Def$^n$: a) A linear form of complex n-space $\mathbb{C}^n$ is polyn of deg 1

in n complex variables $x_1, x_2, \dots, x_n$ of the form

$$\underline{b}^* \underline{x} = \sum_{i=1}^{n} \overline{b_i}\, x_i \quad \text{where} \quad \underline{x} = [x_1 x_2 \cdots x_n]^T$$

$$\underline{b} = [b_1 b_2 \cdots b_n]^T \text{ in } \mathbb{C}^n$$

b) A complex quadratic form on $\mathbb{C}^n$ is a polynomial of deg 2

in n complex variables $x_i$, $i = 1(1)n$ of the form

$$q(x) = x^* A x = \left[\overline{x_1}\ \overline{x_2} \cdots \overline{x_n}\right] \left[a_{ij}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\boxed{x^* = (\overline{x})^T}$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \overline{x_i}\, x_j$$

where $\underline{x} \in \mathbb{C}^n$ & $A = [a_{ij}]$ is an $n \times n$ Hermitian matrix.

For example: $\langle \underline{x}, A\underline{x} \rangle = (A\underline{x})^* x = x^* A x = q(x)$

Remark:- For the def$^n$ Let $A$ be any complex matrix $(n \times n)$

Then for $\underline{x}$ in $\mathbb{C}^n$, the matrix product $x^* A x$ is a complex no.

But for matrix $A$, it's known that $A = B + iC$ for Hermitian matrices

$B$ & $C$ given by $B = \frac{1}{2}(A + A^*)$ and $iC = \frac{1}{2}(A - A^*)$

Hence $x^* A x = x^*(B + iC)x = x^* B x + i x^* C x$

where $x^* B x$ & $x^* C x$ are real nos.

Hence for a Complex quadratic form on $\mathbb{C}^n$, we are only concerned with a Hermitian matrix $A$ so that $x^* A x$ is a real no. for any $\underline{x} \in \mathbb{C}^n$.

**Th. Principal axes Theorem:-**

Let $\underline{x}^* A \underline{x}$ be a complex quadratic form on $\mathbb{C}^n$ with the Hermitian matrix $A$. Then there is a change of co-ordinate of $x$ into

$$y = U^* \underline{x} = [y_1\ y_2 \cdots y_n]^T \quad \text{s.t.} \quad \underline{x}^* A \underline{x} = \lambda_1 |y_1|^2 + \lambda_2 |y_2|^2 + \cdots + \lambda_n |y_n|^2$$

where '$U$' is unitary and '$D$' is diagonal ($U^* A U = D$).

Here the Columns of $U$ form an orthonormal basis of $\mathbb{C}^n$, Called the principal axes of the quadratic form.

$A > 0$ (+ve matrix)

$A \geq 0$ (+ve semidefinite matrix).

**Th.** A sym. $n \times n$ T.F.A.E:-

(a) $A$ is positive Definite

(b) The leading principal submatrix $A_1, A_2, \ldots, A_n$ has (+ve) determinant.

(c) $A$ can be reduced to upper triangular form by using only type III elem. row operations and pivots are all (+ve).

(d) $A$ has a cholesky factorization $LL^t$ (where $L$ is lower triangular with (+ve) diagonal Matrices).

(e) A can be factorized into a product $B^T B$ for some non-singular matrix B.

<u>Proof</u>:- (a) $\Rightarrow$ (b)

A is symmetric (+ve) def, hence its eigen values are all (+ve) where A is non singular (as det A > 0). Let $A_r$ be leading principal submatrix of ~~radius~~ order r. $1 \leq r \leq n$. Let $\underline{x_r} = (x_1, x_2, \dots, x_r)$ be any non zero vector in $\mathbb{R}^n$.

set $\underline{x} = (x_1, x_2, \dots, x_r, 0, 0, \dots, 0) \in \mathbb{R}^n$. since

$$\underline{x}_r^T A_r \underline{x_r} = \underline{x}^T A x > 0$$

it follows that $A_r$ is (+ve) def$^n$. where $A_r > 0$ $\forall r$ $1 \leq r \leq n$

(b) $\Rightarrow$ (c)

From earlier knowledge (Strang) of viability of A = LU decomposition, we know. here $A \overset{r}{\sim} U$ by type-III. operation only and all pivots will be (+ve)

(c) $\Leftrightarrow$ (d)

[Cholesky factorization]:- If A is symmetric (+ve) def., then A can be factored into a product $LL^T$, where L = lower triangular with (+ve) diagonal. Now, we see that

$A = LU = LDV$ . $D$ = diagonal matrix with (+ve) pivots of $A$ featuring in the diagonal positions of $V$, which is still upper triangular (with each diagonal is 1); since $A$ is symmetric.

$A = A^T$ whence

$$LDV = A = A^T = (LDV)^T = V^T D^T L^T = V^T D L^T$$

From uniqueness of $LDV$ decomposition, $V = L^T$

Hence $A = LDL^T$ since diagonal elements of $D$ say $d_1, d_2, \dots, d_n$ are all (+ve), we can take $D^{1/2} = \begin{bmatrix} \sqrt{d_1} & & 0 \\ & \sqrt{d_2} \ddots & \\ 0 & & \sqrt{d_n} \end{bmatrix}$

set, $L_1 = LD^{1/2}$, then, $A = LDL^T = LD^{1/2}D^{1/2}L^T = LD^{1/2}(LD^{1/2})^T$

$= L_1 L_1^T$

$(d) \Rightarrow (e)$

Let $A = LL^T$ If we set $B = L^T$, then $B$ is non singular and $A = LL^T = B^T B$

$(e) \Rightarrow (a)$

$A = B^T B$ $(|B| \neq 0)$. For any $\underline{x} \in \mathbb{R}^n \setminus \{0\}$

set $\underline{y} = B\underline{x}$, since $B$ is non singular, $\underline{y} \neq 0$ & it follows that

$$\underline{x}^T A \underline{x} = \underline{x}^T B^T B \underline{x} = \underline{y}^T \underline{y} = \|y\|^2 > 0$$

$\therefore$ $A$ is (+ve) def. (Proved)

**Remark:-**

Analogous result may fail for a matrix is not (+ve) semidef. ~~semi def~~

Counter example:-

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & 1 & -3 \\ -3 & -3 & 5 \end{bmatrix}$$

Check:- $|A_1| = 1$, $|A_2| = 0$, $|A_3| = 0$

eigen value is $(-1)$. Corresponding eigen vector $(1,1,1)$

take $\underline{x} = (1,1,1)$ & check $\underline{x}^T A \underline{x} = (-3)$

## A note on (+ve) (semi) definite matrices:-

$A_{n\times n}$ (hermitian) :- It's said to be (+ve) def. if $x^* A x > 0$ for all non zero $x \in \mathbb{C}^n$.

