1·3) For a random ~~variable~~ sample X_i ($i=1(1)m$) from an exponential distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta}, & \text{if } 0<x<\infty \\ 0, & \text{otherwise.} \end{cases}$$

where, $0<\theta<\infty$, show that $\sum_i X_i$ is a sufficient statistic for θ.

Ans/. By question, X_i follows exponential distribution with parameter θ ∋ the p.d.f. of X_i is given by

$$f_\theta(x_i) = \begin{cases} \frac{1}{\theta} \cdot e^{-x_i/\theta}, & 0 < x_i < \infty, \ \forall \ i = 1(1)n \\ 0, & \text{otherwise} \end{cases}$$

Being a random sample, X_i's are mutually independent, $\forall \ i = 1(1)n$. Hence, the joint p.d.f. of $X_1, X_2, \ldots, X_n$ is given by

$$f_\theta(x_1, x_2, \ldots, x_n) = \begin{cases} \frac{1}{\theta^n} \cdot e^{-\sum_i x_i/\theta}, & 0 < x_i < \infty \\ 0, & \text{o.w.} \end{cases}$$

$\Rightarrow f_\theta(x_1, x_2, \ldots, x_n) = g(t, \theta)\, h(x_1, x_2, \ldots, x_n)$, where

$g(t, \theta) = \frac{1}{\theta^n} e^{-\sum_i x_i/\theta}$ which is dependent on θ & t

$h(x_1, x_2, \ldots, x_n) = 1$, which is independent of θ. Also $t = \sum_{i=1}^{n} x_i$

As such, the Neyman-Fisher factorisation theorem is satisfied for the statistic $T = \sum_{i=1}^{n} X_i$ & hence, it is sufficient for θ. [Proved].

Let, X_i; $\forall\, 1 \le i \le n$ be a random sample from a distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \theta x^{\theta-1}, & \text{if } 0 < x < 1 \\ 0, & \text{o.w} \end{cases}, \quad \text{where, } 0 < \theta < \infty.$$

Find a sufficient statistic for θ.

Ans/ By question, X_i is a continuous r.v. with p.d.f.

$$f_\theta(x_i) = \begin{cases} \theta\, x_i^{\theta-1}, & \text{if } 0<x<1 \\ 0, & \text{o.w.} \end{cases}, \quad \forall\, i=1(1)n.$$

Being a random sample, X_i's, $\forall\, i=1(1)n$, are mutually independent.
Hence, the joint p.d.f. of $X_1, X_2, \dots, X_n$ is given by,

$$f_\theta(x_1, x_2, \dots, x_n) = \begin{cases} \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}, & 0<x_i<1. \\ 0, & \text{o.w} \end{cases}$$

$$\Rightarrow f_\theta(x_1, x_2, \dots, x_n) = g(t,\theta)\, h(x_1, x_2, \dots, x_n), \text{ where}$$

$g(t,\theta) = \theta^n . (\prod_{i=1}^{n} x_i)^{\theta-1}$ which is dependent on θ, & $t = \prod_{i=1}^{n} x_i$

& $h(x_1, x_2, \dots, x_n) = 1$, which is independent of θ.

$\therefore$ The statistic $T = \prod_{i=1}^{n} x_i$ satisfies the Neyman-Fisher factorisation theorem

& hence it is sufficient for θ.

1.9) Let, $X_1, X_2, \ldots, X_n$ be a random sample from a Gamma dist^n with p.d.f.

$$f_\theta(x) = \begin{cases} \dfrac{\alpha^p}{\Gamma p} e^{-\alpha x} \cdot x^{p-1}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

, where, $\alpha > 0$ & $p > 0$.

Show that, i) $\sum_i x_i$ & $\prod_i x_i$ are jointly sufficient for $\theta = (\alpha, p)$.

