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The arc-sine law

L_{2M} = the epoch of the last visit to zero, up to and including $2M$.

$= \max \{ t : 0 \le t \le 2M \text{ and } S_t = 0 \}$

$L_{2M+1} = L_{2M}$

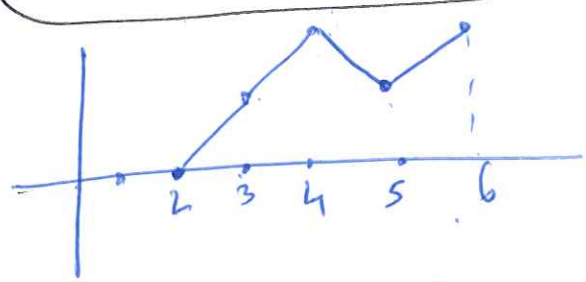
Theorem: The probability mass function for L_{2M} is given by

$$P(L_{2M} = 2K) = u_{2K} \cdot u_{2(M-K)}, \quad K = 0, 1, \ldots, M$$

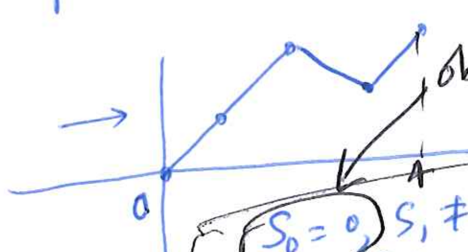
L_{2M}	Prob.
$2K = 0$	$u_0 \cdot u_{2M}$
$2K = 2$	$u_2 \, u_{2M-2}$
$2K = 4$	$u_4 \, u_{2M-4}$
$2K = 6$	$u_6 \, u_{2M-6}$
$\vdots$	$\vdots$

Clarification

$S_2 = 0$, $S_3 \ne 0$, $S_4 \ne 0$, $S_5 \ne 0$, $S_6 \ne 0$



obvious.



$S_0 = 0$, $S_1 \ne 0$, $S_2 \ne 0$, $S_3 \ne 0$, $S_4 \ne 0$

By main theorem

✓ $u_6 = P(S_6 = 0)$

$$P(S_1 \neq 0, S_2 \neq 0, S_3 \neq 0, S_4 \neq 0) = P(S_4 = 0)$$
$$= u_4$$

Therefore

$$P(S_2 = 0, S_3 \neq 0, S_4 \neq 0, S_5 \neq 0, S_6 \neq 0)$$
$$= P(S_1 \neq 0, S_2 \neq 0, S_3 \neq 0, S_4 \neq 0)$$
$$= \underline{\underline{u_4}}$$

$\sin^{-1}$

Proof: (of arc-sine law)

The event $L_{2m} = 2k$ can be written as:

$$\underbrace{S_{2k} = 0}_{A}, \underbrace{S_{2k+1} \neq 0, S_{2k+2} \neq 0, \ldots, S_{2m} \neq 0}_{B}$$

$$P(S_{2k} = 0, S_{2k+1} \neq 0, S_{2k+2} \neq 0, \ldots, S_{2m} \neq 0)$$
$$= P(A \cap B) = P(A) \cdot P(B \mid A)$$
$$= u_{2k} \cdot u_{2m-2k} = u_{2k} \cdot u_{2(m-k)}$$

Why "Arc Sine Law"?

$P(L_{2M} = 2K) = u_{2K} \cdot u_{2(M-K)}$, $K = 0, \dots, M$.

For large K, M.

$u_{2K} \sim \dfrac{1}{\sqrt{\pi K}}$

$\dfrac{u_{2K}}{\frac{1}{\sqrt{\pi K}}} \longrightarrow 1$, as $K \to \infty$

$P(L_{2m} = 2K)$

$\approx \dfrac{1}{\sqrt{\pi K}} \cdot \dfrac{1}{\sqrt{\pi (M-K)}}$

$\therefore$ $P(L_{2m} = 2K) \approx \boxed{\dfrac{1}{M} \cdot \dfrac{1}{\pi \sqrt{\frac{K}{M}\left(1 - \frac{K}{M}\right)}}}$

ASIDE

The function $f(x) = \dfrac{1}{\pi \sqrt{x(1-x)}}$

is infact the pdf on the unit interval $(0 \le x \le 1)$ and the corresponding cdf $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$.