It's called (+ve) semi def. if $x^* A x \geq 0$ $\forall x \in \mathbb{C}^n$

Th. For a Hermitian matrix $A_{n\times n}$ TFAE :

i) $\underline{x}^* A \underline{x} > 0$ $\forall$ non zero $x \in \mathbb{C}^n$

ii) All eigenvalues of $A$ are (+ve)

iii) $\exists$ a non singular (complex) matrix $P$ s.t

$A = P^* I_n P$

($A$ is conjuntive to $I_n$. This is complex counterpart of being congruent over $\mathbb{R}$).

(iv) $A = P^* P$ for some non singular $P$.

Note:- Both Congruence and conjunctivity equivalent relation but this is quite different from the similarity relation. Similar matrices represent same linear transformation w.r.t different choice of basis. Congruent matrices $A$ & $B$ satisfying $B = P^T A P$ and represent the same bilinear form, w.r.t different Basis. In the very special case $P^{-1} = P^T$. congruence and similarity coincides but this is not the case in general.

Proof :- Let $\lambda$ be an eigen value of $A$ with associated eigen vector $x$. Clearly $x \neq 0$. Then $x^* A x = \lambda (x^* x)$ since $x^* x > 0$ then this shows (i) $\Rightarrow$ (ii)

It's known that, if $H$ be Hermitian with inertia $iH = (p, q, t)$ then $H$ is conjunctive to the matrice

$I_p \oplus (-I_q) \oplus O_t$

i.e to a diagonal matrix with $p$ - (+ve) 1, $q$ - (-ve) 1 and $t$ zeros in the non diagonal. so (ii) $\Rightarrow$ (iii)

(iii) $\Rightarrow$ (iv) is trivial.

Finally, if $A = P^* P$ (for $|P| \neq 0$)

then $x^* A x = x^* P^* P x = \|Px\|^2$ since $|P| \neq 0$, $Px$ is non zero for any $x \neq 0$, whence (iv) $\Rightarrow$ (i)

⊞ $A_{n\times n}$ Hermitian then F.A.E:

(i) $x^* A x \geq 0 \ \forall$ nonzero $x \in \mathbb{C}^n$

(ii) all eigen values are non negative

(iii) $A = P^* P$ for some square matrix $P$.

<u>Result</u>:-

(Hadmard inequality)

Let $A$ be $n\times n$ positive semidef. matrix then.

$\det A \leqslant a_{11} a_{22} \cdots a_{nn}$

⊞ <u>To find a square root of (+ve) semidefinite</u>

<u>(Hermitian) matrix $K$.</u>

Let $U$ be unitary matrix which diagonalizes $K$.

and put $U^* K U = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

since $\lambda_i \geq 0$, it has a ~~unique~~ real, non negative sq. root.

Put $D = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$

Then $K = U D^2 U^*$ $\therefore$ $U D U^*$ is a square root of $K$.

<u>Note that</u>:- $UDU^*$ is (+ve) semidef. and if then $K$ is (+ve) semidef.

then so $UDU^*$.

<u>Result 1</u> Let $D = \lambda_1 I_{m_1} \oplus \lambda_2 I_{m_2} \oplus \cdots \oplus \lambda_k I_{m_k}$ where $\lambda_i$ are distinct; $i = 1(1)k$ and $\sum_{i=1}^{k} m_i = n$

Thus an $n\times n$ matrix $A$ commutes with $D$ iff $A$ has the form

$A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ with $A_i$ as $m_i \times m_i$

2. Let $A$ be a diagonalizable matrix with non negative eigen values $\lambda_1, \lambda_2, \cdots, \lambda_n$ then $B$ commutes with $A$ iff $B$ commutes with $A^2$.

Th. If $H$ and $K$ are positive semidefinite hermitian matrices and $H^2 = K^2$ then $H = K$.

Proof :- Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ are the distinct eigen values of $H$. Let $m_i$ be the multiplicity of $\lambda_i$

Let $D = \lambda_1 I_{m_1} \oplus \lambda_2 I_{m_2} \oplus \cdots \oplus \lambda_k I_{m_k}$. Then, $H^2$ has eigen values $\lambda_1^2, \lambda_2^2, \cdots, \lambda_k^2$ with multiplicities $m_1, m_2, \cdots, m_k$ respectively

Since $H^2 = K^2$ and $K$ is also (+ve) semi def., the eigen values of $K$ must be $\lambda_1, \lambda_2, \cdots, \lambda_k$ with multiplicities $m_1, m_2, \cdots, m_k$ Hence $\exists$ unitary matrices $U$ and $V$ s.t $H = UDU^*$, $K = VDV^*$

$\therefore H^2 = K^2$

$\Rightarrow UD^2U^* = VD^2V^*$

and so, $(V^*U)D^2 = D^2(V^*U)$ i.e $V^*U$ commutes with $D^2$.

$\therefore$ By result 1, $V^*U = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ where $A_i$ is $m_i \times m_i$ and so $V^*U$ commutes with $D$ (By result - 2)

Hence, $H = UDU^* = (VV^*)UDU^*(VV^*) = V(V^*U)D(U^*V)V^*$

$= VD(V^*U)(U^*V)V^*$

$= VDV^*$

$= K$

$V$ : Finite Dimensional Inner product space.

and $T : V \to V$ (linear op.)

It's known that

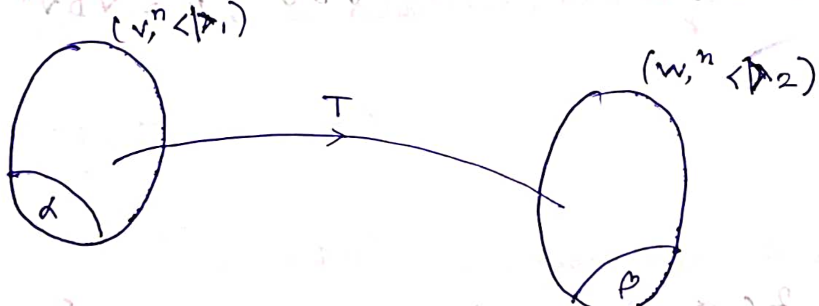
$\exists$ orthogonal basis of $V$ made with eigenvectors of $T$ $\Longleftrightarrow$ $\xrightarrow{V \text{ over } \mathbb{C}}$ $T$ is normal $(TT^* = T^*T)$

$\xrightarrow{V \text{ over } \mathbb{R}}$ $T$ is self Adjoint $(T = T^*)$

where $T^* = \bar{T}^T$, so over $\mathbb{R}$: $T^* = T^T$

## A Generalization

Singular value theorem (for L.T) (holds for entire class of L.T on both complex and real finite dim. inner product space).




(orthonormal basis)

$\alpha$ for $V$ — $\beta$ for $W$

New numerical is variant :- Singular values. (always non negative)

For the relation A. (on L.T) we shall use the term unitary operator (matrix) to include orthogonal operator (matrix). i.e

$\langle T(x), T(y)\rangle = \langle x,y\rangle$ means $T$ is unitary (ortnogonal).

