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Last time: X, Y : continuous random variable.

* $X \perp Y$ (X and Y indep.)

$$X \to f_X \qquad Y \to f_Y$$

If $\boxed{Z = X + Y}$ then the pdf of Z is given by

$$\boxed{\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx \\ &= (f_X * f_Y)(z). \end{aligned}}$$

* $\boxed{Z = \frac{Y}{X}}$ $\quad X \perp Y$.

$$\boxed{f_Z(z) = \int_{-\infty}^{\infty} |x| \cdot f_X(x)\, f_Y(xz)\, dx.}$$

X and Y are independent $\sim N(0,1)$

$Z \sim$ Standard Cauchy distribution.

The general case

Suppose:

- X and Y are jointly distributed CONTINUOUS random variable.
- X and Y are mapped onto U and V

by the <u>transformation</u>

$$\boxed{\begin{aligned} u &= g_1(x,y) \\ v &= g_2(x,y) \end{aligned}}$$

If, $U = X + Y$ then $g_1(x,y) = x+y$

If $V = \frac{Y}{X}$ $g_2(x,y) = y/x$

- and that the transformation can be <u>inverted</u> to obtain

$$\boxed{\begin{aligned} x &= h_1(u,v) \\ y &= h_2(u,v) \end{aligned}}$$

- Assume that g_1, g_2 have continuous partial derivatives and the Jacobian $\neq 0$ for all x, y.

$$\left[\; J(x,y) = \det\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \frac{\partial g_1}{\partial x}\cdot\frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y}\cdot\frac{\partial g_2}{\partial x} \neq 0 \text{ for all } x, y \;\right]$$

Theorem: Under the above assumptions, the joint density of U and V is given by

$$f_{UV}(u,v) = f_{XY}\big(h_1(u,v),\, h_2(u,v)\big)\,\big|J^{-1}\big(h_1(u,v), h_2(u,v)\big)\big|$$

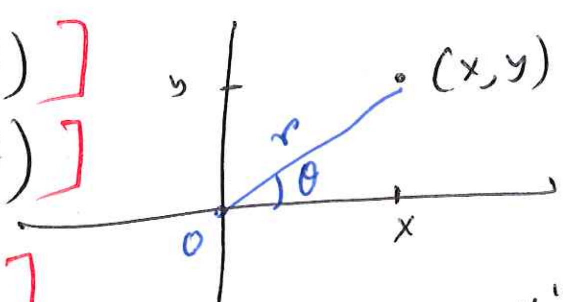
for (u,v) such that $\left.\begin{aligned} u &= g_1(x,y) \\ v &= g_2(x,y) \end{aligned}\right\}$

for some (x,y).

$$f_{UV}(u,v) = 0, \quad \text{otherwise}$$

Example: Cartesian $\longrightarrow$ Polar transformation.

$$x = r\cos\theta \; [= h_1(r,\theta)]$$
$$y = r\sin\theta \; [= h_2(r,\theta)]$$



$$r = \sqrt{x^2+y^2} \; [= g_1(x,y)]$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \; [= g_2(x,y)]$$

$\left\{\begin{aligned} &\theta \rightarrow \text{small "}\theta\text{"} \\ &\Theta \rightarrow \text{big "}\theta\text{"} \end{aligned}\right.$

Question: Suppose we know $f_{XY}(x,y)$

What is the joint distribution in the polar coordinate?

$$\boxed{f_{R\Theta}(r,\theta) = ?}$$

Solution:

$$J(x,y) = \det \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix}$$

$$= \frac{\partial g_1}{\partial x} \cdot \frac{\partial g_2}{\partial y} - \frac{\partial g_1}{\partial y} \cdot \frac{\partial g_2}{\partial x}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{x}{x^2+y^2} - \frac{y}{\sqrt{x^2+y^2}} \cdot \left(-\frac{y}{x^2+y^2}\right)$$

