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MSc Big Data Analytics : Batch 2022-24
DA109: Linear Algebra and Matrix Computation
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Problem set: 2

Please try to solve all the problems [1] alone. If you are stuck, I encourage you first consult with your batch mates and then TA. If you still need help then come to Instructor.

Notations:

- $\cdot$: scalar multiplication
- $\mathbf{Tr}(\mathbf{A})$: *Trace* of a matrix $\mathbf{A}$
- $\mathcal{R}(\mathbf{A})$: *row-space* of a matrix $\mathbf{A}$
- $\mathcal{C}(\mathbf{A})$: *column-space* of a matrix $\mathbf{A}$
- $\mathcal{N}(\mathbf{A})$: *null-space* of a matrix $\mathbf{A}$
- $\rho(\mathbf{A})$: *Rank* of a matrix $\mathbf{A}$
- *g-inverse*: Generalised inverse
- $\mathbf{A}^-$: *g-inverse* of a matrix $\mathbf{A}$

1. How many multiplications (of scalars) are needed to compute the product $\mathbf{AB}$, where $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix?

2. For each of the following matrices, find $\mathbf{A}^k$ for all $k \geq 2$, where $\mathbf{A}^k = \mathbf{A}^{k-1} \times \mathbf{A}$:

   a. $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$

   b. $\begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

   c. $\begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & 1 \\ 0 & 0 & 0 & \cdots & 0 & \alpha \end{bmatrix}$

   d. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ where $\omega = e^{\frac{2\pi i}{3}}$

3. Determine all 2×2 real matrices $\mathbf{A}$ such that :

   a. $\mathbf{A}^2 = \mathbf{0}$ b. $\mathbf{A}^2 = \mathbf{I}$

4. Prove or disprove :

[1] All the problems have been selected by the TA from the Rao & Bhimasankaram book [1] in consultation with the Instructor.

Please turn over...

a. $\mathbf{Axy}^T\mathbf{uv}^T\mathbf{B} = \mathbf{y}^T\mathbf{u} \cdot \mathbf{Axv}^T\mathbf{B}$ where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$ and $\mathbf{v}$ are column vectors.

b. Given any non-null column vector $\mathbf{x}$, there exists a column vector $\mathbf{y}$ such that $\mathbf{y}^T\mathbf{x} = 1$.

c. If $\mathbf{x}$ and $\mathbf{y}$ are column vectors then all columns of $\mathbf{xy}^T$ are scalar multiples of $\mathbf{x}$.

5. If $\mathbf{A}$ is a square matrix of order n such that $a_{ij} = 0$ whenever $i \geq j$, show that $\mathbf{A}^n = \mathbf{0}$.

6. Let $\mathbf{A}$ be a square matrix and $\mathbf{B} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T)$. Prove the following:

   a. $\mathbf{x}^T\mathbf{Bx} = \mathbf{x}^T\mathbf{Ax}$ for all $\mathbf{x} \in n \times 1$.

   b. If $\mathbf{C}$ is a symmetrie matrix such that $\mathbf{x}^T\mathbf{Cx} = \mathbf{x}^T\mathbf{Ax}$ for all $\mathbf{x}$, then $\mathbf{C} = \mathbf{B}$.

7. Let $\mathbf{A}$ be a real $n \times n$ matrix of rank r and $\mathbf{V} = \{\mathbf{X} \in \mathbf{M}_{n\times n}(\mathbf{R}) | \mathbf{AX} = \mathbf{0}\}$ be a vector space. Then find the dimension of $\mathbf{V}$.

8. If $\mathbf{A}$ is an $m \times n$ matrix and if $\mathbf{Ax_1} = \mathbf{0}, \mathbf{Ax_2} = \mathbf{0}, \ldots\ldots, \mathbf{Ax_n} = \mathbf{0}$ for some basis $\{\mathbf{x_1}, \mathbf{x_2}, \ldots\ldots, \mathbf{x_n}\}$ of $\mathbb{F}^\mathbf{n}$, show that $\mathbf{A} = \mathbf{0}$. Can you find any relation with linear transformation from $\mathbb{F}^\mathbf{n} \to \mathbb{F}^\mathbf{m}$?

9. If $\mathbf{y}^T\mathbf{Ax} = \mathbf{0}$ for all $\mathbf{A}$ and if $\mathbf{x} \neq \mathbf{0}$, prove that $\mathbf{y} = \mathbf{0}$.

10. Let $\mathbf{A}$ be an $n \times n$ matrix. If $\mathbf{Tr(AB)} = \mathbf{0}$ for all $n \times n$ matrices $\mathbf{B}$, show that $\mathbf{A} = \mathbf{0}$.

