Introduction to Deep Learning: Perceptron Learning

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Machine Translation

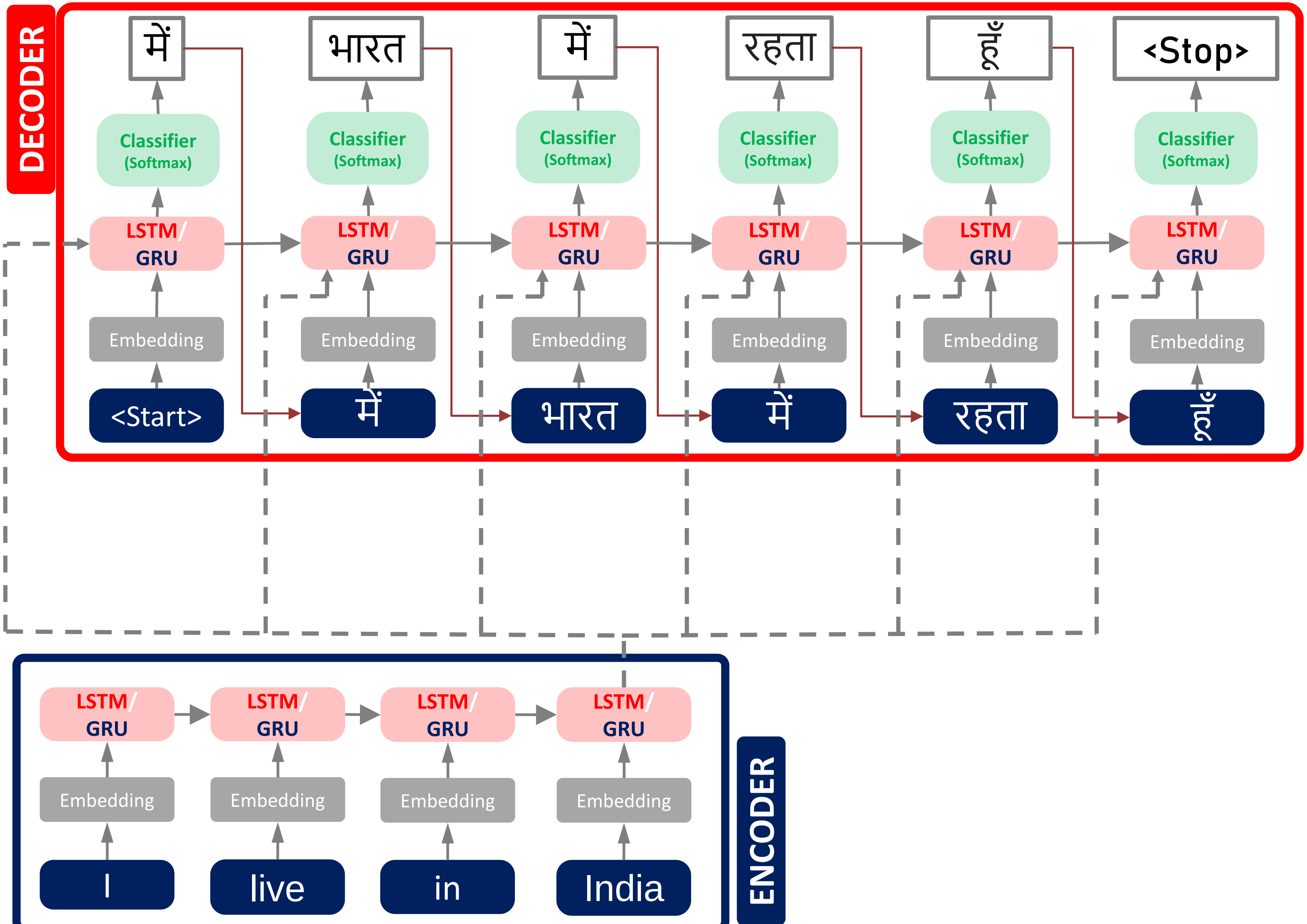
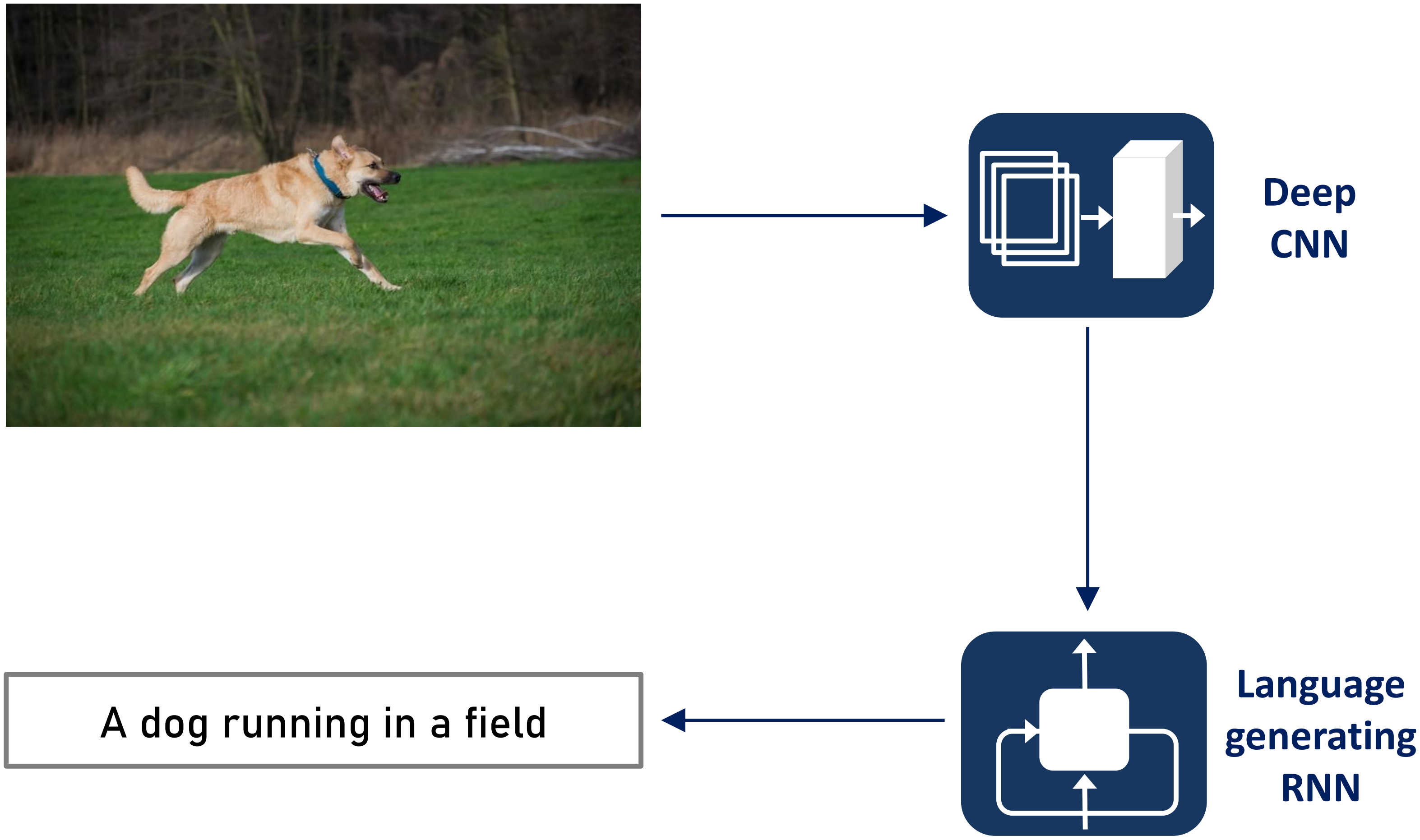
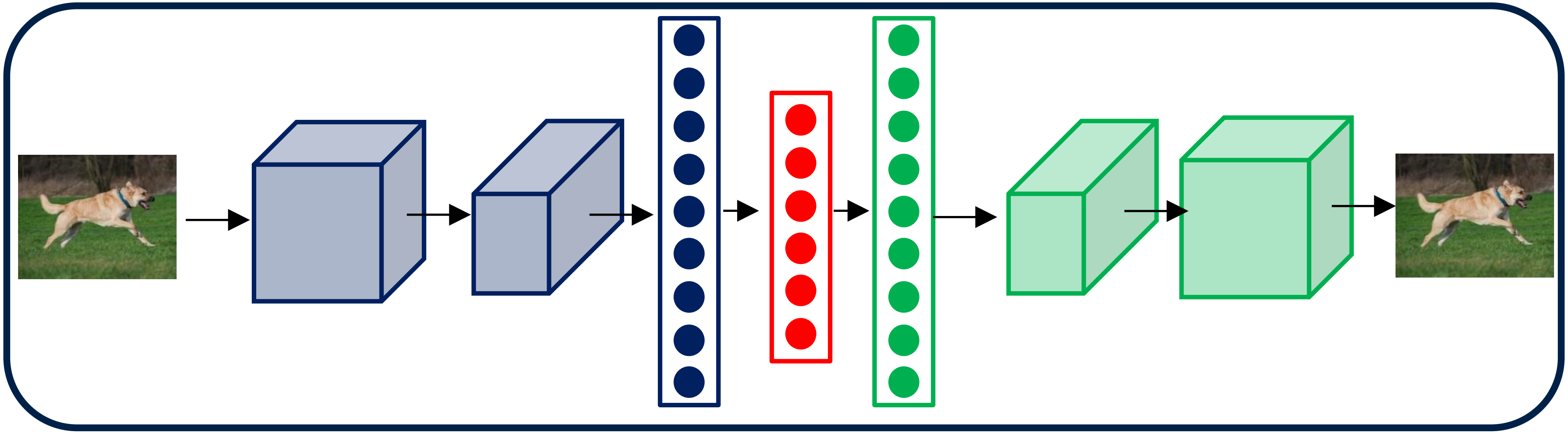


