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Continuation of last class:

$$\lambda = \underset{\substack{\uparrow \\ \alpha}}{1}, \underset{\substack{\uparrow \\ \beta}}{\frac{q}{p}}$$

Case 1: $\alpha \neq \beta \quad \left(\Rightarrow 1 \neq \frac{q}{p} \Rightarrow p \neq q \right)$.

$$h_i = A + B\left(\frac{q}{p}\right)^i \quad \cdots ①$$

$$h_0 = A + B \underset{\substack{\uparrow \\ (\text{Since } h_0 = 1)}}{=} 1 \quad \cdots ②$$

⊗ $(0 \leq h_i \leq 1)$

Subcase 1: $p < q$ (This is the case for most of the casinos)

Therefore, the restriction $0 \leq h_i \leq 1$ for every i

$$\Rightarrow \boxed{B = 0}$$

$\Rightarrow \underline{h_i = A}$ (by ①)

From ②, $A = 1$ $(\because B = 0)$

$\Rightarrow h_i = 1$ for every i

$$\boxed{h_i = P_i(\text{hit } 0) = 1}$$

Subcase 2:

$p > q$

(Unlikely case for any casino)

$$h_i = A + (1-A)\left(\frac{q}{p}\right)^i \quad \left(\text{by ① and ②}\right)$$

$$h_i = \left(\frac{q}{p}\right)^i + A\left(1-\left(\frac{q}{p}\right)^i\right)$$

$h_i \geq 0 \Rightarrow A \geq 0$. ✓

So, the minimal solution is $(A = 0)$

$$h_i = \left(\frac{q}{p}\right)^i$$

Case 2:

$\alpha = \beta \quad (\Rightarrow p = q)$

(Extremely fair casino)

Simple random walk

$\Rightarrow$ for sure it returns to 0.

$\Rightarrow h_i = 0$ for every i

[Another way: Recurrance relation solution:

$h_i = A + Bi \quad \ldots ③, \qquad h_0 = A + B = 1 \quad \ldots ④$

$\alpha = \beta$

$\Rightarrow 1 = \frac{q}{p}$

$0 \leq h_i \leq 1 \Rightarrow \boxed{B = 0}$

$h_i = A, \quad A = 1$

$\Rightarrow h_i = 1$

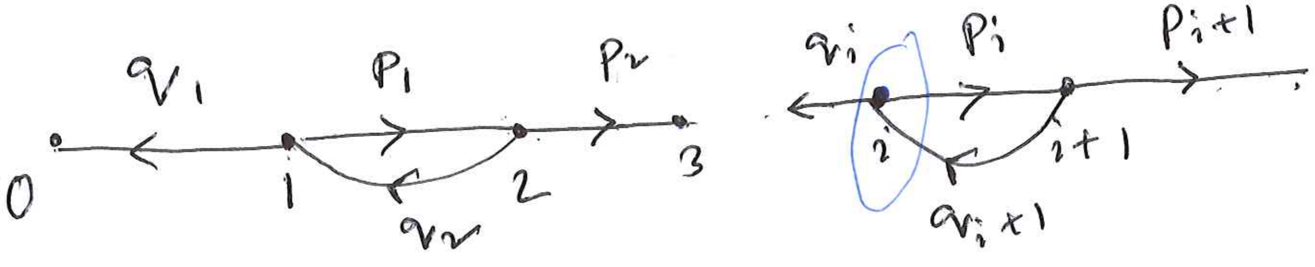
$\Rightarrow \boxed{h_i = P_i(\text{hit } 0) = 1}$]

Choose both roots to be = 1

$\alpha = 1$

Example: (Birth-and-death chain)

Consider the Markov chain: ~~with~~



where $i = 1, 2, \dots,$

$$0 < p_i = 1 - q_i < 1$$

0 is the absorbing state.

Question: Calculate the absorption probability starting from i

(p_i, q_i depends on i)

We want to compute $h_i = P_i(\text{hit } 0)$

$$\| \\ h_i^{\{0\}}$$

$\Rightarrow$ extinction probability starting from i.

$$h_0 = 1 \cdots Ⓐ$$

$$h_i = p_i h_{i+1} + q_i h_{i-1} \cdots Ⓑ \quad i = 1, 2, \ldots$$

Last results of recurrance relation can NOT be used (as p_i, q_i depends on i)

