9/13

Last class:

$$Y = aX + b.$$

$$\boxed{f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).}$$

If $X \sim$ Normal, then $Y \sim$ Normal

Suppose $X \sim \text{Gamma}(\alpha, \lambda)$

$$\boxed{f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}}, \quad x \geq 0.$$

$a > 0$

$b \geq 0$

$$f_Y(y) = \frac{1}{a} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{y-b}{a}\right)^{\alpha-1} e^{-\lambda\left(\frac{y-b}{a}\right)}.$$

$\boxed{b \neq 0 : \; Y \not\sim \text{Gamma}.}$

However, if $\boxed{b = 0}$

$$f_Y(y) = \frac{1}{a} \cdot \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \left(\frac{y}{a}\right)^{\alpha-1} e^{-\lambda\left(\frac{y}{a}\right)}$$

$$= \left(\frac{\lambda}{a}\right)^\alpha \cdot \frac{1}{\Gamma(\alpha)} \cdot y^{\alpha-1} \cdot e^{-\lambda\left(\frac{y}{a}\right)}$$

Hence $\boxed{Y \sim \text{Gamma}\left(\alpha, \frac{\lambda}{a}\right).}$

<u>Cauchy distribution :</u> $X \sim \text{Cauchy}(s, t)$

pdf $\longrightarrow$ $f(x) = \dfrac{1}{s\pi\left(1 + \left(\frac{x-t}{s}\right)^2\right)}$, $x \in \mathbb{R}$

[$t = 0$, $s = 1$: $f(x) = \dfrac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$

Standard Cauchy distribution]

$Y = aX + b$.

$$f_Y(y) = \frac{1}{a} \cdot \frac{1}{s\pi\left(1 + \left(\frac{\frac{y-b}{a} - t}{s}\right)^2\right)}$$

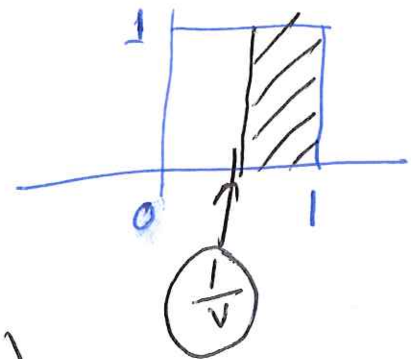
$$= \frac{1}{as\pi\left(1 + \left(\frac{y-b-at}{as}\right)^2\right)}$$

$Y \sim \text{Cauchy}(as, b + at)$

Today:

Example: Let U : uniform random variable on $[0,1]$

$$V = \frac{1}{U}$$



What is the pdf of V.

