12/13

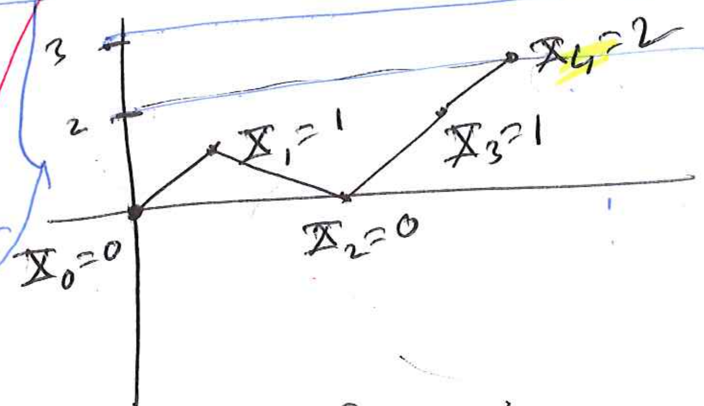
Hitting time and absorption probability

Let $(X_n)_{n \geq 0}$ be a Markov chain with transition matrix P and initial distribution λ.

H^A : random variable.

$$H^A = \inf \{ n \geq 0 : X_n \in A \}$$

(State space of $(X_n)_{n \geq 0}$ is I. Then A is a subset of I)



Example

$A = \{2, 3\}$

$H^A = 4$

Notation:

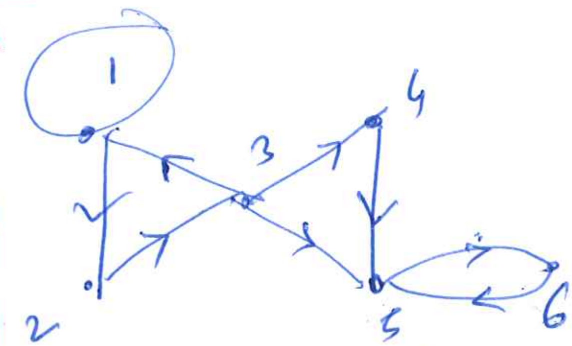
$H^A = \infty$ if the ~~infimum~~ the RHS is empty set.

$\boxed{H^A : \text{hitting time of a subset } A \text{ of } I}$

Notation: The probability that starting from state i, $(X_n)_{n \geq 0}$ ever hits A is

$$h_i^A = P_i(H^A < \infty) = P(H^A < \infty \mid X_0 = i)$$
$$= P_i(\text{hit } A)$$

Definition: If A is a closed class, then h_i^A is called the ABSORPTION PROBABILITY.



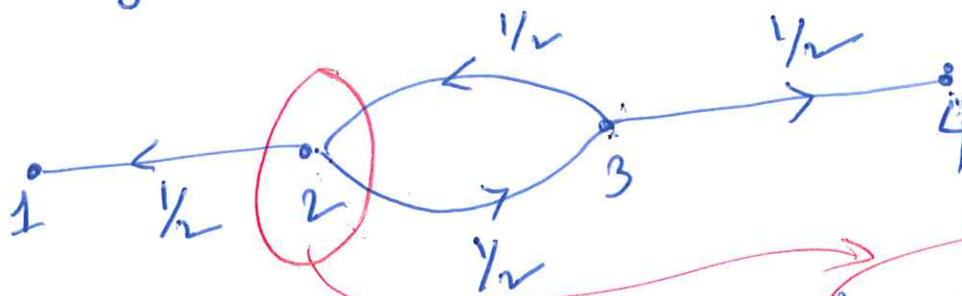
Closed class $\{5, 6\}$

Definition: The mean/expected time taken for $(X_n)_{n \geq 0}$ to reach A is given by (starting at $X_0 = i$)

$$K_i^A = E_i(H^A) = \sum_{n < \infty} n \, P(H^A = n) + \infty \cdot P(H^A = \infty)$$

$$= E_i(\text{time to hit } A)$$

Example: Consider the chain with the following diagram.



"state 2"

$X_6 = 1$
$X_7 = 1$

$\boxed{Q1}$ Starting from "state 2" what is the probability of absorption in "state 4"?

$\boxed{Q2}$ How long does it take until the chain is absorbed in "state 1" or "state 4"?

