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Random Variable

A function (measurable) from Ω (Sample space) to $\mathbb{R}$ (set of real numbers)

Discrete   Continuous.

Example X : $\boxed{\text{Random variable}}$ (discrete)

(Cap/big letters)

X	Prob.
0	1/8
1	3/8
2	3/8
3	1/8
Total	1

$$\begin{cases} P(X = 0) = \frac{1}{8} \\ P(X = 1) = \frac{3}{8} \\ P(X = 2) = \frac{3}{8} \\ P(X = 3) = \frac{1}{8}. \end{cases}$$

X x

X x

~~x~~

Name of a random variable = X

Values of a random variable = x

(small x)

In general a discrete r.v. X

can take values $x_1, x_2, \ldots$

Then we denote

cap → $\boxed{P(X = x_i) = p(x_i)}$ (small)

Note $\sum_{i=1}^{\infty} p(x_i) = 1$ (small)

$\boxed{\text{Probability mass function (p.m.f.)}}$

Cumulative distribution function (cdf)

$$\boxed{F(x) = P(X \leq x), \quad -\infty < x < \infty}$$

Last example: Suppose

X	Prob
0	1/8
1	3/8
2	3/8
3	1/8

$0 < x < 1$, $F(x) = P(X \leq x) = \frac{1}{8}$

$x = 1$, $F(x) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$1 < x < 2$: $F(x) = \frac{1}{2}$