Solution: $\boxed{v \in [1, \infty)}$ $(v \geq 1)$

$$F_V(v) = P(V \leq v)$$

big, little

$$= P\left(\frac{1}{U} \leq v\right)$$

$$= P\left(U \geq \frac{1}{v}\right) \quad (\because v > 0)$$

$$= \int_{\frac{1}{v}}^{1} 1 \, dx = 1 - \frac{1}{v}$$

pdf of U

$$F_V(v) = 1 - \frac{1}{v}$$

$$\boxed{f_V(v) = \frac{1}{v^2}}, \quad 1 \leq v < \infty$$

~~Theorem:~~

Consider a random variable X

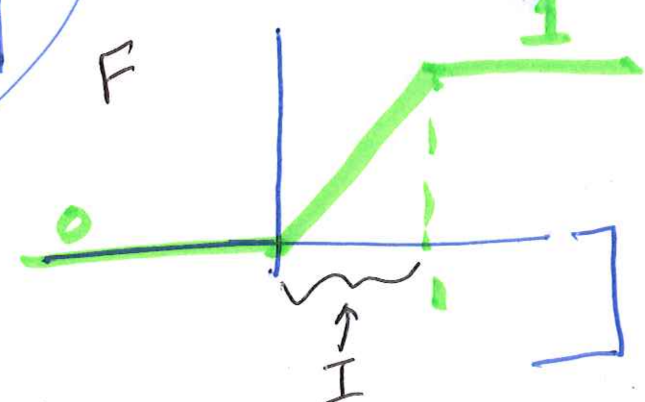
with pdf f

cdf F,

where F is strictly increasing on some interval I, $F = 0$, to the left of I

$F = 1$, to the right of I

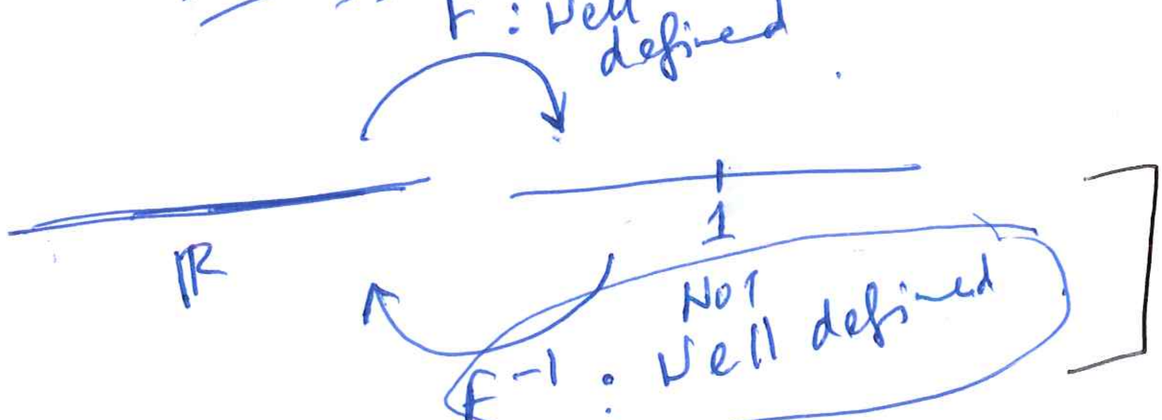
$\boxed{\text{Then } F^{-1}(x) \text{ exists for } x \in I}$



Example:

$F(x) = 1$, for all x.

~~$F^{-1}(1) =$~~ $F^{-1}(1) = \mathbb{R}$.



Theorem: Let $Z = F(X)$

Then $Z \sim \text{Unif}([0,1])$

Proof:

$$\boxed{F_Z(z)} = P(Z \le z)$$

$$= P(F(X) \le z)$$

$$= P(\underbrace{X \le F^{-1}(z)}_{\text{cdf of } X})$$

$Z \in I$

Notation

$$\boxed{P(X \le *) = F(*)}$$

$$= F(F^{-1}(z))$$

$$= \boxed{z}$$

Hence z in $[0,1]$

z can NOT be more than 1 as it is $P(Z \le z)$

$$\boxed{f_Z(z) = 1,} \quad ~~0 \le z \le 1~~ \quad 0 \le z \le 1$$

~~$\int_{-\infty}^{\infty} f_Z(z)\, dz = 1$~~

~~If the function starts @ $z = 0$, then we must have $0 \le z \le 1$~~

F: function that is mentioned earlier except do not think this as a cdf of X.

Theorem: Let U be uniform on $[0,1]$.

$$X = F^{-1}(U)$$

Then the cdf of X is F.

Proof:

$$\boxed{P(X \le x)} = P(F^{-1}(U) \le x)$$
$$= P(U \le F(x))$$
$$= \boxed{F(x)}$$

cdf of U:

$$\boxed{P(U \le x) = x, \quad x \in [0,1]}$$

$$F(z) = z^2, \quad 0 \le z \le 1$$
$$f_z(z) = 2z, \quad 0 \le z \le 1$$

Example: $U \sim Unif([0,1])$

$V = 1 - U$

$$\boxed{P(V \le v)} = P(1 - U \le v) = P(U \ge 1 - v)$$
$$= 1 - \boxed{P(U < 1 - v)}$$
$$= 1 - (1 - v) = \boxed{v}$$

cdf of Unif dbn

$V \sim Unif([0,1])$

Example:

Suppose an experiment is involved with simulation of exponential distributions.

Question: If we have access to a uniform random number generator, how do we generate exponential random numbers.

