_i's has two outcomes say S & F
③ $P(S) = p$ & $P(F) = 1-p$ does not change.

i.e. Sum of n iid Bernoulli (p) random variable's is Binomial (n, p)

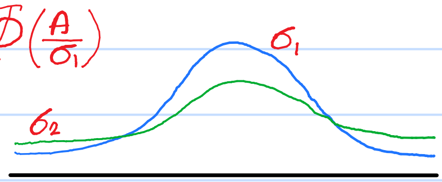
$$X = \begin{cases} d_0 \longrightarrow p \\ d_1 \longrightarrow 1-p \end{cases} \qquad \text{Let, } Y = \frac{X - d_0}{d_1 - d_0} = \begin{cases} 1 \longrightarrow p \\ 0 \longrightarrow 1-p \end{cases}$$

Now, $Y = a_1 X + a_0 \Rightarrow S_n^Y = a_1 S_n^X + n a_0$, $a_0 > 0$

$$\therefore P(S_n^X > S_0^X) = P(a_1 S_n^X + n a_0 > a_1 S_0^X + n a_0) = P(S_n^Y > a_1 S_0^X + n a_0)$$

Again, $$P(S_n^X > S_0^X) = P\left(\frac{S_n^X - E(S_n^X)}{\sqrt{Var(S_n^X)}} > \frac{S_0^X - E(S_n^X)}{\sqrt{Var(S_n^X)}}\right) = P\left(Z > \frac{S_0^X - E(S_n^X)}{\sqrt{Var(S_n^X)}}\right) = 1 - \Phi\left(\frac{S_0^X - E(S_n^X)}{\sqrt{Var(S_n^X)}}\right) \text{[For large } n]$$

$$\begin{matrix} A > 0 \\ \sigma_2 > \sigma_1 \end{matrix} \Rightarrow \frac{A}{\sigma_2} < \frac{A}{\sigma_1} \Rightarrow \Phi\left(\frac{A}{\sigma_2}\right) < \Phi\left(\frac{A}{\sigma_1}\right) \Rightarrow 1 - \Phi\left(\frac{A}{\sigma_2}\right) > 1 - \Phi\left(\frac{A}{\sigma_1}\right)$$



$\boxed{\text{H.W.}}$ $E(S_n \mid S_n > S_0) = ?$

Gambler's Ruin:

① $P(S_n = bS_0$, for the first time before reaching $S_n = 0)$
② Find $E(n \mid S_n = bS_0$, for the first time before reaching $S_n = 0)$
③ $E(n \mid S_n = bS_0$ or $S_n = 0$, for the first time$)$
④ Distribution of $(n \mid S_n = bS_0$ or $S_n = 0$ for the first time$)$

Note: $P(\text{Ruin} \mid S_0) = P(S_n = 0$, before reaching $S_n = bS_0)$
$\phi(c) =$ prob. ruin given $S_0 = C$

Game I:

$$c \begin{cases} \xrightarrow{p} c+1 \\ \xrightarrow{q} c-1 \end{cases} \qquad \phi(c) = p\,\phi(c+1) + q\,\phi(c-1)$$

$$\Rightarrow p(\phi(c+1) - \phi(c)) = q(\phi(c) - \phi(c-1)) \quad [p+q=1]$$

$$\Rightarrow \phi(c+1) - \phi(c) = \frac{q}{p}(\phi(c) - \phi(c-1)) = \left(\frac{q}{p}\right)^c(\phi(1) - \phi(0))$$

Clearly,
$$\begin{aligned}
\phi(c+1)-\phi(c) &= \left(\tfrac{q}{p}\right)^{c}(\phi(1)-\phi(0))\\
\phi(c)-\phi(c-1) &= \left(\tfrac{q}{p}\right)^{c-1}(\phi(1)-\phi(0))\\
&\ \vdots\\
+\quad \phi(1)-\phi(0) &= \left(\tfrac{q}{p}\right)^{0}(\phi(1)-\phi(0))\\
\hline
\phi(c+1)-\phi(0) &= (\phi(1)-\phi(0))\left[1+\tfrac{q}{p}+\dots+\left(\tfrac{q}{p}\right)^{c}\right]\\
&= (\phi(1)-\phi(0))\left[\frac{1-\left(\frac{q}{p}\right)^{c+1}}{1-q/p}\right] \text{ for } p\neq q, \quad (c+1)(\phi(1)-\phi(0)) \text{ for } p=q
\end{aligned}$$

Note that, $\varphi(0)=1$, $\varphi(b)=0$

For $c=b-1$, $\phi(b)-\phi(0)=[\phi(1)-\phi(0)]\left[\dfrac{1-\left(\frac{q}{p}\right)^{b}}{1-q/p}\right]$ for $p\neq q$, $b(\phi(1)-\phi(0))$ for $p=q$

$$\Rightarrow (-1) = (\phi(1)-1)\left[\frac{1-\left(\frac{q}{p}\right)^{b}}{1-\frac{q}{p}}\right] \text{ for } p\neq q, \quad b(\phi(1)-1) \text{ for } p=q$$

$$\Rightarrow \phi(1) = 1-\frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^{b}} \text{ for } p\neq q, \quad \left(1-\frac{1}{b}\right) \text{ for } p=q$$

$$= \frac{\frac{q}{p}-\left(\frac{q}{p}\right)^{b}}{1-(q/p)^{b}} \text{ for } p\neq q, \quad \left(1-\frac{1}{b}\right) \text{ for } p=q$$

Check that, $\phi(c)=\left[\dfrac{\left(\frac{q}{p}\right)^{b}-\left(\frac{q}{p}\right)^{c}}{\left(\frac{q}{p}\right)^{b}-\left(\frac{q}{p}\right)^{a}}\right]$ $\left[\because \phi(c)-\phi(0)=(\phi(1)-\phi(0))\left[\dfrac{1-\left(\frac{q}{p}\right)^{c}}{1-q/p}\right]\right]$ and a (lower target) $< c < b$ (upper target)

$\boxed{\text{H.W.}}$ Find $\phi(c)$ for game II.

