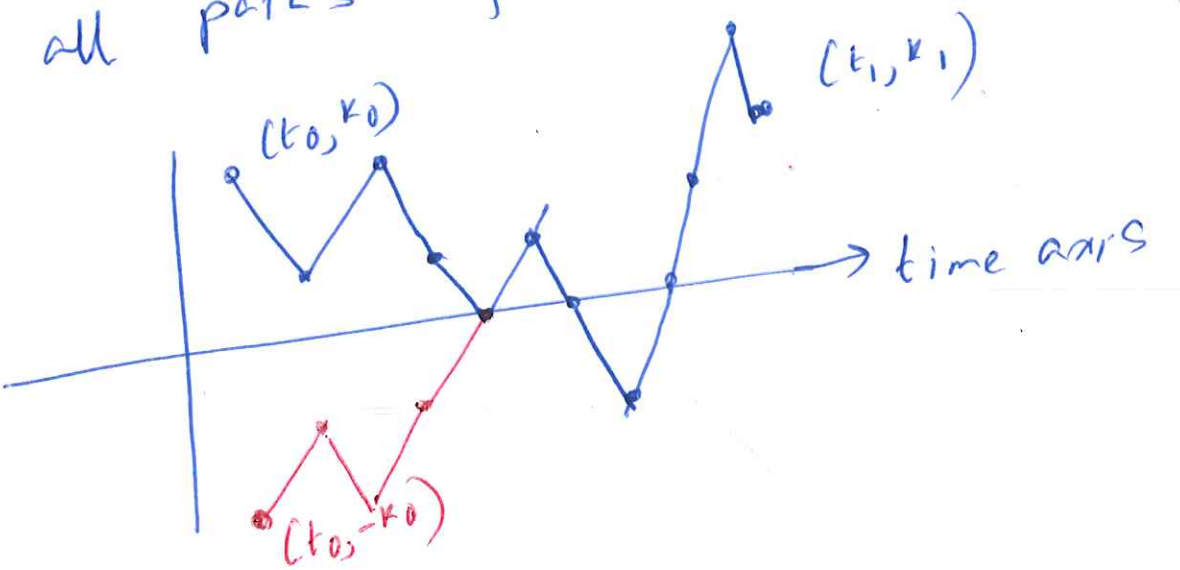
11/17

Last time:

The number of initial segments of paths that reach the reachable point (t, k) is denoted by $N_{t,k}$.

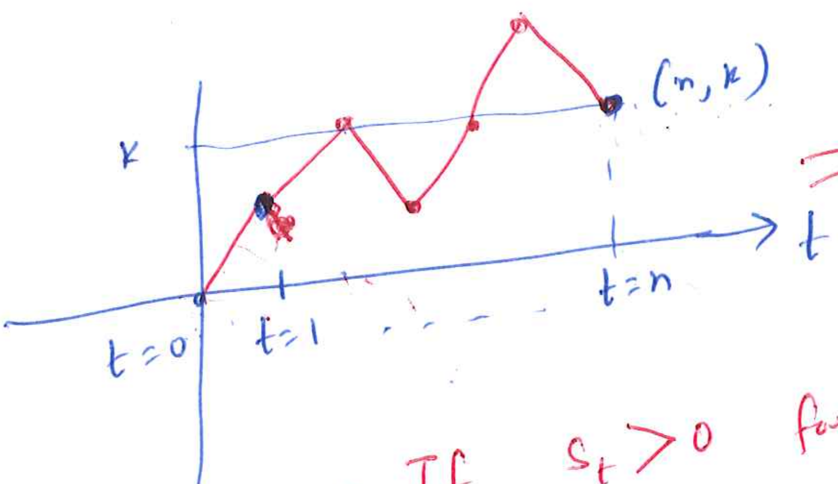
Reflection principle:

Let (t_1, k_1) be reachable from (t_0, k_0) and on the same side of the time axis. Then there is a bijection (one-to-one and onto) between the set of paths from (t_0, k_0) to (t_1, k_1), that meet (touch or cross) the time axis AND the set of all paths from $(t_0, -k_0)$ to (t_1, k_1)



The ballot theorem

If $k > 0$, then there are exactly $\frac{k}{n} N_{n,k}$ paths from the origin to (n, k) satisfying $S_t > 0$, $t = 1, \ldots, n$.



