Q.2) $y_t = \beta_1 \sin(2\pi \omega t) + \beta_2 \cos(2\pi\omega t) + \varepsilon_t + 0.5\, \varepsilon_{t-1}$

$\varepsilon_t \sim WN(0, \sigma^2)$

ω, β_1, β_2 are constants.

Mean:

$E(y_t) = E[\beta_1 \sin(2\pi\omega t) + \beta_2 \cos(2\pi\omega t) + \varepsilon_t + 0.5\,\varepsilon_{t-1}]$

$= \beta_1 \sin(2\pi\omega t) + \beta_2 \cos(2\pi\omega t) + E(\varepsilon_t + 0.5\,\varepsilon_{t-1})$

[∵ Due to linearity of Expectation. and the first two terms are constant]

$= \beta_1 \sin(2\pi\omega t) + \beta_2 \cos(2\pi\omega t) + E(\varepsilon_t) + 0.5\, E(\varepsilon_{t-1})$

Again, $\varepsilon_t \sim WN(0, \sigma^2)$

$\therefore E(\varepsilon_t) = E(\varepsilon_{t-1}) = 0$.

So $E(y_t) = \beta_1 \sin(2\pi\omega t) + \beta_2 \cos(2\pi\omega t)$

Variance:-

$V(y_t) = V[\underbrace{\beta_1 \sin(2\pi\omega t) + \beta_2 \cos(2\pi\omega t)}_{\text{non stochastic}} + \varepsilon_t + 0.5\,\varepsilon_{t-1}]$

$= V(\varepsilon_t + 0.5\,\varepsilon_{t-1})$ [∵ $Var(a+x) = Var(x)$]

$= V(\varepsilon_t) + 0.5^2\, V(\varepsilon_{t-1}) + 2 \times 0.5 \times Cov(\varepsilon_t, \varepsilon_{t-1})$

$= \sigma^2 + 0.25\sigma^2 = 1.25\,\sigma^2$ [∵ $\varepsilon_t \sim WN(0,\sigma^2)$ ⇒ $Cov(\varepsilon_t, \varepsilon_{t-1}) = 0$]

$\gamma(1) = Cov(y_{t+1}, y_t)$

$= Cov(a + \varepsilon_{t+1} + 0.5\,\varepsilon_t,\; b + \varepsilon_t + 0.5\,\varepsilon_{t-1})$ [a, b constants]

$= Cov(a,b) + Cov(a, \varepsilon_t + 0.5\,\varepsilon_{t-1}) + Cov(\varepsilon_{t+1} + 0.5\,\varepsilon_t, b)$

$+ Cov(\varepsilon_{t+1} + 0.5\,\varepsilon_t,\; \varepsilon_t + 0.5\,\varepsilon_{t-1})$

[$\because$ By the distributive property of Covariance]

$= Cov(\varepsilon_{t+1} + 0.5\varepsilon_t, \varepsilon_t + 0.5\varepsilon_{t-1})$ [$\because$ $Cov(a, x) = 0$ as a is const]

$= Cov(\varepsilon_{t+1}, \varepsilon_t) + 0.5\, Cov(\varepsilon_t, \varepsilon_t) + 0.5\, Cov(\varepsilon_{t+1}, \varepsilon_{t-1})$
$+ 0.5^2\, Cov(\varepsilon_t, \varepsilon_{t-1})$

$= 0.5\,\sigma^2$ [$\because$ $Cov(\varepsilon_t, \varepsilon_{t+h}) = 0\ \forall h \in \mathbb{Z}$]

$$\rho(2) = \frac{\gamma(2)}{\gamma(0)}.$$

$\gamma(2) = Cov(Y_{t+2}, Y_t).$

$= Cov(c + \varepsilon_{t+2} + 0.5\,\varepsilon_{t+1}, b + \varepsilon_t + 0.5\,\varepsilon_{t-1})$

$= Cov(\varepsilon_{t+2} + 0.5\,\varepsilon_{t+1}, \varepsilon_t + 0.5\,\varepsilon_{t-1})$

$= Cov(\varepsilon_{t+2}, \varepsilon_t) + 0.5\, Cov(\varepsilon_{t+1}, \varepsilon_t) + 0.5\, Cov(\varepsilon_{t+2}, \varepsilon_{t-1})$
$+ 0.5^2\, Cov(\varepsilon_{t+1}, \varepsilon_{t-1}).$

$= 0.$

So $\rho(2) = \frac{0}{\gamma(0)} = 0$ [$\gamma(0) = Var(Y_t) \neq 0$]

3) x_t, y_t are two stationary process.

$Z_t = x_t + y_t.$

(i) $E(Z_t) = \cancel{E(x_t + x_t)}\ E(X_t + Y_t)$
$= E(X_t) + E(Y_t)$

[independent of t as X_t, Y_t stationary]

- $E(Z_t^2) = E[(X_t+Y_t)^2] = E[X_t^2] + E[Y_t^2] + 2E[X_t, Y_t]$

If $E[X_t, Y_t] < \infty$

then $E[Z_t^2] < \infty$

$X_t ; Y_t$ are stationary process.

- $Cov(Z_{t+h}, Z_t) = Cov(x_{t+h} + y_{t+h}, x_t + y_t)$

$= Cov(x_{t+h}, x_t) + Cov(y_{t+h}, x_t) + Cov(x_{t+h}, y_t)$

$+ Cov(y_{t+h}, y_t)$.

It is independent of t if $Cov(x_{t+h}, y_t)$ and $Cov(y_{t+h}, y_t)$ are independent of t.

If x_t and y_{t+h} are independent then

$Cov(x_t, y_{t+h}) = 0$

$\Rightarrow Cov(Z_{t+h}, Z_t)$ is independent of t.

So Z_t will be stationary based on finite $E[X_t, Y_t]$ and independence of $Cov(x_t, Y_{t+h})$ $\forall h \in \mathbb{Z}$ with t.

If x_t, y_t independent, then Z_t is stationary.