11. Let $\mathbf{A}$ be a square matrix. Show that the columns of $\mathbf{A^3}$ are linear combinations of the columns of $\mathbf{A}$.

12. Find the following products:

    a. $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}$

    b. $\begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix} \mathbf{G} \begin{bmatrix} \mathbf{P} & \mathbf{Q} \end{bmatrix}$

    c. $\begin{bmatrix} \mathbf{P_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_1} \end{bmatrix} \begin{bmatrix} \mathbf{P_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_2} \end{bmatrix} \begin{bmatrix} \mathbf{P_3} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_3} \end{bmatrix}$

13. Consider the real matrix $\mathbf{A} = \begin{bmatrix} 2 & 3 & 7 & 0 \\ 6 & 0 & 0 & 3 \\ 4 & 2 & 0 & 5 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. Find the Row and Column space of $\mathbf{A}$. Also find the dimension of those spaces.

14. Prove that $\rho\left(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}\right) \geq \rho(\mathbf{A}) + \rho(\mathbf{C}), \rho$ denotes the rank of a matrix. Show that strict inequality can occur. Deduce that the rank of an upper triangular matrix is not less than the number of non-zero diagonal elements.

15. If $\mathbf{A}$ is an $m\times n$ matrix with rank r, show that for every k such that $1 \leq k \leq r$, $\mathbf{A}$ has a $k \times k$ submatrix with rank k. Also show that a submatrix of $\mathbf{A}$ formed by k linearly independent rows and k linearly independent columns need not be invertible. What are the possible values for the rank of the submatrix obtained by deleting a row and a column.

16. Write down the conditions to get the *left* and *right inverse* of a matrix respectively. Find the *left or right inverse* of the matrices(which exists):

    a. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 0 & 1 \end{bmatrix}$

    b. $\begin{bmatrix} 4 & 7 & 0 \\ 7 & 8 & 1 \\ 0 & 3 & 1 \end{bmatrix}$

17. Show that a non-invertible matrix $\mathbf{A}$ cannot has both left and right inverse. Also show that A cannot has a unique left inverse.

18. Show that any convex combination of *left inverses* of a matrix also a *left inverse.*

19. Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices such that $\mathbf{AB}$ is diagonal with non-zero diagonal entries. If the diagonal entries of $\mathbf{AB}$ are all equal then show that $\mathbf{A}$ commutes with $\mathbf{B}$.

20. Show that the *rank* of a *symmetric* matrix is the maximum order of a *principal* submatrix which is invertible.

21. Show that a permutation matrices is always invertible.

22. Find 2×2 singular matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ such that $\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is non-singular.

23. Show that $\mathbf{I}_{m \times m}$ is the only non-singular idempotent matrix of order m.

24. Let $\mathcal{N}(\mathbf{A} : \mathbf{B}) = \mathcal{C}\begin{bmatrix} \mathbf{C} \\ \mathbf{D} \end{bmatrix}$, where $\mathbf{AC}$ is defined. Then show that $\mathcal{C}(\mathbf{A}) \cap \mathcal{C}(\mathbf{A}) = \mathcal{C}(\mathbf{AC}) = \mathcal{C}(\mathbf{BD})$.

25. Show that inverse of a *non-singular skew-symmetric* matrix is also *skew-symmetric*.

26. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of orders $m \times n$ and $n \times p$ respectively. Then show that : $\rho(\mathbf{AB}) \geq \rho(\mathbf{A}) + \rho(\mathbf{B}) - n$

27. Prove the following:

    a. Let $\mathbf{A}$ and $\mathbf{B}$ are two matrices with same number of rows. Then show that $\mathcal{C}[\mathbf{A} : \mathbf{B}] = \mathcal{C}(\mathbf{A}) + \mathcal{C}(\mathbf{B})$. And if $\mathbf{A} = \mathbf{BC}$ for some matrix $\mathbf{C}$ then show that $\rho(\mathbf{A} : \mathbf{B}) = \rho(\mathbf{A})$.

    b. If $\mathbf{A}$ and $\mathbf{C}$ have the same number of columns then show that $\mathcal{R}(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}) = \mathcal{R}(\mathbf{A}) + \mathcal{R}(\mathbf{C})$ and $\mathcal{N}(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}) = \mathcal{N}(\mathbf{A}) \cap \mathcal{N}(\mathbf{C})$. If $\mathbf{C} = \mathbf{EA}$ for some matrix $\mathbf{E}$ then show that $\rho(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}) = \rho(\mathbf{A})$.

