~~Econome~~ Finance

Roulette Game

1, 2, 3, ... 36, 0, 00

Even : 2, 4, ..., 36

odd : 1, 3, ..., 35

Game 1

Betting on even / odd

$$\text{Payoff} = \begin{cases} +1 & \text{w.p. } 18/38 \\ -1 & \text{w.p. } 20/38 \end{cases}$$

Expectation $= -1/19$

Game 2

Betting on colours

Col 1	Col 2	Col 3
1	13	25
2	⋮	⋮
⋮	⋮	⋮
12	24	36

Betting on colours

$$\text{Payoff} : \begin{cases} +2 & \text{w.p. } 12/38 \\ -1 & \text{w.p. } 26/38 \end{cases} = -1/19$$

Which one do you bet on ?

No strategy is right or wrong in Finance.

S_0 : The amount of money we started with

$S_n = X_1 + X_2 + \dots + X_n$: Money we have after playing 'n' rounds of a game.

Some Criterias we want to check:

① $P(S_n > b \cdot S_0)$, $b > 1$ or $P(S_n > b_0 + b_1 S_0)$ $b_0 > 0$, $b_1 > 1$

↓

②. $E[S_n / S_n > S_0]$ → more the better.

③. Find n before $S_n = b \cdot S_0$, $b > 1$ or $S_n = 0$

Find n s.t. $S_n = b \cdot S_0$ before $S_n = 0$

lower n is better game

$S \sim \text{Binomial}(n, p)$

S_n: sum of successes out of n independent trials, say $\{X_i\}$'s

- Each X_i has two outcomes, say S & F.
- $P(S) = p$, $P(F) = 1-p$. does not change

Sum of iid Bernoulli (p) r.v.s

$$X = \begin{cases} d_1 \to p \\ d_0 \to 1-p \end{cases} \qquad Y = \frac{X - d_0}{d_1 - d_0} = \begin{cases} 1 \to p \\ 0 \to 1-p \end{cases}$$

$$Y = a_1 X + a_0$$

$$S_n^Y = Y_1 + \cdots + Y_n = a_1 S_n^X + n a_0 \quad , \quad a_1 > 0$$

$$P(S_n^X > S_0^X) = P(a_1 S_n^X + n a_0 > a_1 S_0^X + n a_0)$$

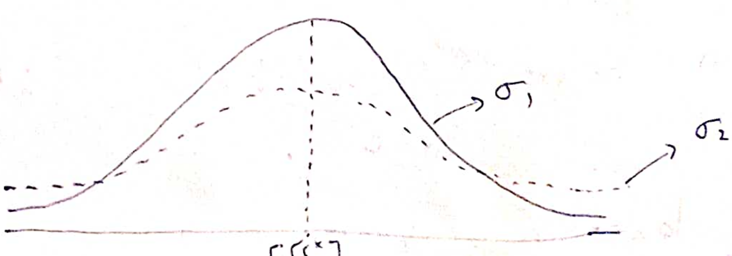
$$= P(S_n^Y > a_1 S_0^X + n a_0)$$

$$P(S_n^X > S_0^X) = P\left(\frac{S_n^X - E[S_n^X]}{\sqrt{Var(S_n^X)}} > \frac{S_0^X - E[S_n^X]}{\sqrt{Var(S_n^X)}} \right)$$

for large values of n

$$P(Z > (\cdot)) = 1 - \Phi\left(\frac{S_0^Y - E[S_n^X]}{\sqrt{Var(S_n^X)}} \right)$$

Abraham De Moivre



Since $E[S_A^X]$ is negative, larger variance is better.
As it has more chance of us getting a +ve return.

HW: $E[S_n / S_n > S_0]$

Gambler's Ruin

1) $P(S_n = bS_0,$ for the first time, before reaching $S_n = 0)$

2) $E[n / S_n = b \cdot S_0$, for the first time, before reaching $S_n = 0]$

3.) $E[n / S_n = b \cdot S_0$ or $S_n = 0$ for the first time$]$

4) Distribution of $n / S_n = b \cdot S_0$ or $S_n = 0$ for the first time.

Gambler's Ruin:

$\phi(c) =$ prob. of ruin given $S_0 = c$

Game 1

$$c \xrightarrow{p} c+1, \qquad c \xrightarrow{q} c-1$$

$$\phi(c) = p \cdot \phi(c+1) + q\phi(c-1)$$

$(p+q=1)$

$$(p+q)\,\phi(c) = p \cdot \phi(c+1) + q\phi(c-1)$$

TPM =
$$\begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \end{array}
\begin{array}{c} 0 \quad \cdots \\ \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ q & 0 & p & & & \\ & q & 0 & p & & \\ & & q & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & p \\ & & & & q & 1 \end{bmatrix} \end{array}$$

$$p\left(\phi(c+1) - \phi(c)\right) = q\left(\phi(c) - \phi(c-1)\right)$$

$$\phi(c+1) - \phi(c) = q/p\left[\phi(c) - \phi(c-1)\right]$$

$$= q^2/p\left[\phi(c-1) - \phi(c-2)\right]$$

$$= (q/p)^c\left[\phi(1) - \phi(0)\right]$$

$$\left.\begin{array}{l} \phi(c+1)-\phi(c) \\ \phi(c)-\phi(c-1) \\ \vdots \\ \phi(1)-\phi(0) \end{array}\right] \text{Add}$$

$$\phi(c+1)-\phi(0) = [\phi(1)-\phi(0)]\left[1+\frac{q}{p}+\dots+\left(\frac{q}{p}\right)^c\right]$$

$$= [\phi(1)-\phi(0)] \cdot \frac{1-(q/p)^{c+1}}{1-q/p} \quad \text{if } p\neq q$$

$$= \phi(1)\cdot(c+1) \quad p=q$$

Note: $\phi(0)=1$, $\phi(b)=0$

b is the upper value (target value)

For $c=b-1$,

$$\Rightarrow \phi(b)-\phi(0) = \begin{cases} \dfrac{1-(q/p)^b}{1-q/p}[\phi(1)-\phi(0)] & p\neq q \\ b[\phi(1)-\phi(0)] & p=q \end{cases}$$

$$\phi(1) = \begin{cases} \dfrac{(q/p)^1-(q/p)^b}{1-(q/p)^b}, & p\neq q \\ \dfrac{b-1}{b}, & p=q \end{cases}$$

Check that

$$\phi(c) = \begin{cases} \dfrac{(q/p)^b-(q/p)^c}{(q/p)^b-(q/p)^0} & p\neq q \\ \dfrac{b-c}{b-0} & p=q \end{cases}$$

if $a\leq c<b$

0 translates to a.

$b=200$
$c=100$
$p=18/38$
$q=20/38$

$$q/p = \frac{20}{38}\times\frac{38}{18} = \frac{10}{9}$$

Ch 1: Random Walk.

Book Bhattacharya & Waymire

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HW: P(Ruin) for game 2. Challenging Problem

Marking:

Final : 50%

Mid Sem: 30%

HW/Project : 20%

12 August

Book : Huang & Litzenberger
Foundations of Finantial Economics

$\phi(c)$: Probability of ruin given that gambler's initial wealth is c.

τ : First time to reach '0' or upper value $(bS_0) = b_0$

$\phi(c) = p\,\phi(c+1) + q\,\phi(c-1)$ why?

$\phi(c) = P(S_\tau = 0 / S_0 = c)$

$= P(\underbrace{S_\tau = 0}_{A}, \underbrace{S_1 = c+1}_{B} / \underbrace{S_0 = c}_{D}) +$
$P(S_\tau = 0, S_1 = c-1 / S_0 = c)$

$= P(S_\tau = 0 / S_1 = c+1, S_0 = c) \cdot P(S_1 = c+1 / S_0 = c)$
$+ P(\cdots\cdots) \times P(S_1 = c-1 / S_0 = c)$

$= P(S_\tau = 0 / S_0 = c+1)$ $\boxed{\text{why?}}$

$P(A \cap B / D)$

$= \dfrac{P(A \cap B \cap D)}{P(D)}$

$P(A / B \cap D) \cdot P(B / D)$

$= \dfrac{P(A \cap B \cap D)}{P(B \cap D)} \cdot \dfrac{P(B \cap D)}{P(D)}$

Markov Chain

$P(X_{n+1}=j / X_n=i, X_{n-1}=k) = P(X_{n+1}=j / X_n=i)$ if X_t is Markov ch

$P(X_{n+1}=j/X_n=i) = P(X_{N+1}=j/X_N=i)$ if MC is stationary.

$P(S_\tau=0/S_1=c+1) = \sum_{n=1}^{\infty} P(S_n=0/\tau=n / S_1=c+1)$

$\phi(c) = P(S_\tau=0 / S_0=c) = \sum_{n=0}^{\infty} P(S_n=0, \tau=n / S_0=c)$

$= \sum_{n=1}^{\infty} P(S_{n-1}=0, \tau=n-1 / S_0=c+1)$ since $\{S_n\}$ has stationarity ~~property~~ transition property

Σ is same because inner prob. is same.

= Put $m=n-1$

$= \sum_{m=0}^{\infty} P(S_m=0, \tau=m / S_0=c+1) = P(S_\tau=0 / S_0=c+1) = \phi(c+1)$

HW. Give your strategy (or utility function) to one of the game rather than the other.

Mortgage. Sub-prime crisis Stress Basel Norms

Mortgage Bag Securities ------ Insurance
CDS - Collateral Something.

Banks : Loan : Interest : Fixed Rate 9 eg: 9% pA

Variable Rate eg: 2% + market rate

Repo Rate

Or a combination : 2 yr fixed 8%, then variable : 2% over the benchmark.

HW

Which loan to take? & why?

Interest Rate - Model it.

Utility Function.

Concave.

① $U(x) = \ln x, \quad x > 0$

② $U(x) = \sqrt{x}$, or $x^{\alpha}, \quad 0 < \alpha < 1$
$x > 0$

③ $U(x) = a - b e^{-cx} \quad b, c, > 0$
$x \geq 0$

Mean-Variance Utility Function ④ $U(x) = ax - bx^2$

Measure of Risk Aversion $R_A = -\frac{U''(x)}{U'(x)}$

① $R_A = -\frac{-1/x^2}{1/x} = 1/x$

Risk Averse people do not prefer a fair game.

Fair game : $E[X] = 0$ i.e., $px_1 + (1-p)x_2 = 0$ simple game

Initial wealth $w_0 = 1$; $1 + x_1$; $1 + x_2$

$$p[U(1+x_1)] + (1-p)[U(1+x_2)] \leq [U\{p_1(1+x_1) + (1-p)(1+x_2)\}]$$

$\|$

$U(1) \geq E[U(1+x)]$

$\|$

$U(p(1+x_1) + (1-p)(1+x_2))$

③ $R_A = -\left[\dfrac{-bc^2 e^{-cx}}{bc e^{-cx}}\right] = c$

② $R_A = \dfrac{1-\alpha}{x}$

④ $R_A = \dfrac{2b}{a-2bc}$

Come up with the best strategy for playing the games or choosing the mortgage.

HW : Till next Friday.

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August 19

HW: state some assumptions & then derive a mathematical expression based upon it & based on the result, choose the strategy.

Renaissance - Stock investment Company
Charges 40% of the profits.

1903-04 Bachelier
1950-52 Markowitz
1965 Fama

1972-73 : Black-Schole-Mert
1978 : Cox-Ross-Rubinstein
1982 : Engle (ARCH)
1986 : Engle Bollerslev GARCH

Bachelier : Martingale / RW / BM
Markowitz : Portfolio - investment
Fama - Efficient Market Investment
Black-schole-- - Option pricing theory
Cox-Ross-Rubenstein - Binomial Model / CRR Model

1996 : LTCM Bankruptcy

Portfolio Investment Theory

N Risky Asset

Returns $R = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix}$

$R_t = \dfrac{P_t - P_{t-1}}{P_{t-1}}$, Daily Return

$\downarrow$

Return on t^{th} Day/unit of time.

Portfolio weights $w = \begin{bmatrix} w_1 \\ \vdots \\ \vdots \\ w_N \end{bmatrix}$

Riskless Asset

Zero Coupon Bond

Risky Asset : Variance is positive.

Assume some distribution of R,

$E[R] = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}$ $Var(R) = V_{N \times N} = ((v_{ij}))$

$v_{ij} = Cov(R_i, R_j)$, $v_{ii} > 0 \ \forall i$

Problem I: ~~Max~~ $E[w'R] \geq b$ & minimize Variance V

Expected Returns

$\min Var(w^T R)$

subject to $\sum w_i = 1$ and $E[w^T R] \geq b$

Problem II: $\max\limits_{w} E[w^T R]$

subject to $Var(w^T R) \leq c$

and $\sum\limits_i w_i = 1$

Risk Taken

I : Non-Conservative

II : Conservative

I is dual of II

Problem III $\max_{w} E[w^T R] - \frac{\gamma}{2} Var(w^T R)$, $\gamma > 0$

γ: Risk premium

Subject to $\sum w_i = 1$

$Var(w^T R) = w^T V w$

$E[w^T R] = w^T \mu$

$f(\underset{\sim}{w}, \lambda_1, \lambda_2) = \frac{1}{2} \underset{\sim}{w}^T V \underset{\sim}{w} - \lambda_1(\underset{\sim}{w}^T \underset{\sim}{\mu} - b) - \lambda_2(\underset{\sim}{w}^T \underset{\sim}{1} - 1)$

Lagrangian

$$\frac{\partial f}{\partial \underset{\sim}{w}} = V\underset{\sim}{w} - \lambda_1 \underset{\sim}{\mu} - \lambda_2 \underset{\sim}{1} \quad \cdots \quad ①$$

$$\frac{\partial f}{\partial \lambda_1} = w^T \mu + b \quad \cdots ② \qquad \Big| \qquad \frac{\partial f}{\partial \lambda_2} = -w^T 1 + 1 \quad \cdots ③$$

$$H = \begin{bmatrix} \left(\left(\frac{\partial^2 f}{\partial w_i \partial w_j}\right)\right)_{N \times N} & \left(\frac{\partial^2 f}{\partial w_i \partial \lambda_1}\right)_{N \times 1} & \left(\frac{\partial^2 f}{\partial w_i \partial \lambda_2}\right)_{N \times 1} \\ \left(\frac{\partial^2 f}{\partial \lambda_1 \partial w_i}\right)_{1 \times N} & \frac{\partial^2 f}{\partial \lambda_1 \partial \lambda_1} & \frac{\partial^2 f}{\partial \lambda_1 \partial \lambda_2} \\ \left(\frac{\partial^2 f}{\partial \lambda_2 \partial w_i}\right)_{1 \times N} & \frac{\partial^2 f}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 f}{\partial \lambda_2 \partial \lambda_2} \end{bmatrix}_{(N+2) \times (N+2)}$$

$$= \begin{bmatrix} V & -\underset{\sim}{\mu} & -\underset{\sim}{1} \\ -\underset{\sim}{\mu}' & 0 & 0 \\ -\underset{\sim}{1}' & 0 & 0 \end{bmatrix}_{(N+2) \times (N+2)}$$

Assume V is positive Definite, $\Rightarrow$ H is non-negative definite

$$\frac{\partial f}{\partial \underset{\sim}{w}} = 0 \Rightarrow V\underset{\sim}{w} = \lambda_1 \mu + \lambda_2 \underset{\sim}{1} = \underbrace{[\underset{\sim}{\mu} \;\; \underset{\sim}{1}]}_{K}{}_{(N\times 2)} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}}_{\underset{\sim}{\lambda}}{}_{(2\times 1)}$$

$$\Rightarrow w = V^{-1}(K\underset{\sim}{\lambda}) \cdots \text{(4)}$$

$$\left.\begin{aligned} \frac{\partial f}{\partial \lambda_1} = 0 &\Rightarrow \underset{\sim}{w}^T \underset{\sim}{\mu} = b \\ \frac{\partial f}{\partial \lambda_2} = 0 &\Rightarrow \underset{\sim}{w}^T \underset{\sim}{1} = 1 \end{aligned}\right\} \quad \begin{aligned} b &= \mu^T w = \mu^T V^{-1} K \underset{\sim}{\lambda} \\ 1 &= 1^T w = 1' V^{-1} K \underset{\sim}{\lambda} \end{aligned}$$

$$\begin{bmatrix} b \\ 1 \end{bmatrix} = (K^T V^{-1} K)\lambda \qquad K^T = \begin{bmatrix} \mu^T \\ 1^T \end{bmatrix}$$

V is positive definite means all eigenvalues > 0.