For $0 \le p \le 1$

$$\boxed{\int_0^p f(x)\,dx = \frac{2}{\pi} \arcsin(\sqrt{p})}$$

($\arcsin = \sin^{-1}$)

Back to simple random walk:

For every $0 < p < 1$, for M large enough

$$P(L_{2M} \leq p \cdot 2M) = \sum_{k < pM} u_{2k} \cdot u_{2(M-k)}$$

Can be shown HW!

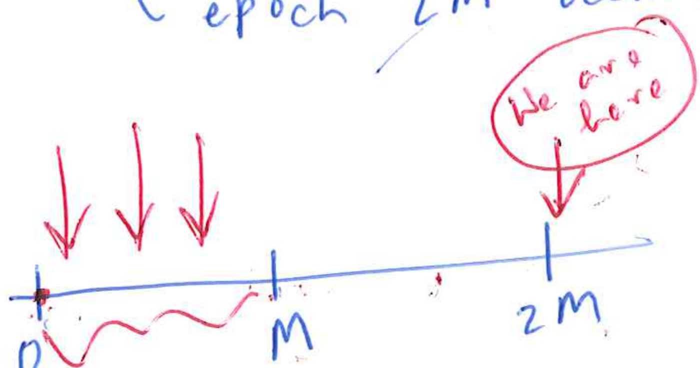
$$\approx \int_0^p f(x)\,dx \quad \left(f(x), \text{ as in "Aside" on last page} \right)$$

$$= \frac{2}{\pi} \arcsin(\sqrt{p})$$

Corollary: For every M,

$$P\left(\begin{array}{l}\text{the latest return to } 0 \text{ through} \\ \text{epoch } 2M \text{ occurs no later than epoch } M\end{array}\right) = P(L_{2M} \leq M)$$

$$\approx \underbrace{\frac{1}{2}}_{\text{by arcsine law with } p=\frac{1}{2}}$$



Proof: $P(L_{2M} \leq M) \approx \frac{2}{\pi} \sin^{-1}\left(\sqrt{\frac{1}{2}}\right)$

$$= \frac{2}{\pi} \cdot \frac{\pi}{4} = \frac{1}{2} \quad //$$

Dual walks

Recall:

$$S_0 = 0$$

$$S_t = X_1 + X_2 + \cdots + X_t$$

where $X_i = \begin{cases} +1, & \text{prob} = \frac{1}{2} \\ -1, & \text{prob} = \frac{1}{2} \end{cases}$

for every i.

Definition:

- Fix a length n.
- Create a NEW random walk S^* of length n by reversing the order of X_t's

$$\left(X_1^* = X_n,\ X_2^* = X_{n-1},\ \ldots,\ X_n^* = X_1\right).$$

$n = 8$

$X_1^* = X_8$

$X_2^* = X_7$

$X_3^* = X_6$

$\ldots X_8^* = X_1$

-
$$\begin{aligned}
\boxed{S_t^*} &= X_1^* + X_2^* + \cdots + X_t^* \\
&= \boxed{X_n + X_{n-1} + \cdots + X_{n-t+1}} \\
&= \boxed{\left(X_n + X_{n-1} + \cdots + X_1\right)} \\
&\quad - \boxed{\left(X_{n-t} + X_{n-t-1} + \cdots + X_1\right)} \\
&= \boxed{S_n - S_{n-t}}.
\end{aligned}$$

$\boxed{t = 1, 2, \ldots, n}$

- This walk is called the dual of S.

$n = 8$

$S_4^* = X_1^* + X_2^* + X_3^* + X_4^*$

$S_4^* = X_8 + X_7 + X_6 + X_5$

$S_4 = X_1 + X_2 + X_3 + X_4$

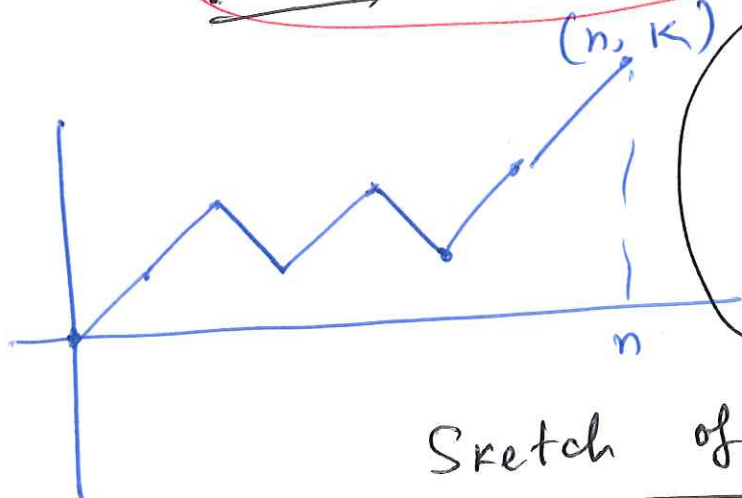
* Every event related to S has a dual event related to S^* that has the same probability.

