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Independent Random Variables

Definition: Random variables $X_1, X_2, \ldots, X_n$ are said to be independent if their joint cdf factors into the product of their marginal cdfs.

$$\boxed{F(x_1, x_2, \ldots, x_n) = F_{X_1}(x_1) F_{X_2}(x_2) \ldots F_{X_n}(x_n)}$$

for all $x_1, x_2, \ldots, x_n$.

[**Reminder:** $F(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n)$

$F_{X_1}(x_1) = P(X_1 \le x_1)$

$F_{X_2}(x_2) = P(X_2 \le x_2)$

...]

Suppose X, Y are two continuous random variables. Then,

$$\boxed{f(x,y) = f_X(x) f_Y(y)} \quad \ldots (1)$$

provided X and Y are independent

[$f(x,y)$ = joint density function
$f_X(x)$ = marginal density for X, $f_Y(y) = \ldots$]

(⇒) (If independent then ① holds)

If X and Y are independent then by definition:

~~$F(x_1, x_2) = F_X$~~

$$F(x,y) = F_X(x) F_Y(y)$$

$$\frac{\partial F(x,y)}{\partial x} = F_Y(y) \frac{\partial F_X(x)}{\partial x} = f_X(x)$$

$$\frac{\partial^2 F(x,y)}{\partial y \, \partial x} = f_X(x) \frac{\partial F_Y(y)}{\partial y} = f_Y(y)$$

So,

$$\boxed{f(x,y) = f_X(x) f_Y(y)}$$

(Done)

- A if and only if B

$A \Rightarrow B$

$B \Rightarrow A$

- $\boxed{\text{IFF}}$
- $\Leftrightarrow$

(⇐) (If ① holds then independent)

$$F(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y) \, dy \, dx$$

$$= \int_{-\infty}^{x} \boxed{\int_{-\infty}^{y}} f_X(x) \boxed{f_Y(y) \, dy} \, dx$$

$$= \left(\int_{-\infty}^{x} f_X(x)\,dx\right)\left(\int_{-\infty}^{y} f_Y(y)\,dy\right)$$

$$= F_X(x) \quad F_Y(y)$$

Hence, $F(x,y) = F_X(x)\,F_Y(y)$

$\therefore$ by 'definition, $X \perp Y$ (Done)

$X \perp Y$,
$X \perp\!\!\!\perp Y$

Remark: It can also be shown that if g and h are "good" (measurable functions) functions, then $Z = g(X)$ and $W = h(Y)$ are independent, provided $X \perp Y$.

Bivariate Normal density: $(\rho = 0)$

$$\boxed{f(x,y)} = \frac{1}{2\pi\,\sigma_X\sigma_Y}\exp\left(-\frac{1}{2}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)\right)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2} = \boxed{f_X(x) f_Y(y)}$$

If X and Y follow a bivariate normal dbn and $\rho = 0$, then X and Y ~~follow~~ are independent.

Example: Suppose that a communication network has the property that if two information arrive within time T, of each other, they "collide" and then have to be retransmitted.

If the times of arrival of the two information are independent ~~a = 10 sec~~, and uniform on $[0, T]$ what is the probability that they collide?