If V is not positive definite, $Var(a^T R) = a^T V a = 0$ for some non-zero a.

$$a^T R = C_0, \Rightarrow \sum a_i R_i = C_0 \Rightarrow a_1 R_1 = C_0 - \sum_{i\neq 1} a_i R_i$$

$$\Rightarrow R_1 = \frac{1}{a_1}\left[C_0 - \sum_{i\neq 1} a_i R_i\right]$$

V pd means $R_j \neq \frac{1}{a_j}\left[C_0 - \sum_{i\neq j} a_i R_i\right]$

K is full Rank means $\underset{\sim}{\mu}$ is not constant vector.

$$K = [\underset{\sim}{\mu} \;\; \underset{\sim}{1}]_{N\times 2} \qquad \rho(K) = 2$$

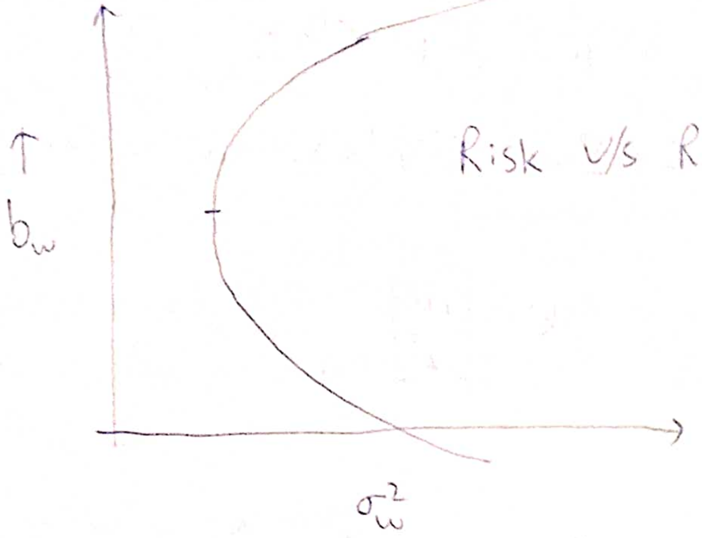
$$\lambda = (K^T V^{-1} K)^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix} \Rightarrow \underset{\sim}{w} = V^{-1} K (K^T V^{-1} K)^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix}$$

Min possible Variance

$$w^T V w = \begin{bmatrix} b \\ 1 \end{bmatrix}^T (K^T V^{-1} K)^{-1} K^T V^{-1} V V^{-1} K (K^T V^{-1} K)^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 1 \end{bmatrix}^T (K^T V^{-1} K)^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix}_{2\times 1} = Ab^2 + Bb + C$$

$A = \Big(\underbrace{(K^T V^{-1} K)^{-1}}_{L}\Big)_{11}$ $\quad B = (L)_{12} + (L)_{21}$ $\quad C = (L)_{22}$



Risk v/s Return Frontier

26 August

Portfolio optimization (continued)

N risky Assets.

$$R = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix} \qquad E[R] = \underline{\mu}$$

$Var[R] = V$ is p.d (assume)

Problem 1;

$$\min_w \frac{1}{2} w^T V w = \frac{1}{2} Var(w^T R)$$

$$\text{s.t.} \quad E[w^T R] = w^T \mu \geq b$$

$$\sum w_i = 1$$

if w_i's can be -ve, Short-selling is allowed.

if $\sum w_i \geq 0$, no shortselling allowed.

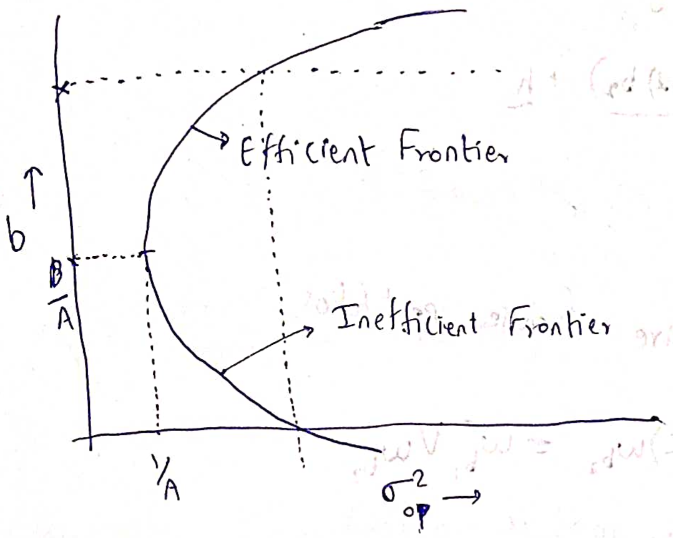
$$\hat{w}_{op} = V^{-1} K (K^T V^{-1} K)^{-1} \begin{bmatrix} b \\ 1 \end{bmatrix} \qquad K = [\mu \;\; 1]_{N \times 2}$$

$$\sigma_{op}^2 = \begin{bmatrix} b \\ 1 \end{bmatrix}^T \underbrace{(K^T V^{-1} K)^{-1}}_{L} \begin{bmatrix} b \\ 1 \end{bmatrix} = L_{11} b^2 + (L_{12} + L_{21}) b + L_{22}$$

$A = 1' V^{-1} 1$
$C = \mu^T V^{-1} \mu$
$B = \mu^T V^{-1} 1$

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \left(\begin{bmatrix} \mu' \\ 1' \end{bmatrix} V^{-1} [\mu \;\; 1] \right)^{-1} = \begin{bmatrix} \overbrace{\mu^T V^{-1} \mu}^{A} & \overbrace{\mu^T V^{-1} 1}^{B} \\ 1^T V^{-1} \mu & \mu^T \overbrace{V^{-1}}^{C} \mu \end{bmatrix} / D$$

where $D^2 = AC - B^2$ (Determinant)



$$\sigma^2_{op}\left[\frac{B}{A}\right] = \frac{1}{D}\left[A\frac{B^2}{A^2} - \frac{2BB}{A} + C\right] = \frac{1}{D}\left(C - \frac{B^2}{A}\right) = \frac{\cancel{(AC-B^2)}}{\cancel{D}A} = \frac{1}{A}$$

Exercise (Kinda)

$$f(w,\delta) = \frac{1}{2}w^T V w - \delta(w^T\mathbb{1} - 1)$$

$$\frac{\partial f}{\partial w} = V\underset{\sim}{w} - \delta\mathbb{1} = 0 \Rightarrow w = \delta V^{-1}\mathbb{1} = \frac{V^{-1}\mathbb{1}}{\mathbb{1}^T V^{-1}\mathbb{1}}$$

$$1 = \mathbb{1}^T w = \delta\mathbb{1}^T V^{-1}\mathbb{1} \qquad \delta = \frac{1}{\mathbb{1}^T V^{-1}\mathbb{1}}$$

Every thing depends only on b:

$$w_{op} = \frac{[V^{-1}\mu \quad V^{-1}\mathbb{1}]}{D}\begin{bmatrix} A & -B \\ -B & C\end{bmatrix}\begin{bmatrix} b \\ 1\end{bmatrix} = \frac{[V^{-1}\mu A - V^{-1}\mathbb{1}B \;\; -V^{-1}\mu B + V^{-1}\mathbb{1}C]}{D}\begin{bmatrix} b \\ 1\end{bmatrix}$$

$$= \frac{[AV^{-1}\mu b - V^{-1}\mathbb{1}Bb - V^{-1}\mu B 1 + V^{-1}\mathbb{1}C1]}{D}\begin{bmatrix} b \\ 1\end{bmatrix}.$$

Question: For a convex combination of w_{op}'s, will we get another w_{op}? (frontier portfolio)

$$w_{op} = \alpha(w_1^{op}) + (1-\alpha)w_2^{op}$$

Yes! it will be optimal

$$\lambda w_1^{*} + (1-\lambda) w_2^{*} = \lambda[g b_1 + h] + (1-\lambda)[g b_2 + h]$$

$$= g\underbrace{(\lambda b_1 + (1-\lambda) b_2)}_{b_\lambda} + h$$

$w_{b_1}^T R$ $\quad w_{b_2}^T R$ both are frontier portfolios.

$$\operatorname{Cov}(w_{b_1}^T R, w_{b_2}^T R) = w_{b_1}^T \operatorname{Var}(R) w_{b_2} = w_{b_1}^T V w_{b_2}$$

$$V w_{b_2} = K (K^T V^{-1} K)^{-1} \begin{bmatrix} b_2 \\ 1 \end{bmatrix}$$

$$w_{b_1}^T V w_{b_2} = \begin{bmatrix} b_1 \\ 1 \end{bmatrix}^T (K^T V^{-1} K)^{-1} (K^T V^{-1} K) (K^T V^{-1} K)^{-1} \begin{bmatrix} b_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 & 1 \end{bmatrix} (K^T V^{-1} K)^{-1} \begin{bmatrix} b_2 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 & 1 \end{bmatrix} \frac{\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}}{D} \begin{bmatrix} b_2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{D}\left[b_1 b_2 A - B[b_1 + b_2] + C\right] \quad \text{(check this)}$$

If $b_2 = B/A$, $\quad \operatorname{Cov} = \frac{1}{D}\left(C - \frac{B^2}{A}\right) = 1/A = \sigma^2_{mvp}$ $\quad$ minimum variance portfolio

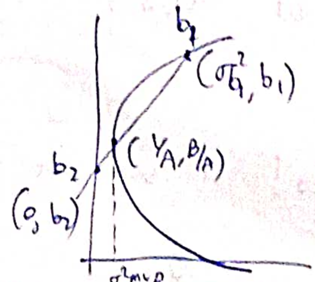
When is the Cov zero?

$$\frac{1}{D}\left[b_1 b_2 A - B(b_1 + b_2) + C\right] = 0 \Rightarrow (b_1 A - B) b_2 = -C + B b_1$$

$$\Rightarrow b_2 = \frac{B b_1 - C}{b_1 A - B}$$

whenever $\operatorname{Cov}(w_{b_1}^T R, w_{b_2}^T R)$ and $b_1 \neq B/A$

Drawing



$$Y - B/A = \frac{b_1 - \frac{B}{A}}{\sigma^2_{b_1} - 1/A}\left(x - 1/A\right)$$

$$y\big|_{x=0} = \frac{b_1 - B/A}{\sigma_{b_1}^2 - 1/A}\left(-1/A\right) + \frac{B}{A} = \frac{\frac{B}{A} - b_1}{A\sigma_{b_1}^2 - 1} + \frac{B}{A} = \dots = \frac{B\sigma_{b_1}^2 - b_1}{A\sigma_{b_1}^2 - 1}$$

$$A\sigma_{b_1}^2 - 1 = \frac{1}{D}\left[A^2 b_1^2 - 2AB b_1 + AC\right] - 1 = \frac{(Ab_1 - B)^2}{D}$$

AB: $\sigma_{b_1}^2 = \frac{1}{D}\left[A b_1^2 - 2B b_1 + C\right]$

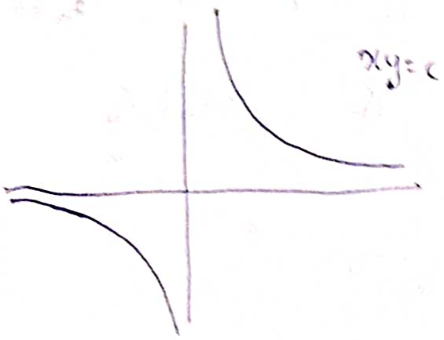
September - 2 Optim package in python

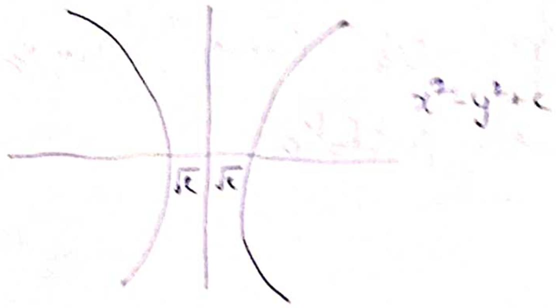
Mean v/s standard deviation

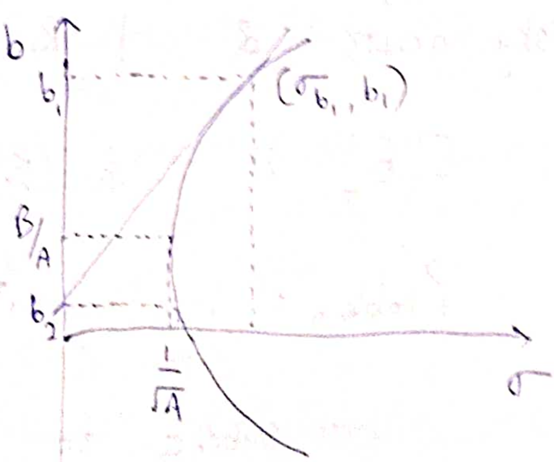
$$xy = c$$

$$\sigma_b^2 = \frac{A}{D}\left(b - \frac{B}{A}\right)^2 + \frac{1}{A} \qquad x^2 = y^2 + c \text{ - Hyperbola}$$

$$x^2 = c_0 (y - d_0)^2 + a_0$$







$2x\,dx = 2c_0(y - d_0)\,d_y$

$x\,d_x = c_0(y - d_0)\,d_y$

$\frac{x}{c_0(y - d_0)} = \frac{d_y}{d_x}$

$$y - b_1 = \left.\frac{dy}{dx}\right|_{(\sigma_{b_1}, b_1)} (x - \sigma_{b_1}) \Rightarrow y - \frac{B}{A} b_1 = \frac{\sigma_{b_1}}{\frac{A}{D}\left(b_1 - \frac{B}{A}\right)} (x - \sigma_{b_1})$$

$$\Rightarrow b_2 = y|_{x=0} = \frac{B}{A} b_1 + \frac{-\sigma_{b_1}^2 D}{A b_1 - B}$$

$$= \frac{B}{A} b_1 - \frac{\frac{A}{D}\left(b_1 - \frac{B}{A}\right)^2 + \frac{1}{A}}{\frac{A}{D}\left(b_1 - \frac{B}{A}\right)} = \frac{B}{A} b_1 - \left(b_1 - \frac{B}{A}\right) - \frac{D}{A(Ab_1 - B)}$$

$$= \frac{1}{A}\left[\frac{ABb_1 - B^2 - (AC - B^2)}{Ab_1 - B}\right]$$

Plot Unrestricted Curve b v/s σ^2, w_i can be

& plot restricted curve, $w_i \geq 0$ & identify

min variance portfolio.

<u>N+1 Assets</u> : N risky assets & 1 Riskless Asset

$$\underset{\sim}{R} = \begin{bmatrix} R_1 \\ \vdots \\ R_N \end{bmatrix} \qquad E[\underset{\sim}{R}] = \underset{\sim}{\mu} \qquad Var(\underset{\sim}{R}) = V$$

Problem : $\min_{w} \frac{1}{2} w^T V w$

$r = 6\%$ per Annum

subject to

$$r_f = \frac{r}{\text{unit of time}}$$

to make it equivalent to $\underset{\sim}{R}$

$$w^T \mu + (1 - \Sigma w_i) r_f \geq b$$

$$\Leftrightarrow w^T \mu + (1 - w^T \mathbb{1}) r_f \geq b$$

$$\Leftrightarrow w^T(\underset{\sim}{\mu} - r_f \mathbb{1}) + r_f \geq b$$

$$f(w) = \frac{1}{2} w^T V w + \lambda \left(b - w^T(\underset{\sim}{\mu} - r_f \underset{\sim}{1}) - r_f \right)$$

$$\frac{\partial f}{\partial w} = Vw - \lambda(\underset{\sim}{\mu} - r_f \underset{\sim}{1}) = 0 \Rightarrow \underset{\sim}{w} = \lambda V^{-1}(\underset{\sim}{\mu} - r_f \underset{\sim}{1}) \quad \cdots ①$$

$$\frac{\partial f}{\partial \lambda} = (b - r_f) - w^T(\underset{\sim}{\mu} - r_f \underset{\sim}{1}) = 0 \Rightarrow w^T(\underset{\sim}{\mu} - r_f \underset{\sim}{1}) = b - r_f \quad \cdots ②$$

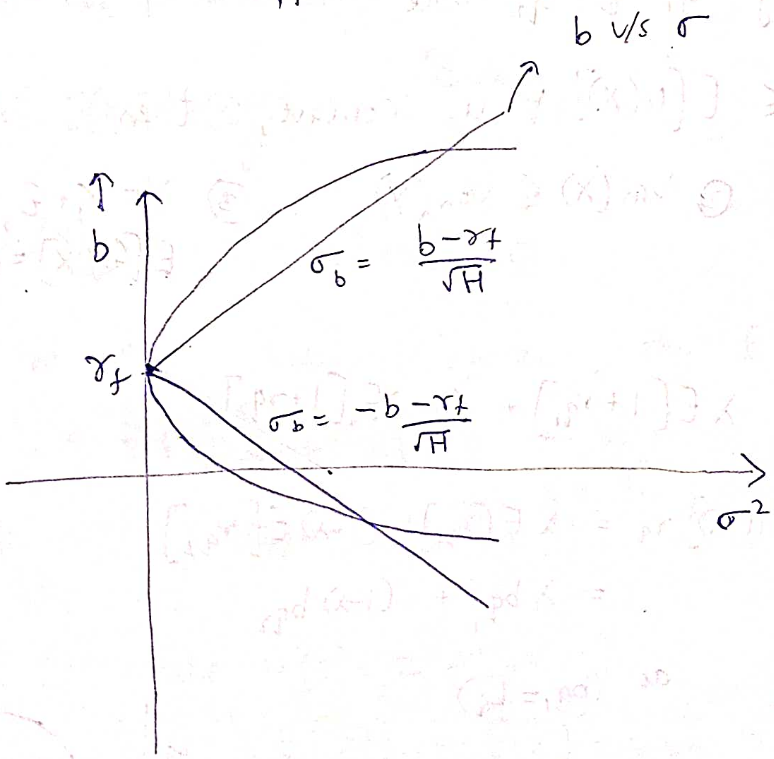
$$\Rightarrow b - r_f = (\underset{\sim}{\mu} - r_f \underset{\sim}{1})^T \underset{\sim}{w} = \lambda \underbrace{(\underset{\sim}{\mu} - r_f \underset{\sim}{1})^T V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{1})}_{H > 0}$$

$\mu \neq r_f 1$ otw. no need to invest in risky asset.

$$\Rightarrow \lambda = \frac{b - r_f}{H}$$

$$\sigma^2_{opt} = \underset{\sim}{w}_{op}^T V \underset{\sim}{w}_{op} = (\underset{\sim}{\mu} - r_f \underset{\sim}{1})^T V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{1}) \frac{(b - r_f)^2}{H^2}$$

$$\Rightarrow \sigma^2_{opt} = \frac{(b - r_f)^2}{H}$$



9 September. Mid Terms - 30% of total marks

Open book, calc. allowed (non-programmable)

HW type questions ~~expo~~ should be able to solve

Two fund Separation

A market is said to have two-fund separation property if $\exists$ two portfolios with returns r_{q_1} & r_{q_2} such that for any other portfolio with return r_q, $\exists\ \lambda$ such that

$$E\left[u(1+r_q)\right] \leq E\left[u(\lambda(1+r_{q_1}) + (1-\lambda)(1+r_{q_2}))\right] \quad \forall u$$

Concave (Risk Averse).

