Example : 17.2. If T_1 and T_2 be two statistics with $E(T_1) = \theta_1 + \theta_2$ and $E(T_2) = \theta_1 - \theta_2$ find the unbiased estimators of θ_1 and θ_2

[W.B.

Solution : Here $E(T_1) = \theta_1 + \theta_2$ ..... (1) and $E(T_2) = \theta_1 - \theta_2$ .....(2).
Now, solving these two we can get the values of θ_1 and θ_2.
Adding (1) and (2) we get, $2\theta_1 = E(T_1) + E(T_2)$

$$\therefore \theta_1 = \frac{E(T_1) + E(T_2)}{2} = E\left(\frac{T_1 + T_2}{2}\right)$$

Similarly, subtracting (2) from (1) we get,
$\theta_1 + \theta_2 - \theta_1 + \theta_2 = E(T_1) - E(T_2)$
$\therefore 2\theta_2 = E(T_1) - E(T_2)$

$$\therefore \theta_2 = \frac{E(T_1) - E(T_2)}{2} = E\left(\frac{T_1 - T_2}{2}\right).$$

Thus, we get $\theta_1 = E\left(\frac{T_1 + T_2}{2}\right)$ and $\theta_2 = E\left(\frac{T_1 - T_2}{2}\right)$.

Hence, the unbiased estimators of θ_1 and θ_2 are respectively,

$\left(\frac{T_1 + T_2}{2}\right)$ and $\left(\frac{T_1 - T_2}{2}\right)$.

Example : 17.3. If T_1 and T_2 be statistics with expectations $E(T_1) = 2\theta_1 + 3\theta_2$ and $E(T_2) = \theta_1 + \theta_2$, find unbiased estimators of parameters θ_1 and θ_2. *[W.B.]*

Solution : Here $E(T_1) = 2\theta_1 + 3\theta_2$, ...(1) and $E(T_2) = \theta_1 + \theta_2$ .... (2). Solving (1) and (2) we get, $\theta_2 = E(T_1) - 2E(T_2) = E[T_1 - 2T_2]$ and $\theta_1 = 3E(T_2) - E(T_1) = E(3T_2 - T_1)$.

Thus, $(3T_2 - T_1)$ and $(T_1 - 2T_2)$ are the unbiased estimators of θ_1 and θ_2 respectively.

Example : 17.5. If T_1, T_2, T_3 are independent unbiased estimators of θ and all have the same variance σ^2, which of the following estimator of θ will you prefer ?

$$\frac{T_1+2T_2+T_3}{4}, \frac{2T_1+T_2+2T_3}{5}, \frac{T_1+T_2+T_3}{3}.$$

Solution : Among the three estimators that one will be preferred which will have the least variance.

Here Var (T_1) = Var (T_2) = Var (T_3) = σ^2 (given).

Now, $\text{Var}\left(\frac{T_1+2T_2+T_3}{4}\right) = \frac{1}{16}$ [var (T_1) + 4 var (T_2) + var (T_3)] as T_1, T_2, T_3 are independent and hence covariance terms vanish.

$$= \frac{1}{16}[\sigma^2+4\sigma^2+\sigma^2] = \frac{6}{16}\sigma^2 = \frac{3}{8}\sigma^2.$$

Similarly, $\text{Var}\left(\frac{2T_1+T_2+2T_3}{5}\right)$ = [4 var (T_1) + var (T_2) + 4 var (T_3)]/25

$$= (4\sigma^2+\sigma^2+4\sigma^2)/25 = \frac{9}{25}\sigma^2$$

and $\text{Var}\left(\frac{T_1+T_2+T_3}{3}\right) = \frac{1}{9}$ [var (T_1) + var (T_2) + var (T_3)]

$$= \frac{1}{9}[\sigma^2+\sigma^2+\sigma^2] = \frac{3\sigma^2}{9} = \sigma^2/3.$$

Among the three estimators $\frac{T_1+T_2+T_3}{3}$ has the least variance and hence it will preferred, which is really the MVU estimator of θ.

(3) Consistency & Efficiency:

These two criteria are concerned with the ~~large~~ large sample behaviour of a statistic. If we consider any estimator T of $\gamma(\theta)$, then from T, we can have a sequence $\{T_n\}$ by varying the no. of ~~obser~~ observations (n). Quite naturally, we expect the distribution of $\{T_n\}$ to be more & more clustered around $\gamma(\theta)$, i.e we would expect the observed values of T to generally differ less & less from $\gamma(\theta)$ with increasing 'n'. An estimator having this property is said to be ~~con~~ consistent.

defn

Normally, an estimator T (based on 'n' sample observations) is said to be a consistent estimator of a parametric function $\gamma(\theta)$ if, given two positive quantities ϵ & η, however small, it is possible to find an n_0 depending on ϵ & η such that

$$P\{|T-\gamma(\theta)| < \epsilon\} \geq 1-\eta, \text{ whenever } n \geq n_0.$$

It can be shown that a set of sufficient conditions for 'T' to be consistent are i) $E(T) \to \theta$ & ii) $V(T) \to 0$, as $n \to \infty$.

