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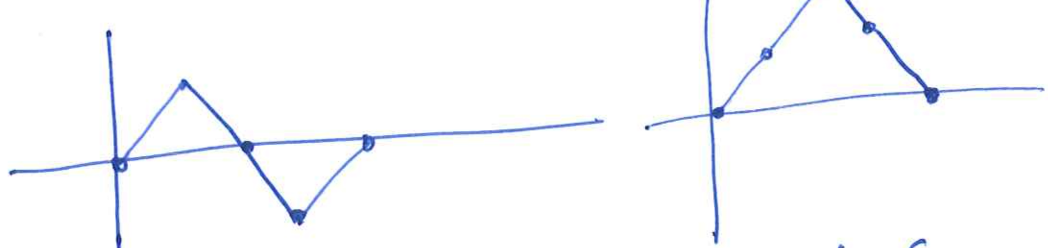
Last time:

Return to zero

$t: \quad S_t = 0$

$$\boxed{\boxed{u_{2m} = P(S_{2m}=0)} \text{ is given by: } \boxed{u_{2m} = \binom{2m}{m}\cdot\frac{1}{2^{2m}}, \quad m \geq 0.}}$$

$m=2$



$$u_4 = P(S_4=0) = \binom{4}{2}\cdot\frac{1}{2^4} = \frac{1\cdot 6}{16} = \boxed{\frac{3}{8}}$$

Main Theorem:

$$u_{2m} \stackrel{\text{def}}{=} \boxed{P(S_{2m}=0)}$$

$$\boxed{= P(S_1 \neq 0, S_2 \neq 0, \dots, S_{2m} \neq 0)}$$

$$= P(S_1 \geq 0, S_2 \geq 0, \dots, S_{2m} \geq 0)$$

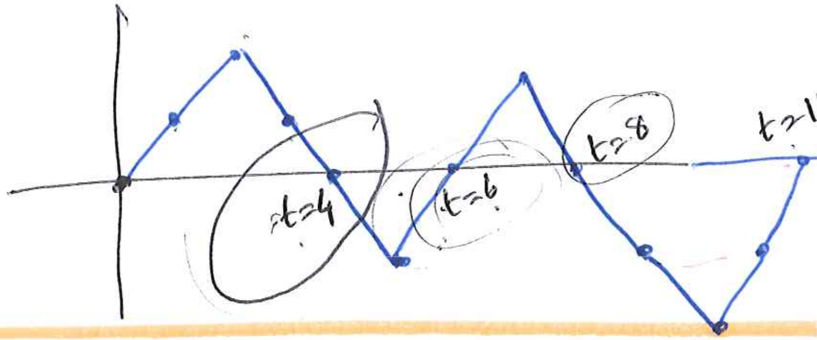
$$= P(S_1 \leq 0, S_2 \leq 0, \dots, S_{2m} \leq 0)$$

$$= 2P(S_1 > 0, S_2 > 0, \dots, S_{2m} > 0)$$

$$= 2P(S_1 < 0, S_2 < 0, \dots, S_{2m} < 0)$$

First Return to zero

after $t=0$



Definition: The first return to zero happens at epoch $t=2M$, if $S_1 \neq 0, \dots, S_{2M-1} \neq 0$.

and $S_t = S_{2M} = 0$.

[Think $t=4$

$S_1 \neq 0, \; S_2 \neq 0, \; S_3 \neq 0$]

$$f_{2M} \stackrel{\text{def}}{=} P\left(S_{2M}=0, \; S_{2M-1} \neq 0, \; S_{2M-2} \neq 0, \dots, S_1 \neq 0\right)$$