$X = x_1$ or x_2, It is fair if $E[X] = p x_1 + (1-p)x_2 = 0$

$$u(1) \geq \cancel{p(x_1)}\ p u(1+x_1) + (1-p) u(1+x_2)$$

$$1 = 1 + p x_1 + (1-p)x_2 = p(1+x_1) + (1-p)(1+x_2)$$

$$u(p(1+x_1) + (1-p)(1+x_2)) \geq p u(1+x_1) + (1-p) u(1+x_2) \quad \text{Concave}$$

This would imply q_1 & q_2 are frontier portfolios.

Suppose: ⓪ $E[u(Y)] \leq E[u(X)]$ $\forall u$ concave, then

① $E[X] = E[Y]$, ② $Var(X) \leq Var(Y)$ ③ $Y = X + \varepsilon$, $E[\varepsilon / X] = 0$

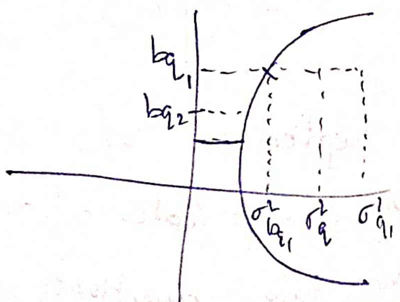
⓪ $\Leftrightarrow$ ③

By ①, $E[1+r_q] = \lambda E[1+r_{q_1}] + (1-\lambda) E[1+r_{q_2}]$

$$[\cancel{\lambda} + \cancel{(1-\lambda)}] E[r_q] = [\lambda + (1-\lambda)] r_q = \lambda E[r_{q_1}] + (1-\lambda) E[r_{q_2}]$$
$$= \lambda b_{q_1} + (1-\lambda) b_{q_2}$$

$\Rightarrow (1-\lambda) b_q = (1-\lambda) b_{q_2}$ as $(b_{q_1} = b_q)$

$\Rightarrow \lambda = 1$, as $b_{q_2} \neq b_q$



$\Rightarrow E[u(1+r_q)] \leq E[u(1+r_{q_1})]$

By ②, $Var(r_{q_1}) \leq Var(r_q)$

which is a contradiction as we assumed

$\sigma_q^2 > \sigma^2$

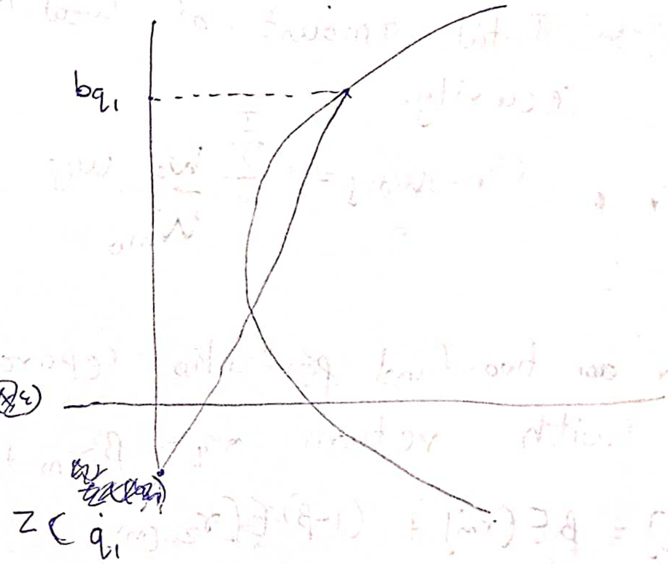
$\lambda w_{q_1}^{\nearrow FP} + (1-\lambda) w_{q_2}^{\nearrow FP}$ will always yield another frontier portfolio.

If we replace w_{q_2} by linear combination of w_{q_1} & $w_{zc(q_1)}$ i.e., $\exists\ \beta$, $w_{q_2} = \beta w_{q_1} + (1-\beta) w_{zc(q_1)}$

$$\Rightarrow w_p = \lambda (w_{q_1}) + (1-\lambda)(\beta) w_{q_1} + (1-\lambda)(1-\beta) w_{zc(q_1)}$$

$$= \underbrace{(\lambda + (1-\lambda)\beta)}_{1-\gamma} w_{q_1} + \underbrace{(1-\lambda)(1-\beta) w_{zc(q_1)}}_{\gamma}$$

w_{q_1}, $w_{zc(q_1)}$ may take up the role of w_{q_1} & w_{q_2}



$y = X + \varepsilon_q$,

$E[\varepsilon / X] = 0$

prove $\mathrm{Cov}(X, \varepsilon) = 0$.

~~$Var(Y) = Var(X) + Var(\varepsilon) + 2Cov(X,\varepsilon)$~~

$\mathrm{Cov}(X,\varepsilon) = E(X\varepsilon) - E(X)E(\varepsilon)$, $E(\varepsilon) = 0$

$$E[E[X\varepsilon / X]] = E[X\, E[\varepsilon / X]] = 0$$

Proof of ①: take $u(z) = z$, then $E(u(Y)) \leq E(u(X))$

$\Rightarrow E[Y] \leq E[X]$ ——— $E[Y] = E[X]$

take $u(z) = -z$, $\Rightarrow E[-Y] \leq E[-X] \Rightarrow E[Y] \geq E[X]$

Proof of ②: take $u(z) = z - \frac{1}{2} z^2$

$$E[u(z)] = E(z) - \tfrac{1}{2} E[z^2] = E[z] - \tfrac{1}{2}\mathrm{Var}(z) + E[z]^2$$

$$= E[z] - \tfrac{1}{2} E[z]^2 - \tfrac{1}{2}\mathrm{Var}(z) = U(E(z)) - \tfrac{1}{2}\mathrm{Var}(z)$$

$$\Rightarrow E[U[X]] = U(E(X)) - \tfrac{1}{2}\mathrm{Var}(X)$$

$$E[U(Y)] = U[E(Y)] - \tfrac{1}{2}\mathrm{Var}(Y)$$

Hence, $E(U(Y)) \leq E[U(X)] \Rightarrow \mathrm{Var}(Y) \geq \mathrm{Var}(X)$

For ③, read Huang & Litzenberger Ch 4.

<u>Market Portfolio</u> : W_0^i = initial wealth invested by ith individual

$W_{mo} = \sum_{i=1}^{I} W_0^i$ = Total wealth invested in the market initially.

$w_{ij} W_0^i$ = Amount of wealth invested by ith individual in the jth security.

I = Total no. of individuals who've invested in the market.

$\sum_{i=1}^{I} w_{ij} W_0^i$ = Total amount of wealth invested in the jth security.

$\parallel$

$w_{mj} W_{mo}$

$$w_{mj} = \sum_{i=1}^{I} \frac{W_0^i}{W_{mo}} w_{ij}$$

$\tilde{w}_m = \sum \alpha_i \tilde{w}_i$ where $\sum \alpha_i = 1$

i.e., under ~~an~~ two-fund ~~portfolio~~ separation property, for any portfolio with return $r_q = \beta r_m + (1-\beta) r_{zc(m)} + \varepsilon$

$\Rightarrow E[r_q] = \beta E[r_m] + (1-\beta) E[r_{zc(m)}]$ where

r_m = return of market portfolio

$r_{zc(m)}$ = return of portfolio which has zero covariance w.r.t. market portfolio.

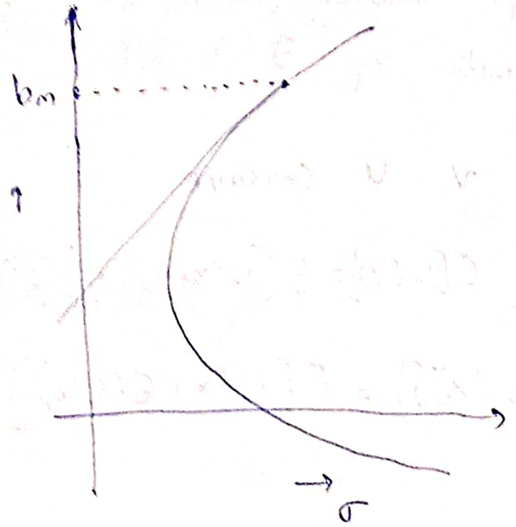
$\rightarrow E(r_q) = E[r_{zc(m)}] + \beta[E[r_m] - E[r_{zc(m)}]]$

How to find β?

$$\min_{\beta} E\left[r_q - r_{zc(m)} - \beta(r_m - r_{zc(m)})\right]^2$$

$$= 2(r_q - r_{zc(m)} - \beta(r_m - r_{zc(m)})) \cdot (r_m - r_{zc(m)}) = 0$$

$$\hat{\beta} = \frac{E[r_q] - E(r_{zc(m)})}{E[r_m] - E(r_{zc(m)})}$$



Eq^n of tangent =

$$y - b_m = \frac{\sigma_m D}{A b_m - B} (x - \sigma_m)$$

$$y\Big|_{x=0} = b_m - \frac{D\sigma_m^2}{A b_m - B}$$

$$= b_m - \frac{D\left[\frac{A}{D}\left(b_m - \frac{B}{A}\right) + \frac{1}{A}\right]}{A\left(b_m - \frac{B}{A}\right)} = b_m - \left(b_m - \frac{B}{A}\right) - \frac{D}{A^2\left(b_m - \frac{B}{A}\right)}$$

$$= \frac{B}{A} - \frac{D}{A(Ab_m - B)} = \frac{1}{A}\, \frac{ABb_m - B^2 - (AC - B^2)}{Ab_m - B}$$

$$= \frac{1}{A}\left[\frac{A(Bb_m - C)}{Ab_m - B}\right] = \frac{Bb_m - C}{Ab_m - B}.$$

Tangent: $y = b_m + \dfrac{D\sigma_m x}{A\left(b_m - \frac{B}{A}\right)} - \dfrac{\sigma_m^2}{A\left(b_m - B/A\right)}$

$$\frac{\sigma_m^2 D}{A(b_m - B/A)} = \frac{AD}{AD}\left(b_m - B/A\right) + \frac{1}{A^2}\left(b_m - B/A\right)$$

$$= \left(b_m - B/A\right) + \frac{D}{A^2}\left(\frac{1}{b_m - B/A}\right)$$

16 September.

Two Fund Separation (continued)

With two fund separation property,

$r_q = r_{zc(p)} + \beta_{qp}\left(r_p - r_{zc(p)}\right) + \varepsilon_q$ where $E[\varepsilon_q / r_{zc(p)} + \beta r_p (\ldots)] = 0$

& $\beta_{qp} = \mathrm{cov}(r_q, r_p)$ & r_p is the return corresponding to a frontier portfolio not equal to mvp

Two fund Separation property: $\exists$ two pf whose returns are r_{p1} & r_{p2}, s.t. for any portfolio, q with r_q $\exists$ λ s.t.

$$E[U(r_q)] \leq E[U(\lambda r_{p1} + (1-\lambda) r_{p2})] \quad \forall\ U \text{ Concave}$$

If $Y = X + \varepsilon$, $E[\varepsilon / X] = 0$, then $E[U(Y)] = E[U(X+\varepsilon)] = E[E[U(X+\varepsilon)$

$$= E[E[U(X+\varepsilon)/X]] \leq E[U(E(X+\varepsilon/X))] = E[U(X + E[\varepsilon/X])]$$

$$= E[U(X)]$$

$$\Rightarrow E[U(Y)] \leq E[U(X)]$$

Converse also holds.
(Rothschild & Stiglitz)

i.e., $E[U(Y)] \leq E[U(X)]$ $\forall$ U Concave.

$\Rightarrow$ $Y = X + \varepsilon$, where $E[\varepsilon/X] = 0$

For $(X,Y) \sim$ BVN, $Y = \alpha + \beta X + \varepsilon$ with $E[\varepsilon / \alpha + \beta X] = 0$ i.e., $E[\varepsilon/X] = 0$

or $X = \alpha' + \beta' X + \varepsilon$ with $E[\varepsilon'/Y] = 0$

or $E[Y/X] = \alpha + \beta X$.

Proof: $Y = \underbrace{E[Y/X]}_{f(X)} + \underbrace{Y - E[Y/X]}_{\varepsilon}$ $\qquad \beta = \dfrac{Cov(X,Y)}{Var(X)}$ $\qquad \alpha = E[Y] - \beta E[X]$

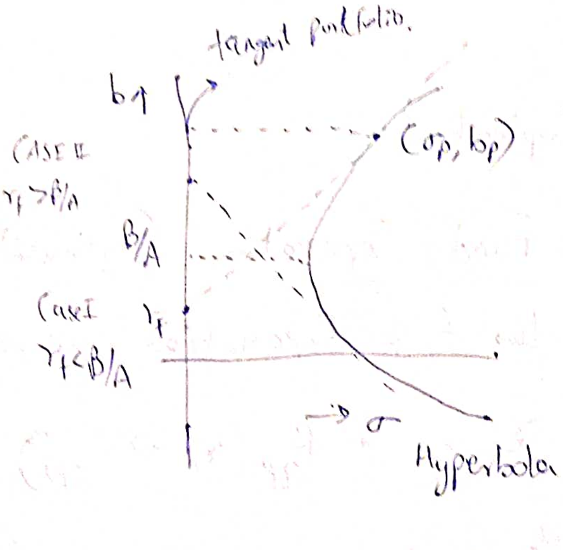
$E[\varepsilon/X] = 0$

Two fund separation for N+1 Assets.

$$r_q = r_f + \beta(r_p - r_f) + \varepsilon$$

$$E[\varepsilon / r_p] = 0 \quad \& \quad \beta = \frac{Cov(r_q, r_p)}{Var(r_p)}$$

$$\sigma_p^2 = \frac{A}{D}\left(b_p - \frac{B}{A}\right)^2 + \frac{1}{A}$$



$$Y - b_p = \frac{D\sigma_p}{(A b_p - B)}(x - \sigma_p)$$ Is this possible $x=0,\ Y = B/A$ or

$x=0,\ Y > B/A$ where $b_p > B/A$

$$y - b_p = \frac{-\sigma_p^2 . D}{A(b_p - \frac{B}{A})} \Rightarrow A = \mu^T V^{-1}\mu \overset{?}{=} 1^T V^{-1} 1 > 0$$

$$\Rightarrow y - b_p = \frac{-D\sigma^2}{A(b_p - B/A)} < 0 = \frac{-D\left[\frac{A}{D}(b_p - B/A) + \frac{1}{A}\right]}{A(b_p - B/A)} = -(b_p - \frac{B}{A}) - \frac{D}{A^2(b_p - B/A)}$$

$$\Rightarrow y - b_p + b_p - B/A = \frac{-D}{A^2(b_p - B/A)} \Rightarrow Y - B/A = \frac{-D}{A^2(b_p - B/A)} \neq 0$$

For $b_p > B/A$, $Y < B/A$ & then $Y \neq B/A$

For $b_p < B/A$, $Y > B/A$

For N+1 Assets, $r_q = r_f + \beta_{qp}(r_p - r_f) + \varepsilon$

$$E[r_q] = r_f + \beta_{qp}(E(r_p) - r_f)$$

tangent

r_p: tangent portfolio returns

$$E[r_q] = (1 - \beta_{qp})r_f + \beta_{qp} E[r_{tangent}]$$

If $0 \leq \beta_{qp} \leq 1$, $\min[E[r_{tangent}], r_f] \leq E[r_q] \leq \max[E(r_{tangent}), r_f]$

For $r_f < B/A$, $\beta_{qp} > 1 \Rightarrow E[r_q] > E[r_{tangent}]$

For $r_f > B/A$, $\beta_{qp} < 0 \Rightarrow E[r_q] > r_f$

Market Pf $\quad W_{mo} = \sum_{i=1}^{I} W_0^i$

Asset allocation

$$(1 - \underset{\sim}{w}^T \underset{\sim}{1}) r_f + \underset{\sim}{w}^T \underset{\sim}{R}$$

w_{ij} = weight of j^{th} security by the i^{th} individual

$\sum_i w_{ij} W_0^i$ = Total amount invested on the j^{th} security

equilibrium = $w_{mj} W_{m0}$ $\Rightarrow$ $w_{mj} = \sum_i w_{ij} \frac{W_0^i}{W_{m0}}$

$\underset{\sim}{w}_m = \sum_i \left(\frac{w_0^i}{W_{m0}}\right) \underset{\sim}{w}_i$ market pf. weights.