Solution: Arrival times of two information are T_1 and T_2

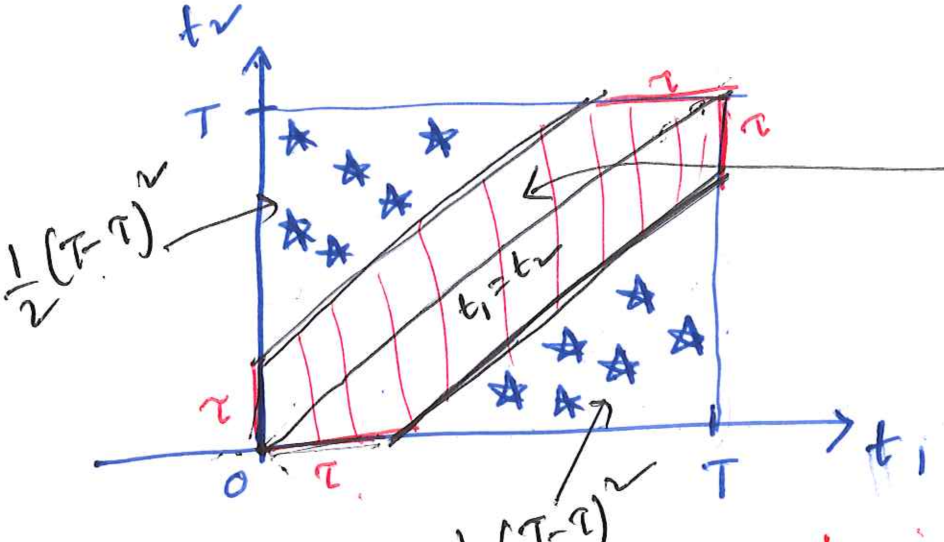
$$T_1 \sim Unif([0,T]) \qquad f_{T_1}(x) = \frac{1}{T}, \ x \in [0,T]$$

$$T_2 \sim Unif([0,T]) \qquad f_{T_2}(x) = \frac{1}{T}, \ x \in [0,T]$$

So, the joint density of T_1, T_2 is:

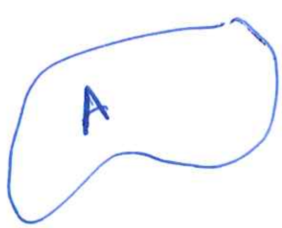
$\cancel{f(x,y)}$ $\because T_1 \perp T_2$

$$f(t_1, t_2) = f_{T_1}(t_1)\, f_{T_2}(t_2) = \frac{1}{T}\cdot\frac{1}{T} = \frac{1}{T^2}$$



For this region ~~that~~ t_1 and t_2 are always $< \tau$ distance

We are really trying to find the area of the unshaded region, and integrate $f(t_1, t_2)$ on that.



$$\iint_A dx\,dy = \text{Area of } A$$

$$\iint_{\bigstar} f(t_1, t_2)\, dt_1\, dt_2, \quad f(t_1,t_2) = \frac{1}{T^2}$$

$$= \frac{1}{T^2} \iint_{\bigstar} dt_1\, dt_2$$

$$= \frac{1}{T^2} (T-\tau)^2$$

← Prob. that information will NOT collide

The probability that the information will collide

$$= \boxed{1 - \frac{1}{T^2}(T-\tau)^2}$$

Conditional Distributions (Discrete)

If X, Y are jointly distributed discrete random variables.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$\boxed{P(X = x_i \mid Y = y_j)} = \boxed{\frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}}$$

$X: x_1, x_2, \ldots \ldots$

$Y: y_1, y_2, \ldots - - -$

$$= \boxed{\frac{p_{XY}(x_i, y_j)}{p_Y(y_j)}}$$

joint frequency function

marginal frequency of Y.

NOTATION

$$\boxed{p_{X|Y}(x_i | y_j)}$$

Example:

$X \backslash Y$	0	1	2	3
0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	0
1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

The conditional frequency function of X given $\boxed{Y=1}$

$$P_{X|Y}(0\,|\,1) = \frac{P_{XY}(0,1)}{P_Y(1)} = \frac{\left(\frac{2}{8}\right)}{\left(\frac{3}{8}\right)} = \frac{2}{3}$$

$$P_{X|Y}(1\,|\,1) = \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{\left(\frac{1}{8}\right)}{\left(\frac{3}{8}\right)} = \frac{1}{3}$$