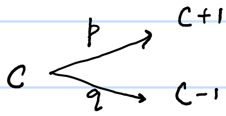
<u>Book:</u> Bhattacharya & Waymire ; <u>Contact:</u> gkb.isical@gmail.com

<u>Final:</u> 50% , <u>Midterm:</u> 30% , <u>H.W./Project:</u> 20%

12/08/2023

<u>Book:</u>

$\varphi(c)$ = Prob. of ruin given that gambler's initial wealth is c.



$$\phi(c)=p\cdot\varphi(c+1)+q\,\phi(c-1)$$

τ = first time to reach '0' (a) or upper value $(b_0\cdot S_0)_{b_0}$

$$\therefore \phi(c)=P(S_\tau=0\mid S_0=c)=P(S_\tau=0, S_1=c+1\mid S_0=c)+P(S_\tau=0, S_1=c-1\mid S_0=c)$$

$$= P(S_\tau=0\mid S_1=c+1, S_0=c)\cdot P(S_1=c+1\mid S_0=c)+P(S_\tau=0\mid S_1=c-1, S_0=c)\cdot P(S_1=c-1\mid S_0=c)$$

$$\left[\because P(A\cap B\mid D)=\frac{P(A\cap B\cap D)}{P(D)}=\frac{P(A\cap B\cap D)}{P(B\cap D)}\cdot\frac{P(B\cap D)}{P(D)}=P(A\mid B\cap D)\cdot P(B\mid D)\right]$$

$$= P(S_\tau=0\mid S_1=c+1)\cdot P(S_1=c+1\mid S_0=c)+P(S_\tau=0\mid S_1=c-1)\cdot P(S_1=c-1\mid S_0=c) \quad \left[\because P \text{ is Markov}\right]$$

Note that, $\left[\varphi_c=P(S_\tau=0\mid S_0=c)=\sum_{n=0}^{\infty}P(S_n=0,\tau=n\mid S_0=c)\right]$

$$P(S_\tau=0\mid S_1=c+1)=\sum_{n=1}^{\infty}P(S_n=0,\tau=n\mid S_1=c+1)=\sum_{n=1}^{\infty}P(S_{n-1}=0,\tau=n-1\mid S_0=c+1)$$

$$=\sum_{m=0}^{\infty}P(S_m=0,\tau=m\mid S_0=c+1)\quad [m=n-1] \quad = P(S_\tau=0\mid S_0=c+1)=\varphi(c+1)$$

[Since, S_n has stationary transition probability]

H.W. Give your strategy (or. Utility func.) to one of games rather than the other. (Roulette Games)

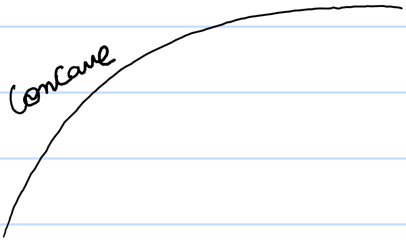
1. Sub prime mortage crisis
2. Stress Test
3. Prime borrowers / Prime lenders
4. Loan to Value Ratio (LTV)
5. Mortgage Backed Security
6. Foregin Reserve.
7. (RIB) Residual Interest Bond.

Bank
→ Fixed Rate (10%)
→ Variable Rate (+2% ↑ over Benchmark)
→ Repo Rate
→ Reverse Repo Rate

H.W. which one would you take and why? For a loan of 10 lakhs for 20 years (Repayment / Interest only) [2/3/5 year fixed 9%]

Utility Function :

Concave



1. $u(x) = \log x \;, x>0$
2. $u(x) = \sqrt{x}$ or x^{α} , $0<\alpha<1, x\geq 0$
3. $u(x) = a - be^{-cx}$, $b,c>0, x\geq 0$
4. $u(x) = ax - bx^2$

(R_A)

1. $-\left(\dfrac{-1/x^2}{1/x}\right) = \dfrac{1}{x}$
2. $-\left(\dfrac{\alpha(\alpha-1)x^{\alpha-2}}{\alpha x^{\alpha-1}}\right) = \dfrac{1-\alpha}{x}$
3. $-\left(\dfrac{-bc^2}{bc}\right) = c$
4. $-\left(\dfrac{-2b}{a-2bx}\right) = \dfrac{2b}{a-2bx}$

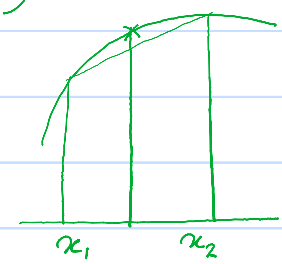
Measure of Risk Aversion $(R_A) = -\dfrac{u''(x)}{u'(x)}$

Note that,
for a fair game $E(x)=0$ i.e. $px_1+(1-p)x_2=0$ (Simple Game)
For risk aversion fair game is not likely.

$N_0 = 1$, $(1+x_1)$ or $(1+x_2)$

$$u\big(p(1+x_1)+(1-p)(1+x_2)\big) \geq pu(1+x_1)+(1-p)(1+x_2)$$

$$\Rightarrow u\big(1+px_1+(1-p)x_2\big) = u(1) \geq E(u(1+x))$$



Come up with the best strategy for playing the games or choose the mortgage.

$$\text{Return } (R_t) = \frac{P_t - P_{t-1}}{P_{t-1}}$$

19/08/2023

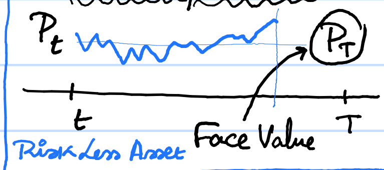
1903-1904	Bachelier	Martingle/RW/BM
1950-1952	Malcowitz	Portfolio Investment
1965	Fama	Efficient Market Hypothesis
1972-1973	Black-Schole-Merton	Option Pricing Theory
1978	Cox-Ross-Rubinstein	Binomial Model / CRR Model
1982	Engle	ARCH
1986	Eng	GARCH
1996-1997	LTCM Bankcrupcy	

Portfolio Investment Theory:

