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Stochastic Process

Simple Random Walk

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables.

$(t : \text{integer})$

$$X_t = \begin{cases} +1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}$$

(For every t)

$$E(X_t) = +1\left(\frac{1}{2}\right) - 1\left(\frac{1}{2}\right) = 0$$

$$Var(X_t) = E(X_t^2) - \cancel{E(X_t)}^2 = 1 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$$

W. Feller

An introduction to prob. theory and its applications

Feller "epoch"

$\downarrow$

particular moment t.

$\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$

Set of non-negative integers.

For each t: integer (positive).

Random variable ← $S_t \stackrel{def}{=} X_1 + X_2 + \cdots + X_t$

$(S_0 \stackrel{def}{=} 0)$

The sequence of $\{S_0, S_1, \ldots, S_t, \ldots\}$

$\Downarrow$

Discrete-time stochastic process.

Known as: SIMPLE RANDOM WALK (on integers)

$S \to$ big "s"

Paths as the sample space:

Paths $\downarrow$ small "s"

$$\left.\begin{array}{l} X_1 = +1 \\ X_2 = -1 \\ X_3 = -1 \\ X_4 = +1 \\ X_5 = +1 \\ X_6 = +1 \end{array}\right\}$$

$S_0 = 0$ (True always).

$S_1 = X_1 = +1$

$S_2 = X_1 + X_2 = +1 - 1 = 0$

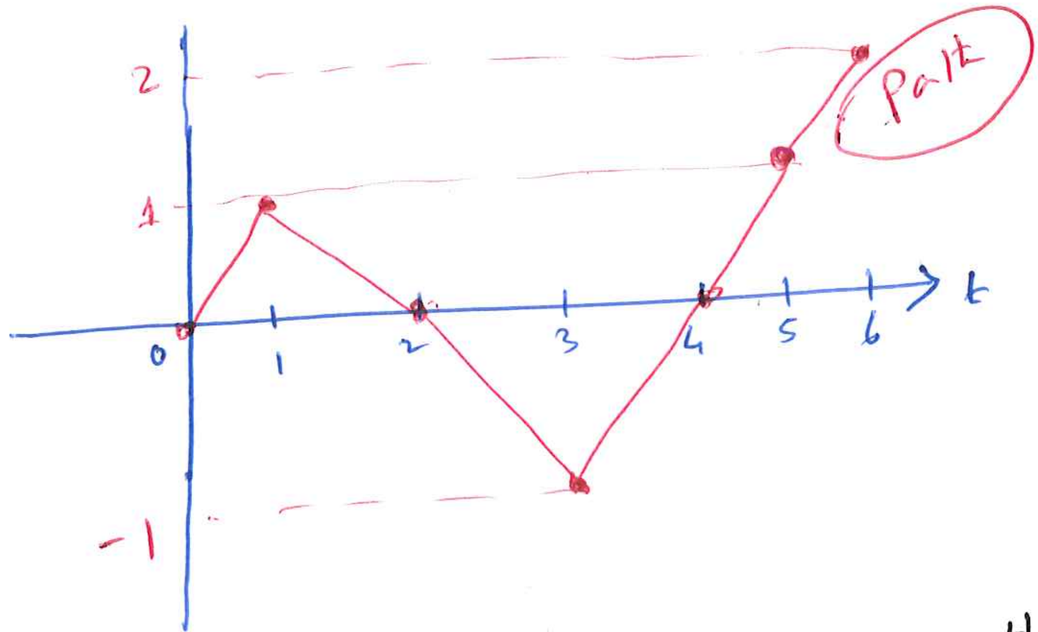
$S_3 = X_1 + X_2 + X_3 = -1$

$S_4 = X_1 + X_2 + X_3 + X_4 = 0$

$S_5 = X_1 + \cdots + X_5 = +1$

$S_6 = X_1 + \cdots + X_6 = +2$

. . . . .



Question: How many paths the random walk may take through epoch t?

Answer $= 2^t$

Each path has equal probability.

So, ~~the~~ the probability of each of the 2^t paths ~~has~~ is $= \frac{1}{2^t}$

Notation: Let us say that the path s visits K at epoch t if

$$\boxed{s_t = K}$$

[**Example:** For the last case: $s_6 = 2$]

X : r.v.

x : value of r.v.