Image captioning



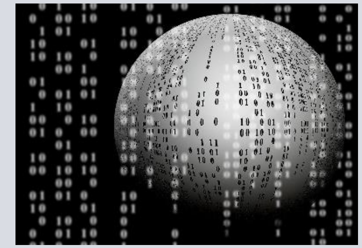
Convolutional Autoencoder



The rise of Deep Learning ....

BIG DATA

- World is data rich!
- Deep learning needs big datasets.



HARDWARE

- Graphics Processing Units

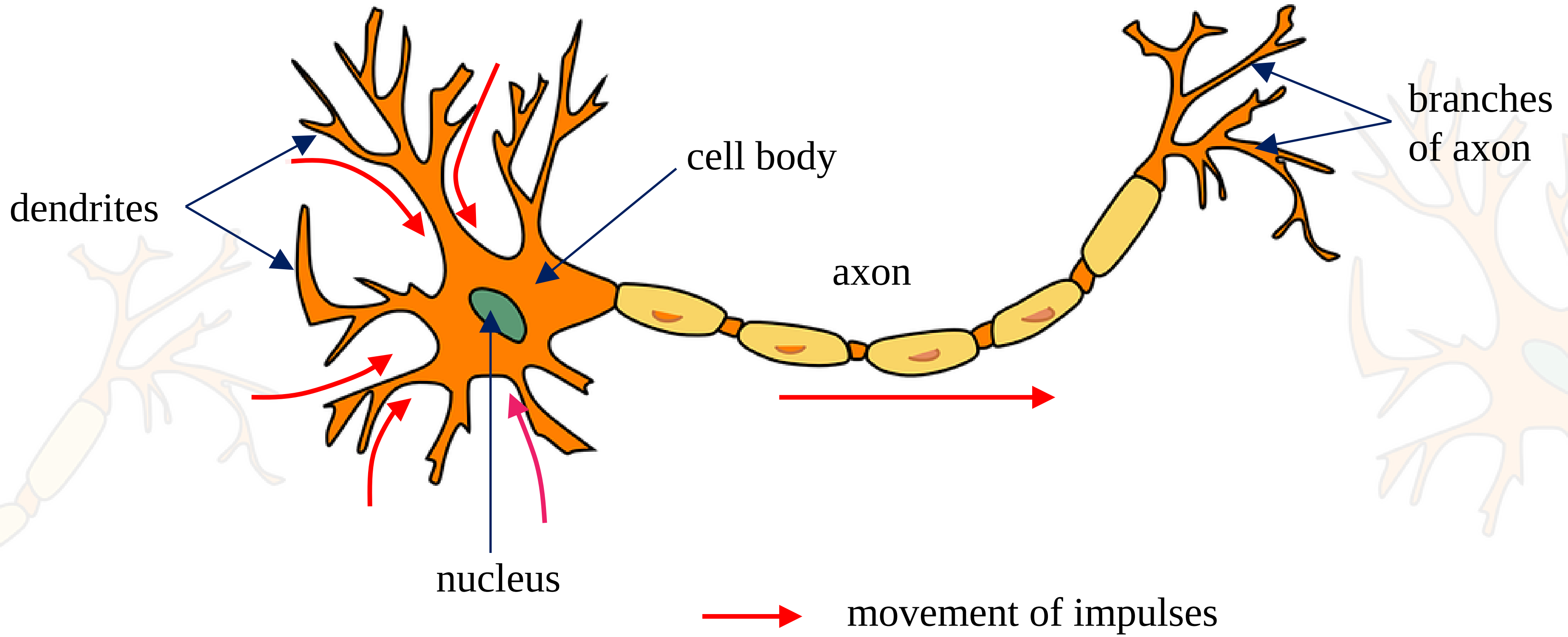
SOFTWARE

- Open source toolboxes
- Efficient implementations

TensorFlow PyTorch

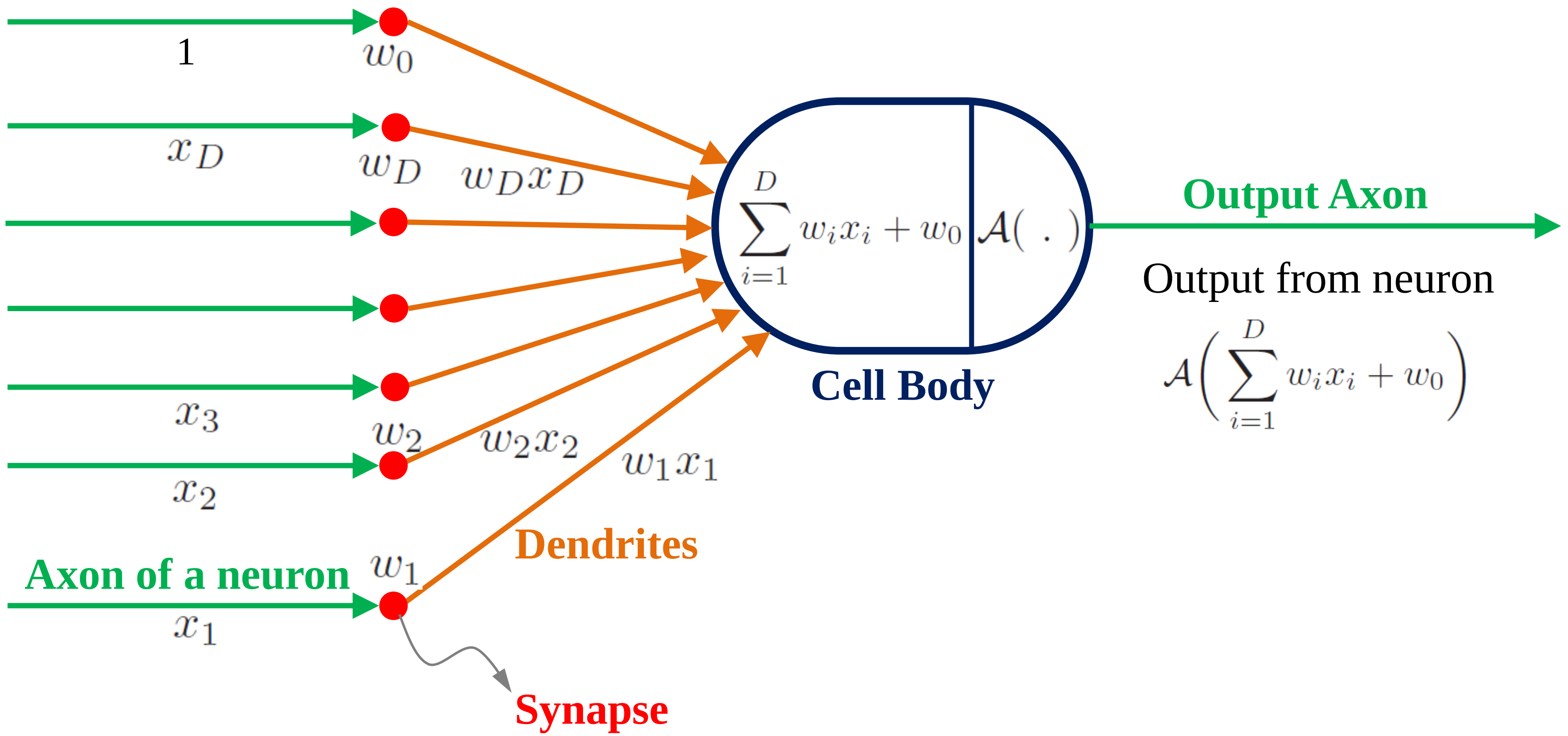
Neuron

- The brain is composed of densely interconnected network of neurons.
- Each neuron has a body, axon, synapses and dendrites.
- A neuron fires if the sum of the weighted signals is greater than a threshold.
- Signals propagate along neurons via axons, synapses and dendrites.