N risky assets, Returns $\underset{\sim}{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}$, $R_t = R(t) = \dfrac{P_t - P_{t-1}}{P_{t-1}}$ $\boxed{eg.}$ $\left.\begin{matrix} P_t = 110 \\ P_{t-1} = 100 \end{matrix}\right\} \Rightarrow R_t = 10\%$

Return on t^{th} until of time

Zero Coupon Bond



Portfolio Weights $(\underset{\sim}{\omega}) = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix}$

$E(\underset{\sim}{R}) = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$, $\mu_i = E(R_i)$, $Var(\underset{\sim}{R}) = V_{N\times N} = ((v_{ij}))$

$v_{ij} = Cov(R_i, R_j)$, $v_{ii} > 0 \ \forall i$

$Var(\underset{\sim}{\omega}'\underset{\sim}{R}) = \sum_j \sum_k \omega_{ij} v_{jk} \omega_{kl}$

$= \underset{\sim}{\omega}' \underset{\sim}{V} \underset{\sim}{\omega}$

$E(\underset{\sim}{\omega}'\underset{\sim}{R}) = \underset{\sim}{\omega}'\underset{\sim}{\mu}$

Problem: ① $E(\underset{\sim}{\omega}'\underset{\sim}{R}) \geq b$, $\min_{\underset{\sim}{\omega}} Var(\underset{\sim}{\omega}'\underset{\sim}{R})$ subject to $\sum \omega_i = 1$

② $\max_{\underset{\sim}{\omega}} E(\underset{\sim}{\omega}'\underset{\sim}{R})$ subject to $Var(\underset{\sim}{\omega}'\underset{\sim}{R}) \leq c$ and $\sum \omega_i = 1$

③ $\max_{\underset{\sim}{\omega}} E(\underset{\sim}{\omega}'\underset{\sim}{R}) - \frac{\gamma}{2} Var(\underset{\sim}{\omega}'\underset{\sim}{R})$ subject to $\sum \omega_i = 1$, $\gamma > 0$

$$f(\underset{\sim}{\omega}, \lambda_1, \lambda_2) = \frac{1}{2} \underset{\sim}{\omega}' V \underset{\sim}{\omega} - \lambda_1(\underset{\sim}{\omega}'\underset{\sim}{\mu} - b) - \lambda_2(\underset{\sim}{\omega}'\underset{\sim}{1} - 1) \qquad [\underset{\sim}{1} = [1, 1, \ldots, 1]']$$

$$\underbrace{\frac{\partial f}{\partial \underset{\sim}{\omega}} = V\underset{\sim}{\omega} - \lambda_1 \underset{\sim}{\mu} - \lambda_2 \underset{\sim}{1}}_{①} = \begin{pmatrix} \frac{\partial f}{\partial \omega_1} \\ \vdots \\ \frac{\partial f}{\partial \omega_N} \end{pmatrix}, \quad \underbrace{\frac{\partial f}{\partial \lambda_1} = -\omega'\mu + b}_{②}, \quad \underbrace{\frac{\partial f}{\partial \lambda_2} = -\omega'\underset{\sim}{1} + 1}_{③}$$

$$\text{Hessian } (H) = \begin{bmatrix} \left[\left(\dfrac{\partial^2 f}{\partial \omega_i \partial \omega_j}\right)\right]_{N\times N} & \left(\dfrac{\partial^2 f}{\partial \omega_i \partial \lambda_1}\right)_{N\times 1} & \left(\dfrac{\partial^2 f}{\partial \omega_i \partial \lambda_2}\right)_{N\times 1} \\ \left(\dfrac{\partial^2 f}{\partial \omega_i \partial \lambda_1}\right)'_{1\times N} & \dfrac{\partial^2 f}{\partial \lambda_1 \partial \lambda_1} & \dfrac{\partial^2 f}{\partial \lambda_1 \partial \lambda_2} \\ \left(\dfrac{\partial^2 f}{\partial \omega_i \partial \lambda_2}\right)'_{1\times N} & \dfrac{\partial^2 f}{\partial \lambda_1 \partial \lambda_2} & \dfrac{\partial^2 f}{\partial \lambda_2 \partial \lambda_2} \end{bmatrix}_{(N+2)\times(N+2)} = \begin{bmatrix} V & -\underset{\sim}{\mu} & -\underset{\sim}{1} \\ -\underset{\sim}{\mu}' & 0 & 0 \\ -\underset{\sim}{1}' & 0 & 0 \end{bmatrix}$$

Assume V as positive definite matrix $\Rightarrow$ H is non-negative definite

$$\frac{\partial f}{\partial \underset{\sim}{\omega}} = 0 \Rightarrow V\underset{\sim}{\omega} = \lambda_1 \underset{\sim}{\mu} + \lambda_2 \underset{\sim}{1} = [\underset{\sim}{\mu} : \underset{\sim}{1}]_{N\times 2} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}_{2\times 1} \Rightarrow \underset{\sim}{\omega}' = V^{-1}(K\underset{\sim}{\lambda}) \text{ ——— ④}$$

$[K = [\underset{\sim}{\mu} : \underset{\sim}{1}]]$

$$\left.\begin{matrix} \dfrac{\partial f}{\partial \lambda_1} = 0 \Rightarrow \omega'\mu = b \\ \dfrac{\partial f}{\partial \lambda_2} = 0 \Rightarrow \omega'\underset{\sim}{1} = 1 \end{matrix}\right\} \begin{matrix} b = \underset{\sim}{\mu}'\underset{\sim}{\omega} = \underset{\sim}{\mu}' V^{-1} K \underset{\sim}{\lambda} \\ 1 = \underset{\sim}{1}'\underset{\sim}{\omega} = \underset{\sim}{1}' V^{-1} K \underset{\sim}{\lambda} \end{matrix}$$

$$\therefore \begin{pmatrix} b \\ 1 \end{pmatrix} = (K'V^{-1}K)\lambda = \left(\begin{bmatrix} \underset{\sim}{\mu}' \\ \underset{\sim}{1}' \end{bmatrix} V^{-1} [\underset{\sim}{\mu} \ \underset{\sim}{1}]\right) \underset{\sim}{\lambda}$$

Linearly Dependent:

$U_1, U_2, \cdots, U_N$ are said to be linearly dependent if $\sum a_i U_i = 0$ for some $(a_1, a_2, \dots, a_N)$ such that not all of a_i's are zero.

