$$T^*T = \begin{bmatrix} |\lambda_1|^2 & * & * \\ * & |t_{12}|^2 + |\lambda_2|^2 & * \\ * & * & |t_{13}|^2 + |t_{23}|^2 + |\lambda_3|^3 \end{bmatrix}$$

$$\therefore tr(T^*T) = \sum_{i=1}^{n} |\lambda_i|^2 + \sum_{i<j} |t_{ij}|^2.$$

Generalized Inverse :-

Recall that, $A_{m\times n}$ is left invertible $\Leftrightarrow N_A = \{0\} \Leftrightarrow r(A) = n$ (full col. rank)

$A_{m\times n}$ is right invertible $\Leftrightarrow \mathcal{C}(A) = \mathbb{F}^m \Leftrightarrow r(A) = m$ (full row rank)

Defn:- 3.1

$A_{m\times n}$ matrix over F.

a) A matrix $X^0 \in F^{n\times m}$ is said to be generalized inverse (g-inverse) of A if $AA^0A = A$. [If A is invertible sq. matrix then $A^0 = A^{-1}$]

(b) A g-inverse of 'A' is called 'reflexive g-inverse' (or, psudo inverse) if it also satisfies $A^0AA^0 = A^0$. [i.e here $AA^0A = A$ & $A^0AA^0 = A^0$]

c) A g-inverse of 'A' is said to be 'minimum norm g-inverse' if it satisfies $(A^0A)^* = A^0A$ [i.e here $AA^0A = A$ & $(A^0A)^* = A^0A$]

(d) A g-inverse of 'A' is said to be 'least square g-inverse' if it satisfies $\to (AA^0)^* = AA^0$ [i.e here $AA^0A = A$ & $(AA^0)^* = AA^0$]

(f) e) $\underline{\text{Th.}}$ Moore-Penrose Inverse :- For $A_{m\times n}$ $\exists$ exactly one matrix $A^+ \in \mathbb{F}^{n\times m}$ that meets all the four conditions —

i) $AA^+A = A$

ii) $A^+AA^+ = A^+$

iii) $(A^+A)^* = A^+A$

iv) $(AA^+)^* = AA^+$

[The proof of Th.] (MP Inverse) :-

Proof :- Uniqueness :- If possible both G_1 and G_2 satisfy the given conditions:-

To show $G_1 = G_2$

Now,

$$G_1 = G_1AG_1 = G_1 G_1^* A^* = G_1 G_1^* A^* G_2^* A^* = G_1 G_1^* A^* A G_2$$
$$= G_1 A G_1 A G_2$$
$$= G_1 A G_2$$
$$= G_1 A G_2 A G_2$$
$$= G_1 A A^* G_2^* G_2$$
$$= A^* G_1^* A^* G_2^* G_2$$
$$= A^* G_2^* G_2$$
$$= G_2 A G_2$$
$$= G_2$$

Existance :— Let $A = BC$ be a rank factorization, Then it can be 'easily' verified that, $B^+ = (B^*B)^{-1}B^*$, $C^+ = C^*(CC^*)^{-1}$

and then $A^+ = C^+B^+$ [Have to verify !] Home work

1) B^+ is a Moore Penrose Inverse:-

i) To show:- $BB^+B = B$

$\therefore BB^+B = B(B^*B)^{-1}B^*B$

$= B.I$

$= B$ (Proved)

ii) To show $B^+BB^+ = B^+$

$\therefore B^+BB^+ = (B^*B)^{-1}B^*B.(B^*B)^{-1}B^*$

$= I.(B^*B)^{-1}B^*$

$= B^+$

iii) To show, $(B^+B)^* = B^+B$

$\therefore (B^+B)^* = \left((B^*B)^{-1}B^*B\right)^*$

$= I^* = I = (B^*B)^{-1}B^*B$

$= B^+B$

iv) To show $(BB^+)^* = BB^+$

$\therefore (BB^+)^* = \left(B(B^*B)^{-1}B^*\right)^* = B\left((B^*B)^{-1}\right)^*B^*$

$= B\left((B^*B)^*\right)^{-1}B^*$

$= B(B^*B)^{-1}B^*$

$= B.B^+$ (Proved)

for c^+

1stly. we have to show that

i) $cc^+c = c$

$\therefore$ Now, $cc^+c = cc^*(cc^*)^{-1}c$

$= I\,c = c$ (Proved)

ii) $c^+cc^+ = c^+$

Now, $c^+cc^+ = c^*(cc^*)^{-1}c.\,c^*(cc^*)^{-1}$

$= c^*(cc^*)^{-1}.I = c^+$

iii) $(c^+c)^* = c^+c$

$= \left(c^*(cc^*)^{-1}c\right)^*$

$= c^*\left((cc^*)^{-1}\right)^*c$

$= c^*\left[(cc^*)^*\right]^{-1}c$

$= c^*(cc^*)^{-1}c$

$= c^+c$ (Proved)