Goal: Find h_i

Define:

$$\boxed{u_i} = h_{i-1} - h_i \quad \text{by Ⓑ}$$

$$= h_{i-1} - (p_i h_{i+1} + q_i h_{i-1})$$

$$= (1-q_i) h_{i-1} - p_i h_{i+1}$$

$$(\because p_i + q_i = 1)$$

$$= p_i h_{i-1} - p_i h_{i+1}$$

$$✓ = \boxed{p_i (h_{i-1} - h_{i+1})} \cdots Ⓒ$$

Similarly

$$\boxed{u_{i+1}} = h_i - h_{i+1} \quad \text{by Ⓑ}$$

$$= p_i h_{i+1} + q_i h_{i-1} - h_{i+1}$$

$$= q_i h_{i-1} - (1-p_i) h_{i+1}$$

$$✓ = \boxed{q_i (h_{i-1} - h_{i+1})} \cdots Ⓓ$$

By Ⓒ and Ⓓ

$$p_i u_{i+1} = q_i u_i$$

$$\Rightarrow \quad u_{i+1} = \left(\frac{q_i}{p_i}\right) u_i \quad \cdots\cdots \text{Ⓔ}$$

by Ⓔ

$$= \left(\frac{q_i}{p_i}\right)\left(\frac{q_{i-1}}{p_{i-1}}\right) u_{i-1}$$

$$= \cdots$$

$$= \left(\frac{q_i \, q_{i-1} \cdots q_1}{p_i \, p_{i-1} \cdots p_1}\right) u_1$$

$$= \gamma_i u_1$$

$$\left(\text{where, } \gamma_i = \frac{q_i \, q_{i-1} \cdots q_1}{p_i \, p_{i-1} \cdots p_1}\right)$$

So, $$\boxed{u_{i+1} = \gamma_i u_1} \quad \cdots\cdots \text{Ⓕ}$$

Then ✓ $$u_1 + u_2 + \cdots + u_i$$

$$= (h_0 - \cancel{h_1}) + (\cancel{h_1} - \cancel{h_2}) + \cdots + (\cancel{h_{i-1}} - h_i)$$

(Use definition $u_i = h_{i-1} - h_i$)

$$= h_0 - h_i \quad \cdots\cdots \text{Ⓖ}$$

By Ⓕ

by Ⓕ by Ⓕ

✓ $$u_1 + u_2 + \cdots + u_i = \gamma_0 u_1 + \gamma_1 u_1 + \cdots + \gamma_{i-1} u_1$$

(with $\gamma_0 = 1$)

$$= \left(\sum_{n=0}^{i-1} \gamma_n\right) u_1 \quad \cdots Ⓗ$$

So, by Ⓖ:

$$h_i = h_0 - (u_1 + u_2 + \cdots + u_i)$$

by Ⓐ

$$= 1 - (u_1 + u_2 + \cdots + u_i)$$

by Ⓗ

$$= 1 - \left(\sum_{n=0}^{i-1} \gamma_n\right) u_1$$

So, $$\boxed{h_i = 1 - A(\gamma_0 + \cdots + \gamma_{i-1})}$$

(where $A = u_1$)

A : not determined yet.

Case 1: $\sum_{n=0}^{\infty} \gamma_n = \infty$, then since $0 \le h_i \le 1$

$\Rightarrow A = 0.$

$\Rightarrow h_i = 1 \Rightarrow$ Population will go extinct with prob=1 //

$\boxed{\text{Case 2}} \rightarrow \sum_{n=0}^{\infty} \gamma_n < \infty$

Then $0 \le h_i \le 1$

$$\Rightarrow 0 \le \underbrace{1 - A(\gamma_0 + \cdots + \gamma_{i-1})}_{} \le 1$$

$$A(\gamma_0 + \cdots + \gamma_{i-1}) \geq 0$$

$$\Rightarrow \boxed{A \geq 0} \checkmark$$

So, we want $A \geq 0$.

$$0 \le \overset{h_i}{1 - A(\gamma_0 + \cdots + \gamma_{i-1})}$$

$\Rightarrow$ when $i \to \infty$

$$0 \le 1 - A \sum_{n=0}^{\infty} \gamma_n$$

✓

$$\Rightarrow A \le \frac{1}{\sum_{n=0}^{\infty} \gamma_n}$$

So, minimality of h_i $\Rightarrow$ $\boxed{A = \frac{1}{\sum_{n=0}^{\infty} \gamma_n}}$

Hence, $h_i = 1 - \frac{1}{\sum_{n=0}^{\infty} \gamma_n} \sum_{n=0}^{i-1} \gamma_n$

⑧

$$h_i = \frac{\sum_{n=0}^{\infty} \gamma_n - \sum_{n=0}^{i-1} \gamma_n}{\sum_{n=0}^{\infty} \gamma_n}$$

$$\Rightarrow \boxed{h_i = \frac{\sum_{n=i}^{\infty} \gamma_n}{\sum_{n=0}^{\infty} \gamma_n}}$$

Extinction probability