12/1

Last time:

Theorem: Let $(X_n)_{n \geq 0}$ be Markov $(\underset{(1 \times N)}{\lambda}, \underset{(N \times N)}{P})$.

Then, for all $n, m \geq 0$.

✓ ① $P(X_n = j) = (\lambda P^n)_j$

✓ ② $P_i(X_n = j) = P(X_{n+m} = j \mid X_m = i)$

$= p_{ij}^{(n)} \leftarrow$ ((i, j) entry of P^n)

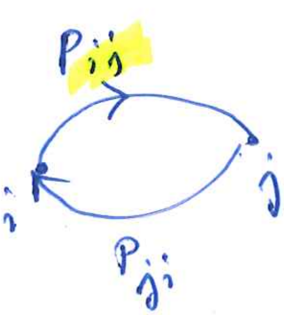
State i, j

X_0	Prob.
i_0	λ_{i_0}
i_1	λ_{i_1}
i_2	λ_{i_2}
$\vdots$	$\vdots$

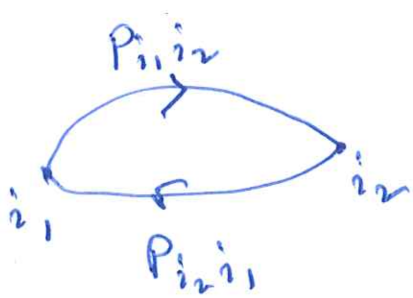
Initial distribution.

$\lambda = (\lambda_{i_0}, \lambda_{i_1}, \ldots, \ldots)$

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & & & \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{pmatrix}$$

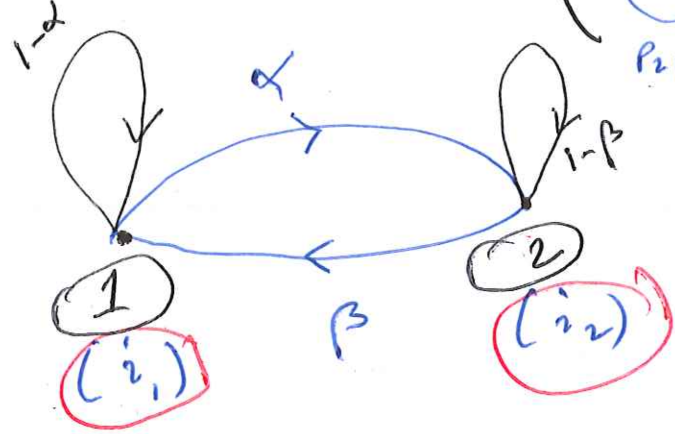


$P_{i_1 i_2}$



Example: The most general two-state Markov chain has transition matrix of the form

$$P = \begin{array}{cc} p_{11} & p_{12} \\ \left(\begin{matrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{matrix}\right) & \\ p_{21} & p_{22} \end{array}$$

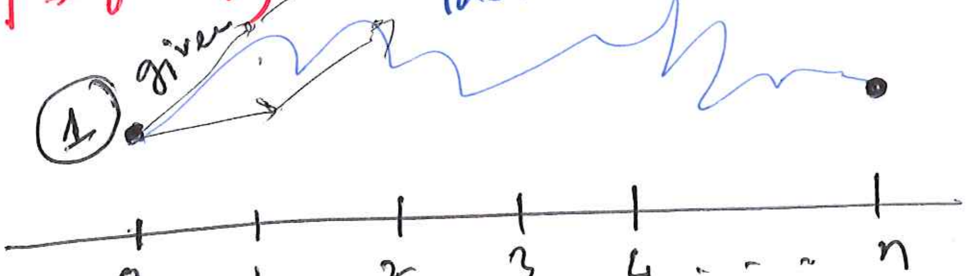


Question: What is the probability that this Markov chain $i_1 \longrightarrow i_1$ after n-steps.

$$P_{i_1}(X_n = i_1) = \boxed{p^{(n)}_{11}} \quad \text{(by the last Theorem)}$$

$$P(X_n = i_1 \mid X_0 = i_1)$$

Two entries of state
$i_1 \longrightarrow 1$
$i_2 \longrightarrow 2$



So, the goal is to compute

$$P_{11}^{(n)} \leftarrow P^n \text{ (2x2 matrix)}$$

$\downarrow$

(1, 1) entry.

To do so:

$$P^{n+1} = P^n \cdot P$$

$$\begin{pmatrix} P_{11}^{(n+1)} & P_{12}^{(n+1)} \\ P_{21}^{(n+1)} & P_{22}^{(n+1)} \end{pmatrix} = \begin{pmatrix} P_{11}^{(n)} & P_{12}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} \end{pmatrix} \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Notation Notation Given

$$P_{11}^{(n+1)} = (1-\alpha) P_{11}^{(n)} + \beta P_{12}^{(n)} \quad \ldots (1)$$

Also, $P_{11}^{(n)} + P_{12}^{(n)} = 1 \quad \ldots (2)$

Solving (1) and (2) to eliminate $P_{12}^{(n)}$ we get

$$\boxed{\begin{aligned} P_{11}^{(n+1)} &= (1-\alpha-\beta) P_{11}^{(n)} + \beta \\ P_{11}^{(0)} &= 1 \end{aligned}} \quad \ldots (*)$$

This is a recurrance relation on $P_{11}^{(n)}$, $n = 0, 1, 2, \ldots$.

