10/20

Example: Consider a joint density of X and Y

$$f(x,y) = (2x + 2y - 4xy), \quad 0 \le x \le 1,\ 0 \le y \le 1$$

Find $Cov(X, Y)$.

Solution: $Cov(X, Y) = E(XY) - E(X) \cdot E(Y)$

$$\boxed{E(XY)} = \int_{y=0}^{1} \int_{x=0}^{1} xy \, f(x,y) \, dx \, dy = \int_{y=0}^{1} \int_{x=0}^{1} x y (2x + 2y - 4xy) \, dx \, dy$$

$$= \int_{y=0}^{1} y \cdot \left(2 \cdot \frac{x^3}{3} + \cancel{2} \cdot y \cdot \frac{x^2}{\cancel{2}} - 4y \cdot \frac{x^3}{3} \right) \Bigg|_0^1 dy$$

$$= \int_{y=0}^{1} y \left(\frac{2}{3} + y - \frac{4}{3} y \right) dy$$

$$= \frac{2}{3} \cdot \frac{y^2}{2} + \frac{y^3}{3} - \frac{4}{3} \cdot \frac{y^3}{3} \Bigg|_0^1$$

$$= \frac{2}{6} + \frac{1}{3} - \frac{4}{9}$$

$$= \frac{3 + 3 - 4}{9} = \boxed{\frac{2}{9}}$$

$$E(X) = \int_0^1 x \cdot f_X(x)\, dx \qquad f_X(x) = \int_0^1 f(x,y)\, dy$$

$$E(Y) = \int_0^1 y \cdot f_Y(y)\, dy \qquad f_Y(y) = \int_0^1 f(x,y)\, dx$$

$$f_X(x) = \int_0^1 \boxed{(2x + 2y - 4xy)}\, dy$$

$$= 2xy + \cancel{2}\frac{y^2}{\cancel{2}} - 4x \cdot \frac{y^2}{2} \Big|_{y=0}^{1}$$

$$= (2x + 1 - 2x) = 1,$$

Hence, $f_X(x) = 1, \quad 0 \le x \le 1$

So, $X \sim \text{Unif}([0,1])$

Similarly we can show
$Y \sim \text{Unif}([0,1])$

$$E(X) = \int_0^1 x \cdot 1 \cdot dx = \frac{1}{2}$$

$$E(Y) = \int_0^1 y \cdot 1 \cdot dy = \frac{1}{2}$$

Hence: $\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$

$$= \frac{2}{9} - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \boxed{-\frac{1}{36}}$$

Example: Find ρ for the last example.

Solution:

$$\boxed{\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}}$$

We found $Cov(X,Y) = -\frac{1}{36}$ ← (last example)

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \int_0^1 x^2 \underbrace{f_X(x)}_{=1}\, dx - \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Similarly $Var(Y) = \frac{1}{12}$

Hence

$$\rho = \frac{\left(-\frac{1}{36}\right)}{\sqrt{\frac{1}{12} * \frac{1}{12}}} = 12\left(-\frac{1}{36}\right)$$

So, $\boxed{\rho = -\frac{1}{3}}$

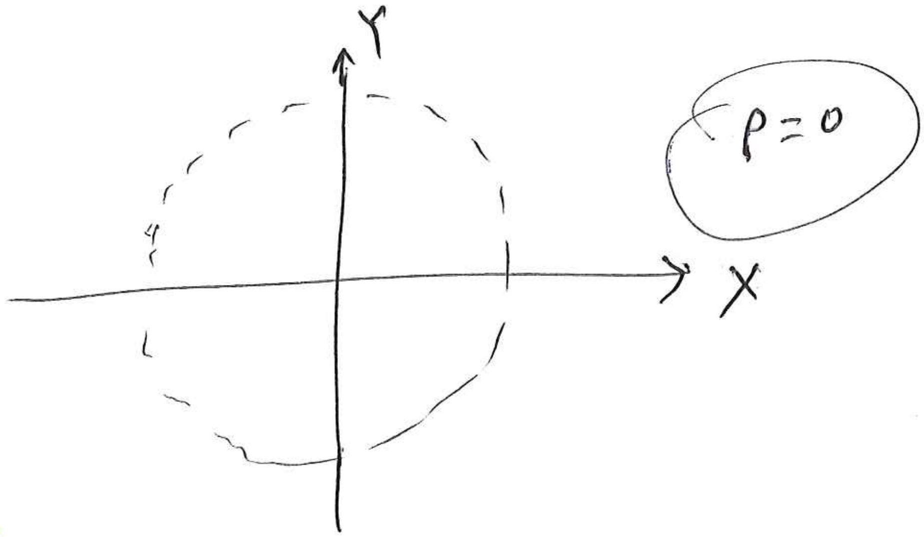
Correlation coefficient

Remark:

$\rho > 0 \Rightarrow$ positive <u>linear</u> correlation

$\rho < 0 \Rightarrow$ negative <u>linear</u> correlation.

$\rho = 0 \Rightarrow$ zero <u>linear</u> correlation.



Remark:

If $X \perp Y$ then $Cov(X, Y) = 0$ $(\Rightarrow \rho = 0)$

(As, $E(XY) = E(X) \cdot E(Y)$ when $X \perp Y$)

$\boxed{\text{Independence} \Rightarrow \text{Uncorrelated}}$

The converse is NOT true.

If $Cov(X, Y) = 0 \not\Rightarrow X \perp Y$.
$(\Rightarrow \rho = 0)$

$\boxed{\text{Uncorrelated} \not\Rightarrow \text{Independence}}$

If $X \perp Y$

$$f(x,y) = f_X(x) f_Y(y)$$

$$\left(\Rightarrow \quad E(XY) = E(X)\,E(Y) \right)$$

But $E(XY) = E(X) \cdot E(Y) \not\Rightarrow X \perp Y$.

Example: Suppose $X \sim \text{Unif}[-1,1]$, $Y = |X|$

$$Y = \begin{cases} -X, & \text{when } X \le 0 \\ X, & \text{when } X \geq 0 \end{cases}$$

It is easy to check: (Check it!)

- $Y \sim \text{Unif}[0,1]$

$$F(y) = P(Y \le y) = \cdots$$

$$f(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

- $E(XY) = 0$ [How: $E(XY \mid X \le 0) = -\frac{1}{3}$

$E(XY \mid X > 0) = +\frac{1}{3}$

Law of total expectation

$E(XY) = 0$]

- $E(X) = 0, \quad E(Y) = \frac{1}{2}$

Hence $\quad \operatorname{Cov}(X, Y) = E(XY) - E(X)E(Y)$

$$= 0 - 0\left(\tfrac{1}{2}\right) = 0$$

So, X and Y are uncorrelated.

However X and Y are dependent.
(as the joint density is NOT uniform)

Remark: Suppose X, Y are Normal

Then $X \perp Y \iff X$ and Y are uncorrelated

Example: Find the variance of a Binomial distribution

Solution: $X \sim \text{Binomial}(n, p)$

~~$p(x) = ($~~

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$k = 0, 1, 2, \ldots, n.$

$$\operatorname{Var}(X) = E(X^2) - (E(X))^2 = E(X^2) - (np)^2$$

Another way:

Think $X = X_1 + X_2 + \cdots + X_n$

Sum of n- independent Bernoulli distributions

$$X_i = \begin{cases} +1, & \text{with prob} = p \\ 0, & \text{with prob} = 1-p. \end{cases}$$

For $i = 0, 1, \ldots, n$

$E(X_i) = 1 * p + 0 * (1-p) = p.$

(Last time: $\boxed{E(X)} = E(X_1) + E(X_2) + \cdots + E(X_n)$

$= \underbrace{p + p + \cdots + p}_{n\text{-times}} = \boxed{np}.$

Remark: We did not need independence of X_i's to get this)

$Var(X) = Var(X_1) + Var(X_2) + \cdots + Var(X_n)$

ONLY TRUE If $X_1, X_2, \ldots, X_n$ are independent

$= n\, Var(X_i)$

(as $Var(X_1) = Var(X_2) = \cdots = Var(X_n)$)

Last time:

$Var(X + Y)$

$= Var(X) + Var(Y) + 2\, Cov(X, Y)$

So, $X \perp Y$

$Var(X+Y)$

$= Var(X) + Var(Y)$

For $X_i \sim \text{Bernoulli}(p)$

$$Var(X_i) = \boxed{E(X_i^2)} - (p)^2$$

$$= \boxed{1^2 \cdot p + 0^2 \cdot (1-p)} - p^2$$

$$= p - p^2 = p(1-p)$$

[For every $i = 0, 1, 2, \ldots, n$]

