Time Series

Sudipta Das

Assistant Professor,
Department of Computer Science,
Ramakrishna Mission Vivekananda Educational & Research Institute

Outline I

1. ACF and PACF of Stationary Time Series
   - ACF of Stationary Time Series
   - PACF of Stationary Time Series

ACF of Stationary Time Series I

- The autocorrelation function (ACF) of a stationary process, X_n, denoted as $\rho(h)$, for $h = 0, 1, 2, \ldots,$ is defined as follows

$$\begin{aligned} \rho(h) &= cor(X_{n+h}, X_n) \\ &= \frac{E(X_{n+h}X_n)}{\sqrt{E[X_{n+h}^2]E[X_n^2]}} \end{aligned}$$

- Remarks
  - The autocorrelation matrix R_n is positive definite for all n, where

$$R_n = \begin{bmatrix} 1 & \rho(1) & \cdots & \rho(n-1) \\ \rho(1) & 1 & \cdots & \rho(n-2) \\ \vdots & \vdots & \vdots & \vdots \\ \rho(n-1) & \rho(n-2) & \cdots & 1 \end{bmatrix}$$

ACF of MA(q) process I

- q-order moving average or MA(q) process:

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \ldots, \theta_q$ are real valued constants

  - ACF

$$\rho(h) = \begin{cases} \frac{1}{(1+\theta_1^2+\cdots+\theta_q^2)} \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & \text{if } |h| \leq q, \\ 0, & \text{if } |h| > q. \end{cases}$$

    - where θ_0 is defined to be 1

- ACF of $MA(q)$ process is **ZERO** for lags greater than q.
  - Cut-off to zero after lag q

ACF of AR(1) process I

- 1st order autoregressive or AR(1) process:

$$X_t = \phi X_{t-1} + Z_t,\ t = 0, \pm 1, \ldots,$$

  where $\{Z_t\} \sim WN(0, \sigma^2)$ and $|\phi| < 1$

- The ACF of an $AR(1)$ process

$$\rho(h) = \phi^{|h|}$$

  - Tails off to zero

ACF of ARMA(1) process I

- 1st order ARMA or ARMA(1,1) process:

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2), |\phi| < 1, Z_t$ is uncorrelated with X_s for each $s < t$ and $\phi + \theta \neq 0$

- The ACF of an $ARMA(1,1)$ process

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \frac{(\theta+\phi)(1-\phi^2)+(\theta+\phi)^2\phi}{(1-\phi^2)+(\theta+\phi)^2}, & \text{if } h = \pm 1 \\ \phi^{|h|-1}\rho(1), & \text{if } |h| \geq 2. \end{cases}$$

  - Tails off to zero

- The partial autocorrelation function (PACF) of a stationary process, X_n, denoted as $\alpha(h)$, for $h = 0, 1, 2, \ldots$ is defined as follows

$$\alpha(0) = 1, \alpha(1) = \rho(1)$$

and

$$\alpha(h) = cor(X_{n+h} - X_{n+h}^{n+1,n+h-1}, X_n - X_n^{n+1,n+h-1}), h \geq 2$$

- Remarks
  - The PACF, $\alpha(h)$, is the correlation between X_{n+h} and X_n with the linear dependence of $\{X_{n+1}, \ldots, X_{n+h-1}\}$ on each, removed.
  - Both $(X_{n+h} - X_{n+h}^{n+1,n+h-1})$ and $(X_n - X_n^{n+1,n+h-1})$ are uncorrelated with $\{X_{n+1}, \ldots, X_{n+h-1}\}$.
  - If the process X_n is Gaussian, then

$$\alpha(h) = cor(X_{n+h}, X_n | X_{n+1}, \ldots, X_{n+h-1}).$$

    - That is, $\alpha(h)$ is the correlation coefficient between X_{n+h} and X_n in the bivariate distribution of (X_{n+h}, X_n) conditional on $\{X_{n+1}, \ldots, X_{n+h-1}\}$.

PACF of Stationary Time Series III

- Theorem: $\alpha(h) = \phi_{hh}$
  - Recall, ϕ_{hh} is the last element of the vector $\phi_{\mathbf{h}}$ and $\Gamma_h \phi_{\mathbf{h}} = \gamma_{\mathbf{h}}$
- Proof:-
  - Forward MSE:

$$E\left[\left(X_{n+h} - \sum_{i=1}^{h-1} a_i X_{n+h-i}\right)^2\right]$$

  - Normal Equations

$$E\left[\left(X_{n+h} - \sum_{i=1}^{h-1} a_i X_{n+h-i}\right) X_{n+h-j}\right] = 0, \text{ for } j = 1, \ldots, h-1$$