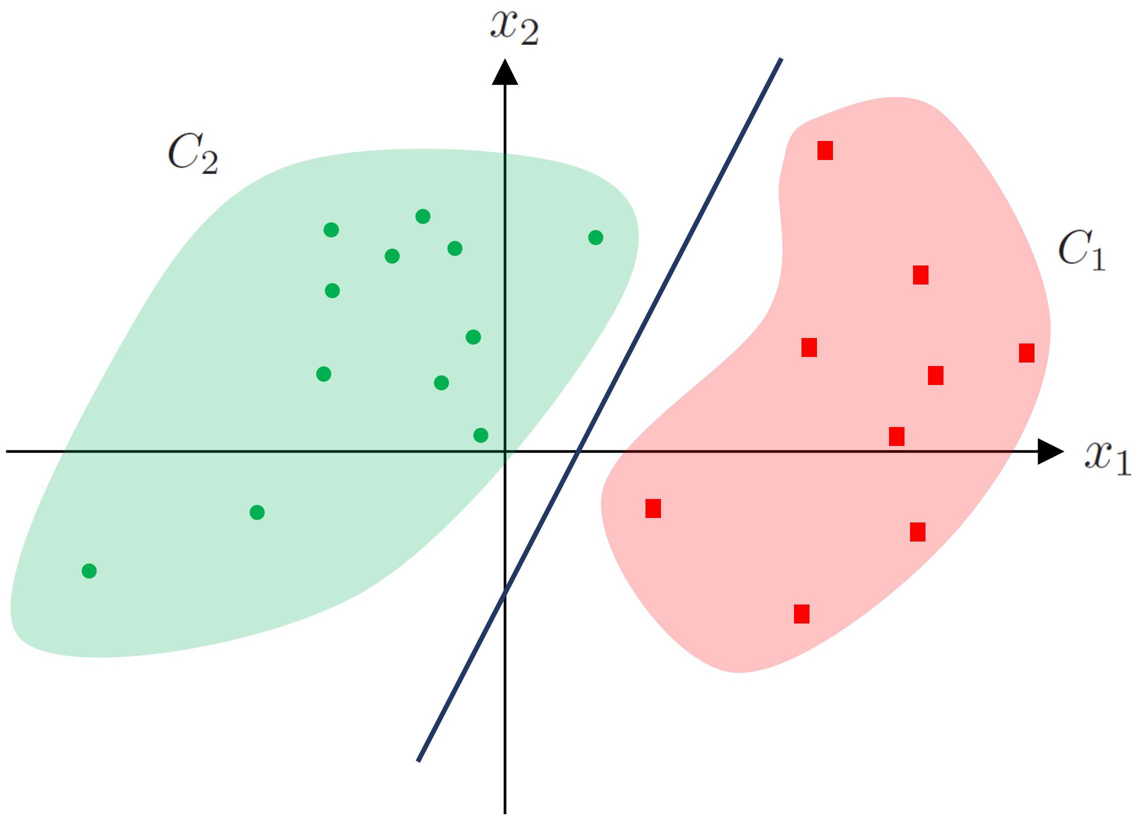


Mathematical model of a neuron

- The threshold activation function "fires" if the weighted sum of the inputs and bias exceeds a certain threshold.