S : random walk

s : path of random walk

Aside:

$$E(S_t) = E\left(\sum_{i=1}^{t} X_i\right) = \sum_{i=1}^{t} \cancelto{0}{E(X_i)} = 0$$

$$Var(S_t) = Var\left(\sum_{i=1}^{t} X_i\right) = \sum_{i=1}^{t} \cancelto{1}{Var(X_i)} = t$$

($\because$ X_i's are independent)

So, by $\boxed{\text{central limit theorem}}$:

$$\frac{S_t - 0}{\sqrt{t}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty.$$

i.e.,

$$\frac{S_t}{\sqrt{t}} \xrightarrow{d} N(0,1) \quad \text{as } t \to \infty.$$

Back to main topic:

Characterization of reachable points.

$\cancel{S_1 = 5}$, $\cancel{S_1 = 0}$

$S_{200} = 23$

Theorem: In order for (t, k) $\;(S_t = k)$ to be reachable, there MUST be non-negative integers p and m,

where p : # of $+1$s
m : # of -1s

such that $p + m = t$, and $\boxed{p - m = k}$

So that,

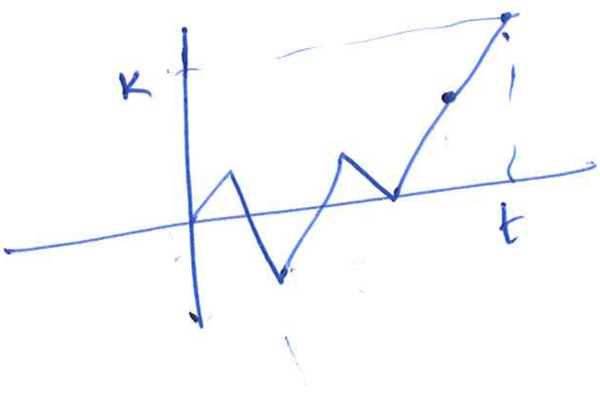
$$\left.\begin{aligned} p &= \frac{t+k}{2} \\ m &= \frac{t-k}{2} \end{aligned}\right\} \quad \cdots (1)$$

Proof: total positive value $= +1 \cdot p$
total negative value $= -1 \cdot m$

Total value $= p - m$

Want this $= k$.

Hence $p - m = k$
$p + m = t$ (obvious)



Solving these we obtain (1).

Observation:

- Reachability implies both $(t+k)$ and $(t-k)$ must be even

$\Rightarrow$ $\boxed{\text{t and k MUST have the same parity (even/odd)}}$

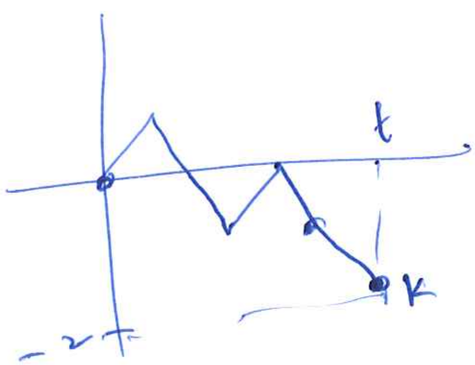
- We must also have

$t+k \geq 0 \Rightarrow t \geq -k$
$t-k \geq 0 \Rightarrow t \geq k$

$\Rightarrow \boxed{t \geq |k|}$

(k can be both + or −

But $t \geq 0$)



Example:

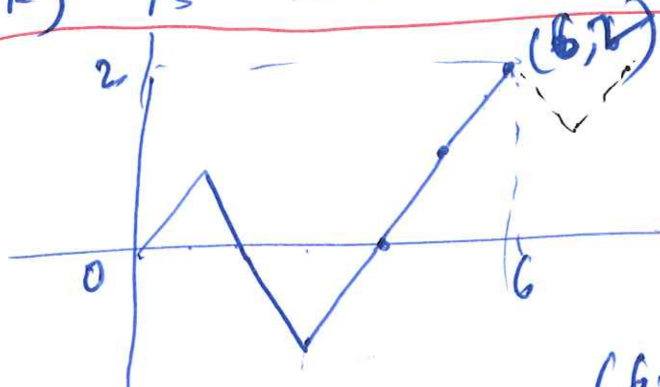
$S_{200} = 23 \rightarrow$ NO NOT reachable.