iv) $(cc^+)^* = cc^+$

$\therefore (cc^+)^* = \left(cc^*(cc^*)^{-1}\right)^* = I^* = cc^+$

Now, we have to show it for A:-

i) $AA^+A = A$

$\therefore AA^+A = AC^+B^+A$

$= \cancel{A}\, BC\, C^+B^+\, BC$

$= B \cdot I_K \cdot I_K \cdot C$

$= B \cdot I_K C$

$= B_{m\times K}\, I_{K\times K}\, C_{K\times m}$

$= B_{m\times K} \cdot C_{K\times m} = (BC)_{m\times m} = A$ (Proved)

ii) $A^+AA^+ = A^+$

$\therefore A^+AA^+ = C^+B^+BCC^+B^+ = C^+ I_K I_K B^+$

$= C^+B^+$

$= A^+$

iii) $(A^+A)^* = A^+A$

$\therefore (A^+A)^* = \cancel{B}(C^+B^+BC)^*$

$= (C^+I_KC)^*$

$= (C^+_{m\times K}\, C_{K\times m})^*$

$= C^*. (C^+)^*$

$= C^*.(C^*(CC^*)^{-1})^*$

$= C^*((CC^*)^{-1})^*\, C$

$= C^*((CC^*)^*)^{-1}\, C = C^*(CC^*)\, C$

$= C^+C$

$= C^+I_KC = C^+B^+BC$

$= A^+A$ (P)

(iv) $(AA^+)^* = AA^+$

Now, $(AA^+)^* = (BCC^+B^+)^*$

$= (BI_kB^+)^*$

$= (BB^+)^*$

$= (B((B^*B)^{-1}B^*))^* B^*$

$= B((B^*B)^{-1})^* . B^*$

$= B.((B^*B)^*)^{-1} . B^*$

$= B(B^*B)^{-1} B^*$

$= B.B^+$

$= BI_kB^+ = BCC^+B^+ = AA^+$ (Proved)

Remark:- If G is any g-inverse of A, then GAG is a pseudoinverse of A.

Proof:- G is any g inverse of A.

$\therefore AGA = A$

~~GAG = G = a pseudoinverse~~

Now,

$AGAGA = AGA = A$

again, $(GAG)A(GAG) = GAG$ ~~[as GAG = G]~~

$= G(AGA)GAG = GAGAG = G(AGA)G = GAG$.

$\therefore GAG$ is a pseudoinverse of A (Proved)

$AGA = A$

we're to prove

$GAG = B$

$BAB = B$

$(GAG).A(GAG)$

$= G(AGA)(GAG)$

$= GAGAG = GAG = B$

(iv) $(AA^+)^* = AA^+$

Now, $(AA^+)^* = (BCC^+B^+)^*$

$$= (BI_kB^+)^*$$

$$= (BB^+)^*$$

$$= \left(B\left((B^*B)^{-1}B^*\right)\right)^* B^*$$

$$= B\cdot B\left((B^*B)^{-1}\right)^* \cdot B^*$$

$$= B\cdot\left((B^*B)^*\right)^{-1} \cdot B^*$$

$$= B\,(B^*B^{**})^{-1}\, B^*$$

$$= B\cdot B^+$$

$$= BI_kB^+ = BCC^+B^+ = AA^+ \quad \text{(Proved)}$$

Remark :- If G is any g-inverse of A, then GAG is a psudoinverse of A.

Proof :- G is any g inverse of A.

$\therefore AGA = A$

~~GAG = G = a psudoinverse~~

Now,

$$A\underline{GAG}A = AGA = A$$

$AGA = A$

we're to prove

$GAG = B$

$BAB = B$

$(GAG)A(GAG)$

$= G(AGA)(GAG)$

$= GAGAG = \underline{GAG = B}$

again, $(GAG)A(GAG) = GAG$ ~~[as GAG = G]~~

$= G(AGA)GAG = GAGAG = G(AGA)G = GAG$.

$\therefore GAG$ is a psudoinverse of A (Proved)

2. If A is left invertible then every left inverse of A is P.I

3. A is right invertible, then every ~~rt~~ inverse of A is P.I

2) A is L.I

$\therefore$ L.I of $A = (A^*A)^{-1}A^*$

$\therefore$ Now,

$$A(A^*A)^{-1}A^*A = A$$

again,

$$\because (A^*A)^{-1}A^*A(A^*A)^{-1}A^* = (A^*A)^{-1}A^*$$

$\therefore$ Left ~~invertible~~ inverse is P.I.

3) Let C is Right inverse of A.

$\therefore AC = I$

$\therefore ACA = A$

$CAC = C$.

