9/20

Joint Distribution

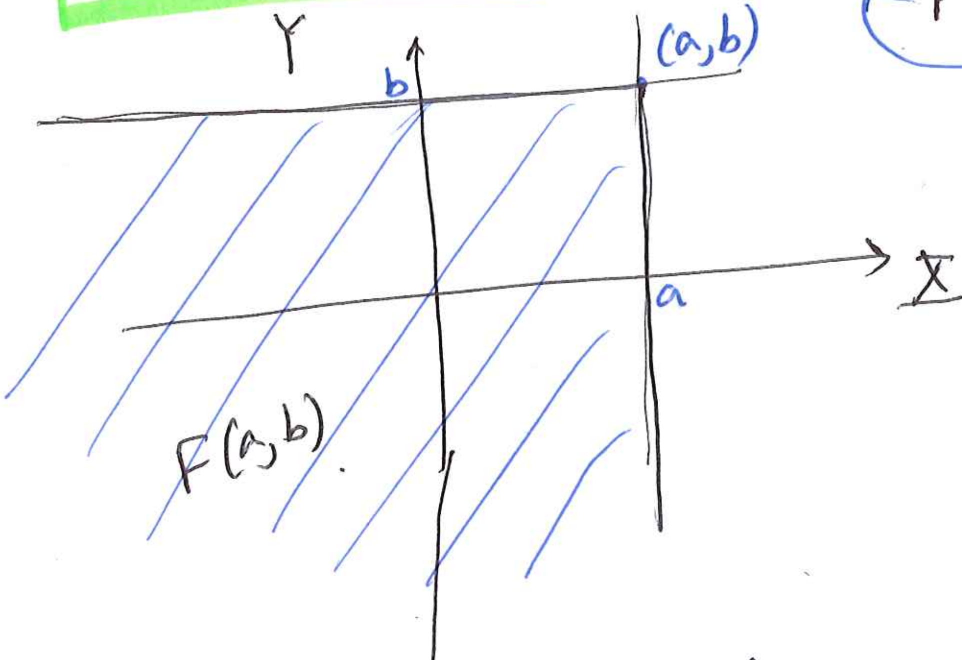
So far : r.v. X → Discrete (pmf)
→ Continuous (pdf)

(cdf)

Two random variables X and Y.

The cdf (for two variable)

$$\boxed{F(x,y) = P(\underbrace{X \le x,\ Y \le y}_{F(a,b)})}$$



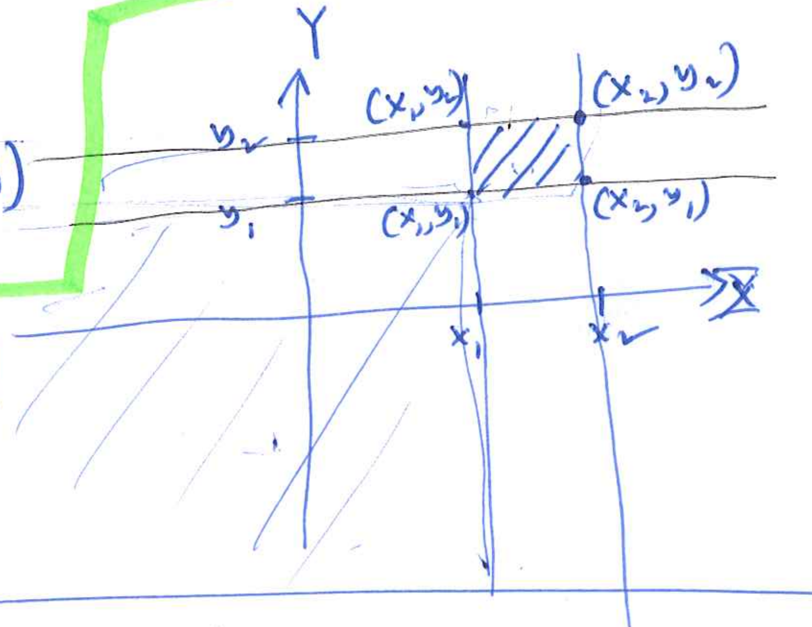
Multiple variable: Joint cdf is:

$$F(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$$

Compute (in terms of joint cdf)

$$\boxed{\begin{aligned} &P(x_1 \le X \le x_2,\ y_1 \le Y \le y_2) \\ &= F(x_2, y_2) - F(x_2, y_1) \\ &\quad - F(x_1, y_2) + F(x_1, y_1) \end{aligned}}$$

General formula!



