# Time Series

Sudipta Das

Assistant Professor,
Department of Computer Science,
Ramakrishna Mission Vivekananda Educational & Research Institute

# Outline I

1. Stationary Process
   - Strict Stationary Process
   - (Weak) Stationary Process
   - Linear Process
     - Revisiting ARMA Process

# Stationary Process I

- Going beyond *i.i.d* stochastic process (time series)
- Stationary Process
    - Strict Stationary Process
    - Weak Stationary Process

# Strict Stationary Process I

- $\{X_t\}$ is a strictly stationary process if

$$(X_{i_1}, X_{i_2}, \dots, X_{i_n})' \stackrel{D}{=} (X_{i_1+h}, X_{i_2+h}, \dots, X_{i_n+h})'$$

for all integers $h$ and $n \geq 1$.

  - i.e.

$$f_{X_{i_1}, X_{i_2}, \dots, X_{i_n}}(x_1, \dots, x_n) = f_{X_{i_1+h}, X_{i_2+h}, \dots, X_{i_n+h}}(x_1, x_2 \dots, x_n)$$

for all integers $h$ and $n \geq 1$.

# Strict Stationary Process II

- Properties of a Strictly Stationary Process $\{X_t\}$ :
  - The random variables $X_t$ are identically distributed
    - Not necessarily independent
  - An *i.i.d* sequence is also strictly stationary
  - $(X_t, X_{t+h})' \overset{D}{=} (X_1, X_{1+h})'$ for all integers $t$ and $h$.

# (Weak) Stationary Process I

- $\{X_t\}$ is a (weakly) stationary process if all of the following three conditions hold
  1. $E[X_t^2] < \infty$
     - Finite second order moment
  2. $\mu_X(t)$ is independent of $t$,
     - where $\mu_X(t)$ is the **mean function** of $\{X_t\}$ and is defined as
       $$\mu_X(t) = E(X_t)$$
  3. $\gamma(t+h, t)$ is independent of $t$ for each $h$,
     - where $\gamma(t+h, t)$ is the **covariance function** of $\{X_t\}$ and is defined as

$$\begin{aligned}\gamma(t+h, t) &= \mathrm{Cov}(X_{t+h}, X_t) \\ &= E\left[(X_{t+h} - \mu_X(t+h))\,(X_t - \mu_X(t))\right]\end{aligned}$$

- Remarks:
  - If $\{X_t\}$ is strictly stationary and $E[X_t^2] < \infty$ for all $t$, then $\{X_t\}$ is also weakly stationary
  - In this course, whenever we use the term stationary we shall mean weakly stationary, unless we specifically indicate otherwise.
  - We use the term covariance function with reference to a stationary time series $\{X_t\}$ we shall mean the function $\gamma_X$ of one variable defined by

$$\gamma_X(h) := \gamma_X(h, 0) = \gamma_X(t + h, t)$$

  - The function $\gamma_X(\cdot)$ will be referred to as the **autocovariance function** (ACVF) of $X_t$ and $\gamma_X(h)$ as its value at *lag h*.

# (Weak) Stationary Process III

- Formally, the **autocovariance function** (ACVF) of $\{X_t\}$ at lag $h$ is defined as

$$\gamma_X(h) = E\left[(X_{t+h} - \mu_X)(X_t - \mu_X)\right]$$

  - Note that

$$\gamma_X(0) \geq 0$$

    and

$$\gamma_X(h) = \gamma_X(-h)$$

# (Weak) Stationary Process IV

- The **autocorrelation function** (ACF) of $\{X_t\}$ at lag $h$ is defined as

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)}$$

- Basic Properties of $\rho(\cdot)$:
  - $|\rho(h)| \leq 1$
  - $\rho(\cdot)$ is even, i.e., $\rho(h) = \rho(-h)$ for all $h$.

