so $\alpha = 1/5$, $\beta = 4/5$, $\gamma = -2/5$ and

$$p_{11}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left\{\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right\}.$$

More generally, the following method may in principle be used to find a formula for $p_{ij}^{(n)}$ for any M-state chain and any states i and j.

(i) Compute the eigenvalues $\lambda_1, \dots, \lambda_M$ of P by solving the characteristic equation.

(ii) If the eigenvalues are distinct then $p_{ij}^{(n)}$ has the form

$$p_{ij}^{(n)} = a_1\lambda_1^n + \dots + a_M\lambda_M^n$$

for some constants $a_1, \dots, a_M$ (depending on i and j). If an eigenvalue λ is repeated (once, say) then the general form includes the term $(an+b)\lambda^n$.

(iii) As roots of a polynomial with real coefficients, complex eigenvalues will come in conjugate pairs and these are best written using sine and cosine, as in the example.

Exercises

1.1.1 Let $B_1, B_2, \dots$ be disjoint events with $\bigcup_{n=1}^\infty B_n = \Omega$. Show that if A is another event and $\mathbb{P}(A|B_n) = p$ for all n then $\mathbb{P}(A) = p$.

Deduce that if X and Y are discrete random variables then the following are equivalent:

(a) X and Y are independent;

(b) the conditional distribution of X given $Y = y$ is independent of y.

1.1.2 Suppose that $(X_n)_{n\geq 0}$ is Markov (λ, P). If $Y_n = X_{kn}$, show that $(Y_n)_{n\geq 0}$ is Markov (λ, P^k).

1.1.3 Let X_0 be a random variable with values in a countable set I. Let $Y_1, Y_2, \dots$ be a sequence of independent random variables, uniformly distributed on $[0,1]$. Suppose we are given a function

$$G : I \times [0,1] \to I$$

and define inductively

$$X_{n+1} = G(X_n, Y_{n+1}).$$

Show that $(X_n)_{n\geq 0}$ is a Markov chain and express its transition matrix P in terms of G. Can all Markov chains be realized in this way? How would you simulate a Markov chain using a computer?

Suppose now that $Z_0, Z_1, \ldots$ are independent, identically distributed random variables such that $Z_i = 1$ with probability p and $Z_i = 0$ with probability $1-p$. Set $S_0 = 0$, $S_n = Z_1 + \ldots + Z_n$. In each of the following cases determine whether $(X_n)_{n\geq 0}$ is a Markov chain:

(a) $X_n = Z_n$, (b) $X_n = S_n$,
(c) $X_n = S_0 + \ldots + S_n$, (d)$X_n = (S_n, S_0 + \ldots + S_n)$.

In the cases where $(X_n)_{n\geq 0}$ is a Markov chain find its state-space and transition matrix, and in the cases where it is not a Markov chain give an example where $P(X_{n+1} = i | X_n = j, X_{n-1} = k)$ is not independent of k.

1.1.4 A flea hops about at random on the vertices of a triangle, with all jumps equally likely. Find the probability that after n hops the flea is back where it started.

A second flea also hops about on the vertices of a triangle, but this flea is twice as likely to jump clockwise as anticlockwise. What is the probability that after n hops this second flea is back where it started? [*Recall that* $e^{\pm i\pi/6} = \sqrt{3}/2 \pm i/2$.]

1.1.5 A die is 'fixed' so that each time it is rolled the score cannot be the same as the preceding score, all other scores having probability 1/5. If the first score is 6, what is the probability p that the nth score is 6? What is the probability that the nth score is 1?

Suppose now that a new die is produced which cannot score one greater (mod 6) than the preceding score, all other scores having equal probability. By considering the relationship between the two dice find the value of p for the new die.

1.1.6 An octopus is trained to choose object A from a pair of objects A, B by being given repeated trials in which it is shown both and is rewarded with food if it chooses A. The octopus may be in one of three states of mind: in state 1 it cannot remember which object is rewarded and is equally likely to choose either; in state 2 it remembers and chooses A but may forget again; in state 3 it remembers and chooses A and never forgets. After each trial it may change its state of mind according to the transition matrix

State 1	$\frac{1}{2}$	$\frac{1}{2}$	0
State 2	$\frac{1}{2}$	$\frac{1}{12}$	$\frac{5}{12}$
State 3	0	0	1.