Ans/- The joint distribution of $X_1, X_2, \dots, X_n$ is given by,

$$f_{\underset{\sim}{\theta}}(x_1, x_2, \dots, x_n) = \frac{\alpha^{pn}}{(\Gamma p)^n} \cdot e^{-\alpha \sum_i x_i} \prod_{i=1}^{n} x_i^{p-1} \dots (*).$$

i) When α & p are both the unknown parameters of the popl^n, then from $(*)$ we have, $f_{\underset{\sim}{\theta}}(x_1, x_2, \dots, x_n) = g(t_1, t_2, \underset{\sim}{\theta})\, h(x_1, x_2, \dots, x_n)$,

where, $g(t_1, t_2, \underset{\sim}{\theta}) = \frac{\alpha^{pn}}{(\Gamma p)^n} e^{-\alpha t_1} (t_2)^{p-1}$, which is dependent on α & p,

t_1 & t_2 being $\sum_i x_i$ & $\prod_i x_i$ respectively.

$h(x_1, x_2, \dots, x_n) = 1$ which is independent of α & p.

$\therefore$ The statistic $T_1 = \sum_i X_i$ & $T_2 = \prod_i X_i$ are jointly sufficient for $\theta = (\alpha, p)$.

As such $T_1^* = \frac{1}{n} \sum_i^n X_i$ = Sample mean & $T_2^* = \left(\prod_{i=1}^n X_i\right)^{1/n}$ = Sample G.M. are also jointly sufficient for $\underset{\sim}{\theta}$ ($\bar{x}$ ...) as T_1^* & T_2^* are 1:1 transformation from (T_1, T_2).

ii) When [illegible]

1.10) Let $X_i (i=1(1)n)$ be a random sample from a beta distribution with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, & \text{if } 0 < x < 1 \\ 0, & \text{o.w.} \end{cases}$$

where $\alpha > 0$, $\beta > 0$.

i) If α, β are both unknown, get sufficient statistics for (α, β).

ii) If α is known, get a sufficient statistic for β.

iii) If β is known, get a sufficient statistic for α.

Ans. We know that, $B(\alpha,\beta) = \dfrac{\Gamma\alpha\,\Gamma\beta}{\Gamma\alpha+\beta}$.

$\therefore$ $x_1, x_2, \ldots, x_n$ being a random sample from a beta distⁿ, their joint p.d.f. is given by

$$f_\theta(x_1, x_2, \ldots, x_n) = \begin{cases} \dfrac{(\Gamma\alpha+\beta)^n}{(\Gamma\alpha)^n (\Gamma\beta)^n} \cdot \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \left\{\prod_{i=1}^{n} (1-x_i)\right\}^{\beta-1} & , 0 < x < 1, \ \alpha > 0, \beta > 0. \\ 0 & , \text{o.w} \end{cases} \quad \ldots\ldots (*)$$

i) If, α, β are both unknown then, (*) gives,

$f_\theta(x_1, x_2, \ldots, x_n) = g(t, \theta)\, h(x_1, x_2, \ldots, x_n)$ where

$g(t,\theta) = \dfrac{(\Gamma\alpha+\beta)^n}{(\Gamma\alpha)^n (\Gamma\beta)^n} \cdot t_1^{\alpha-1} t_2^{\beta-1}$, where, $t_1 = \prod_{i=1}^{n} x_i$, $t_2 = \prod_{i=1}^{n} (1-x_i)$.

& $g(t, \theta)$ is dependent on both α & β.

Also, $h(x_1, x_2, \ldots, x_n) = 1$, which is independent of α & β.

$\therefore$ The statistics $T_1 = \prod_{i=1}^{n} X_i$ & $T_2 = \prod_{i=1}^{n} (1-X_i)$ jointly satisfy the Neyman-Fisher factorisation theorem & as such they are jointly sufficient for (α, β).

ii) If α is known, & β is the only unknown parameter of the popl^n, then from (*), we have,

$f_\theta(x_1, x_2, \cdots, x_n) = g(t, \theta)\, h(x_1, x_2, \cdots, x_n)$, where,

$g(t, \overset{\beta}{\theta}) = \left(\frac{\Gamma \alpha+\beta}{\Gamma \beta}\right)^n t^{\beta-1}$, $t = \prod_{i=1}^{n}(1-x_i)$, $g(t, \overset{\beta}{\theta})$ being ~~depe~~ dependent on β.

& $h(x_1, \cdots, x_n) = \frac{1}{(\Gamma \alpha)^n}\left(\prod_{i=1}^{n} x_i\right)^{\alpha-1}$, which is independent of β.