Proof:

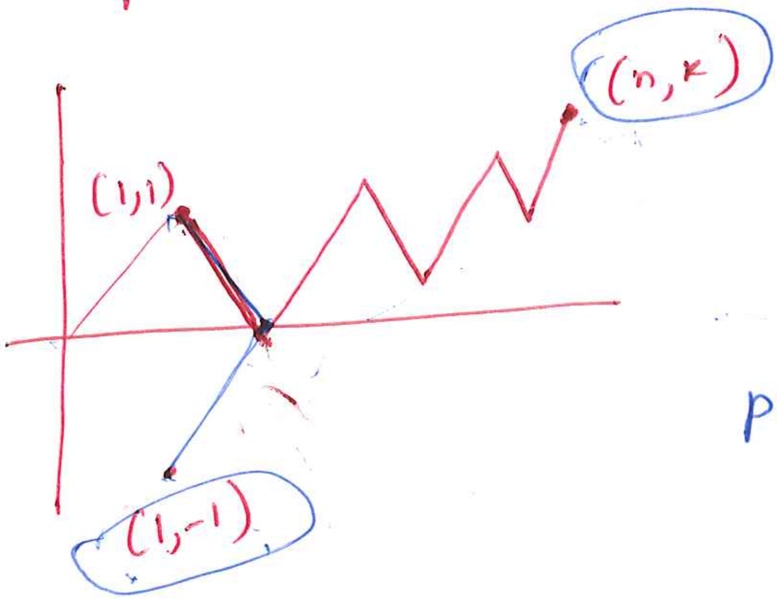
- If $S_t > 0$ for all $t = 1, \ldots, n$, then $S_1 = +1$
- How many paths from $(1, 1)$ to (n, k)?

All possible paths $\rightarrow$ $N_{n-1, k-1}$

(Some of these paths may touch or cross t-axis. They do NOT satisfy $S_t > 0$)

$\rightarrow$ If they do that they do NOT satisfy $S_t > 0$.

• How many of these ($N_{n-1,k-1}$) paths touch the time axis?



||

By reflection principle, this same as the number of paths from (1, −1) to (n, k)

$N_{n-1,k+1}$

• Thus the number of paths from (1, 1) to (n, k) that do NOT touch the time axis = $N_{n-1,k-1} - N_{n-1,k+1}$

Required number

• Let p and m be defined as

$\boxed{\begin{array}{l} p: \text{\# of } +1\text{s} \\ m: \text{\# of } -1\text{s} \end{array}}$

in the last class

So,
$$p + m = n$$
$$p - m = K$$

$$\Rightarrow \boxed{n + k = 2p} \quad \ldots\ldots (*)$$

"trite calculation" (Feller)

$$N_{n-1,k-1} - N_{n-1,k+1} = \binom{n-1}{\frac{n+k-2}{2}} - \binom{n-1}{\frac{n+k}{2}}$$

Aside:

$$N_{t,k} = \binom{t}{\frac{t+k}{2}}$$

change to p and m

$$= \binom{m+p-1}{p-1} - \binom{m+p-1}{p}$$

$$= \frac{(m+p-1)!}{(p-1)!\ m!} - \frac{(m+p-1)!}{p!\ (m-1)!}$$

$$= p \cdot \frac{(m+p-1)!}{m!\ p!} - \frac{m\ (m+p-1)!}{m!\ p!}$$

$$= (p-m)\ \frac{(m+p-1)!}{m!\ p!}$$

$$= \left(\frac{p-m}{p+m}\right) \frac{(m+p)!}{m!\ p!}$$

$$= \left(\frac{p-m}{p+m}\right) \cdot \binom{m+p}{p}$$

back to n and k

$$= \frac{k}{n} N_{n,k}$$

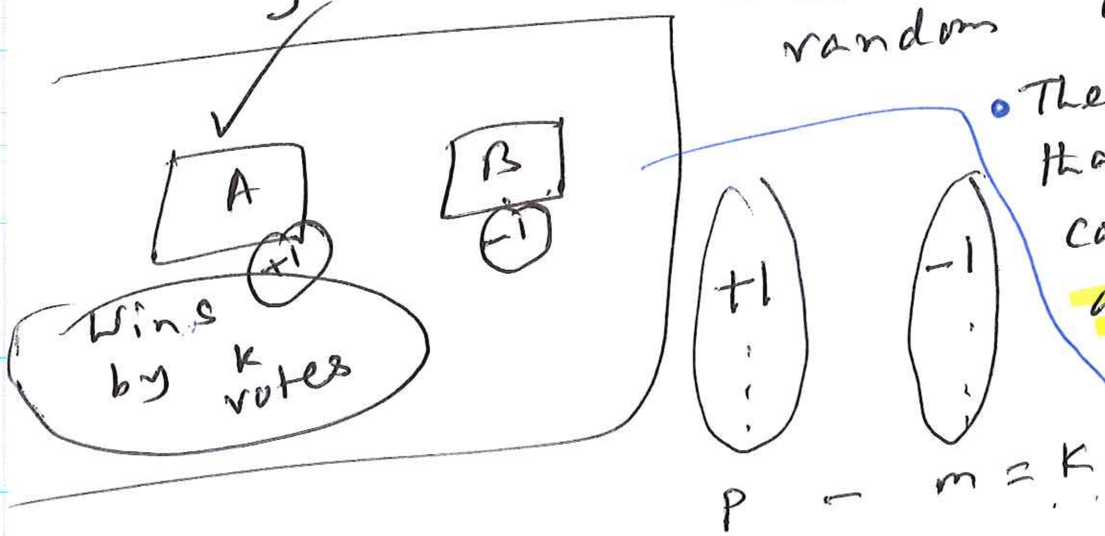
Why ballot theorem?

Two different versions of the same theorem:

1) • Suppose an election with n ballots cast has one candidate winning by k-votes:

• Count the votes in random order:

• The probability that the winning candidate **always leads** $= \frac{k}{n}$.



[Proof: Total number of possibilities $= N_{n,k}$

Total number of cases when the $s_t > 0$, $t = 1, \ldots, n$ is $\frac{k}{n} N_{n,k}$. (by the last theorem)

Hence probability $= \dfrac{\frac{k}{n} N_{n,k}}{N_{n,k}} = \dfrac{k}{n}$]

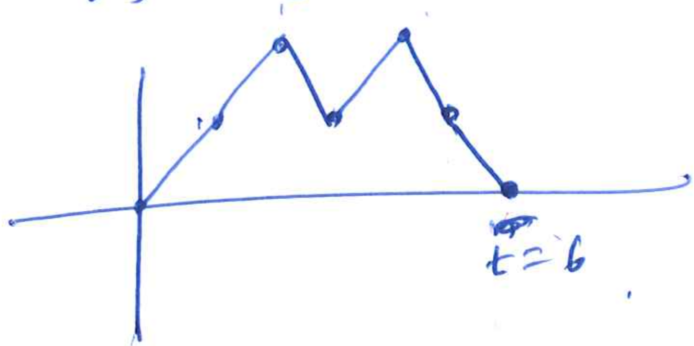
Version 3: Suppose an election has one candidate getting p votes, and the other getting m votes with $p > m$.

Count the votes in random order.

The probability that the winning candidate always leads $= \dfrac{p-m}{p+m}$.

Return to zero:

Definition: We say that the walk "equalizes" or "returns to zero" at epoch t, if $S_t = 0$.



Epoch t MUST be even in this case. Let $t = 2M$

The number of paths from

$(0,0) \longrightarrow (\underbrace{2M}_{t}, 0) = N_{2M,0}$

Total number of paths for $t = 2M$

$= 2^{2M}$

Probability that the path returns to zero after $t = 2M$ is

within

$$\boxed{u_{2M} = \frac{N_{2M,0}}{2^{2M}} = \frac{1}{2^{2M}} \binom{2M}{M}}$$

$(M \geq 0)$

with $\boxed{u_0 = 1}$

Aside: Stirling's formula:

$$n! = e^{-n} \cdot n^n \sqrt{2\pi n} \, (1 + \varepsilon_n)$$

where $\varepsilon_n \to 0$ as $n \to \infty$

It can be shown using Stirling's formula that $\boxed{u_{2M} \sim \frac{1}{\sqrt{\pi M}}}$

$\left(\text{i.e., } \frac{u_{2M}}{\frac{1}{\sqrt{\pi M}}} \to 1 \text{ as } M \to \infty \right)$

Main Lemma:

The following are equal:

$$u_{2m} \equiv P(S_{2m}=0)$$

Notation

$$= P(S_1 \neq 0, S_2 \neq 0, \dots, S_{2m} \neq 0)$$

$$= P(S_1 \geq 0, S_2 \geq 0, \dots, S_{2m} \geq 0)$$

$$= P(S_1 \leq 0, S_2 \leq 0, \dots, S_{2m} \leq 0)$$

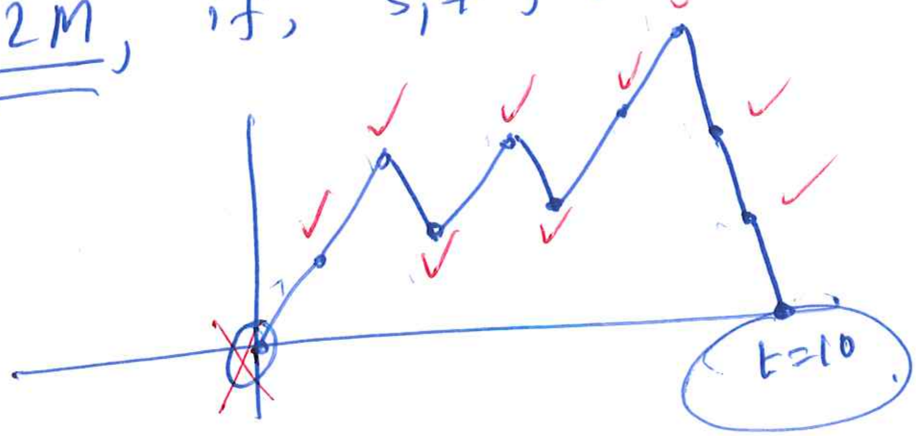
$$= 2P(S_1 > 0, S_2 > 0, \dots, S_{2m} > 0)$$

$$= 2P(S_1 < 0, S_2 < 0, \dots, S_{2m} < 0).$$

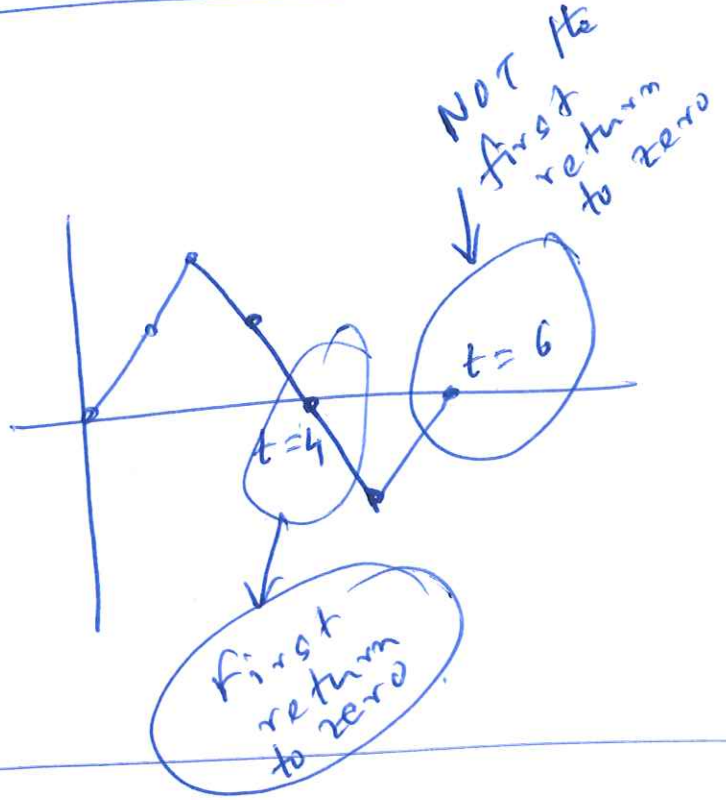
Back to "return to zero":

"First" return to zero: The first return to zero happens at epoch $t = 2m$, if, $S_1 \neq 0, S_2 \neq 0, \dots, S_{2m-1} \neq 0$.

and, $S_{2m} = 0$



Let $f_t = f_{2M}$ ("first") denote the probability of this above event.



Aside:

$u_{2M} = P(S_{2M} = 0)$

$f_{2M} = P(S_{2M} = 0,\ S_{2M-1} \neq 0,\ S_{2M-2} \neq 0,\ \dots,\ S_1 \neq 0)$

Question: How do we compute f_{2M}?