4. Although there are matrices which are neither left nor right invertible, every matrix A has a psudoinverse.

5. A^0 is a psudo inverse of A iff A is a psudoinverse of A^0.

<u>Result</u>:— Let A, G be matrices of order $m \times n$ and $n \times m$ respectively. Then the following conditions are equivalent :-

i) G is a g-inverse of A.

ii) For any $y \in C(A)$, $x = Gy$ is a solution of $Ax = y$.

Proof:-

(i) $\Rightarrow$ (ii)

Any $y \in C(A)$ is of the form.

$y = Az$ for some z

Then $A(Gy) = AGAz = (AGA)z = Az = y$.

(ii) $\Rightarrow$ (i)

$AGy = y$ for any 'y' $\in C(A)$ we have

$AGAz = Az$ for all z in particular we choose z as e_i; then

ith column of AGA and A becomes equal. Hence $AGA = A$ Q.E.D

~~Hence~~ How to find g-inverse of a given matrix

Method-I Let $A = BC$ be a rank factorization.

We know that B admits a left inverse B_l & C admits a right inverse.

C_r then $G = C_r B_l$ is a g-inverse of A

Since $AGA = BC(C_r B_l).BC = BC = A$.

Method-II If $\rho(A) = r$ (rank), we know that $\exists$ non singular matrices P, Q such that $A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q$. It can be verified that for any U, V, W of appropriate sizes $\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}$ is a g-inverse.

of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ then $G = Q^{-1}\begin{bmatrix} I_r & U \\ V & W \end{bmatrix} P^{-1}$ is a g-inverse of A.

$AGA = A$ (check!)

Method-3

$\rho(A) = r$ (rank), choose any $r \times r$ non singular submatrix of A. for

for convenience let us assume,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{where } |A_{11}| \neq 0$$

Since $\rho(A) = r$, there exists a matrix X s.t $A_{12} = A_{11}X$, $A_{22} = A_{21}X$.

(Similar to linear depending columns for elements) Now it can be verified that the $n \times m$ matrix G defined as

$$G = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{is a g inverse of A.}$$

Q. Find two different G inverse of $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 4 \\ -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 6 \end{bmatrix}$.

Q. Find the Moore penrose inverse of $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$.

Q $A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

Rank of A = 1

$\therefore A = B_{2\times1} C_{1\times2} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$

$\therefore G = C^+ B^+$

$(B^* B)^{-1} B^*$

$\therefore B^+ = \cancel{B^*(BB^*)^{-1}} = \left([2\ 3] \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)^{-1} [2\ 3]$

$= \cancel{[2\ 3] \left[\begin{bmatrix} 2 \\ 3 \end{bmatrix} [2\ 3] \right]^{-1}} =$

$= \cancel{[2\ 3] \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}^{-1}}$

$=$

$$B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$\therefore$ B is left invertible

C is right "

$$\therefore (B^*B)^{-1} B^* = \left(\begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= (4+9)^{-1} \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{13} & \frac{3}{13} \end{bmatrix}$$

and

$$C^+ = C^* (C C^*)^{-1}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} 1/5$$

$$= \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$\therefore A^+ = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} \begin{bmatrix} \frac{2}{13} & \frac{3}{13} \end{bmatrix} = \begin{bmatrix} \frac{2}{65} & \frac{3}{65} \\ \frac{4}{65} & \frac{6}{65} \end{bmatrix}$$

$$= \frac{1}{65} \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 4 \\ -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 6 \end{bmatrix}$$

$$\underset{\substack{R_2 \sim R_2 - 2R_1 \\ R_3 \sim R_3 + R_1 \\ R_4 = R_4 + 2R_1}}{\sim} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 2 & 0 & 10 \end{bmatrix} \underset{R_4 = R_4 - 2R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A = BC$

$$\therefore B = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 1 \\ -2 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 4 \end{bmatrix}$$

$$\therefore B^{\dagger} = (B^{*} B)^{-1} B^{*}$$

$$= \left(\underset{2\times4}{\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}} \underset{4\times2}{\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ -1 & 1 \\ -2 & 2 \end{bmatrix}} \right)^{-1} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{pmatrix} 1+4+1+4 & -1-4 \\ -1-4 & 1+4 \end{pmatrix}^{-1} \begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$= \underset{2\times2}{\begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix}} \underset{2\times4}{\begin{bmatrix} 1 & 2 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 10 & 20 & -10-5 & -20-10 \\ -5 & -10 & 5+5 & 10+10 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 20 & -15 & -30 \\ -5 & -10 & 10 & 20 \end{bmatrix} = 5 \begin{bmatrix} 2 & 4 & -3 & -6 \\ -1 & -2 & 2 & 4 \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

rank = 1

$\therefore A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \underset{R_2 - R_1}{\overset{B}{\sim}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$

$\therefore B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$\therefore B^{+} = (B^{*}B)^{-1} B^{*}$

