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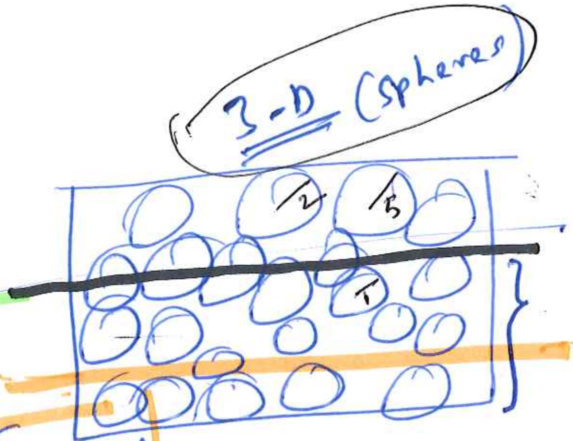
Metallography

An application of quantitative microscopy.

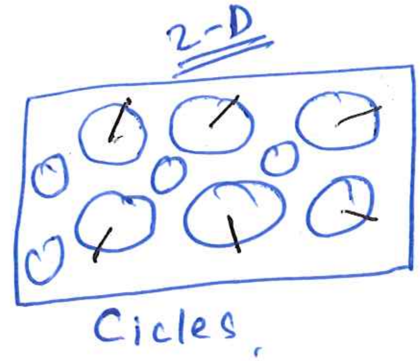
→ Spherical particles are dispersed in a medium.

→ Density function of the radii of the spheres can be denoted as $f_R(r)$

(R: random variable that gives the value of radius.)

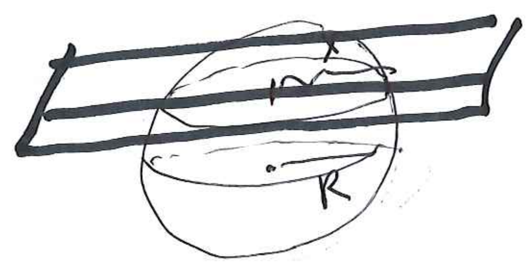


→ When the medium is sliced, two-dimensional circular cross sections of the spheres are observed.

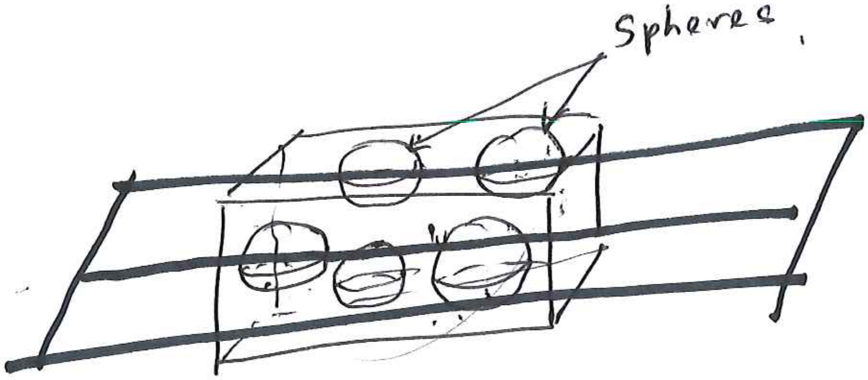


→ let the density function of the radii of these circles be denoted by $\boxed{f_X(x)}$.

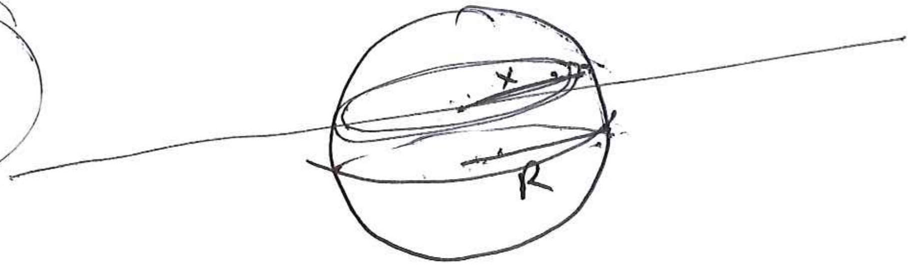
X: random variable that gives the value of radius of the cross sectional circles.