$t = 200$ (even)
$k = 23$ (odd)

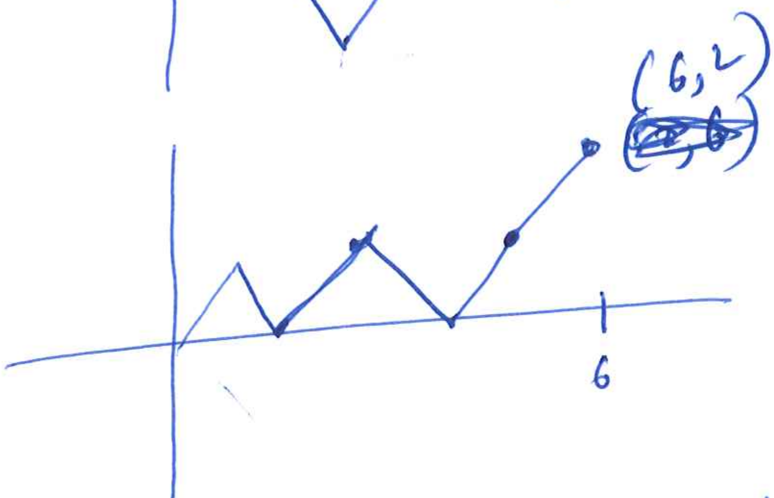
Definition: The number of initial segments of paths that reach the reachable point (t, k) is denoted by $N_{t,k}$

[If (t, k) is NOT reachable, $N_{t,k} = 0$]

Example:



$(6,2)$ is reachable



(Another way to reach $(6,2)$)

Question: How many total ways to reach $(6,2)$?

Notation: $N_{6,2}$

$$\cancel{\binom{2\cdot(t+k)-1}{t+k-1}}$$

Theorem: If (t,k) is reachable.

$$N_{t,k} = \binom{t}{\frac{t+k}{2}} = \binom{t}{\frac{t-k}{2}}$$

p = # of $+1$s

m = # of -1s.

Proof: If (t,k) is reachable, there MUST be (by the last Theorem) integers p and m, such that ① (pg-5) is satisfied.

Since the p ~~and~~ $+1$s and m -1s can be arranged in any order,

$$N_{t,k} = \binom{p+m}{p} \left(= \binom{p+m}{m}\right)$$

$$= \binom{t}{\frac{t+k}{2}} = \binom{t}{\frac{t-k}{2}}$$

by ①

by ①

Hence $N_{t,k} = \binom{t}{\frac{t+k}{2}} = \binom{t}{\frac{t-k}{2}}$ //

Define

$$P_{t,k} = P(S_t = k)$$

Then,

$$P_{t,k} = \frac{\text{\# of paths that reach } (t,k)}{\text{Total \# of paths until epoch } t.}$$

$$= \frac{\binom{t}{\frac{t+k}{2}}}{2^t}$$

So,

$$\boxed{P_{t,k} = \frac{1}{2^t}\binom{t}{\frac{t+k}{2}}}$$

if (t,k) is reachable.

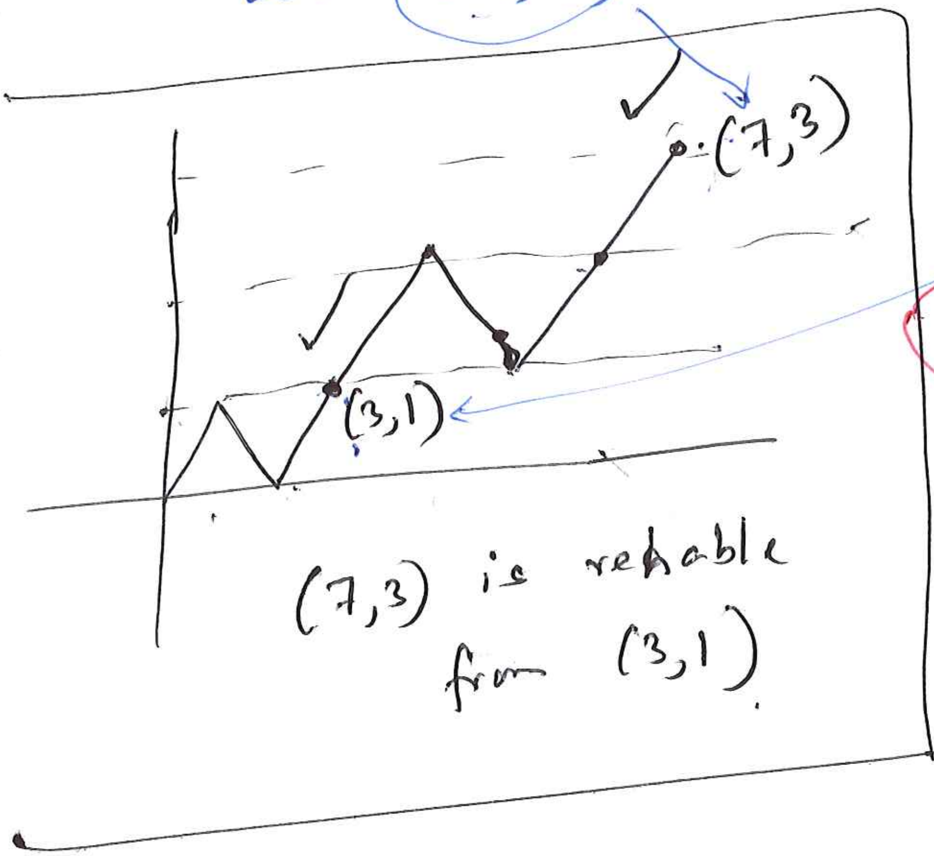
Corollary: If (t_1, k_1) is reachable from (t_0, k_0) then the number of sample paths connecting them is

