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Law of total expectation

$$E(Y) = E(E(Y|X))$$

Example: • Suppose that in a system a component and a back-up unit both have mean lifetimes (expected value) equal to μ.

• If the component fails, the system automatically substitutes the back-up unit but there is probabability p that something will go wrong and it will fail to do so and

Let T be the total lifetime and let $\boxed{X = 1}$ if the substitution of the back-up takes place successfully, and $\boxed{X = 0}$, if it does NOT.

Question: What is the expected lifetime of the system?

Solution:

	Prob
$E(T \mid \underbrace{X=0}_{\text{Back-up NOT working}}) = \mu$	p
$E(T \mid \underbrace{X=1}_{\text{Back-up IS working}}) = 2\mu$	$1-p$

$$E(T) = E\big(E(T \mid X)\big)$$

law of total expectation

discrete random variable

$$= \mu \cdot p + 2\mu(1-p)$$

$$= \boxed{\mu(2-p)}$$

Example: Random Sums

$$T = \sum_{i=1}^{N} X_i$$

(N: can take only ~~int~~ positive integer values)

where N is a random variable with finite expectation, and

X_i are . . are independent of N and have the common mean $E(X)$

(i.e., $E(X_1) = E(X_2) = \cdots = E(X)$)

N	T	Prob.
1	X_1	$P(N=1)$
2	$X_1 + X_2$	$P(N=2)$
3	$X_1 + X_2 + X_3$	$P(N=3)$
	$\vdots$	$\vdots$

[Example: An insurance company may receive N claims in a given period of time, and the amount of individual claim may be modeled by random variables $X_1, X_2, \ldots$

Total claim would be

$$T = \sum_{i=1}^{N} X_i$$

(If $N=0$, $T=0$)

Question: What is the $E(T)$?

[Wrong: $T = X_1 + X_2 + \cdots + X_{N}$ ← random.

$$E(T) = E(X_1) + E(X_2) + \cdots + E(X_N)$$
$$= N \, E(X)$$]

random.

A number (NOT a random variable)

$E(T|N)$ → random variable

Correct answer:

$$\boxed{E(T) = E\left(E(T|N)\right)}$$

$$E(T|N=n) = E(X_1 + X_2 + \cdots + X_n) = n\,E(X),$$

Hence, $E(T|N) = N\,E(X)$ → Random variable

So, $E(T) = E(N\,E(X))$

$$\boxed{E(T) = E(X) \cdot E(N)}$$

Theorem:

$$\boxed{Var(Y) = Var\big(E(Y|X)\big) + E\big(Var(Y|X)\big)}$$

Example: Suppose $T = \sum_{i=1}^{N} X_i$ random

N : random variable, with finite expectation (and it takes positive integer values)

X_i are INDEPENDENT random variables, with the same mean $E(X)$ and same variance $Var(X)$ and $\boxed{Var(N) < \infty}$

$Var(T) = \ ?$

⑥

$$Var(T) = E\left(Var(T|N)\right) + Var\left(E(T|N)\right)$$

n ← fixed

$$(T|N=n) = \sum_{i=1}^{n} X_i$$

$$= X_1 + X_2 + \dots + X_n$$

N ← given

$$T|N = \sum_{i=1}^{N} X_i$$

N is provided.

$$Y = aX$$

$$Var\,Y = a^2\,Var(X)$$

$$E(T|N) = N\,E(X)$$ → number

$$\boxed{Var\left(E(T|N)\right) = E(X)^2\,Var(N)} \quad \dots ①$$

$$Var(T|N) = Var(X_1 + X_2 + \dots + X_N)$$

N fixed

As $X_1, X_2, \dots$ are independent

$$= \underbrace{Var(X_1)}_{Var(X)} + \underbrace{Var(X_2)}_{Var(X)} + \dots + \underbrace{Var(X_N)}_{Var(X)}$$

$$= N \cdot Var(X)$$

$$\boxed{E\left(Var(T|N)\right) = Var(X) \cdot E(N)} \quad \dots ②$$

By (1) and (2)

$$\boxed{Var(T) = Var(X)E(N) + E(X)^2 \, Var(N)}$$

Variance of random sum of independent r.v.'s

[Aside: If N is NOT random.

$N = n \leftarrow$ a number

$T = X_1 + X_2 + \cdots + X_n$

$Var(T) = n\,Var(X)$

~~Aggre~~, Agrees with the above formula, as in this case

$Var(N) = 0$ $\quad E(N) = n$]

The Moment generating function (mgf)

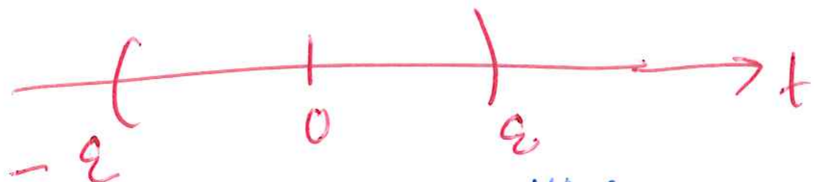
X is random variable

mgf of X is $\quad M(t) = E(e^{tX})$

[In the discrete case: $M(t) = \sum_x e^{tx} P(X = x)$

In the continuous case: $M(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$ $\leftarrow$ pdf of X]

$\boxed{\text{Property: If the moment-generating function exists for } t \text{ in an open-interval containing } 0\text{, it UNIQUELY determines the probability distribution.}}$



Definition: $\boxed{E(X^r)}$: (r : positive integer)

$= r^{th}$ moment of X.