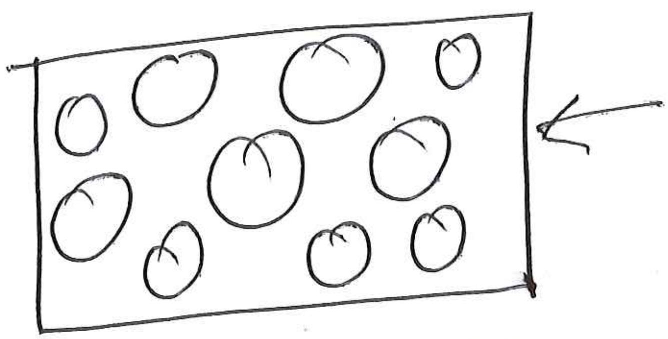


$f_{\underline{R}}(r)$



$f_{\underline{X}}(x)$

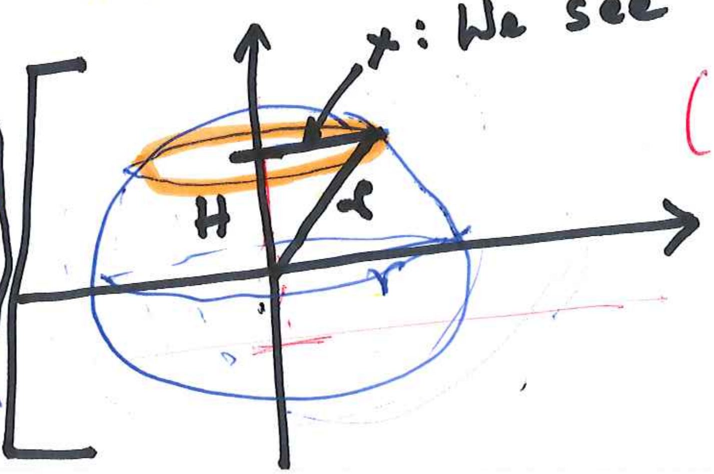




To derive the relationship, we assume that the cross-sectioning plane is chosen at random, fix $R = r$

Let's think of spheres of <u>radius $= r$</u>

(original picture we can NOT see)



(H = abs. value of height).

Want to find $f_{X|R}(x|r)$.

$$\boxed{F_{X|R}(x|r)} = P(X \le x) \quad \text{(for the picture)}$$

$$= P\left(\sqrt{r^2 - H^2} \le x\right) \quad x \geq 0$$

$$= P\left(r^2 - H^2 \le x^2\right)$$

$$= P\left(\sqrt{r^2 - x^2} \le H\right)$$

$$= 1 - P\left(H < \sqrt{r^2 - x^2}\right)$$

$|x| \le |y|$

$\Rightarrow |x|^2 \le |y|^2$

$x^2 < 3$

$\Rightarrow |x| < \sqrt{3}$

Assumption: H: follows uniform dbn on $[0, r]$

$$= \boxed{1 - \frac{\sqrt{r^2 - x^2}}{r}}$$

$P(\sqrt{3} \le \cancel{X})$

$= P(3 \le X^2)$

~~$x \in [-\sqrt{3}, \sqrt{3}]$~~

~~$x \in [0, \sqrt{3}]$~~

$x^2 > 3$

$\Rightarrow x > \sqrt{3}$ or $\cancel{x < -\sqrt{3}}$

$$f_{X|R}(x|r) = \frac{\cancel{2}x}{\cancel{2}r\sqrt{r^2 - x^2}}, \quad 0 \le x \le r$$

Marginal density of X:

$$\boxed{f_X(x)} = \boxed{\int_{-\infty}^{\infty} \boxed{f_{X|R}(x|r)} \cdot \boxed{f_R(r)}\, dr}$$

$$\boxed{f_X(x) = \int_{x}^{\infty} \frac{x}{r\sqrt{r^2-x^2}} \cdot f_R(r)\, dr.}$$

~~Radon~~ Radon Transform

Functions of jointly distributed random variables

Sum: X, Y discrete random variables with joint frequency function $p(x,y)$

Let $Z = X + Y$

Goal: Frequency function (pmf) of Z.

Solution: $X = x, \quad Y = z - x$

$$\boxed{P(Z = z) = \underbrace{\sum_{x=-\infty}^{\infty} p(x, z-x)}_{}}$$

Z: random variable; z: number

Written in terms of sum of joint frequency functions.

Suppose $X \perp Y$

$$\boxed{P(Z = z) = P_Z(z) = \sum_{x=-\infty}^{\infty} P_X(x)\, P_Y(z-x)}$$

Notation

Continuous case: • X, Y are continuous r.v. with joint density $f(x, y)$

$$Z = X + Y.$$

Goal: Find the pdf of Z.

