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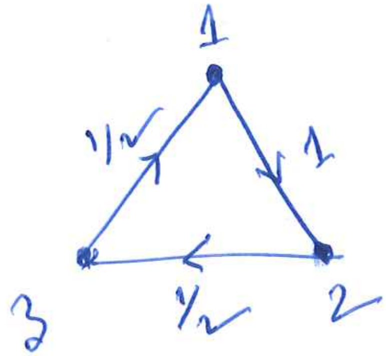
Example (started last time)



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Question: $1 \xrightarrow[\text{n-steps}]{} 1$

probability? $P_{11}^{(n)}$

Found last time: eigenvalues of P.

are $1, \frac{i}{2}, -\frac{i}{2}$

$$P = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} U^{-1}$$

$$P^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & (i/2)^n & 0 \\ 0 & 0 & \left(-\frac{i}{2}\right)^n \end{pmatrix} U^{-1}$$

$$\boxed{P_{11}^{(n)} = a \cdot 1 + b \cdot \left(\frac{i}{2}\right)^n + c \cdot \left(-\frac{i}{2}\right)^n}$$

for some constants a, b, c.

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$$\left(= e^{i\frac{\pi}{2} + 2\pi k i}\right)$$

$$\Rightarrow \quad i^n = e^{i\frac{n\pi}{2}}$$

$$\Rightarrow \quad \left(\frac{i}{2}\right)^n = \left(\frac{1}{2}\right)^n \cdot e^{\frac{in\pi}{2}}$$

$$-i = e^{-i\pi/2}$$

$$\Rightarrow \quad (-i)^n = e^{-in\pi/2}$$

$$\Rightarrow \quad \left(-\frac{i}{2}\right)^n = \left(\frac{1}{2}\right)^n e^{-\frac{in\pi}{2}}$$

So,

$$P_{11}^{(n)} = a + b\left(\frac{1}{2}\right)^n e^{i\frac{n\pi}{2}} + c\left(\frac{1}{2}\right)^n e^{-i\frac{n\pi}{2}}$$

$$= a + b\left(\frac{1}{2}\right)^n\left(\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}\right)$$

$$+ c\left(\frac{1}{2}\right)^n\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\boxed{P_{11}^{(n)} = a + \left(\frac{1}{2}\right)^n\left(\beta\cos\frac{n\pi}{2} + \gamma\sin\frac{n\pi}{2}\right)} \quad \cdots (*)$$

TRUE for $n = 0, 1, 2, \cdots$

where $\boxed{\begin{aligned}\beta &= b + c \\ \gamma &= i(b - c)\end{aligned}}$

To find $p_{11}^{(n)}$, it is sufficient to find a, β, γ in $(*)$

$p_{11}^{(0)} = 1$

$p_{11}^{(1)} = p_{11} = 0$

$p_{11}^{(2)} = 0$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & * \\ & * & \\ * & & \end{pmatrix}_{3\times 3}$$

$p_{11}^{(2)}$

Plug in $n=0,1,2$ in $(*)$

$$1 = a + \beta$$
$$0 = a + \frac{1}{2}\gamma$$
$$0 = a + \frac{1}{4}(-\beta)$$

Solving

$$\boxed{a = \frac{1}{5}, \quad \beta = \frac{4}{5}, \quad \gamma = -\frac{2}{5}}$$

Hence $(*)$ gives

$$\boxed{p_{11}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right)}$$

$n = 0, 1, 2, \ldots$

Example: Consider a Markov Chain $(X_n)_{n\geq 0}$ on three states $S=\{1,2,3\}$

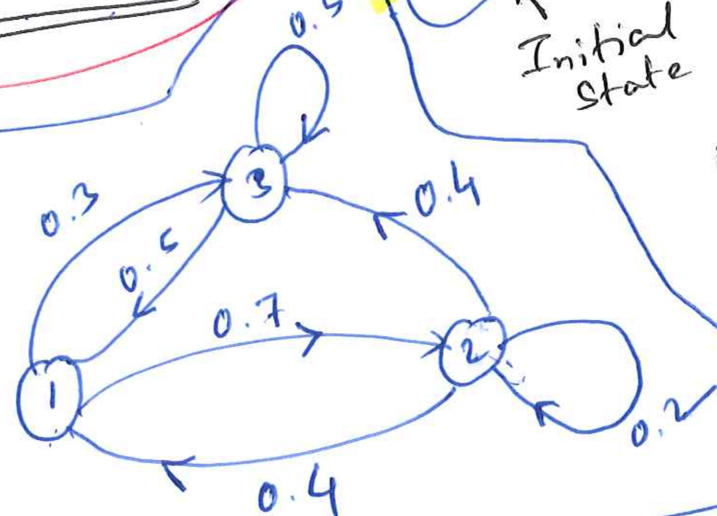
Let $P = \begin{pmatrix} 0 & 0.7 & 0.3 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0 & 0.5 \end{pmatrix}$

