Ramakrishna Mission Vivekananda Educational and Research Institute
Problems - Random walk

1. Let S_n be a simple random walk (i.e., $S_n = \sum_{i=1}^{n} X_i$, for $X_i = \pm 1$, with (each) probability $1/2$ and X_i's are independent.). Fix a number σ and define the process

$$P_n = e^{\sigma S_n} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^n .$$

Show that $E(P_{n+1}|S_n) = P_n$. (i.e., when you know the random walk upto epoch n, the expected value of P_{n+1} is simply P_n.)

2. Fix an integer m. Let τ_m denote the first time the random walk reaches level m. That means, $\tau_m = \min\{n : S_n = m\}$. The random variable τ_m is called the *first passage time* of the random walk to level m. From the last problem it is clear that

$$1 = P_0 = E(P_{\min\{n,\tau_m\}}),$$

for every n, τ_m. Take the limit $n \to \infty$ to show that when $\tau_m < \infty$,

$$E\left[e^{\sigma m} \left(\frac{2}{e^{\sigma} + e^{-\sigma}} \right)^{\tau_m} \right] = 1.$$

3. From the result in last problem show that for non-zero integer m,

$$E(\alpha^{\tau_m}) = \left(\frac{1 - \sqrt{1 - \alpha^2}}{\alpha} \right)^{|m|},$$

for all $\alpha \in (0, 1)$.