Hence, under regular estimation case for $Bin(m, p)$ dist^n, $\frac{x}{m}$ is MVBE as well as MVUE for 'p'.

Violation of Regularity Conditions

Illustration: Suppose, $X_1, X_2, \ldots, X_n \overset{iid}{\sim} R(0, \theta)$. Then show that, here it is possible to have an unbiased estimator of θ based on $X_{(n)}$ & CRLB $>$ V(MVUE).

Proof: Since, here, $X_i \overset{iid}{\sim} R(0, \theta)$, $\forall\, i = 1(1)n$, so, their common p.d.f. is

$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta;\ \theta > 0. \\ 0, & \text{o.w} \end{cases}$$

Now $F_{X_{(n)}}(x) = P[X_{(n)} \le x] = P[\max\{X_1, \ldots, X_n\} \le x]$.

$= P[X_1 \le x, X_2 \le x, \ldots, X_n \le x] = P[X_1 \le x]^n$; $\because$ X_i's are iid, $\forall\, i = 1(1)n$.

$= F^n_{X_1}(x) = \frac{x^n}{\theta^n}$ [$\because$ X_i's are iid with common c.d.f. $F(x) = \frac{x}{\theta}$].

$$\therefore f_{X_{(n)}}(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & 0 < x < \theta \\ & \theta > 0 \\ 0, & \text{o.w.} \end{cases}$$

$$\therefore E(X_{(n)}) = \int_0^\theta x \cdot \frac{n}{\theta^n} \cdot x^{n-1}\, dx = \frac{n}{\theta^n} \int_0^\theta x^n\, dx = \frac{n}{\theta^n} \cdot \frac{x^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1} \cdot \frac{\theta^{n+1}}{\theta^n}$$

$$\therefore E(X_{(n)}) = \frac{n}{n+1}\theta. \Rightarrow E\left[\frac{n+1}{n} X_{(n)}\right] = \theta.$$

$$\therefore E[X_{(n)}^2] = \frac{n}{\theta^n} \int_0^\theta x^2 x^{n-1}\, dx = \frac{n}{\theta^n} \cdot \frac{x^{n+2}}{n+2}\Big|_0^\theta = \frac{n}{n+2} \cdot \theta^2.$$

$$\therefore V(X_{(n)}) = \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = n\theta^2\left[\frac{1}{n+2} - \frac{n}{(n+1)^2}\right] = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

$$\therefore V\left[\frac{n+1}{n} X_{(n)}\right] = \frac{(n+1)^2}{n^2} V(X_{(n)}) = \frac{(n+1)^2}{n^2} \cdot \frac{n\theta^2}{(n+2)(n+1)^2} = \frac{\theta^2}{n(n+2)}$$

Now, $$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ & \theta > 0 \\ 0, & \text{o.w.} \end{cases}$$

$\therefore \ln f_\theta(x) = -\ln\theta \quad \therefore \frac{\partial}{\partial\theta}\ln f_\theta(x) = -\frac{1}{\theta}.$

$\therefore E\left[\left(\frac{\partial}{\partial\theta}\ln f_\theta(X)\right)^2\right] = E\left[\left(-\frac{1}{\theta}\right)^2\right] = \frac{1}{\theta^2}.$

Now, choosing $\gamma(\theta) = \theta$, $\forall \theta > 0$, we have $\gamma'(\theta) = 1$.

$$\therefore CRLB = \frac{[\gamma'(\theta)]^2}{n E\left[\left\{\frac{\partial}{\partial\theta}\ln f_\theta(X)\right\}^2\right]} = \frac{1}{n \times \frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

Again, $V\left(\frac{n+1}{n} \cdot X_{(n)}\right) = \frac{\theta^2}{n(n+2)} = \frac{\theta^2}{n} \cdot \frac{1}{n+2} < \frac{\theta^2}{n} = CRLB$, $\because 0 < \frac{1}{n+2} < 1$ as $n > 0$ i.e. $(n+2) > 2$

$\therefore V(\text{unbiased estimator of } X_{(n)}) < CRLB$.

Hence, it is obvious that, $V[\text{MVUE of } \theta \text{ (if exists)}] < CRLB$.