Hence $\boxed{Var(X) = n \cdot p(1-p)}$

Binomial(n, p)

[Aside: In general

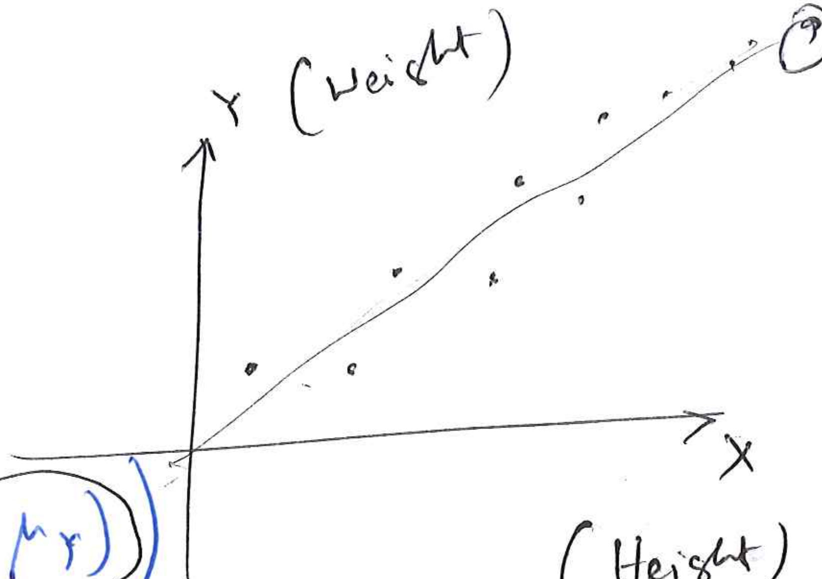
$$Var(a + b_1 X_1 + b_2 X_2 + \cdots + b_n X_n)$$

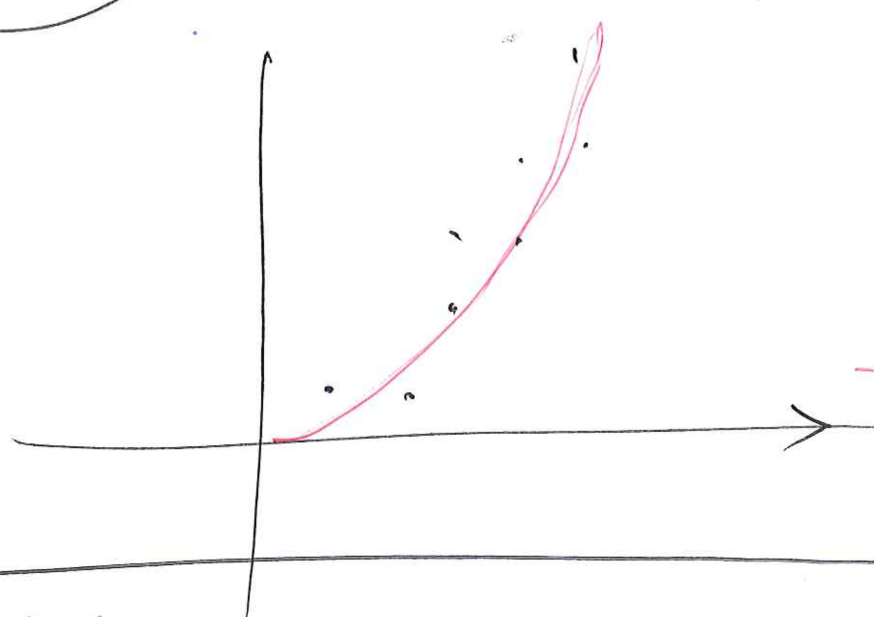
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j Cov(X_i, X_j)$$]

When $i = j$ $\quad Cov(X_i, X_i) = Var(X_i)$

[$$Var\left(a + \sum_{i=1}^{n} b_i X_i\right) = \sum_{i=1}^{n} b_i^2 Var(X_i) + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} b_i b_j Cov(X_i, X_j)$$]

$$\mathrm{Cov}(X, Y) = E\big((X - \mu_X)(Y - \mu_Y)\big)$$





Conditional Expectation

Suppose that X and Y are discrete random variables and the conditional frequency function of Y given x is $p_{Y|X}(y|x)$