Def$^n$ :- $V, W$ finite dim. inner product space with inner products $\langle\,|\,\rangle_1$, $\langle\,|\,\rangle_2$ respectively, and $T: V \to W$ be a L.T. A function $T^*: W \to V$ is called an 'adjoint of $T$' if

$$\langle T(x), y\rangle_2 = \langle x, T^*(y)\rangle_1 \quad \forall\, x \in V \text{ and } y \in W$$

Result:-

i) $T^*$ is unique and linear

ii) $[T^*]_\gamma^\beta = \left[[T]_\beta^\gamma\right]^*$

iii) $\langle T^*(x), y\rangle_1 = \langle x, T(y)\rangle_2 \;\forall\, x \in W$ and $y \in V$

iv) $r(T^*) = r(T)$ ; $[r : \text{rank}]$

(v) $\forall\, x \in V$, $T^*T(x) = \theta$ iff $T(x) = \theta$.

Def$^n$ :- $T: V \to V$ is called (+ve) definite [(+ve) semidef] if $T$ is self adjoint i.e Hermitian, and $\langle T(x), x\rangle > 0$ [respectively $\langle T(x), x\rangle \geq 0$] $\forall\, x \neq \theta$.

Result :- For $T: V \to W$ it can be easily seen that

i) $T^*T$ and $TT^*$ are (+ve) semi-def.

ii) $r(T^*T) = r(TT^*) = r(T)$

Indeed, $\langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle \geq 0$ and

$$(T^*T)^* = T^*(T^*)^* = T^*T$$

while $r(T^*) = r(T)$ indicates
rank $(T^*T)$ = rank $(TT^*)$ = rank $(T)$

Th. Singular Value Theorem for L.T :-

Let $T: V \to U$ of rank '$r$' Then $\exists$ orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ for $V$ and $\{u_1, u_2, \ldots, u_m\}$ for $U$ and and (+ve) scalers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ s.t

$$T(v_i) = \begin{cases} \sigma_i u_i & \text{if } 1 \leq i \leq r \\ 0 & \text{if } i > r \end{cases} \quad (i)$$

Conversely, suppose that the preceding conditions are satisfied. Then for $1 \leq i \leq n$, $v_i$ is an eigen vector for $T^* T$ with corresponding eigenvalue $\sigma_i^2$ if $1 \leq i \leq r$, and 0 if $i > r$. There scalers are uniquely determined by T.

Proof:- $(T^* T)$ is a (+ve) semidef linear operation of rank $r$ on V. Hence $\exists$ an orthonormal basis for V, say $\{v_1, v_2, \ldots, v_n\}$ consisting of eigenvectors of $(T^* T)$ with corresponding eigen value $\lambda_i$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and $\lambda_r > 0$ and $\lambda_i = 0$ for $i > r$

For $1 \leq i \leq r$ we define $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{1}{\sigma_i} T(v_i)$

we now show that $\{u_1, u_2, \ldots, u_r\}$ is orthonormal subset of $U$

suppose $1 \leq i, j \leq r$ then

$$\langle u_i, u_j \rangle = \left\langle \frac{1}{\sigma_i} T(v_i), \frac{1}{\sigma_j} T(v_j) \right\rangle = \frac{1}{\sigma_i \sigma_j} \langle T^* T(v_i), v_j \rangle$$

$$= \frac{1}{\sigma_i \sigma_j} \langle \lambda_i v_i, v_j \rangle = \frac{\lambda_i}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = \frac{\sigma_i^2}{\sigma_i \sigma_j} \delta_{ij}$$

$= \frac{\sigma_i}{\sigma_j} \delta_{ij}$

where clearly $\{u_1, u_2, \ldots, u_r\}$ is orthonormal. This set can be extended to an orthonormal basis $\{u_1, u_2, \ldots, u_r, \ldots, u_m\}$ for $U$, Clearly $T(v_i) = \sigma_i u_i$, $1 \leq i \leq r$ if $i > r$ then $(T^*T)(v_i) = \theta$

Which gives $T(v_i) = \theta$.

To establish uniqueness :-

Suppose that $\{v_1, v_2, \ldots, v_n\}$, & $\{u_1, u_2, \ldots, u_m\}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ satisfy the properties stated in first part of (a). Then for $1 \leq i \leq m$, $1 \leq j \leq n$.

$\langle T^*(u_i), v_j \rangle = \langle u_i, T(v_j) \rangle = \begin{cases} \sigma_i & \text{if } i = j \leq r \\ 0 & \text{otherwise} \end{cases}$ $\quad [\because T(v_j) = \sigma_j u_j]$

and hence for any $1 \leq i \leq m$

$T^*(u_i) = \sum \langle T^*(u_i), v_j \rangle v_j$ [$\because$ $v_j$'s are orthonormal basis of $V$]

$= \begin{cases} \sigma_i v_j & 1 \leq j \leq r \\ 0 & \text{otherwise} \end{cases}$

so, for $i \leq r$

$T^*T(v_i) = T^*(\sigma_i u_i) = \sigma_i T^*(u_i) = \sigma_i^2 v_i$

for $i > r$

$T^*T(v_i) = T^*(\theta) = \theta$

$\because$ each $v_i$ is an eigenvector of $T^*T$ with corresponding eigenvalue $\sigma_i^2$ if $i \leq r$

and is 0 if $i > r$.

**Def$^n$**:- The unique scalers are called the singular values of T

If r is less than both m and n, the term 'singular value' is extended to include $\sigma_{r+1} = \cdots = \sigma_K = 0$ when $K = \min\{m,n\}$

**Remark**:- Though singular value of $T: V \to U$ are uniquely determined by T, the corresponding orthonormal basis for V & U are not ~~necessar~~ necessarily unique (as there may be more than one orthonormal basis of eigenvectors of $T^*T$)

**Ex.**

Let $P_2(\mathbb{R})$ and $P_1(\mathbb{R})$ be polyn spaces with inner product defined by $\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)\,dt$

Let $T: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ given by $T(f(x)) = f'(x)$. Find orthnormo. basis $\beta = \{v_1, v_2, v_3\}$ for $P_2(\mathbb{R})$ and $\gamma = \{u_1, u_2\}$ for $P_1(\mathbb{R})$ s.t $T(v_i) = \sigma_i u_i$ and $T(v_3) = 0$ where $\sigma_2 \geq \sigma_1 > 0$ are non zero singular values of T.

Legendre polyn (orthonormal basis for $P_1(\mathbb{R})$)

$$\alpha = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2-1) \right\} \text{ for } P_2(\mathbb{R})$$

$$\alpha' = \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x \right\} \text{ for } P_1(\mathbb{R})$$

$T: P_2(\mathbb{R}) \to P_1(\mathbb{R})$

$c_1 R_1,\ c_2 R,\ R_1$
$0\ R_2\ c_2R_2\ c_3R_3$
$c_1 R_3\ c_2R_3\ c_3R_3$

Let $A = [T]_{\alpha}^{\alpha'} = \begin{bmatrix} 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{15} \end{bmatrix}_{2\times 3}$ $\quad \begin{bmatrix} 0 & 0 \\ \sqrt{3} & 0 \\ 0 & \sqrt{15} \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 \end{bmatrix}$

$A^tA$ $\quad \lambda_1 = 15,\ \lambda_2 = 3,\ \lambda_3 = 0$ (listed in decending order)

$$A^tA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

eigenvectors corresponding to $\lambda_1 = 15 = e_3 = (0, 0, 1)$.