Hence, the statistic $T = \prod_{i=1}^{n}(1-x_i)$ satisfies the Neyman-Fisher factorisation theorem & as such it is sufficient for β.

(ii) If β is known & 'α' is the only unknown parameter of the pop^n, then, from (*), we have,

$$f_{\alpha}(x_1, x_2, \dots, x_n) = g(t, \alpha)\, h(x_1, x_2, \dots, x_n), \text{ where,}$$

$g(t, \alpha) = \left(\frac{\overline{|\alpha+\beta}}{\overline{|\alpha}}\right)^n t^{\alpha-1}$, with $t = \prod_i (x_i)$, is dependent on α ;

$h(x_1, x_2, \dots, x_n) = \frac{1}{(\overline{|\beta})^n} \cdot \left\{\prod_{i=1}^{n} (1-x_i)\right\}^{\beta-1}$ is independent of α.

$\therefore$ The statistic $T = \prod_{i=1}^{n} X_i$ satisfies the Neyman-Fisher factorisation theorem & as such it is sufficient for α.

1.8) Show that for a random sample of size 'n' from a dist^n with p.d.f.

$$f_\theta(x) = \begin{cases} \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2}, & \theta_1 < x < \infty \\ 0, & \text{o.w.}, \end{cases}$$

where, $\theta = (\theta_1, \theta_2)$ & $-\infty < \theta_1 < \infty$, $0 < \theta_2 < \infty$, the statistics $X_{(1)}$ & $\sum_i X_i$ are jointly sufficient for θ_1 & θ_2.

A/ ~~The~~ Let, $X_1, X_2, \ldots, X_n$ be a random sample of size 'n' from the dist^n & $x_{(1)}, \ldots, x_{(n)}$ be the corresponding order statistic.
Then, the joint p.d.f. of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is given by

$$f_\theta(x_1, x_2, \ldots, x_n) = \frac{n!}{\theta_2^n} \cdot e^{-\sum_{i=1}^{n}(x_{(i)} - \theta_1)/\theta_2}, \quad \theta_1 < x_{(1)} < \cdots < x_{(n)} < \infty$$

$$= \frac{n!}{\theta_2^n} \cdot e^{-\frac{1}{\theta_2}\left[\sum_i x_{(i)} - n\theta_1\right]}, \quad \theta_1 < x < \infty \quad \text{---}(*)$$

where, $\phi(a, b) = \begin{cases} 1, & \text{if } b \geq a \\ 0, & \text{o.w} \end{cases}$ $\quad \phi(x_{(1)}, \theta)\cancel{\phi(x_{(n)}, \infty)} \ldots (*)$,

$\therefore$ (*) can be written as

$$f_\theta(x_1, x_2, \ldots, x_n) = g(T_1, T_2, \theta = (\theta_1, \theta_2))\, h(x_1, x_2, \ldots, x_n),$$

where, $g(T_1, T_2, \theta) = \frac{1}{\theta_2^n} \cdot \left(e^{-\sum_i x_{(i)}/\theta_2 + \frac{n\theta_1}{\theta_2}}\right) \phi(x_{(1)}, \theta)$
$= \frac{1}{\theta_2^n}\left(e^{-T_1/\theta_2 + \frac{n\theta_1}{\theta_2}}\right)\phi(T_2, \theta)$, $T_1 = \sum_i x_i$.

~~$A = \{(x_1, x_2, \ldots, x_n) : \theta_1 < \min x_i < \infty\}$.~~

& $h(x_1, x_2, \ldots, x_n) = \cancel{\phi(x_{(n)}, \infty)}\ n!$ $\quad T_1 = \sum_i x_{(i)}$ & $T_2 = x_{(1)}$.

~~Here, since, $\theta_1 < x < \infty$, so the minimum value of x is ~~less~~ greater than θ_1~~

~~the minimum value of x, ie $x_{(1)}$ characterises θ_1.~~

$\therefore$ ~~The minimum value of~~

$\therefore$ Neyman-Fisher Factorisation theorem is satisfied by $T_1 = \sum_i X_{(i)} = \sum_i X_i$ & $T_2 = X_{(1)}$

& hence, they are jointly sufficient for θ_1 & θ_2. [Proved].