Example:

Simple random walk

Dual

$$P\left(S_n = k,\ S_1 > 0, \ldots, S_{n-1} > 0\right) = P\left(S_n^* = k,\ S_n^* > S_1^*,\ S_n^* > S_2^*,\ \ldots,\ S_n^* > S_{n-1}^*\right)$$



Note $S_n = S_n^*$ for a fixed n

Sketch of proof: $S_n^* > S_1^*$

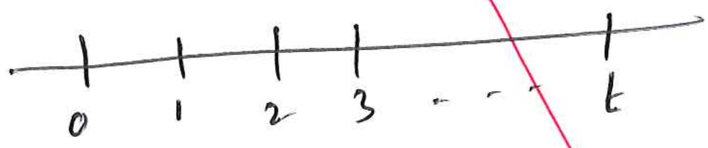
$\Rightarrow S_n > S_n - S_{n-1}$

$\Rightarrow S_{n-1} > 0$ //

"First visit" problem

We know:

$$P(S_t = k) = \frac{N_{t,k}}{2^t} = \binom{t}{\frac{t-k}{2}} \cdot \frac{1}{2^t}$$



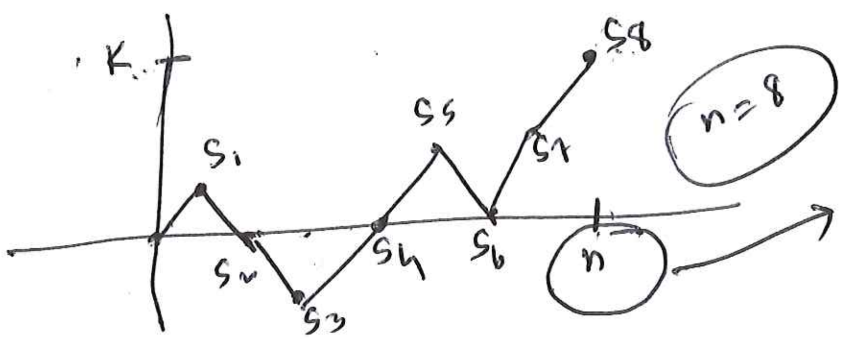
provided (t, k) is reachable from the origin.

- $k > 0$
- Assume that (n, k) is reachable from the origin. (i.e., $n - k \geq 0$ and $n - k$ is even)

What is the probability that the first visit to k happens at epoch n?

We want to find the prob. of

$$A = (S_1 < S_n,\ S_2 < S_n, \ldots, \ , S_{n-1} < S_n, S_n = k)$$



First visit to k.

Consider the dual walk S^*.

Then define

$$B = (S_1^* > 0, S_2^* > 0, \dots, S_{n-1}^* > 0, S_n^* = k).$$

Aside:
For fixed n.
$S_n^* = S_n$

$$\Big[\; S_1^* > 0 \;\Rightarrow S_n - S_{n-1} > 0 \;\Rightarrow \underline{S_n > S_{n-1}} \quad \Big| \quad S_2^* > 0 \quad S_n - S_{n-2} > 0 \quad S_n > S_{n-2} \quad \dots \;\Big]$$

So, B is the same event A in dual space.

$$\underbrace{P(A)}_{\text{We want this!}} = P(B)$$

$$= P(S_1^* > 0, S_2^* > 0, \dots, S_{n-1}^* > 0, \underline{S_n^* = k})$$

by Ballot theorem.

$$= \boxed{\frac{\frac{k}{n} \cdot N_{n,k}}{2^n}}$$

Theorem: Suppose $K > 0$

$$\boxed{P\left(\text{the first visit to } K \text{ occurs at epoch } n\right) = \frac{K}{n} \cdot \binom{n}{\frac{n-K}{2}} \cdot \frac{1}{2^n}} \quad \cdots (*)$$

provided $n-K$ is a non-negative even integer. (Otherwise, it is zero)

Last time:

For every integer K, with prob. $= 1$, the random walk visits K.

~~infinitely~~

In $(*)$ write $n = 2m + K$, $K \geq 1$

Then, the last probability can be written as

$$\boxed{\sum_{m=0}^{\infty} \frac{K}{2m+K} \cdot \binom{2m+K}{m} \cdot \frac{1}{2^{2m+K}} = 1}$$

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