Independent means they are not dependent.

$$Var(\underset{\sim}{a}'\underset{\sim}{R}) = \underset{\sim}{a}'\underset{\sim}{R}\underset{\sim}{a} = 0$$

Dependent in terms of market return means, $\underset{\sim}{a}'\underset{\sim}{R} = C_0$ (deterministic return)

$$\Rightarrow \sum a_i R_i = C_0 \Rightarrow a_K R_K = C_0 - \sum_{i \neq K} a_i R_i \Rightarrow R_K = \frac{1}{a_K}\Big[C_0 - \sum_{i \neq K} a_i R_i\Big]$$

i.e. return of one asset can be determinded by all other asset returns.

If V is positive definite matrix $\Rightarrow V^{-1}$ is PD also

$\Leftrightarrow a'Va > 0 \;\; \forall a \neq 0 \;\Leftrightarrow\; Q'VQ$ is PD, Provided Q is full rank

$\Leftrightarrow a'Q'VQa > 0 \;\Leftrightarrow\; y'Vy > 0$ [taking $y = Qa$]

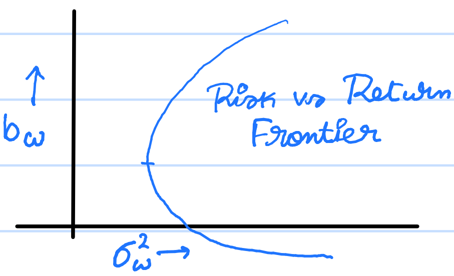
K is full rank means that $\underset{\sim}{\mu}$ is not a constant vector in terms of financial domain.

$$\underset{\sim}{\lambda} = (K'V^{-1}K)^{-1}\begin{pmatrix} b \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{\omega} = V^{-1}K(K'V^{-1}K)^{-1}\begin{pmatrix} b \\ 1 \end{pmatrix}$$

$$\text{Variance} = \underset{\sim}{\omega}'V\underset{\sim}{\omega} = \begin{pmatrix} b \\ 1 \end{pmatrix}'(K'V^{-1}K)^{-1}K'V^{-1}VV^{-1}K(K'V^{-1}K)\begin{pmatrix} b \\ 1 \end{pmatrix} = \underset{1\times 2}{(b \;\; 1)}\underset{2\times 2}{(K'V^{-1}K)^{-1}}\underset{2\times 1}{\begin{pmatrix} b \\ 1 \end{pmatrix}} = Ab^2 + Bb + C$$

Find:

$$A = ((K'V^{-1}K)^{-1})_{11}$$
$$B = ((K'V^{-1}K)^{-1})_{21} + ((K'V^{-1}K)^{-1})_{12}$$
$$C = ((K'V^{-1}K)^{-1})_{22}$$



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Portfolio Optimization:

N risky assets, $\underset{\sim}{R} = \begin{pmatrix} R_1 \\ \vdots \\ R_N \end{pmatrix}$, $E(\underset{\sim}{R}) = \underset{\sim}{\mu}$, $Var(\underset{\sim}{R}) = V$ (assumed P.D.)

Allowed short selling

Problem I: $\min\limits_{\omega} \frac{1}{2}\omega'\underset{\sim}{R}\omega = \frac{1}{2}Var(\omega'\underset{\sim}{R})$ subjected to $E(\omega'\underset{\sim}{R}) = \omega'\underset{\sim}{\mu} \geq b$, $\sum \omega_i = 1$

$[w_i \geq 0]$
No short selling

We have,

$$\underset{\sim}{\omega}_{op} = V^{-1}K(K'V^{-1}K)\begin{pmatrix} b \\ 1 \end{pmatrix} \quad \text{where } K = \underset{N\times 2}{[\underset{\sim}{\mu} : \underset{\sim}{1}]}$$

$$\sigma^2_{op} = \begin{pmatrix} b \\ 1 \end{pmatrix}'(K'V^{-1}K)^{-1}\begin{pmatrix} b \\ 1 \end{pmatrix} = L_{11}b^2 + (L_{12}+L_{21})b + L_{22} = \frac{Ab^2 - 2Bb + C}{D}$$

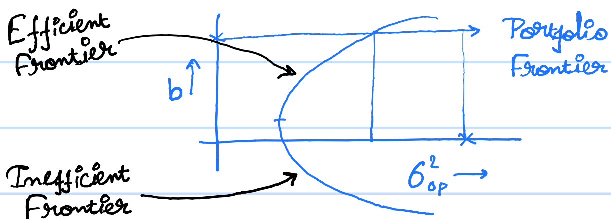
$$\text{where, } L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \left(\begin{bmatrix} \underset{\sim}{\mu}' \\ \underset{\sim}{1}' \end{bmatrix} V^{-1}[\underset{\sim}{\mu} : \underset{\sim}{1}]\right)^{-1} = \begin{bmatrix} \underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu} & \underset{\sim}{\mu}'V^{-1}\underset{\sim}{1} \\ \underset{\sim}{1}'V^{-1}\underset{\sim}{\mu} & \underset{\sim}{1}'V^{-1}\underset{\sim}{1} \end{bmatrix}^{-1} = \begin{bmatrix} \underset{\sim}{1}'V^{-1}\underset{\sim}{1} & -\underset{\sim}{\mu}'V^{-1}\underset{\sim}{1} \\ -\underset{\sim}{1}'V^{-1}\underset{\sim}{\mu} & \underset{\sim}{\mu}'V^{-1}\underset{\sim}{\mu} \end{bmatrix} \Big/ D$$

A, B, C

where, $D = AC - B^2$

$$\text{and } \frac{1}{D}\begin{bmatrix} b \\ 1 \end{bmatrix}'\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}\begin{bmatrix} b \\ 1 \end{bmatrix} = \frac{[bA - B1 \quad -bB + C]}{D}\begin{bmatrix} b \\ 1 \end{bmatrix} = \frac{1}{D}[Ab^2 - bB - bB + C]$$
$$= \frac{1}{D}[Ab^2 - 2bB + C]$$

$$x'Lx = \sum_{i,j} x_i L_{ij} x_j = \sum_i L_{ii} x_i^2 + \sum_{i \neq j} x_i L_{ij} x_j$$



Pay off $= X_1 + X_2 + \dots + X_m$, X_i = Pay-off the i^{th} game given first $i-1$ games are lost

Initial bet $= C_0$

$$X_i = \begin{cases} -2^{i-1}C_0 & \text{if you lose} \\ 2^{i-1}C_0 & \text{if you win} \end{cases}$$

Game is over when you win first

$n_0 \to$ Time you won.