Goal: $P_{11}^{(n)} = \ldots$.

Aside: (Solution of recurrance relation)

If, $x_{n+1} = a x_n + b$

Then,

$$\boxed{\begin{aligned} x_n &= A a^n + \frac{b}{1-a}, \quad a \neq 1 \\ &\text{(for a some constant } A) \\ \text{and} \quad x_n &= x_0 + nb, \quad a = 1. \end{aligned}}$$

In (*): $\underbrace{P_{11}^{(n+1)}}_{x_{n+1}} = \underbrace{(1-\alpha-\beta)}_{a} \underbrace{P_{11}^{(n)}}_{x_n} + \underbrace{\beta}_{b}$

Case 1: $1 - \alpha - \beta \neq 1$
i.e., $\alpha + \beta \neq 0$

$$P_{11}^{(n)} = A(1-\alpha-\beta)^n + \frac{\beta}{\alpha+\beta} \quad \ldots (3)$$

We know

$$P_{11}^{(0)} = 1 \quad \ldots (4)$$

Put $n=0$, in (3) : (use (4))

$$1 = A + \frac{\beta}{\alpha+\beta}$$

$$\Rightarrow \quad A = 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}.$$

So, (3) gives.

(5) $$\boxed{P_{11}^{(n)} = \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n + \frac{\beta}{\alpha+\beta}, \quad \text{when } \alpha+\beta \neq 0}$$

Case 2: $1-\alpha-\beta = 1$

$\Rightarrow \alpha+\beta = 0$

$\alpha = \beta = 0 \Longrightarrow P_{11}^{(n)} = P_{11}^{(0)} + n\beta$

$\beta = 0$

$[\, P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,]$

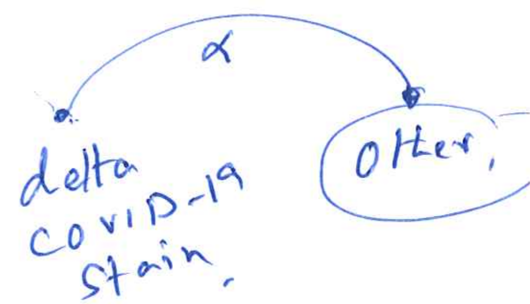
$P_{11}^{(n)} = P_{11}^{(0)} + 0$

$= 1$

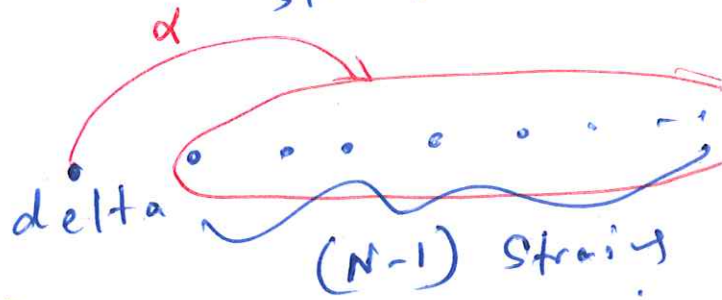
So, $\boxed{P_{11}^{(n)} = 1 \quad \text{if } \alpha = \beta = 0}$

Example: (Virus mutation)

- Suppose a virus can exist in N different strains.
- In each generation a virus may either stays the same, or with probability α mutates to another strain (which is chosen at random)