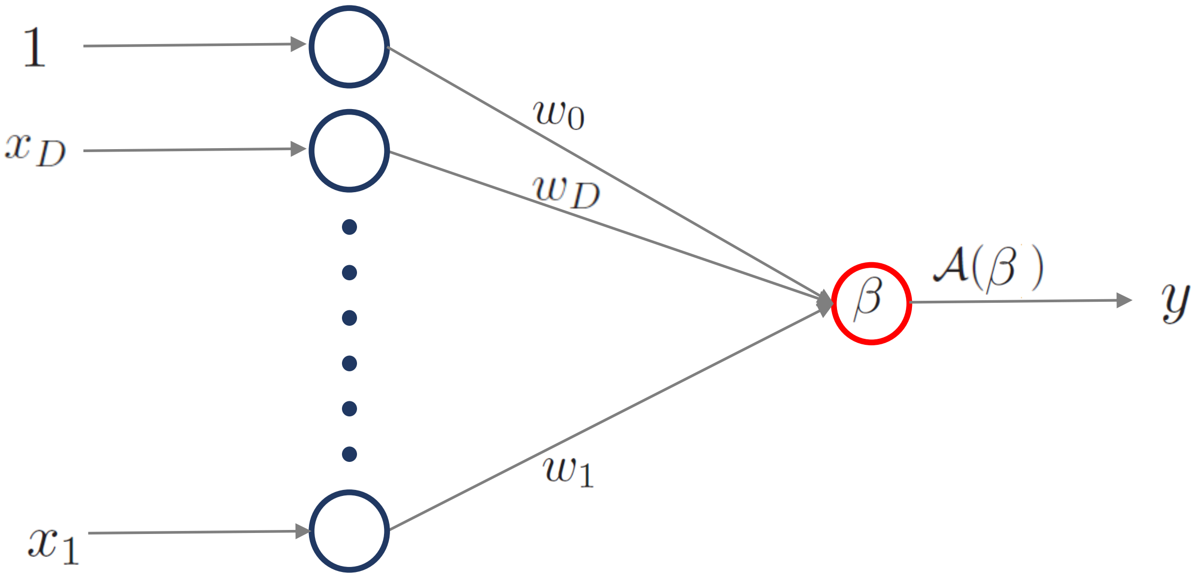


Perceptron Learning



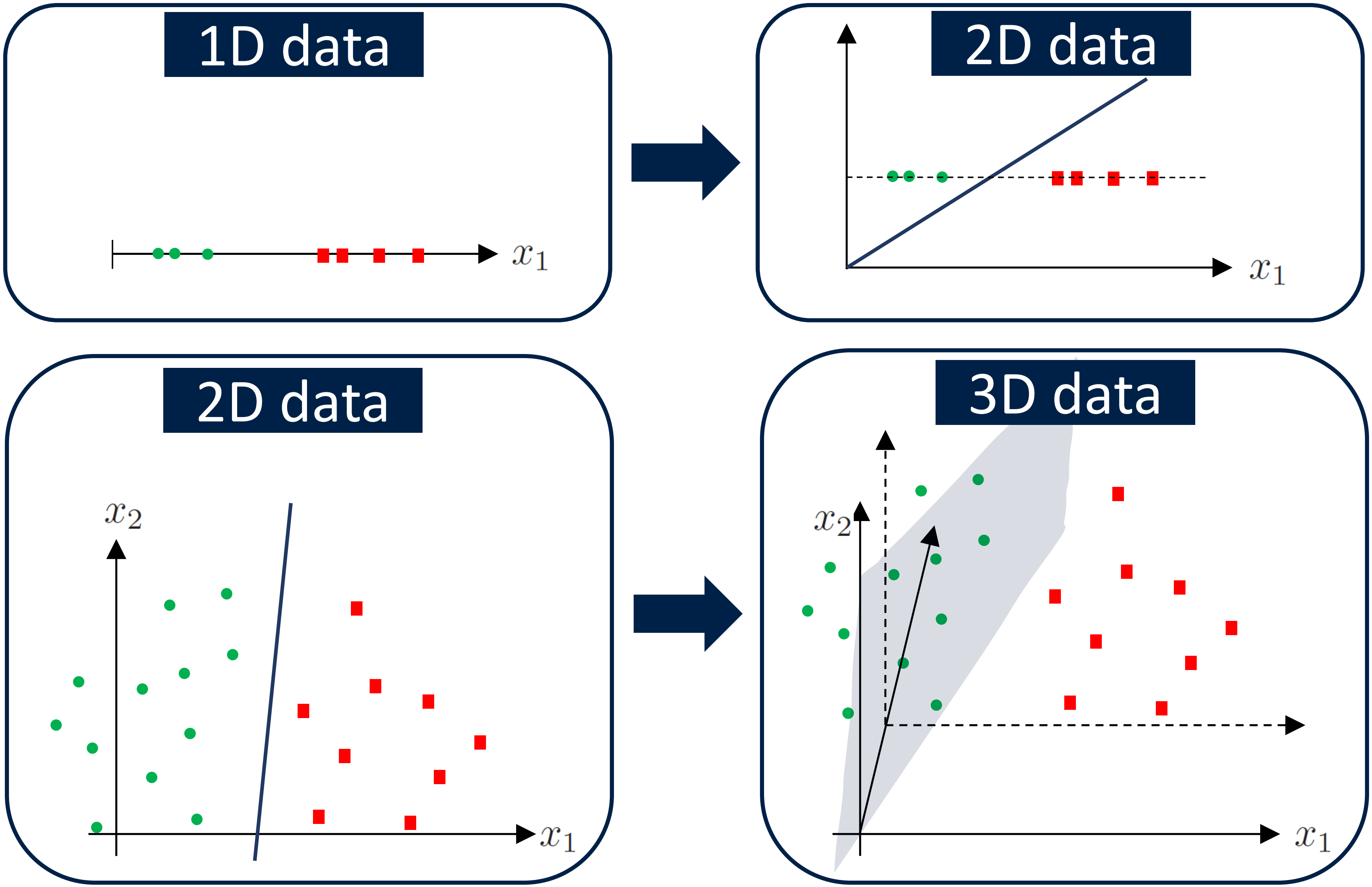
1D Perceptron

- Single layer feed-forward network – only 1 layer of weights is used.



$$\beta = \mathbf{w}^T \mathbf{x}$$

Augmenting input space



Functional margin

- Correct classification:
  - If $\mathbf{x} \in y = 1$, then $\mathbf{w}^{\mathrm{T}}\mathbf{x} > 0$
  - If $\mathbf{x} \in y = -1$, then $\mathbf{w}^{\mathrm{T}}\mathbf{x} < 0$

- The two equations can be jointly written as: $y\mathbf{w}^{\mathrm{T}}\mathbf{x} > 0$

- The functional margin $\widehat{\gamma}_n$ of an example $(\mathbf{x}^{(n)}, y^{(n)})$ with respect to the hyperplane $\mathbf{w}$ is

$$\widehat{\gamma}_n = y^{(n)}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(n)}\right)$$

- $+$ve $\widehat{\gamma}_n$ means the example is correctly classified.
- $-$ve $\widehat{\gamma}_n$ means the example is incorrectly classified.

Geometric margin

- Geometric margin γ_n of an example is the signed perpendicular distance of the point to the hyperplane

$$\gamma_n = \frac{\mathbf{w}^{\mathrm{T}}\mathbf{x}^{(n)}}{||\mathbf{w}||}$$

- Margin is defined as the minimum of the geometric margin

$$\gamma = \min_{\mathcal{D}} |\gamma_n|$$

Approach

- Perceptron algorithm checks if all the training examples are correctly classified with respect to a hyperplane.
  - If an example is misclassified, then the hyperplane is updated.

- If after k updates, a misclassified example $(\mathbf{x}^{(n)}, y^{(n)})$ is encountered, then $\mathbf{w}^{(k)}$ is updated as

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}$$

- What happens if an example is misclassified?
  - If $(\mathbf{x}^{(n)}, y^{(n)})$ is misclassified (after k updates to weights), then

$$\widehat{\gamma}_n = y^{(n)}(\mathbf{w}^{(k)})^{\mathrm{T}}\mathbf{x}^{(n)} < 0.$$

  - Updated weights: $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}$

Approach

– New functional margin $\widehat{\gamma}_n^{(\text{new})}$ is then

$$
\begin{aligned}
\widehat{\gamma}_n^{(\text{new})} &= y^{(n)}\left(\mathbf{w}^{(k+1)}\right)^{\mathrm{T}}\mathbf{x}^{(n)} \\
&= y^{(n)}\left(\mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}\right)^{\mathrm{T}}\mathbf{x}^{(n)} \\
&= y^{(n)}\left(\mathbf{w}^{(k)}\right)^{\mathrm{T}}\mathbf{x}^{(n)} + \left(y^{(n)}\right)^{2}||\mathbf{x}^{(n)}||^{2} \\
&\geq y^{(n)}\left(\mathbf{w}^{(k)}\right)^{\mathrm{T}}\mathbf{x}^{(n)} \\
&\geq \widehat{\gamma}_n^{(\text{old})}
\end{aligned}
$$

- So the new hyperplane $\mathbf{w}^{(k+1)}$ has a larger functional margin than older one $\Rightarrow$ better classification of $(\mathbf{x}^{(n)}, y^{(n)})$.