Discrete random variables

A fair coin is tossed 3 times.

X : # of heads on the first toss.

Y : total # of heads

$P(X=0, Y=0)$ $\quad$ $P(X=0, Y=2)$

(frequency)

Joint probability function ~~distribution~~ for X and Y

$x \backslash y$	0	1	2	3	
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0	$\frac{4}{8}$
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{4}{8}$
	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

$\Omega = \{$ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT $\}$

Suppose we wish to find the frequency function of Y from the joint ~~distribution~~ frequency.

Marginal frequency

$$\boxed{P(Y=0)} = P(Y=0, X=0) + P(Y=0, X=1)$$
$$= p_Y(0) = \frac{1}{8} + 0 = \frac{1}{8}$$

$$\boxed{P(Y=1)} = P(Y=1, X=0) + P(Y=1, X=1)$$
$$= p_Y(1) = \frac{2}{8} + \frac{1}{8} = \frac{3}{8}.$$

Similarly:

$$\boxed{P(Y=2)} = \frac{3}{8} \qquad p_Y(2)$$
$$\boxed{P(Y=3)} = \frac{1}{8} \qquad = p_Y(3)$$

In general:

$$p_Y(y_i) = P(Y = y_i)$$

(little p) (big P)

$$= \sum_{j=1}^{n} p(x_j, y_i)$$

(little p)

joint frequency

$$\boxed{p(x_j, y_i) = P(X = x_j, Y = y_i)}$$

Similarly

$$p_X(x_i) = \sum_{j=1}^{m} p(x_i, y_j)$$

In case of several random variables $(\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_m)$

Write:

$$\boxed{p(x_1, x_2, \ldots, x_m) = P(\underline{X}_1 = x_1, \underline{X}_2 = x_2, \ldots, \underline{X}_m = x_m)}.$$

[Joint ~~distribution~~ frequency function].

The marginal frequency function of $\underline{X}_1$ is given by:

$$\boxed{p_{\underline{X}_1}(x_1) = \sum_{x_2, \ldots, x_m} p(x_1, x_2, \ldots, x_m)}$$

$x_1, x_2, \ldots, x_m$ ~~all take n values~~.

$x_1 \to n_1$ values

$x_2 \to n_2$ values

$x_m \to n_m$ values.

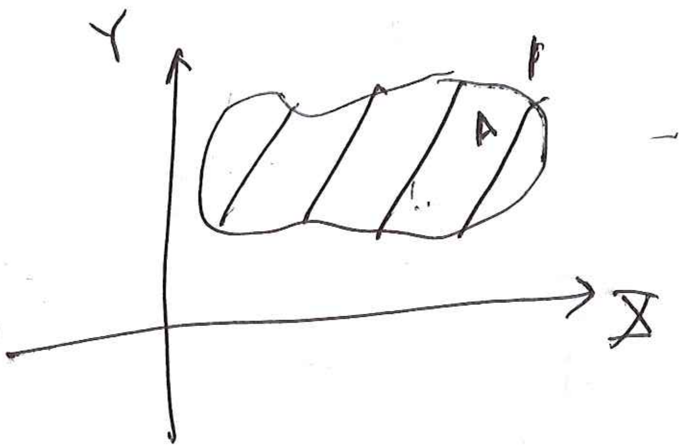
$$p_{\underline{X}_1}(x_1) = \sum_{x_2} \sum_{x_3} \sum_{x_4} \cdots \sum_{x_m} p(x_1, x_2, \ldots, x_m).$$