Theorem: $f_{2M} = u_{2M-2} \quad u_{2M} \stackrel{\checkmark}{=} \dfrac{1}{2M-1} u_{2M}$

$$\stackrel{\checkmark}{=} \frac{1}{2M-1} \binom{2M}{M} \cdot \frac{1}{2^{2M}}$$

Note: $u_{2M} = \binom{2M}{M} \cdot \frac{1}{2^{2M}}$

$(M = 1, 2, \dots)$

Proof:

$$f_{2m} \overset{def}{=} P(S_{2m}=0,\ S_{2m-1}\neq 0,\ S_{2m-2}\neq 0,\ \dots,\ S_1\neq 0)$$

$$(S_{2m}=0,\ S_{2m-1}\neq 0,\ S_{2m-2}\neq 0,\ \dots,\ S_1\neq 0) = \underbrace{(S_{2m-1}\neq 0,\ S_{2m-2}\neq 0,\ \dots,\ S_1\neq 0)}_{A} \setminus \underbrace{(S_{2m}\neq 0,\ S_{2m-1}\neq 0,\ \dots,\ S_1\neq 0)}_{B}$$

Red

$S_{2m}=0$

$S_{2m}\neq 0$

Green

Since,

$$B \subset A$$

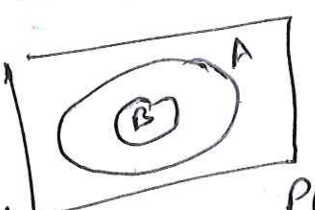
$$P(A\setminus B) = P(A) - P(B)$$

So, (*)

$$P(S_{2m}=0,\ S_{2m-1}\neq 0,\ \dots,\ S_1\neq 0) = P(A) - P(B)$$

anyway non-zero

Aside: $B \subset A$



$$P((A\setminus B)\cup B) = P(A)$$

$$P(A\setminus B) + P(B) = P(A)$$

$$P(A\setminus B) = P(A) - P(B)$$

$$A = (S_{2m-1}\neq 0,\ S_{2m-2}\neq 0,\ \dots,\ S_1\neq 0)$$

$$= (S_{2m-2}\neq 0,\ S_{2m-3}\neq 0,\ \dots,\ S_1\neq 0)$$

$$P(A) = P(S_{2m-2}\neq 0,\ S_{2m-3}\neq 0,\ \dots,\ S_1\neq 0)$$

$$= u_{2m-2}$$

by main Theorem

$$P(B) = P(S_{2m} \neq 0, S_{2m-1} \neq 0, \ldots, S_1 \neq 0)$$

by main Theorem

$$= u_{2m}$$

So, by (*) (last page)

$$f_{2m} = P(S_{2m} = 0, S_{2m-1} \neq 0, S_{2m-2} \neq 0, \ldots, S_1 \neq 0)$$

$$= u_{2m-2} - u_{2m}$$

(Done one part!)

$$f_{2m} = u_{2m-2} - u_{2m}$$

$$u_{2m} = \binom{2m}{m} \cdot \frac{1}{2^{2m}}$$

$$= \binom{2m-2}{m-1} \cdot \frac{1}{2^{2m-2}} - \binom{2m}{m} \frac{1}{2^{2m}}$$

$$= \frac{(2m-2)!}{(m-1)!\,(m-1)!} \cdot \frac{1}{2^{2m-2}} - \frac{(2m)!}{m!\,m!} \cdot \frac{1}{2^{2m}}$$

$$= \frac{4m^{\cancel{2}}}{\cancel{2m}(2m-1)} \frac{(2m)!}{m!\,m!} \frac{1}{2^{2m}} - \frac{(2m)!}{m!\,m!} \cdot \frac{1}{2^{2m}}$$

$$= \left(\frac{2m}{2m-1} - 1\right) \binom{2m}{m} \cdot \frac{1}{2^{2m}} = \frac{1}{2m-1} \cdot \boxed{u_{2m}}$$

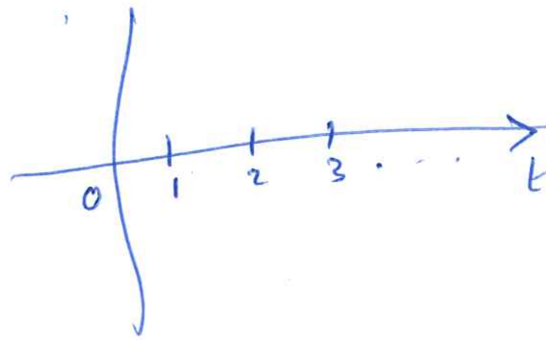
(Done)

Corollary: With $\boxed{\text{probability } 1,}$ the simple random walk returns to 0.