Assume that Markov Chain starts at State 2 $\lambda = (0, 1, 0)$

Initial state prob.

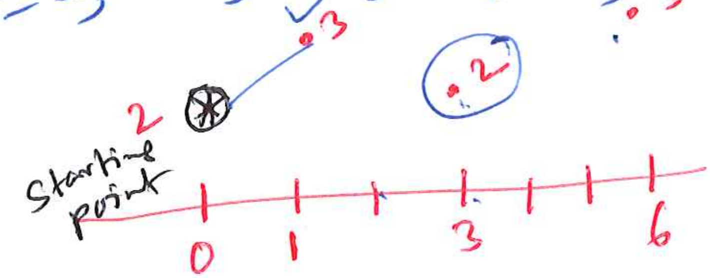
λ	prob
λ_1	$P(X_0=1)$
λ_2	$P(X_0=2)$
λ_3	$P(X_0=3)$

Diagram:



Compute: (1) $P(X_1=3, X_3=2, X_6=3)$

Solution:



$$P(X_1=3, X_3=2, X_6=3) = P_{23} \; P_{32}^{(2)} \; P_{23}^{(3)}$$

Start at 2
after 1 step you
end up at 3.

$$= (0.4)(0.35)(0.412)$$

$$\approx \boxed{0.06}$$

P_{23}

Check:

$$P^2 = \begin{pmatrix} 0.43 & 0.14 & 0.43 \\ 0.28 & 0.32 & 0.4 \\ 0.25 & 0.35 & 0.4 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.271 & 0.329 & 0.4 \\ 0.328 & 0.26 & 0.412 \\ 0.34 & 0.245 & 0.415 \end{pmatrix}$$

Already happened

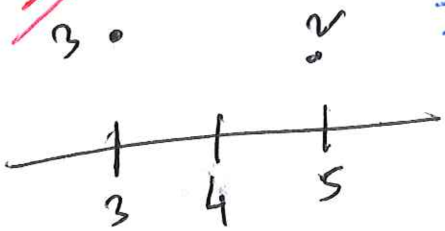
(2) Compute $P(X_5 = 2 \mid X_2 = 1, X_3 = 3)$

Given

Method 1

Markov Property

$$= P(X_5 = 2 \mid X_3 = 3)$$

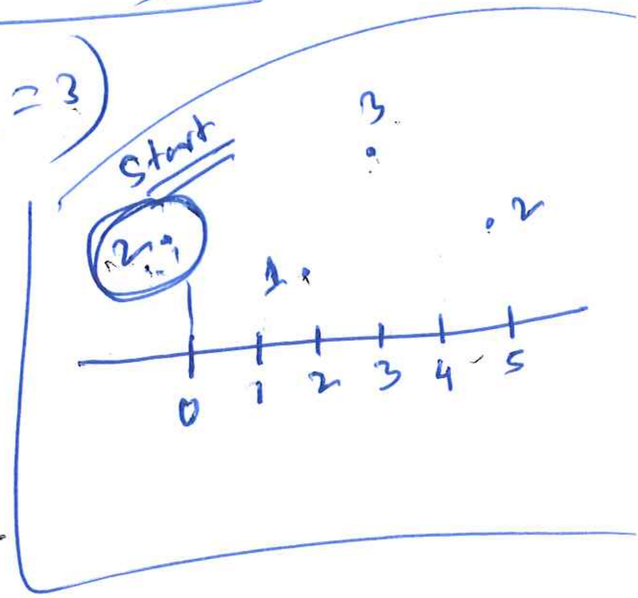


$$= P_{32}^{(2)} = \boxed{0.35}$$

Method 2

$$P(X_5=2 \mid X_2=1, X_3=3)$$

$$= \frac{P(X_5=2, X_2=1, X_3=3)}{P(X_2=1, X_3=3)}$$



$$= \frac{\cancel{P_{21}^{(2)} P_{13}}\, P_{32}^{(2)}}{\cancel{P_{21}^{(2)} P_{13}}}$$