$\lambda_2 = 3, \quad e_3 = (0,1,0)$

$\lambda_3 = 0; \quad e_1 (1,0,0)$

which happened to be orthonormal in $\mathbb{R}^3$, Now transleting every thing ~~to~~ into the context of $T$, $P_2(\mathbb{R})$, $P_1(\mathbb{R})$

$v_1 = \sqrt{\frac{5}{8}}(3x^2-1)$, $v_2 = \sqrt{\frac{3}{2}}\,x$, $v_3 = \frac{1}{\sqrt{2}}$ where $\beta = (v_1, v_2, v_3)$ counting the eigenvettors of. is an orthonormal basis of $P_2(\mathbb{R})$

$T^*T$ with corrs. eigen evalues $\lambda_1 = 15$, $\lambda_2 = 3$, $\lambda_3 = 0$

Now, define $\sigma_1 = \sqrt{\lambda_1} = \sqrt{15}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{3}$, all non-zero singular values of $T$,

Take $u_1 = \frac{1}{\sigma_1} T(v_1)$

$$= \frac{1}{\sqrt{15}} \times \frac{1}{2}\sqrt{\frac{5}{2}} \cdot 6x = \frac{\sqrt{3}}{\sqrt{2}}\,x$$

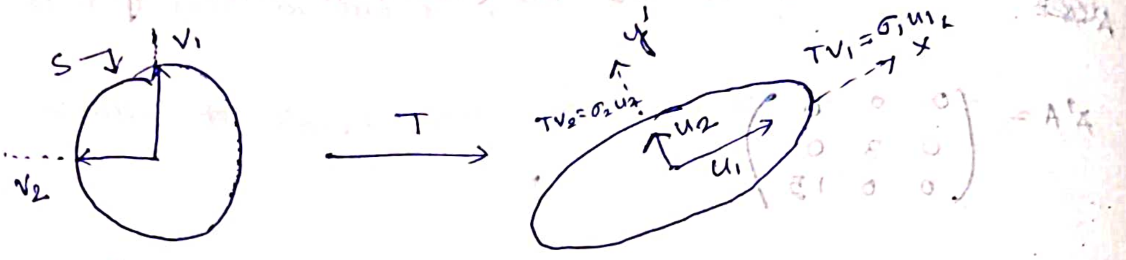
$$u_2 = \frac{1}{\sigma_2} T(v_2) = \frac{1}{\sqrt{3}} \cdot \sqrt{\frac{3}{2}} = \frac{1}{\sqrt{2}}$$

$\therefore\ \gamma = \{u_1, u_2\}$ orthonormal basis of $P_1(\mathbb{R})$.

Ex. $T$ : invertible lin. op.

$S$ : unit circle on $\mathbb{R}^2$, given by $S = \{x \in \mathbb{R}^2 ; \|x\| = 1\}$

Describe $S' = T(S)$ (by Th-1.5).




$v = x_1 v_1 + x_2 v_2$.

So it must have singular values $\sigma_1 \geq \sigma_2 > 0$

Let $\alpha = \{v_1, v_2\}$ and $\beta = \{u_1, u_2\}$ or.no. basis of $\mathbb{R}^2$

So that $T(v_1) = \sigma_1 u_1$

$T(v_2) = \sigma_2 u_2$

## Singular Value Decomposition of a Matrix:-

Part-I SVD of a square, non-singular matrix.

$A \in \mathbb{C}^{n \times n}$. Then $A^*A$ is Hermitian and (+ve) definite (in this case)

$\exists$ an orthonormal basis of $\mathbb{C}^n$ corresponding to the eigenvectors of $A^*A$ and the eigenvectors of $A^*A$ are known to be (+ve)

i.e $\exists$ unitary matrix $V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $D \in \mathbb{R}^{n \times n}$ (with the diagonal entries) s.t $A^*A = VDV^*$

i.e Diagonal entries of $D$ are (+ve). We may write them as $\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2$. Also by appropriatingely ording the columns of $V$ (i.e eigenvectors of $A^*A$), we can arrange in a way so that

$\sigma_1^2 \geq \sigma_2^2 \geq \sigma_3^2 \geq \dots \geq \sigma_n^2 > 0$

We now write $v_1, v_2, \cdots, v_n$ for columns of $V$ ; $V = [v_1 | v_2 | \cdots | v_n]$

$$D = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix}$$

We consider the vectors $Av_1, Av_2, \cdots, Av_n \in \mathbb{C}^n$

Indeed $\langle Av_i, Av_j \rangle = \langle A^* A v_i, v_j \rangle_{\mathbb{C}^n}$

$$= \langle \sigma_i^2 v_i, v_j \rangle_{\mathbb{C}^n} = \sigma_i^2 \langle v_i, v_j \rangle_{\mathbb{C}^n}$$

$$= \begin{cases} \sigma_i^2, & i = j \\ 0 & i \neq j \end{cases}$$

This calculation shows that $\|Av_i\| = \sigma_i$

We therefore define $u_i = \sigma_i^{-1} A v_i$, and ~~calculate~~ conclude that $\{u_1, u_2, \cdots, u_n\}$ is another orthonormal basis for $\mathbb{C}^n$.

If $U = [u_1 | u_2 | \cdots | u_n]$ and

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$ then $Av_i = \sigma_i v_i$ ; $i = 1, 2, \cdots, n$

which implies that $AV = U\Sigma$ ($\because$ columns of $U$ are orthonormal $U$ = unitary) $\therefore$ whence $A = U\Sigma V^*$ —— (i)

(i) is called the SVD of A. The nos. $\sigma_1, \sigma_2, \cdots, \sigma_n$ are called singular values of A., $v_1, v_2, \cdots, v_n$ are called Right singular vectors, $u_1, u_2, \cdots, u_n$ are called Left singular vectors.

Part II :- SVD for singular matrix.

Suppose now $A \in \mathbb{C}^{n \times n}$ is singular Then $A^* A$ is (+ve) semidef$^n$, (not (+ve) definite), Whence $A^* A$ has one or 0's eigen values However $A^* A$ is symmetric and it has orthonormal eigen vectors $v_1, v_2, \dots, v_n$ and corresponding eigen values $\sigma_1^2, \dots, \sigma_n^2$ where $\sigma_i^2 \geq 0$ $\forall i = 1(1)n$. Here we can order the eigenvectors in such a way that $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots \geq \sigma_r > 0$ and

$$\sigma_{r+1} = \sigma_{r+2} = \dots = \sigma_n = 0$$

Here also $v_1, v_2, \dots, v_n$ are right singular ~~value~~ vectors of $A$ and $\sigma_1, \sigma_2, \dots, \sigma_n$ are singular values

However there is a crucial difference in defining the left singular vectors!

