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Probability + Stochastic Process
✓ ✓

Chapter 4

30] Find $E\left(\frac{1}{X+1}\right)$, where $X \sim \text{Poisson}(\lambda)$

(Done in HW)

48] Let X, Y and Z be uncorrelated random variables with variances σ_X^2, σ_Y^2 and σ_Z^2, repectively.

Let $U = Z + X$
$V = Z + Y$

Find $Cov(U, V)$ and ρ_{UV}.

Solution: $Cov(U,V) \overset{def}{=} E(UV) - E(U)E(V)$

$$\boxed{Cov(U,V) = E(U - E(U))(V - E(V))}$$

$= E((Z+X)(Z+Y)) - E(Z+X)E(Z+Y)$

$= E(Z^2 + ZY + ZX + XY)$
$- (E(Z)^2 + E(X)E(Z) + E(Y)E(Z) + E(X)E(Y))$

$$= E(Z^2) + E(ZY) + E(ZX) + E(XY) - E(Z)^2 - E(X)E(Z) - E(X)E(Y) - E(Y)E(Z)$$

$$= Var(Z) + Cov(Z,Y) + Cov(X,Z) + Cov(X,Y)$$

$$= Var(Z),$$

$\because$ X, Y, Z are uncorrelated

$$Cov(Z,Y) = Cov(X,Z) = Cov(X,Y) = 0$$

$$= \sigma_Z^2$$

$$\boxed{Cov(Z+X, Z+Y) = \sigma_Z^2}$$

$$\boxed{\rho_{UV} = \frac{Cov(U,V)}{\sqrt{Var(U)\,Var(V)}} = \frac{\sigma_Z^2}{\sqrt{(\sigma_Z^2+\sigma_X^2)(\sigma_Z^2+\sigma_Y^2)}}}$$

$$Var(U) = Var(Z+X) = Var(Z) + Var(X) = \sigma_Z^2 + \sigma_X^2 \quad (\because Cov(Z,X) = 0)$$

$$Var(V) = Var(Z+Y) = \sigma_Z^2 + \sigma_Y^2 \quad (\because Cov(Z,Y) = 0)$$

58] Let X and Y be jointly distributed random variables with correlation coefficient ρ_{XY}.

Define (strandardize) random variables.

$$\tilde{X} = \frac{X - E(X)}{\sqrt{Var(X)}}, \quad \tilde{Y} = \frac{Y - E(Y)}{\sqrt{Var(Y)}}$$

Show that $Cov(\tilde{X}, \tilde{Y}) = \rho_{XY}$

Solution:

$$Cov(\tilde{X}, \tilde{Y}) \cancel{\overset{def}{=} E((\tilde{X}\tilde{Y}) - E(\tilde{X}}$$

$$= Cov\left(\frac{X - E(X)}{\sqrt{Var(X)}}, \frac{Y - E(Y)}{\sqrt{Var(Y)}}\right)$$

$$= \frac{1}{\sqrt{Var(X)Var(Y)}} Cov\left(X - E(X), Y - E(Y)\right)$$

$$= \frac{1}{\sqrt{Var(X)Var(Y)}} \left[Cov(X, Y) - \underbrace{Cov(X, E(Y))}_{0} - \underbrace{Cov(E(X), Y)}_{0} + \underbrace{Cov(E(X), E(Y))}_{0} \right]$$

$$= \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}} = \rho_{XY}$$

Aside

$$Cov(X, a) = E((X - E(X))(a - a)) = 0$$

67] A fair coin is tossed n times and the number of heads, N, is counted. The coin is then tossed N more times. Find the expected total # of heads generated by this process.

$\boxed{n = 100}$

70 Heads

$N = 70$

Total $= 100 + 70$
$= 170$

Question: How many head

$70 + x_1$

N

$\boxed{n = 100}$

40 Heads

$N = 40$

Total $= 100 + 40$
$= 140$

Question: How many heads

$40 + X$

N

Solution: Let X denoted the # of heads in the 2nd stage of the process

So, we want to compute

$$E(N + X) \; \cancel{\left(= E(N) + E(X)\right)} \quad \cancel{N \tfrac{1}{2}}$$

For the first stage (n toss with N heads) distributed

# of heads are binomially distributed with parameters $\left(n, \frac{1}{2}\right)$

$$\boxed{E(N) = n \cdot \frac{1}{2} = \frac{n}{2} \quad \ldots (1)}$$

Now, $E(N + X) = E\left(E(N + X \mid N)\right)$ fix.

(law of total expectation)

$$= E\left(E(N \mid N) + E(X \mid N)\right)$$

$$= E\left(N + E(X \mid N)\right)$$ fixed

$$= E\left(N + N \cdot \frac{1}{2}\right)$$

$$= E\left(\frac{3N}{2}\right)$$

$$= \frac{3}{2} E(N)$$

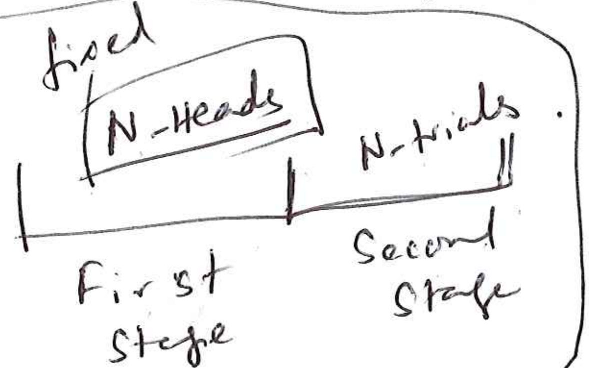
$$= \frac{3}{2}\left(\frac{n}{2}\right) \quad \text{(by (1))}$$

$$= \boxed{\frac{3}{4} n}$$

$E(N \mid N)$ (fixed)

