Ramakrishna Mission Vivekananda Educational and Research Institute
Probability and Stochastic Processes 2022
Midterm Exam

Name: ________________________

- Answer **ANY 10** problems. If you answer more than 10 problems, CLEARLY indicate which 10 problems you would like to be graded. Otherwise, the first 10 problems will be graded.
- For full credit you must show all the details of your work and justify all the answers.

1. (10 points) What are the three axioms of a probability measure? Suppose B is an event, with the probability of B, $P(B) > 0$. Show that the "set function" $Q(A)$ satisfies the (three) axioms for a probability measure, where $Q(A) = P(A|B)$.

2. (10 points) If X is a geometric random variable with $p = 0.5$, for what value of k is $P(X \leq k) \approx 0.99$?

3. (10 points) Find the median of a random variable X that follows exponential distribution with parameter λ.

4. (10 points) Suppose that X has the density function $f(x) = cx^2$, for $0 \leq x \leq 1$, and $f(x) = 0$, otherwise. (a) Find c. (b) What is $P(0.1 \leq X \leq 0.5)$?

5. (10 points) Find the probability density function of a random variable X that is given by $X = Z^2$, where $Z \sim N(0, 1)$. What is a popular name for such X?

6. (10 points) (a) If $Y = aX+b$, where X is a random variable and $a > 0$, and b are constants, show that the relation between the probability density functions of X and Y is: $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$. (b) If in (a) $X \sim N(\mu, \sigma^2)$, what is the distribution for Y?

7. (10 points) Let X and Y have the joint density $f_{XY}(x, y) = \frac{6}{7}(x + y)^2$, $0 \leq x \leq 1$, $0 \leq y \leq 1$. (a) Find $P(X > Y)$. (b) Find the marginal densities of X and Y. Are X and Y independent?

8. (10 points) Suppose that a communication network has the property that if two pieces of information arrive within time τ of each other, they "collide" and have to be retransmitted. If the times of arrival of the two pieces of information are independent and uniform on $[0, T]$, what is the probability that they collide?

9. (10 points) The lifetime of an electronic component follows an exponential distribution with parameter λ. Suppose that there is an independent and identical backup component available for an electronic system (i.e., the electronic system consists of two aforementioned electronic components). The electronic system operates as long as one of the components is functional. What is the probability distribution of the lifetime of this electronic system?

10. (10 points) Find $E\left(\frac{1}{(X+1)}\right)$, where $X \sim \text{Poisson}(\lambda)$.

11. (10 points) A coin (with probability of head $= p$) is tossed n times, and the number of heads, N, is counted. The coin is then tossed N more times. Find the expected total number of heads generated by this process.

12. (10 points) Let X be a continuous random variable with mean μ, and variance σ^2. Then for any $t > 0$, $P(|X - \mu| > t) \leq \frac{\sigma^2}{t^2}$.

SOLUTION (Exam 1)

1) • Write three axioms.
- ① $P(\Omega) = 1$
- ② If $A \subset \Omega$, $P(A) \geq 0$
- ③ If $A_1 \cap A_2 = \phi$, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$

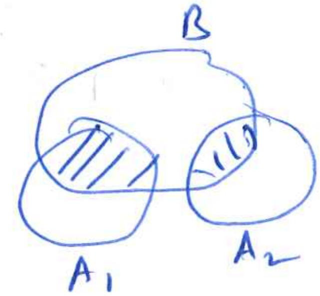
•
$$Q(A) = P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

① $$Q(\Omega) = P(\Omega \mid B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1 \checkmark$$

② If $A \subset \Omega$
$$Q(A) = P(A \mid B) = \frac{P(A \cap B)}{P(B)} \geq 0$$
($\because$ P is a probability)

③ If $A_1 \cap A_2 = \phi$
$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

$$Q(A_1 \cup A_2) = P(A_1 \cup A_2 \mid B)$$
$$= \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$
$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$



Disjoint sets

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)}$$
$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} = Q(A_1) + Q(A_2)$$

2) If $X \sim \text{Geometric}(p) \longrightarrow$ (In general. For our problem $p = 0.5$)

then $P(X = k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$

Therefore

$$P(X \le k) = \sum_{n=1}^{k} p(1-p)^{n-1}$$

$$= p \sum_{n=1}^{k} (1-p)^{n-1}$$

$$= p \cdot \frac{1-(1-p)^k}{1-(1-p)}$$

$$= 1-(1-p)^k$$

<u>Want:</u>

$$1-(1-p)^k = 0.99$$

so, $(1-p)^k = 0.01$

$p = 0.5$ So, $(0.5)^k = 0.01$

$$\Rightarrow \boxed{k = 6.64}$$

3) $X \sim Exp(\lambda)$

So, its pdf:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

(for $\lambda > 0$)