Under some assumptions one can show r_m = market corresponding to the tangent portfolio

CASE III $r_f = B/A$ $E[(1-\underset{\sim}{w}^T\underset{\sim}{1})r_f + (\underset{\sim}{w}^T\underset{\sim}{R})] \geq b$, $\min_w \underset{\sim}{w}^T V \underset{\sim}{w}$

$$\underset{\sim}{w}_{op} = \left(\frac{b_p - r_f}{H}\right) V^{-1}(\underset{\sim}{\mu} - r_f \underset{\sim}{1})$$

$$\underset{\sim}{1}^T \underset{\sim}{w}_{op} = \left(\frac{b_p - r_f}{H}\right) \underset{\sim}{1}^T V^{-1}(\underset{\sim}{\mu} - r_f\underset{\sim}{1}) = \frac{b_p - r_f}{H}\left(\underset{\sim}{1}^T V^{-1}\underset{\sim}{\mu} - r_f \underset{\sim}{1}^T V^{-1}\underset{\sim}{1}\right)$$

$$= [\quad][B - r_f A] = 0 \Rightarrow w^T 1 = 0 \ , \ \sum_i w_i = 0$$

Mid Term:

a) i) $\mu = [0.06 \quad 0.04 \quad 0.02]$ $V = \begin{bmatrix} 0.83 & 0.49 & 0.57 \\ 0.49 & 1.46 & 0.76 \\ 0.57 & 0.76 & 0.63 \end{bmatrix}$

$\min_w \frac{1}{2} w^T V w$ s.t. $\sum w_i = 1$

$$\frac{V^{-1}1}{1^T V^{-1} 1} = \frac{\begin{bmatrix} 4.13 & 1.5 & -5.543 \\ 1.5 & 2.381 & -4.23 \\ -5.543 & -4.23 & 11.715 \end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix} 0.87 \\ -0.351 \\ 1.934\end{bmatrix}}{[1 \ 1 \ 1]\begin{bmatrix} 0.087 \\ -0.351 \\ 1.934\end{bmatrix} = 1.67}$$

$$w = \frac{1}{1.67}\begin{bmatrix} 0.087 \\ -0.351 \\ 1.934\end{bmatrix} = \begin{bmatrix} 0.052 \\ -0.21 \\ 1.1581\end{bmatrix}$$

ii) Use Python, GPP

iii) Return for zero tangent portfolio.

$$b = \frac{B \times 0.05 - C}{A \times 0.05 - B}$$

$A = 1^T V^{-1} 1$ $B = \mu^T V^{-1} 1$ $C = \mu^T V^{-1} \mu$

$D = AC - B^2$

$$\mu^T V^{-1} = \begin{bmatrix} 0.06 & 0.07 & 0.02 \end{bmatrix} \begin{bmatrix} 4.13 & 1.5 & -5.573 \\ 1.5 & 2.317 & -4.238 \\ -5.573 & -4.238 & 11.715 \end{bmatrix} = \begin{bmatrix} 0.197 & 0.101 & -0.265 \end{bmatrix}$$

$C = 0.011$ $\quad A = 1.67$ $\quad K = [\mu : 1] = \begin{bmatrix} 0.03 & 1 \\ 0.12 & 1 \\ 0.07 & 1 \end{bmatrix}$

$B = -0.03$

$$W_{zc}(P) = V^{-1}K(K^TV^{-1}K)^{-1} \begin{bmatrix} b_{zc}(P) \\ 1 \end{bmatrix}$$

Q2.b) Suppose from the market with 3 risky assets & one risk free asset w/ return ~~10%~~ 8%. You would like to have a portfolio whose expected return is 10%. How would you invest s.t. var is min.

1) Allowing Shortsale

$\underset{\sim}{w} = \lambda V^{-1}(\mu - r_f \underset{\sim}{1})$ $\quad \lambda = \frac{b - r_f}{H}$ $\quad H = (\underset{\sim}{\mu} - r_f \underset{\sim}{1})^T V^{-1} (\underset{\sim}{\mu} - r_f \underset{\sim}{1})$

$$H = \begin{bmatrix} -0.05 & 0.04 & -0.01 \end{bmatrix} \begin{bmatrix} 6.393 & -1.127 & -1.49 \\ -1.127 & 1.445 & -0.592 \\ -1.49 & -0.592 & 1.92 \end{bmatrix} \begin{bmatrix} -0.05 \\ 0.04 \\ -0.01 \end{bmatrix}$$

$$= \begin{bmatrix} -0.34981 & 0.12 & 0.03162 \end{bmatrix} \begin{bmatrix} -0.05 \\ 0.04 \\ -0.01 \end{bmatrix} = 0.0062$$

$$\lambda = \frac{0.1 - 0.08}{0.0062} = 3.13$$

So, $\underset{\sim}{w} = 3.23 \begin{bmatrix} 6.393 & - & - \\ & - & - \end{bmatrix} \begin{bmatrix} -0.05 \\ 0.04 \\ \vdots \end{bmatrix} \cdots$

30/9/23

$X_1, \ldots, X_n$ i.i.d. $\qquad \hat{\theta} = \varphi(X_1, \ldots, X_n)$

$\qquad \hat{\theta}_{-i} = \varphi(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$

Jackknife

With R. Jagannath & T. Ma, 2009 Management Science,

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} = \frac{P_t}{P_{t-1}} - 1 \Rightarrow \ln(1+R_t) = \ln(P_t) - \ln(P_{t-1})$$

$|R_t| \approx 0$

$\approx R_t$

$\ln(1+x) \approx x$

Thus return vector may be stationary being (approximately) diff. of log prices.

$$\underset{\sim}{\varphi} = \begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_N \end{bmatrix} \to \text{Strategy}$$

Let $\underset{\sim}{S}$ be the price vector

$$\underset{\sim}{S} = \begin{bmatrix} S_0 \\ \vdots \\ S_N \end{bmatrix}$$

Initial investment value of the portfolio: $V_0 = \underset{\sim}{\varphi_0}' \underset{\sim}{S_0}$

$V_n = \underset{\sim}{\varphi_n}^T \underset{\sim}{S_n}$ = value of the portfolio at the end of time n

$$\varphi = \begin{bmatrix} 1.5 \\ \vdots \\ 2.6 \\ \vdots \\ 7 \end{bmatrix}$$

How many units of the stock to buy?

decimal means asset is liquid (can be bought in parts).

Self Financing Strategy.

$[\underset{\sim}{\varphi_0} \ldots \underset{\sim}{\varphi_n}]$ is called a self-financing strategy if

$\underset{\sim}{\varphi_{k-1}}' S_{k-1} = \underset{\sim}{\varphi_k}' S_{k-1} + C_{k-1}$ where C_{k-1} is the

~~assumption~~ consumption out at the end of time unit $k-1$.

$\underset{\sim}{\phi_k}$ is decided based on the data available upto time $k-1$. That's why this is a predictable process.

C_{k-1} is the consumption taken out at the end of time unit $k-1$.

Then, $V_n = \underset{\sim}{\phi_n} \cdot \underset{\sim}{S_n} = \phi_0 \cdot S_0 + \sum_{k=1}^{n} \left[\underset{\sim}{\phi_k} \cdot \underset{\sim}{S_k} - \underset{\sim}{\phi_{k-1}} \, \underset{\sim}{S_{k-1}} \right]$

i.e.,

$a_n = (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_1 - a_0) + a_0$

Using Self-financing property, $\underset{\sim}{\phi_{k-1}} \cdot \underset{\sim}{S_{k-1}} = \underset{\sim}{\phi_k} \cdot \underset{\sim}{S_{k-1}} - C_{k-1}$

we get, $V_n = V_0 + \sum_{k=1}^{n} \phi_k \cdot (\underset{\sim}{S_k} - \underset{\sim}{S_{k-1}}) - \sum_{k=1}^{n} C_{k-1}$

When no money is taken out, then the term $\sum_{k=1}^{n} C_{k-1}$ is zero.

It's called Arbitrage pricing but actually No Arbitrage pricing. — GKB

Info upto time $k-1$

$E[S_k / \mathcal{F}_{k-1}] = \underset{\sim}{S_{k-1}}$ then $V_n = V_0$

Continuous Compounding

$r\%$ at end of time 't', Initial amount $= P$

If paid n times, $P \cdot \left(1 + \frac{r}{n}\right)^{nt}$

If paid half yearly, $t = 1$ (year) $P \cdot \left(1 + \frac{r}{2}\right)^{2}$

Zero-Coupon Bond

Maturity : T

risk-free interest rate.

$~~P_t = P_T~~$ $P_T = P_t \left(e^{r(T-t)}\right)$

Maturity Value

HW: ₹ 1000 Zero Coupon Bond. Find the interest rates, $P_0, P_1, P_{1.5}, P_{2.5}, P_3, P_4, P_{4.5}$ for five years bond.

Interest Rates are:

0	1	2	3	4
6	6.5	7	6.5	7

Notation: $\tilde{S}_k = (1+r_f)^{-k} S$

This is F, not H

Assumption : $E[\tilde{S}_1 / \mathcal{H}_0] = \tilde{S}_0 = S_0$

i.e., $E[S_1 / \mathcal{H}_0] = r S_0$

Also, $E[S_k / \mathcal{H}_{k-1}] = r_f S_{k-1}$ Remains same / Assumption.

Then, $E[\tilde{S}_k / \mathcal{H}_{k-1}] = \tilde{S}_{k-1}$

Because, $E[r_f^{-k} \cdot S_k / \mathcal{H}_{k-1}] = r_f^{-k} E[S_k / \mathcal{H}_{k-1}] = r_f^{-k} \cdot r_f S_{k-1}$

$= r_f^{-(k-1)} S_{k-1} = \tilde{S}_{k-1}$

→ If this holds, then $\{\tilde{S}_k\}$ is called a Martingale w.r.t. $\{\mathcal{H}_k\}$

Martingale Pricing

Let $\{X_n\}$ be a sequence of random variables defined on a probability space $(\Omega, \underset{\text{events}}{\mathcal{F}}, P)$. Let $\{\mathcal{F}_n\}$ be a sequence of information set or sets of events up to time n. That means $\mathcal{F}_{n-1} \subseteq \mathcal{F}_n \ \forall n$.

Also, knowledge of $\mathcal{F}_n$ determines everything about $X_1, \dots, X_n$.

Assume further $E|X_n| < \infty$.

Then $\{X_n\}$ is said to be a martingale w.r.t $\{\mathcal{F}_n\}$

if $E[X_{n+1}/\mathcal{F}_n] = X_n \ \forall n$.

if $E[X_{n+1}/\mathcal{F}_n] \geq X_n$, Submartingale

$\leq$, Supermartingale.

Examples

Let $\{Y_i\}$ are iid r.v. s.t. $P(Y_i = +1) = p$ & $P(Y_i = -1) = 1-p = q$.

Let $X_n = X_0 + Y_1 + \dots + Y_n$, s.t. X_0 is independent of $\{Y_i\}$'s

Then $\{X_n\}$ is a martingale w.r.t. $\{\mathcal{F}_n = \sigma(Y_1, \dots, Y_n)\}$ provided $p = q$

$\sigma(Y_1, \dots, Y_n) =$ ~~σ~~ σ field, set of all possible events generated by $(Y_1, \dots, Y_n)$.

① $Z_n = (q/p)^{X_n}$. Then $\{Z_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

$E[Z_{n+1}/\mathcal{F}_n] = E[Z_n (q/p)^{Y_{n+1}} / \mathcal{F}_n]$ as $Z_{n+1} = (q/p)^{X_n + Y_{n+1}}$

$= Z_n (q/p)^{Y_{n+1}}$

$$E[X_{n+1}/\mathcal{F}_n] = E[X_n + Y_{n+1}/\mathcal{F}_n] = X_n + E[\cancel{X_n}\, Y_{n+1}/\mathcal{F}_n] = X_n + E[Y_{n+1}]$$

$$= X_n + p - q \qquad = X_n \quad \text{if} \quad p = q.$$

$$E\left[Z_n \left(\tfrac{q}{p}\right)^{Y_{n+1}} / \mathcal{F}_n\right] = Z_n E\left[\left(\tfrac{q}{p}\right)^{Y_{n+1}} / \mathcal{F}_n\right] = Z_n E\left[\left(\tfrac{q}{p}\right)^{Y_{n+1}}\right]$$

$$= Z_n\left[\left(\tfrac{q}{p}\right)^{1} \times p + \left(\tfrac{q}{p}\right)^{-1} q\right] = Z_n [q + p] = Z_n$$

τ = 1st time to reach a or b starting from $a < c < b$.

Suppose $E[Z_\tau / \mathcal{F}_0] = Z_0$

$$\Rightarrow \left(\tfrac{q}{p}\right)^a \underbrace{P(X_\tau = a)}_{P_r} + \left(\tfrac{q}{p}\right)^b \underbrace{P(X_\tau = b)}_{1-P_r} = \left(\tfrac{q}{p}\right)^c$$

$$\Rightarrow P_r\left[\left(\tfrac{q}{p}\right)^a - \left(\tfrac{q}{p}\right)^b\right] = \left[\tfrac{q}{p}\right]^c - \left[\tfrac{q}{p}\right]^b$$

$$\Rightarrow P_r = \frac{\left(\tfrac{q}{p}\right)^c - \left(\tfrac{q}{p}\right)^b}{\left(\tfrac{q}{p}\right)^a - \left(\tfrac{q}{p}\right)^b} = \frac{\left(\tfrac{q}{p}\right)^b - \left(\tfrac{q}{p}\right)^c}{\left(\tfrac{q}{p}\right)^b - \left(\tfrac{q}{p}\right)^a}, \quad p \neq q.$$

H.W.

3. $W_n = X_n^2 - n$

Show $\{W_n\}$ is a martingale wrt $\{\mathcal{F}_n\}$ if $p=q$.

4. Let $U_n = X_n - (p-q)n$. S.T. $\{U_n\}$ is a martingale w.r.t. $\{\mathcal{F}_n\}$

5. $\tilde{V}_n = r_f^{n} V_n$ is a martingale w.r.t. $\mathcal{F}_n = \sigma(\tilde{S}_0, \dots, \tilde{S}_n)$

Book : Bhattacharya & Waymire Stochastic Processes and their Applications.

Ch 1 Sections of Random Walk & Section 13

Theorem : Doob's optional stopping Theorem.

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7 October

$V_m = \underset{\sim}{\varphi_m} \cdot \underset{\sim}{S_m} = V_0 + \sum_{k=1}^{n} \varphi_k (S_k - S_{k-1})$ wherever $\{\varphi_k\}$ are self-financing strategy $\underset{\sim}{\varphi_{k-1}} \underset{\sim}{S_{k-1}} = \underset{\sim}{\varphi_k} \underset{\sim}{S_{k-1}} - \underset{\substack{\| \\ 0}}{C_{k-1}}$ for no consumption.

Assume, $E[\underset{\sim}{\tilde{S}_k} / \mathcal{F}_{k-1}] = \underset{\sim}{\tilde{S}_{k-1}} \quad \forall k$ where,

$\underset{\sim}{\tilde{S}_k} = (1+r_f)^{-k} \underset{\sim}{S_k}$ $\Rightarrow$ $\{S_k\}$ is a martingale w.r.t $\{\mathcal{F}_k\}$.