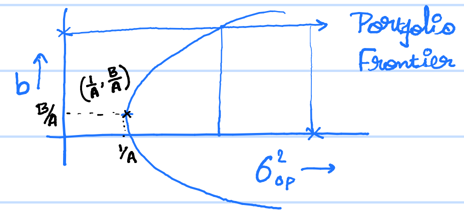
$X_1 + \dots + X_{n_0-1} = -C_0\left(1 + \dots + 2^{n_0-2}\right) = -C_0\left(2^{n_0-1} - 1\right)$, $X_{n_0} = C_0 2^{n_0-1}$

$$\sigma^2_{op} = \frac{Ab^2 - 2Bb + C}{D}$$

Let, $f(b) = \frac{1}{D}\left(Ab^2 - 2Bb + C\right)$

$\Rightarrow f'(b) = 0 \Rightarrow 2Ab - 2B = 0 \Rightarrow \boxed{b = \frac{B}{A}}$

$\therefore \sigma^2_{op} = \frac{1}{D}\left[A\frac{B^2}{A^2} - 2\frac{B}{A}B + C\right]$
$= \frac{1}{D}\left[\frac{B^2}{A} - 2\frac{B^2}{A} + C\right]$
$= \frac{1}{D}\left[C - \frac{B^2}{A}\right] = \frac{1}{D}\left[\frac{AC - B^2}{A}\right] = \frac{1}{A}$



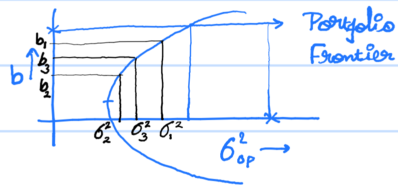
Reduced Problem, $\min\limits_{\underset{\sim}{w}} \frac{1}{2}\underset{\sim}{w}'V\underset{\sim}{w}$ subjected to $\sum w_i = 1$

$f(\underset{\sim}{w}, \delta) = \frac{1}{2}w'Vw - \delta\left(w'\underset{\sim}{1} - 1\right)$ [Lagrange Multiplier]

$\frac{\partial f}{\partial \underset{\sim}{w}} = V\underset{\sim}{w} - \delta\underset{\sim}{1} = 0 \Rightarrow \underset{\sim}{w} = \delta V^{-1}\underset{\sim}{1} = \frac{V^{-1}\underset{\sim}{1}}{\underset{\sim}{1}'V^{-1}\underset{\sim}{1}}$ $\left[\because \underset{\sim}{1}'\underset{\sim}{w} = \delta\underset{\sim}{1}'V^{-1}\underset{\sim}{1} = 1 \Rightarrow \delta = \frac{1}{\underset{\sim}{1}'V^{-1}\underset{\sim}{1}}\right]$

$\underset{\sim}{w}_{op} = \frac{\left[V^{-1}\underset{\sim}{\mu} : V^{-1}\underset{\sim}{1}\right]}{D}\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}\begin{pmatrix} b \\ 1 \end{pmatrix} = \frac{1}{D}\left[AV^{-1}\underset{\sim}{\mu}\cdot b - BV^{-1}\underset{\sim}{\mu} - BV^{-1}\underset{\sim}{1}b + V^{-1}\underset{\sim}{1}C\right]$

$= \frac{1}{D}\left[\left(AV^{-1}\underset{\sim}{\mu} - BV^{-1}\underset{\sim}{1}\right)b + \left(CV^{-1}\underset{\sim}{1} - BV^{-1}\underset{\sim}{\mu}\right)\right] = g\,b + h$ (say)



We can get an $w^{op} = \alpha w_1^{op} + (1-\alpha) w_2^{op}$ [Convex Portfolio]

$= \alpha\left[g b_1 + h\right] + (1-\alpha)\left[g b_2 + h\right] = g\left[\alpha b_1 + (1-\alpha) b_2\right] + h = g b + h$ [taking $b = \alpha b_1 + (1-\alpha) b_2$]

Frontier Portfolio

Again, note that, $Var\left(\alpha\sigma_1^2 + (1-\alpha)\sigma_2^2\right) > Var\left(w'\underset{\sim}{R}\right)$

Frontier Portfolio, $w'_{b_1}\underset{\sim}{R}$, $w'_{b_2}\underset{\sim}{R}$

$Cov\left(w'_{b_1}\underset{\sim}{R}, w'_{b_2}\underset{\sim}{R}\right) = w'_{b_1} Var(\underset{\sim}{R})\, w_{b_2} = w'_{b_1} V w_{b_2}$

$V\underset{\sim}{w}_{b_2} = K\left(K'V^{-1}K\right)^{-1}\begin{pmatrix} b_2 \\ 1 \end{pmatrix} \Rightarrow w'_{b_1}Vw_{b_2} = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}'\left(K'V^{-1}K\right)^{-1}\left(K'V^{-1}K\right)\left(K'V^{-1}K\right)^{-1}\begin{pmatrix} b_2 \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}'\left(K'VK\right)^{-1}\begin{pmatrix} b_2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} b_1 \\ 1 \end{pmatrix}'\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}\begin{pmatrix} b_2 \\ 1 \end{pmatrix} \Big/ D = \frac{1}{D}\left[b_1 b_2 A - B(b_1 + b_2) + C\right]$

