8/25

Last time : law of total probability

U: upper occupation
M: middle - - .
L: lower - - .

U_1, M_1, L_1 } father
U_2, M_2, L_2 } son.

Transition (probability) matrix

	U_2	M_2	L_2
U_1	0.45	0.48	0.07
M_1	0.05	0.70	0.25
L_1	0.01	0.50	0.49

$P(U_2|U_1)$

$P(U_2|M_1)$

$P(U_2|L_1)$

Last time: given,

$$\left.\begin{aligned} P(U_1) &= 0.10 \\ P(M_1) &= 0.40 \\ P(L_1) &= 0.5 \end{aligned}\right\}$$

we found $P(U_2) = 0.07$

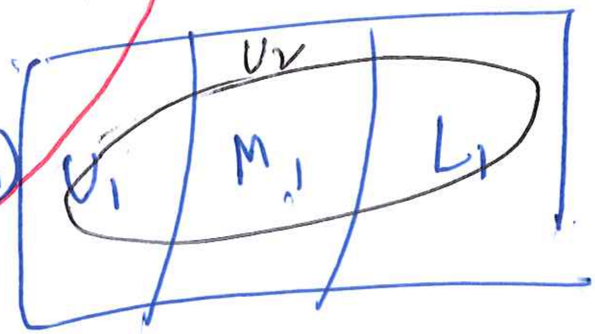
Suppose all these information are given.
Can you compute $P(U_1|U_2)$?

$$P(U_1|U_2) = \frac{P(U_1 \cap U_2)}{P(U_2)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(U_1)P(U_2|U_1)}{P(U_2|U_1)P(U_1) + P(U_2|M_1)P(M_1) + P(U_2|L_1)P(L_1)}$$

Law of total probability.



In our context:

$$P(U_1|U_2) = \frac{(0.10)(0.45)}{(0.10)(0.45) + (0.05)(0.40) + (0.01)(0.50)}$$

$$= 0.64$$

Bayes' Rule

Let A and $B_1, \dots, B_n$ are events where the B_i s are disjoint, $\bigcup_{i=1}^{n} B_i = \Omega$, and $P(B_i) > 0$ for all i

fixed index

Then, $$\boxed{P(B_j|A) = \frac{P(B_j)P(A|B_j)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}}$$

Example: "Lie-detector test" example

(Polygraph test)

Test

$+$: polygraph test is positive $\Rightarrow$ the subject is lying

$-$: polygraph test is negative $\Rightarrow$ the subject is telling a truth

Actuality

T : The ~~person~~ subject is actually telling a truth.

L : The subject is actually lying.

By experiments

$P(+|L) = 0.88$, $P(-|L) = 0.12$

$P(-|T) = 0.86$, $P(+|T) = 0.14$

Suppose for a particular question

$P(T) = 0.99$, $P(L) = 0.01$

Question:

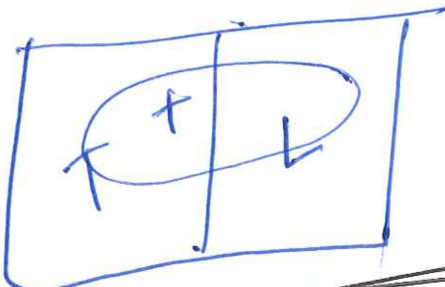
$$P(T \mid +)$$

~~Bayes~~

Bayes' Rule

$$= \frac{P(T)\,P(+\mid T)}{P(+\mid T)P(T) + P(+\mid L)\,P(L)}$$

$$= \frac{(0.99)(0.14)}{(0.14)(0.99) + (0.88)(0.01)}$$



$$= \boxed{0.94}$$

Independence of events

A: H, T — Flipping a coin

B: 1, 2, 3, 4, 5, 6 — Throwing a die.

