Department of Computer Science

Ramakrishna Mission Vivekananda Educational Research Institute, Belur Math

Optimization for ML –BDA 2022

Problem Set on Convexity of Functions

1. *Inverse of an increasing convex function*:
Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse function, i.e., the function with domain $(f(a), f(b))$, and $g(f(x)) = x$ for $a < x < b$. Is g convex/concave?

2. *Monotone mappings*:
A function ψ: $\mathbb{R}^n \rightarrow \mathbb{R}$ is called monotone if for all $\mathbf{x}, \mathbf{y} \in$ dom ψ, $(\psi(\mathbf{x}) - \psi(\mathbf{y}))^T(\mathbf{x} - \mathbf{y}) \geq 0$.
Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable, show that ∇f is monotone.

3. For each of the following functions determine whether it is convex, concave, or neither.
(a) $f(x) = e^x - 1$ on $\mathbb{R}$.
(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
(d) $f(x_1, x_2) = x_1 / x_2$ on $\mathbb{R}^2_{++}$.
(e) $f(x_1, x_2) = x^2_1 / x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$
(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where, $0 \leq \alpha \leq 1$, on $\mathbb{R}^2_{++}$.

4. *Products and ratios of convex functions:* Show that
(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.
(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.
(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

5. *Strong, strict convexity of functions:* For each function below, determine whether it is convex, strictly convex, strongly convex or none of the above.
(a) $f(x) = (x_1 - 3x_2)^2$
(b) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2$
(c) $f(x) = (x_1 - 3x_2)^2 + (x_1 - 2x_2)^2 + x_1^3$
(d) $f(x) = |x|$, $x \in \mathbb{R}$.
(e) $f(x) = ||x||$, $x \in \mathbb{R}^n$

6. Lipschitz continuity and smoothness of functions: Show that
(a) $f(x) = |x|$ and $f(x) = \log(1 + e^x)$ are both 1-Lipschitz over $\mathbb{R}$,
(b) $f(x) = x^2$ is not ρ-Lipschitz over $\mathbb{R}$ for any $\rho > 0$, but is ρ-Lipschitz over set $C = \{x| \, |x| \leq \rho/2\}$
(c) $f(x) = x^2$ is 2- smooth and $f(x) = \log(1 + e^x)$ is (¼) smooth [Hint: show $f'(x)$ is ¼-Lipschitz]

7. Composition of Lipschitz and Smooth functions
(a) Let $f(x) = g_1(g_2(x))$, where g_1 is ρ_1-Lipschitz and g_2 is ρ_2-Lipschitz. Then f is $\rho_1\rho_2$-Lipschitz. In particular, if g_2 is the linear function, $g_2(x) = <\mathbf{v};\mathbf{x}> + b$, for some $\mathbf{v} \in \mathbb{R}^n$ and $b \in \mathbb{R}$., then f is $(\rho_1 ||\mathbf{v}||)$-Lipschitz.
(b) Let $f(\mathbf{w}) = g(<\mathbf{w};\mathbf{x}> + b)$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is β-smooth and $\mathbf{x} \in \mathbb{R}^n$ and $b \in \mathbb{R}$, then f is $(\beta ||\mathbf{x}||^2)$-smooth.

Exercise problems of Chapter 3 of Nonlinear Programming by Bazaraa et al.

[3.1] Which of the following functions is convex, concave, or neither? Why?

a. $f(x_1, x_2) = 2x_1^2 - 4x_1x_2 - 8x_1 + 3x_2$

b. $f(x_1, x_2) = x_1 e^{-(x_1+3x_2)}$

c. $f(x_1, x_2) = -x_1^2 - 3x_2^2 + 4x_1x_2 + 10x_1 - 10x_2$

d. $f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1^2 + x_2^2 + 2x_3^2 - 5x_1x_3$

[3.2] Over what subset of $\{x : x > 0\}$ is the univariate function $f(x) = e^{-ax^b}$ convex, where $a > 0$ and $b \geq 1$?