Let's put, $b_2 = \frac{B}{A}$ ie. $w'_{b_1}Vw_{b_2} = \frac{1}{D}\left[b_1\cdot\frac{B}{A}A - Bb_1 - B\frac{B}{A} + C\right] = \frac{1}{D}\left[C - \frac{B^2}{A}\right] = \frac{1}{A}$

$Var\left(\alpha r_1 + (1-\alpha) r_2\right) = \alpha^2 Var(r_1) + (1-\alpha)^2 Var(r_2) + 2\alpha(1-\alpha)Cov(r_1, r_2) = \alpha^2\sigma_1^2 + (1-\alpha)\sigma^2_{mvp} + 2\alpha(1-\alpha)\sigma^2_{mvp}$

$= \alpha^2\sigma_1^2 + (1-\alpha)(1+\alpha)\sigma^2_{mvp}$ [Taking $r_2 = r_{mvp}$]

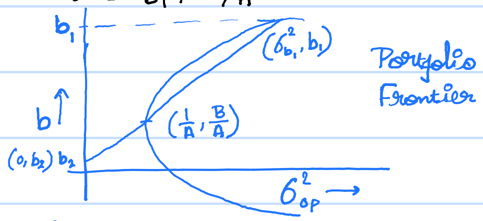
Again,

$Cov\left(w'_{b_1}\underset{\sim}{R}, w'_{b_2}\underset{\sim}{R}\right) = \frac{1}{D}\left[b_1 b_2 A - B\left[b_1 + b_2\right] + C\right] = 0 \Rightarrow (b_1 A - B) b_2 = -C + Bb_1 \Rightarrow \boxed{b_2 = \frac{Bb_1 - C}{b_1 A - B}}$

whenever, $C_w\left(w'_{b_1}\underset{\sim}{R}, w'_{b_2}\underset{\sim}{R}\right) = 0$
and $b_1 \neq B/A$

$\underset{\sim}{w}_{op} = \frac{\left[V^{-1}\underset{\sim}{\mu} : V^{-1}\underset{\sim}{1}\right]}{D}\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}\begin{pmatrix} b \\ 1 \end{pmatrix}$

Thus, the eq. of the line $\Rightarrow \left(y - \frac{B}{A}\right) = \frac{b_1 - \frac{B}{A}}{\sigma^2_{b_1} - \frac{1}{A}}\left(x - \frac{1}{A}\right)$



$\Rightarrow y_{x=0} = \frac{B}{A} + \frac{\frac{B}{A} - b_1}{A\sigma^2_{b_1} - 1} = \frac{1}{A\sigma^2_{b_1} - 1}\left[\sigma^2_{b_1}B - \frac{B}{A} + \frac{B}{A} - b_1\right] = \frac{B\sigma^2_{b_1} - b_1}{A\sigma^2_{b_1} - 1}$

$\therefore B\sigma^2_{b_1} - b_1 = \frac{1}{D}\left[ABb_1^2 - 2B^2b_1 + BC\right] - b_1 = \frac{1}{D}\left[ABb_1^2 - 2B^2b_1 + BC - b_1AC + b_1B^2\right] = \frac{1}{D}(Ab_1 - B)(Bb_1 - C)$

$\therefore A\sigma^2_{b_1} - 1 = \frac{1}{D}\left[A^2b_1^2 - 2ABb_1 + AC\right] - 1 = \frac{1}{D}\left[A^2b_1^2 - 2ABb_1 + AC - AC + B^2\right] = \frac{1}{D}(Ab_1 - B)^2$

$\therefore y_{x=0} = \frac{(Bb_1 - C)}{(Ab_1 - B)}$

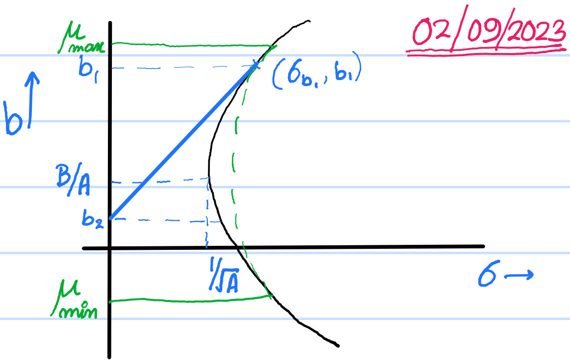
Collecting Data: ① Banking Sector, Pharma, Hi Tech, Agriculture others.
② 2 (stock/prices) for each of 2 years daily data
③ For 1.5 years data $\hat{\mu} = \bar{x}$, $\hat{V}$
④ Check 1 month performance with your w_b then change/update and go on.
(one week)
⑤ Find Total Sum of Squares of the date (6 TSS).
⑥ Compare this 3 months update & its Sum of Squares & no update and its sum of squares of the date.

Mean vs Standard Deviation:

02/09/2023

$$\sigma_b^2 = \frac{A}{D}\left(b - \frac{B}{A}\right)^2 + \frac{1}{A} \quad , \quad x^2 = c_0\left(y - d_0\right)^2 + a_0$$

$$\Rightarrow 2x\,dx = 2c_0(y-d_0)\,dy \Rightarrow \frac{dy}{dx} = \frac{x}{c_0(y-d_0)}$$



For a portfolio with return $b_1 \neq B/A$ the return of corresponding zero cov (unique) portfolio b_2 is given by $\boxed{b_2 = \frac{Bb_1 - C}{Ab_1 - B}}$

$$y - b_1 = \left(\frac{dy}{dx}\right)_{(\sigma_{b_1}, b_1)}(x - \sigma_{b_1})$$

$$\Rightarrow \left(y - \frac{B}{A}\right) = \frac{\sigma_{b_1}}{\frac{A}{D}\left(b_1 - \frac{B}{A}\right)}(x - \sigma_{b_1})$$

Similarly,

$$b = y_{x=0} = \frac{B}{A} + \frac{(-\sigma_{b_1}^2)D}{(b_1 - B/A)A} = \frac{B}{A} - \frac{\left[\frac{A}{D}\left(b_1 - \frac{B}{A}\right)^2 + \frac{1}{A}\right]}{\frac{A}{D}\left(b_1 - B/A\right)}$$

$$= b_1 - \left(b_1 - \frac{B}{A}\right) - \frac{D}{A(Ab_1 - B)} = \frac{1}{A}\frac{[ABb_1 - B^2 - (AC - B^2)]}{Ab_1 - B} = \frac{Bb_1 - C}{Ab_1 - B}$$

$\sum \omega_i = 1$, $\omega'\mu = b$, $\omega_i \geq 0$; ω_i can take values between μ_{min} and μ_{max}.