28. For $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, show that the rank of $\begin{bmatrix} \mathbf{A} & \mathbf{I} \\ \mathbf{I} & \mathbf{B} \end{bmatrix}$ is n iff $\mathbf{A} = \mathbf{B}^{-1}$.

29. Show that Null space of a matrix is not altered by premultiplying any non-singular matrix.

30. Show that if $\mathbf{A}^2 = \mathbf{A}^3$ and $\rho(\mathbf{A}) = \rho(\mathbf{A}^2)$ then $\mathbf{A} = \mathbf{A}^2$. Show also that the condition $\rho(\mathbf{A}) = \rho(\mathbf{A}^2)$ cannot be dropped even for a 2×2 matrix.

31. For each positive integer k, find a matrix $\mathbf{A}$ of order $2k$ and rank k such that $\mathbf{A}^2 = \mathbf{0}$.

32. Prove that $\rho(\mathbf{PAQ}) = \rho(\mathbf{A})$, iff $\rho(\mathbf{A}) = \rho(\mathbf{PA}) = \rho(\mathbf{AQ})$.

33. Prove that for any idempotent matrix $\mathbf{A}$, $\text{Rank}(\mathbf{A}) = \text{Tr}(\mathbf{A})$.

34. If $(\mathbf{P}, \mathbf{Q})$ is a rank factorization of $\mathbf{A}$ then show that $\mathcal{C}(\mathbf{P}) = \mathcal{C}(\mathbf{A})$, $\mathcal{R}(\mathbf{Q}) = \mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{Q}) = \mathcal{N}(\mathbf{A})$.

35. Show that a matrix $\mathbf{A}$ is of rank 1 if and only if $\mathbf{A} = \mathbf{x}\mathbf{y}^T$ for some column vectors $\mathbf{x}$ and $\mathbf{y}$.

36. If $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{B})$ and $\mathcal{R}(\mathbf{A}) \subseteq \mathcal{R}(\mathbf{D})$ then prove that $\mathbf{A} = \mathbf{BCD}$ for some matrix $\mathbf{C}$.

37. Using a rank-factorization of the middle matrix, deduce Frobenius inequality from Sylvester's inequality.

38. Let $(\mathbf{P}, \mathbf{Q})$ be a rank-factorization of $\mathbf{A}$. Then $\mathbf{A}$ is projector iff $\mathbf{QP} = \mathbf{I}$.

39. Let $\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 6 \\ 2 & 6 & 4 \end{bmatrix}$.

    a. Obtain rank-factorization of $\mathbf{A}$ and $\mathbf{B}$. Hence show that $\rho(\mathbf{A}) = \rho(\mathbf{B}) = 1$ and $\rho(\mathbf{A} + \mathbf{B}) = \rho(\mathbf{A} + \mathbf{B})$

    b. Find a matrix $\mathbf{C}$ with rank 1 such that $\mathbf{A} + \mathbf{B} + \mathbf{C}$ is non-singular.

    c. Find $(\mathbf{A} + \mathbf{B} + \mathbf{C})^{-1}$ and verify that $\mathbf{A}(\mathbf{A} + \mathbf{B} + \mathbf{C})^{-1}\mathbf{A} = \mathbf{A}$ and $\mathbf{A}(\mathbf{A} + \mathbf{B} + \mathbf{C})^{-1}\mathbf{B} = \mathbf{0}$.

Please turn over...

40. Determine all projectors of order 2×2 over $\mathbb{R}$.

41. If $\rho(\mathbf{A}) = \rho(\mathbf{A^2})$ and $\mathbf{AB} = \mathbf{BA} = \mathbf{0}$, prove that $\rho(\mathbf{A} + \mathbf{B}) = \rho(\mathbf{A}) + \rho(\mathbf{B})$. Show that none of $\mathbf{AB} = \mathbf{0}$ and $\mathbf{BA} = \mathbf{0}$ can be dropped here.

42. Let $\mathbf{A}$ and $\mathbf{B}$ be projectors of the same order. Then show that $\mathbf{A} + \mathbf{B}$ is a projector iff $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{B})$ and $\mathcal{C}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{A})$.

43. Suppose the matrix $\mathbf{A}$ satisfies $\mathbf{A}^2 - (a+1)\mathbf{A} + a\mathbf{I} = \mathbf{0}$. Show that $\mathbf{A}$ is invertible. Take $\mathbf{A} = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ and find the inverse of $\mathbf{A}$.

44. Find the inverse of $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 1 & 1 & 1 & 2 & -1 \\ 0 & -1 & 2 & 1 & 3 \end{bmatrix}$ by using suitable partition.