If $r=1$: $E(X) =$ expected value

1^{st} moment of X

Definition: $\boxed{E\left(\left(X - E(X)\right)^r\right)}$

$= r^{th}$ central moment of X

If $r=2$: $E\left(\left(X - E(X)\right)^2\right) = Var(X)$

2^{nd} central moment of X

<u>Theorem:</u> If $M(t)$ is defined near 0, then

mgf

$$M^{(r)}(0) = E(X^r) = r^{th}\text{-moment},$$

$$e^{tX} = 1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \cdots$$

$$M(t) = E(e^{tX}) = 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

$$\boxed{E(X) = \sum_{x=0}^{\infty} x \cdot P(X = x)}$$

<u>Example:</u> Poisson distribution (Discrete)

$$X \sim \text{Poisson}(\lambda)$$

mgf $\rightarrow$ $M(t) = \exp(\lambda(e^t - 1))$

<u>Example:</u> (Continuous)

$$Z \sim N(0,1)$$

$$M(t) = e^{t^2/2}$$

Example:

$$X \sim N(\mu, \sigma^2)$$

$$M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

Limit Theorems

Suppose n: fixed.

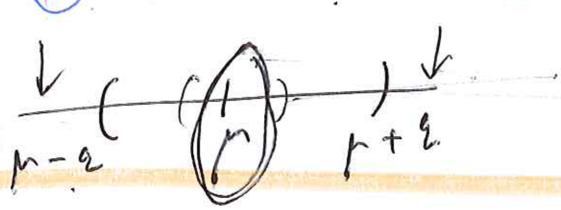
$$\boxed{\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

X_i's are a sequence of $\boxed{\text{INDEPENDENT}}$ random variables with $\left.\begin{array}{l} E(X_i) = \mu \\ Var(X_i) = \sigma^2 \end{array}\right\}$

Theorem: (Weak) law of large number:

For any $\varepsilon > 0$

$$P(|\overline{X}_n - \mu| > \varepsilon) \to 0 \text{ as } n \to \infty.$$



Proof:

$$E(\overline{X}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \underbrace{E(X_i)}_{\mu}$$

$$= \mu$$

$\overline{X}_1 = X_1$

$\overline{X}_2 = \frac{X_1 + X_2}{2}$

$\overline{X}_3 = \frac{X_1 + X_2 + X_3}{3}$

$\vdots$

$$Var(\overline{X}_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \underbrace{Var(X_i)}_{\sigma^2} \quad (\because X_1, X_2, \ldots \text{ are independent})$$

$$= \frac{\sigma^2}{n}$$

[Chebyshev's inequality:

$$P(|\overline{X} - \underset{E(\overline{X})}{\mu}| > \varepsilon) \le \frac{\sigma^2}{\varepsilon^2}$$

$var(\overline{X})$

Use Chebyshev's inequality for $\overline{X}_n$

$$P\left(\left|\overline{X}_n - E(\overline{X}_n)\right| > \varepsilon\right) \le \frac{Var(\overline{X}_n)}{\varepsilon^2}$$

done above   done above

$$P\left(\left|\overline{X}_n - \mu\right| > \varepsilon\right) \le \frac{\frac{\sigma^2}{n}}{\varepsilon^2}$$

Hence

$$P\left(\left|\overline{X}_n - \mu\right| > \varepsilon\right) \le \frac{\sigma^2}{n\varepsilon^2} \to 0$$

as $n \to \infty$

(for every $\varepsilon > 0$) //

Definition: (Convergence in probability)

If a sequence of random variables $\{X_n\}$ is such that $P(|X_n - \alpha| > \varepsilon) \to 0$ as $n \to \infty$ for every $\varepsilon > 0$

then $X_n \to \alpha$ in probability.

Monte-Carlo integration

Goal: Want to compute

$$I(f) = \int_0^1 f(x)\, dx \quad \ldots\ldots (*)$$

Suppose we can NOT be evaluated directly

Technique: (1) Generate independent uniform random variables on [0,1]

$$X_1, X_2, \ldots, X_n.$$

(2) Compute $\frac{1}{n}\sum_{i=1}^{n} f(X_i)$

(f(x) is provided in (*))

(3) By the ~~law~~ (weak) law of learge number) when n is LARGE

this sum should be close to

$$E(f(X)) = \int_0^1 f(x)\, 1\, dx$$

pdf of unif[0,1]

So, $$E\left(f(X)\right) = \int_0^1 f(x)\,dx$$

$$\approx \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

weak law of large numbers

(for large n)