$\boxed{\text{H.W.}}$ Plot the restricted curve and unrestricted curve and identify min-var portfolio.

In real life, we may have (lower) $l_i \leq \omega_i \leq u_i$ (upper), $\sum \omega_i = 1$.

N+1 Assets: N risky and 1 riskless asset.

$\underset{\sim}{R} = (R_1, \ldots, R_N)'$, r_f (risk free) $= \dfrac{r}{\text{unit of time to make it equivalent } \underset{\sim}{R}}$ $\boxed{\text{eg.}}$ 6% per annum

$E(\underset{\sim}{R}) = \underset{\sim}{\mu}$, $Var(\underset{\sim}{R}) = V$

Problem: $\min\limits_{\omega} \frac{1}{2}\omega' V \omega$ subject to $\underset{\sim}{\omega}'\underset{\sim}{\mu} + (1 - \sum\omega_i) r_f \geq b \Leftrightarrow \underset{\sim}{\omega}'\underset{\sim}{\mu} + (1 - \underset{\sim}{\omega}'\underset{\sim}{1}) r_f \geq b$

$$\Leftrightarrow \underset{\sim}{\omega}'(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) + r_f \geq b$$

Solution: $f(\omega) = \frac{1}{2}\omega' V \omega + \lambda\left(b - \omega'(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) - r_f\right)$

$$\Rightarrow \frac{\partial f}{\partial \omega} = V\omega - \lambda(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) = 0 \Rightarrow \underset{\sim}{\omega} = \lambda V^{-1}(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) \quad \text{——— (1)}$$

$$\frac{\partial f}{\partial \lambda} = (b - r_f) - \omega'(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) = 0 \Rightarrow \omega'(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) = b - r_f \quad \text{——— (2)}$$

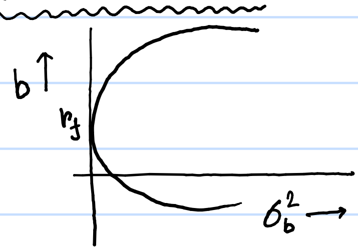
$$(2) \Rightarrow b - r_f = (\underset{\sim}{\mu} - r_f\underset{\sim}{1})'\underset{\sim}{\omega} = \lambda\underbrace{(\underset{\sim}{\mu} - r_f\underset{\sim}{1})' V^{-1}(\underset{\sim}{\mu} - r_f\underset{\sim}{1})}_{= H \text{ (say)} > 0}$$

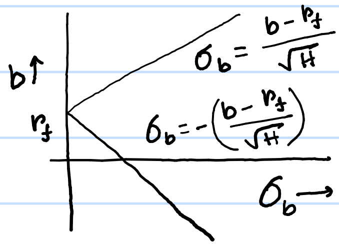
[$\underset{\sim}{\mu} \neq r_f\underset{\sim}{1}$ otherwise no need to invest in risky assets]

$$\Rightarrow \boxed{\lambda = \frac{b - r_f}{H}}$$

$$\sigma^2_{optimal} = \underset{\sim}{\omega}'_{op} V \underset{\sim}{\omega}_{op} = (\underset{\sim}{\mu} - r_f\underset{\sim}{1})' V^{-1}(\underset{\sim}{\mu} - r_f\underset{\sim}{1})\frac{(b - r_f)^2}{H^2} = \frac{(b - r_f)^2}{H}$$

Mean-Variance Plot:





Prices of 10 years of Govt. Bond

Assignment:

Invest in N+1 assets and do it allowing (1) shortsale and without shortsale (2) min investment $\geq$ 20% of investment in Govt. Bond, (3) $\omega \leq 0.15$

Two Fund Separation (Continued):

16/09/2023

With the two fund separation property,

$r_q = r_{zc(p)} + \beta_{qp}(r_p - r_{zc(p)}) + \varepsilon$, where $E(\varepsilon \mid r_{zc(p)} + \beta_{qp}(r_p - r_{zc(p)})) = 0$ & $\beta_{qp} = Cov(r_q, r_p)$

and r_p is the return, correspondingly to a frontier portfolio not equal to mvp

$\exists$ two portfolio whose returns are r_p and r_{pq} such that for any portfolio ε with r_q $\exists\ \lambda$ such that

$E(u(r_q)) \le E(u(\lambda r_{p_1} + (1-\lambda) r_{p_2}))\ \forall\ u$ concave

If, $Y \overset{d}{=} X + \varepsilon$, $E(\varepsilon \mid X) = 0$

then $E[u(Y)] = E[u(X+\varepsilon)] = E[E[u(X+\varepsilon) \mid X]] \le E[u[E[(X+\varepsilon) \mid X]]] = E[u(X + E(\varepsilon \mid X))] = E[u(X)]$

concave [Jensen's Inequality]

$\Rightarrow E[u(Y)] \le E(u(X))\ \forall$ concave u

Note that, the converse also holds (Rothschild & Stiglitz) ie. $E(u(Y)) \le E(u(X))\ \forall\ u$ concave $\Rightarrow Y = X + \varepsilon$, where $E(\varepsilon \mid X) = 0$