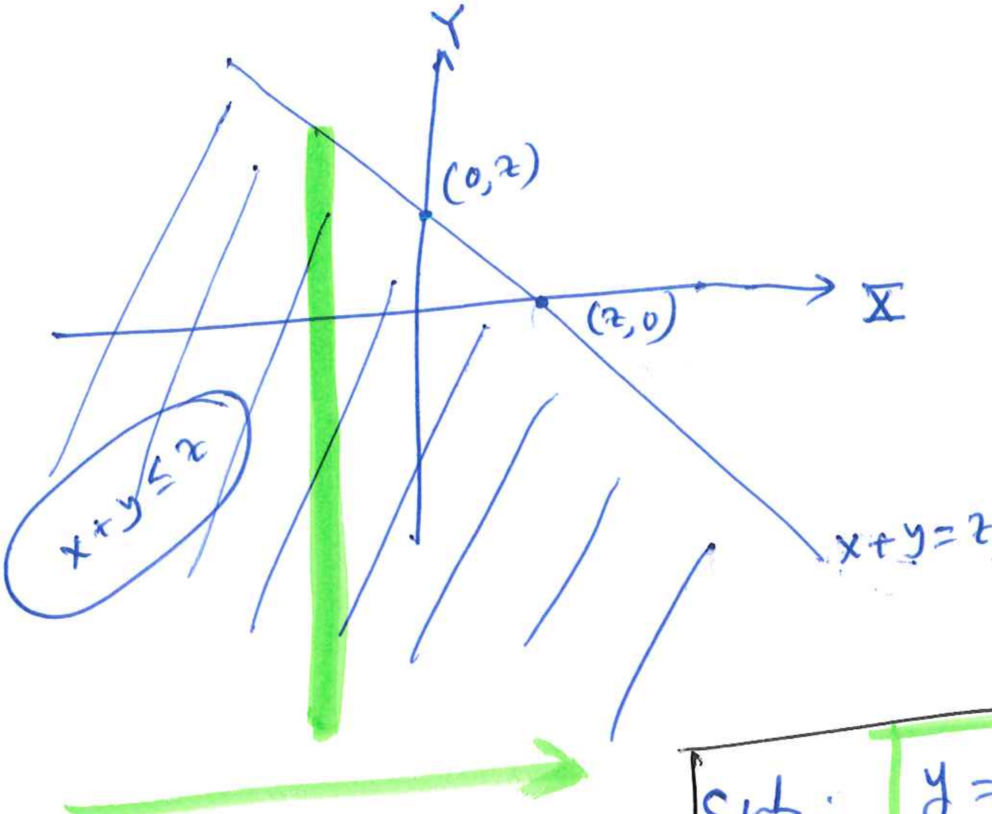
Solution:

$$F_Z(z) = P(Z \le z)$$

number

$$= P(X + Y \le z)$$

$$= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f(x, y)\, dy\, dx \qquad (*)$$



$$\boxed{\text{Sub: } \boxed{y = v - x} \Rightarrow \boxed{v = y + x}} \quad dv = dy$$

$$(*) = \int_{x=-\infty}^{\infty} \int_{v=-\infty}^{z} f(x, v-x)\, dv\, dx$$

Fubini's Theorem

$$\boxed{F_Z(z) = \int_{v=-\infty}^{z} \left(\int_{x=-\infty}^{\infty} f(x, v-x)\, dx \right) dv}$$

Differentiate w.r.t. z:

$$f_Z(z) = \int_{x=-\infty}^{\infty} f(x, z-x)\, dx$$

$$\boxed{f_Z(z) = \int_{x=-\infty}^{\infty} f(x, z-x)\, dx}$$

Aside:

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

$$F'(x) = f(x)$$

$$\boxed{X \perp Y}$$

$$f(x, z-x) = f_X(x)\, f_Y(z-x)$$

Then
$$\boxed{f_Z(z) = \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx}$$

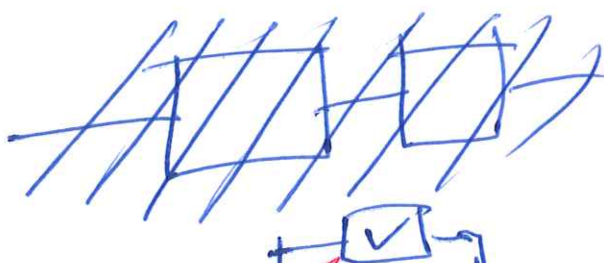
$\Downarrow$

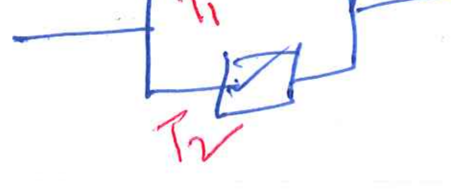
the CONVOLUTION of f_X and f_Y.