$x = 2$: $F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8}$

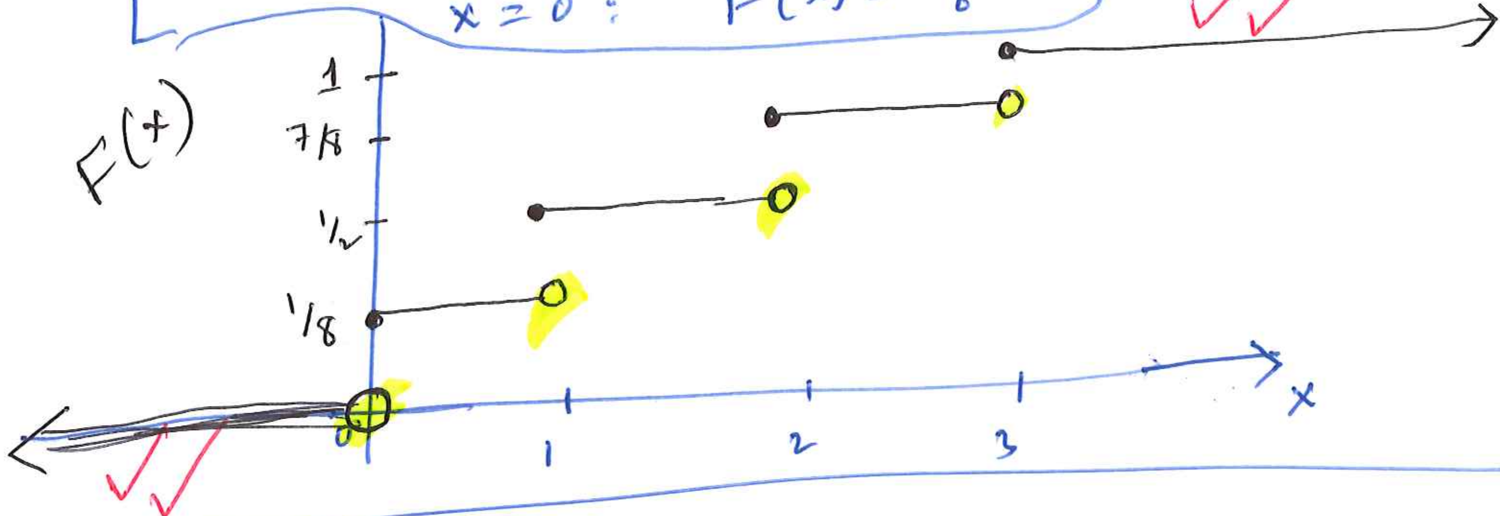
$2 < x < 3$: $F(x) = \frac{7}{8}$

$x = 3$: $F(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1$

$x > 3$: $F(x) = 1$

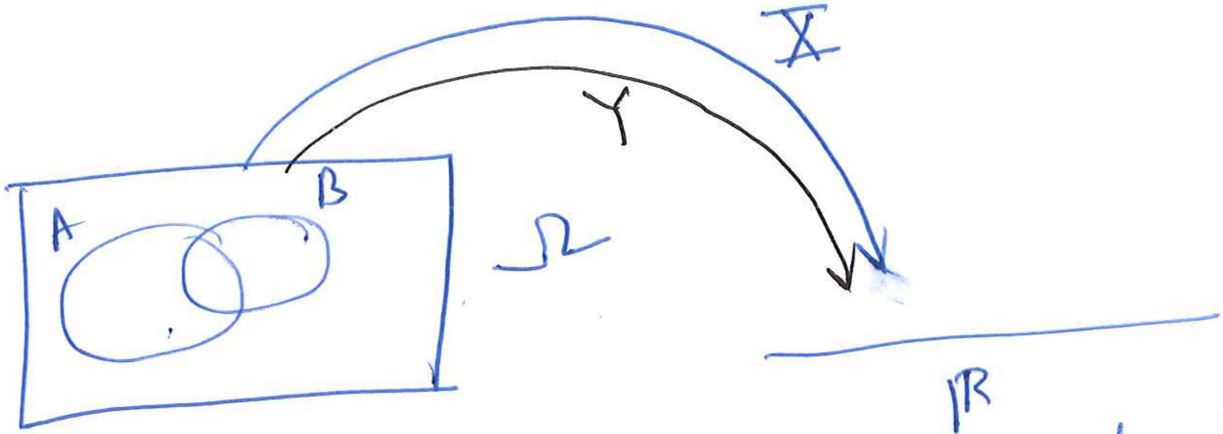
$x < 0$: $F(x) = 0$

$x = 0$: $F(x) = \frac{1}{8}$



In general

$$\left.\begin{aligned} \lim_{x \to -\infty} F(x) &= 0 \\ \lim_{x \to +\infty} F(x) &= 1 \end{aligned}\right\}$$



$A \perp B$:

$P(A \cap B) = P(A)\,P(B)$ ✓✓

Suppose X and Y are two random variables.

$X = x_1, x_2, \ldots$

$Y = y_1, y_2, \ldots$

$X \perp Y$ if

$$P(X = x_i \text{ and } Y = y_j) = P(X = x_i)\,P(Y = y_j)$$

Examples of discrete r.v.

- Bernoulli
- Binomial. ✓
- Geometric
- Negative-binomial.
- Hypergeometric
- Poisson. ✓

1 Bernoulli Random Variable

$X = 0, 1$

0: Prob $= 1-p$; 1: Prob $= p$.

X	Prob.
0	$1-p$
1	p

p.m.f.

$$P(X=0) = 1-p = p(0)$$
$$P(X=1) = p = p(1)$$
$$p(x) = 0 \text{ if } x \neq 0 \text{ or } x \neq 1$$

Example: Tossing a coin (fair)

$X = 0$ if H, Prob $= \frac{1}{2}$; $X = 1$ if T, Prob $= \frac{1}{2}$

X	Prob.
✓ 0	$1/2$
✓ 1	$1/2$

(2) Binomial Random Variable

- Suppose n independent experiments. (n = fixed)
- Each experiment results in a "success" with prob $= p$. and a "failure" with prob $= (1-p)$.

The total number of successes $= X$

We say X follows a binomial random variable with parameters

$$X \sim \text{Bin}(n, p)$$

- What is the pmf of X?

$$\boxed{P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}}$$

S F S F S F

($k = 0, 1, 2, 3, \ldots, n$)

Same thing!

$$\boxed{p(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n}$$

Example:

- If a single bit (0 or 1) is transmitted over a noisy channel, it has prob. $= p$ of being incorrectly transmitted. $p = 0.1$

- To improve the reliability of the transmission, the bit is transmitted $n = 5$ times.

Send $\{0, 1\}$ → Channel. → Receive

Prob. of detecting the bit correctly $= 1 - p$

Question: What is the prob. that the message is correctly received?

