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# Discrete random variables

## The Poisson Distribution

For this random variable $(X)$, the pmf is given by

$$\boxed{P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, 2, \ldots \quad (\lambda > 0)}$$

[Check: Is this really a pmf?

- $P(X = k) > 0$, $k = 0, 1, 2, \ldots$
- $\sum_{k=0}^{\infty} P(X = k) \stackrel{\text{WANT}}{=} 1$

$$\text{LHS} = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\left[\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)\right.$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1 = \text{RHS} \,/\!/ \,]$$

**Goal:** Poisson random variable

$$\boxed{X \sim \text{Poisson}(\lambda)}$$

[ Remember: $Y \sim \text{Binomial}(n, p)$ ]

→ The Poisson distribution can be derived as the limit of a binomial distribution:

$$\left.\begin{array}{l} n \to \infty \\ p \to 0 \end{array}\right\}$$

such that $np = \lambda$

Binomial | Poisson

**Proof:** Binomial pmf $(Y \sim \text{Binomial}(n,p))$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k}$$

Try to get rid of $p$:

$$p(k) = \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$p(k) = \frac{n!}{k!\,(n-k)!}\,\frac{\lambda^k}{n^k}\left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-k}$$

[So far we used $np = \lambda$]

As $n \to \infty$

$\frac{\lambda}{n} \to 0$

$\frac{n!}{(n-k)!\,n^k} \to 1$

$$\frac{n!}{(n-k)!} = \underbrace{(n)(n-1)(n-2)\cdots(n-k+1)}_{k\text{-terms}}$$

$$\frac{n!}{(n-k)!\,n^k} = \left(\frac{n}{n}\right)\frac{(n-1)}{n}\frac{(n-2)}{n}\cdots\frac{(n-k+1)}{n}$$

$$= 1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)$$

$$\longrightarrow 1$$

As $n \to \infty$,

As $\boxed{n \to \infty}$

$$\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\left(1-\frac{\lambda}{n}\right)^{-k} = 1$$

So, as $n \to \infty$ and $np = \lambda$ ($\Rightarrow p \to 0$)

$$p(k) \longrightarrow \frac{\lambda^k e^{-\lambda}}{k!} = \text{pmf of Poisson}(\lambda)$$ //

**Example:** Two dice are rolled 100 times

$\Omega = \{11, 12, 13, 14, 15, 16,$
$21, 22, 23, 24, 25, 26,$
$31, 32, 33, 34, 35, 36,$
$41, 42, 43, 44, 45, 46,$
$51, 52, 53, 54, 55, 56,$
$61, 62, 63, 64, 65, 66$

Denote $X$ : the number of double sixes.

$$p = \frac{1}{36}$$

$$n = 100$$

$$X \sim \text{Binomial}\left(100, \frac{1}{36}\right)$$

**Question:** What is the probability of getting $X = 0, 1, 2, 3, 4, 5, 6$.

| $X = k$ | Binomial | Poisson approximation |
|---|---|---|
| 0 | 0.0596 | 0.0620 |
| 1 | 0.1705 | 0.1725 |
| 2 | 0.2414 | 0.2397 |
| 3 | 0.2255 | 0.2221 |
| 4 | ~~0.0858~~ 0.1564 | 0.1544 |
| 5 | 0.0858 | 0.0858. ✓ |
| 6 | 0.0389 | 0.0398 |

$$\binom{n}{k} p^k (1-p)^{n-k}$$

$$\frac{e^{-\lambda} \lambda^k}{k!}$$ with $\lambda = np$

$$np = 100 * \frac{1}{36} = 2.78.$$

We pretend $\left.\begin{array}{l} n \text{ is big} \\ p \text{ is small.} \\ np = 2.78 = \lambda \end{array}\right\}$

$X \sim \text{Poisson}(\lambda)$

$$P(X = 0)$$

$$= \frac{e^{-2.78} \cdot (2.78)^0}{0!}$$

$$= 0.0620$$

(pmf for Poisson($\lambda$) $= \frac{e^{-\lambda} \lambda^k}{k!}$)

$$P(X = 1) = \frac{e^{-2.78} \cdot (2.78)^1}{1!} = 0.1725$$

$$P(X = 2) = \frac{e^{-2.78} (2.78)^2}{2!} = 0.2397.$$

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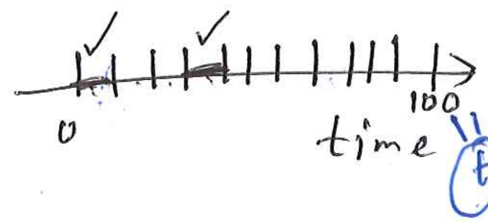
Assumptions underlying the Poisson distribution:

1. What happens in one subinterval is independent of what happens in other subintervals.

② The probability of an event is the same in each (equal) subintervals.