Continuous random variables

Suppose X and Y are continuous random variables with a joint cdf $F(x,y)$

Their $\boxed{\text{JOINT DENSITY FUNCTION}}$ is piecewise continuous function of two variables, $\boxed{f(x,y)}$, such that for any "reasonale" two dimensional set A:

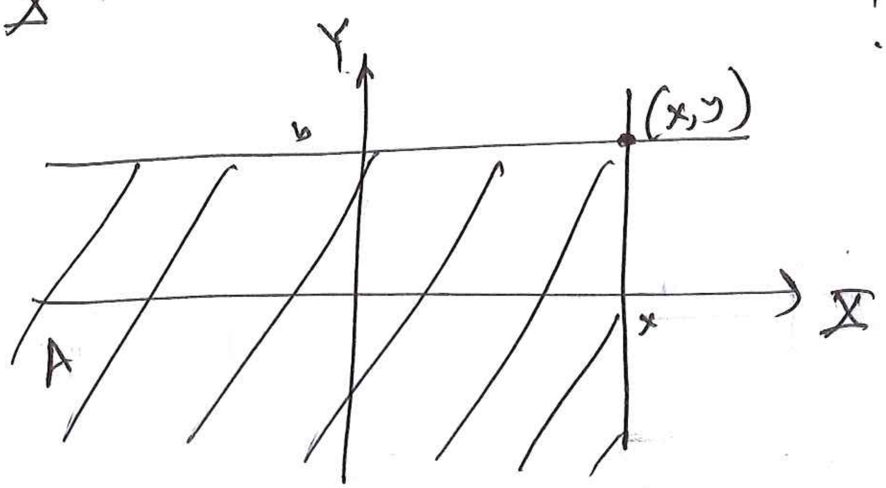
$$\boxed{P((X,Y) \in A) = \iint_A f(x,y)\, dy\, dx}$$



In particular

$$A = \{ (X,Y) \mid X \le x,\ Y \le y \}$$

$$[P(A) = F(x,y)]$$



A: described above

$$F(x,y) = P\left((X, Y) \in A\right) \stackrel{\text{def}}{=} \iint_A f(x,y)\,dy\,dx$$

$$= \iint_A f(u,v)\,du\,dv.$$

$$\boxed{F(x,y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u,v)\,dv\,du}$$

$$\Downarrow$$

$$\boxed{\frac{\partial^2}{\partial x \partial y} F(x,y) = f(x,y)}$$

whenever the derivative is defined.

Aside: (one-variable)

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

cdf ↑ ↑ pdf

$$F'(x) = f(x)$$

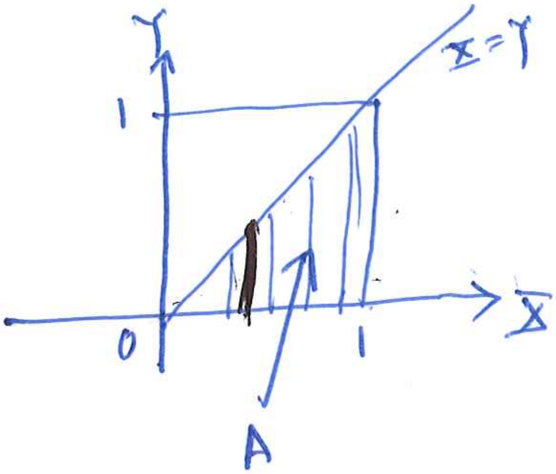
Example: Consider $f(x,y) = \frac{12}{7}(x^2 + xy)$, $0 \le x \le 1$, $0 \le y \le 1$

(joint density function)

Find $P(X > Y)$

Solution:

$$P(X > Y) = \iint_A f(x,y)\,dy\,dx$$



$$= \int_{x=0}^{1}\int_{y=0}^{x} \frac{12}{7}\left(x^2 + xy\right)\,dy\,dx$$

$$= \frac{12}{7}\int_{x=0}^{1} \left(x^2 y + x\frac{y^2}{2}\right)\Bigg|_{y=0}^{x}\,dx$$

$$= \boxed{\frac{9}{14}}$$

Marginal density:

$$f_X(x) = \int_{y=-\infty}^{\infty} f(x,y)\,dy$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f(x,y)\,dx$$