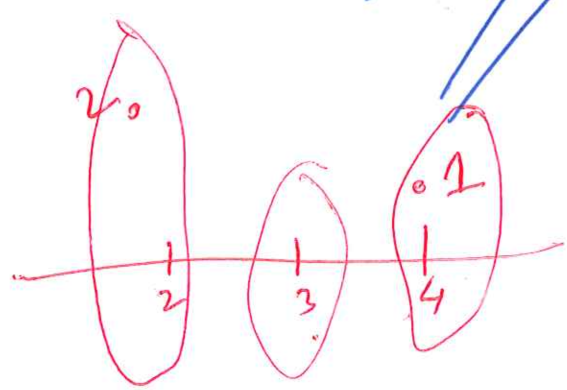
$$= P_{32}^{(2)} = \boxed{0.35}$$

③ Compute:

$$P(X_3=2 \mid X_2=3, X_4=1)$$

$$= \frac{P(X_3=2, X_2=3, X_4=1)}{P(X_2=3, X_4=1)}$$

$$= \frac{P_{23}^{(2)}\, \overset{=0}{P_{32}}\, P_{21}}{P_{23}^{(2)}\, P_{31}^{(2)}} = 0$$



4. Compute
$$P(X_2=3, X_4=1 \mid X_3=2)$$

$$= \frac{P(X_2=3, X_4=1, X_3=2)}{P(X_3=2)} = 0$$

by 3

P_{32}

Example (A non-Markov Chain):

A box contains one red ball (R) and one green ball. (G)

At each time step, one ball is drawn at random, its color is noted and then, together with one additional ball of the SAME color, put back into the box.

Define the stochastic (random) process $(Y_n)_{n\geq 1}$ with state space $S=\{0,1\}$:

Let $Y_n=1$, if the n^{th} ball drawn is red (R)

$Y_n=0$, if the n^{th} ball drawn is green (G)

THEN

$(Y_n)_{n\geq 0}$ is NOT a Markov Chain.

State
||
Value of discrete r.v.

Proof:

$$P(Y_3=1 \mid Y_1=1, Y_2=1)$$

$$= \frac{P(Y_3=1, Y_1=1, Y_2=1)}{P(Y_1=1, Y_2=1)} = \frac{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)} = \frac{3}{4}$$

$$P(Y_3=1 \mid Y_1=0, Y_2=1)$$

$$= \frac{P(Y_3=1, Y_1=0, Y_2=1)}{P(Y_1=0, Y_2=1)} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{2}{4}\right)}{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{1}{2}$$

Hence $P(Y_3=1 \mid Y_1=1, Y_2=1)$

$$\neq P(Y_3=1 \mid Y_1=0, Y_2=1)$$

So, it is NOT memory-less

$\Rightarrow$ is is NOT Markov!

Example:

Conside the Markov chain of three states $S=\{1,2,3\}$ that has the transition matrix

$$P=\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/3 & 0 & 2/3 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

~~Find~~ If $P(X_1=1)=P(X_1=2)=\frac{1}{4}$

Find $P(X_1=3, X_2=2, X_3=1)$

Solution:

Initial distribution: $(1, 0, 0)$

$P(X_0=1)$ $P(X_0=2)$ $P(X_0=3)$

$$P(X_1=3, X_2=2, X_3=1)$$

$$= P_{13}\cdot P_{32}\cdot P_{21} = \frac{1}{4} * \frac{1}{2} * \frac{1}{3}$$

$$= \boxed{\frac{1}{24}}$$

Think: Can you solve this problem if π is NOT given,

Yes! $P(X_1=3) = 1 - P(X_1=1) - P(X_1=2)$

$= 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$

In that case

$P(X_1=3, X_2=2, X_3=1) = P(X_1=3) \cdot P_{32} \cdot P_{21}$

$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3}$

$= \frac{1}{12}$

(when X_0 distribution is NOT provided, this is the best we can get)

Treat X_1 as the initial distribution