③ Events do not happen simultaneously.

↳ We can model the process by Poisson distribution with parameter $= (\lambda t)$

$$\text{pmf} = \frac{e^{-\lambda t} \cdot (\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots$$

Example: Analysis of telephone system

The number of calls coming into an exchange during a unit of time may be modeled as a Poisson random variable


(If the exchange services a large number of customers, who act more or less independently)

Example: Suppose an exchange receives telephone calls as a Poisson $\boxed{\text{process}}$ with $\lambda = 0.5/\text{min}$.

The number of calls in a 5-min interval follows a Poisson distribution with parameter

$$\lambda t = 5(0.5) = 2.5 .$$

Probability that NO calls in 5-min.

$$P(X=0) = \frac{e^{-2.5}(2.5)^0}{0!} = e^{-2.5} = 0.082 .$$

Prob. of exactly one call in 5-min

$$P(X=1) = \frac{e^{-2.5}(2.5)^1}{1!} = 2.5 . e^{-2.5} = 0.205$$

* To model the # of $\alpha$-particles from a radioactive source.

Examples: (Problem 2.20 ← New HW set)

If $X$ is a geometric random variable with $p = 0.5$, for what value of $k$ is

$$P(X \leq k) \approx 0.99.$$

Solution: [ Remember: $P(X=k) = p(1-p)^{k-1}$, $k = 1, 2, \cdots$ ]

$$P(X \leq \overset{\text{fixed}}{k}) = \sum_{n=1}^{k} p(1-p)^{n-1}$$

Want: $\sum_{n=1}^{k} p(1-p)^{n-1} = 0.99$.

What is $k$?

$$p \sum_{n=1}^{k} (1-p)^{n-1} = 0.99.$$

$$\Rightarrow \cancel{p} \cdot \frac{1-(1-p)^k}{\cancel{1-(1-p)}} = 0.99$$

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$$

$1-(1-p)^k = 0.99$

$(1-p)^k = 0.01$

$(0.5)^k = 0.01$

$k = 6.64$

~~6.64~~

Answer $k \approx 7$

Example (2.26 from book)

Solution:

$$n = 5 * 52 * 10 = 2600.$$

$$p = \frac{1}{10000} = 0.0001$$

"Success" = Stuck at ~~an~~ the elevator

$$\left[ \binom{2600}{0} (0.0001)^0 (0.9999)^{2600} \right].$$

$$\lambda = np = 2600 * 0.0001 = 0.26.$$

(Poisson approximation of Binomial)

$$P(X=0) = \frac{e^{-0.26} \cdot (0.26)^0}{0!} = 0.7711$$

$$P(X=1) = \frac{e^{-0.26} \cdot (0.26)^1}{1!} = 0.2005.$$

$$P(X=2) = \frac{e^{-0.26} \cdot (0.26)^2}{2!} = 0.0261$$

## Problem 2.11 (book)

Consider the binomial dbn with n-trials and prob. = p of success on each trial.

For what value of k is $P(X = k)$ maximized?

**Solution:** $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

Let

$$c = \frac{P(X = k+1)}{P(X = k)} = \frac{\binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}}{\binom{n}{k} p^k (1-p)^{n-k}}$$

$$= \frac{\frac{\cancel{n!}}{(k+1)!\,(n-k-1)!} \cdot p^{k+1} \cdot \cancel{(1-p)^{n-k-1}}}{\frac{\cancel{n!}}{k!\,(n-k)!}\,\cancel{p^k} \cdot (1-p)^{\cancel{n-k}}}.$$

$$c = \frac{(n-k)\,p}{(k+1)(1-p)}$$

(~~b~~ When $c \geq 1$)

$$1 \leq \frac{(n-k)p}{(k+1)(1-p)}$$

(then $P(X=k)$ is increasing w.r.t. $k$)

$$(k+1)(1-p) \leq (n-k)p$$

$$k - \cancel{kp} + 1 - p \leq np - \cancel{pk}$$

$$\boxed{k \leq np - (1-p)}$$

$\Downarrow$

$P(X=k+1) \geq P(X=k)$

The max will be attained at $k+1$

$$np - (1-p) + 1$$

$$= np + p$$

$$= (n+1)p$$

Round to the nearest integer (floor sense)

$$= \lfloor (n+1)p \rfloor$$

**Notation:**

$[2.89] = 2$

$\lceil 2.89 \rceil = 3$

$\lfloor 2.89 \rfloor = 2$

$$\boxed{\text{Mode} = \lfloor (n+1)p \rfloor}$$

# Problem 2.23 (book)

In a sequence of independent trials with probo = p of success, what is the prob. that there are r successes before the $k^{th}$ failure

Solution: Negative binomial (Think "success" as "failure")

$$\binom{r+k-1}{r} p^r (1-p)^k$$