(1+1=2 dimensional)

[This is also true for $\boxed{1+2 \text{ dimension}}$ (Szegö)

But NOT true for 1+3 dimensions or any other dimension.]



Proof: P(Simple random walk $\boxed{\text{returns}}$ to 0)

$$= f_2 + f_4 + f_6 + \cdots\cdots$$

$$= \sum_{m=1}^{\infty} f_{2m}$$

by the last theorem

$$= \sum_{m=1}^{\infty} \left(u_{2m-2} - u_{2m} \right)$$

$$= u_0 = 1$$ //

$$\begin{array}{l} u_0 - \cancel{u_2} \\ \cancel{u_2} - \cancel{u_4} \\ \cancel{u_4} - \cancel{u_6} \\ \cancel{u_6} - \cancel{u_8} \\ \vdots \\ \hline u_0 \end{array}$$

Theorem: Let W denote the epoch of the first return to zero.
$(W = 2, 4, 6, \ldots)$

Then $E(W) = \infty$.

W	Pr.
2	f_2
4	f_4
6	f_6
8	f_8
⋮	⋮

Proof:

$$E(W) = \sum_{M=1}^{\infty} (2M) \cdot f_{2M}$$

$$= \sum_{M=1}^{\infty} 2M \cdot \frac{1}{2M-1} u_{2M}$$

$$= \sum_{M=1}^{\infty} \frac{2M}{2M-1} \cdot u_{2M} \quad \cdots (1)$$

We wrote before:

$$u_{2M} \sim \frac{1}{\sqrt{\pi M}}$$

$$\Rightarrow \frac{u_{2M}}{\frac{1}{\sqrt{\pi M}}} \to 1 \text{ as } M \to \infty.$$

$$\frac{\frac{2M}{2M-1} u_{2M}}{\frac{1}{\sqrt{\pi M}}} \xrightarrow[M\to\infty]{} 1$$

$$\sum_{m=1}^{\infty} \frac{1}{\sqrt{\pi m}} \geq \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \frac{1}{m} \to \infty$$

Harmonic Series.

When $m \geq 1$

$m \geq \sqrt{m}$

$\frac{1}{m} \leq \frac{1}{\sqrt{m}}$

$\frac{1}{\sqrt{\pi}\, m} \leq \frac{1}{\sqrt{\pi m}}$

$\Rightarrow \sum_{m=1}^{\infty} \frac{1}{\sqrt{\pi m}}$ is also divergent.

So, by Limit comparison test ~~...~~.

$\sum_{m=1}^{\infty} \frac{2m}{2m-1} u_{2m}$ is also $\to \infty$.

Hence, by (1): $E(N) = \infty$. //

Corollary: [With probability 1, the simple random walk returns to zero]

Already proved!

$\Downarrow$

$\boxed{\text{Consequently}}$, with probability 1, it returns to zero infinitely often !!

Definition: (Recurrence)

The value k is recurrent if

$$P(S_t = k, \text{ infinitely often}) = 1$$

[We have just seen,

$\boxed{0 \text{ is a recurrent value of simple random walk}}$]

Corollary: For every integer k, with probability 1, the random walk visits k. $\boxed{\text{Consequently,}}$ ~~with~~ each integer k is a recurrent value of simple random walk.