$\Rightarrow \{\tilde{V}_k = \underset{\sim}{\varphi_k} \cdot \underset{\sim}{\tilde{S}_k}\}$ is a martingale w.r.t. $\{\mathcal{F}_k\}$

Check ① if $\{\tilde{V}_k\}$'s have finite expectation which should follow from finite expectation of $\{\underset{\sim}{\tilde{S}_k}\}$ and $\{\varphi_k\}$

② $E[\tilde{V}_{k+1} / \mathcal{F}_k] = \tilde{V}_k$

Note: $\tilde{V}_n = \varphi_n \cdot \tilde{S}_n = \tilde{V}_0 + \sum_{k=1}^{n} (\varphi_k \tilde{S}_k - \varphi_{k-1} \tilde{S}_{k-1})$ & $\varphi_{k-1} S_{k-1} = \varphi_k S_{k-1}$

$\Rightarrow (1+r_f)^{-(k-1)} \varphi_{k-1} S_{k-1} = (1+r_f)^{-(k-1)} \varphi_k S_{k-1}$

i.e., $\varphi_{k-1} \tilde{S}_{k-1} = \varphi_k \tilde{S}_{k-1}$

$\Rightarrow \tilde{V}_n = V_0 + \sum_{k=1}^{n} \varphi_k (\tilde{S}_k - \tilde{S}_{k-1})$

Then $\tilde{V}_{k+1} = \tilde{V}_0 + \sum_{j=1}^{k+1} \varphi_j (\tilde{S}_j - \tilde{S}_{j-1}) = \tilde{V}_k + \varphi_{k+1}(\tilde{S}_{k+1} - \tilde{S}_k)$

Note: $\tilde{V}_n = (1+r_f)^{-n} V_n$ $\qquad \mathcal{F}_k = F_k$

$\Rightarrow E[\tilde{V}_{k+1} / \mathcal{F}_k] = E[\tilde{V}_k / \mathcal{F}_k] + E[\varphi_{k+1}(\tilde{S}_{k+1} - \tilde{S}_k) / \mathcal{F}_k]$

$= \tilde{V}_k + \varphi_{k+1} \cdot \underbrace{E[\tilde{S}_{k+1} - \tilde{S}_k / \mathcal{F}_k]} = \tilde{V}_k$

Constant when given $\mathcal{F}_k$

Self financing Strategy hence predictable using $\mathcal{F}_k$

This is 0.

$E[\tilde{S}_{k+1} / \mathcal{F}_k] - E[\tilde{S}_k / \mathcal{F}_k]$

$= \tilde{S}_k - \tilde{S}_k = 0$

H.W.

Find $E[T / X_0 = c]$ where T is the first time to reach 'a' or 'b'. $(a < c < b)$

Also show that martingale property

① $\{X_n^2 - n\}$ if $p = q$

②. $\{X_n - (p-q)n\}$

Doob's optional stopping Theorem.

Let $\{X_n\}$ be a martingale and τ be a stopping time w.r.t. $\{\mathcal{F}_n\}$. Then $E[X_\tau / \mathcal{F}_0] = X_0$

A stopping time τ is a non-negative integer valued random variable w.r.t. $\{\mathcal{F}_n\}$ if $\{\tau = k\} \in \mathcal{F}_k$

$\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$

eg: First time temp reaches 40°C

Second time Stock price reaches 1000.

Not a stopping time : Last time temp. reaches 50°C, we need information till ∞, info cannot be known by knowing info upto some finite time $\mathcal{F}_k$.

$E[X_n / \mathcal{F}_0] = X_0 \quad \forall n$

$E[X_n / \mathcal{F}_{n-1}] = X_{n-1} \Rightarrow E[E[X_n / \mathcal{F}_{n-1}] / \mathcal{F}_{n-2}] = E[X_{n-1} / \mathcal{F}_{n-2}] = X_{n-2}$

Taking $E[\cdot / \mathcal{F}_{n-2}]$ both sides.

$\Rightarrow E[X_n / \mathcal{F}_{n-2}] = X_{n-2}$

$E[Y] = E[Y / \mathcal{F}_{trivial}] \qquad \mathcal{F}_{trivial} = \{\phi, \Omega\}$

trivial sigma field.

In martingale process, $E[X_n / \mathcal{F}_0] = X_0$ is satisfied.

Doob's th. has τ random. Expected X for given random time.

→ Provided

1) $P(\tau < \infty) = 1$

2) $E[|X_\tau|] < \infty$

3) $E[X_m I_{\{\tau > m\}}] \longrightarrow 0$ as $m \to \infty$

$E[I_{\{\tau > m\}}] = P(\tau > m) \longrightarrow 0$

τ: First time to reach a or b starting at $a < c < b$.

$a \leq X_m \leq b$ as $\{\tau \geq m\}$

$\Rightarrow |X_m| \leq \max(|a|, |b|)$

Arbitrage

A market is said to have arbitrage if $\exists$ a self-finan. strategy φ_1 such that $\varphi_1 S_0 = 0$ and $V_1 = \varphi_1 \underset{\sim}{S_1} \geq 0$ $\forall$ scenario i.e., for all possible values of $\underset{\sim}{S_1}$.

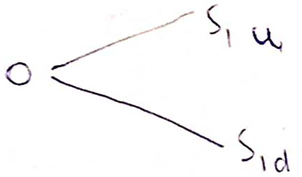
and > 0 for some values of $\underset{\sim}{S_1}$, i.e., for some scenarios.

$$\boxed{0 = \varphi_0 \cdot S_0 = \varphi_1 S_0}$$

or $\varphi_1 S_0 < 0$ for some

& $\underset{\sim}{\varphi_1} \underset{\sim}{S_1} \geq 0$ $\forall$ scenarios.

Example :



Risky Asset

Risk-free Asset

$1 \longrightarrow 1 + r_f$

$$\begin{bmatrix} \varphi_{11}, & \varphi_{12} \end{bmatrix} \begin{bmatrix} 1 \\ S_0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1+r_f & 1+r_f \\ S_1u & S_1d \end{bmatrix}$$

$\varphi_{11} \to x$, $\varphi_{12} \to y$, $S_0 \to a$, $S_1u \to u$, $S_1d \to d$

$$[x \ y] \begin{bmatrix} 1 \\ a \end{bmatrix} = 0 \Rightarrow x + ay = 0 \Rightarrow y = -x/a$$

$$[x \ y] \begin{bmatrix} 1+r_f & 1+r_f \\ u & d \end{bmatrix} \geq 0 \Rightarrow (1+r_f)x + u\left(-\frac{x}{a}\right) \geq 0 \quad \text{--- (1)}$$

$$(1+r_f)x + d\left(-\frac{x}{a}\right) \geq 0 \quad \text{--- (2)}$$

$$(1) \Rightarrow (1+r_f)x \geq \frac{u}{a}x \Rightarrow (1+r_f) \geq \frac{u}{a} \quad \boxed{\text{if } x > 0}$$

$$(2) \Rightarrow (1+r_f) \geq d/a \Rightarrow (1+r_f) > \frac{d}{a} \quad \text{as} \quad u/a > \frac{d}{a}$$

arbitrage opportunities
since Risk-free asset performs uniformly better, than all possible scenarios of risky asset.

If $x < 0$,

$\frac{d}{a}$: per unit of risky asset.

Same for u/a.

① $\Rightarrow$ $(1 + r_f) \leq \frac{u}{a}$

② $\Rightarrow$ $1 + r_f \leq \frac{d}{a} < \frac{u}{a}$

$\Rightarrow$ arbitrage opportunities.

Since risk-free asset is underperforming than risky assets uniformly (i.e., for all scenarios).

Definition : Market is said to have no arbitrage if there does not exist any self financing strategy φ, that allows arbitrage opportunities.

Theorem: First fundamental theorem of Finance.

Market has no arbitrage iff $\exists$ a strictly positive probability vector $\underset{\sim}{p}$ such that $E_{\underset{\sim}{p}}[\tilde{S}_1 \mid \mathcal{F}_0] = \underset{\sim}{\tilde{S}_0} = S_0$

HW: Assume no arbitrage in binomial Model example and find a $\underset{\sim}{p}$ vector that assures $E_p[\tilde{S}_1 \mid \mathcal{F}_0] = \tilde{S}_0$

Due : Next Friday (13th oct)

Turn ↷

13 October.

Project - Send a write-up by this weekend.

Topics - Collect Finantial Data & use any ML techniques or statistical techniques to do various things (any one) like risk assessment, price prediction, comparing monthly & weekly moving averages.

- Can build a factor model (sector-wise) analysis. Check whether news has any impact on the stocks (factors).
- Currency exchange rates
- Loan defaulter prediction.
- Predict the other markets, depending on other markets.

eg: London → USA → Japan   Europe Market, etc.

Any ~~country~~ company affected in USA might have same effect in Tokyo market.

- Political Business Cycle - Country doing good → price goes up as demand goes up

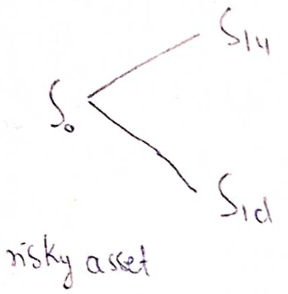
↓

Central bank raises interest to combat inflation → Big Projects shut off as they can't afford loans → Layoffs in companies → Growth reduced

↓

inflation goes down
recession prominent → interest rates are lowered.

Example :



$1 \longrightarrow 1+r_f$

risk-free asset.

$$\begin{bmatrix} 1+r_f & 1+r_f \\ S_{1u} & S_{1d} \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f)$$

$E_0 [\tilde{S}_1 / \mathcal{F}_0] = \tilde{S}_0$ $\qquad$ $E_0 [S_1 / \mathcal{F}_0] = S_0 (1+r_f)$

$P_u + P_d = 1$

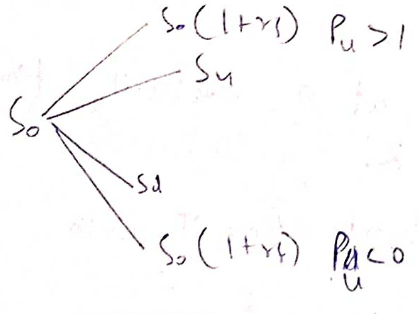
$S_u P_u + S_d (1 - P_u) = S_0 (1+r_f)$

$\Rightarrow P_u (S_u - S_d) = S_0 (1+r_f) - S_d$

$P_u = \dfrac{S_0(1+r_f) - S_d}{S_u - S_d}$

$\Rightarrow P_d = \dfrac{S_u - S_0(1+r_f)}{S_u - S_d}$

If $P_u > 1$ or $P_u < 0$ then there are arbitrage opportunities.



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eg : $S_0 = 50$; $S_{1u} = 55$; $S_{1d} = 45$; $r_f = 12\%$; $S_0(1+r_f) = 56$

$(a, -1)\begin{pmatrix} 1 \\ S_0 \end{pmatrix} = 0 \Rightarrow a - 50 = 0 \Rightarrow a = 50$

Short sell risky asset, say 1 unit (-1)

so, we get $S_0(1+r_f) = 56$ $\qquad$ $56 - 55 = 1 > 0$

$56 - 45 = 11 > 0$

eg: $S_{1u} = 55$ $S_{1d} = 53$ $r_f = 6\%$ $S_0(1+r_f) = 53$

Borrow from bank (sell risk-free asset).

$(a, 1)\begin{bmatrix}1\\ S_0\end{bmatrix} = 0$ $a + 50 = 0 \Rightarrow a = -50$

$55 - 53 = 2 > 0$
$53 - 53 = 0$

$\underset{\sim}{\varphi} \cdot \underset{\sim}{S_0} = 0$ $\underset{\sim}{\varphi} \cdot \underset{\sim}{S_1} \geq 0$ with at least one scenario to have positive gain.

Sterling 1990-1992 Pound, USD, Portugal, George Soros

Replicable Strategies.

$\underset{\sim}{\varphi_1} \cdot \underset{\sim}{S_1} = f_1$ $\{\varphi\}$ is called replicable strategy

<u>Option</u>: It is a contract which gives the holder a right to buy or sell

Call option (European): Gives right to buy, at a maturity time T, with prefixed price K.

Put option: Gives right to sell, at a maturity time T. with prefixed price K.

Eg: Airlines Company holding a call option whose giver (writer) is an oil company.

eg: 1000 barrels @ 100 \$ each for T = 3 months.

The contracts have a price.

European option: Only used at end of maturity American: Can be used anytime before maturity.

f_0: price of the contract at time $t=0$.

f_t: price of the contract at time $t \leq T$.

S_T: Value of the asset on which the contract was done at time T.

$$f_T = (S_T - K)^+ \qquad x^+ = \begin{cases} x & \text{if } x<0 \\ 0 & \text{otw} \end{cases}$$

Why are contracts priced: If price $=0$, there is a positive probability to make a loss to the writer.

No arbitrage $\Leftrightarrow \exists\ \mathbb{P}$ s.t. $E_{\mathbb{P}}(S_1 / \mathcal{F}_0) = \underset{\sim}{S_0}(1+r_f)$

$$\underset{\sim}{S_0} = \begin{bmatrix} 1 \\ S_1 \\ \vdots \\ S_N \end{bmatrix} \qquad N=1, \quad \begin{bmatrix} 1 \\ S_{01} \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1+r_f & 1+r_f \\ S_u & S_d \end{bmatrix}}_{D} \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f)$$

All cases of S_1

$$E_{\mathbb{P}}[f_1 / \mathcal{F}_0] = \begin{bmatrix} f_u & f_d \end{bmatrix} \begin{bmatrix} p_u \\ p_d \end{bmatrix} = \varphi D p = \varphi \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f) \overset{?}{=} f_0(1+r_f)$$

Yes

If $[\varphi_{01}, \varphi_{11}] \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f) \lessgtr f_0(1+r_f)$ then arbitrage opportunities.

So, $E_{\mathbb{P}}[f_1 / \mathcal{F}_0] = f_0(1+r_f) \Rightarrow f_0 = \frac{1}{1+r_f} E_{\mathbb{P}}[f_1 / \mathcal{F}_0]$.

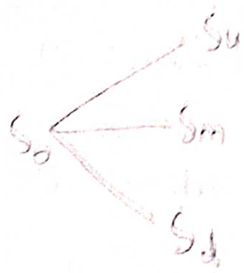
In a ~~no arbitrage~~ market if every claim of a derivative (or contingent) is replicable then it is called complete.

A no arbitrage market is complete iff the strictly positive martingale measure $\mathbb{P}$ is unique.

Second Fundamental Theorem of Finance

Oct 18.