$\Downarrow$

$$\boxed{f_Z(z) = \left(f_X * f_Y\right)(z)}$$

Example:
- Suppose that the lifetime of a component is $\sim Exp(\lambda)$
- Suppose that there is an identical and independent back up component is available
- The system operates as long as one of the components is functional.





Question: What is the distribution of the lifetime of this system?

So we are really looking for the pdf of $S = T_1 + T_2$.

$$f_S(s) = f_{T_1} * f_{T_2}(s)$$

$$= \int_{x=0}^{x=s} \underbrace{\left(\lambda e^{-\lambda x}\right)}_{f_{T_1}(x)} \underbrace{\left(\lambda e^{-\lambda(s-x)}\right)}_{f_{T_2}(s-x)} dx$$

$$= \lambda^2 \int_0^s e^{-\lambda s}\, dx = \boxed{\lambda^2 s \cdot e^{-\lambda s}}$$

$\sim \text{Gamma}(2, \lambda)$

$$\boxed{\begin{aligned} f_{T_1}(x) &= \lambda e^{-\lambda x}, \quad x > 0 \\ &= 0, \quad \text{otherwise} \end{aligned}}$$

$$f_{T_2}(s-x) = \begin{cases} \lambda e^{-\lambda(s-x)}, & s-x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda e^{-\lambda(s-x)}, & s > x \\ 0, & \text{otherwise} \end{cases}$$

$X_1, X_2, \ldots, X_n$ independent and identically distributed. $\Rightarrow$ $\boxed{\text{i.i.d.}}$

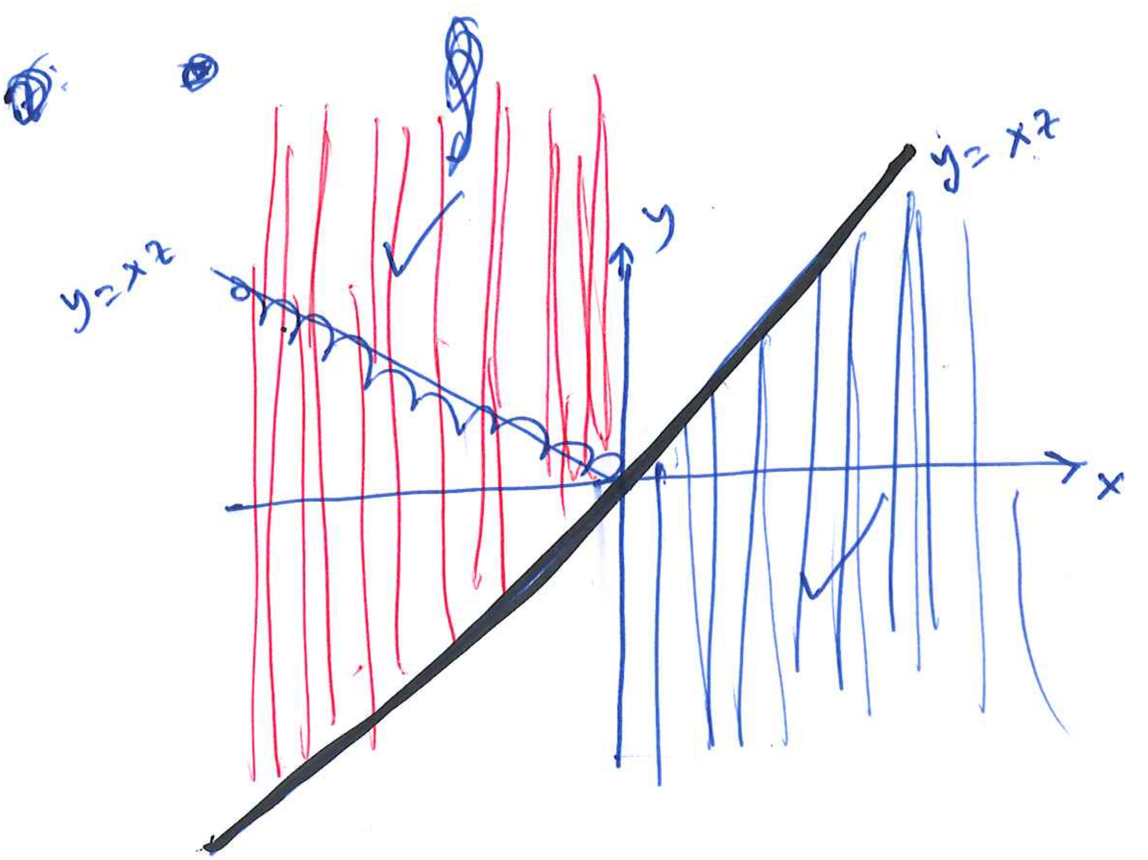
Quotient: (Two continuous random variables)