Algorithm

Initialize: $\mathbf{w} = 0$

while (not converged) **then**

$k = 0$

for $n = 1, 2, .., N$ **do**

if $y^{(n)}((\mathbf{w}^{(k)})^{\mathrm{T}}\mathbf{x}) \leq 0$ **then**

$\mathbf{w} \longleftarrow \mathbf{w} + y^{(n)}x^{(n)}$

$k \longleftarrow k + 1$

end if

end for

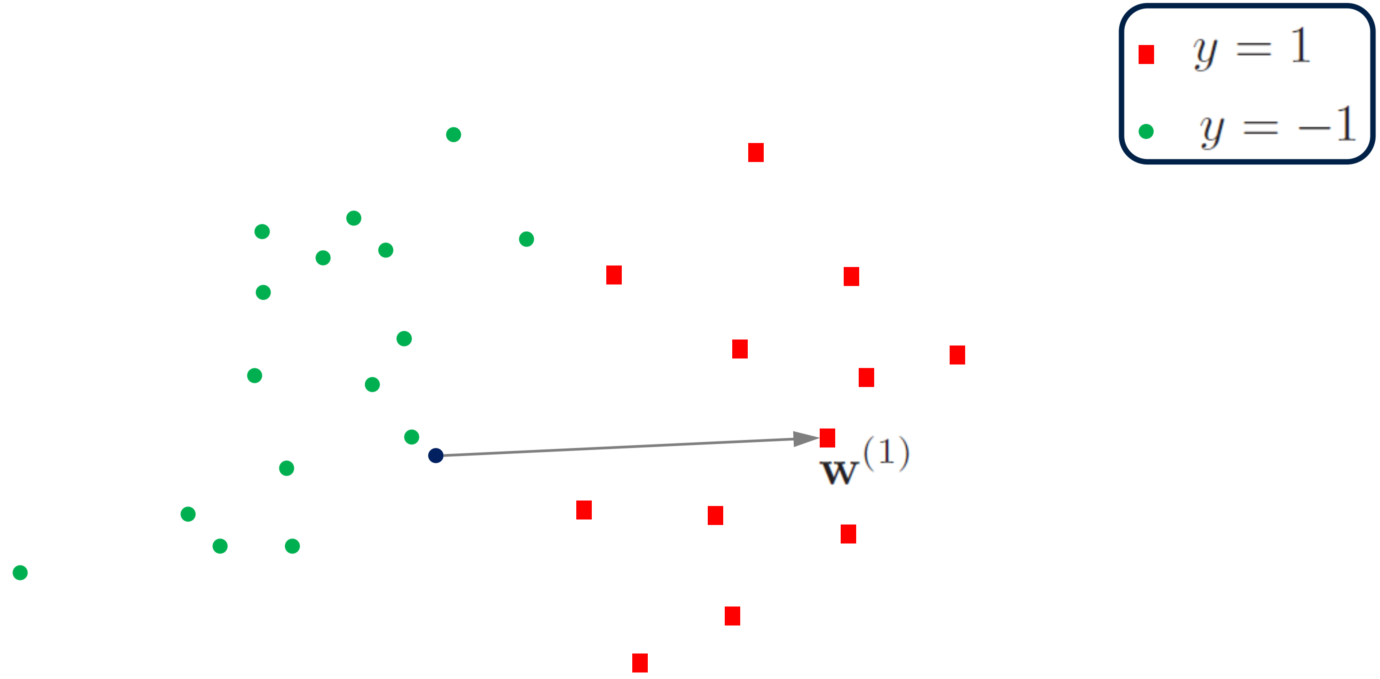
if $k = 0$ **then**

break

end if

end loop

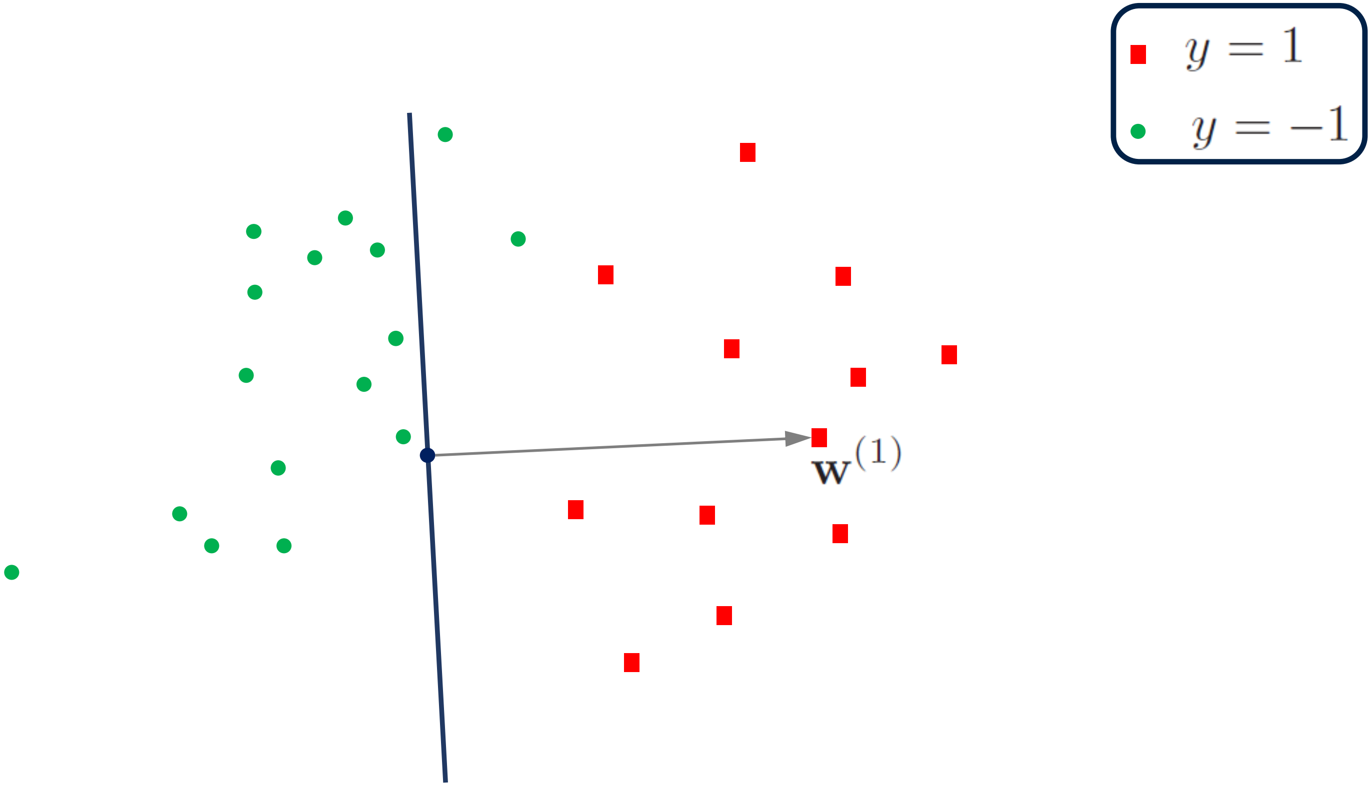
Visualization



Weight update rule:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}$$

Visualization



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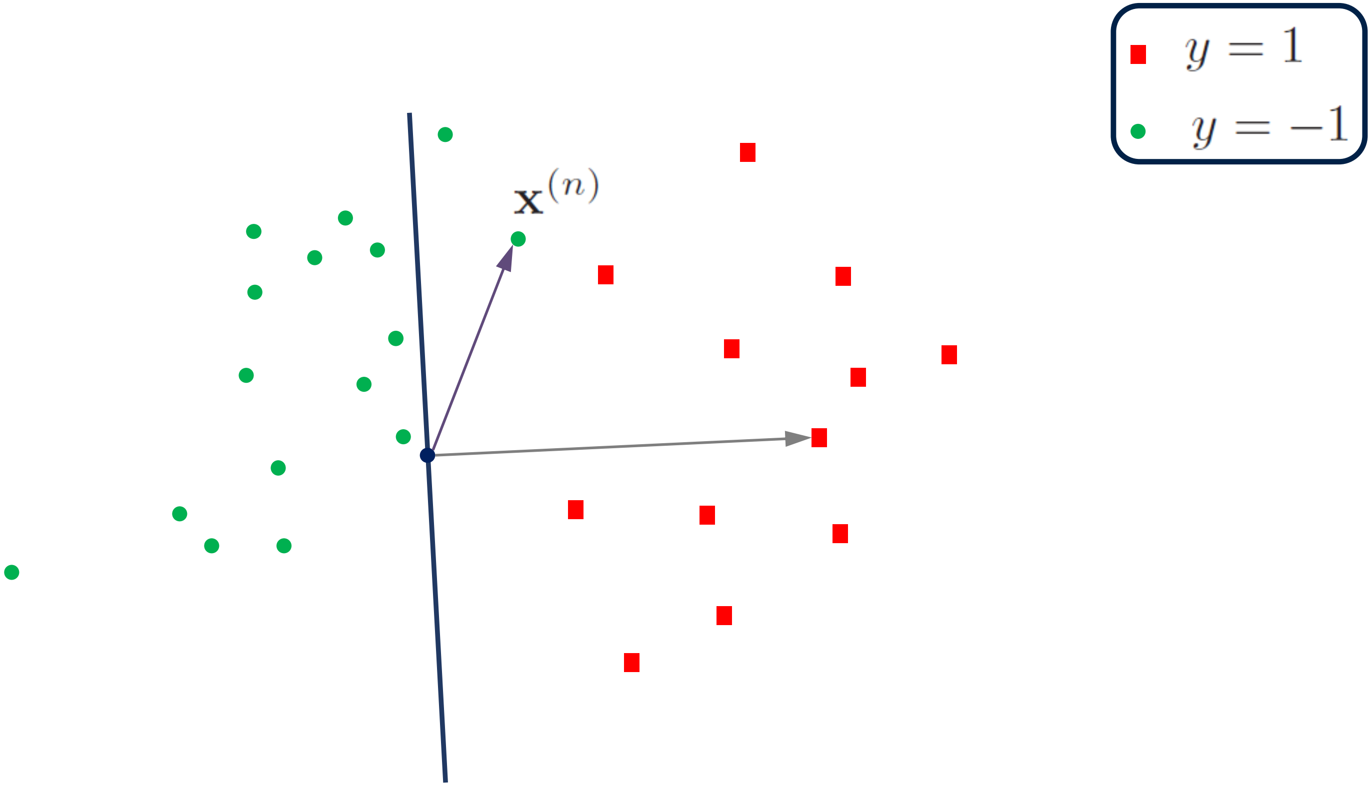

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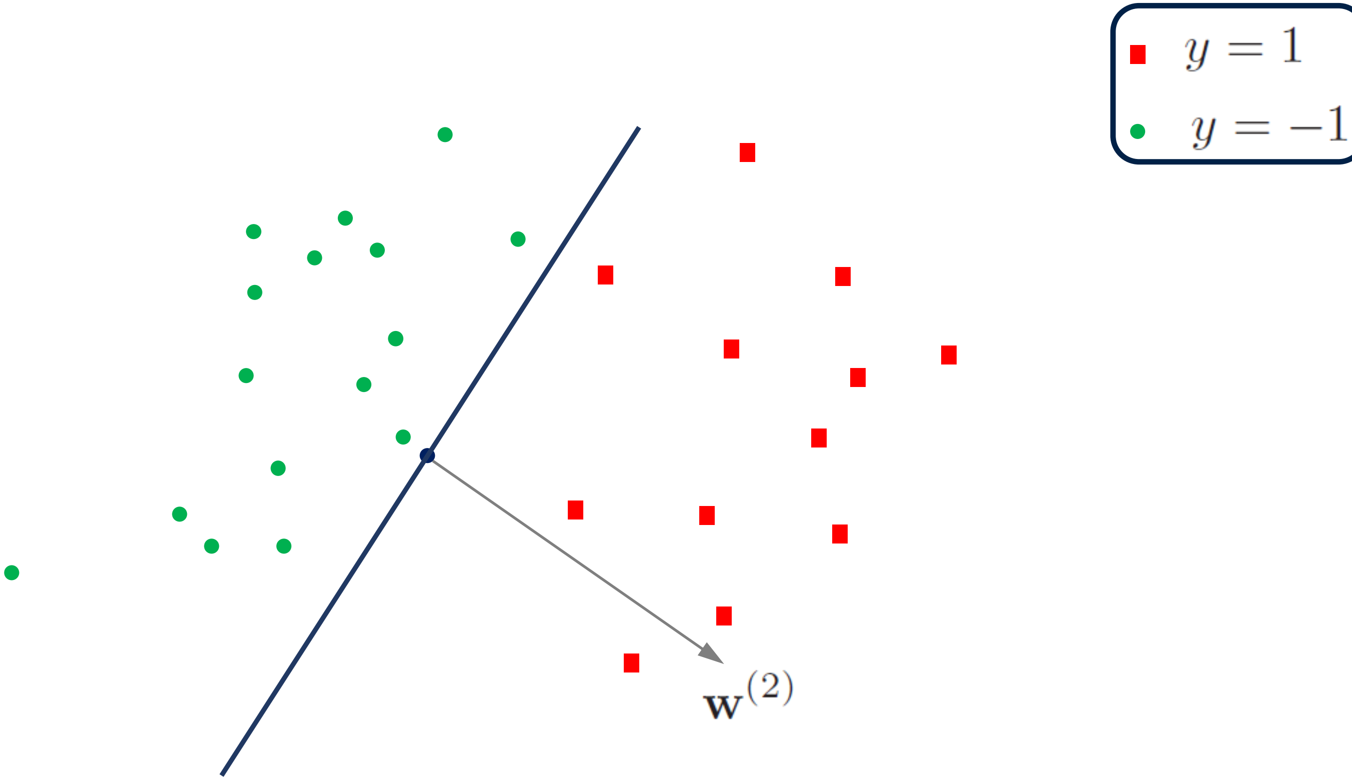
Visualization



Weight update rule: $\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}$

Convergence Theorem