$= \cancel{\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}\right)^{-1} B^{*}}$

$\left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \end{bmatrix}.$

$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$C^{+} = C^{*}(CC^{*})^{-1}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^{-1}$

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$\therefore A^{+} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

1. If G is a g-inverse of A then $\operatorname{rank}(A) = \operatorname{rank}(AG)$

$= \operatorname{rank}(GA)$

Proof :

$\operatorname{rank}(A) = \operatorname{rank}(AGA) \leq r(AG) \leq r(A)$

where $r(A) = r(AG)$

2. G is a g-inverse of A then $\operatorname{rank} A \leq \operatorname{rank} G$.

3. Further more equality holds iff G is reflexive.

G is a g-inverse of A

$\therefore AGA = A$

$\therefore \operatorname{rank}(A) = \operatorname{rank}(AGA) \leq r(AG) \leq r(G)$

$\therefore r(A) \leq r(A$

Th.

Every matrix $A \in \mathbb{F}^{p\times q}$ admits a psudoInverse $A^0 \in \mathbb{F}^{q\times p}$. Moreover if rank $A = r > 0$ and SVD of A is expressed generically as

$$A = V \begin{bmatrix} D & O_{r\times(q-r)} \\ O_{(p-r)\times r} & O_{(p-r)\times(q-r)} \end{bmatrix} V^* \quad \text{---(1)}$$

where $V \in F^{p\times p}$ and $U \in F^{q\times q}$ are unitary

and $D = \mathrm{diag}\{s_1, s_2, \dots, s_r\}$ is invertible, then

$$A^0 = U \begin{bmatrix} D^{-1} & B_1 \\ B_2 & B_2 D B_1 \end{bmatrix} V^*$$ is a psudoinverse of A,

for any choice of $B_1 \in \mathbb{F}^{r\times(p-r)}$ & $B_2 \in \mathbb{F}^{(q-r)\times r}$. Furthermore every psudoinverse A^0 of A can be expressed in this way.

Proof:- If $A = O_{p\times q}$ then $A^0 = O_{q\times p}$ and here it's the P.I. of A

Suppose $\mathrm{rank}(A) = r > 0$ & note that every matrix $\tilde{A} \in \mathbb{F}^{q\times p}$ can be written in the form $\tilde{A} = U \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} V^*$ where U, V are unitary as in (1) above and R_{ij}'s are of compatible size. The Component $A\tilde{A}A = A$ is not iff

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$ (check!)

i.e iff $R_{11} = D^{-1}$, Next fixing $R_{11} = D^{-1}$ we see that the

1000
135
270
300
240
810

The other condition $\tilde{A} A \tilde{A} = \tilde{A}$ is met iff

$$\begin{bmatrix} D^{-1} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^{-1} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} D^{-1} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$$

i.e iff $R_{21} D R_{12} = R_{22}$

Result:-

Let $A \in \mathbb{F}^{p \times q}$ & Let A^0 is a psudoinverse of A Then:-

i) $C(AA^0) = C(A)$ ii) $N(AA^0) = N(A^0)$ (iii) $\dim C(A) = \dim C(A^0)$

Proof:- $C(AA^0) \subseteq C(A) = C(AA^0A) \subseteq C(AA^0)$ whence

(i) follows

Again, $N(A^0) \subseteq N(AA^0) \subseteq N(A^0AA^0) = N(A^0)$, whence (ii)

followes.

Finaly, by rank nulity Theorem,

$p = \dim N(A^0) + \dim C(A^0)$

$= \dim N(AA^0) + \dim C(A^0)$ — (a)

and $p = \dim N(AA^0) + \dim C(AA^0)$

$= \dim N(AA^0) + \dim C(A)$ — (b)

Hence from (a), (b) $\Rightarrow$

$\dim C(A) = \dim C(A^0)$

Note:- $\text{rank}(A) = \text{rank}(A^0)$

Result :- i) $\mathbb{F}^p = C(A) \oplus N(A^o)$ (ii) $\mathbb{F}^q = C(A^o) \oplus N(A)$

Proof :- we observe that AA^o and A^oA are both projections

Since $(AA^o)(AA^o) = AA^o$ and $(A^oA).(A^oA) = A^oA$.

Then as $C(P) \oplus N(P) = \mathbb{F}^k$ for any projection $P \in \mathbb{F}^{k \times k}$

$\mathbb{F}^p = C(AA^o) \oplus N(AA^o)$ and

$\mathbb{F}^q = C(A^oA) + N(A^oA)$.