45. Reduce the following square matrice to its *Hermite canonical form* (HCF) and hence determine the rank: $\begin{bmatrix} 2 & 4 & 3 & 1 \\ 1 & 2 & 5 & 0 \\ 3 & 6 & 0 & 5 \\ 4 & 8 & 1 & 2 \end{bmatrix}$

46. If $\mathbf{H}$ is a matrix in HCF, show that the non-null columns of $\mathbf{I}$ - $\mathbf{H}$ form a basis for $\mathcal{N}(\mathbf{H})$.

47. Prove or disprove: if $\mathbf{H_1}$ and $\mathbf{H_2}$ are $n \times n$ matrices in HCF, $\mathbf{H_1 H_2}$ is in HCF.

48. If a system $\mathbf{Ax} = \mathbf{b}$ of linear equations over IR has two different solutions $\mathbf{u}$ and $\mathbf{v}$, show that there exist infinitely many solutions.

49. Let $\mathbf{A}$ be an $m \times n$ matrix. Are the following statements true?

    a. $\mathbf{Ax} = \mathbf{b}$ has a solution for all $\mathbf{b}$ if $m < n$.

    b. $\mathbf{Ax} = \mathbf{b}$ has at most one solution if $m > n$.

50. Let $\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -3 & -1 & 1 & 0 \\ 2 & 0 & 2 & 1 \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. Find the solution of $\mathbf{Ly} = \mathbf{c}$ and using it find the solution of $\mathbf{Ax} = \mathbf{c}$ where $\mathbf{A} = \mathbf{LU}$.

51. Show that $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$ iff $\mathbf{A}$ is of full row rank.

52. Obtain a system $\mathbf{Ax} = \mathbf{b}$ for which $\begin{bmatrix} 1 + \alpha + 3\beta \\ 2 + 3\alpha \\ 1 + 8\beta \\ \alpha + 5\beta \end{bmatrix}$ is a general solution, where α and β are arbitrary scalars.

53. Let $\mathbf{A}$ be a real $m \times n$ matrix. Show that $\mathbf{A^T A x} = \mathbf{A^T b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$. Show also that if $\mathbf{Ax} = \mathbf{b}$ is consistent, then the solution sets of the two systems are the same.

54. Consider the lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ in $\mathbb{R}^2$. Let $\mathbf{A}$ be the 2×2 matrix with (a_i, b_i) as the i^{th} row, $i = 1, 2$. Let $\mathbf{B}$ be the 2×3 matrix with (a_i, b_i, c_i) as the i^{th} row, $i = 1, 2$. Show the following:

    a. The lines are identical iff $\rho(\mathbf{B}) = 1$.

    b. The lines are parallel but not identical iff $\rho(\mathbf{A}) = 1$ and $\rho(\mathbf{B}) = 2$.

    c. The lines intersect but are not identical iff $\rho(\mathbf{A}) = 2$.

55. Show that $\mathbf{c^T} \in \mathcal{R}(\mathbf{A})$ iff $\mathbf{c^T u} = \mathbf{0}$ for all $\mathbf{u} \in \mathcal{N}(\mathbf{A})$.

56. If $\mathbf{G_1}$ and $\mathbf{G_2}$ are two g-inverses of $\mathbf{A}$, show that $\alpha\mathbf{G_1} + (1-\alpha)\mathbf{G_2}$ is a g-inverse of $\mathbf{A}$ for all $\alpha \in \mathbb{F}$

57. Show that $\rho(\mathbf{A}) = tr(\mathbf{GA})$ if $\mathbf{G}$ is a g-inverse of $\mathbf{A}$.

58. Let $\mathbf{A}$ be an $m \times n$ matrix. Show that an $n \times m$ matrix $\mathbf{G}$ is a g-inverse of $\mathbf{A}$ iff $\rho(\mathbf{I} - \mathbf{GA}) = n - \rho(\mathbf{A})$.

59. If $\mathbf{R}(\mathbf{A}) \subseteq \mathbf{R}(\mathbf{X})$, show that $\rho(\mathbf{X}) = \rho(\mathbf{XA^-A}) + \rho(\mathbf{X}(\mathbf{I} - \mathbf{A^-A})) = \rho(\mathbf{A}) + \rho(\mathbf{X}(\mathbf{I} - \mathbf{A^-A}))$.

60. Find when the system:

$$\begin{aligned} &x + y + z = 1 \\ &\alpha x + \beta y + \gamma z = \epsilon \\ &\alpha^3 x + \beta^3 y + \gamma^3 z = \epsilon^3 \end{aligned}$$

is consistent and find a general solution whenever it is consistent.

References

[1] A. Ramachandra Rao and P Bhimasankaram. *Linear Algebra*. Hindustan Book Agency, 2nd edition, 2000.