Cumulative distribution function:

$$F(x) = \int_{-\infty}^{x} f(u)\, du$$

If $\underline{x \geq 0}$ $\quad F(x) = \int_{-\infty}^{0} 0 \, du + \int_{0}^{x} \lambda e^{-\lambda u} \, du$

$$= (1 - e^{-\lambda x})$$

If $\underline{x < 0}$ $\quad F(x) = \int_{-\infty}^{x} 0 \, du = 0$

So,
$$F(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Median of $X \sim Exp(\lambda)$ is the $x = x_p$ such that $F(x_p) = \frac{1}{2}$

$$\Rightarrow 1 - e^{-\lambda x_p} = \frac{1}{2}$$

$$\Rightarrow e^{-\lambda x_p} = \frac{1}{2}$$

$$\Rightarrow \cancel{-}\lambda x_p = \ln\left(\tfrac{1}{2}\right) = \cancel{-}\ln 2$$

$$\Rightarrow \boxed{x_p = \frac{\ln 2}{\lambda}}$$

4) (a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_{0}^{1} c x^2\,dx = 1 \Rightarrow c\frac{x^3}{3}\Big|_0^1 = 1$

$\Rightarrow \frac{c}{3} = 1$

$\Rightarrow \boxed{c = 3}$

(b) cdf of X is

$$F_X(x) = \int_{-\infty}^{x} f(u)\,du$$

$$\Rightarrow F(x) = \begin{cases} 0, & x < 0 \\ x^3, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Now,

$$P(0.1 \le X \le 0.5) = F(0.5) - F(0.1)$$
$$= (0.5)^3 - (0.1)^3$$
$$= \boxed{0.124}$$

Another way for part (b):

$$P(0.1 \le X \le 0.5) = \int_{0.1}^{0.5} 3x^2\,dx = x^3\Big|_{0.1}^{0.5} = 0.124$$

c (by part (a))

5) $Z \sim N(0,1)$

$X = Z^2$

cdf of X: $F_X(x) = P(X \le x)$

$= P(Z^2 \le x)$

$= P(-\sqrt{x} \le Z \le \sqrt{x})$, $\boxed{x \geq 0}$

$= \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$

(where Φ is the cdf of Z (std. normal dbn))

To find the pdf of X:

$$f_X(x) = \frac{d}{dx}\left(F_X(x)\right) = \frac{d}{dx}\left(\Phi(\sqrt{x}) - \Phi(-\sqrt{x})\right)$$

$$= \frac{1}{2}x^{-1/2}\Phi'(\sqrt{x}) + \frac{1}{2}x^{-1/2}\Phi'(-\sqrt{x})$$

$$= \frac{1}{2}x^{-1/2}\phi(\sqrt{x}) + \frac{1}{2}x^{-1/2}\phi(-\sqrt{x})$$

$\Phi'(u) = \phi(u)$

std. normal cdf

std. normal pdf

$\phi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

$$\boxed{\phi(-u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(-u)^2}{2}} = \phi(u)}$$

$$= x^{-1/2}\phi(\sqrt{x}) \quad \left(\because \phi(\sqrt{x}) = \phi(-\sqrt{x})\right)$$

Hence, $f_X(x) = x^{-1/2}\frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^2}{2}}$, $\boxed{x \geq 0}$

$$\Rightarrow \boxed{f_X(x) = \frac{x^{-1/2}}{\sqrt{2\pi}} e^{-x/2}, \ x \geq 0}$$

Known as Chi-square distribution

6) (a) $Y = aX + b$, $a > 0$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$

$$= P\left(X \le \frac{y-b}{a}\right) \quad (\because a > 0)$$

$$= F_X\left(\frac{y-b}{a}\right)$$

cdf of Y

cdf of X

So, by differentiating,

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right)$$

$$\Rightarrow \quad f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \quad \text{(Done)}$$

(b) If $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

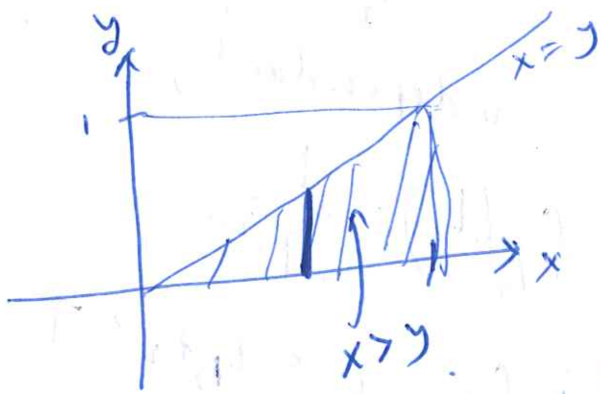
So, $$f_Y(y) \underset{\text{by (a)}}{=} \frac{1}{a\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-b-a\mu}{a\sigma}\right)^2},$$

$$-\infty \le y < \infty$$

7)

$$f_{XY}(x,y) = \frac{6}{7}(x+y)^2, \quad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}$$

(a) $P(X > Y) = \int_{x=0}^{1} \int_{y=0}^{x} f(x,y)\, dy\, dx$



$$= \int_{x=0}^{1} \left(\int_{y=0}^{x} \frac{6}{7}(x+y)^2\, dy \right) dx$$

$$= \int_{x=0}^{1} \frac{6}{7} \cdot \left(\frac{1}{3}(x+y)^3 \Big|_{y=0}^{x} \right) dx$$

$$= \int_{x=0}^{1} \frac{6}{7} * \frac{1}{3} \cdot (8x^3 - x^3)\, dx$$

$$= \int_{x=0}^{1} 2x^3\, dx = 2 \cdot \frac{x^4}{4} \Big|_0^1 = \boxed{\frac{1}{2}}$$