Now $\mathbb{F}^p = C(A) \oplus N(A^o)$

$\mathbb{F}^q = C(A^o) \oplus N(A)$ [Note A^o is a P.I of A iff A is a P.I of A^o]

Construction of M.P Inverse through SVD:-

Theorem :- Let $A \in \mathbb{F}^{p \times q}$ then there exists exactly one matrix $A^+ \in \mathbb{F}^{q \times p}$ s.t $AA^+A = A$, $A^+AA^+ = A^+$, $(AA^+) = (AA^+)^*$, $(A^+A) = (A^+A)^*$ [Rogers Penrose / 26 July 1954]

Proof :- If $A = O_{p \times q}$ then the matrix $A^+ = O_{q \times p}$ clearly it meets all the conditions.

If rank $A = r > 0$ and $A = V \begin{bmatrix} D & O_{r \times (q-r)} \\ O_{(p-r) \times r} & O_{(p-r)(q-r)} \end{bmatrix} U^*$

$= V_1 D V_1^*$ (say) where

$V=[V_1\ V_2]$ and $U=[U_1\ U_2]$ are unitary matrices with first block V_1 & U_1 of size $p\times r$ and $q\times r$ respectively and $D=\text{diag}\{s_1, s_2, \ldots, s_r\}$ is a diagonal matrix based on no zero singular values $s_1 \geq s_2 \geq \ldots \geq s_r > 0$ then the matrix $A^{+} \in \mathbb{F}^{q\times p}$ defined by the formula

$$A^{+} = U\begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} V^{*} = U_1 D^{-1} V_1^{*}$$ mets all the 4 Conditions of the theorem.

It remains to check the uniqueness

Let $B, C \in \mathbb{F}^{q\times p}$ both satisfy 4 conditions of the theorem.

Let $Y = B^{*} - C^{*}$ Then

$$A = ABA = A(BA)^{*} = AA^{*}B^{*}$$

$$A = ACA = A(CA)^{*} = AA^{*}C^{*}$$ together imply

$0 = AA^{*}Y$ and hence $\mathcal{C}(Y) \subseteq \mathcal{N}(AA^{*}) = \mathcal{N}(A^{*})$

On the other hand the formulas $B = BAB = B(AB)^{*} = BB^{*}A^{*}$

$C = CC^{*}A^{*}$ [Similarly]

$\therefore Y = B^{*} - C^{*} = A(BB^{*} - CC^{*})$ and

Hence, $\mathcal{C}(Y) \subseteq \mathcal{C}(A)$

Now note that $\mathcal{C}(A) \oplus \mathcal{N}(A^{*}) = \mathbb{F}^{p}$ for any $A \in \mathbb{F}^{p\times q}$

whence, the direct sum indicates $\mathcal{C}(A) \cap \mathcal{N}(A^{*}) = \{0\}$

This shows that $\mathcal{C}(Y) \subseteq \mathcal{C}(A) \cap \mathcal{N}(A^{*}) = \{0\} \Rightarrow Y = \{0\}$ i.e

$B^{*} = C^{*}$ whence $B = C$

<u>Remark:-</u> AA^+ & A^+A are both orthogonal projections w.r.t standard innerproduct (improving upon AA^0, A^0A which are ordinary projection]

<u>Result:-</u> Let $A^+ \in \mathbb{F}^{q \times p}$ be the MP inverse of $A \in \mathbb{F}^{p \times q}$ then

i) $\mathbb{F}^p = \mathcal{C}(A) \dot{\oplus} \mathcal{N}(A^+)$ (orthogonal compliment $\dot{\oplus}$)

ii) $\mathbb{F}^q = (\mathcal{C}A^+) \dot{\oplus} \mathcal{N}(A)$

ex.

Ex. 1. Find A^+ for $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

2. Find A^+ via rank factorization, $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

3. Prove that any square matrix A can be written as VBW where V and W are unitary and B is diagonal.

4. Find A^+ for $A = \begin{bmatrix} 5 & 1 & 3 \\ 0 & 0 & 2 \\ 10 & 2 & 4 \end{bmatrix}$ via rank factorization

5) If $A \in \mathbb{C}^{p\times q}$ and $b \in \mathbb{C}^p$ then prove that,

$$\|Ax-b\|^2 = \|Ax - AA^+b\|^2 + \|(I_p - AA^+)b\|^2 \text{ for every } x \in \mathbb{C}^q$$

6) Exhibit matrices A and B s.t. AB is defined but $(AB)^+ \neq B^+A^+$

7) Let V and W be finite dimensional innerproduct spaces & let $T: V \to W$ be linear, Prove that -

i) if T is one to one then T^*T is invertible and $T^+ = (T^*T)^{-1}T^*$

ii) if T is onto then TT^* is invertible and $T^+ = T^*(TT^*)^{-1}$

$x \in \mathbb{C}^q$
$q\times 1$

5) $A \in \mathbb{C}^{p\times q}$, $b \in \mathbb{C}^p$, $x \in \mathbb{C}^q$

$\therefore Ax \in \mathbb{C}^{p\times 1}$

$\therefore \|Ax - b\|^2 = Ax$

$(Ax - AA^+b)(b - AA^+b)$

$= Axb - AxAA^+b - AA^+b^2$

$+ AA^+b \cdot AA^+b$

$= AA^+(Ax - b) - (I_p - AA^+)b$

Now AA^+

5)