We find the conditional frequency function of X given $Y=1$,

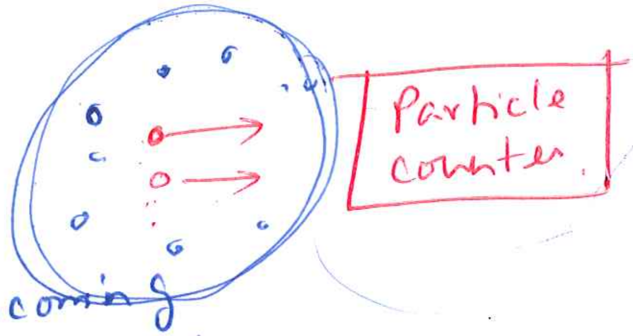
$X \,|\, (Y=1)$

is a random varible

$X \mid Y=1$	Prob
0	$\frac{2}{3}$
1	$1/3$

Example:

- Suppose that a "particle counter" is imperfect and independently detects each incoming particle with probability $= p$.



- If the distribution of the # of incoming particles in a unit time is a Poisson distribution with parameter λ.

What is the distribution of the number of counted particles?

Solution: Let N denote the TRUE number of particles, and X the counted number.

$$P(\underset{\text{counted}}{X} = \underset{\text{fixed}}{k}) = \sum_{n=\cancel{0}k}^{\infty} P(\underset{\text{true}}{N} = n) \cdot P(X = k \mid N = n)$$

$$= P(X = k, N = n)$$

$$N \sim \text{Poisson}(\lambda)$$

$$P(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$P(\underline{X=k} \mid \underline{N=n}) = \binom{n}{k} p^k (1-p)^{n-k}$$

So, $$P(X=k) = \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{\cancel{n!}} \binom{n}{k} p^k (1-p)^{n-k}.$$

$$\binom{n}{k} = \frac{\cancel{n!}}{(n-k)!\,.\,k!}$$

$$= \frac{e^{-\lambda}}{k!} p^k \sum_{n=k}^{\infty} \lambda^n \frac{(1-p)^{n-k}}{(n-k)!}$$

$$= \frac{e^{-\lambda}}{k!} (p\lambda)^k \sum_{n=k}^{\infty} \lambda^{n-k} \cdot \frac{(1-p)^{n-k}}{(n-k)!}$$

$j = n-k$

$$= \frac{e^{-\lambda}(p\lambda)^k}{k!} \boxed{\sum_{j=0}^{\infty} \frac{\lambda^j (1-p)^j}{j!}}$$

$$\boxed{e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}}$$

$$= \frac{e^{-\lambda}(p\lambda)^k}{k!} \boxed{\sum_{j=0}^{\infty} \frac{(\lambda(1-p))^j}{j!}}$$

$$= \frac{e^{-\lambda}(p\lambda)^k}{k!} e^{\lambda(1-p)} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

So, $P(\underline{X} = k) = e^{-(\lambda p)} \dfrac{(\lambda p)^k}{k!}$,

$k = 0, 1, 2, \ldots$

So, $\underline{X} \sim \text{Poisson}(\lambda p)$

[Another example:
N: # of traffic accidents.
$\underline{X}$: # of fatal accidents.]

Continuous Case:

$\underline{X}$ and Y are jointly continuous random variables,

$$f_{Y|\underline{X}}(y|x) = \frac{f_{\underline{X}Y}(x,y)}{f_{\underline{X}}(x)},$$

joint density

Conditional density

$0 < f_{\underline{X}}(x) < \infty$

$$\boxed{f_{XY}(x,y) = f_{Y|X}(y|x) \cdot f_X(x)}$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x)\, f_X(x)\,dx}$$

Example:

$$\begin{cases} f_{XY}(x,y) = \lambda^2 e^{-\lambda y}, & 0 \le x \le y \\ f_X(x) = \lambda e^{-\lambda x}, & x \ge 0. \\ f_Y(y) = \lambda^2 y e^{-\lambda y}, & y \ge 0 \end{cases}$$

$$f_{Y|X}(y|x) = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, \quad y \ge x \ge 0$$

$$f_{X|Y}(x,y) = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 \cdot y e^{-\lambda y}} = \frac{1}{y}, \quad 0 \le x \le y$$