Solution:

Denote:

$$\begin{cases} h_i = P_i(\text{hit } 4) \left(= P_i(H^{\{4\}} < \infty) \right) \\ K_i = E_i(\text{time to hit } \{1,4\}) \end{cases}$$

Goal: $\boxed{Q1}$: to find h_2 ✓

$\boxed{Q2}$: to find K_2

Q1

✓ $h_1 = P(\text{starting } 1, \text{ we end up at } 4)$

$= 0$

✓ $h_4 = P(\text{starting } 4, \text{ we end up at } 4)$

$= 1$

Mark on property intuitively used

✓ $h_2 = \frac{1}{2}\cancelto{0}{h_1} + \frac{1}{2}h_3 = \frac{1}{2}h_3 \quad \cdots (1)$

✓ $h_3 = \frac{1}{2}\cancelto{1}{h_4} + \frac{1}{2}h_2 = \frac{1}{2} + \frac{1}{2}h_2 \quad \cdots (2)$

$\Rightarrow$ by (1) and (2)

$h_2 = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}h_2\right)$

$\Rightarrow h_2 = \frac{1}{4} + \frac{1}{4}h_2$

$\Rightarrow \frac{3}{4}h_2 = \frac{1}{4}$

$\Rightarrow \boxed{h_2 = \frac{1}{3}}$

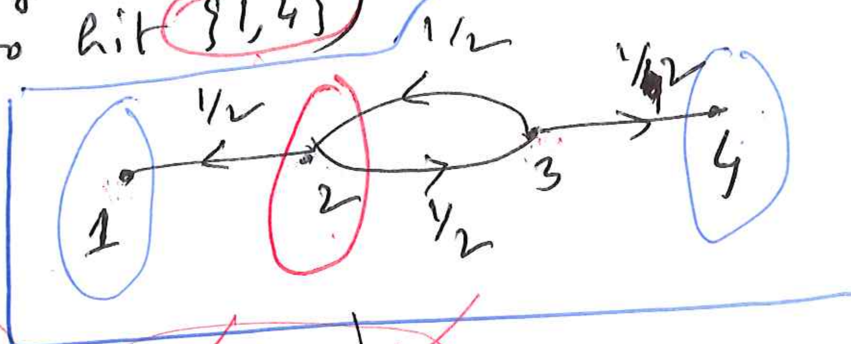
Q2

$K_1 = E$ (starting at 1, time to hit $\{1,4\}$)

$= 0$

$K_4 = E$ (starting at 4, time to hit $\{1,4\}$)

$= 0$



$$K_2 = 1 + \frac{1}{2}K_3 + \frac{1}{2}\underbrace{K_1}_{0}$$

1 step to move from $2 \to 3$ or $2 \to 1$

$$K_2 = 1 + \frac{1}{2}K_3 \quad \cdots (3)$$

$$K_3 = 1 + \left(\frac{1}{2}\underbrace{K_4}_{0} + \frac{1}{2}K_2\right)$$

$$K_3 = 1 + \frac{1}{2}K_2 \quad \cdots (4)$$

Solve (3) and (4):

$$K_2 = 1 + \frac{1}{2}\left(1 + \frac{1}{2}K_2\right)$$

$$\Rightarrow \frac{3}{4}K_2 = \frac{3}{2} \Rightarrow \boxed{K_2 = 2}$$

Theorem: The vector of hitting probabilities $h^A = (h_1^A, h_2^A, \dots, h_N^A)$

$P(\text{starting at } 1, \text{ hit } A)$

$I = \{1, 2, \dots, N\}$ states

is the MINIMUM NON-NEGATIVE solution to the system of linear equations:

$$\begin{cases} h_i^A = 1, & \text{for } i \in A \\ h_i^A = \sum_{j=1}^{N} p_{ij} h_j^A, & \text{for } i \notin A \end{cases}$$

The last exaple (in view of this Theorem)