Computation

$$\frac{\partial g_1}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}$$

$$\frac{\partial g_1}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

~~Similarly compute~~ Computation

~~$\frac{\partial g_2}{\partial x}, \frac{\partial g_2}{\partial y}$~~

$$\frac{\partial g_2}{\partial x} = \frac{\left(-\frac{y}{x^2}\right)}{1+\left(\frac{y}{x}\right)^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial g_2}{\partial y} = \frac{\left(\frac{1}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$J(x,y) = \frac{\cancel{(x^2+y^2)}}{\sqrt{x^2+y^2}\cdot\cancel{(x^2+y^2)}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$J^{-1}(x,y) = \sqrt{x^2+y^2}$$

Using the formula from the last Theorem:

$$f_{R\Theta}(r,\theta) = f_{XY}(r\cos\theta, r\sin\theta)\left|\left(\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\right)\right|$$

$$\Rightarrow \boxed{f_{R\Theta}(r,\theta) = r\, f_{XY}(r\cos\theta, r\sin\theta)}$$

Example: Suppose that X_1 and X_2 are independent standard normal random variables and

$$Y_1 = X_1 \quad (= g_1(X_1, X_2))$$

$$Y_2 = X_1 + X_2 \quad (= g_2(X_1, X_2))$$

Question: What is the joint density for Y_1 and Y_2

Solution: $J(x,y) = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1 - 0 = 1$

$$X_1 = Y_1 \quad (= h_1(Y_1, Y_2))$$

$$X_2 = Y_2 - Y_1 \quad (= h_2(Y_1, Y_2))$$

So, by the last Theorem:

$$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(\underset{x_1}{y_1}, \underset{x_2}{y_2 - y_1}) \, |1|$$

Jacobian $= 1$

Hence $J^{-1}(\ldots) = 1$

As $X_1 \perp X_2$

$$= f_{X_1}(y_1) \cdot f_{X_2}(y_2 - y_1)$$

$$= \boxed{\frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y_1^2}{2}}} \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{(y_2 - y_1)^2}{2}}}$$

$$\boxed{f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(2y_1^2 + y_2^2 - 2y_1 y_2)} \quad \cdots (1)}$$

REMEMBER: (Bivariate Normal density function)

$$f(x, y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y}\right]\right) \quad \cdots (2)$$

$x \to y_1$, $\mu_X \to \mu_{Y_1}$, $y \to y_2$, $\mu_Y \to \mu_{Y_2}$, $\sigma_X \to \sigma_{Y_1}$, $\sigma_Y \to \sigma_{Y_2}$

Compare the above two forms.
(Keep in mind for comparison in 2 take $x = y_1$, $y = y_2$
$X = Y_1$, $Y = Y_2$)

$$\sigma_{Y_1} \cdot \sigma_{Y_2} \cdot \sqrt{1-\rho^2} = 1 \quad \ldots\ldots (3)$$

$$\boxed{\mu_{Y_1} = \mu_{Y_2} = 0}$$

Compare the coefficient of y_1:

$$\sigma_{Y_1}^2 (1-\rho^2) = \frac{1}{2} \quad \ldots (4)$$

Compare the coefficient of y_2: $\sigma_{Y_2}^2(1-\rho^2) = 1 \ldots (5)$

(3) ÷ (4)

$$\frac{\cancel{\sigma_{Y_1}^2}\, \sigma_{Y_2}^2 \cancel{(1-\rho^2)}}{\cancel{\sigma_{Y_1}^2}\, \cancel{(1-\rho^2)}} = 2$$

$$\boxed{\sigma_{Y_2}^2 = 2}$$

Put in (5) to get $(1-\rho^2) = \frac{1}{2}$

$$\boxed{\rho^2 = \frac{1}{2}}$$

By (3) using all these we have.

$$\sigma_{Y_1} \cdot \sqrt{2} \cdot \sqrt{\frac{1}{2}} = 1$$

$$\Rightarrow \boxed{\sigma_{Y_1} = 1}$$

Hence what we found in (1) is indeed a bivariate normal density function with parameters:

$$\boxed{\begin{array}{ll} \sigma_{Y_1}^2 = 1 & \mu_{Y_1} = 0 \\ \sigma_{Y_2}^2 = 2 & \mu_{Y_2} = 0 \\ \multicolumn{2}{c}{\rho^2 = \frac{1}{2}} \end{array}}$$