# (Weak) Stationary Process: Examples I

- Some Elementary Stationary processes
  1. iid noise: $\{X_t\} \sim IID(0, \sigma^2)$, with $E(X_t^2) = \sigma^2 < \infty$
     - ACVF:
     $$\gamma_X(t+h, t) = \gamma_X(h) = \begin{cases} \sigma^2, & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases}$$
     - ACF
     $$\rho_X(t+h, t) = \rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

2. White Noise: $\{X_t\} \sim WN(0, \sigma^2)$
   - It's a sequence of uncorrelated random variables, each with zero mean and variance $\sigma^2$,
   - ACVF:

$$\gamma_X(h) = \begin{cases} \sigma^2, & \text{if } h = 0, \\ 0 & \text{if } h \neq 0. \end{cases}$$

   - ACF

$$\rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0, & \text{if } h \neq 0. \end{cases}$$

# (Weak) Stationary Process: Examples III

Note that every $IID(0, \sigma^2)$, sequence is $WN(0, \sigma^2)$ but the converse is not true, e.g.
Let $\{Z_t\}$ be iid$\sim N(0, 1)$ noise and define

$$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even,} \\ (Z_{t-1}^2 - 1)/\sqrt{2} & \text{if } t \text{ is odd.} \end{cases}$$

Here, that $\{X_t\}$ is $WN(0, 1)$ but not $iid(0, 1)$.

3. First-order moving average or MA(1) process:

$$X_t = Z_t + \theta Z_{t-1}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta$ is real valued constant

- $EX_t = 0$
- ACVF

$$\gamma_X(h) = \begin{cases} \sigma^2(1+\theta^2), & \text{if } h = 0, \\ \sigma^2\theta, & \text{if } h = \pm 1 \\ 0, & \text{if } |h| \geq 2. \end{cases}$$

- ACF

$$\rho_X(h) = \begin{cases} 1, & \text{if } h = 0, \\ \theta/(1+\theta^2), & \text{if } h = \pm 1 \\ 0, & \text{if } |h| \geq 2. \end{cases}$$

4. First-order autoregressive or AR(1) process:

$$X_t = \phi X_{t-1} + Z_t, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2), |\phi| < 1$ and $Z_t$ is uncorrelated with $X_s$ for each $s < t$

- $EX_t = 0$
- ACVF

$$\gamma(h) = \phi^{|h|} \frac{\sigma^2}{1 - \phi^2}$$

- ACF

$$\rho(h) = \phi^{|h|}$$

5. First-order ARMA or ARMA(1,1) process:

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2), |\phi| < 1, Z_t$ is uncorrelated with $X_s$ for each $s < t$ and $\phi + \theta \neq 0$

- $EX_t = 0$
- ACVF

$$\gamma_X(h) = \begin{cases} \sigma^2 \left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right], & \text{if } h = 0, \\ \sigma^2 \left[\theta + \phi + \frac{(\theta+\phi)^2 \phi}{1-\phi^2}\right], & \text{if } h = \pm 1 \\ \phi^{|h|-1} \gamma_X(1), & \text{if } |h| \geq 2. \end{cases}$$

- Is Random Walk a stationary process?

$$X_t = X_{t-1} + Z_t,$$

where $Z_t \sim IID(\mu, \sigma^2)$

# (Weak) Stationary Process: Examples VIII

- NO
  - $\mu_X(t) = \mu t$
  - $Cov(X_m, X_n) = \min(m, n) \times \sigma^2$

# (Weak) Stationary Process: Examples IX

- Three higher order stationary processes
  1. $q$-order moving average or MA($q$) process:

     $$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, t = 0, \pm 1, \ldots,$$

     where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \ldots, \theta_q$ are real valued constants
  2. $p$-order autoregressive or AR($p$) process:

     $$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, t = 0, \pm 1, \ldots,$$

     where $\{Z_t\} \sim WN(0, \sigma^2)$, $Z_t$ is uncorrelated with $X_s$ for each $s < t$ and all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ lie outside the unit circle.

3. ARMA$(p, q)$ process:

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, with all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ lie outside the unit circle and the polynomials $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ and $(1 + \theta_1 z + \ldots + \theta_q z^q)$ have no common factors.

# Linear Process I

- Linear processes: It includes the class of autoregressive moving-average (ARMA) process,
- Definition: The process $\{X_t\}$ is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

for all $t$, where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$

# Linear Process II

- Alternate representation by backward shift operator:

$$X_t = \Psi(B)Z_t,$$

where $\Psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$

  - The operator $\Psi(B)$ can be thought of as a linear filter, which when applied to the white noise "input" series $\{Z_t\}$ produces the "output" $\{X_t\}$

- Note: every second-order stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component.

