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Exercise: If X_1 is uniform on $[0,1]$ and $X_2 | X_1$ also uniform on $[0, X_1]$

find the joint and marginal distribution of X_1 and X_2.

Solution: Joint distribution of X_1 and X_2:

$$\boxed{f_{X_1 X_2}(x_1, x_2)} = f_{X_1}(x_1) \cdot f_{X_2|X_1}(x_2|x_1)$$

$$f_{X_1}(x_1) = \begin{cases} 1, & 0 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= 1 * \cdot \frac{1}{x_1}$$

$$= \boxed{\frac{1}{x_1}}$$

$$f_{X_2|X_1}(x_2|x_1) = \begin{cases} \frac{1}{x_1}, & 0 < x_2 < x_1 \\ 0, & \text{otherwise} \end{cases}$$

$0 < x_1 < 1$

$0 < x_2 < x_1$

Aside:

$X \sim \text{Unif}([a,b])$

$f(x) = \frac{1}{b-a}$, in $[a,b]$

Solution:

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{x_1}$$

$0 < x_1 < 1$

$0 < x_2 < x_1$

Marginal distribution for X_1:

Given

$$f_{X_1}(x_1) = 1 \,, \quad 0 < x_1 < 1$$

Another way:

$$f_{X_1}(x_1) = \int_0^{x_1} f_{X_1 X_2}(x_1, x_2)\, dx_2$$

$$= \int_0^{x_1} \frac{1}{x_1}\, dx_2 = 1$$

when $0 < x_1 < 1$

Marginal distribution for X_2:

$$f_{X_2}(x_2) = \int_{x_2}^{1} f(x_1, x_2)\, dx_1$$

$$= \int_{x_2}^{1} \frac{1}{x_1}\, dx_1$$

$$= \ln(x_1) \Big|_{x_2}^{1}$$

$$= \cancel{\ln(1)}^{0} - \ln(x_2)$$

$$\boxed{f_{X_2}(x_2) = -\ln(x_2) \,, \quad 0 < x_2 < 1}$$

Example: Let X and Y be independent std. normal variables.

Find the density of $Z = X + Y$.

~~Solution: $f_Z(z) =$ Answer $Z \sim N(0, 2)$ (done in class before!)~~

Aside: $Z = \frac{Y}{X}$ and $X, Y \sim N(0,1)$. Then $Z \sim$ Std. Cauchy

Solution:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\, dx$$

$(\because X \perp Y)$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2}\, dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(x^2 + (z-x)^2\right)\right) dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(2x^2 - 2xz + z^2\right)\right) dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(2\left(x - \frac{1}{2}z\right)^2 + \frac{z^2}{2}\right)\right) dx$$

$$\boxed{\exp(x) = e^x}$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{4}z^2\right) \boxed{\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left(\frac{x-\frac{1}{2}z}{1/\sqrt{2}}\right)^2\right) dx}$$

$$= \frac{1}{2\pi} \exp\left(-\frac{1}{4}z^2\right) \cdot \sqrt{2\pi \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 1}$$

pdf of Normal

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \sqrt{2\pi\sigma^2}$$

$$\cancel{= \frac{1}{\sqrt{2\pi\left(\sqrt{2}\right)^2}}}$$

$N(\mu, \sigma^2)$

$$= \frac{1}{\sqrt{2\pi\left(\sqrt{2}\right)^2}} \exp\left(-\frac{1}{2}\left(\frac{z}{\sqrt{2}}\right)^2\right)$$

So, this is the pdf of Normal $\left(0, (\sqrt{2})^2\right)$

$\Rightarrow$ Normal $(0, 2)$

Expected Value

Definition: (Discrete)

If X is a discrete random variable with probability mass function (p.m.f) $p(x)$, the expected value of X is:

$$\boxed{E(X) = \sum_i x_i \, p(x_i) \quad \cdots (1)}$$

provided $\sum_i |x_i| \, p(x_i) < \infty \quad \cdots (2)$

Aside: $(2) \Rightarrow E(X) < \infty$

$$|E(X)| = \left| \sum_i x_i \, p(x_i) \right| \leq \sum_i |x_i \, p(x_i)|$$

Triangle inequality.

$$= \sum_i |x_i| \, p(x_i) < \infty$$

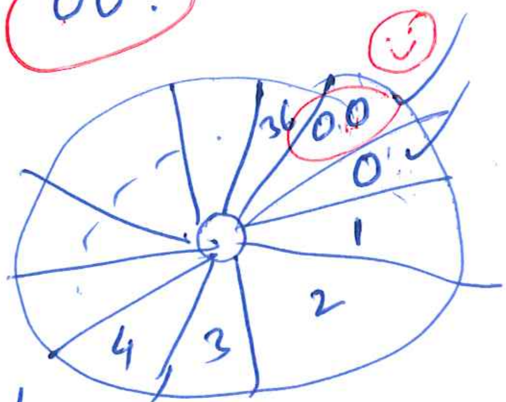
by (2).