$= E(N)$ fixed

$= N$



47] Show that
$$\mathrm{Cov}(X, Y) \le \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}.$$

Solution:

$$-1 \le \rho_{XY} \le 1.$$ (Done in class)

$$\frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} \le 1$$

$$\Rightarrow \mathrm{Cov}(X, Y) \le \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}$$

$$|\langle f, g\rangle| \le \|f\|\,\|g\|$$

Distribution Derived ~~for~~ from The Normal Distribution

* $Z \sim N(0,1)$ $\boxed{\chi^2\text{-distribution}}$

$U = Z^2$

$U =$ chi-squared distribution with 1 degree of freedom

Suppose $X \sim N(\mu, \sigma^2)$

$$\frac{(X-\mu)}{\sigma} \sim N(0,1)$$

$$\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_1$$

* If $U_1, U_2, \dots, U_n$ are independent

χ^2_1 (a chi-square distribution with 1 degree of freedom)

then $V = U_1 + U_2 + \dots + U_n$

V: chi-square distribution with n-degrees of freedom.

(χ^2_n)

Definition: If ① $Z \sim N(0,1)$

② $U \sim \chi^2_n$

③ $Z \perp U$

then the distribution of $\dfrac{Z}{\left(\sqrt{\frac{U}{n}}\right)} \sim$ $\boxed{\text{t-distribution with n-degrees of freedom.}}$

The probability density function of such t-distribution is given by:

$$f(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\ \Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{t^2}{n}\right)^{-\left(\frac{n+1}{2}\right)},$$

$$-\infty < t < \infty.$$

Definition:

1. $U \perp V$
2. $U \sim \chi^2_m$
   $V \sim \chi^2_n$

Then

$$W = \frac{\left(\frac{U}{m}\right)}{\left(\frac{V}{n}\right)} \sim$$ F-distribution, with (m, n) degrees of freedom.

Notation: $F_{m,n}$.

Sample mean and Sample Variance

Let $X_1, X_2, \cdots, X_n$, be independent $N(\mu, \sigma^2)$ random variables.

(sample from a normal distribution)

Then,

$$\boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

← called SAMPLE mean

it is a random variable.

μ: Population mean

$$\boxed{S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

← called SAMPLE variance

(σ^2: Population variance)

$$E\left(\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = \sum_{i=1}^{n} E\left(X_i^2 - 2X_i\bar{X} + \bar{X}^2\right)$$

$$= (\star)$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu$$

$$= \frac{n\mu}{n}$$

$$= \mu$$

$$X_1, X_2, \ldots \sim N(\mu, \sigma^2)$$

$$\parallel$$

$$X$$

$$E(X_i^2) = E(X^2)$$

$$i = 1, 2, \ldots, n$$

$$E(\bar{X}^2) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

$$= \frac{1}{n^2} \cdot E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right)$$

$$\boxed{\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{j=1}^{n}\sum_{i=1}^{n} a_i a_j}$$

So,

$$\sum_{i=1}^{n} E(X_i^2) = n \cdot \frac{1}{n^2} E\left(\sum_{i=1}^{n}\sum_{j=1}^{n} X_i X_j\right)$$

$\uparrow$ numbers

$$\sum_{i=1}^{n} E\left(X_i \bar{X}\right) = \sum_{i=1}^{n} E\left(X_i \frac{1}{n}\sum_{j=1}^{n} X_j\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(X_i X_j\right)$$

(¥) gives

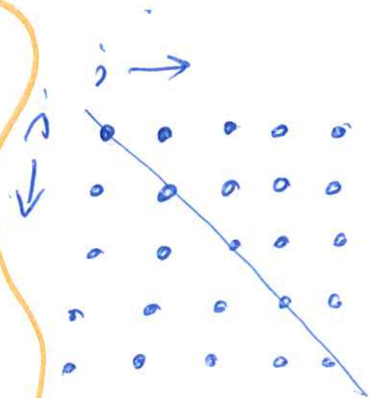
$$E\left(\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2\right)$$

$$= n \cdot E\left(X^2\right) - \frac{2}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(X_i X_j\right)$$

$$+ \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(X_i X_j\right)$$

$$= n\, E\left(X^2\right) - \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(X_i X_j\right)$$

$$= n\, E\left(X^2\right) - \frac{1}{n}\sum_{\substack{i=1 \\ i \neq j}}^{n}\sum_{j=1}^{n} E\left(X_i\right) E\left(X_j\right) - \frac{1}{n}\sum_{i=1}^{n} E\left(X_i^2\right)$$



$$= n\,E(\bar{X}^2) - \frac{1}{n}\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\mu^2 - \cancel{\frac{n}{n}}\,E(\bar{X}^2)$$

$$= n\,E(\bar{X}^2) - E(\bar{X}^2) - \frac{\mu^2}{n}(n^2-n)$$

$$= n\,E(\bar{X}^2) - E(\bar{X}^2) - \mu^2(n-1)$$

$$= (n-1)\left(E(\bar{X}^2) - \mu^2\right) = (n-1)\,\sigma^2$$

So,

$$E\left(\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = (n-1)\,\sigma^2$$

$$\boxed{E\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right) = \sigma^2}$$

$$E(*) = \sigma^2$$

(Unbiased estimator of σ^2) $\Rightarrow$ Sample variance is an UNBIASED ESTIMATOR of population variance

However, we have already seen $E(\overline{X}) = \mu$.

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ Sample mean is an unbiased estimator of population mean