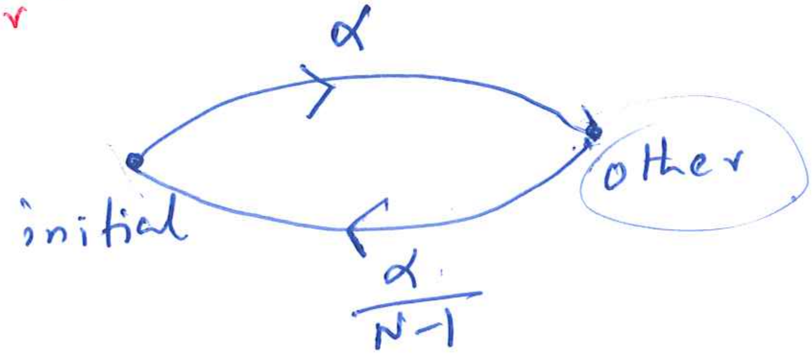


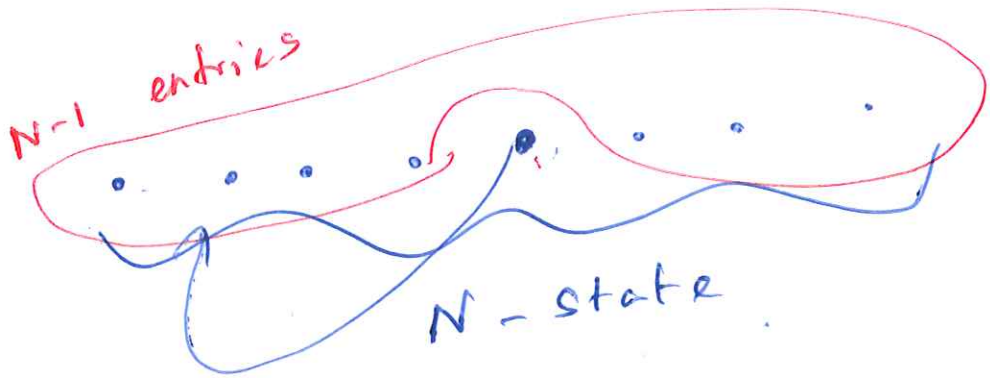
Question: What is the probability that the strain in the n^{th} generation is the same as that in the 0^{th} generation?



Solution: We can model this process as an N-state ~~chan~~ Markov chain. ~~Otherwise~~ If the 0th generation.

State is denoted as "initial" and any other state is denoted as "other"





α

Referring back to the last problem, here $\beta = \frac{\alpha}{N-1}$

$(\alpha, \beta \neq 0)$

(By the last problem)

$$\boxed{P_{11}^{(n)} = \frac{\alpha}{\alpha+\beta}\left(1-\alpha-\beta\right)^n + \frac{\beta}{\alpha+\beta}}$$

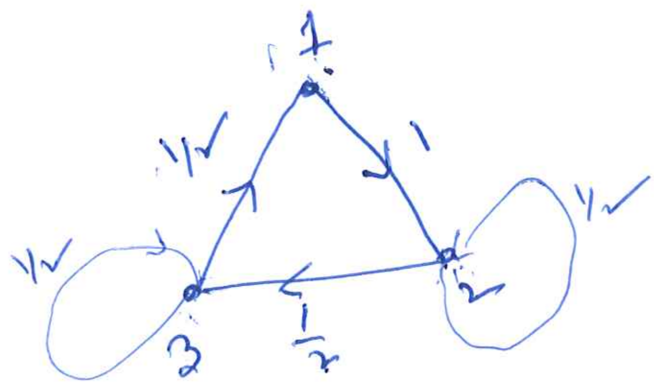
by [illegible]

$$= \frac{\alpha}{\alpha + \frac{\alpha}{N-1}}\left(1 - \alpha - \frac{\alpha}{N-1}\right)^n + \frac{\frac{\alpha}{N-1}}{\alpha + \frac{\alpha}{N-1}}$$

$$\boxed{p_{11}^{(n)} = \left(\frac{N-1}{N}\right)\left(1-\frac{\alpha N}{N-1}\right)^{n} + \frac{1}{N}}$$

[State 1 (or i_1) : intial
State 2 (or i_2) : other]

Example: Consider the state-diagram (Markov-diagram)



with transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Question: Find $P_{11}^{(n)}$

Solution:

STEP 1: Find the eigenvalues of P.

$$\begin{vmatrix} 0-\lambda & 1 & 0 \\ 0 & \frac{1}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{4}(1-\lambda)(4\lambda^2+1) = 0$$

$$\Rightarrow \lambda = 1, \pm\frac{i}{2} \quad (i=\sqrt{-1})$$

$\Rightarrow$ The matrix is diagonalizable

STEP 2: So, there is an invertible matrix U such that

$$P = \underline{U} \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} \underline{U^{-1}}$$

So, in our case

$$P^n = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{i}{2}\right)^n & 0 \\ 0 & 0 & \left(-\frac{i}{2}\right)^n \end{pmatrix} U^{-1}.$$

$$P = U \cdot D U^{-1}$$

$$P^2 = U D^2 U^{-1}$$

$$P^3 = U D^3 U^{-1}$$

$$\vdots$$

$$P^n = U D^n U^{-1}$$

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$$

$$D^n = \begin{pmatrix} d_1^n & 0 & 0 \\ 0 & d_2^n & 0 \\ 0 & 0 & d_3^n \end{pmatrix}$$

STEP 3:

$$P^n = \begin{pmatrix} p_{11}^{(n)} & p_{12}^{(n)} & p_{13}^{(n)} \\ p_{21}^{(n)} & p_{22}^{(n)} & p_{23}^{(n)} \\ p_{31}^{(n)} & p_{32}^{(n)} & p_{33}^{(n)} \end{pmatrix}$$

We want this!

$$\boxed{p_{11}^{(n)} = a \cdot 1 + b \left(\frac{i}{2}\right)^n + c \cdot \left(-\frac{i}{2}\right)^n.}$$

for some constants a, b, c.

What are a, b, c? $\longrightarrow$ Next Class

$$\cancel{\frac{i}{2} = \frac{1}{2}(i) = \frac{1}{2} e^{i \frac{n\pi}{2}} = \frac{1}{2}\left(\cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2}\right)}$$