Trinomial model



$$\begin{bmatrix} 1+r_f & 1+r_f & 1+r_f \\ S_u & S_m & S_d \end{bmatrix} \begin{bmatrix} P_u \\ P_m \\ P_d \end{bmatrix} = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f)$$

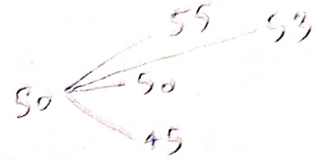
$$E\left[\underset{\sim}{S_1} \mid \mathcal{F}_0\right] = \underset{\sim}{S_0}(1+r_f)$$

$S_0 = 50$ $\quad r_f = 6\%$

$S_u = 55$

$S_m = 50$

$S_d = 45$



$S_0(1+r_f) = 53$

Assume $P_d = 0$

Soln 1

$$P_u = \frac{S_0(1+r_f) - S_m}{S_u - S_m}$$

$$P_m = \frac{S_u - S_0(1+r_f)}{S_u - S_m}$$

Denote this Solution as $\underset{\sim}{p}^1$ & the 2nd one as $\underset{\sim}{p}^2$

Assume $P_m = 0$

$$P_u = \frac{S_0(1+r_f) - S_d}{S_u - S_d}, \quad P_d = \frac{S_u - S_0(1+r_f)}{S_u - S_d}$$

Define $\underset{\sim}{p}^\alpha = \alpha p^1 + (1-\alpha) p^2 > 0$ for $\alpha \in (0,1)$ infinitely many solutions.

$$p^1 = \begin{bmatrix} \frac{53-50}{55-50} = 3/5 \\ 2/5 \\ 0 \end{bmatrix}, \quad p^2 = \begin{bmatrix} \frac{53-45}{55-45} = \frac{8}{10} \\ 0 \\ 2/10 = 1/5 \end{bmatrix}$$

for $\alpha = \frac{1}{3}$, $\underset{\sim}{p}^{\alpha} = \begin{bmatrix} \frac{1}{3}\times\frac{3}{5} \\ \frac{1}{3}\times\frac{2}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3}\times\frac{4}{5} \\ 0 \\ \frac{2}{3}\times\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{11}{15} \\ \frac{2}{15} \\ \frac{2}{15} \end{bmatrix}$

for $\alpha = \frac{1}{2}$, $p^{(1/2)} = \left(\begin{bmatrix} 3/5 \\ 2/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/5 \\ 0 \\ 1/5 \end{bmatrix} \right) \frac{1}{2} = \begin{bmatrix} 7/10 \\ 2/10 \\ 1/10 \end{bmatrix}$

$$E_{p(1/2)} \left[\underset{\sim}{S_1} / \mathcal{F}_0 \right] = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r_f)$$

$$LHS = \begin{bmatrix} 1.1 r_f \\ 55\times\frac{11}{15} + 50\times\frac{2}{15} + 45\times\frac{2}{15} \end{bmatrix} = \begin{bmatrix} 1.06 \\ 53 \end{bmatrix}$$

Call option:

$f_u = (S_u - K) = 3$

$f_m = (50-52)^+ = 0$

$f_d = (45-52)^+ = 0$

$K = 52$

(can take anything) K value

$$E_{p(1/3)} \left[f_1 / \mathcal{F}_0 \right] \overset{?}{=} f_0 (1+r_f) \rightarrow 3\times\frac{11}{15} + 0\times\frac{2}{15} + 0\times\frac{3}{15} = \frac{11}{5} = 2.2$$

$$E_{p(1/2)} \left[f_1 / \mathcal{F}_0 \right] = 3 \times \begin{bmatrix} 7/10 \\ 2/10 \\ 1/10 \end{bmatrix} = \frac{21}{10} = 2.1 \neq 2.2$$

Cannot find f_0 uniquely in this case

$$\underset{\sim}{\Phi_1}^T \underset{\sim}{S_1} = f_1 \quad \text{i.e.;} \quad \begin{bmatrix} \Phi_{01} & \Phi_{11} \end{bmatrix} \begin{bmatrix} 1+r_f & 1+r_f & 1+r_f \\ S_u & S_m & S_d \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix}$$

For $A\underset{\sim}{x}=b$ to check consistency, $Rank[A:b] = Rank(A)$

For $x^T A = b$, to check consistency, $Rank\begin{bmatrix} A \\ \hdashline b \end{bmatrix} = Rank(A)$

In this case, $Rank\begin{bmatrix} A \\ \hdashline b \end{bmatrix} = 3 \neq 2$

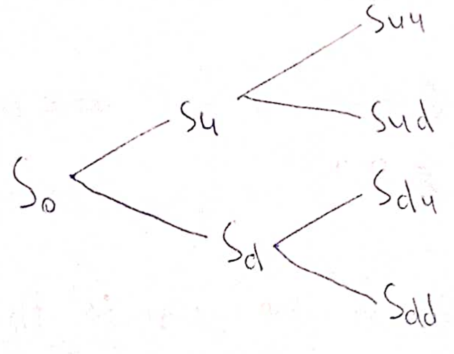
$E_{\mathbb{Q}}[\underset{\sim}{S_1}/\mathcal{F}_0] = S_0(1+r_f)$, That may not give $E_{\mathbb{Q}}[f_1/\mathcal{F}_0] = f_0/$

RHS may not be unique for different $\mathbb{Q}$ that solves $E_{\mathbb{Q}}[\underset{\sim}{S_1}/\mathcal{F}_0] = S_0($

Incompleteness of the market with M scenarios with N+1 assets.

$\mathbb{Q}: \underset{\sim}{S_1} = f$ i.e., $\underbrace{\underset{\sim}{x}^T}_{1\times(N+1)} \underbrace{A}_{(N+1)\times M} = \underbrace{\underset{\sim}{b}}_{1\times M}$ and $Rank\begin{bmatrix} A \\ \hdashline b \end{bmatrix} \neq Rank[A]$

CRR Model



$S_u = S_0 \cdot u$ $\quad S_d = S_0 \cdot d$

$S_{ud} = S_0 \cdot u \cdot d = S_u \cdot d = S_d \cdot u = S_{du}$

$S_{dd} = S_d \cdot d = S_0 d^2$

$S_{uu} = S_0 \cdot u^2$

$S_n = S_0 u^k d^{n-k}$, $k = 0, \dots, n$

$\ln S_n = \ln S_0 + Y_1 + \dots + Y_n$, where $Y_i = \ln u$ or $\ln d$

Y_is are iid r.v.s

$E[Y_i] = P_u \cdot \ln u + P_d \cdot \ln d$

$$P_u = \frac{S_0(1+r_f) - S_d}{S_u - S_d} = \frac{S_0[1+r_f-d]}{S_0[u-d]} = \frac{1+r_f-d}{u-d}$$

$$\Rightarrow P_d = \frac{u-(1+r_f)}{u-d}$$

$$P_{uu} = \frac{S_u(1+r_f) - S_{ud}}{S_{uu} - S_{ud}} = \frac{S_u(1+r_f-d)}{S_u(u-d)} = \frac{1+r_f-d}{u-d} = P_u \equiv p$$

$P_d = 1-p.$

$E[Y_i] = P_u \ln u + P_d \ln d = p\ln u + (1-p)\ln d$

$= (2p-1)\ln u$

Assumption

$d = 1/u$

$d = 1/u$

Check $Var(Y_i) = 1-(\ln u)^2 + (1-p)(\ln d)^2$

Nov 4th Mid Sem (31/60)

$$\begin{bmatrix} 1.5 & 0.75 \\ 0.75 & 1 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 1 & -.75 \\ -.75 & 1.5 \end{bmatrix}}{1.5 - 0.5625}$$

Bach:

$$\begin{bmatrix} \overbrace{\mu^T V^{-1} \mu}^{AC} & \overbrace{\mu^T V^{-1} 1}^{B} \\ \mu^T V^{-1} 1 & \underbrace{1^T V^{-1} 1}_{A} \end{bmatrix} = \begin{bmatrix} C & B \\ B & A \end{bmatrix}$$

or

$$\underbrace{\begin{bmatrix} A & -B \\ -B & C \end{bmatrix}}_{D} = \frac{\begin{bmatrix} 1^T V^{-1} 1 & -\mu^T V^{-1} 1 \\ -\mu^T V^{-1} 1 & \mu^T V^{-1} \mu \end{bmatrix}}{B^2 - AC}$$

$$\begin{bmatrix} C & B \\ B & A \end{bmatrix} = L^{-1}$$

Q2. Mortgage.

2% 1 year 19 years 5.5% P = 1,000,000

4% 3 years 17 years 5.5%

x = installment of loan per month.

$$\left[\left[P\left(1+\frac{r}{12}\right)-x\right]\left(1+\frac{r}{12}\right)-x\right]\left\{\left(1+\frac{r}{12}\right)-x\right\} \cdots = 0$$

$$\Rightarrow P\left(1+\frac{r}{12}\right)^{(12\times 20)=n} - x\left[\left(1+\frac{r}{12}\right)^{n-1}+\left(1+\frac{r}{12}\right)^{n-2}+\cdots+1\right]=0$$

$$\Rightarrow P\left(1+\frac{r}{12}\right)^{n} = \frac{x\left[\left(1+\frac{r}{12}\right)^{n}-1\right]}{\left(1+\frac{r}{12}-1\right)} \Rightarrow x = \frac{P\left(1+\frac{r}{12}\right)^{n}\times\frac{r}{12}}{\left(1+\frac{r}{12}\right)^{n}-1}$$

For 1 year: $x_0 = 104{,}721.515592$

$$P_1 = P\left(1+\frac{r_0}{12}\right)^{12} - x_0\left[\frac{\left(1+\frac{r_0}{12}\right)^{12}-1}{r_0/12}\right]$$

After 1 year.

$$x_1 = \frac{P_1\left(1+\frac{r_1}{12}\right)^{m}\times\frac{r_1}{12}}{\left(1+\frac{r_1}{12}\right)^{m}-1}$$

$r_1 = 5.5\%$

$$P_1 = P_0\left(1+\frac{r_0}{12}\right)^{12} - x_0\frac{\left[\left(1+\frac{r_0}{12}\right)^{12}-1\right]}{r_0/12}$$

Thursday online class

8am - 10am

$S_n: X_1 \cdots X_n$

$P(S_n > 0)$, $P(S_n > c)$ $\quad P\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} > \frac{c - n\mu}{\sqrt{n\sigma^2}}\right) \approx \Phi\left(\frac{c - n\mu}{\sqrt{n\sigma}}\right)$

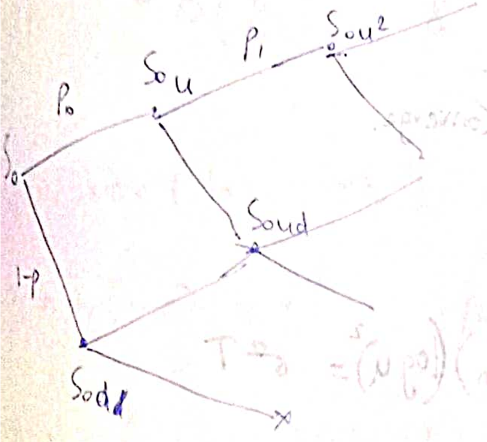
HW: Do the exam questions & submit it by next Thursday.

RR Model. $\quad$ n Step Model.

umption: Jump sizes are fixed & independent of previous posⁿ

w to calculate p : Assume martingale, no arbitrage.

$$\begin{bmatrix} 1+r & 1+r \\ S_u & S_d \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r)$$



$$p_0 = \frac{S_0(1+r) - S_d}{S_u - S_d} = \frac{S_0(1+r-d)}{S_0(u-d)} = \frac{1+r-d}{u-d}$$

$$p_1 = \frac{S_0u(1+r) - S_0ud}{S_0u^2 - S_0ud} = \frac{S_0u[1+r-d]}{S_0u(u-d)} = \frac{(1+r)-d}{u-d} = p_0$$

$S_n: S_0 u^k d^{n-k}$ $\quad$ for $k = 0, \dots, n$

$\ln S_n = \ln S_0 + k \ln u + (n-k) \ln d$

$$Y_i := \begin{cases} \ln u & \text{, with prob. } p \\ \ln d & \text{, w.p. } 1-p \end{cases}$$

i^{th} step

$$R_1 = \frac{S_1 - S_0}{S_0} = \frac{S_1}{S_0} - 1 \quad . \quad \ln(1+R_1) = \ln\left(\frac{S_1}{S_0}\right) = \ln S_1 - \ln S_0$$

$$\approx R_1 = \begin{cases} \ln u \\ \ln d \end{cases}$$

Small values of R

$d = \frac{1}{u} \Rightarrow$ Extra Assumption for Symmetry.

$$E\left[\sum Y_i\right] = n(2p-1)\ln u$$

$$Var(Y_i) = 4p(1-p)(\ln u)^2 = E[Y_i^2] - E[Y_i]^2$$

$$= p(\ln u)^2 + (1-p)(\ln u)^2 - (2p-1)^2(\ln u)^2 = (\log u)^2\left[1-(2p-1)^2\right]$$

$$= 4p(1-p)(\log u)^2.$$

$$Var\left(\sum Y_i\right) = n \cdot 4p(1-p)(\log u)^2 \propto \text{Time } T$$

Volatility $\swarrow$ $\sigma^2 T$

p should be $1/2$ otherwise var won't converge.

$$n 4p(1-p)(\log u)^2 \longrightarrow \sigma^2 T$$

$$P_n = \left(\frac{1}{2} + \delta_n\right) \qquad \left(\frac{1}{4} - \delta_n\right) \Big| \; 4n\,(1-4\delta_n)(\log u)^2 = \sigma^2 T$$

$$1 - P_n = \left(\frac{1}{2} - \delta_n\right)$$

$$\Rightarrow (n - 4n\delta_n)(\log u)^2 = \sigma^2 T \qquad n(\log u)^2 = \sigma^2 T$$

$$(\log u)^2 = \frac{\sigma^2 T}{n} \Longrightarrow \log u = \sigma\sqrt{T/n} \Rightarrow u_n = e^{-\sigma\sqrt{T/n}}$$

$$u_n = 1 + \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\frac{T}{n} + o\left(\frac{T}{n}\right)$$

$$d_n = 1 - \sigma\sqrt{\frac{T}{n}} + \frac{\sigma^2}{2}\frac{T}{n} + o\left(\frac{T}{n}\right)$$

$$u_n - d_n = 2\sigma\sqrt{T/n} + o(T/n)$$

$$\boxed{P_n = \frac{1}{2} + \frac{\left(r - \frac{\sigma^2}{2}\right)T/n}{2\sigma\sqrt{T/n}} + o(\cdot)}$$

9^{th} November.

$S_n = S_0 u^k d^{n-k}$ $\quad k = 0, 1, \dots, n.$ $\qquad \ln S_n = \ln S_0 + Y_1 + \dots + Y_n$

where $Y_i = \begin{cases} \ln u & \text{w.p. } p \\ \ln d & \text{w.p. } 1-p \end{cases}$ $\qquad \Rightarrow E[Y_i] = p\ln u + (1-p)\ln d$

Y_i's are iid

If we assume $d = 1/u$, $E[Y_i] = (2p-1)\ln u$

$\Rightarrow E[\ln S_n - \ln S_0 / \mathcal{F}_0] = nE[Y_i] = n(2p-1)\ln u$

n - times

p_n, u_n $\qquad \delta_n \to \frac{\delta_T}{n}$

ideal $p_n \to 1/2$ $\quad u_n \to 1$

$(2p_n - 1) = O(1/\sqrt{n})$ & $\ln u_n = O(1/\sqrt{n})$, $nE[Y_i]$ might converge to a limit.

Again, $Var(\ln S_n - \ln S_0 / \mathcal{F}_0) = n Var(Y_i) = n\left[p(\ln u)^2 + (1-p)(\ln u)^2 - E[Y_i]^2\right]$

$= n\,4p(1-p)(\ln u)^2$, $\qquad p_n = \frac{1}{2} + \delta_n$ then $1 - p_n = \frac{1}{2} - \delta_n$

$\Rightarrow Var(\ln S_T - \ln S_0 / \mathcal{F}_0) \propto T$ (assumption as in Simple Random Walk)

$\Rightarrow n\,4p(1-p)(\ln u)^2 = \sigma^2 T$

$\Rightarrow n\,4\left(1/4 - \delta_n^2\right)(\ln u_n)^2 = \sigma^2 T \qquad = n(\ln u_n)^2 - 4n\delta_n^2(\ln u_n)^2 = \sigma^2 T$

$\Rightarrow n(\ln u_n)^2 = \sigma^2_T$ as $n\delta_n^2(\ln u_n)^2 \to 0$

$\Rightarrow \ln u_n = \sqrt{\frac{\sigma^2 T}{n}} = \sigma\sqrt{\frac{T}{n}}$ $\qquad \Rightarrow u_n = e^{\sigma\sqrt{T/n}}$

$d_n = e^{-\sigma\sqrt{T/n}}$

$$p_n = \frac{1+\frac{r_T}{n} - d_n}{u_n - d_n}, \quad u_n = 1 + \sigma\sqrt{T/n} + \frac{\sigma^2}{2}\cdot\frac{T}{n} + o(1/n)$$

small 'o'.

