Assignment-1

1. Answer the followings :

   i) By which property of a vector space V you will explain that $0 + 0 = 0$ where 0 is the zero vector of V.

   ii) Prove that :
   a) $0.x = 0$, where $x, 0 \in V$ and 0 is the zero vector.
   b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.

2. What is the Parallelogram Law of Vector Addition? Consider the vector space $\mathbb{R}^2$, add the vectors $(3, 1)$ and $(2, 1)$ in $\mathbb{R}^2$ by this law and explain it by a picture.

3. Prove that the set of all $m \times n$ matrices over a field $\mathbb{F}$ denoted by $M_{m\times n}(\mathbb{F})$ is a vector space under usual matrix addition and matrix multiplication.

4. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and c is an element of F. Define, $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2b_2)$ and $c(a_1, a_2) = (ca_1, ca_2)$. Is V is a vector space under this operations? Justify your answer.

5. Consider the set of all polynomials with coefficients from a field $\mathbb{F}$ with usual polynomial addition and multiplication denoted by $P(\mathbb{F})$. Is $P(\mathbb{F})$ a vector space over $\mathbb{F}$? Justify your answer.

6. Define the Subspace of a Vector Space. Consider $\mathbb{R}^2$, prove that $W = \{(a, a) : a \in \mathbb{R}^2\}$ is a subspace of $\mathbb{R}^2$. Draw the subsapce in $\mathbb{R}^2$. What are the basis and dimension of W?

7. Which of the following sets are the subspaces of $\mathbb{R}^3$?
   (a) $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) : \lambda, \mu \in \mathbb{R}^2\}$
   (b) $B = \{(\lambda^2, -\lambda^3, 0) : \lambda \in \mathbb{R}\}$
   (c) Let $\gamma \neq 0$ be in $\mathbb{R}$. $C = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_1 - 2\xi_2 + 3\xi_3 = \gamma\}$
   (d) $D = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 : \xi_2 \in \mathbb{Z}\}$

8. Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

9. The set of all differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ where $\mathbb{R}^{(0,3)}$ is the set of real-valued functions on the interval (0, 3) if and only if b = 0.

10. Let V be a vector space and $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

11. Answer the following:
    (a) What are the standard bases and dimensions for the following vector spaces: $\{0\}$, $\mathbb{R}^n$, $P_n(\mathbb{R})$, $M_{m\times n}(\mathbb{R})$, $P(\mathbb{R})$, Space of all $n \times n$ symmetric matrices.
    (b) How do you relate the bases of $M_{m\times n}(\mathbb{R})$ and $\mathbb{R}^k$ for any k? Logically or Computationally find a non-standard basis of the subspace of all the upper triangular matrices W of $M_{m\times n}(\mathbb{R})$. Also find the dimension of W.

12. Find the bases and dimensions of the following subspaces of $\mathbb{R}^n$:
    (a) $W_1 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : a_1 + a_2 - a_3 = 0\}$
    (b) $W_2 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : a_{11} + 2a_{21} = 0, a_{27} = a_4 = a_9\}$
    (c) $W_3 = \{(a_1, a_2, ..., a_n) \in \mathbb{R}^n : m$ constraints are given, $m \leq n\}$. What happens if we assume $m \geq n$?

13. Let $S = \{x_1, x_2, ..., x_n\}$ be a linearly independent subset of a vector space V over the field $\mathbb{Z}_2$. How many elements are there in $span(S)$? Justify your answer.

14. If V and W are vector spaces over $\mathbb{F}$ for which $|V| = |W|$ then does it mean that $dim(V) = dim(W)$?

15. Choose $x = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$. It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2). Those 24 vectors, including x itself, span a subspace S. Find specific vectors x so that the dimension of S is: (a) zero, (b) one, (c) three, (d) four.

16. Let V be a real vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less. Let t be a fixed real number and define $g_1(x) = 1$, $g_2(x) = x + t$, $g_3(x) = (x + t)^2$. Prove that $B = \{g_1, g_2, g_3\}$ is a basis for V. If $f(x) = c_0 + c_1x + c_2x^2$ then what are the coordinates of f in this ordered basis B?

17. Let $v = (x_1, x_2)$ and $w = (y_1, y_2)$ be vectors in $\mathbb{R}^2$ such that $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$ and $x_1y_1 + x_2y_2 = 0$. Prove that $B = \{v, w\}$ is a basis for $\mathbb{R}^2$. Find the coordinates of the vector (a, b) in the basis B . Can you interpret the conditions geometrically?

18. Let V be the vector space over the complex numbers of all functions from $\mathbb{R}$ into $\mathbb{C}$, i.e., the space of all complex-valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$, $f_3(x) = e^{-ix}$.
    (a) Prove that f_1, f_2, and f_3 are linearly independent.
    (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$, $g_3(x) = \sin x$. Find an invertible 3×3 matrix P such that $g_i = \sum_{i=1}^{3} P_{ij} f_i$.

19. Let W_1 and W_2 be two finite dimensional subspaces of a vector space V.Then prove the following:
    (a) $W_1 + W_2$ is the smallest subspace of V that contains both W_1 and W_2.
    (b) $dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$.
    (c) If $V = W_1 + W_2$ then $V = W_1 \oplus W_2$ if and only if $dim(V) = dim(W_1) + dim(W_2)$.

20. Suppose V is a vector space over $\mathbb{R}$. Prove that V cannot be written as the set-theoretic union of a finite number of proper subspaces.