Theorem:

$$\boxed{\begin{aligned} f_{2M} &= u_{2M-2} - u_{2M} \\ &= \frac{1}{2M-1} u_{2M} = \frac{1}{2M-1}\binom{2M}{M}\cdot\frac{1}{2^{2M}} \end{aligned}}$$

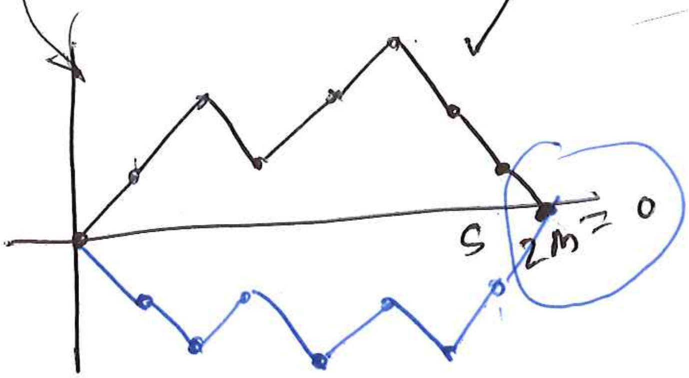
with $M = 1, 2, \dots$

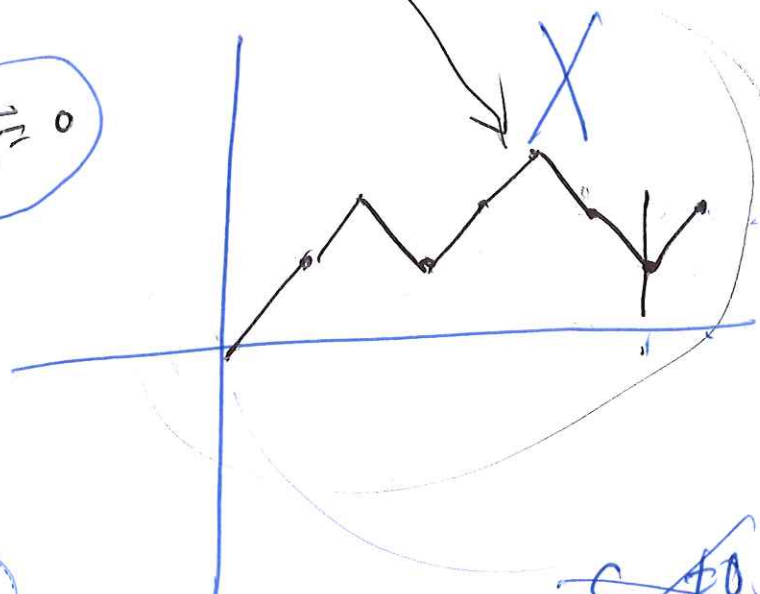
$f_0 = 0$ (convention)

Proof: The event that the first return to zero occurs at epoch $= 2m$

$$(S_1 \neq 0, S_2 \neq 0, \dots, S_{2m-2} \neq 0, S_{2m-1} \neq 0, S_{2m} = 0)$$

$$= (S_1 \neq 0, \dots, S_{2m-2} \neq 0, S_{2m-1} \neq 0) \setminus (S_1 \neq 0, \dots, S_{2m} \neq 0)$$





Since

$$(S_1 \neq 0, \dots, S_{2m} \neq 0) \subset (S_1 \neq 0, \dots, S_{2m-2} \neq 0).$$

$\cancel{S_{49} \neq 0}$

$S_{50} = 0$

$$P\left((S_1 \neq 0, \ldots, S_{2m-2} \neq 0) \setminus (S_1 \neq 0, \ldots, S_{2m} \neq 0)\right)$$

$$= P(S_1 \neq 0, \ldots, S_{2m-2} \neq 0) - P(S_1 \neq 0, \ldots, S_{2m} \neq 0).$$

$P(A \setminus B) = P(A) - P(B)$

Provided $B \subset A$

By the "Main Lemma"

$$= u_{2m-2} - u_{2m}$$

$$u_{2m-2} = \frac{(2m-2)!}{(m-1)!\,(m-1)!} \cdot \frac{1}{2^{2m-2}} = \frac{4m^2}{2m(2m-1)} \binom{2m}{m}$$

So, $u_{2m-2} - u_{2m} = \left(\frac{2m}{2m-1} - 1\right) u_{2m}$

$\Longrightarrow$ The results //