9/8

Continuous Random variable

- → Uniform distribution
- → Exponential distribution.

Exponential distribution:

Last Class

pdf $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$(\lambda > 0)$

$\boxed{X \sim \text{Exp}(\lambda)}$

Median $= \dfrac{\ln 2}{\lambda}$

Today: Exponential distribution is used to model waiting time / life time.

Convention: Replace x by t

Assumption: We consider modeling the lifetime of an iPhone. $\sim \text{Exp}(\lambda)$

Question: Suppose that this iPhone has lasted a length of time s, we wish to compute.

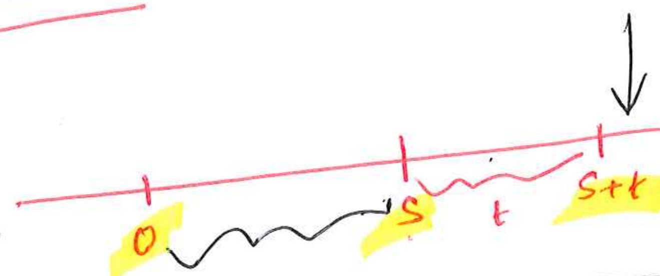
the probability that it will last for at least t more time units.

$$T \sim \mathrm{Exp}(\lambda)$$

[If (for example) $s = 9$ months, $t = 6$ months]

Goal: $P(T > t+s \mid T > s)$

$$= \frac{P(T > t+s \cap T > s)}{P(T > s)}$$



$$= \frac{P(T > t+s)}{P(T > s)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1 - P(T \le t+s)}{1 - P(T \le s)}$$

$$F(x) = P(X \le x)$$

For $T \sim \mathrm{Exp}(\lambda)$

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

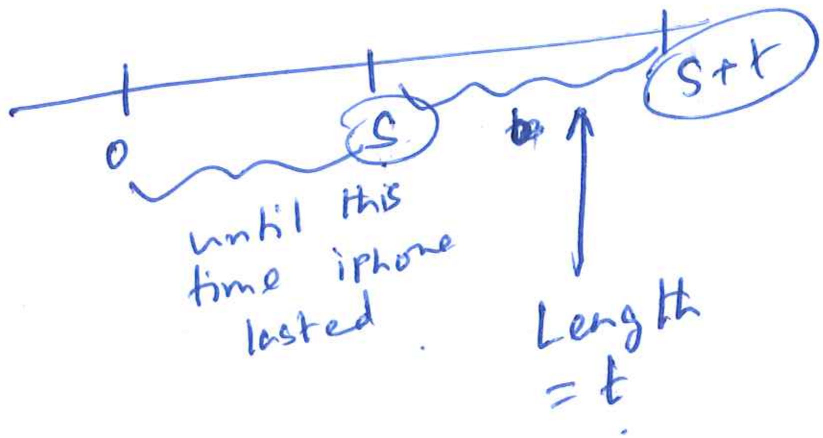
$$= \frac{1 - F(t+s)}{1 - F(s)}$$

$$= \frac{1 - \left(1 - e^{-\lambda(t+s)}\right)}{1 - \left(1 - e^{-\lambda s}\right)}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = \boxed{e^{-\lambda t}}$$

What we found:

$$\boxed{P(T > t + s \mid T > s) = e^{-\lambda t}}$$



~~iPhone last 9~~

- iPhone Lasted for 9 months.
  → Prob. that the iPhone will last another 6 months $= e^{-6\lambda}$.
- iPhone lasted for 100 months
  → Prob. that the iPhone will last another 6 months $= e^{-6\lambda}$

This is called $\boxed{\text{memoryless property}}$

Conclusion: Exponential distribution is memoryless.

Gamma distribution

$\Gamma(x) = (x-1)!$ x: positive integer.

$$\boxed{\Gamma(x) = \int_0^\infty e^{-u} u^{x-1}\, du, \quad x > 0}$$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-u} u^{-1/2}\, du = \sqrt{\pi}$$

$$\Gamma(x+1) = x\,\Gamma(x).$$

Gamma distribution has the density function

$$g(t) = \begin{cases} \dfrac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot t^{\alpha-1} \cdot e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$\boxed{\alpha, \lambda > 0}$

- $\alpha = 1$: $\quad g(t) = \begin{cases} \lambda e^{-\lambda t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

$\Rightarrow \mathrm{Exp}(\lambda)$

- α : shape parameter.
  λ : scale parameter.

Normal distribution

Gaussian distribution

Pdf of ~~Normal~~ Normal dbn:

$$\boxed{f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty}$$

where $\mu \in \mathbb{R}$, $\sigma > 0$.