$\langle AA^+(Ax-b), (I_p - AA^+)b \rangle$

$= \langle (Ax-b), (AA^+)^* (I_p - AA^+) b \rangle$

$= \langle (Ax-b), (AA^+)(I_p - AA^+) b \rangle$

$= \langle (Ax-b), (AA^+ - AA^+AA^+) b \rangle$

$= \langle (Ax-b), (AA^+ - AA^+) b \rangle$

$= \langle (Ax-b), 0 \cdot b \rangle$

$= 0$

---

3) $\because$ AA^* and A^*A are both Hermitian & both have same eigenvalues, so we have unitary matrices P & Q s.t

$$P AA^* P^* = D = Q A^* A Q^*$$

where D = Diagonal matrix with common eigen values in the diagon

This means

$$Q^* P AA^* P^* Q = AA^*$$

i.e. $\exists$ unitary matrix $T = (Q^* P)$ T s.t. $AA^* T^* = A^* A$

Hence $(TA)(TA)^* = (TA)^*(TA)$ whence (TA) in normal,

and therefore unitarily diagonalizable, i.e

$$W(TA)W^* = B$$

i.e $(WT) A W^* = B$ i.e $V_1 A W^* = B$

i.e $A = V B W$ (Proved)

Exercise :-

Show that i) $A^+ A^{+*} A^* = A^+ = A^* A^{+*} A^+$

ii) $A^+ A A^* = A^* = A^* A A^+$

$A^+ (AA^+)^*$

$= A^+ A A^+$

$= A^+$

again, $A^* A^{+*} A^+$

$= (A^+ A)^* A^+$

$= A^+ A A^+$

$= A^+$

Note :- (From sum no 5) If exhibits the fact that the least approximation to 'b' that we can hope to get by vectors of the form Ax is obtained by choosing 'x' so that, $Ax = AA^+b$. This is quite reasonable, since AA^+b is equal to the orthogonal projection of b onto $C(A)$.

2. The particular choice $x = A^+b$ has one more feature.

$\|A^+ b\| = \min \{ \|y\| : y \in \mathbb{C}^q \,\&\, Ay = AA^+b \}$. To verify, observe that if y is any vector for which $Ay = AA^+b$, then $y - A^+b \in N(A)$ i.e., $y = A^+b + u$ $\exists\, u \in N(A)$. Therefore since $\langle A^+b, u\rangle = \langle A^+AA^+b, u\rangle = \langle A^+b, A^+Au\rangle = \langle A^+b, 0\rangle = 0$

it follows that $\|y\|^2 = \|A^+b\|^2 + \|u\|^2$. Thus $\|y\| \geq \|A^+b\|$ with equality following iff $y = A^+b$.

$\underline{A^+A A^*} = (A^+A)^* A^* = A^* A^{+*} A^* = (A A^+ A)^*$

$= A^* (AA^+)^* = A^*$

$\underline{= A^* A A^+}$ $=$ $A^*AA^+AA^+$

$= A^*AA^+$

2. Show that i) $A^{++} = A$ ii) $(A^+)^* = (A^*)^+$

iii) If $|A| \neq 0$, $A^+ = A^{-1}$, (iv) $(\lambda A)^+ = \lambda^+ A^+$ (where $\lambda^+ = \lambda^{-1}$ i.e $\lambda \neq 0$
$= 0$ if $\lambda = 0$)

(v) a) $(AA^*)^+ = A^{+*} A^+$

b) $(A^* A)^+ = A^+ A^{*+}$.

(vi) If U and V are unitary $(UAV)^+ = V^* A^{-1} U^*$.

(vii) If A is normal, $A^+A = AA^+$; $(A^n)^+ = (A^+)^n$

(viii) A, A^*A, A^+, A^+A all have same rank and that rank is equal to trace of (A^+A).

(v) a) $(AA^*)^+ = \cancel{(A A^+ A A^*)^+}$

$= \cancel{(A A^+ A A^+ A A^*)^+}$

$= A A^+ A A^*$

$= (AA^*)(AA^*)^{+*}(AA^*)^+$

$= AA^*(AA^*)^+$

---

v) a) $(AA^*)^+ = (AA^*)^*((AA^*)^+)^*(AA^*)^+$

$= AA^*((AA^*)^*)^+(AA^*)^+$

$A^{+*}A^+$

$= AA^*(AA^*)^+(AA^*)^+$

$((A^+)^* A^+)^*$

$=$

$=$

$I = AA^*(AA^*)^+ \Rightarrow (AA^*)^+ = (A^{*} A^*)^{-1}$

$= A^{*+} A^+$.