$$N_{t_1-t_0,\, k_1-k_0}$$

(Check it!)

The reflection principle

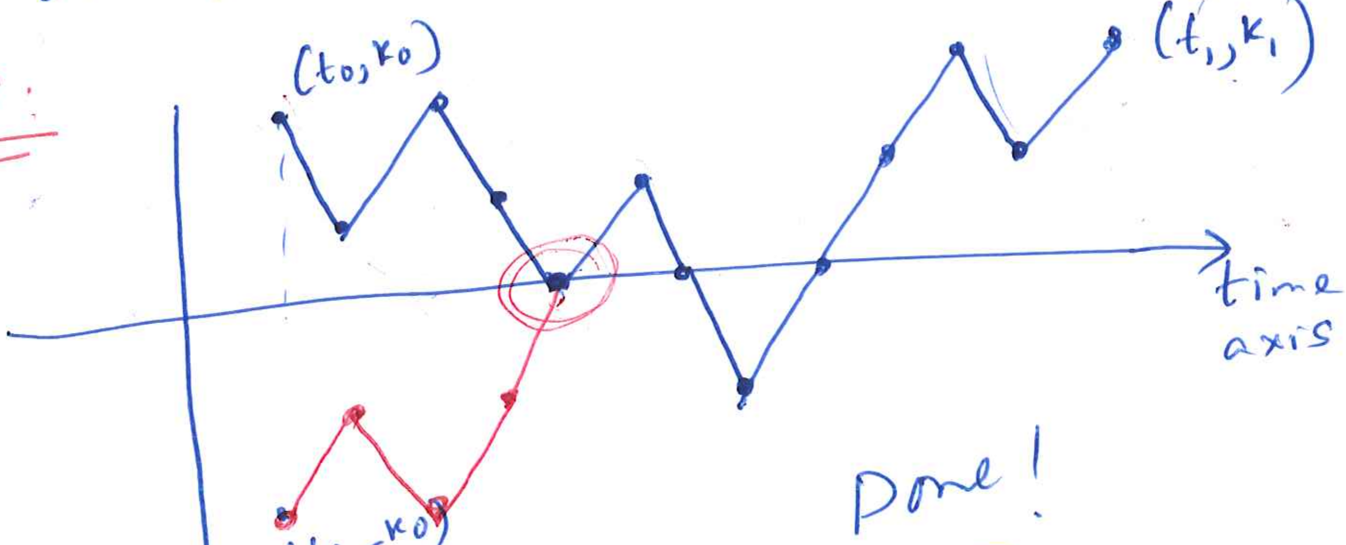
Let (t_1, k_1) be reachable from (t_0, k_0) **AND** on the **SAME SIDE** of the time axis.



(7,3) is rechable from (3,1).

THEN: there is a one-to-one correspondence between the set of paths from (t_0, k_0) to (t_1, k_1) that meet (touch or cross) the time axis and the set of all paths from $(t_0, -k_0)$ to (t_1, k_1).

Proof:



Done!

Naive Set Theory

P. Halmos

The Ballot Theorem

~~Th~~ If $k > 0$, then there are exactly $\frac{k}{n} N_{n,k}$ paths from the origin to (n, k) satisfying $S_t > 0$, $t = 1, \ldots, n$.