We've $A^* A v_i = 0$ ; $i = r+1, r+2, \dots, n$

Which in turn implies $A v_i = 0$ ; $i = (r+1)(1)n$ so we may define

$u_i = \sigma_i^{-1} A v_i$, $i = 1, 2, \dots, r$ and these vectors are orthonormal just as define, However we cannot define $u_{r+1}, u_{r+2}, \dots, u_n$ by this formula

However note that we need for $\{u_1, u_2, \dots, u_n\}$ is to be an orthonormal basis for $\mathbb{C}^n$ and we can always extend $\{u_1, u_2, \dots, u_r\}$ to that (thanks to Gram-Schimidth) so, we choose just any vectors $u_{r+1}, u_{n+2}, \dots, u_n$ is $\mathbb{C}^n$ s.t $\{u_1, u_2, \dots, u_n\}$ may be an orthonormal basis of $\mathbb{C}^n$

defining

$U = [u_1 | u_2 | \cdots | u_n]$

$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ and

$V = [v_1 | v_2 | \cdots | v_n]$ as before and we have

$A = U \Sigma V^*$

Find SVD of $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

$c_1R_1 \quad c_2R_1 \quad c_3R_1$
$c_1R_2 \quad c_2R_2 \quad c_3R_2$
$c_1R_3 \quad c_2R_3 \quad c_3R_3$

$A^* = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$A^*A = \begin{bmatrix} 1+4+1 & 0 & 1+4+1 \\ 0 & 0 & 0 \\ 1+4+1 & 0 & 1+4+1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix}$

Now $\lambda_1 = 0$ is an eigen value of $A^*A$.

$\therefore \lambda_1 + \lambda_2 + \lambda_3 = 12 \qquad \lambda_2 + \lambda_3 = 12$

$\lambda_1 \lambda_2 \lambda_3 = 0 \qquad \lambda_2 = 12 - \lambda_3$

$\lambda_2 = 0, \ \underline{\lambda_3 = 12}$.

$\therefore$ eigen vectors of $A^*A$ with respect to $\lambda_1 = 0$ is

$c_1R_1$
$c_1R_2$
$c_1R_3$

$\begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix} X = 0 \qquad x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 6x_1 + 6x_3 \\ 0 \\ 6x_1 + 6x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 = -x_3$

$v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$\therefore u_1 = \frac{1}{\sqrt{12}} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{12}} \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ \sqrt{2} + \sqrt{2} \\ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{\sqrt{12}} \begin{bmatrix} \sqrt{2} \\ 2\sqrt{2} \\ \sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{12}} \\ \frac{2\sqrt{2}}{\sqrt{12}} \\ \frac{\sqrt{2}}{\sqrt{12}} \end{bmatrix} = \frac{1}{\sqrt{12}} y_1$$

we may choose.

$y_2, y_3$ in $\mathbb{R}^3$ s.t $\{y_1, y_2, y_3\}$ is orthogonal set and convert them to $\{U_1, U_2, U_3\}$ as an orthonormal basis of $\mathbb{R}^3$.

$$y_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$y_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$y_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$y_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Thus we have

$$A = U \Sigma V^*$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}^2$

$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ non diagonalizable. $[T(x) = Ax]$

SVD

different bases (orthonormal ~~basis~~) made of singular vectors.

Finite precision Arithmetic.

## SVD for general matrices :-

$A \in \mathbb{C}^{m \times n}$

every such matrix has a SVD

**Th:-** $A \in \mathbb{C}^{m \times n}$ Then $\exists$ unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Sigma \in \mathbb{C}^{m \times n}$, with non negative diagonal entries $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_{\min(m,n)} \geq 0$ s.t

$$A = U \Sigma V^*$$

**Proof:-** We first assume that $m \geq n$. Let $D \in \mathbb{R}^{n \times n}$, $V \in \mathbb{C}^{n \times n}$ from the Spectral Decomposition of $A^* A \in \mathbb{C}^{n \times n}$. s.t we have

$A^* A = V D V^*$, Since $A^* A$ is (+ve) semidef the diagonals of $D$ are non negative, and we can assume that the columns of $V$ (i.e the eigenvectors of $A^* A$) are so ordered that

$$D = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_r^2 \end{bmatrix} \quad \text{when } \underbrace{\sigma_1^2 \geq \sigma_2^2 \geq \cdots \geq \sigma_r^2}_{\text{(+ve) eigen values of } A^* A} > \sigma_{r+1} = \cdots = \sigma_n = 0$$

We now define $u_1, u_2, \dots, u_r \in \mathbb{C}^m$ by. $u_i = \frac{1}{\sigma_i} A v_i$ ; $i = 1, 2, \dots, r$

As before, we can see that $\{u_1, u_2, \dots, u_r\}$ is orthogonal. If $r < m$, then let $u_{r+1}, \dots, u_m$ be so chosen that $\{u_1, u_2, \dots, u_m\}$ be orthogonal basis for $\mathbb{C}^m$ and define $U = [u_1, u_2, \dots, u_m] \in \mathbb{C}^{m \times m}$

clearly $U$ is unitary. Finally define $\Sigma \in \mathbb{C}^{m \times n}$ by

$$\Sigma_{ij} = \begin{cases} \sigma_i, & i = j \\ 0, & i \neq j \end{cases}$$

Hence $A = U \Sigma V^*$ so eventually $AV = U\Sigma$

but $j^{th}$ column of $AV$ is $Av_j = \begin{cases} \sigma_j u_j ; & j = 1, 2, \dots, r \\ 0 ; & j = r+1, \dots, n \end{cases}$

While the $j^{th}$ column of $U\Sigma$ is

$$\sum_{i=1}^{m} \Sigma_{ij} u_i = \Sigma_{jj} u_j = \begin{cases} \sigma_j u_j, & j = 1, 2, \dots, r \\ 0 ; & j = r+1, \dots n \end{cases}$$

$[\because \Sigma_{ij} = 0 \text{ for } i \neq j]$

Case-II :- If $n > m$, then the result proved above applies to $A^* \in \mathbb{C}^{n \times m}$, so $\exists$ unitary matrices $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\hat{\Sigma} \in \mathbb{C}^{n \times m}$ with entries $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \sigma_{r+1} = \dots$

$\dots = \sigma_m = 0$. s.t.

$A^* = V \hat{\Sigma} U^*$

Taking adjoint $A = U \hat{\Sigma} V^*$ $[\because \hat{\Sigma}$ comprise of real numbers$]$

$= U \Sigma V^*$ [taking $\hat{\Sigma}$ as $\Sigma$] — as defined.