Suppose

$$Z = Y/X$$

Goal: Find the pdf of Z, when the joint density $f(x, y)$ of X and Y is given.

$$F_Z(z) = P(Z \le z)$$
$$= P\left(\frac{Y}{X} \le z\right)$$



$\frac{Y}{X} \le z$ $X > 0$ $Y \le zX$
$\frac{Y}{X} \le z$ $X < 0$ $Y \ge zX$

$$F_Z(z) = \int_{x=-\infty}^{0} \left(\int_{y=xz}^{\infty} f(x,y)\, dy \right) dx + \int_{x=0}^{\infty} \left(\int_{y=-\infty}^{xz} f(x,y)\, dy \right) dx.$$

Substitute. $\boxed{v = \frac{y}{x} \qquad dv = \frac{1}{x} dy}$ for the inner integrals

as $x<0$

$$= \boxed{\int_{x=-\infty}^{0} \left(\int_{v=z}^{-\infty} x\, f(x, xv).\, dv \right) dx} + \int_{x=0}^{\infty} \left(\int_{v=-\infty}^{z} x\, f(x, xv)\, dv \right) dx$$

$$= \int_{x=-\infty}^{0} \int_{v=-\infty}^{z} (-x)\, f(x, xv)\, dv\, dx + \int_{x=0}^{\infty} \int_{v=-\infty}^{z} x\, f(x, xv)\, dv\, dx$$

$$= \int_{x=-\infty}^{0} \left(\int_{v=-\infty}^{z} |x|\, f(x, xv)\, dv \right) dx + \int_{x=0}^{\infty} \left(\int_{v=-\infty}^{z} |x|\, f(x, xv)\, dv \right) dx$$

$$= \int_{x=-\infty}^{\infty} \left(\int_{v=-\infty}^{z} |x| \, f(x, xv) \, dv \right) dx$$

$$\boxed{F_Z(z) = \int_{v=-\infty}^{z} \left(\int_{x=-\infty}^{x} |x| \, f(x, x.v) \, dx \right) dv}$$

Differentiating w.r.t. z:

$$\boxed{f_Z(z) = \int_{x=-\infty}^{\infty} |x| . f(x, xz) \, dx}$$

Example: Suppose $X \perp Y$

$X, Y \sim N(0,1)$

a.s. (almost surely) statement

$$\boxed{Z = \frac{Y}{X}, \quad X \neq 0 \text{ with Probability} = 1}$$

Question: What is $f_Z(z)$?

$$f_Z(z) = \int_{x=-\infty}^{\infty} |x|\, f(x, xz)\, dx$$

$X \perp Y$

$$= \int_{x=-\infty}^{\infty} |x| \cdot f_X(x)\, f_Y(xz)\, dx$$

$$= \int_{x=-\infty}^{\infty} |x| \left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right)\left(\frac{e^{-\frac{x^2 z^2}{2}}}{\sqrt{2\pi}}\right) dx$$

$$= \frac{1}{2\pi} \int_{x=-\infty}^{\infty} |x| \cdot e^{-\frac{x^2}{2}(1+z^2)}\, dx$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

when $f(x)$ is even

$$f(-x) = f(x)$$

$$= \frac{1}{\cancel{2}\pi} \cancel{2} \int_{x=0}^{\infty} |x| \cdot e^{-\frac{x^2}{2}(1+z^2)}\, dx$$

$$= \frac{1}{\pi} \int_{x=0}^{\infty} |x| \cdot e^{-\frac{x^2}{2}(1+z^2)}\, dx$$

Use sub $u = x^2$

$$= \frac{1}{2\pi} \int_{0}^{\infty} e^{-\frac{u}{2}(z^2+1)}\, du.$$

Check it!

$$= \boxed{\frac{1}{\pi(z^2+1)}}, \quad -\infty < z < \infty$$

$$f_Z(z) = \frac{1}{\pi(z^2+1)} \Rightarrow$$ Standard Cauchy dbn