<u>Note</u>: Th   ears here as ~~the~~ all the regularity conditions of C.R. inequality are not satisfied ~~here~~ for $R(0, \theta)$. Specially here, the domain of positive probability density depends on the unknown parameter θ ($\because 0 < x < \theta$)...

Rao-Blackwellization & MVUE.

Where, CR-inequality is applicable under some regularity conditions, ie under a no. of stringent (कठोर) conditions, Rao-Blackwellization is applicable in a much more relaxed situation. Moreover, here we can have MVUE (if exists) directly from the theorem.

Rao-Blackwellization Theorem:

Suppose, $U = U(X_1, X_2, \ldots, X_n)$ is an unbiased estimator of an estimable parametric function $\gamma(\theta)$, $\forall \theta \in \Theta$.

Suppose, $T = T(X_1, X_2, \ldots, X_n)$ is a sufficient statistic for $\theta \in \Theta$. Then, the estimator $\phi(T) = E(U|T)$ is also an unbiased estimator of $\gamma(\theta)$, $\forall \theta \in \Theta$ & $V[\phi(T)] \leq V_\theta(U)$, $\forall \theta \in \Theta$.

Proof: According to the definition, we have,

$$E_\theta[\phi(T)] = E_\theta[E_\theta(U|T)] = E_\theta(U) = \gamma(\theta), \ \forall \theta \in \Theta$$

$\because U$ is an u.e. of $\gamma(\theta)$.

$\therefore \phi(T)$ is also an u.e. of $\gamma(\theta)$, $\forall \theta \in \Theta$.

Now, $V_\theta(U) = E_\theta[V_\theta(U|T)] + V_\theta[E_\theta(U|T)]$

$= E_\theta[V_\theta(U|T)] + V_\theta[\phi(T)]$.

$\Rightarrow$

$\Rightarrow V_\theta[\phi(T)] \leq V_\theta(U)$, $\forall\, \theta \in \Theta$, as $E[V(U|T)] \geq 0$,

with sign of equality iff $E_\theta[V(U|T)] = 0$.

$\Rightarrow E_\theta[V(U|T)] = 0. \Rightarrow E_\theta[E_\theta\{(U - E(U|T))^2 | T\}] = 0.$

$\Rightarrow E_\theta[\{U - E(U|T)\}^2] = 0 \Rightarrow E_\theta[\{U - \phi(T)\}^2] = 0. \iff U - \phi(T) = 0$, w.p. 1,

i.e. $U = \phi(T)$, w.p. 1

$= E(U|T)$, $\forall\, \theta \in \Theta$, w.p. 1.

Applications:

(1) Suppose, $X_1, X_2, \ldots, X_n \overset{iid}{\sim} P(\lambda)$. Then find an MVUE of

$$P_\lambda[X=k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \lambda > 0 \; = \gamma(\lambda), \; \lambda > 0.$$

Solution: Let us first define an unbiased estimator of $\gamma(\lambda)$ as,

$$U = \begin{cases} 1, & \text{if } X_1 = k \\ 0, & \text{o.w.} \end{cases}$$

$\therefore E(U) = 1 \cdot P[X_1 = k] + 0 \cdot P[X_1 \neq k] = P[X_1 = k] = \frac{e^{-\lambda}\lambda^k}{k!} = \gamma(\lambda)$, $\because X_1 \sim P(\lambda)$.

Now, $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ, where, $X_i \sim P(\lambda)$, $\forall\, i = 1(1)n$.

Now, from Rao-Blackwellization theorem [To be proved], we have the MVUE of $\gamma(\theta)$ as,

$$\phi(T) = E(U|T) = E[U|T=t] = \frac{E(U, T=t)}{P(T)}$$

$$= \frac{1 \times P[X_1 = k, T=t] + 0 \times P[X_1 \neq k, T=t]}{P[T=t]} = \frac{1 \times P[X_1 = k, \sum_{i=1}^{n} X_i = t]}{P[\sum_{i=1}^{n} X_i = t]}$$

$$= \frac{P[X_1 = k, \sum_{i=2}^{n} X_i = t-k]}{P(\sum_{i=1}^{n} X_i = t)} = \frac{P(X_1 = k) \cdot P(\sum_{i=2}^{n} X_i = t-k)}{P[\sum_{i=1}^{n} X_i = t]}$$

$$= \frac{\frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{e^{-(n-1)\lambda}\{(n-1)\lambda\}^{t-k}}{(t-k)!}}{\frac{e^{-n\lambda}(n\lambda)^t}{t!}} = \frac{t!}{k!(t-k)!} \cdot \frac{(n-1)^{t-k}}{n^t} = \binom{t}{k}\frac{(n-1)^{t-k}}{n^t}, \; t = k, k+1, \ldots$$