It is in state 1 before the first trial. What is the probablity that it is in state 1 just before the $(n+1)$th trial ? What is the probability $P_{n+1}(A)$ that it chooses A on the $(n+1)$th trial ?

So, if y is non-negative,

$$y_i \geq \mathbb{P}_i(H^A \geq 1) + \ldots + \mathbb{P}_i(H^A \geq n)$$

and, letting $n \to \infty$,

$$y_i \geq \sum_{n=1}^{\infty} \mathbb{P}_i(H^A \geq n) = \mathbb{E}_i(H^A) = x_i.$$

□

Exercises

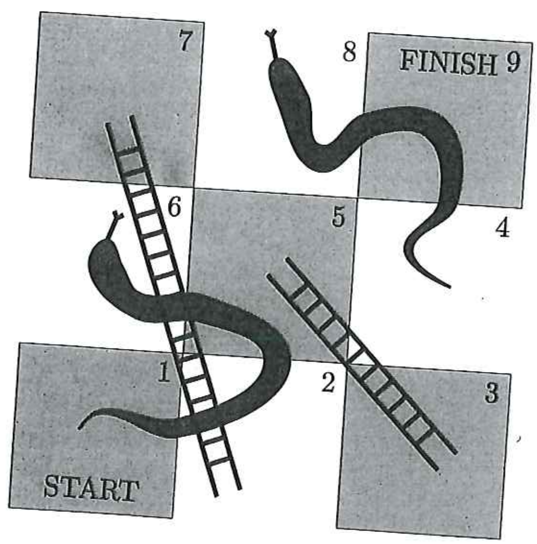
1.3.1 Prove the claims (a), (b) and (c) made in example (v) of the Introduction.

1.3.2 A gambler has £2 and needs to increase it to £10 in a hurry. He can play a game with the following rules: a fair coin is tossed; if a player bets on the right side, he wins a sum equal to his stake, and his stake is returned; otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has £5 or less, and otherwise stakes just enough to increase his capital, if he wins, to £10.

Let $X_0 = 2$ and let X_n be his capital after n throws. Prove that the gambler will achieve his aim with probability 1/5.

What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

1.3.3 A simple game of 'snakes and ladders' is played on a board of nine squares.



At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail. How many turns on average does it take to complete the game?

What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

1.3.4 Let $(X_n)_{n\geq 0}$ be a Markov chain on $\{0, 1, \dots\}$ with transition probabilities given by

$$p_{01} = 1, \quad p_{i,i+1} + p_{i,i-1} = 1, \quad p_{i,i+1} = \left(\frac{i+1}{i}\right)^2 p_{i,i-1}, \quad i \geq 1\,.$$

Show that if $X_0 = 0$ then the probability that $X_n \geq 1$ for all $n \geq 1$ is $6/\pi^2$.

1.4 Strong Markov property

In Section 1.1 we proved the Markov property. This says that for each time m, conditional on $X_m = i$, the process after time m begins afresh from i. Suppose, instead of conditioning on $X_m = i$, we simply waited for the process to hit state i, at some random time H. What can one say about the process after time H? What if we replaced H by a more general random time, for example $H-1$? In this section we shall identify a class of random times at which a version of the Markov property does hold. This class will include H but not $H-1$; after all, the process after time $H-1$ jumps straight to i, so it does not simply begin afresh.

A random variable $T : \Omega \to \{0, 1, 2, \dots\} \cup \{\infty\}$ is called a *stopping time* if the event $\{T = n\}$ depends only on $X_0, X_1, \dots, X_n$ for $n = 0, 1, 2, \dots$. Intuitively, by watching the process, you know at the time when T occurs. If asked to stop at T, you know when to stop.

Examples 1.4.1

(a) The *first passage time*

$$T_j = \inf\{n \geq 1 : X_n = j\}$$

is a stopping time because

$$\{T_j = n\} = \{X_1 \neq j, \dots, X_{n-1} \neq j, X_n = j\}.$$

(b) The first hitting time H^A of Section 1.3 is a stopping time because

$$\{H^A = n\} = \{X_0 \notin A, \dots, X_{n-1} \notin A, X_n \in A\}.$$