If A and B are independent:

$$P(A \mid B) = P(A)$$

$$P(B \mid A) = P(B) \;//$$

$$\boxed{P(A) = P(A|B)} = \frac{P(A\cap B)}{P(B)}, \quad P(B) \neq 0$$

A and B are indep.

$$\boxed{P(A\cap B) = P(A).P(B)}$$

Definition:

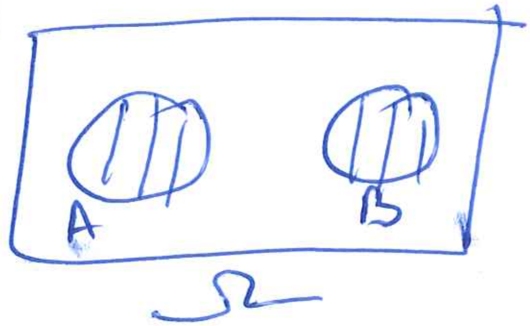
A and B are said to be independent if $P(A\cap B) = P(A).P(B)$

Example: A and B are mutually exclusive ($A\cap B = \phi$), and A, B both have > 0 probability.

Then $P(A\cap B) = P(\phi) = 0$

$P(A).P(B) > 0$

So, $P(A\cap B) \neq P(A).P(B)$



So, A and B are NOT independent.

Example:

A card is selected randomly from a deck. Let A denote the event that it is an ace ("A") and D the event that it is diamond ($\diamondsuit$)

Question: is $A \perp D$?

$\boxed{A \perp D: \text{ A and D are independent}}$

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

$$P(D) = \frac{13}{52} = \frac{1}{4}$$

$$P(A) \cdot P(D) = \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52}$$

$$P(A \cap D) = \frac{1}{52}$$

So, $P(A \cap D) = P(A) \cdot P(D)$

Hence $A \perp D$ ✓

~~Gambler's Ruin~~

Next point:

A "fair coin" $\left(\begin{array}{l} P(H) = \frac{1}{2} \\ P(T) = \frac{1}{2} \end{array} \right)$

is tossed twice.

A: Event that we get "H" on the first toss.

B: Event that we get "H" on the 2nd toss.

C: We get exactly one head.

$$\{ HH, HT, TH, TT \} = \Omega$$

$$P(A) = \frac{2}{4} = \frac{1}{2}, \quad P(B) = \frac{2}{4} = \frac{1}{2}$$

$$P(C) = \frac{2}{4} = \frac{1}{2}$$

Pairwise independence

$$P(A) \cdot P(B) = \frac{1}{4} \qquad P(A \cap B) = \frac{1}{4}$$

$A \perp B$ ✓

$$P(B)\,P(C) = \frac{1}{4}, \qquad P(B \cap C) = \frac{1}{4}$$

$B \perp C$ ✓

$A \perp C$ ✓

$$P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$\underset{\overset{\|}{0}}{P(\phi)} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\boxed{\begin{array}{l} A = \{HH, HT\} \\ B = \{TH, HH\} \\ C = \{HT, TH\} \\ A \cap B \cap C = \phi \end{array}}$$

Definition: A collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if for any subcollection $A_{i_1}, A_{i_2}, \ldots, A_{i_m}$,

$$P(A_{i_1} \cap \ldots \cap A_{i_n}) = P(A_{i_1}) \ldots P(A_{i_n})$$

For the last example

$$P(A \cap B \cap C) \neq P(A) \cdot P(B) \cdot P(C)$$

So, A, B, C are NOT mutually indep.

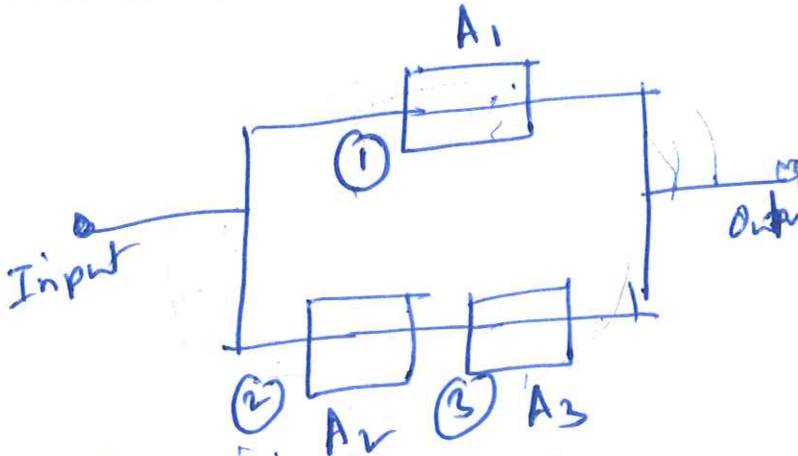
Example:

Consider a circuit

A_1: (1) works

A_2: (2) works

A_3: (3) works



~~Quest~~ • Suppose A_1, A_2, A_3 are mutually independent

So, $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

• Assume that $P(A_i) = p$, $i = 1, 2, 3$

Question: F: current flows through the circuit

what if $P(F)$

Solution:

$$F = A_1 \cup (A_2 \cap A_3)$$

$$P(F) = P(A_1 \cup (A_2 \cap A_3))$$

$$= P(A_1) + P(A_2 \cap A_3) - P(A_1 \cap (A_2 \cap A_3))$$

As the events are mutually indep.

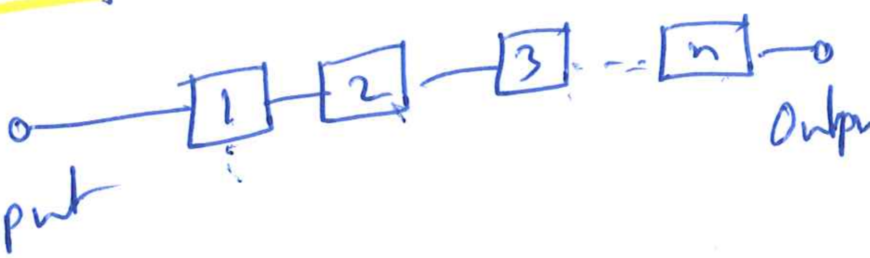
$$= P(A_1) + P(A_2).\,P(A_3) - P(A_1)P(A_2)P(A_3)$$

$$P(G \cup F) = P(G) + P(F) - P(G \cap F)$$

So, $\boxed{P(F) = p + p^2 - p^3}$

[Aside:
Mutual independence
$\Rightarrow$ Pairwise independence
But the other way is NOT
necessarily true.]

Example: Suppose that a system consists of components in series, so the system fails, if any one component fails.
If there are n mutually independent components and each fails with prob. $= p$;



Question: What is the probability that the system will fail.

Answer:

$$P(F) = 1 - (\text{Circuit works})$$

$$= 1 - (\text{When "n" components are working})$$

$$= 1 - P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$= 1 - P(A_1)\,P(A_2) \cdots P(A_n)$$

[Each A_i fails with prob. $= p$.
So, $P(A_i) = 1 - p$]
$i = 1, 2, \ldots, n$.

$$= 1 - (1-p)^n$$

For 3 events. (Mutual indep)

$$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$$

$$\left\{\begin{array}{l} P(A \cap B) = P(A)\,P(B) \\ P(B \cap C) = P(B).P(C) \\ P(C \cap A) = P(C).P(A) \end{array}\right.$$

Random Variables

Fair coin	Real numbers	Prob
$\Omega = \{$ H, H	$\to 2$	$1/4$
H T	$\to 1$	$1/4$
T H	$\to -1$	$1/4$
T T $\}$	$\to -2$	$1/4$

A random variable is a function from Ω to $\mathbb{R}$.

Two kinds

Discrete

(Random variable that takes only finitely many or countbly infinite number of values)

$\{1, 2, 3, 4, \ldots\}$

Continuous

(Radom variable that takes uncountably infinite numbe of values)

Example: Discrete

X	Prob.
1	1/4
2	1/2
7	1/8
6 7	1/8
Total	1

x	Prob.
1	$1/2^1$
2	$1/2^2$
3	$1/2^3$
4	$1/2^4$
⋮	⋮
Total.	$\frac{1}{2}+\frac{1}{2^2}+\dots = \frac{1/2}{1-1/2} = 1$

Aside

$\aleph_0$ "aleph-nought"

c : continuum

$c > \aleph_0$