$(AA^*)^+ = A^{+*} A^+$

$(AA^*)^+$

$= (AA^* A A^+)^+$

$= (AA^* (A^{+*} A^*)^+$

$= \cancel{(A(AA^+A)^*)^+}$

$=$

5) a) $(AA^*) A^{+*} A^+ (AA^*)$

$= AA^* A^{+*} A^+ AA^* = AA^* A^{+*} A^+ AA^*$

$= A(A^+A)^* (A^+A)A^* = AA^+AA^*$

$= AA^*$

$= A(AA^*A^{+*} AA^+)^*$

$= A$

2) $A^{+*}A^+ (AA^*) A^{+*} A^+$

$= A^{+*} A^+ AA^* A^{+*} A^+$ (3) $(AA^* A^{+*} A^+)^*$

$= A^{+*} A^+ A A^+$ $= (AA^+)^*$

$= A^{+*} A^+$ $= AA^+$

$= AA^* A^{+*} A^+$

4) $(A^{+*} A^+ AA^*)^*$ $= (A^{+*} A^*)^* = (AA^+)^*$

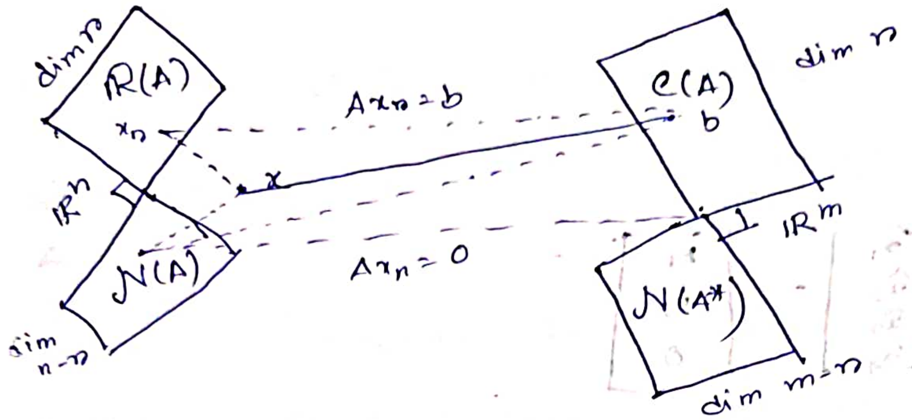
$= AA^* A^{+*} A$

$= AA^+ A^{+*} A^*$

$=$

$=$

Big Picture (Strang) through least Square :-



$$Ax = A(x_r + x_n) = Ax_r$$

i.e $Ax = b \equiv Ax_r = b$

retangular system $Ax = b$

least square solution comes from

Normal eqn $A^TA\hat{x} = A^Tb$.

Remark-I If A has dependent columns then A^TA is not invertible & so $\hat{x}$ is not determined. Any vector in the null space can not be added to $\hat{x}$

Now we choose a 'least' (shortest/smallest) $\hat{x}$ for every $Ax = b$

$Ax = b$ has two possible difficulties—

i) Dependent rows ii) Dependent columns.

i) with Dependent rows $Ax = b$ may have no solution. This happens when b is outside C(A). Indeed of $Ax = b$ we solve $A^TA\hat{x} = A^Tb$

ii) If A have dependent columns, then $\hat{x}$ will not be unique. we have to choose a particular solution of $A^TA\hat{x} = A^Tb$ and we choose the shortest.

The optimal solution of $Ax=b$ is the min. length solution of

$A^T A \hat{x} = A^T b$ we call it $\rightarrow x^+$

Example:-

$A\hat{x} = p$

$$A = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix}$$

The cols of A and $\hat{}$ with 0, Thus in the C(A) the closest vector to $b = (b_1, b_2, b_3)$ is $p = (b_1, b_2, 0)$.

The error in $Ax=b$ cannot be universal. But the error in first two will be zero. if we take $\hat{x}_1 = \frac{b_1}{\sigma_1}$, $\hat{x}_2 = \frac{b_2}{\sigma_2}$

Now, we face the second difficulty. To make $\hat{x}$ as small as possible, we choose the totally arbitary $\hat{x}_3$ & $\hat{x}_4$ to be zero.

