Rest

Tests Concerning a Univariate Normal ~~Distribution~~ Population

Suppose, we have a population characterised by $N(\mu, \sigma^2)$ distribution. One may then consider the following testing problems.

(1) To test $H_0: \mu = \mu_0$

against $H_{11}: \mu > \mu_0$

$H_{12}: \mu < \mu_0$

$H_{13}: \mu \neq \mu_0$,

where, μ_0 is some specified value of μ.

For the purpose of the above testing we draw a random sample of size 'n' from the given population distribution. Let $x_1, x_2, \ldots, x_n$ be the sample observations. Let us write

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ (sample mean)}$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \text{ (sample variance).}$$

Here, we shall be inclined to reject H_0 iff the ~~as~~ observed value of $\bar{x}$ is too large or too small compared to μ_0. But whether the difference between $\bar{x}$ & μ_0 is large is to be judged in comparison with the standard error of $\bar{x}$ because a large difference when the standard error is small, may not be considered as so large when the standard error is too large.

Here, one may consider two cases.

Case-I: σ^2 is Known

In this case, one may consider

$$t = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma}$$

as our test statistic.

We know that, under H_0, $\bar{x} \sim N(\mu_0, \frac{\sigma^2}{n})$. As such, under H_0, t follows a standard normal distribution. The test procedure may, then, be given as below:

a) We reject H_0 against H_{11} iff the observed value of $\bar{x}$ is too large compared to μ_0, ie iff $t > \tau_\alpha$, where τ_α is the upper α-point of a standard normal distribution.

b) We reject H_0 against H_{12} iff the observed value of $\bar{x}$ is too ~~larg~~ small compared to μ_0 ie iff $t < -\tau_\alpha$, $-\tau_\alpha$ being the lower α-point of a standard normal distribution.

c) We reject H_0 against H_{13} iff the observed value of $\bar{x}$ is too large or too small compared to μ_0 ie iff $t > \tau_{\alpha/2}$ or $t < -\tau_{\alpha/2}$, ie iff $|t| > \tau_{\alpha/2}$, $\tau_{\alpha/2}$ being the upper $\alpha/2$ point of a $N(0,1)$ distribution.

In all the above three situations α is the desired level of significance.

Case - II : 'σ' is unknown [Student's 't'-distn]

Since, σ^2 is unknown, we replace it by its unbiased estimator s^2 & consider as our test statistic $t = \frac{\sqrt{n}(\bar{x} - \mu_0)}{s}$

Now, $\frac{\sqrt{n}(\bar{x}-\mu_0)}{s} = \frac{\sqrt{n}(\bar{x}-\mu_0)}{\sigma} \Big/ \sqrt{\frac{(n-1)s^2}{\sigma^2} \Big/ (n-1)}$ which is of the form

$\tau \Big/ \sqrt{\chi^2_{n-1} / (n-1)}$, where, τ is a ~~std~~ standard normal variate & χ^2_{n-1} is a χ^2 variate with $(n-1)$ d.f.

As such, under H_0, t follows a 't' distribution with $(n-1)$ d.f.

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a) We reject H_0 against H_{11} iff $t > t_{\alpha, n-1}$; $t_{\alpha, n-1}$ being the upper α-point of a t distribution with d.f. $(n-1)$.

b) We reject H_0 against H_{12} iff the observed value of 't' is less than $-t_{\alpha, n-1}$; $-t_{\alpha, n-1}$ being the lower α-point of a t-distribution with $(n-1)$ d.f.

c) We reject H_0 against H_{10} iff $t > t_{\alpha/2, n-1}$ or $t < -t_{\alpha/2, n-1}$ ie iff $|t| > t_{\alpha/2, n-1}$, $t_{\alpha/2, n-1}$ being the upper $\alpha/2$ point of a t distribution with $(n-1)$ d.f.

Here, in each case, 'α' is the desired level of significance.

<u>Note</u>: Here, the t-test performed is called a student's 't'-test.

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To test $H_0 : \sigma = \sigma_0$ (or, $\sigma^2 = \sigma_0^2$)

against $H_{11} : \sigma > \sigma_0$ (or, $\sigma^2 > \sigma_0^2$)

$H_{12} : \sigma < \sigma_0$ (or, $\sigma^2 < \sigma_0^2$)

$H_{13} : \sigma \neq \sigma_0$ (or, $\sigma^2 \neq \sigma_0^2$).