For $(X, Y) \sim N_2(\mu, \Sigma)$ then $Y = \alpha + \beta X + \varepsilon$ with $E(\varepsilon \mid \alpha + \beta X) = 0$ ie. $E(\varepsilon \mid X) = 0$

or $X = \alpha' + \beta' Y + \varepsilon'$ with $E(\varepsilon' \mid Y) = 0$

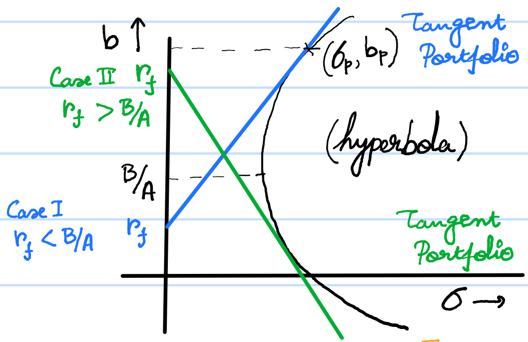
$Y = \underbrace{E(Y \mid X)}_{f(X)} + \underbrace{Y - E(Y \mid X)}_{\varepsilon}$, $E(\varepsilon \mid X) = 0$, for normal $\beta = \dfrac{Cov(X,Y)}{Var(X)}$, $\alpha = E(Y) - \beta E(X)$

For bivariate Normal, $E(Y \mid X) = \alpha + \beta X$ with $\beta = \dfrac{Cov(X,Y)}{Var(X)}$, $\alpha = E(Y) - \beta E(X)$

With two fund separation property,

for $N+1$ assets, $r_q = r_f + \beta_{qp}(r_p - r_f) + \varepsilon$, $E(\varepsilon \mid r_p) = 0$ & $\beta = \dfrac{Cov(r_q, r_p)}{Var(r_p)}$

$\sigma_p^2 = \frac{A}{D}\left(b_p - \frac{B}{A}\right)^2 + \frac{1}{A}$



$x^2 = c_0(y - b_0)^2 + a_0 \Rightarrow 2x\,dx = 2c_0(y - b_0)\,dy$

$\Rightarrow \left(\dfrac{dy}{dx}\right)_{(\sigma_p, b_p)} = \dfrac{x}{c_0(y - b_0)} = \dfrac{\sigma_p}{\frac{A}{D}\left(b_p - \frac{B}{A}\right)} \Rightarrow (y - b_p) = \dfrac{D\sigma_p}{(Ab_p - B)}(x - \sigma_p)$

Is this property to have $\boxed{y = B/A,\ x = 0}$ or $\boxed{x = 0,\ y > B/A \text{ where } b_p > B/A}$

$y - b_p = \dfrac{-D\sigma_p^2}{A(b_p - B/A)} \Rightarrow (y - b_p)(b_p - B/A) = -\dfrac{D\sigma_p^2}{A} < 0$ $[\because D = AC - B^2 > 0,\ A = \underset{\sim}{1}'V'\underset{\sim}{1} > 0]$

$\Rightarrow (y - b_p) = \dfrac{-D\left[\frac{A}{D}(b_p - B/A) + \frac{1}{A}\right]}{A(b_p - B/A)} = -(b_p - B/A) - \dfrac{D}{A^2(b_p - B/A)}$

$\Rightarrow y - b_p + b_p - B/A = -\dfrac{D}{A^2(b_p - B/A)} \Rightarrow (y - B/A) = -\dfrac{D}{A^2(b_p - B/A)} \ne 0$ then $y \ne B/A$

Clearly then, For $b_p > B/A \Rightarrow y < B/A$

and, For $b_p < B/A \Rightarrow y > B/A$

Again, With two fund separation property,

for $N+1$ assets, $r_q = r_f + \beta_{qp}(r_p - r_f) + \varepsilon$, $E(\varepsilon \mid r_p) = 0$ & $\beta = \dfrac{Cov(r_q, r_p)}{Var(r_p)}$

$\Rightarrow E(r_q) = r_f + \beta_{qp}(E(r_p) - r_f) \Rightarrow E(r_q) = (1 - \beta_{qp}) r_f + \beta_{qp} E(r_t)$

tangent portfolio

$0 \le \beta_{qp} \le 1 \Rightarrow \min(E(r_{tg}), r_f) \le E(r_q) \le \max(E(r_{tg}), r_f)$ tangent

For $r_f < B/A$, $\beta_{qp} > 1 \Rightarrow E(r_q) > E(r_{tg})$

For $r_f > B/A$, $\beta_{qp} < 0 \Rightarrow E(r_q) > r_f$

Market Portfolio, $W_{m_0} = \sum_{i=1}^{I} w_0^i$

Asset Allocation,

$(1 - \underset{\sim}{w}'\underset{\sim}{1}) r_f + \underset{\sim}{w}' R$

w_{ij} = weight on j^{th} security by the i^{th} individual

then $\sum_i w_{ij} W_0$ = total amount invested on the j^{th} security

Equilibrium = $\omega_{mj} W_{m_0}$

$\Rightarrow \omega_{mj} = \sum_i \omega_{ij} \dfrac{W_0^i}{W_{m0}} \quad \Rightarrow \quad \underset{\sim}{\omega}_m = \sum_i \left(\dfrac{W_0^i}{W_{m0}}\right) \omega_i$ = Market Portfolio Weights.

Under same assumption one can show r_m = market is corresponds to the tangent portfolio.

$\boxed{\text{Case III}}$

$r_f = B/A \quad , \quad \omega_{op} = \left(\dfrac{b_p - r_f}{H}\right) V^{-1} \left(\underset{\sim}{\mu} - r_f \underset{\sim}{1}\right)$

$\underset{\sim}{1}' \omega_{op} = \left(\dfrac{b_p - r_f}{H}\right) \underset{\sim}{1}' V^{-1} \left(\underset{\sim}{\mu} - r_f \underset{\sim}{1}\right) = \left(\dfrac{b_p - r_f}{H}\right) \left(\underset{\sim}{1}' V^{-1} \underset{\sim}{\mu} - r_f \underset{\sim}{1}' V^{-1} \underset{\sim}{1}\right) = \left(\dfrac{b_p - r_f}{H}\right) (B - r_f A)$

when we put $r_f = B/A$ then $\underset{\sim}{1}' \omega_{op} = 0$ i.e. the sum of all weights are zero.