I want to ~~set~~ send $= 0$

Send $\{0, 0, 0, 0, 0\}$

Channel ↓

Receive $\{0, 0, 1, 1, 0\}$ = Detect as "0"

$0 \to \{0, 0, 1, 1, 0\}$ ✓ $= 0$

$0 \to \{1, 1, 0, 0, 1\}$ ✗ $= 1$

I want to $\boxed{\text{send} = 0}$

Send $\{0, 0, 0, 0, 0\}$

Channel $\downarrow$

Receive $\{1, 1, 0, 0, 1\}$

Detect as "1"

Solution: The correct way to detect the message is to have 0, 1, 2 bit error.

$$\binom{5}{0} p^0 (1-p)^5 + \binom{5}{1} p^1 (1-p)^4 + \binom{5}{2} p^2 (1-p)$$

(p = prob. of error)

$$= (0.1)^0 (0.9)^5 + 5 (0.1)^1 (0.9)^4 + 10 (0.1)^2 (0.9)$$

$$= \boxed{0.9914}$$

$X \sim Bin(5, 0.1)$

Remark: ~~Bin~~ Bernoulli and Binomial

$X_1, X_2, \ldots, X_n$ are independent Bernoulli ~~distribution~~ random variables.

with $\left.\begin{array}{l} P(X_i = 1) = p \\ P(X_i = 0) = 1-p \end{array}\right\}$ for $i = 1, 2, \ldots, n.$

$$\boxed{Y = X_1 + X_2 + \cdots + X_n}$$

$$\boxed{Y \sim Bin(n, p)}$$

③ Geometric random variable

- Also contructed from independent Bernoulli trials, but from an infinite sequence.
- On each trial. "success" prob = p

$\underline{X}$ = total # of trials <u>up to and including</u> the <u>first success</u>.

Suppose the deal is
→ get $\underline{X} = k$ trials ~~successes~~ up to and including the first success.

$$\boxed{P(\underline{X} = k) = (1-p)^{k-1}\, p\,, \quad k = 1,2,3,\dots}$$

F F F F S

[Is this really a pmf?
We must have
$$\sum_{k=1}^{\infty} P(\underline{X} = k) = 1 \quad ??$$

<u>Yes:</u>
$$\sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{k=1}^{\infty} (1-p)^{k-1}$$
$$= p\left(1 + (1-p) + (1-p)^2 + \dots\right)$$
$$= p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1$$]

4 Negative binomial random variable

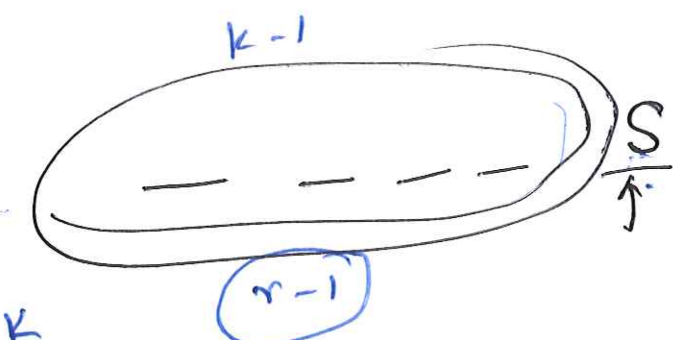
- Generalization of geometric random variable
- Suppose that a sequence of independent trials (each with prob of success $= p$) is performed until there are r successes in total.

$$\boxed{P(\underline{X} = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}}$$

$k = 1, 2, \ldots \quad r < k$

$k = r, r+1, r+2, \ldots$

$P($~~The # of~~ k trials are needed to get r successes with the last trial is a success$)$



Example: How many trials are needed to get 3 "Heads"

How many $= k$

$3 = r$

$$P(\underline{X} = 10) = \binom{10-1}{3-1} p^3 \cdot (1-p)^7$$

$$P(X = 15) = \binom{15-1}{3-1} p^3 (1-p)^{12}$$

✗ H H H T T

✓ H T H T H

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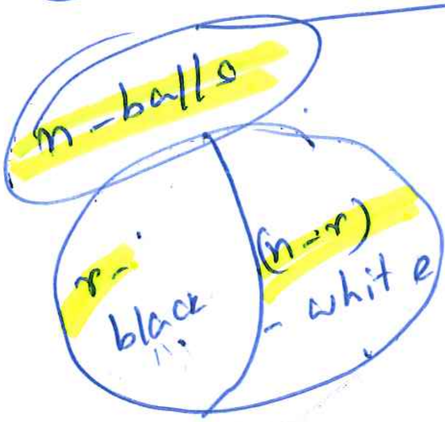
5 Hypergeometric distribution



X: the # of black balls drawn when taking m balls w/o replacement.

$$\boxed{P(X = k) = \frac{\binom{r}{k}\binom{n-r}{m-k}}{\binom{n}{m}}}$$

$$\cancel{\frac{\binom{m}{k}\binom{n-m}{m-k}}{\binom{n}{m}}}$$

$X \sim HyperG.(r, n, m)$ //