Reminder:

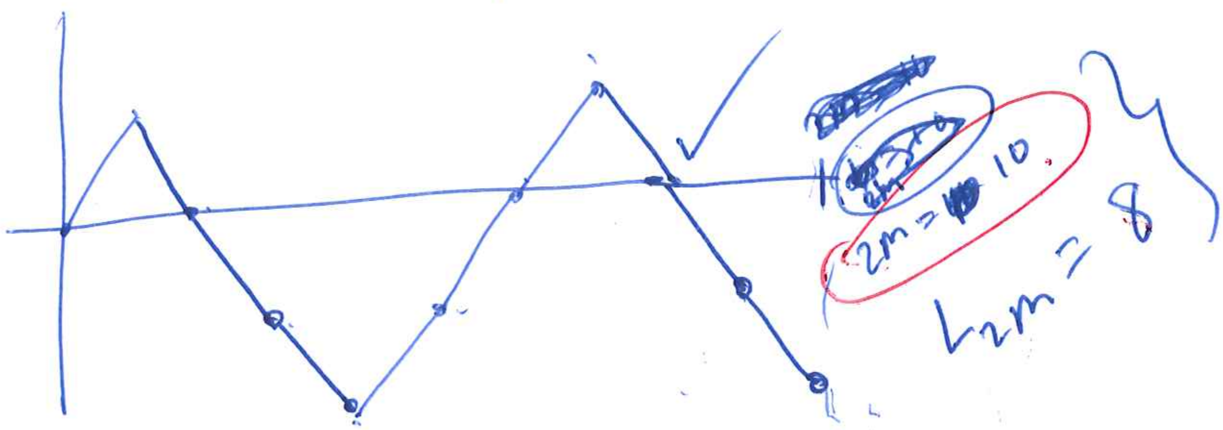
$$u_{2m} = P(S_{2m} = 0)$$

$$f_{2m} = P(S_{2m} = 0, S_{2m-1} \neq 0, S_{2m-2} \neq 0, \ldots, S_1 \neq 0)$$

Definition:

For each M, define a random variable,

L_{2M} = the <u>epoch</u> of the last <u>visit to zero,</u> up to and including epoch $2M$.



$$L_{2m} = \max \left\{ t : \begin{array}{c} 0 \le t \le 2M \\ \text{and} \quad S_t = 0 \end{array} \right\}$$

The Arc-Sine Law for last return:

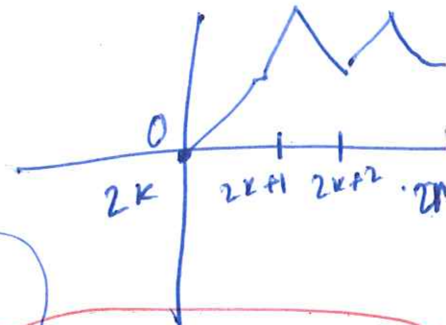
The probability mass function for L_{2m} is given by

$$\boxed{P(L_{2m} = 2K) = u_{2K} \cdot u_{2(M-K)} ,}$$

$K = 0, \dots, M.$

Proof: The event $L_{2m} = 2k$ can be written as:

$$\underbrace{S_{2k} = 0}_{A},\ \underbrace{S_{2k+1} \neq 0,\ S_{2k+2} \neq 0,\ S_{2k+3} \neq 0, \dots, S_{2m} \neq 0}_{B}$$



We want $P(L_{2M} = 2k)$

$$= P(A \cap B)$$

$$= P(A) \cdot P(B \mid A)$$

$$= P(S_{2k} = 0)\, P\left(\begin{array}{l}\text{think the starting}\\ \text{point is } 2k\\ \text{and the rest}\\ \text{are non-zero}\end{array}\right)$$

by defn   main theorem

$$= u_{2k} \cdot u_{2(m-k)}$$

$L_{2m+1} = L_{2m}$

Hence done!

Note: $L_{2m+1} = L_{2m}$