Probability and Stochastic Processes

Indranil Sen Gupta.

indranil.sengupta@ndsu.edu

Sample Space

Doing an experiment

- Flipping a coin {Head, Tail}
- Throwing a die. {1, 2, 3, 4, 5, 6}

A set of all possible outcomes of an experiement is called the sample space.

<u>Book:</u>

Mathematical Statistics and Data Analysis

— John Rice.

<u>Example:</u> <u>Earthquake modeling</u>

We try to model the length of time between two successive earthquakes.

Sample space = All positive ($t \geq 0$) numbers.

Ω

$$\Omega = \{ t \mid t \geq 0 \}$$

<u>Example:</u> <u>Tossing two coins</u>

Head = H
Tail = T.

$$\Omega = \{ HH, \; HT, \; TH, \; TT \}$$

~~There.~~
Three coins

$$\Omega = \{HHH,\ HHT,\ HTH,\ HTT,\ THH,\ THT,\ TTH,\ TTT\}$$

Events: Subset of a sample space.

Example: Tossing three coins and getting first outcome = H.

$$E = \{HHH,\ HHT,\ HTH,\ HTT\}$$

Probability

A probability measure on Ω is a function P from subsets of Ω to the real numbers that satisfies the following axioms:

(1) $P(\Omega) = 1$

(2) If $A \subset \Omega$, then $P(A) \geq 0$

(3) If A_1 and A_2 are disjoint $(A_1 \cap A_2 = \phi)$

Then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

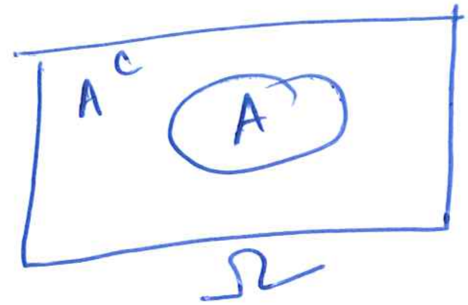
[If $A_1, A_2, \ldots, A_n, \ldots$ are mutually disjoint sets then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$]

Property 1:

$$P(A^c) = 1 - P(A)$$

Proof:



$$\Omega = A \cup A^c$$

$$P(\Omega) = P(A \cup A^c)$$

|| by (3)

by (1) ||

$$1 = P(A) + P(A^c)$$

$$\boxed{P(A^c) = 1 - P(A)}$$

(Done)

Property 2: $P(\phi) = 0$

Proof: Note that $\Omega^c = \phi$

So, by "Property 1"

$$P(\Omega^c) = 1 - P(\Omega)$$

// by (1)

$$= 1 - 1$$

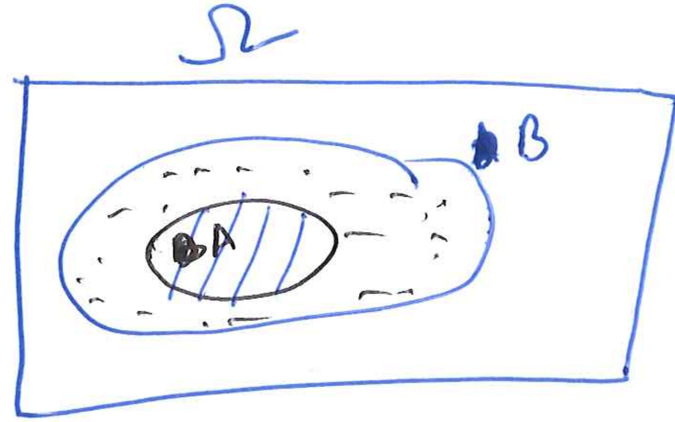
$$P(\phi) = 0$$

(Done)

Property 3:

If $A \subset B$, then $P(A) \leq P(B)$.

Proof:



$$B = A \cup (B \cap A^c)$$

(Disjoint)

$$P(B) = P(A \cup (B \cap A^c))$$

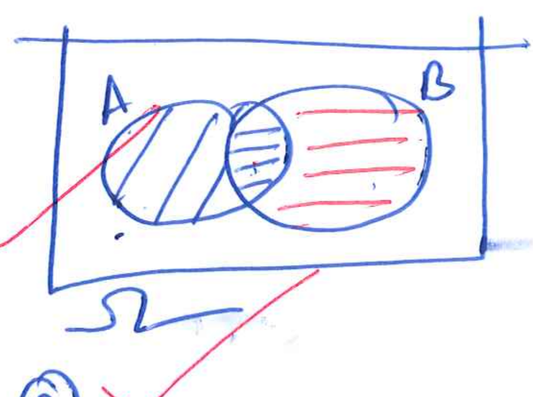
$$\underset{(\text{by } (3))}{=} P(A) + \underbrace{P(B \cap A^c)}_{\geq 0 \ (\text{by } (2))}$$

$$\Rightarrow \boxed{P(B) \geq P(A)}$$

(Done)

Property 4: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof:



$$A = (A \setminus B) \cup (A \cap B)$$

$$P(A) = P(A \setminus B) + P(A \cap B) \quad \ldots (1)$$

$$B = (B \setminus A) \cup (A \cap B)$$

$$P(B) = P(B \setminus A) + P(A \cap B) \quad \ldots (2)$$

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A)$$

$$P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$$

$$= \cancel{(P(A) - P(A \setminus B)) + P}$$

by 1 by 2

$$= (P(A) - \cancel{P(A \cap B)}) + \cancel{P(A \cap B)} + (P(B) - P(A \cap B))$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$ (Done)

Computing Probabilities

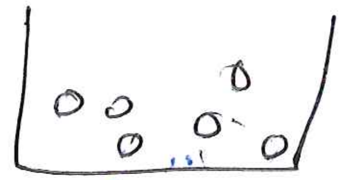
If A can occur in any of $\boxed{n}$ mutually exclusive ways out of a total $\boxed{N}$ ways, then

$$P(A) = \frac{n}{N}$$

$$\boxed{P(A) = \frac{\text{number of ways A can occur}}{\text{total number of outcomes}}}$$

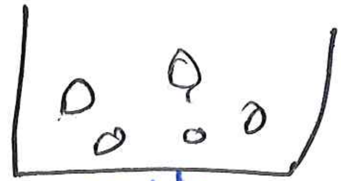
Example:

A black urn contains 5 red and 6 green balls.