# Linear Process III

Remarks

1. The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ (absolute summability) ensures that the infinite sum

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j},$$

converges (with probability one)

- Sketch of proof:-

Let $X_t^n = \sum_{j=-n}^{n} \psi_j Z_{t-j}$, and for small $\epsilon > 1$, define

$$A_n(\epsilon) = \left\{ |X_t^n - X_t| > \epsilon \right\} = \left\{ \left| \sum_{|j|>n} \psi_j Z_{t-j} \right| > \epsilon \right\}.$$

By Chebyshev's inequality,

$$P(A_n) \leq E\left[ \left| \sum_{|j|>n} \psi_j Z_{t-j} \right|^2 \right] / \epsilon^2.$$

# Linear Process V

Thus,

$$
\begin{aligned}
\sum_{n=1}^{\infty} P(A_n) &\leq \frac{1}{\epsilon^2} \sum_{n=1}^{\infty} E\left|\sum_{|j|>n} \psi_j Z_{t-j}\right|^2 \\
&< \sum_{n=1}^{\infty} E\left|\sum_{|j|>n} \psi_j Z_{t-j}\right|^2 \\
&= \sum_{n=1}^{\infty} E\left|\sum_{|i|>n} \sum_{|k|>n} \psi_i \psi_k \left(Z_{t-i} Z_{t-k}\right)\right| ; \text{ as } |ab| = |a||b| \\
&\leq \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i||\psi_k| E\left|\left(Z_{t-i} Z_{t-k}\right)\right|\right] ; \text{ by triangular inequality} \\
&\leq \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i||\psi_k| E\left|\left(Z_{t-i}^2\right)^{1/2}\right| E\left|\left(Z_{t-k}^2\right)^{1/2}\right|\right] ; \text{ by Cauchy-Schwarz inequality} \\
&\leq \sigma^2 \sum_{n=1}^{\infty} \left[\sum_{|i|>n} \sum_{|k|>n} |\psi_i||\psi_k|\right] ; \text{ Stationarity} \\
&= \sigma^2 \sum_{n=1}^{\infty} \left[\sum_{|j|>n} |\psi_j|\right]^2 < \infty; \text{ absolute summability}
\end{aligned}
$$

Therefore, by Borel Cantelli Lemma

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P\left(A^{(S)}\right) = 0, \text{ where } A^{(S)} = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m = \limsup_n A_n$$

- Event $A^{(S)}$ is called the lim sup event of the infinite sequence $\{A_n\}$.
- Event $A^{(S)}$ occurs if and only if for all $n \geq 1$, there exists an $m \geq n$ such that $A_m$ occurs,
- equivalently, Event $A^{(S)}$ occurs if and only if infinitely many of the $A_n$ occur.

By definition of limit, $\omega \in \left\{\lim_n X_n = X\right\}$ if and only if for all $u \geq 1$ there exists $n \geq 1$ such that for every $m \geq n$, $|X_m(\omega) - X(\omega)| \leq \frac{1}{u}$. Equivalently, it holds if and only if

$$\omega \in \cap_{u=1}^{\infty} \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} \left[A_m\left(\frac{1}{u}\right)\right]^c = \left(\cup_{u=1}^{\infty} \limsup_n A_n\left(\frac{1}{u}\right)\right)^c.$$

Thus,

$$\begin{aligned} P\left(\omega : \lim_n X_t^n(\omega) = X_t(\omega)\right) = P\left(\left(\cup_{u=1}^{\infty} \limsup_n A_n(1/u)\right)^c\right) &= 1 - P\left(\cup_{u=1}^{\infty} \limsup_n A_n(1/u)\right) \\ &\geq 1 - \sum_{u=1}^{\infty} P\left(\limsup_n A_n(1/u)\right) = 1 \end{aligned}$$

Hence, $X_t^n \overset{a.s.}{\to} X_t$.