To sum up the whole state of affairs :-

Th: Let $A \in \mathbb{C}^{p \times q}$ be a vector of rank $r$, with singular values $s_1, s_2, \dots, s_q$ and let $D = \text{diag}\{s_1, s_2, \dots, s_r\} \in \mathbb{R}^{r \times r}$ with $s_1 \geq s_2 \geq \dots \geq s_r > 0$. Then $\exists$ a unitary matrix $V \in \mathbb{C}^{p \times p}$ and a unitary matrix $U \in \mathbb{C}^{q \times q}$ s.t

$$A = \begin{cases} V \begin{bmatrix} D_{r \times r} & O_{r \times (q-r)} \\ O_{(p-r) \times r} & O_{(p-r) \times (q-r)} \end{bmatrix} U^* & \text{if } r < \min\{p, q\} \\ V \begin{bmatrix} D_{r \times r} \\ O_{(p-r) \times r} \end{bmatrix} U^* & \text{if } r = q < p \\ V \begin{bmatrix} D_{r \times r} & O_{r \times (q-r)} \end{bmatrix} U^* & \text{if } r = p < q. \\ V D_{r \times r} U^* & \text{if } r = p = q \end{cases}$$

Remark :- 1. $A \in \mathbb{R}^{p \times q}$ then the unitary matrices $U$ and $V$ may be choosen to have real entries, i.e to be orthogonal matrix

Note :- When $A$ is real not only eigen values but also eigen vectors of $A^*A = A^TA$ are real. So the SVD eleminates the need to move to the complex domain as might be necessary when spectral resolution is used.

**Remark:-** It's helpful to examine carefully the matrices $U$, $\Sigma$ and $V$ in the three cases for $A^{m\times n}$, $m=n$, $m>n$, $m<n$.

Case-I

If $m=n$ then $U, \Sigma, V$ are all square and of same size

Case-II :-

If $m>n$ then $\Sigma \in \mathbb{C}^{m\times n}$ can be partitioned as

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ \hdashline 0 \end{bmatrix}$$ we can also write $U = [U_1 \mid U_2]$ where

$$U_1 = [u_1, u_2, \dots, u_n] \in \mathbb{C}^{m\times n}, \quad U_2 = [U_{n+1}, \dots, U_m] \in \mathbb{C}^{m\times(m-n)}$$

we obtain

$$A = U\Sigma V^* = [U_1 \mid U_2] \begin{bmatrix} \Sigma_1 \\ \hdashline 0 \end{bmatrix} V^* = \underline{\underline{U_1 \Sigma_1 V^*}}$$

Case-III

If $m<n$ then $\Sigma$ has the form $\Sigma = [\Sigma_1 \mid 0]$ where

$\Sigma_1 \in \mathbb{C}^{m\times m}$ has diagonal entries $\sigma_1, \sigma_2, \dots, \sigma_m$ and $\underline{0}$ is $m\times(n-m)$

zero matrix.

we write $V = [V_1 \mid V_2]$ where $V_1 = [v_1, v_2, \dots, v_m] \in \mathbb{C}^{n\times m}$.

$V_2 = [v_{m+1}, v_{m+2}, \dots, v_n] \in \mathbb{R}^{n\times(n-m)}$, and we obtained

$$A = U\Sigma V^* = U[\Sigma_1 \mid 0] \begin{bmatrix} V_1^* \\ \hdashline V_2^* \end{bmatrix} = \underline{\underline{U\Sigma_1 V_1^*}}$$

Are called Reduced SVD.

$A = U\Sigma_A V^*$ (for $m\times n$ matrix of rank $r$)

Another way to reduce the SVD:

$$A = U_r \hat{\Sigma}_A V_r^* \quad \text{.....(i)}$$

$U_r$ : $m\times r$ with orthonormal columns

$\hat{\Sigma}_A$ : is an $r\times r$ diagonal matrix with (+ve) diagonal entries.

$V_r^*$ : is $r\times n$ with orthonormal rows.

(Since only first $r$ diagonal entries of $\Sigma_A$ are non zero, the last $(m-r)$ col's of $U$ and last $(n-r)$ rows of $V^*$ are superfluous.

Remark:- 1.2.1 we may also decompose (i) into a sum of '$r$' rank one matrices.

Let $u_i$ denote the $i$th col of $U$ and $v_i^*$ is $i$th row of $V^*$, then,

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots + \sigma_r u_r v_r^* \quad \text{(ii)}$$

where each $u_i v_i^*$ is a rank 1 matrix.

Note:- $A = U\Sigma_A V^*$ (where $\Sigma_A$ is uniquely determined by $A$ but $U, V$ are not).

The equation, $AV = U\Sigma_A$ tells us how the transformation $A$ acts.

$A(\text{col } j \text{ of } V) = \sigma_j (\text{col } j \text{ of } U)$.

Observation:-

We want to show that; the largest singular value $\sigma_1$ is the largest factor by which the length of any unit vector is multiplied.

Theorem:- $A_{m\times n}$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ then maximum of $\|Ax\| = \sigma_1$, $\|x\| = 1$

Proof:- It's easy to see from. $AV = U\Sigma_A$ it follows clearly that if $x$ is the first col. vector of $V$, then $\|x\| = 1$ and $\|Ax\| = \sigma_1$

Now, Let $x$ is any vector in $\mathbb{C}^n$ of unit length, from the S.V.D $A = U\Sigma_A V^*$, we have $Ax = U\Sigma_A (V^* x)$, set $y = V^* x$ since $V$ is unitary $\|y\| = 1$, in the product $\Sigma_A y$, coordinate $i$ of $y$ gets multiplied by $\sigma_i$ hence $\|\Sigma_A y\| \leq \sigma_1 \|y\| = \sigma_1$ since $U$ is unitary. $\|Ax\| = \|U\Sigma_A y\| = \|U\Sigma_A V^* x\| = \|\Sigma_A y\| \leq \sigma_1$

~~The Spero~~

The spectral norm (or operator norm) of a $m\times n$ matrix $A$ is $\max \|Ax\| = \sigma_1$, $\|x\| = 1$

It's denoted by $\|A\|_2$

Note:- we have $\|Ax\| \leq \|A\|_2 \|x\|$ $\forall x$, when $\|Av\| = \|\lambda v\| = |\lambda| \|v\|$ so $|\lambda| \leq \|A\|_2$ for any eigen value $\lambda$. Hence $\rho(A) \leq \|A\|_2 = \sigma_1$ where $\rho(A) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$ is called the "spectral radius of $A$".

**Note:-**

Frobenious norm of A :- $\|A\|_F = \left(\text{trace}(A^*A)\right)^{1/2}$

$$= \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$

we usually work with $\|A\|_F^2 = $ trace of $(A^*A)$

Since, $\|A\|_F = \|UAV^*\|_F$ for any unitary $U, V$

we have. $\|A_F\|^2 = \|\Sigma_A\|_F^2 = \sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2$

**Remarks:-**

Since $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ while all vectors $u_i$ and $v_i$ are of length 1, the sum in (ii) above we have, the earlier terms in the sum (i.e terms corresponding to larger singular values) make larger contribution to A.