Black

A White urn contains 3 red and 4 green balls.



White

If you choose a red ball, you get a prize!

If the black urn is chosen

prob. of getting red ball $= \frac{5}{11} \simeq 0.45$

If the white urn is chosen

prob. of getting red ball $= \frac{3}{7} \simeq 0.42$

A ~~black~~ white urn contains 9 red balls, and 5 green balls.

~~Black~~ 2 White

$\rightarrow P(R) = 9/14 \approx 0.643$

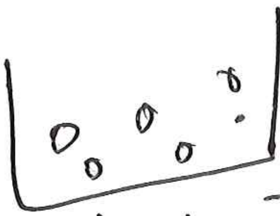
A ~~white~~ black urn contains 6 red balls and 3 green balls

Black 2

$\rightarrow P(R) = \frac{6}{9} \approx 0.667$

Final step:

Mix the balls of black urns in a single black urn



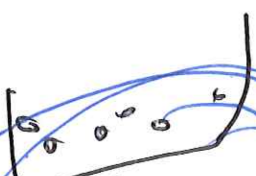
Black

5 red balls + 6 green balls
6 red balls + 3 green balls

11 red balls + 9 green balls.

$P(R) = \frac{11}{20} = 0.55$

Do the same for the white urn



White

3 red balls + 4 green balls
9 red balls + 5 green balls

12 red balls + 9 green balls

$P(R) = \frac{12}{21} = 0.57$

Choose White urn

Multiplication Principle

Example:

In a class 12 white students, 18 black students. A teacher selects 1 white and 1 black student to act as a class representative. How many ways?

$$12 * 18 = \boxed{216}$$

In general:

If one experiment has m outcomes and another experiment has n possible outcomes, then there are $\underline{mn}$ outcomes for the $\underline{\text{two}}$ experiments,

Example: 8-bit binary word.
→ How many?

$$\underline{2} * \underline{2} * \underline{2} * \underline{2} * \underline{2} * \underline{2} * \underline{2} * \underline{2} = 2^8 = 256$$

Permutation:

For a set of size n, and a sample of size r, there are ${}^nP_r$ different ORDERED samples without replacemt.

$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}} = n.(n-1)(n-2)\ldots\ldots(n-r+1)$$

Example: 10 children

five of Them are to chosen and lined up.

How many ways?

$${}^{10}P_5 = 10 * 9 * 8 * 7 * 6 = \underline{30240}$$

$\underline{10}$ $\underline{9}$ $\underline{8}$ $\underline{7}$ $\underline{6}$

Example: (Birthday - Problem)

Suppose that a room contains 23 students. What is the probability that at least two of them have a common birthday?

Solution: A : ~~None~~ Required

A^c = No two students will have same birthday

A^c can happen in how many ways? $^{365}P_{23}$

The way all possible outcome can happen is $= (365)^{23}$

$$P(A^c) = \frac{^{365}P_{23}}{(365)^{23}} = \frac{\frac{365!}{(365-23)!}}{(365)^{23}}$$

$$P(A) = 1 - P(A^c) = 1 - (\ldots) = 0.507$$

In general

n: students

Question: What is the probabitity that at least two of them have a common birthday?

Answer: A: event in question.

A^c: None of the students share birthday.

$$P(A^c) = \frac{{}^{365}P_n}{(365)^n}$$

$$\boxed{P(A) = 1 - P(A^c) = 1 - \frac{{}^{365}P_n}{(365)^n}}$$

n	$P(A)$
4	0.016
16	0.284
23	0.507
32	0.753
40	0.891
56	0.988

License plate problem

ABC 719

How many distinct such plates are possible?

~~777 000~~

ABC 123

$26^3 * 10^3 = 17,576,000$

Question: If all sequences of six characters are equally likely, what is the probability that the license plate for a new car will contain NO duplicate letters or numbers?

~~AAB123~~

~~ABC112~~

Answer:

$$P(A) = \frac{{}^{26}P_3 * {}^{10}P_3}{17576000} = 0.64$$

The outcomes favorable to event

Total possible outcomes.

Combination

The number of **UNORDERED** samples of **r objects** from **n objects** **without** replacement is

$$\binom{n}{r} = \left({}^{n}C_{r}\right)$$

$$= \frac{n!}{r!\,(n-r)!}$$

Example: **"California jackpot"**

Until 1991: **Jackpot** if 6 #s from 1 to 49 were chosen correctly.

: number

$\{7, 6, 8, 23, 48, 12\}$ ✓

$\{n_1, n_2, n_3, n_4, n_5, n_6\}$

There are $\binom{49}{6}$ ways of choosing 6 numbers from 49 numbers.

$$= 13,983,816$$

$$\text{Chance} = \frac{1}{13,983,816}$$