# Linear Process VII

2. The condition $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ ensures that $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ therefore, the series $\sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$ converges in mean square to $X_t$, i.e.,

$$X_t^n = \sum_{j=-n}^{n} \psi_j Z_{t-j} \stackrel{m.s.}{\rightarrow} X_t$$

# Linear Process VIII

3. In generally, let $\{Y_t\}$ be a stationary process with mean 0 and covariance function $\gamma_Y$. If $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the process

$$X_t = \Psi(B)Y_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j},$$

is also stationary with mean 0 and autocovariance function as

$$\begin{aligned}
\gamma_X(h) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E[Y_{t-j} Y_{t+h-k}] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h-k+j) \\
&= \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2, \ \text{(if } X_t \text{ is linear)}
\end{aligned}$$

# Linear Process IX

4. The filters of the form $\alpha(B) = \sum_{j=-\infty}^{\infty} \alpha_j B^j$ and $\beta(B) = \sum_{j=-\infty}^{\infty} \beta_j B^j$ with absolutely summable coefficients can be applied successively to a stationary series $\{Y_t\}$ to generate a new stationary series

$$
\begin{aligned}
W_t &= \sum_{j=-\infty}^{\infty} \alpha_j \left( \sum_{k=-\infty}^{\infty} \beta_k Y_{(t-j)-k} \right) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_j \beta_k Y_{(t-j)-k} \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \alpha_{j-k} \beta_k Y_{t-j}, \text{replacing } j \text{ by } j-k \\
&= \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j}
\end{aligned}
$$

- Therefore, $\psi_j = \sum_{k=-\infty}^{\infty} \alpha_{j-k}\beta_k = \sum_{k=-\infty}^{\infty} \alpha_k \beta_{j-k}$
- Alternate form $W_t = \alpha(B)\beta(B)Y_t = \beta(B)\alpha(B)Y_t = \psi(B)Y_t$

# Linear Process X

- Forms of (stable) linear process:
  - Causal: A linear process $\{X_t\}$ is causal if $X_t$ can be expressed in terms of the current and past values $Z_s, s \leq t$,

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

  - Invertible: A linear process $\{X_t\}$ is invertible if $Z_t$ can be expressed in terms of the current and past values $X_s, s \leq t$,

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}.$$

# Revisiting ARMA Proces I

- Let $X_t$ be an ARMA(p,q) process

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$$

  where $\{Z_t\} \sim WN(0, \sigma^2)$, and the polynomials $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ and $(1 + \theta_1 z + \ldots + \theta_q z^q)$ have no common factors.

- Representing $X_t$ as linear process

$$X_t = (1 - \phi_1 B - \cdots - \phi_p B^p)^{-1}(1 + \theta_1 B + \cdots + \theta_q B^q) Z_t$$

# Revisiting ARMA Proces II

- Condition for stability of $X_t$:

  The coefficients of linear process expression of $X_t$ $\left(= \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}\right)$ are absolutely summable.

- Equivalent Condition:

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \text{ for all } |z| = 1$$

  - No roots of $\phi(z)$ on the unit circle

# Revisiting ARMA Proces III

- Example: AR(1)

$$X_t - \phi X_{t-1} = Z_t$$

  - if $|\phi| < 1$, $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ is stable
  - if $|\phi| > 1$, $X_t = \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$ is stable

# Revisiting ARMA Proces IV

- Condition for causality of $X_t$:
  Process $X_t$ can be expressed in terms of the current and past values $Z_s, s \leq t$, (i.e., $X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$)
- Equivalent Condition

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0 \text{ for all } |z| < 1$$

  - No roots of $\phi(z)$ inside the unit circle

# Revisiting ARMA Proces V

- Example: AR(1)

$$X_t - \phi X_{t-1} = Z_t$$

  - if $|\phi| < 1$, $X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}$ is stable and causal
  - if $|\phi| > 1$, $X_t = \sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}$ is stable but non-causal

# Revisiting ARMA Proces VI

- Condition for invertibility of $X_t$:
  Process $Z_t$ can be expressed in terms of the current and past values $X_s, s \leq t$, (i.e., $Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$)
- Equivalent Condition

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \neq 0 \text{ forall } |z| < 1$$

  - No roots $\theta(z)$ inside the unit circle