This suggests, that to approximate A with a matrix of lower rank (i.e one with lower terms of the sum) one should use the terms corresponding to larger singular values and drop the terms with smaller singular values

Let $1 \leq k \leq r$

From, $A = U\Sigma_A V^*$ Let $U_K$ the $m \times k$ matrix consisting of first k columns of $U$ and Let $V_K^*$ be the $k \times n$ matrix consisting of the first k rows of $V^*$, Set $A_K = U_K \,\text{diag}(\sigma_1, \sigma_2, \ldots \sigma_K) V_K^*$

$$= \sum_{i=1}^{k} \sigma_i u_i v_i^*$$

It can be shown that of all matrices of rank K, the matrix $A_K$ is closest to A, where the distance is measured by Frobenious norm.

Let $\Sigma_K$ denote $m \times n$ matrix with $\sigma_1, \ldots, \sigma_K$ is the first K diagonal and 0 elsewhere

$$A_k = U\Sigma_K V^*$$

and $\|A - A_K\|_F^2 = \|U(\Sigma_A - \Sigma_K)V^*\|_F^2$

$$= \|\Sigma_A - \Sigma_K\|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

We now state (without proof) the following th.

**Th** Let $A_{m\times n}$, $r(A) = r$ with SVD, $A = U\Sigma_A V^* = \sum_{j=1}^{K} \sigma_i u_i v_i^*$

For $1 \leq K \leq r$, set $A_K = \sum_{i=1}^{K} \sigma_i u_i v_i^*$ Then for any matrix B of rank at most k, $\|A - A_K\|^2 = \sum_{i=k+1}^{r} \sigma_i^2 \leq \|A - B\|_F^2$

⇓ ⇓ ⇓

**Application in Image processing.**

Q Find S.V.D of $A = \begin{pmatrix} 1+i & 1 \\ 1-i & -i \end{pmatrix}$

**Remark:-**

$A \in \mathbb{C}^{p\times q}$, rank$(A) = r$, Then non zero singular values of A coincides with the non zero singular values of $A^*$

$$s_j(A) = s_j(A^*) \quad \text{for } j = 1, 2, \ldots, r.$$

**Note:-** It can be shown that, $A \in \mathbb{C}^{m\times n}$ has exactly r (+ve) singular values, Then $r(A) = r$

2. $A \in \mathbb{C}^{n\times n}$, $s_j(A)$, $s_j(B)$, $j=1,2,\ldots,n$ are singular values:

1) $s_j(A) = s_j(A^*)$

2) $s_j(BA) \leq \|B\| s_j(A)$

3) $\prod_{j=1}^{k} s_j(AB) \leq \prod_{j=1}^{k} s_j(A) \prod_{j=1}^{k} s_j(B)$

4) $\sum_{j=1}^{k} s_j(A+B) \leq \sum_{j=1}^{k} s_j(A) + \sum_{j=1}^{k} s_j(B)$

Remark :- Here $\|B\| = \|B\|_{2,2}$ (operator norm)

Defn :- $\|A\|_{s,t} = \max \{ \|Ax\|_t : x \in \mathbb{C}^q$ and $\|x\|_s \leq 1 \}$ $1 \leq s,t \leq \infty$

In particular $\|x\|_2 = \left\{ \sum_{j=1}^{n} |x_j|^2 \right\}^{1/2}$

3.

(a) Singular values of $A$ and $PAQ$ are identical for any orthogonal matrices $P$, $Q$.

(b) $A$ is symm, then singular values of $A$ are the absolute values of its eigenvalues.

(c) $A$ is (+ve) semi def. then the singular values are same as eigenvalues.

$$A = \begin{bmatrix} 1+i & 1 \\ 1-i & -i \end{bmatrix}$$

$$\therefore A^* = (\overline{A})^T = \begin{bmatrix} 1-i & 1 \\ 1+i & i \end{bmatrix}^T$$

$$= \begin{bmatrix} 1-i & 1+i \\ 1 & i \end{bmatrix}$$

$$\therefore A^*A = \begin{bmatrix} 1-i & 1+i \\ 1 & i \end{bmatrix} \begin{bmatrix} 1+i & 1 \\ 1-i & -i \end{bmatrix}$$

$c_1R_1$ $c_2R_1$
$c_1R_2$ $c_2R_2$

$i^2 - i^2 + 1 = 3$

$$= \begin{bmatrix} \cancel{i^2+1+1} \; \cancel{-2i-i} & \cancel{2i+i} \\ & \cancel{i^2+1+1} \end{bmatrix}$$

$\overset{8}{\cancel{16}} - (4 + \cdots$

$$= \begin{bmatrix} \cancel{8} & \cancel{3i} \\ \cancel{-3i} & \cancel{3} \end{bmatrix} \begin{bmatrix} 4 & 2-2i \\ 2+2i & 2 \end{bmatrix}$$

$\therefore$ eigen values :- 0, 6

$\therefore$ eigen vectors :- $AX = \lambda X = 0$

$$\Rightarrow \begin{bmatrix} 4 & 2-2i \\ 2+2i & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4x_1 + 2x_2 - 2x_2 i \\ 2x_1 + 2x_1 i + 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + x_2 - x_2 i = 0$

$x_1 + x_1 i + x_2 = 0$

$2x_1 + x_2(1-i) = 0$

$x_1(1+i) + x_2 = 0$

$$2x_1 + x_2(1-i) = 0$$

$$x_1(1+i) + x_2(1-i) = 0$$

$$\begin{bmatrix} i-1 & 1-i \\ 1 & 1 \end{bmatrix}$$

$$2x_1 = (i-1)x_2$$

$$\therefore x_1 = \left(\frac{i-1}{2}\right)x_2$$

$$\therefore x_2 \begin{bmatrix} \left(\frac{i}{2} - \frac{1}{2}\right) \\ 1 \end{bmatrix} \Rightarrow \lambda = 0 \Rightarrow x_1 = (i-1, 1)$$

$$\lambda = 6 \; ; x_2 = (1-i, 1)$$

$$\therefore \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1-i}{\sqrt{6}} \\ \frac{1+i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}^*$$

2. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

<u>Remark</u>:- Some inside observations:-

<u>Real</u>:- $A = U \Sigma V^T$ = (orth) (diag) (orth)

A. Columns of $U_{m \times m}$ are eigenvectors of $AA^T$

" " $V_{n \times n}$ " " of $A^T A$

Non zero diagonal entries of $\Sigma$ are (+ve) square root of non zero eigenvalues of both $AA^T$ and $A^T A$.