Solution: cdf of exponential distribution

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$

$$y = 1 - e^{-\lambda x}$$

$$\Rightarrow e^{-\lambda x} = 1 - y$$

$$\Rightarrow -\lambda x = \ln(1-y)$$

$$\Rightarrow x = -\frac{1}{\lambda}\ln(1-y)$$

$\boxed{x \leftrightarrow y}$

$$y = \boxed{-\frac{1}{\lambda}\ln(1-x) = F^{-1}(x)}$$

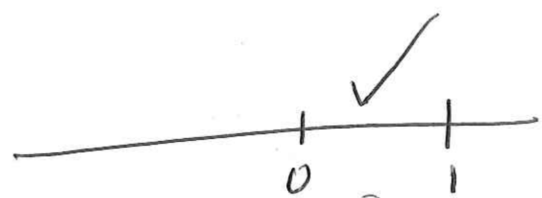
So, $E \sim Exp(\lambda)$, then: $\boxed{E} = -\frac{1}{\lambda}\ln(1-U) = \boxed{-\frac{1}{\lambda}\ln V}$, $V \sim Unif[0,1]$

Example: Suppose that X has density
$$f(x) = \begin{cases} c x^2, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(#40, Problem set 2)

- Find c:

$$\int_{-\infty}^{\infty} f(x) = 1$$



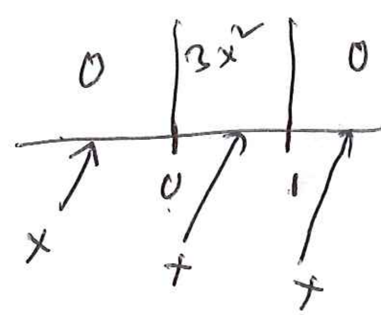
$$\Rightarrow \int_{-\infty}^{0} \cancelto{0}{0}\, dx + \int_{0}^{1} c x^2\, dx + \int_{1}^{\infty} \cancelto{0}{0}\, dx = 1$$

$$c \cdot \frac{1}{3} = 1$$

$$\Rightarrow \boxed{c = 3}$$

- Find the cdf of X:

$$F_X(x) = \int_{-\infty}^{x} f(x)\, dx$$



$$F_{X}(x) = \begin{cases} 0, & x \le 0 \\ \int_0^x 3x^2\,dx = x^3, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

- What is $P(0.1 \le X \le 0.5)$

$$= \cancel{P(0} \; F(0.5) - F(0.1)$$

$$= (0.5)^3 - (0.1)^3$$

$$= \boxed{0.124}$$

Example (Problem #47 Problem Set 2)

If $\alpha > 1$, show that the gamma density has a maximum at $\frac{\alpha - 1}{\lambda}$

Solution:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x \ge 0$$

$$f'(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \left[(\alpha-1) x^{\alpha-2} e^{-\lambda x} - \lambda e^{-\lambda x} \cdot x^{\alpha-1} \right]$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \left[(\alpha-1) x^{\alpha-2} - \lambda x^{\alpha-1} \right] e^{-\lambda x}$$

So, $f'(x) = 0$ if

$$(\alpha - 1)x^{\alpha-2} - \lambda x^{\alpha-1} = 0.$$

$$\Rightarrow \quad x = \frac{\alpha - 1}{\lambda}$$

Check it: $f''\left(\frac{\alpha-1}{\lambda}\right) < 0$.

So, $x = \frac{\alpha - 1}{\lambda}$ is the value that makes $f(x)$ maximum.

Example
Problem Set 2
Problem #52

Suppose that in a certain population, individual's heights are approximated $\sim N(70, 3^2)$

[in inches]

(a) What proportion of the population is over 6ft tall? (72 in)

Solution:

$$P(X > 72)$$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{72-\mu}{\sigma}\right)$$

$$= P\left(\frac{X-70}{3} > \frac{72-70}{3}\right)$$

Std. Normal. Z

$$= P(Z > 0.67)$$

$$= 1 - P(Z \leq 0.67)$$

$$= 1 - \Phi(0.67) \quad [\Phi : \text{cdf of } Z]$$

standard table

$$= 1 - 0.7486$$

$$= \boxed{0.2514}$$

(b) What is the distribution of heights if they are expressed in centimeters?

Solution:

$\cancel{Y = aX}$

$\boxed{1 \text{ in} = 2.54 \text{ cm}}$

$\boxed{Y = 2.54 X}$

$X \sim N(70, 3^2)$

Question: What is the distribution for Y ?

In general: $Y = aX + b, \quad a > 0$

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{1}{2.54} \cdot \frac{1}{3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\frac{y}{2.54} - 70}{3}\right)^2}$$

$$= \frac{1}{(2.54 * 3)\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y - 70*2.54}{2.54*3}\right)^2}$$

Hence $Y \sim N\left(70 * 2.54, (2.54 * 3)^2\right)$

$$\Rightarrow \boxed{Y \sim N(177.8, 58.06)}$$