$\therefore \phi(T) = \binom{T}{k}\frac{(n-1)^{T-k}}{n^T}$ is the MVUE of $\gamma(\theta) = \frac{e^{-\lambda}\lambda^k}{k!}$

K!

(2) Let, $X_1, X_2, \ldots, X_n \overset{iid}{\sim} Bin(m, p)$. Then, find an MVUE of $q^m + \binom{m}{1} q^{m-1} p + \binom{m}{2} q^{m-2} p^2$.

Solution: Since, $X_i \overset{iid}{\sim} Bin(m, p)$, $\forall\ i = 1(1)n$, so ~~[illegible]~~.

$P[X_1 \leq 2] = P[X_1 = 0] + P[X_1 = 1] + P[X_1 = 2]$.

$= \binom{m}{0} p^0 q^{m-0} + \binom{m}{1} p^1 q^{m-1} + \binom{m}{2} p^2 q^{m-2}$

$= q^m + \binom{m}{1} p q^{m-1} + \binom{m}{2} p^2 q^{m-2}$

$\therefore$ Here, we have to find the MVUE of $P[X_1 \leq 2] = \gamma(p)$.

13c Let us now define, $U = \begin{cases} 1, & \text{if } X_1 \le 2 \\ 0, & \text{o.w.} \end{cases}$

$\therefore E_p(U) = 1 \times P[X_1 \le 2] = \gamma(p), \quad \forall\, p \in (0,1).$

$\therefore$ U is an u.e. of $\gamma(p)$.

Now, we know that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for 'p'.

$\therefore$ According to the Rao-Blackwellization theorem (To be proved), we have, $\phi(T) = E(U|T)$ is the MVUE of $\gamma(p)$.

Again, $\phi(t) = E(U|T=t) = \dfrac{E(U, \sum_{i=1}^{n} X_i = t)}{P[\sum_{i=1}^{n} X_i = t]}$

$$= \frac{1 \times P[X_1 \le 2, \sum_{i=1}^{n} X_i = T] + 0 \times P[X_1 \not\le 2, \sum_{i=1}^{n} X_i = T]}{P[\sum_{i=1}^{n} X_i = t]}$$

$$= \frac{P[X_1 = 0, \sum_{i=1}^{n} X_i = t] + P[X_1 = 1, \sum_{i=1}^{n} X_i = t] + P[X_1 = 2, \sum_{i=1}^{n} X_i = t]}{P[\sum_{i=1}^{n} X_i = t]}.$$

$$= \frac{P[X_1 = 0, \sum_{i=2}^{n} X_i = t] + P[X_1 = 1, \sum_{i=2}^{n} X_i = t-1] + P[X_1 = 2, \sum_{i=2}^{n} X_i = t-2]}{P[\sum_{i=1}^{n} X_i = t]}.$$

$\ldots + P(X_1 = 1) P(\sum^{n} X_i = t-1) + P(X_1 = 2) P(\sum^{n} X_i = t-2)$

$$= \frac{P\left[X_1=0, \sum_{i=2}^{n} X_i = t\right] + P\left[X_1=1, \sum_{i=2}^{n} X_i = t-1\right] + P\left[X_1=2, \sum_{i=2}^{n} X_i = t-2\right].}{P\left[\sum_{i=1}^{n} X_i = t\right]}$$

$$= \frac{P(X_1=0)\, P\left(\sum_{i=2}^{n} X_i = t\right) + P(X_1=1)\, P\left(\sum_{i=2}^{n} X_i = t-1\right) + P(X_1=2)\, P\left(\sum_{i=2}^{n} X_i = t-2\right)}{P\left(\sum_{i=1}^{n} X_i = t\right).}$$