[Note: A consistent estimator need not be unbiased. It can be shown that $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$ is a consistent estimator of the population variance σ^2 although it is biased.]

In any particular situation, there may be a no. of consistent estimators. In fact, if we can have atleast one consistent estimator, then, from it, we can form an infinite no. of other consistent estimators. So, to make a choice among them, we first confine ourselves to the case, where, the large sample distribution, called assymptotic distribution, of the estimator is normal. Then, naturally, we consider that consistent estimator to be the best for which the assymptotic variance (i.e variance of the large sample distribution) is minimum. This estimator T is said to be efficient.

defn

Thus, by definition, a consistent estimator 'T' is said to be efficient for a parametric function $\gamma(\theta)$ if its assymptotic distribution is normal & if the assymptotic variance of 'T' is less than or equal to assymptotic variance of any other estimator T' which, too, is consistent & assymptotically normally distributed.

.3. Consistent Estimator

An estimator T consistently estimates the population parameter θ if the following conditions are satisfied :

(i) T_n is an estimator based on a sample size n.

(ii) $E(T_n) \to \theta$ and var $(T_n) \to 0$ as $n \to \infty$. *If an estimator is unbiased it is always consistent but a consistent estimator may not be unbiased.*

Example : 17.6. In a simple random sampling with replacemrnt show that sample variance is a consistent estimator of population variance.

Solution : Since $x_i - \bar{x} = [(x_i - \mu) - (\bar{x} - \mu)]$

$\therefore (x_i - \bar{x})^2 = (x_i - \mu)^2 - 2(x_i - \mu)(\bar{x} - \mu) + (\bar{x} - \mu)^2$

or. $$\sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu)\sum_{i=1}^{n}(x_i - \mu) + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \mu)^2 - 2(\bar{x} - \mu).n(\bar{x} - \mu) + n(\bar{x} - \mu)^2$$

$$\therefore \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(x_i - \mu)^2 - 2n(\bar{x} - \mu)^2 + n(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}(x_i - \mu)^2 - n(\bar{x} - \mu)^2$$

Now, $$E\left[\sum_{i=1}^{n}(x_i - \bar{x})^2\right] = E\left[\sum_{i=1}^{n}(x_i - \mu)^2 - n(\bar{x} - \mu)^2\right]$$

$$= \sum_{i=1}^{n}E(x_i - \mu)^2 - nE(\bar{x} - \mu)^2$$

$$= \sum_{i=1}^{n}E(x_i - E(x_i))^2 - nE[\bar{x} - E(\bar{x})]^2$$

[Since $E(x_i) = E(\bar{x}) = \mu$]

$$= \sum_{i=1}^{n}\mathrm{Var}(x_i) - n\,\mathrm{Var}(\bar{x}) = \sum_{i=1}^{n}\sigma^2 - n\sigma^2/n$$

[Since $\mathrm{Var}(x_i) = \sigma^2$ and $\mathrm{Var}(\bar{x}) = \sigma^2/n$]

$= n\sigma^2 - n\sigma^2/n = (n-1)\sigma^2.$

Now, $E(s^2) \to \sigma^2$ as $n \to \infty$.

Here the sample variance s^2 is a consistent estimator of the population variance σ^2.

But, $$E\left[\sum_{i=1}^{n}(x_i-\bar{x})^2\right]=(n-1)\,\sigma^2$$

$$\therefore\ E\left[\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{n-1}\right]=\sigma^2$$

$$\therefore\ E[s'^2]=\sigma^2 \text{ where } s'^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2.$$

$s'^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$ is an unbiased estimator of the population variance σ^2. Thus, an unbiased estimator is consistent but a consistent estimator may not be unbiased.

Example : 17.7 Let $t_1, t_2, \ldots\ldots, t_k$ be the mutually independent and unbiased estimators μ with variances $v_1, v_2, \ldots\ldots, v_k$ respectively. Consider a linear function

$T = a + \sum_{i=1}^{k} b_i t_i$ where $a, b_1, b_2, \ldots. b_k$ are constants. Find out the conditions on $a, b_1, b_2 \ldots\ldots b_k$ required to make T as an unbiased estimator.

Solution :

Let $T = a + \sum_{i=1}^{k} b_i t_i$

$$\therefore E(T) = E\left\{a + \sum_{i=1}^{k} b_i t_i\right\} = E(a) + \sum_{i=1}^{k} b_i E(t_i)$$

$$= a + \sum_{i=1}^{k} b_i \mu, \text{ Since } E(t_i) = \mu \text{ for } i = 1, 2, \ldots., k. \quad \therefore E(T) = a + \mu \sum_{i=1}^{k} b_i$$

Now, $E(T) = \mu$ *i.e.*, T is an unbiased estimator of μ if $a = 0$ and $\sum_{i=1}^{k} b_i = 1$. These are the conditions for unbiasedness.