[3.3] Prove or disprove concavity of the following function defined over $S = \{(x_1, x_2) : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$:

$$f(x_1, x_2) = 10 - 3(x_2 - x_1^2)^2.$$

[3.4] Over what domain is the function $f(x) = x^2(x^2 - 1)$ convex? Is it strictly convex over the region(s) specified? Justify your answer.

[3.5] Show that a function f: $R^n \to R$ is affine if and only if f is both convex and concave. [A function f is *affine* if it is of the form $f(\mathbf{x}) = \alpha + \mathbf{c}^t\mathbf{x}$, where a is a scalar and $\mathbf{c}$ is an n-vector.]

[3.37] Let $f(x_1, x_2) = e^{2x_1^2 - x_2^2} - 3x_1 + 5x_2$. Give the linear and quadratic approximations of f at (1, 1). Are these approximations convex, concave, or neither? Why?

Problems on convexity preserving operations

[3.8] Let $f_1, f_2, \ldots, f_k$: $R^n \to R$ be convex functions. Consider the function f defined by $f(\mathbf{x}) = \sum_{j=1}^{k} \alpha_j f_j(\mathbf{x})$, where $\alpha_j > 0$ for $j = 1, 2, \ldots, k$. Show that f is convex. State and prove a similar result for concave functions.

[3.10] Let h: $R^n \to R$ be a convex function, and let g: $R \to R$ be a nondecreasing convex function. Consider the composite function f: $R^n \to R$ defined by $f(\mathbf{x}) = g[h(\mathbf{x})]$. Show that f is convex.

[3.16] Let g: $R^m \to R$ be a convex function, and let $\mathbf{h} : R^n \to R^m$ be an affine function of the form $\mathbf{h}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{b}$ is an $m \times 1$ vector. Then show that the composite function f: $R^n \to R$ defined as $f(\mathbf{x}) = g[\mathbf{h}(\mathbf{x})]$ is a convex function. Also, assuming twice differentiability of g, derive an expression for the Hessian of f.

[3.24] Let f be a convex function on R^n. Prove that the set of subgradients of f at a given point forms a closed convex set.

Problems on composition of convex functions

1. Prove the following
   a) $f(\mathbf{x}, u, v) = -\,\mathrm{sqrt}(uv - \mathbf{x}^T\mathbf{x})$ on **dom** f = $\{(\mathbf{x}, u, v) \mid uv > \mathbf{x}^T\mathbf{x},\ u, v > 0\}$ is convex
   b) $f(\mathbf{x}, u, v) = -\log(uv - \mathbf{x}^T\mathbf{x})$ on **dom** f = $\{(\mathbf{x}, u, v) \mid uv > \mathbf{x}^T\mathbf{x},\ u, v > 0\}$ is convex.
   c) $f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$, where $\mathbf{A} \in \mathbb{R}^{\mathbf{m}\times\mathbf{n}}$, $b \in \mathbb{R}^{\mathbf{m}}$, and $\|.\|$ is a norm on $\mathbb{R}^{\mathbf{m}}$.

Problems on Log-concavity

1. Show that the following functions are log-concave.

   (a) Logistic function: $f(x) = e^x/(1+e^x)$ with **dom** f = $\mathbb{R}$.

   (b) Harmonic mean: $f(x) = \dfrac{1}{1/x_1 + \cdots + 1/x_n}$ on **dom** f = $\mathbb{R}^n_{++}$

   (c) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is log-concave and $a \geq 0$, then the function $g = f - a$ is log-concave, where **dom** $g = \{\mathbf{x} \in \mathbf{dom}\, f \mid f(\mathbf{x}) > a\}$

2. Log-convexity of moment functions: Suppose f : $\mathbb{R} \to \mathbb{R}$ is nonnegative with $\mathbb{R}_+ \subseteq \mathbf{dom}\, f$. For $x \geq 0$ define

$$\phi(x) = \int_0^\infty u^x f(u)\, du.$$

   Show ϕ is a log-convex function.

3. Show that the cumulative distribution function of a Gaussian random variable is log-concave. Also show the cumulative distribution function of any log-concave probability density is log-concave