Marginal cdf:

$\cancel{f_X(x)}$

$$F_X(x) = P(X \le x)$$

$$F_X(x) = \int_{u=-\infty}^{x}\int_{v=-\infty}^{\infty} f(u,v)\,dv\,du$$

$$F_Y(y) = \int_{v=-\infty}^{y}\int_{u=-\infty}^{\infty} f(u,v)\,dv\,du$$

Example: For the last joint density function, compute the marginal density of X. $\left(f_X(x) \right)$

$$\boxed{f_X(x)} = \int_{y=-\infty}^{\infty} f(x, y)\, dy$$

fixed

$$= \int_{y=0}^{1} \frac{12}{7}(x^2 + xy)\, dy \quad \left[\because f(x,y) = 0 \text{ when } y \notin [0,1] \right]$$

$$= \frac{12}{7}\left(x^2 y + x\frac{y^2}{2} \right)\Bigg|_{y=0}^{1}$$

$$= \boxed{\frac{12}{7}\left(x^2 + \frac{x}{2} \right)}, \quad 0 \le x \le 1.$$

Similarly the marginal density of Y.

$$\boxed{f_Y(y)} = \int_{x=-\infty}^{\infty} f(x, y)\, dx$$

fixed

$$= \int_{x=0}^{1} \frac{12}{7}(x^2 + xy)\, dx$$

$$= \frac{12}{7}\left(\frac{x^3}{3} + \frac{x^2}{2} y \right)\Bigg|_{x=0}^{1}$$

$$= \boxed{\frac{12}{7}\left(\frac{1}{3} + \frac{y}{2} \right)}, \quad 0 \le y \le 1 //$$

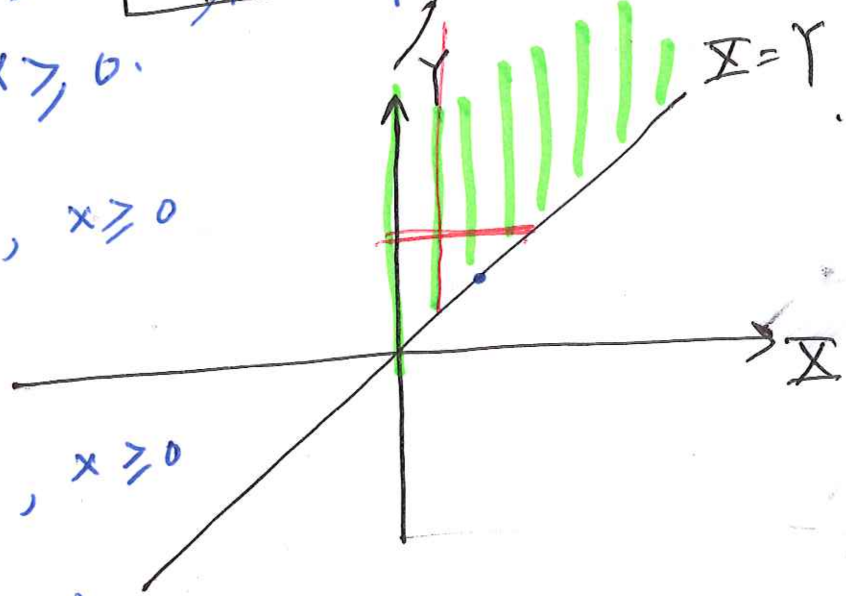
Example: Consider the following joint density:

$$\boxed{f(x,y) = \begin{cases} \lambda^2 e^{-\lambda y}, & 0 \le x \le y \\ & \lambda > 0. \\ 0, & \text{elsewhere} \end{cases}}$$