Theorem: If the training data is linearly separable with margin γ by a unit norm hyperplane $\mathbf{w}^*$ ($||\mathbf{w}^*|| = 1$) with $w_0 = 0$, then perceptron converges after $1/\gamma^2$ mistakes during training (assuming $||\mathbf{x}|| < 1 \;\; \forall \mathbf{x}$).

- Note, for any datatset the inputs $\mathbf{x}$ can be scaled as $\mathbf{x}/\max_{n=1:N} ||\mathbf{x}^{(n)}||$ so as to achieve $||\mathbf{x}|| < 1 \;\; \forall \mathbf{x}$.

- Consider a hyperplane $\mathbf{w}^*$ such that the following conditions hold:
  - $y^{(n)}\left(\mathbf{x}^{(n)}\right)^{\mathrm{T}}\mathbf{w}^* > 0 \; \forall \left(\mathbf{x}^{(n)}, y^{(n)}\right) \in \mathcal{D}$
  - $||\mathbf{w}^*|| = 1$

- Will check the effect of weight update, i.e. $\mathbf{w}$ to $\mathbf{w} + y\mathbf{x}$, on $\mathbf{w}^{\mathrm{T}}\mathbf{w}^*$ and $\mathbf{w}^{\mathrm{T}}\mathbf{w}$.