$$= \frac{\binom{m}{0} p^0 q^{m-0} \times \binom{\overline{n-1}\,m}{t} p^t q^{\overline{n-1}\,m-t} + \binom{m}{1} p^1 q^{m-1} \times \binom{\overline{n-1}\,m}{t-1} p^{t-1} q^{\overline{n-1}\,m-t+1} + \binom{m}{2} p^2 q^{m-2} \times \binom{\overline{n-1}\,m}{t-2} p^{t-2} q^{\overline{n-1}\,m-t+2},}{\binom{nm}{t} p^t q^{nm-t}}$$

$$= \frac{q^{nm-t}\left[\binom{m}{0}\binom{\overline{n-1}\,m}{t} + \binom{m}{1}\binom{\overline{n-1}\,m}{t-1} + \binom{m}{2}\binom{\overline{n-1}\,m}{t-2}\right].}{\binom{nm}{t}}$$

$$\therefore \phi(T) = \frac{\binom{m}{0}\binom{\overline{n-1}\,m}{T} + \binom{m}{1}\binom{\overline{n-1}\,m}{T-1} + \binom{m}{2}\binom{\overline{n-1}\,m}{T-2}}{\binom{nm}{T}}$$

is the MVUE of $\gamma(p) = P(X_1 \le 2) = q^m + \binom{m}{1} q^{m-1} p + \binom{m}{2} q^{m-2} p.$

③ Let, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} P(\lambda)$. Then find an MVUE of $(1-e^{-\lambda})$.

Solution: Since, by question, $X_1, X_2, \cdots, X_n \overset{iid}{\sim} P(\lambda)$, so the common p.d.f. of X

$$f_\lambda(x) = \begin{cases} \frac{e^{-\lambda} \cdot \lambda^x}{x!}, & x = 0, 1, 2, \cdots ; \lambda > 0. \\ 0, & o.w. \end{cases}$$

$\therefore P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-\lambda} = \gamma(\lambda).$

Now, let us define $U = \begin{cases} 1, & \text{if } X_1 \geq 1 \\ 0, & \text{o.w.} \end{cases}$

$\therefore E(U) = 1 \times P(X_1 \geq 1) + 0 \times P(X_1 \ngeq 1) = \gamma(\lambda), \ \forall \ \lambda > 0.$

$\therefore$ U is an unbiased estimator of $\gamma(\lambda)$.

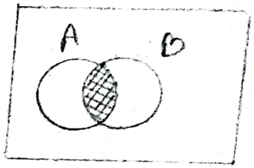
Now, we know that $T = \sum_{i=1}^{n} X_i$ is a sufficient statistic for λ.

$\therefore$ According to the Rao-Blackwellization theorem [to be proved],

$\phi(T) = E(U|T)$ is the MVUE of $\gamma(\lambda)$.

Method → (1) Now, $\phi(t) = E[U \mid T = t] = \dfrac{E[U, \sum_{i=1}^{n} X_i = t]}{P[\sum_{i=1}^{n} X_i = t]}$

$$= \frac{1 \times P(X_1 \geq 1, \sum_{i=1}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)} \quad \cancel{= 1 - P(X_1 = 0, \sum_{i=1}^{n} X_i =}$$



$$= \frac{P[\sum_{i=1}^{n} X_i = t] - P(X_1 = 0, \sum_{i=1}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)}, \quad \because P(A \cap B) = P(B) - P(A^c \cap B).$$

$$= 1 - \frac{P(X_1 = 0, \sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)} = 1 - \frac{P(X_1 = 0) P(\sum_{i=2}^{n} X_i = t)}{P(\sum_{i=1}^{n} X_i = t)}, \quad \because X_i\text{'s are iid, } \forall i \ldots$$

$$= 1 - \frac{e^{-\lambda} \cdot e^{-(n-1)\lambda} \cdot \{(n-1)\lambda\}^t / t!}{e^{-n\lambda} \cdot (n\lambda)^t / t!} = 1 - \frac{(n-1)^t}{n^t}, \quad t = 1, 2, \ldots$$

$\therefore \phi(T) = 1 - \dfrac{(n-1)^T}{n^T}$ is the MVUE of $\gamma(\lambda) = 1 - e^{-\lambda}$.