The minimum length (norm) solution is

$$x^+ = \begin{bmatrix} b_1/\sigma_1 \\ b_2/\sigma_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sigma_1 & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$x^+ = A^+ b$ is shortest (where A^+ is the MP inverse) This suggest a formula for Σ^+ and x^+ for any diagonal matrix

$$\Sigma : \Sigma = \begin{pmatrix} \sigma_1 & & 0 \\ & \sigma_2 \ddots & \\ 0 & & \sigma_n \ddots \end{pmatrix}_{m \times n}, \quad \Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \frac{1}{\sigma_2} \ddots & \\ 0 & & \ddots \end{bmatrix}_{n \times m}$$

Note:- Here $(\Sigma^+)^+ = \Sigma$ which may look like $(A^{-1})^{-1} = A$ but our Σ is not invertible

Now, we find x^+ in general case:-

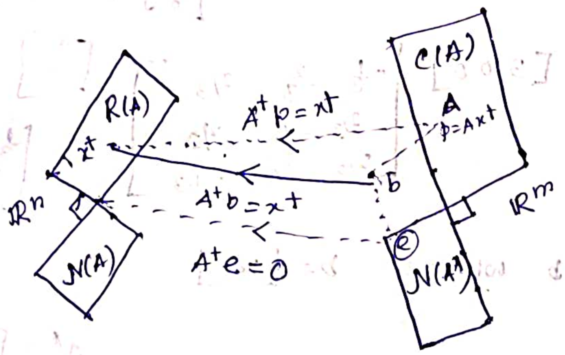
Claim:- the shortest solution x^+ is always in the row space of A.

Remember, $\hat{x} = x_r + x_n$. Following are important

1. The row space component also solves $A^TA\hat{x} = A^Tb$ (because $Ax_n = 0$)

2. The components are orthogonal, $\|\hat{x}\|^2 = \|x_r\|^2 + \|x_n\|^2$
so, $\hat{x}$ is shortest when $x_n = 0$ (i.e $\hat{x} \in R(A)$)

3. All solutions of $A^TA\hat{x} = A^Tb$ have the same x_r. This vector is x^+.



Example:-
$Ax = b$ is $-x_1 + 2x_2 + 2x_3 = 18$, Find ~~[illegible]~~ min. length solution.

Sol^n:- According to our theory, shortest solution should be in row space of $A = [-1\ 2\ 2]$. A multiple of that row which satisfies the equation (by observation)

$x^+ = (-2, 4, 4)$ [There are 'longer' sol's like $(-2,5,3), (-2,7,1), (-6,3,3)$ etc. but they all have non zero components from the null space. $(0,-1,+1)$, $(0,-3,3)$, $(4,1,1)$ respectively). The matrix that produces x^+ from $b = [18]$ is the M.P Inverse A^+ of A.

A is 1×3 so A^+ will be 3×1

$$A^+ = \begin{bmatrix}-1 & 2 & 2\end{bmatrix}^+$$

$$= \begin{bmatrix}-1/9 \\ 2/9 \\ 2/9\end{bmatrix}$$

$-1 \quad 2 \quad 2$

whence $A^+b = \begin{bmatrix}-2 \\ 4 \\ 4\end{bmatrix}$

$B \quad 1 \quad [3\ 0\ 0]$

$(B^* B)^{-1} B^* = 1.$

$e^* (ee^*)^{-1}$

$= \begin{bmatrix}3\\0\\0\end{bmatrix} \left([3\ 0\ 0]_{1\times3} \begin{bmatrix}3\\0\\0\end{bmatrix} \right)^{-1}$

$= \begin{bmatrix}3\\0\\0\end{bmatrix} 1/9$

$= \begin{bmatrix}1/3\\0\\0\end{bmatrix} \times 1$

Ex. $A = [-1\ 2\ 2] = U\Sigma V^T = [1]\ [3\ 0\ 0] \begin{bmatrix}-\frac{1}{3} & \frac{2}{3} & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3\end{bmatrix}$

Find A^+ hence find $x^+ = A^+ b$ where $b = [18]$

$A^* = \begin{bmatrix}-1\\2\\2\end{bmatrix} [-1\ 2\ 2]$

$= \begin{bmatrix}1 & -2 & -2 \\ -2 & 4 & 2 \\ -2 & 4 & 4\end{bmatrix}$

$(1,0,0)$

$A^+ = V\Sigma^+ U^T$

$$= \begin{bmatrix}-\frac{1}{3} & 2/3 & 2/3 \\ \frac{2}{3} & -\frac{1}{3} & 2/3 \\ \frac{2}{3} & 2/3 & -\frac{1}{3}\end{bmatrix} \begin{bmatrix}1/3 \\ 0 \\ 0\end{bmatrix} [1] = \begin{bmatrix}-\frac{1}{9} \\ \frac{2}{9} \\ \frac{2}{9}\end{bmatrix}$$

1. (Check!) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

Check $(AB)^+ \neq B^+ A^+$

2. If a $m \times n$ matrix Q has orthogonal columns, what is Q^+.

3. Find SVD and MP inverse of of $m \times n$ matrix O with proper justification.

Note :- If $A \in K^{m \times n}$, $B \in K^{n \times p}$ and if –

i) $A^* A = I_n$ (i.e. A has orthonormal columns)

or ii) $A A^* = I_n$ (i.e. B has orthonormal rows)

or iii) A has full col. rank & B has full row rank

or iv) $B = A^*$ then $(AB)^+ = B^+ A^+$.