B. For (+ve) def. matrix $\Sigma$ and $\Lambda$ and $U\Sigma V^T$ is identical $Q\Lambda Q^T$

For other symmetric matrix any (-ve) eigen values of $\Lambda$ become positive in $\Sigma$ for complex matrices. $\Sigma$ remains real, let $U$ and $V$ become unitary.

c) $U$ and $V$ given orthogonal bases for all the four fundamental subspaces

(i) First $r$ col of $U$ : $C(A)$

ii) last $(m-r)$ col of $U$ : $N(A^T)$

iii) first $r$ col. of $V$ : $R(A)$

iv) last $(n-r)$ cols of $V$ : $N(A)$

<u>Defn</u>:- the ratio $\frac{\sigma_{max}}{\sigma_{min}}$ is called condition number of an invertible $n\times n$ matrix.

<u>Remark</u> :-

A: arbitary Diagonalizable matrix

Then its eigen decomposition $A = V\Lambda V^{-1}$ is unrelated to its SVD of A.

However if A is normal, we may relate any eigen decomposition of A to a 'SVD' of 'A'.

$$A = V\Lambda V^T = \sum_{n=1}^{K} \lambda_n v_n v_n^T \quad \text{—(i)}$$

Without loss of generality we can order the eigen values in decreasing magnitude i.e $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_K|$

However still $V\Lambda V^T$ may not be a SDV.

($\because$ Still some of the eigenvalues may be (-ve) or even imaginary)

so $\Sigma \neq \Lambda$

To construct an SVD of A in terms of $V$ and $\Lambda$ we use the fact that $\lambda_n = \text{sign}(\lambda_n)|\lambda_n|$ where $\text{sign}(|z|e^{i\angle z}) = e^{i\angle z}$ for $z \in \mathbb{C}^n$

$$A = V \Lambda V^T = \sum_{n=1}^{K} \lambda_n v_n v_n^T = \sum_{n=1}^{K} (\text{sign}(\lambda_n) v_n) |\lambda_n| v_n^T \quad \text{—(ii)}$$

So, when A is normal with eigen vectors in $V$ and eigenvalues in $\Lambda$ with decending magnitude, a SVD of A is:-

$$A = U \Sigma V^T \text{ where } U = VS, \quad S = \text{diag}\{\text{sign}(\lambda_n)\}$$

$$\Sigma = \text{diag}\{|\lambda_n|\}$$

This SVD is not unique.

However it does not follow that all ~~sin~~ singular vectors of a normal matrix A, all eigen vectors [In fact not that have in the proposed SVD, that matrix $V$ of right singular vectors consists entirely of eigen vectors of A and the matrix $U$ of Left singular vectors also consists of entirely the eigen vectors of A and we have $u_n = \text{sing}(\lambda_n) v_n$]

$$\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix}$$

Example:-

Consider $A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, This $x'$ is a ~~s~~ right singular vector as $A^T A x = 9x$, but $x$ is not an eigen vector.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{V^T}$$ an elementary eigenvalue Decomposition.

$$\therefore A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ SVD, type-I using $V$ and $S$, $u_n = \text{sing}\{\lambda_n\} v_n$

$$= U \Sigma V^T$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ SVD of type-II, unrelated to eigen vectors of A.

<u>Question</u>:- For what class of matrices does these exists a SVD in which $U = V$?

<u>Soln</u>:- Firstly, if $n \neq m$ and $U, V$ are of different ~~siq~~ size. So we must focus on square matrices $(n = m)$ If $A = V \Sigma V^T$ then Clearly, A is Hermitian Symmetric.

But is symmetry is a sufficient condition for possibly $U = V$? <u>NO</u>

To have $U = V$, we need A to be Hermitian symmetric and to have non negative eigenvalues (i.e a (+ve) semidef. matrix (as well).)

<u>Ex</u>. $A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (eigendecomposition)

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$ (SVD type-I)

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ (SVD-II)

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (SVD-III)

Lemma :-

If $A = BB^T$ for any matrix $B$, then $A$ is (+ve) semidefinite

Indeed $x^T A x = x^T B B^T x = \|B^T x\| \geq 0$

Th - If $A = BB^T$ for any matrix $B$ then $A = U \Sigma U^T$ with $\Sigma_{ii} \geq 0$

In words, for such matrices an eigen decomposition (with real, non negative) eigen values in decreasing order is also a SVD.

Proof:-

Let $B = U \Sigma_B V^T$ denote a SVD of $B$

Recall that $\Sigma_B$ is real and non negative

Then, $A = BB^T = U \Sigma_B (V^T V) \Sigma_B^T U^T = U \Sigma_B \Sigma_B^T U^T = U \Sigma U^T$ where

$\Sigma = \Sigma_B \Sigma_B^T$ in diagonal with entries $\sigma_K^2$ (and some zeros if $B$ is "tall") So, when $A = BB^T$ then eigenvalues of $A$ are the sqrt. of eigen values of $B$ and have real and non negative.

$U = V$

$\Sigma = \Lambda$

(*) $\lambda_{(n)}$ | $n^{th}$ largest magnitude eigenvalue

i.e $\sigma_n(A) = \lambda_n(A) \geq 0$

Vendiagram of matrices and Decomposition :-

| Regular SVD | Square | Diagonalizable |
|---|---|---|
| $A = U \Sigma V^*$ | $AV = V\Lambda$ | $A = V \Lambda V^{-1}$ |

iff col's of $V$ are lin. independent. (*)

Normal

$A = V \Lambda V^*$, $V$ is unitary, $\sigma_n = |\lambda_n|$

Hermitian (+ve) Semidef.

$\Lambda$ real $\quad A \geq 0$

$\Sigma = \Lambda \geq 0$

$U = V$

Rectangular $\supseteq$ Square $\supset$ Diagonalizable $\supset$ Normal $\supset$ Hermitian $\supset$ (+ve) semidefinite.

Polar Decompocition :-

$z = \rho e^{i\theta}$, where $\rho = |z|$ (a (+ve) number) and $e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ lies on unit circle

Matrix analoge :- A non singular matrix, $A = UH$ where $U$: unitary matrix, $H$: (+ve) def Hermitian matrix.

Theorem:- A non Singular matrix, $A = UH$ where $U$: unitary and $H$: (+ve) def. Hermitian. (for non-negative A, both $U$ and $H$ are uniquely determi-ned by A)

Proof:- Let $K = A^*A$, Then K is (+ve) definite and Hence has a (+ve) def. square root H st, $H^2 = A^*A$. Now, $U = AH^{-1}$, we claim that $U$ is unitary. Indeep $U^*U = (AH^{-1})^*(AH^{-1}) = (H^{-1})^* A^* A H^{-1} = H^{-1}H^2H^{-1}$
$= I$

So, $A = UH$, where $U$ is unitary and $H$ is (+ve) def Hermitian matrix

To prove the uniqueness, suppose $A = U_1H_1$ and $A = U_2H_2$. where $U_i$'s are Unitary and $H_i$'s are (+ve) def. Hermitian.

Then $A^*A = H_1^2 = H_2^2$ whence from a theorem done earlier $H_1 = H_2 = H$ (say) Then $U_1H = U_2H$ since H is non singulare so $U_1 = U_2$