Real numbers

μ: mean
σ: standard deviation

Notation:

$X \sim N(\mu, \sigma^2)$

$X \sim N(5, 7)$
$\sigma = \sqrt{7}$

Special case:

When $\underbrace{\mu = 0, \quad \sigma = 1}$

$\Downarrow$

Standard normal distribution.

$Z \sim N(0, 1)$

cdf:

$$P(X \leq x)$$

If we consider a $Z \sim N(0,1)$

then cdf of Z is denoted as

$$\boxed{\Phi(z) = P(Z \leq z)}$$

$$\boxed{\begin{aligned}\phi(z) &= \text{pdf of } Z \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad \cancel{-\infty < x <} \quad -\infty < z < \infty\end{aligned}}$$

Show that $\displaystyle\int_{-\infty}^{\infty} \underbrace{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}_{f(x)} dx = 1$

Functions of a random variable

X is a random variable

Y is another random variable

constructed as: $Y = aX + b$, $a > 0$

8 **Question:** We know the cdf of X F_X

What can we say about the cdf of Y? F_Y

Answer:

$$F_Y(y) = P(Y \le y)$$
$$= P(aX + b \le y)$$
$$= P\left(X \le \frac{y-b}{a}\right)$$
$$= F_X\left(\frac{y-b}{a}\right)$$

cdf: $F_Y(y) = F_X\left(\frac{y-b}{a}\right)$

pdf: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right)$
$$= \frac{1}{a} F_X'\left(\frac{y-b}{a}\right)$$
$$= \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$\boxed{f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)}$$

Suppose $X \sim N(\mu, \sigma^2)$

$Y = aX + b \quad (a > 0)$

<u>Question</u>: What is the pdf of Y?

$$\boxed{f_Y(y)} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

just found!

$$= \frac{1}{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{y-b}{a} - \mu}{\sigma}\right)^2}$$

$$\boxed{f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

$$\boxed{= \frac{1}{\sigma a\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2}}$$

<u>Question</u>: What kind of distribution is this Y?

$$f_Y(y) = \frac{1}{(\sigma a)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-(b+a\mu)}{\sigma a}\right)^2}$$

$$\boxed{Y \sim N(b + a\mu, \sigma^2 a^2)}$$

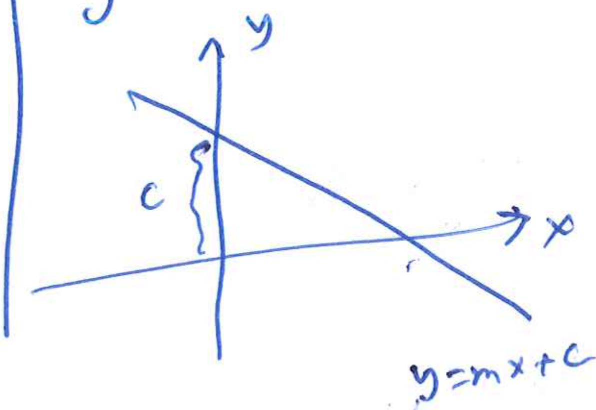
<u>Proposition</u>: If $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$ then

$$Y \sim N(a\mu + b, a^2\sigma^2)$$ //

$$Y = a X + b.$$

(Linear Transformation)

$y = mx + c$



Consider $\boxed{X \sim N(\mu, \sigma^2)}$

$$\boxed{Z = \frac{X - \mu}{\sigma}} = \left(\frac{1}{\sigma}\right) X + \left(-\frac{\mu}{\sigma}\right)$$

$$(Y = a X + b)$$

Question:

What is the distribution for Z.

$$Z \sim N\left(\frac{1}{\sigma}\mu - \left(\frac{\mu}{\sigma}\right), \frac{1}{\sigma^2}\sigma^2\right)$$