$a_n = o(1/n)$ means $na_n \to 0$.

$$d_n = 1 - \sigma\sqrt{T/n} + \frac{\sigma^2}{2}\frac{T}{n} + o(1/n)$$

$$\Rightarrow u_n - d_n = 2\sigma\sqrt{\frac{T}{n}} + o(1/n) \quad [\because 2o(1/n) \simeq o(2/n) = o(1/$$

Now, $$\frac{1+r_T}{n} - d_n = \frac{r_T}{n} + \sigma\sqrt{\frac{T}{n}} - \frac{\sigma^2}{2}\frac{T}{n} + o(1/n)$$

$$= \left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + \sigma\sqrt{T/n} + o(1/n)$$

$$\Rightarrow p_n = \frac{\left(r-\frac{\sigma^2}{2}\right)T/n + \sigma\sqrt{T/n} + o(1/n)}{2\sigma\sqrt{\frac{T}{n}} + o\left(\frac{1}{n}\right)} = \left[\frac{1}{2} + \frac{\left(r - \frac{\sigma^2}{2}\right)}{2\sigma}\sqrt{\frac{T}{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]$$

$\cdot(1-o$

Since $$\frac{1}{2\sigma\sqrt{\frac{T}{n}}\left(1+o\left(\frac{1}{\sqrt{n}}\right)\right)} = \frac{1}{2\sigma\sqrt{\frac{T}{n}}}\left(1 - o\left(\frac{1}{\sqrt{n}}\right)\right)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = 1 - x[1 - x + x^2 - x^3 + \cdots]$$

$$= 1 - \frac{x}{1+x} = 1 - \frac{o\left(\frac{1}{\sqrt{n}}\right)}{1+o\left(\frac{1}{\sqrt{n}}\right)} = 1 - o\left(\frac{1}{\sqrt{n}}\right)$$

$$\Rightarrow p_n = \frac{1}{2} + \frac{\left(r - \frac{\sigma^2}{2}\right)}{2\sigma}\sqrt{\frac{T}{n}} + o\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2} + \delta_n$$

$$\Rightarrow n\mathrm{Var}(Y_i) = n4p_n(1-p_n)(\log u_n)^2$$

$$= n\cdot 4\left(\frac{1}{4} - \delta_n^2\right)\left(\sigma\sqrt{\frac{T}{n}}\right)^2 = n\sigma^2\frac{T}{n} - n\delta_n^2\sigma^2 T/n$$

$$= \sigma^2 T - \delta_n\sigma^2 T \to \sigma^2 T$$

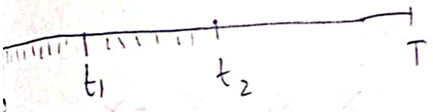
$n \to \infty$, since $\delta_n \to 0$ as $n \to \infty$.

$$n E[Y_1] = n(2p_n - 1)\ln u_n = n\left[\frac{(r - \frac{\sigma^2}{2})}{\sigma}\sqrt{\frac{T}{n}} + o\left(\frac{1}{\sqrt{n}}\right)\right]\sigma\sqrt{\frac{T}{n}}$$

$$n\left[\left(r - \frac{\sigma^2}{2}\right)\frac{T}{n} + o\left(\frac{1}{n}\right)\right] = \left(r - \frac{\sigma^2}{2}\right)T + o(1) \to \left(r - \frac{\sigma^2}{2}\right)T$$

$$\ln S_T - \ln S_0 \to N\left(\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right) \quad \text{as } \sum_{i=1}^{n} Y_i \sim N\left(r - \frac{\sigma^2}{2}, \sigma^2 T\right)$$

iid by CLT.



$$\ln S_{t_1} - \ln S_0 \,\Big|\, \mathcal{F}_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)t_1, \sigma^2 t_1\right)$$

$$\ln S_{t_2} - \ln S_{t_1} / \mathcal{F}_{t_1} \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(t_2 - t_1), \sigma^2(t_2 - t_1)\right)$$

$$\ln S_{t_i} - \ln S_{t_{i-1}} \,\Big/\, \mathcal{F}_{t_{i-1}} \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}), \sigma^2(t_i - t_{i-1})\right)$$

For $0 \le t_0 < t_1 < \cdots < t_n \le T$

$(\ln S_{t_i} - \ln S_{t_{i-1}})$ are indep for all i $\sim$ Normal

$X_t = \ln S_t$ and assume continuity of X_t in t. it behaves like Brownian Motion.

$X \sim N(\mu, \sigma^2)$, then $X = \mu + \sigma Z$ where $Z \sim N(0,1)$

$\ln S_t - \ln S_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma^2 t\right)$

$X_t - X_0 \sim \nearrow$ $\Rightarrow X_t - X_0 = \left(r - \frac{\sigma^2}{2}\right)t + \sigma Z_t$, $Z_t \sim N(0,1)$

$\{Z_t\} \sim$ Brownian Motion $\{B_t\}$

$$e^{X_t} = \cancel{\ln} S_t \qquad S_0\, e^{(r-\frac{\sigma^2}{2})t + \sigma B_t}$$

Geometric Brownian motion.

18th November.

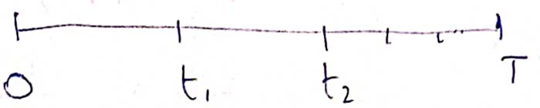
Behaviour of $\{\log S_t\}$

$$\log S_t - \log S_0 \,/\, \mathcal{F}_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)t,\ \sigma^2 t\right).$$

For $X_t = \log S_t$ we can also observe $(X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_k}$ are independent for any $t_0 < t_1 < \cdots < t_k$. for $k \geq 1$.

$$\log S_{t_1} - \log S_{t_0} = \sum Y_i \qquad Y_i = \begin{cases} \log u_n & \text{w.p. } p_n \\ \log d_n & \text{w.p. } 1-p_n. \end{cases}$$

If we divide the subinterva into n parts.



- Assuming continuous sample path
- Independent Increment $[(X_{t_1} - X_{t_0}, \ldots, X_{t_k} - X_{t_{k-1}})$ are independent.$]$
- $X_t - X_s \,/\, \mathcal{F}_0 \sim N\left(\left(r - \frac{\sigma^2}{2}\right)(t-s),\ \sigma^2(t-s)\right)$

Brownian Motion if satisfy above 3 properties. $\{X_t\} \sim$ Brownian

Norbert Weiner. Brownian Motion is also called Wiener Process.

Standard Brownian Motion $\{B_t\}$ has the property

- $B_0 = 0$ and has continuous sample paths
- Independent increment i.e., $(B_{t_1} - B_{t_0}, \ldots, B_{t_k} - B_{t_{k-1}})$ are independent for any $t_0 < t_1 < \ldots < t_k$ $\forall$ $k \geq 1$
- $B_t - B_s / \mathcal{F}_s \sim N(0, t-s)$; $\mathcal{F}_s = \sigma\{B_u : 0 \leq u \leq s\}$

i.e.; $\mu = 0$, $\sigma = 1$

Suppose $X \sim N(\mu, \sigma^2)$ then $X = \mu + \sigma Z$ where $Z \sim N(0,1)$

Similarly, $X_t = \mu_t + \sigma B_t$

$$\Rightarrow \frac{S_t}{S_0} = e^{X_t - X_0} = e^{\mu t + \sigma B_t} \qquad \left[\because S_0 = \mu \cdot 0 + \sigma \cdot 0 = 0 \right.$$

Geometric Brownian Motion.

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

Checking for Martingale

$$E[S_t / \mathcal{F}_0] = S_0 E\left[e^{\mu t + \sigma B_t}\right] = S_0 e^{\mu t} E\left[e^{\sigma B_t}\right]$$

$$= S_0 e^{\mu t + \frac{\sigma^2 t}{2}} = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \frac{\sigma^2 t}{2}} = S_0 e^{rt}$$

explain?

$B_t \sim N(0, t)$

$E[e^{B_t}] = e^{t/2}$

$X_t \sim N(\mu, \sigma^2)$

$E[e^{sX}] = e^{\mu s + \frac{\sigma^2 s^2}{2}}$

Similarly, $E(S_t / \mathcal{F}_u) = S_u e^{r(t-u)}$ ④ ~~ⓐ~~

Then $\{\tilde{S}_t = e^{-rt} S_t\}$ is martingale ⓒ

t $E[e^{-rt} S_t / \mathcal{F}_u] = e^{-ru} S_u$ ⑤

then b then c

Under No Arbitrage, Two Assets $\underset{60}{A} > \underset{50}{B}$ Given $A_T = B_T$

If this happens, sell A_0 & buy B_0 Initial gain $10 \to 10e^{rt}$
risk-free. Cannot happen under NA Put it in Bank, conto.

$$\varphi_1 = \Delta = \frac{f_u - f_d}{S_u - S_d}$$

$$\varphi_0(1+r) + \varphi_1 S_u = f_u$$

$$\varphi_0(1+r) + \varphi_1 S_d = f_d$$

$$\Rightarrow \varphi_1(S_u - S_d) = f_u - f_d$$

$$\Rightarrow \varphi_1 = \frac{f_u - f_d}{S_u - S_d} = \Delta \quad \text{hedging parameter.}$$

$r_{pf} = f - \Delta S$ risk-neutral.

For continuous time, $\varphi_1 \to \frac{\partial f}{\partial S}$ $\left(f - \frac{\partial f}{\partial S} S\right)$

Call option $f_T = (S_T - K)^+$

$$f_0 e^{rT} = E_{rn}(f_T / \mathcal{F}_0) = E[(S_T - K)^+ / \mathcal{F}_0]$$

$$= \int_K^\infty (u-K) f_{S_T}(u)\, du$$

$(S_T - K)^+ = S_T - K$, $S_T > K$

$u = S_T = S_0 e^{\mu t + \sigma Z}$

$\log S_T - \log S_0 = \mu t + \sigma Z$ $Z \sim N(0,t)$

$y = \log u - \log S_0$; $dy = \frac{1}{u} du$ $\Rightarrow$ $du = u\,dy$ $S_0 e^y dy$

$\log u = y + \log S_0$, $u = S_0 e^y$

Transformation

$$\int (S_0 e^y - K) f_y(u)\, S_0 e^y dy$$

$$f_{S_T} dS_T = f_y dy$$

$$\text{egral} = \int_{S_0 e^y > K} S_0 e^y f_Y(y)\, dy - K \int_{S_0 e^y > K} f_Y(y)\, dy$$

$$\underbrace{S_0 e^y > K}_{\left\{ y > \ln\left(\frac{K}{S_0}\right) \right\}}$$

$$) = \frac{e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}$$

$N(\mu t, \sigma^2 t)$.

$$(Y > \ln \frac{K}{S_0}) = K\, P\left(\frac{Y - \mu t}{\sigma \sqrt{t}} > \frac{\ln K/S_0 - \mu t}{\sigma \sqrt{t}} \right)$$

$$K\left(1 - \Phi\left(\frac{\ln K/S_0 - \mu t}{\sigma\sqrt{t}}\right)\right) = K\, \Phi\left(\frac{\ln S_0/K + \mu t}{\sigma \sqrt{t}}\right)$$

$$1^{st} \text{ part} = S_0 \int_{y > \ln\left(\frac{K}{S_0}\right)} \frac{e^y \cdot e^{-\left(\frac{(y-\mu t)^2}{2\sigma^2 t}\right)}}{\sigma\sqrt{2\pi t}}\, dy$$

Complete the Square

$$\text{onent} = -\left[\frac{y^2 + \mu^2 t^2 - 2y\mu t - 2\sigma^2 t y}{2\sigma^2 t}\right]$$

$$= -\left[\frac{y^2 - 2y(\mu t + \sigma^2 t) + \mu^2 t^2}{2\sigma^2 t}\right]$$

$$= -\frac{(y - (\mu t + \sigma^2 t))^2}{2\sigma^2 t} + \frac{(\mu t + \sigma^2 t)^2 - \mu^2 t^2}{2\sigma^2 t}$$

$$\text{erm} = \frac{\mu^2 t^2 + (\sigma^2 t)^2 + 2\mu t \sigma^2 t - \mu^2 t^2}{2\sigma^2 t} = \frac{\sigma^2 t}{2} + \mu t$$

$$\frac{\sigma^2 t}{2} + \boxed{rt - \frac{\sigma^2 t}{2}}_{\to \mu t} = rt.$$

$$1^{st} \text{ part} = S_0 e^{rt} \int_{y > \ln \frac{K}{S_0}} \frac{e^{-\frac{(y-(\mu t + \sigma^2 t))^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}} \, dy$$

$$\tilde{y} \sim N(\underbrace{\mu t + \sigma^2 t}_{rt + \frac{\sigma^2 t}{2}}, \sigma^2 t)$$

$$\Rightarrow P\left(\tilde{y} > \ln \frac{K}{S_0}\right)$$

$$= 1 - \Phi\left(\frac{\ln \frac{K}{S_0} - \left(rt + \frac{\sigma^2 t}{2}\right)}{\sigma\sqrt{t}}\right)$$

$$= \Phi\left(\frac{\ln \frac{S_0}{K} + \left(rt + \frac{\sigma^2 t}{2}\right)}{\sigma\sqrt{t}}\right) \sim d_{1T}$$

$$\Rightarrow f_0 e^{rT} = S_0 e^{rT} \Phi(d_{1T}) + K \Phi(d_{2T})$$

$$\Rightarrow f_0 = S_0 \Phi(d_{1T}) + K e^{-rT} \Phi(d_{2T})$$

$$K \Phi\left(\frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}\right) \sim d_{2T} \qquad d_{2T} = d_{1T} - \sigma\sqrt{t}$$

<u>H.W.</u> Calculate the put price.
Check the put-call parity.

$S_0 = 60 \quad r = 6\% \quad \sigma = 25\% \quad K = 62 \quad T = 3$ months

$\Phi(d_{1T}) = 0.468 \qquad \Phi(d_{2T}) = 0.4188$

$S_0 \times 0.468 - K e^{-rT} \times 0.4188$

$= 2.5089$

Black Scholes Merton
?

15th November.   Next class Thursday Morning online 8-10 (Final)

rivative Pricing

evaluate the derivative uniquely in a No Arbitrage market
plicability of the (contingent) claim is needed.

W: Give example of a derivative in a NA but incomplete
market ~~which~~ ~~is~~ ~~replicable~~. whose contingent claim is ~~whose~~
eplicable.

merican / Asian Options.

merican Options

all: It gives you right to buy certain asset with exercise
price K at any time on or before the maturity date, say T.

ut: Similar: It gives you right to sell ...... K at any time on or
before maturity writing date.

---

xample on how to price such an option.

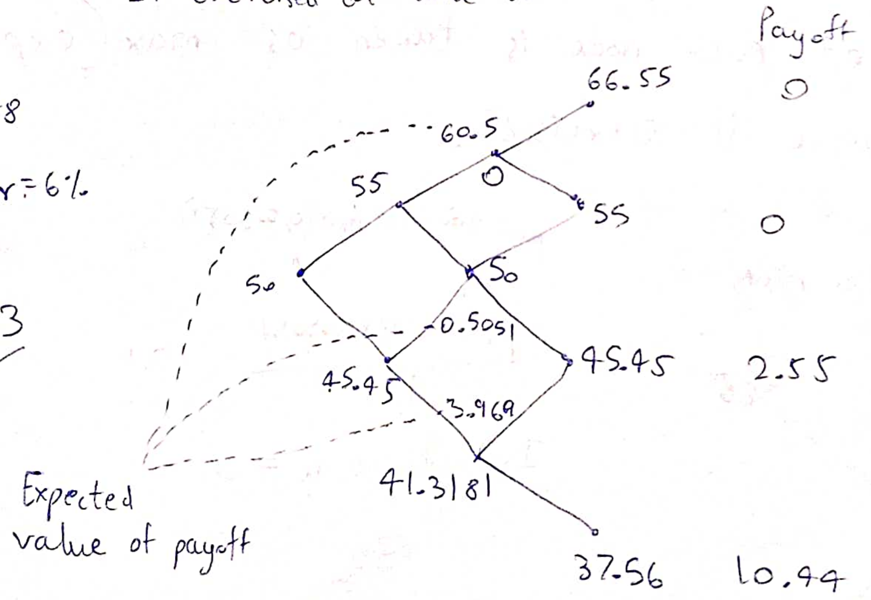
ut option payoff = $\max(K - S_t, 0)$ $0 \le t \le T$
If exercised at time t.

$S_0 = 50$ $K = 48$

$u = 10\% = 1.1$ // $r = 6\%$

$d = \frac{1}{u}$

T = 3 years.

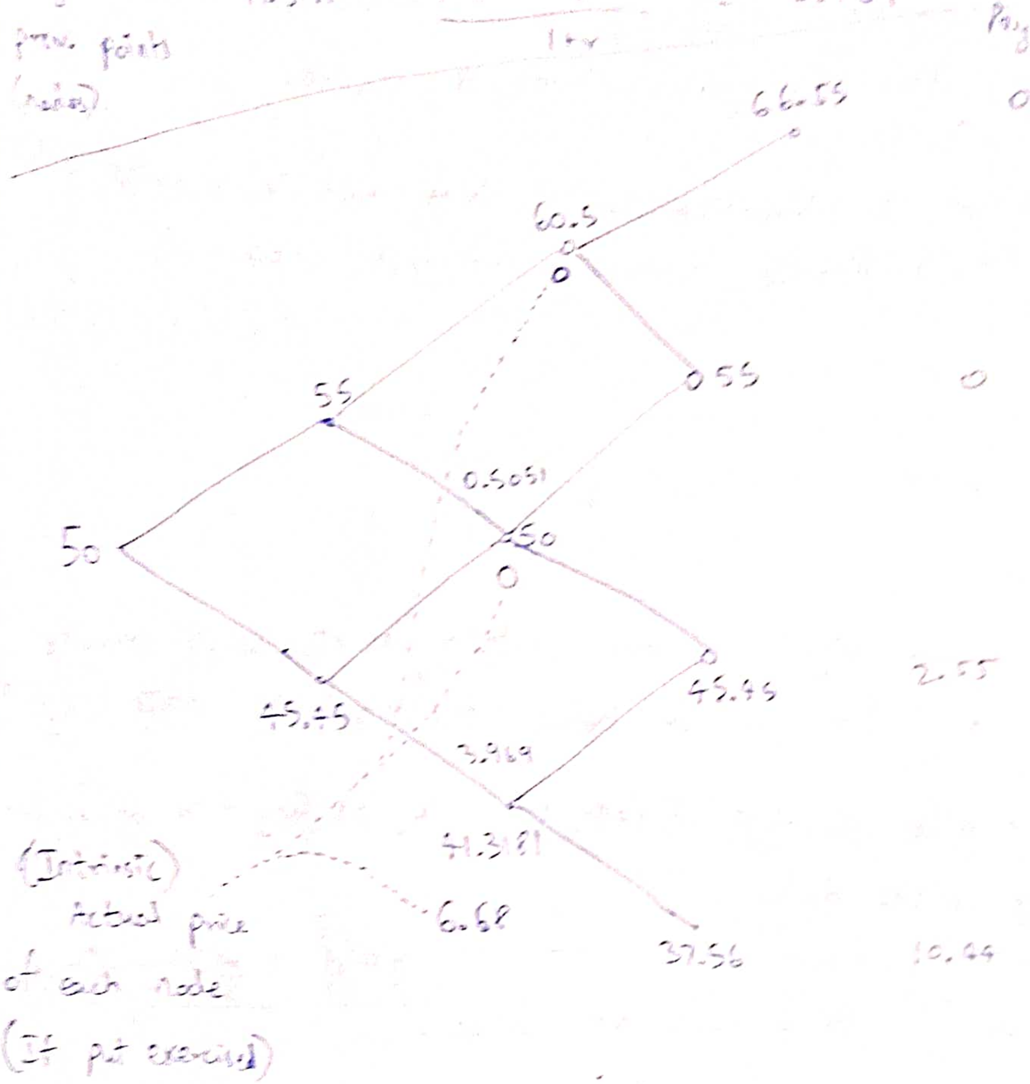


$$p = \frac{(1+r)-d}{u-d} = \frac{1.06 - \frac{1}{u}}{u - \frac{1}{u}} = \frac{(1.1)(1.06)-1}{u^2-1} = \frac{0.166}{0.21} = 0.79$$

$$\text{Payoff} = \frac{(p\times 0 + q\times 0)}{1+r} = 0 \quad ; \quad \text{for } 50 \text{ (middle)} \quad \frac{p\times 0 + q\times 2.55}{1+r} = 0.505$$

Expected Payoff at prev. points (nodes)

$$41.3181 = \frac{p\times 2.55 + q\times 10.44}{1+r} = 3.969$$



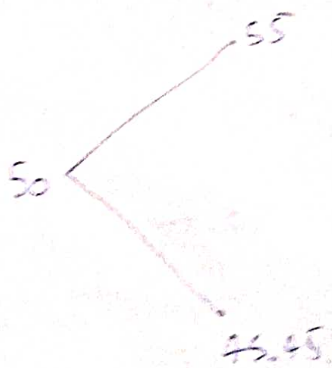
Price at each node is taken as max(expected value, the value if exercised).