Here, σ_0 is a specified value of σ.

Let us draw a random sample of size 'n' from the given distribution.

Let $x_1, x_2, \ldots, x_n$ be the sample observations. Let us write

$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ (Sample mean)

$s_0^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)^2$ [Sample variance when μ is known]

$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$ [Sample variance when μ is unknown].

[Rough: $E(s^2) = E\left[\frac{1}{n}\sum_{i=1}^{n} (x_i - \mu)^2\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \mu^2\right] = \frac{1}{n}\sum_{i=1}^{n} E(x_i^2) - E(\mu^2)$

$= \cancel{\frac{1}{n}\sum_{i=1}^{n} V(x_i)} = \frac{1}{n}\sum_{i=1}^{n}\left[V(x_i) + \{E(x_i)\}^2\right] - \mu^2$

$= \frac{1}{n}\sum_{i=1}^{n}(\sigma^2 + \mu^2) - \mu^2 = \sigma^2 + \mu^2 - \mu^2 = \sigma^2$.

Hence, $s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$ is an unbiased estimator of σ^2 when μ is unknown.]

Since the hypothesis concerns the population variance, it is natural to assume that the test should be based on the sample variance.

Case → I: μ is known:

In this case, the sample variance is appropriately taken as s_0^2. In fact, we take as our test statistic $t = \frac{n s_0^2}{\sigma_0^2}$, ~~which~~ which under H_0 follows a χ^2 distribution with 'n' d.f., as

$$t = \frac{n s_0^2}{\sigma_0^2} = \sum_{i=1}^{n} \left(\frac{x_i - \mu}{\sigma_0}\right)^2$$

is the sum of squares of 'n' independent $N(0,1)$ variates under H_0.

The test procedure is, then, given as below:

a) We reject H_0 against H_{11} iff $t > \chi^2_{\alpha;n}$, where $\chi^2_{\alpha;n}$ is the upper α-point of a χ^2 distribution with 'n' d.f.

b) We reject H_0 against H_{12} iff $t < \chi^2_{1-\alpha;n}$, where $\chi^2_{1-\alpha;n}$ is the lower α-point of a χ^2 distribution with 'n' d.f.

c) We reject H_0 against H_{13} iff $t > \chi^2_{\alpha/2;n}$ or, at $t < \chi^2_{1-\alpha/2;n}$, where $\chi^2_{\alpha/2;n}$ & $\chi^2_{1-\alpha/2;n}$ are respectively the upper & lower $\frac{\alpha}{2}$ points of a χ^2 distribution with n d.f., where 'α' is a desired level of significance.

Case → II : 'μ' is unknown :

In this case, the sample variance ~~is~~ is appropriately taken as

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2 .$$

We know that, $\sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma}\right)^2 \sim \chi^2_{n-1}$. As such, under H_0, $\sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_0}\right)^2 \sim \chi^2_{n-1}$.

Hence, we ~~need~~ may take as our test statistic

$$t = \sum_{i=1}^{n} \left(\frac{x_i - \bar{x}}{\sigma_0}\right)^2 = \frac{(n-1)s^2}{\sigma_0^2}$$ which under H_0 follows a χ^2_{n-1} distribution.

The test procedure is then given as below :

a) We reject H_0 against H_{11} iff $t > \chi^2_{\alpha; n-1}$, where, $\chi^2_{\alpha; n-1}$ is the upper α-point of a χ^2_{n-1} distribution.

b) We reject H_0 against H_{12} iff $t < \chi^2_{1-\alpha; n-1}$, where, $\chi^2_{1-\alpha; n-1}$ is the

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lower α-point of a χ^2_{n-1} distribution.

c) We reject H_0 against H_{13} iff $t > \chi^2_{\frac{\alpha}{2};n-1}$ or $t < \chi^2_{1-\frac{\alpha}{2};n-1}$,

where, $\chi^2_{\frac{\alpha}{2};n-1}$ & $\chi^2_{1-\frac{\alpha}{2};n-1}$ are the ~~lower~~ upper & lower α-point of a χ^2_{n-1} distribution respectively.