$$\Rightarrow Z \sim N(0, 1) \Rightarrow \text{Std. normal distribution}$$

$$\boxed{F_X(x)} = P(X \le x)$$

$$= P\left(\frac{X - \mu}{\sigma} \le \frac{x - \mu}{\sigma}\right)$$

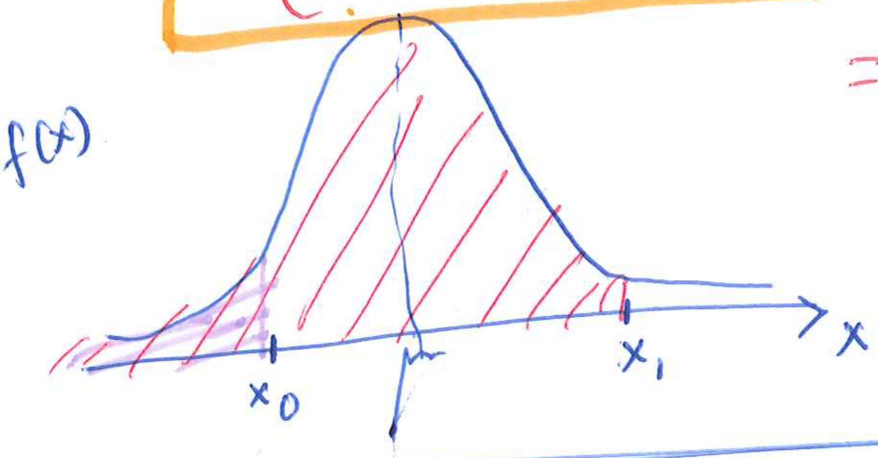
$$= P\left(Z \le \frac{x - \mu}{\sigma}\right) = \boxed{\Phi\left(\frac{x - \mu}{\sigma}\right)}$$

Check: How to find these online

Remark: $X \sim N(\mu, \sigma^2)$

$$\boxed{P(x_0 < X < x_1)} = F_X(x_1) - F_X(x_0)$$

$$= \boxed{\Phi\left(\frac{x_1-\mu}{\sigma}\right) - \Phi\left(\frac{x_0-\mu}{\sigma}\right)}$$



Example: For an exam (200 marks)

$$\left.\begin{aligned}\mu &= 100\\ \sigma &= 15\end{aligned}\right\}$$

No class on next Thursday!

X = Score in the exam
$\sim N(100, 15^2)$

$$P(120 < X < 130) = P\left(\frac{120-100}{15} < \frac{X-100}{15} < \frac{130-100}{15}\right)$$

$$= P(1.33 < Z < 2)$$

$$= \Phi(2) - \Phi(1.33)$$

$$= 0.9772 - 0.9082$$

$$= \boxed{0.069}$$

$$P(120 < X < 130) = F_X(130) - F_X(120) = \int_{-\infty}^{130} \ldots dx - \int_{-\infty}^{120} \ldots dx$$

Example: $X \sim N(\mu, \sigma^2)$

$$P(|X - \mu| < \sigma)$$

$$= P(-\sigma < X - \mu < \sigma)$$

$$= P\left(-1 < \frac{X-\mu}{\sigma} < 1\right)$$

$$= P(-1 < Z < 1)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \boxed{0.68}$$

Remember

- $|x-a| < b$
- $-b < x-a < b$
- $(x-a)^2 < b^2$

$|x-a| > b$

$(x-a) > b$ or
$(x-a) < -b$

* A normal r.v. is within 1 standard deviation of its mean with probability 0.68.

$$P(-\sigma < X - \mu < \sigma) = 0.68$$

$$\Rightarrow \boxed{P(\mu - \sigma < X < \mu + \sigma) = 0.68}$$

$$\Rightarrow P(X \in [\mu - \sigma, \mu + \sigma]) = 0.68$$

Example:

$$\cancel{X \sim N(\mu, \sigma^2)}$$

$$Z \sim N(0,1)$$

Let, $Y = Z^2$

Question: Find the dbn of Y.

Answer:

$$\boxed{F_Y(y)} = P(Y \le y)$$

cdf of Y

$$= P(Z^2 \le y)$$

$$= P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= \boxed{\Phi(\sqrt{y}) - \Phi(-\sqrt{y})}$$

$Z^2 \le y$

$\Rightarrow -\sqrt{y} \le Z \le \sqrt{y}$

To find the pdf of Y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\right)$$

$$= \frac{1}{2} y^{-1/2} \Phi'(\sqrt{y}) + \frac{1}{2} y^{-1/2} \Phi'(-\sqrt{y})$$

$$= \frac{1}{2} y^{-1/2} \phi(\sqrt{y}) + \frac{1}{2} y^{-1/2} \phi(-\sqrt{y})$$

$$= y^{-1/2} \phi(\sqrt{y})$$

$\Phi' = \phi$

Φ: cdf of Z; ϕ: pdf of Z

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\phi(-z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Hence,

$$f_Y(y) = y^{-1/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}}, \quad y \geq 0.$$

$$\Rightarrow \boxed{f_Y(y) = \frac{y^{-1/2}}{\sqrt{2\pi}} e^{-\frac{y}{2}}}, \quad y \geq 0$$

$\Downarrow$

Chi-square distribution.

(χ^2 dbn)