Problem Sheet

1. Prove that any two finite dimensional vector space are isomorphic if and only if they have equal dimension.

2. Give an example of distinct linear transformations T and U such that $N(T) = N(U)$ and $R(T) = R(U)$. Where $R(T)$ and $N(T)$ are Range of T and Null space of T respectively.

3. Suppose $U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}$. Find a subspace W of $\mathbb{F}^4$ such that $U \bigoplus V = \mathbb{F}^4$

4. Let W_1, W_2, W_3 be three distinct subspaces of $\mathbb{R}^{10}$ with each W_i has dimension 9. If $W = W_1 \bigcap W_2 \bigcap W_3$ then prove that $7 \leq dim(W) \leq 8$.

5. Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be a linear transformation such that $T(A) = 0$ whenever A is symmetric or skew symmetric. Find $Rank(T)$.

6. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be linear.
   (a) Prove that if $dim(V) < dim(W)$, then T cannot be onto.
   (b) Prove that if $dim(V) > dim(W)$, then T cannot be one-to-one.

7. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}^3$ be two linear transformations such that $S \circ T = I$ on $\mathbb{R}^3$. Then show that $T \circ S$ is neither one-one nor onto.

8. T and S be two linear operators on $\mathbb{R}^n$ such that $ST = TS = 0$ and $T + S$ is invertible. Then show that:
   (a) $Rank(T) + Rank(S) = n$
   (b) $Nullity(T) + Nullity(S) = n$
   [Hint : Use $Rank(T + S) \leq Rank(T) + Rank(S) \leq Rank(TS) + n$]

9. A linear transformation T rotates each vector in $\mathbb{R}^2$ clockwise through an angle θ. Find the matrix of T w.r.t the standard ordered basis.

10. Let $T : \mathbb{R}^4 \to \mathbb{R}^5$ such that $Tx = 0$ iff $x = 0$. Find $Rank(T)$.

11. Let V be a vector space and $T : V \to V$ be a linear transformation. Prove that the following two statements about T are equivalent:
    (a) The intersection of the range of T and the null space of T is the zero subspace of V.
    (b) If $T(Ta) = 0$, then $Ta = 0$.

12. Let $T : V \to W$ be a linear transformation. Then prove that T is non-singular if and only if T carries each linearly independent subset of V onto a linearly independent subset of w.

13. Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Is T invertible? If so, find a rule for T^{-1} like the one which defines T.

14. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = O$ but $UT \neq O$.
15. If W is a k-dimensional subspace of an n-dimensional vector space V, then prove that W is the intersection of n - k hyperspaces in V.
16. Prove the followings:
    (a) The subspaces $\{0\}$, V, R(T), and N(T) are all T-invariant.
    (b) If W is T-invariant, prove that T_W is linear.
17. Let V be a finite-dimensional vector space and $T : V \to V$ be linear. Then if $V = R(T) + N(T)$ prove that $V = R(T) \bigoplus N(T)$.
18. A function $T : V \to W$ between vector spaces V and W is called additive if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.
19. Let V be a vector space and W be a subspace of V. Define the mapping $\eta : V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.
    (a) Prove that η is a linear. Find $N(\eta)$.
    (b) Suppose that V is finite-dimensional. Then find the relation between $dim(V)$, $dim(W)$, and $dim(V/W)$.
20. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $R(T) \cap R(U) = \{0\}$, prove that $\{T, U\}$ is a linearly independent subset of $\mathcal{L}(V, W)$.
21. Let S be a subset of V and S^0 is the annihilator of S. Then prove that:
    (a) S^0 is a subspace of $\mathcal{L}(V)$.
    (b) If S_1 and S_2 are subsets of V such that $S_1 \subseteq S_2$ then prove that $S_2^0 \subseteq S_1^0$.
    (c) For any two subspaces V_1 and V_2 of V prove that $(V_1 + V_2)^0 = V_1^0 \cap V_2^0$.
22. Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then prove that $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$.
23. Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let $T : V \to W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.
24. Let $T : V \to Z$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping $\bar{T} : V/N(T) \to Z$ by $\bar{T}(v + N(T)) = T(v)$ for any coset $v + N(T)$ in $V/N(T)$. Then prove that $\bar{T}$ is well-defined, linear and an isomorphism.
25. Suppose V & W be two finite dimensional vector spaces. Let $T : V \to W$ be a linear transformation and $T' : W' \to V'$ be the dual map of T. Then prove the following:
    (a) $(N(T))^0 = V'$ if and only if $N(T) = \{0\}$.
    (b) $Rank(T) + Nullity(T') = dim(W)$.
    (c) $(R(T))^0 = N(T')$
26. Show that $W = \{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : Nullity(T) > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

27. Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap N(T) = \{0\}$ and $R(T) = \{Tu : u \in U\}$.

28. Suppose U, V and W are arbitrary subspaces of a finite dimensional vector space. Then prove that:

    (a) $U \cap (V + W) \supset (U \cap V) + (U \cap W)$.

    (b) $(U \cap V) + W \subset (U + W) \cap (V + W)$.

29. Suppose V is finite dimensional and $v_1, v_2, ..., v_n \in V$. Define a linear map $\Psi : V' \to \mathbb{R}^n$ by $\Psi(\phi) = (\phi(v_1), ..., \phi(v_n))$. Prove that:

    (a) $v_1, v_2, ..., v_n$ spans V if and only if Ψ is injective.

    (b) $\{v_1, v_2, ..., v_n\}$ is linearly independent if and only if Ψ is surjective.

30. For any positive integer m :

    (a) Prove that $\{1, (x - 5), ..., (x - 5)^m\}$ is a basis for $\mathcal{P}_m(\mathbb{R})$

    (b) What is the dual basis of the basis given in (a)?