Goal was to find $h_2 = h_2^{\{4\}}$

$= P(\text{starting at } 2, \text{ hit } 4)$

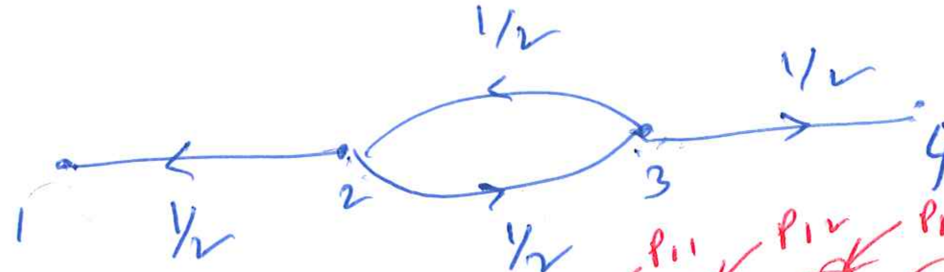
$I = \{1, 2, 3, 4\}$

$A = \{4\}$

Using the above theorem

$$\boxed{h_4 = h_4^{\{4\}} = 1}$$

.... (1)



$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

p_{11} p_{12} p_{13} p_{14} p_{23} p_{24} p_{21} p_{22}

$$h_1 = p_{11} h_1 + \cancel{p_{12}}^{\,0} h_2 + \cancel{p_{13}}^{\,0} h_3 + \cancel{p_{14}}^{\,0} h_4$$

$$\boxed{h_1 = h_1} \quad \ldots (2)$$

MINIMUM NON-NEGATIVE

$$h_1 = 0$$

$$h_2 = p_{21} h_1 + \cancel{p_{22}}^{\,0} h_2 + p_{23} h_3 + \cancel{p_{24}}^{\,0} h_4$$

$$\boxed{h_2 = \frac{1}{2} h_1 + \frac{1}{2} h_3} \quad \ldots (3)$$

$$h_3 = \cancel{p_{31}}^{\,0} h_1 + p_{32} h_2 + \cancel{p_{33}}^{\,0} h_3 + p_{34} h_4$$

$$\Rightarrow \boxed{h_3 = \frac{1}{2} h_2 + \frac{1}{2} h_4} \quad \ldots (4)$$

Solving all these 4 equations find (like before) $h_2 = \frac{1}{3}$ //

Aside on recurrance relation

$$a x_{n+1} + b x_n + c x_{n-1} = 0 \quad \cdots \quad (*)$$

$n = 1, 2, \ldots\ldots$

$\{x_0, x_1, x_2, \ldots\ldots\}$

a, b, c : constants.

To ~~sto~~ solve

Characteristic equation.

$$\boxed{\begin{array}{c} a\lambda^2 + b\lambda + c = 0 \\ \text{Roots} : \alpha, \beta . \end{array}}$$

Then the solution of $(*)$ is

$$x_n = \begin{cases} A\alpha^n + B\beta^n, & \text{if } \alpha \neq \beta. \\ (A + nB)\alpha^n, & \text{if } \alpha = \beta. \end{cases}$$

Note: $x_0 = \begin{cases} A + B, & \text{if } \alpha \neq \beta. \\ A, & \text{if } \alpha = \beta. \end{cases}$

Gambler's Ruin

Imagine that you are at a casino, with $\$ i$ and gamble $\$ 1$ at a time. $\rightarrow$ Stake

Probability p of doubling your stake is p.

Probability of losing $= q = 1 - p$.

The resources of the casino are regarded as infinite.

Question: What is the probability that you leave broke?

The transition probability matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & & \\ q & 0 & p & 0 & \cdots & \\ 0 & q & 0 & p & \cdots & \\ 0 & 0 & q & 0 & p & \cdots \end{pmatrix}.$$

Question: What is

$h_i^{\{0\}} = \boxed{h_i = P_i(\text{hit } 0)}$

$$\begin{cases} h_0 = 1 \\ h_i = p \cdot h_{i+1} + q\, h_{i-1}, \quad i = 1, 2, \ldots \end{cases}$$

$\downarrow$ Solve this recurrance relat-.

Ch. equation:

$$\lambda = p\lambda^2 + q\,1$$

$$p\lambda^2 - \lambda + (1-p) = 0$$

$$\Rightarrow p(\lambda^2 - 1) - (\lambda - 1) = 0$$

$$\Rightarrow p(\lambda+1)(\lambda-1) - (\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(p\lambda + p - 1) = 0$$

$p + q = 1$

$$\Rightarrow (\lambda-1)(p\lambda - q) = 0$$

$$\Rightarrow \lambda = 1, \frac{q}{p}$$

$\alpha \qquad \beta$.

$\boxed{\text{Next class}}$