The conditional expectation of Y given $X = x$ is

$$E(Y|X=x) = \sum_y y \, p_{Y|X}(y|x)$$

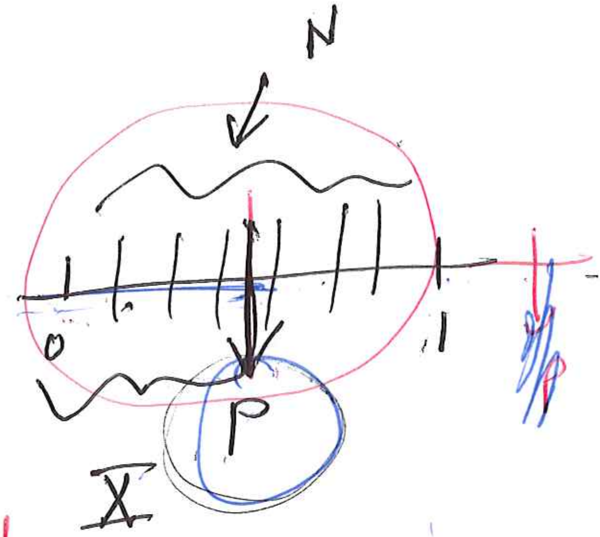
- For the continuous case

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy$$

Example: • Consider a $\boxed{\text{Poisson process}}$ on $[0,1]$ with mean λ.

• Let N be the number of events in $[0,1]$.

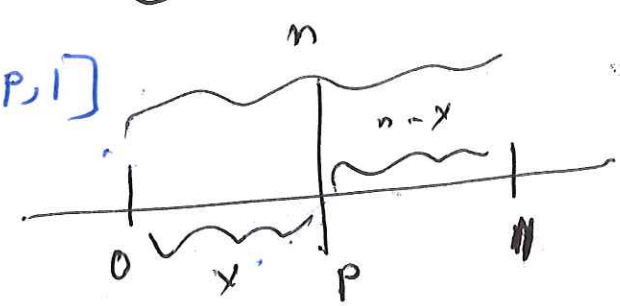
• Let X be the number of events in $[0,p]$.



Find the conditional distribution and conditional expectation (mean) of X given $N=n$.

Solution: Joint distribution : $P(X=x, N=n)$

= x events in $[0,p]$ and $(n-x)$ events in $[p,1]$



In between $[0, p]$ we

have Poisson process with parameter $= \lambda(p-0)$ $= \lambda p$

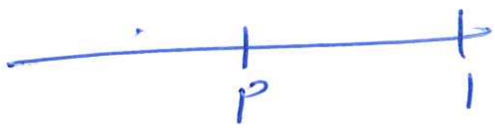
Aside:

$X \sim \text{Poisson}(\lambda)$

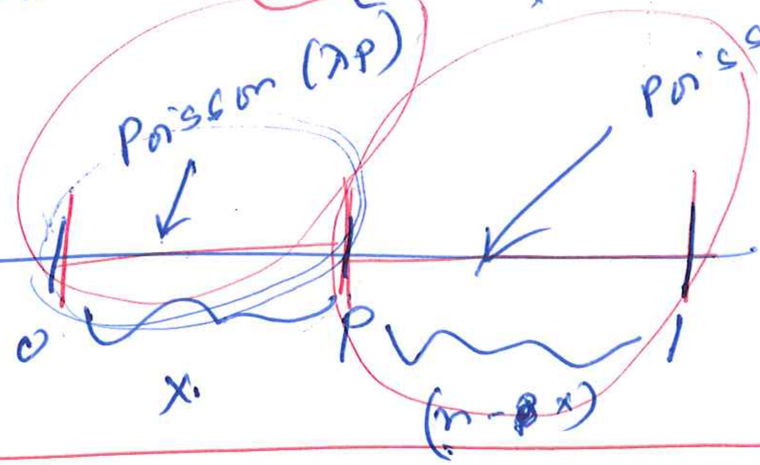
Poisson distribution

$X \sim$ Poisson process with mean λ.