Find the marginal densities

$$\boxed{f_X(x)} \overset{def}{=} \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=x}^{\infty} f(x,y)\, dy, \quad x \geq 0.$$



$$= \int_{y=x}^{\infty} \lambda^2 e^{-\lambda y}\, dy, \quad x \geq 0$$

$$= \lambda^{\cancel{2}} \left. \frac{e^{-\lambda y}}{-\cancel{\lambda}} \right|_{y=x}^{\infty}, \quad x \geq 0$$

$0 \le X \le y.$

$$= \cancel{\lambda} - \lambda\left(0 - e^{-\lambda x}\right), \quad x \geq 0$$

$$= \boxed{\lambda e^{-\lambda x}, \quad x \geq 0}$$

$\Rightarrow$ Exponential distribution!

$$\boxed{f_Y(y)} \stackrel{def}{=} \int_{x=-\infty}^{\infty} f(x,y)\,dx \quad , \quad y \geq 0$$

$$= \int_{x=0}^{y} \underbrace{\lambda^2 e^{-\lambda y}} \,dx \quad , \quad y \geq 0$$

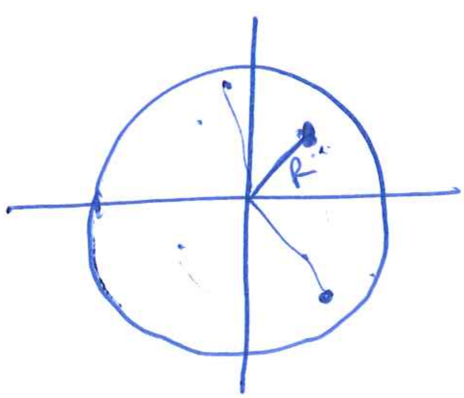
$$= \lambda^2 e^{-\lambda y} \int_{x=0}^{y} dx \quad , \quad y \geq 0$$

$$= \boxed{\lambda^2 \, y \, e^{-\lambda y} \quad , \quad y \geq 0.}$$

$\Rightarrow$ Gamma distribution

Example: A point is chosen randomly in a disk of radius 1.

(disc) (Area = π)



$$\boxed{f(x,y) = \begin{cases} \dfrac{1}{\pi} , & \text{if } x^2+y^2 \leq 1 \\ 0 , & \text{otherwise.} \end{cases}}$$

[R : distance of the point from the origin

R : random variable

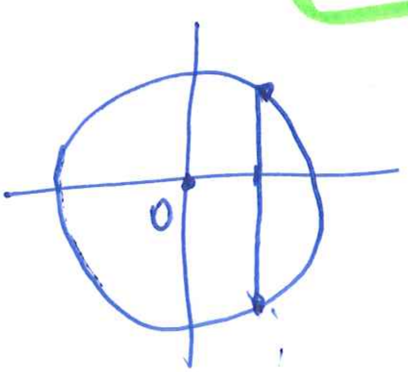
$0 \le R \le 1$]

Marginal density of the x-coordinate of the random point:

$$\boxed{f_X(x)} \stackrel{def}{=} \int_{y=-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy$$

$$= \boxed{\frac{2\sqrt{1-x^2}}{\pi}, \quad -1 \le x \le 1}$$



$x^2 + y^2 = 1$

$y = \pm\sqrt{1-x^2}$

Similarly

$$\boxed{f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}, \quad -1 \le y \le 1}$$

Bivariate Normal Density

(Joint density function)

$$\boxed{f(x,y) = \frac{1}{2\pi \sigma_X \sigma_Y \sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)} \left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right) \right]}$$

where,

$$-\infty < \mu_X < \infty$$

$$-\infty < \mu_Y < \infty$$

$$\sigma_X > 0, \quad \sigma_Y > 0, \quad -1 < \rho < 1$$

- The marginal distribution of X

$$\boxed{f_X(x)} = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \boxed{\frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}$$

So, marginal distribution of X $\sim N(\mu_X, \sigma_X^2)$

Similarly

$$\boxed{f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}}$$

So, marginal distribution of Y

$$\sim N(\mu_Y, \sigma_Y^2)$$