$q = 0.21$

$p = 0.79$

$\boxed{r \text{ is take yearly}}$

For earlier nodes:



max(0,0) max(0,0.5051)

$$\frac{p\times 0 + q\times 0.5051}{1+r} = 0.1$$

Intrinsic value = 0

$$\frac{p\times 0.5051 + q\times 6.68}{1+r} = 1.7$$

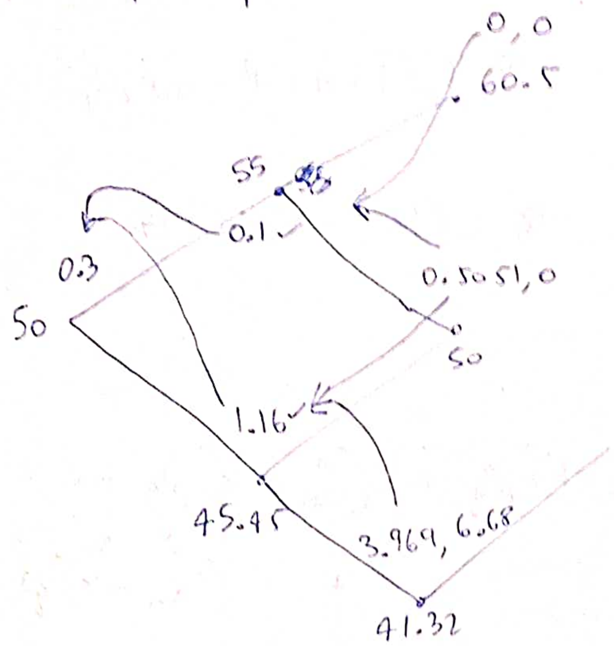
Intrinsic value

$= K - S_t$

$= 48 - 45.45 = \boxed{2.55}$

For initial node: Expected payoff = $\frac{p \times 0.1 + q \times 2.55}{1.1} = 0.58$.

For European options:



50 European option : 0.3

American option : 0.58

HW : Calculate the price of the call options. (American)

It would be same as european ones & it can be proven.

- For cont. case, solve PDE & numerically get derivative.

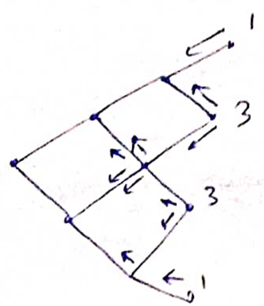
Asian options

Call : Payoff = max (S_{ave} − K, 0) . Where S_{ave} may be arithmetic or geometric mean.

$S_{max} - S_{avg}$ can also be possibility. or $S_{ave} - S_{min}$

Put : Payoff = max (K − S_{ave}, 0)

paths = 2^T



A.M.

S_{avg}

① $S_0 \cdot \frac{(1+u+u^2+u^3)}{4}$

② $S_0(1+u+u^2+u^2d)/4$

③ $S_0(1+u+ud+u^2d)/4$

④ $S_0(1+d+ud+u^2d)/4$

⑤ $S_0(1+u+ud+ud^2)/4$

⑥ $S_0(1+d+ud+ud^2)/4$

⑦ $S_0(1+d+d^2+ud^2)/4$

⑧ $S_0(1+d+d^2+d^3)/4$

GM can be done similarly.

$$(\overset{1}{S}_{ave}-K)^+p^3 + (\overset{2}{S}_{ave}-K)^+p^2q + (\overset{3}{S}_{ave}-K)^+p^2q + (\overset{4}{S}_{ave}-K)^+p^2q + \dots + (\overset{8}{S}_{ave}-K)^+q$$

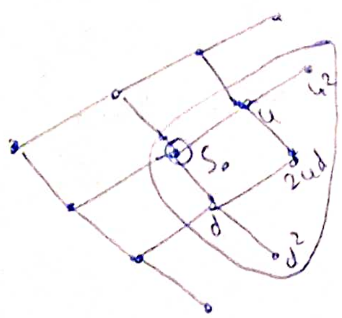
$$\Downarrow$$

$$E[(S_{ave}-K)^+]$$

$(\ )^+ \to \max(x,0)$ positive ~~for~~ value.

$$\text{Price at } 0 = \frac{1}{(1+r)^3} E[(S_{ave}-K)^+]$$

Price at intermediate node: Consider the subgraph originating from & take avg or whatever.



HW: Calculate Asian Put and Call option prices at each node.

Absent: Anirban, Debajyoti, Subhajyoti, Sourish, Nirmalya, Shahreyar.

$-S^1_{ave}) = 0$

$-S^2_{ave}) = 0$

$-S^3_{ave}) = 0$

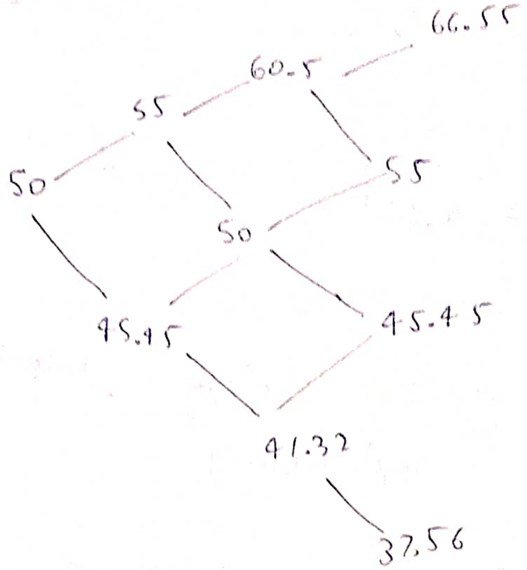
$-S^4_{ave}) = 0$

$-S^5_{ave}) = 0$

$-S^6_{ave}) > 0$

$-S^7_{ave}) > 0$

$-S^8_{ave}) > 0$



$S^6_{ave} = (50 + 45.45 + 50 + 45.45)/4 = 47.725$

$\Rightarrow K - S^6_{ave} = 0.275 \qquad pq^2$

$S^7_{ave} = (50 + 45.45 + 41.32 + 45.45)/4 = 45.555$

$\Rightarrow K - S^7_{ave} = 2.445 \qquad pq^2$

$S^8_{ave} = (50 + 45.45 + 41.32 + 37.56)/4 = 43.5825$

$\Rightarrow K - S^8_{ave} = 4.4175 \qquad q^3$

$E[(K-S_{ave})^+] = 0.275 \times pq^2 + 2.445\, pq^2 + 4.4175 q^3$

$= 0.13567$

ian put price at time $0 = \frac{1}{(1+r)^3} E[(K-S_{ave})^+] = 0.1139.$

Geometric Mean

$ve = \sqrt[4]{u^2 d^4 S_0^4} = u^{1/2} d S = S d^{1/2} = 47.67$

$= \sqrt[4]{S_0^4 u d^5} = S_0 \cdot d = 45.45$

$= \sqrt[4]{S_0^4 d^6} = S_0 d^{3/2} = 43.339$

$K - S^{6}_{ave} = 0.33$

$K - S^{7}_{ave} = 2.55$

$K - S^{8}_{ave} = 4.661$

$$E\left[(K - S_{ave})^{+}\right] = \frac{0.33\, pq^2 + 2.55\, pq^2 + 4.661\ldots}{(1+r)^3}$$

$$= \frac{0.1435}{(1+r)^3} = 0.12$$

Why is $\underbrace{\text{replicable strategy}}_{\text{replicability}}$ important for pricing?

$$\begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} \begin{bmatrix} 1+r & 1+r \\ S_u & S_d \end{bmatrix} = \begin{bmatrix} f_u & f_d \end{bmatrix}$$

Replicability.

Unique Sol^n exists iff $S_u \neq S_d$

For NA, $\exists$ (P_u, P_d) unique, ~~this gives~~ such that $\{S_T\}$ is a discounted Martingale.

$$\begin{bmatrix} 1+r & 1+r \\ S_u & S_d \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r)$$

$$E\left[S_1 / \mathcal{F}_0\right] = S_0 (1+r).$$

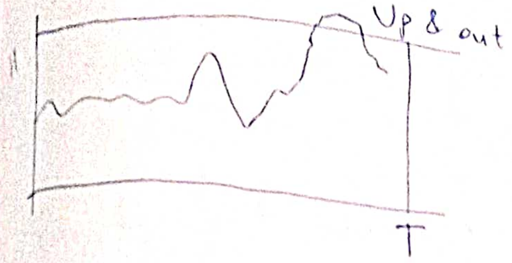
Now, $$\begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} \begin{bmatrix} 1+r & 1+r \\ S_u & S_d \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = \begin{bmatrix} f_u & f_d \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = f_u P_u + f_d P_d$$
$$= E\left[f_1 / \mathcal{F}_0\right]$$

$$\begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} P_u \\ P_d \end{bmatrix} = \begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} \begin{bmatrix} 1 \\ S_0 \end{bmatrix} (1+r) = (1+r)\left[\varphi_0 + \varphi_1 S_0\right] = (1+r) f_0$$

Exotic Options:

American, Asian Barrier Options, Call on a Call, call on a put, etc.

Quanto options
Currency options



Cease to exist if it goes above a prefixed barrier.

Similarly, down & out ; up & in

Up & in could be useful for oil companies or other sellers.
Up & in becomes valid when the underlying asset price goes above certain predefined level. Similar for down & in.

Currency options

$P_{0(USA)} \longrightarrow P_{t\,(USA)} \overset{\text{expected growth}}{=\!=} P_{0(US)}\, e^{r_U t}$

$P_{0(IN)} \longrightarrow P_{t\,(IN)} \overset{\text{expected growth}}{=\!=} P_{0(IN)}\, e^{r_I t}$

If S_0 is amount of money you pay to get foreign (purchasing power)

$$S_0 = \frac{P_{0(IN)}}{P_{0(US)}} \longrightarrow \frac{P_{0I}\, e^{r_I t}}{P_{0U}\, e^{r_U t}} \Rightarrow S_t \overset{exp}{=\!=} S_0 e^{(r_I - r_U)t}$$

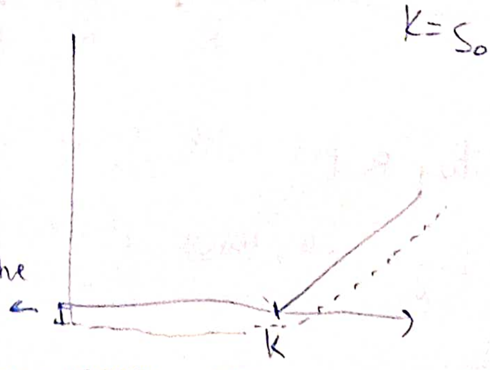
$\Rightarrow$ distribution of $S_t = N\left(\left\{(r_I - r_U) - \frac{\sigma^2}{2}\right\}t,\ \sigma^2 t\right)$

going through Binomial argument, where σ is volatility of currency price.

Trading Strategies

$K = S_0$

Bull Market: Buy a call

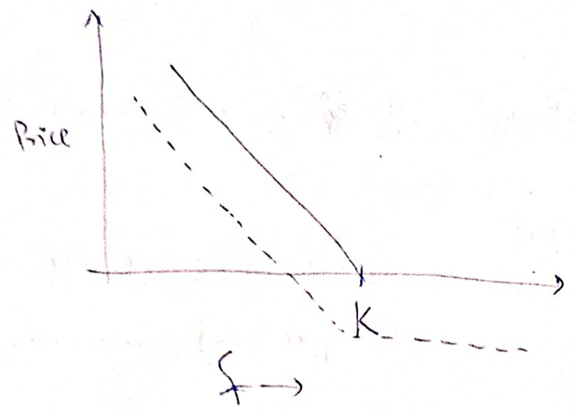


price of the option.

Dotted line is profit line

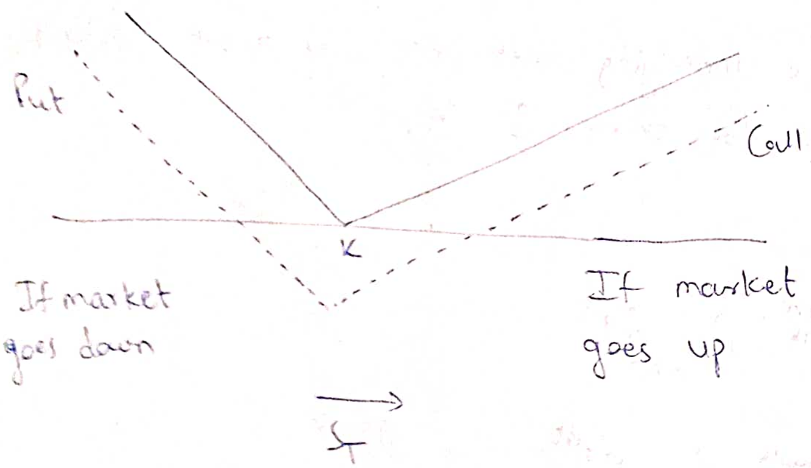
Bear Market: Buy a put

At $K=S_0$,



Volatile Market: Buy a call and a put at the same price with $K=S_0$

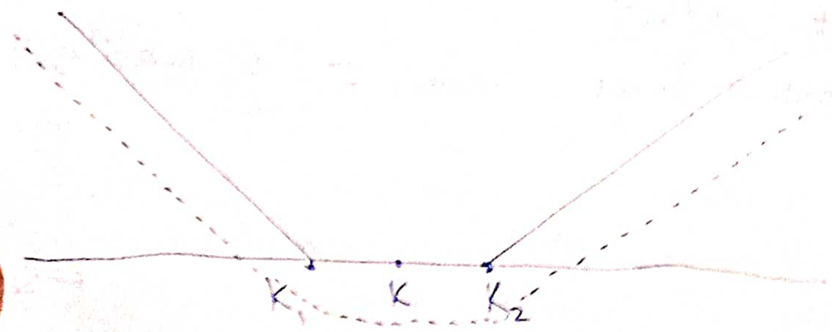
Straddle (The cha on left



Dull Market - Nothing happens

Strangle (Volatile market)

$K=S_0$



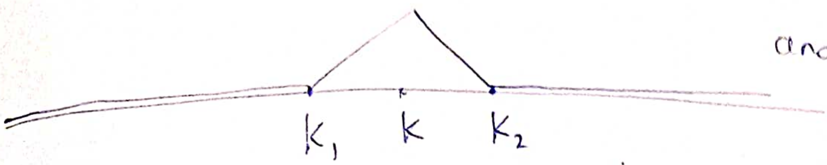
Buy a put with strike price K_1

Buy a call with strike price K_2

Dull Market.

Butterfly Spread

$K = S_0$.



Write a put with K_1 another with K_2 and sell.

Buy two put with strike price K.

HW.

1) Find initial cash flow or gain

2) Draw the profit line.