$X \sim \text{Poisson}(\lambda T)$



Between $[p, 1]$ we have Poisson process with parameter $= \lambda(1-p)$



Remark: Poisson random variables over two non-overlapping intervals are independent

$$P_{XN}(x,n) = \boxed{\frac{(p\lambda)^x \cdot e^{-p\lambda}}{x!}} \cdot \boxed{\frac{(\lambda(1-p))^{n-x} \cdot e^{-(1-p)\lambda}}{(n-x)!}}$$

We know:

$$P_{X|N}(x|n) = \frac{P_{XN}(x,n)}{P_N(n)} \xrightarrow{} \text{Given to be a Poisson distribution with parameter } \lambda(1-0) = \lambda$$

$$P_N(n) = P(N=n)$$

$$= \frac{\frac{(p\cancel{\lambda})^x \cancel{e^{-p\lambda}}}{x!} \cdot \frac{(\cancel{\lambda}(1-p))^{n-x} \cancel{e^{-(1-p)\lambda}}}{(n-x)!}}{\frac{\cancel{e^{-\lambda}} \cdot \cancel{\lambda^n}}{n!}}$$

$$= \frac{n!}{x!\,(n-x)!} p^x \cdot (1-p)^{n-x}$$

$$\boxed{P_{X|N}(x|n) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \ldots, n}$$

$$E(X \mid N=n) = \sum_{x=0}^{n} x \cdot P_{X|N}(x|n)$$

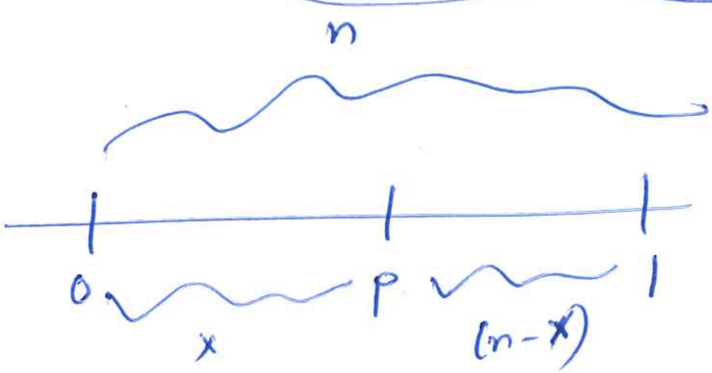
$$= \sum_{x=0}^{n} x \cdot \left(\binom{n}{x} p^{x} (1-p)^{n-x} \right)$$

$$= np$$

Next:

$$P_{X|N}(x|n) = \binom{n}{x} p^{x} (1-p)^{n-x}$$

$x = 0, 1, \ldots, n$



$$E(X \mid N=n) = \sum_{x} x \cdot P_{X|N}(x|n)$$

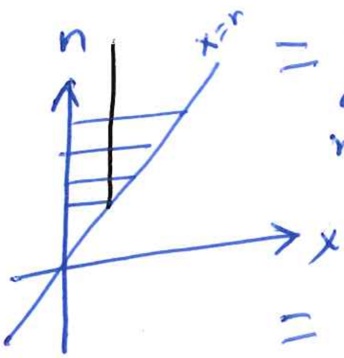
- Given different "N = n" the expectation changes.
- We can have different probabilities for different $N=n$
- With different prob. we have $\boxed{E(X \mid N=n)}$

~~Theorem~~:

random variable

$$E\Big(E(X \mid N=n)\Big)$$

$$= \sum_{n=0}^{\infty} \boxed{E(X \mid N=n)}\, P(N=n)$$



$$= \sum_{n=0}^{\infty} \boxed{\left(\sum_{x=0}^{n} x \cdot P_{X|N}(x|n) \right)} P(N=n)$$

$$= \sum_{x=0}^{\infty} x \sum_{n=x}^{\infty} P_{X|N}(x|n)\, P(N=n)$$

Law of Total probability

$$= \sum_{x=0}^{\infty} x \cdot P(X=x)$$

$\left(P_{X|N}(x|n) = 0 \text{ when } n < x \right)$

$$= E(X)$$

Theorem:

$$\boxed{E(Y) = E\Big(E(Y \mid X)\Big)}$$