  - Solution:

$$\gamma_{\mathbf{h-1}} = \Gamma_{h-1} \mathbf{a}_{\mathbf{h-1}}$$

PACF of Stationary Time Series IV

- Backward MSE:

$$E\left[\left(X_n - \sum_{i=1}^{h-1} b_i X_{n+i}\right)^2\right]$$

- Normal Equations

$$E\left[\left(X_n - \sum_{i=1}^{h-1} b_i X_{n+i}\right) X_{n+j}\right] = 0, \text{ for } j = 1, \ldots, h-1$$

- Solution

$$\gamma_{\mathbf{h-1}} = \Gamma_{h-1}\mathbf{b}_{\mathbf{h-1}}$$

- Therefore,

$$\mathbf{a}_{\mathbf{h-1}} = \mathbf{b}_{\mathbf{h-1}} = \phi_{\mathbf{h-1}}$$

- As a result,

$$
\begin{aligned}
\alpha(h) &= cor(X_{n+h} - X_{n+h}^{n+h-1,n+1}, X_n - X_n^{n+1,n+h-1}) \\
&= \frac{E\left[\left(X_{n+h} - \phi'_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)\left(X_n - \phi'_{\mathbf{h-1}}\mathbf{X_{n+1,n+h-1}}\right)\right]}{\sqrt{E\left[\left(X_{n+h} - \phi'_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)^2\right] E\left[\left(X_n - \phi'_{\mathbf{h-1}}\mathbf{X_{n+1,n+h-1}}\right)^2\right]}} \\
&= \frac{E\left[\left(X_{n+h} - \phi'_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)\left(X_n - \phi'^{(\mathbf{r})}_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)\right]}{\sqrt{E\left[\left(X_{n+h} - \phi'_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)^2\right] E\left[\left(X_n - \phi'^{(\mathbf{r})}_{\mathbf{h-1}}\mathbf{X_{n+h-1,n+1}}\right)^2\right]}} \\
&= \frac{\gamma(h) - \phi'_{\mathbf{h-1}}\gamma^{(\mathbf{r})}_{\mathbf{h-1}} - \phi'^{(\mathbf{r})}_{\mathbf{h-1}}\gamma_{\mathbf{h-1}} + \phi'_{\mathbf{h-1}}\Gamma_{h-1}\phi^{(\mathbf{r})}_{\mathbf{h-1}}}{\sqrt{\left[\gamma(0) - \phi'_{\mathbf{h-1}}\Gamma_{h-1}\phi_{\mathbf{h-1}}\right]\left[\gamma(0) - \phi'^{(\mathbf{r})}_{\mathbf{h-1}}\Gamma_{h-1}\phi^{(\mathbf{r})}_{\mathbf{h-1}}\right]}} \\
&= \frac{\gamma(h) - \phi'_{\mathbf{h-1}}\gamma^{(\mathbf{r})}_{\mathbf{h-1}} - \phi'_{\mathbf{h-1}}\gamma^{(\mathbf{r})}_{\mathbf{h-1}} + \phi'_{\mathbf{h-1}}\gamma^{(\mathbf{r})}_{\mathbf{h-1}}}{\gamma(0) - \phi'_{\mathbf{h-1}}\Gamma_{h-1}\phi_{\mathbf{h-1}}} \\
&= \frac{\gamma(h) - \phi'_{\mathbf{h-1}}\gamma^{(\mathbf{r})}_{\mathbf{h-1}}}{v_{h-1}} = \phi_{hh}
\end{aligned}
$$

PACF of AR(p) process I

- p-order autoregressive or AR(p) process:

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2),\ Z_t$ is uncorrelated with X_s for each $s < t$ and all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ lie outside the unit circle.

- PACF of causal $AR(p)$
  - For $h \geq p$ the best linear predictor of X_{h+1} in terms of $1, X_1, \ldots, X_h$ is
  $$X_{h+1} = \phi_1 X_h + \phi_2 X_{h-1} + \cdots + \phi_p X_{h+1-p}.$$
  Since the coefficient ϕ_{hh} of X_1 is ϕ_p if $h = p$ and 0 if $h > p$, we conclude that the
  $$\alpha(h) = \phi_p \text{ for } h = p$$
  and
  $$\alpha(h) = 0 \text{ for } h > p$$
- PACF of a causal $AR(p)$ process is **ZERO** for lags greater than p.
  - Cut-off to zero after lag p

PACF of MA(1) process I

- 1st order moving average or MA(1) process:

$$X_t = Z_t + \theta Z_{t-1}, \ t = 0, \pm 1, \ldots,$$

  where $\{Z_t\} \sim WN(0, \sigma^2)$ and θ is a real constant.
- The PACF of an $MA(1)$ process

$$\alpha(h) = \phi_{hh} = -(-\theta)^h / (1 + \theta^2 + \cdots + \theta^{2h}).$$

  - Tails off to zero

ACF & PACF of Stationary Time Series I

- Behavior of the ACF and PACF for Causal and Invertible ARMA Models

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off