—x—

Method → (2)

④ Let, $x_1, x_2, \cdots, x_n \overset{iid}{\sim} G(\alpha, p)$. Then find CRLB of an unbiased estimator of 'p'.

Ans/ ∵, X_i's are iid, so, their common p.d.f. is given by

$$f_p(x) = \begin{cases} \dfrac{\alpha^p}{\Gamma p} e^{-\alpha x} \cdot x^{p-1}, & x > 0 \\ & \alpha, p > 0. \\ 0, \text{ o.w.} \end{cases}$$

$$\therefore \ln f_p(x) = p \ln \alpha - \ln \Gamma p - \alpha x + (p-1) \ln x.$$

$$\therefore \frac{\partial \ln f_p(x)}{\partial p} = \ln \alpha - \frac{\partial}{\partial p}(\ln \Gamma p) + \ln x.$$

Now, $\Gamma p = \int_0^\infty e^{-u} \cdot u^{p-1} du$

$$\therefore \frac{\partial}{\partial p} \Gamma p = \int_0^\infty e^{-u} \cdot u^{p-1} (\ln u)\, du.$$

$$\therefore \frac{\partial}{\partial p} \ln \Gamma p = \frac{1}{\Gamma p} \cdot \frac{\partial}{\partial p} \Gamma p = \frac{1}{\Gamma p} \int_0^\infty e^{-u} \cdot u^{p-1} (\ln u)\, du.$$

$= E(\ln U \mid U \sim \Gamma(1,p))$, ∵ $f(u) = \frac{1}{\Gamma p} \cdot e^{-u} \cdot u^{p-1}$, $u > 0$, $p > 0$

$$\therefore \frac{\partial^2}{\partial p^2} \ln f_p(x) = -\frac{\partial}{\partial p}\left[\frac{\partial}{\partial p} \ln \Gamma p\right] = -\frac{\partial}{\partial p}\left[\frac{1}{\Gamma p} \int_0^\infty e^{-u} \cdot u^{p-1} \ln(u)\, du\right]$$

$$\cancel{= -\left[-\frac{1}{(\Gamma p)^2} \int_0^\infty e^{-u} \ln u \cdot e^{-u} \cdot u^{p-1} du\right.}$$

$$= -\left[-\frac{1}{(\Gamma p)^2} \cdot \left(\frac{\partial}{\partial p} \Gamma p\right) \cdot \int_0^\infty \ln u\, e^{-u} \cdot u^{p-1} du + \frac{1}{\Gamma p} \int_0^\infty (\ln u)^2 \cdot e^{-u} \cdot u^{p-1} du\right]$$

$$= \frac{1}{(\Gamma p)^2} \left[\int_0^\infty \ln u\, e^{-u} \cdot u^{p-1} du\right]^2 - \frac{1}{\Gamma p} \int_0^\infty (\ln u)^2 \cdot e^{-u} \cdot u^{p-1} du$$

$$= \left(\frac{1}{\Gamma p} \int_0^\infty \ln u\, e^{-u} \cdot u^{p-1} du\right)^2 - \frac{1}{\Gamma p} \int_0^\infty (\ln u)^2 \cdot e^{-u} \cdot u^{p-1} du.$$

$$\cancel{= \frac{1}{\Gamma p} \int_0^\infty \ln u} = [\{E^2(\ln U) - E[(\ln U)^2]\} \mid U \sim \Gamma(1,p)]$$

$$= -V(\ln U \mid U \sim \Gamma(1,p)).$$

Now, choosing $\gamma(p) = p$, ie $\gamma'(p) = 1$, we have,

$$CRLB = \frac{\{\gamma'(p)\}^2}{-nE\left[\frac{\partial^2}{\partial p^2} \ln f_p(X)\right]} = \frac{1}{-nE[-V(\ln U \mid U \sim \Gamma(1,p))]} = \frac{1}{nV(\ln U)},$$

where, $U \sim \Gamma(1,p)$.

∴ Here, CRLB of an unbiased estimator of 'p' is $\frac{1}{nV(\ln U)}$, where $U \sim \Gamma(1,p)$