Convergence Proof

$$
\begin{aligned}
(\mathbf{w}^{(k+1)})^T \mathbf{w}^* &= (\mathbf{w}^{(k)} + y^{(n)} \mathbf{x}^{(n)})^T \mathbf{w}^* \\
&= (\mathbf{w}^{(k)})^T \mathbf{w}^* + y^{(n)} ((\mathbf{x}^{(n)})^T \mathbf{w}^*) \\
&\geq (\mathbf{w}^{(k)})^T \mathbf{w}^* + \gamma \\
&\geq (\mathbf{w}^{(k-1)})^T \mathbf{w}^* + 2\gamma \\
&. \\
&. \\
&\geq k\gamma \\
||(\mathbf{w}^{(k+1)})^T \mathbf{w}^*|| &\geq k\gamma \\
||(\mathbf{w}^{(k+1)})^T|| ||\mathbf{w}^*|| &\geq k\gamma \\
||(\mathbf{w}^{(k+1)})|| &\geq k\gamma \quad (\text{as} \quad ||\mathbf{w}^*|| = 1)
\end{aligned}
$$

- This is a lower bound on the weights.

Convergence Proof

$$
\begin{aligned}
||\mathbf{w}^{(k+1)}||^2 &= ||\mathbf{w}^{(k)} + y^{(n)}\mathbf{x}^{(n)}||^2 \\
&= ||\mathbf{w}^{(k)}||^2 + ||y^{(n)}\mathbf{x}^{(n)}||^2 + 2((\mathbf{w}^{(k)})^T\mathbf{x}^{(n)})y^{(n)} \\
&= ||\mathbf{w}^{(k)}||^2 + ||\mathbf{x}^{(n)}||^2 + 2((\mathbf{w}^{(k)})^T\mathbf{x}^{(n)})y^{(n)} \\
&\leq ||\mathbf{w}^{(k)}||^2 + ||\mathbf{x}^{(n)}||^2 \quad (\text{as } \ \mathbf{x}^{(n)} \text{ is misclassified, so}((\mathbf{w}^{(k)})^T\mathbf{x}^{(n)})y^{(n)} \leq 0) \\
&\leq ||\mathbf{w}^{(k)}||^2 + 1 \quad (\text{as}\, ||\mathbf{x}^{(n)}|| \leq 1) \\
&\leq ||\mathbf{w}^{(k-1)}||^2 + 1 + 1 \\
&\,. \\
&\,. \\
&\leq k
\end{aligned}
$$

- This is an upper bound on the weights.

Convergence Proof

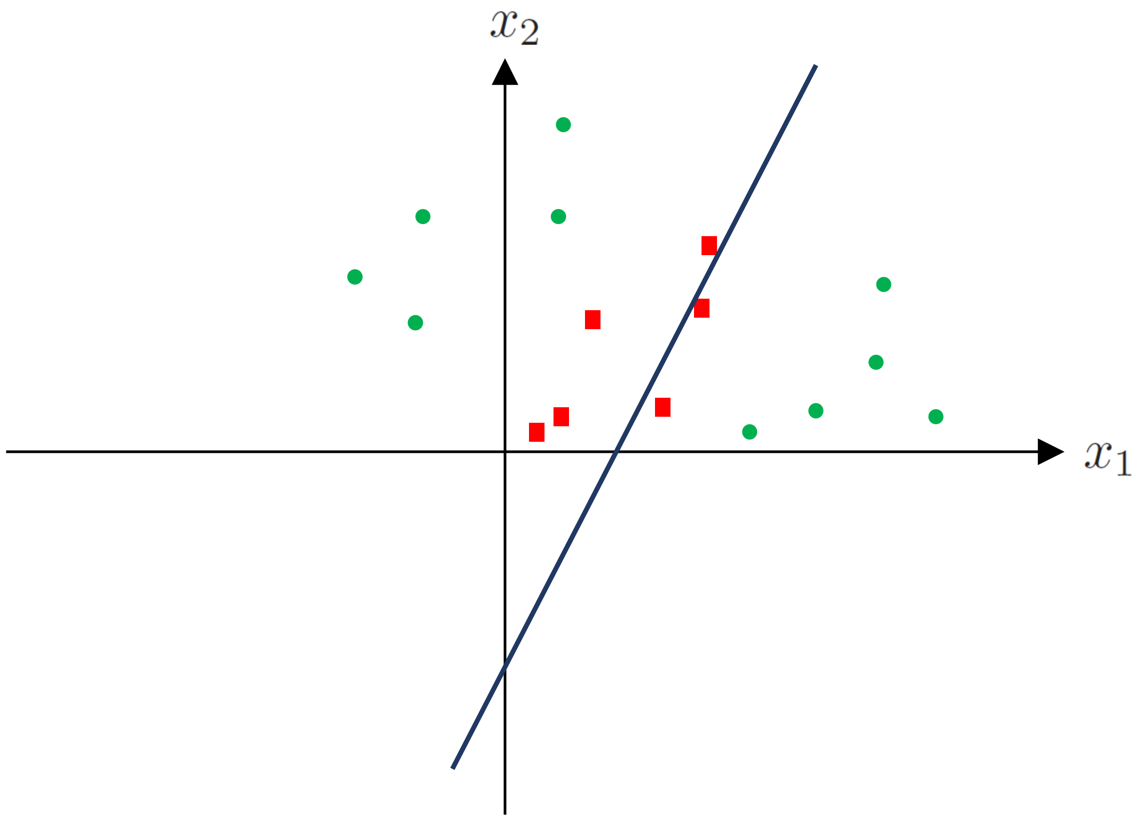
- Using the two inequalities (lower and upper bounds), we have

$$k^2\gamma^2 \leq ||\mathbf{w}^{(k+1)}||^2 \leq k$$

$$\boxed{k \leq \frac{1}{\gamma^2}}$$

- The number of updates k is bounded from above by a constant.
  - The convergence rate is independent of the dimensionality of the dataset D and the number of examples N.

Limitations



XOR function

- XOR data is not linearly separable.

x_1	x_2	XOR	Class label
0	0	0	$\mathcal{C}_2$
0	1	1	$\mathcal{C}_1$
1	0	1	$\mathcal{C}_1$
1	1	0	$\mathcal{C}_2$