Hence $|E(X)| < \infty$

$\Rightarrow E(X)$ is finite.

Example: Roulette

A roulette wheel number 1 through 36 as well as 0 and 00.



If you bet \$1 that an ODD number comes up, you win or lose \$1 according to whether or not that event occurs.

X : ~~net~~ gain

$X = +1$ with prob. $= \frac{18}{38}$

$X = -1$ with prob $= \frac{20}{38}$

The expected value of X:

$$E(X) = (+1)\frac{18}{38} + (-1)\frac{20}{38} = -\frac{1}{19}$$

Expected value = "Avergage"

(Statistical)

$$1, 2, 3, 4$$

$$\frac{1+2+3+4}{4}$$

X	Prob.
1	1/4
2	1/8
3	1/4
4	3/8

$$\frac{1}{2}-\frac{1}{8}=\frac{3}{8}$$

$E(X) =$

$$1\left(\frac{1}{4}\right)+2\left(\frac{1}{8}\right)+3\left(\frac{1}{4}\right)+4\left(\frac{3}{8}\right)$$

Expectation of a geometric random variable

Discrete $k=1,2,3,\ldots$

$$P(X=k)=q^{k-1}\,p \qquad (q=1-p)$$

$$E(X)=\sum_{k=1}^{\infty} k\left(q^{k-1}p\right)=p\sum_{k=1}^{\infty}k\,q^{k-1}$$

value prob.

$$=p\sum_{k=1}^{\infty}\frac{d}{dq}\left(q^{k}\right)$$

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$$= p \cdot \frac{d}{dq} \sum_{k=1}^{\infty} q^k$$

$$= p \cdot \frac{d}{dq}\left(\frac{1}{1-q}\right)$$

$$= p \cdot \frac{1}{(1-q)^2} = p \cdot \frac{1}{p^2}$$

$$= \boxed{\frac{1}{p}}$$

$$S = \sum_{k=1}^{\infty} k q^{k-1} = 1 + 2q + 3q^2 + 4q^3 + \cdots$$

$$qS = \qquad q + 2q^2 + 3q^3 + \cdots$$

$$(-) \qquad (-)$$

$$(1-q)S = 1 + q + q^2 + q^3 + \cdots$$

$$(1-q)S = \frac{1}{1-q}$$

$$S = \frac{1}{(1-q)^2}$$

Hence, $E(X) = p \cdot S = p \cdot \frac{1}{(1-q)^2} = p \cdot \frac{1}{p^2} = \boxed{\frac{1}{p}}$

Summary: $X \sim$ Geometric (prob of success $= p$)

$$E(X) = \frac{1}{p}$$

Poisson distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad k = 0, 1, 2, \ldots$$

$$E(X) = \sum_{k=0}^{\infty} k \cdot \left(\frac{e^{-\lambda}\lambda^k}{k!}\right)$$

$$= e^{-\lambda}\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = e^{-\lambda}\lambda \sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}$$

$u = k-1$

$$= e^{-\lambda}\lambda \sum_{u=0}^{\infty}\frac{\lambda^u}{u!}$$

$$= e^{-\lambda}\cdot\lambda\cdot e^{\lambda} = \boxed{\lambda}$$

Summary: $X \sim$ Poisson (λ)

$$E(X) = \lambda$$

$$\boxed{\begin{aligned} &X \sim \text{Binomial}(n, p) \\ &E(X) = np. \end{aligned}}$$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

~~$E(X) =$~~

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \cdots$$

$$= np$$ (Homework)

Definition (Continuous random variable):

If X is a continuous random variable with density $f(x)$, then

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx \quad \cdots (1)$$

provided $\int_{-\infty}^{\infty} |x| \cdot f(x)\, dx < \infty$.

If (1) diverges, the expectation is undefined.

Normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$$E(X) = \int_{-\infty}^{\infty} x . f(x)\,dx = \int_{-\infty}^{\infty} x . \frac{1}{\sigma\sqrt{2\pi}} . e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Sub:
$u = x - \mu$
$du = dx$

$$= \int_{-\infty}^{\infty} (u+\mu) . \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma^2}} du$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} u . e^{-\frac{u^2}{2\sigma^2}} du}_{0} + \frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du$$

$f(u) = u . e^{-\frac{u^2}{2\sigma^2}}$
$f(-u) = -f(u)$
Hence $f(u)$ is odd

Aside:

$\int_{-\infty}^{\infty} f(x)\,dx = 0$

$f(x)$: odd function

$f(-x) = -f(x)$

$$= \mu . \boxed{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du} = 1$$

$$= \boxed{\mu}$$

Standard Cauchy distribution

We found pdf of Standard Cauchy

$$f(x) = \frac{1}{\pi} \cdot \left(\frac{1}{1+x^2}\right), \quad -\infty < x < \infty.$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

odd function.

$$= \int_{-\infty}^{\infty} \boxed{x \cdot \left(\frac{1}{\pi} \cdot \frac{1}{1+x^2}\right)}\, dx$$

$$= 0$$

Even function

But $\int_{-\infty}^{\infty} |x| \cdot f(x)\, dx = \int_{-\infty}^{\infty} |x| \cdot \left(\frac{1}{\pi} \cdot \frac{1}{1+x^2}\right) dx$

$$\int_{-\infty}^{\infty} f(x)\, dx = 2 \int_{0}^{\infty} f(x)\, dx$$

If $f(x)$ is even i.e., $f(-x) = f(x)$

$$= \frac{2}{\pi} \int_{0}^{\infty} |x| \cdot \frac{1}{1+x^2}\, dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2}\, dx$$

$$= \frac{\cancel{x}}{\pi} \int_1^{\infty} \frac{1}{u} \, \frac{du}{\cancel{x}}$$

$$= \frac{1}{\pi} \ln u \Big|_1^{\infty}$$

$$\longrightarrow \infty$$

Sub.

$u = 1 + x^2$

$du = 2x \, dx$

When $x = 0, \; u = 1$

$x = \infty, \; u = \infty$

Hence, $\displaystyle\int_{-\infty}^{\infty} |x| \, f(x) \, dx \longrightarrow \infty$

(Though, $E(X) = \displaystyle\int_{-\infty}^{\infty} x \, f(x) \, dx = 0$)

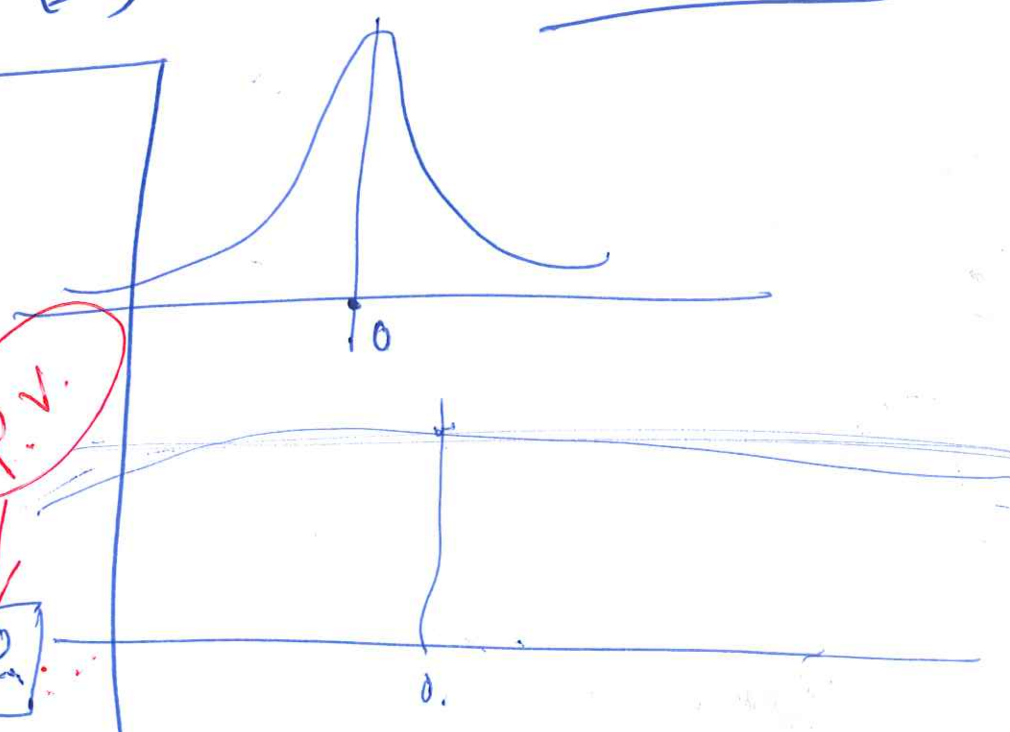
We say $E(X)$ fails to exist.

$$PV\left(\int_{-\infty}^{\infty} x^3 \, dx \right)$$

$$= \lim_{R \to \infty} \int_{-R}^{R} x^3 \, dx$$

P.V.

$$= \lim_{R \to \infty} 0 = \boxed{0}$$



$$\int_{-\infty}^{\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_{0}^{\infty} x^3\,dx$$

$$\rightarrow (\infty + \infty)\ ?.$$