(b) Marginal density for X:

$$\boxed{f_X(x) = \int_{y=0}^{1} \frac{6}{7}(x+y)^2\, dy = \frac{2}{7}(3x^2 + 3x + 1), \quad 0 \le x \le 1}$$

Marginal density for Y: (by symmetry)

$$\boxed{f_Y(y) = \frac{2}{7}(3y^2 + 3y + 1), \quad 0 \le y \le 1}$$

8) Arrival times of two information are T_1 and T_2

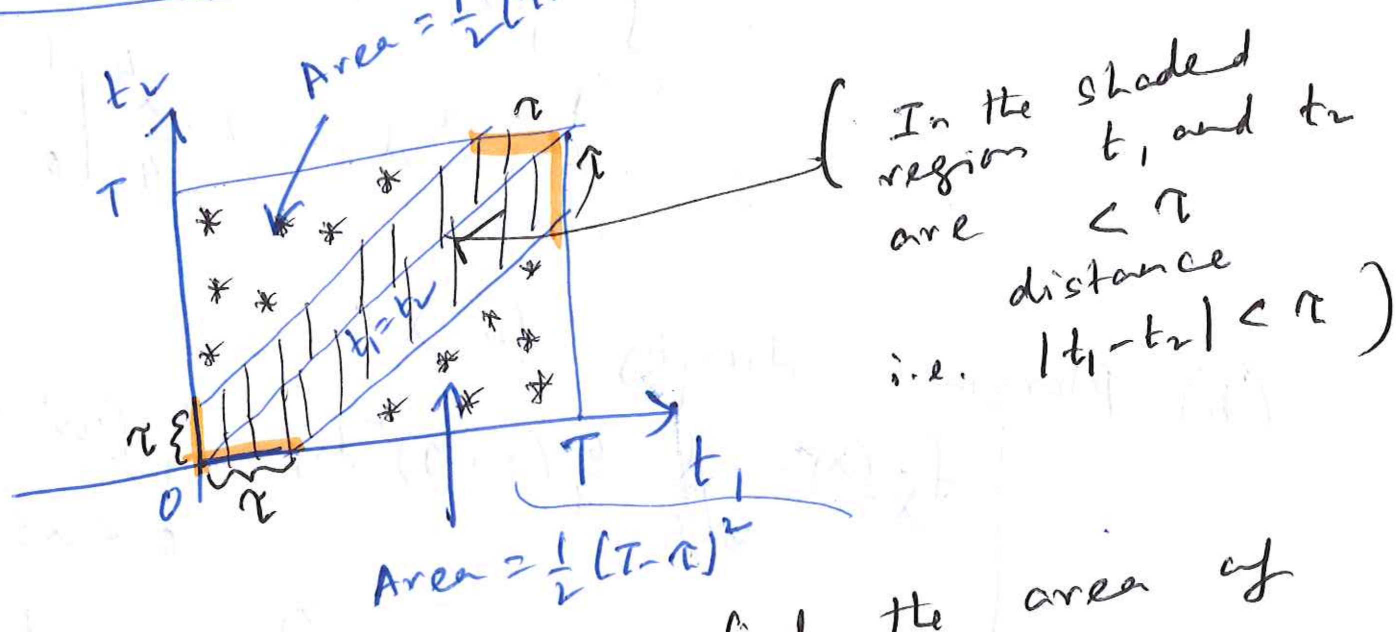
$T_1 \sim \text{Uniform}([0,T])$, $f_{T_1}(x) = \frac{1}{T}$, $x \in [0,T]$

$T_2 \sim \text{Uniform}([0,T])$, $f_{T_2}(x) = \frac{1}{T}$, $x \in [0,T]$.

Since, T_1 and T_2 are independent,

$$\underbrace{f(t_1,t_2)}_{\text{Joint density of } T_1 \text{ and } T_2} = f_{T_1}(t_1)\, f_{T_2}(t_2) = \frac{1}{T^2}$$

for, $0 \le t_1, t_2 \le T$



We are trying to find the area of the "$***$" region $= \iint_{***} f(t_1,t_2)\,dt_1\,dt_2$

$= \frac{1}{T^2}$ (Area of the $***$ region)

$$= \frac{1}{T^2}\cdot\left(2 * \frac{1}{2}(T-\tau)^2\right)$$

$$= \frac{(T-\tau)^2}{T^2}$$

Hence the probability that the information collide

$$= \boxed{1 - \frac{(T-\tau)^2}{T^2}}$$

9) We want to find the pdf of $S = T_1 + T_2$ where T_1, T_2 are i.i.d $\sim Exp(\lambda)$

CONVOLUTION

$$\boxed{f_S(s)} = \left(f_{T_1} * f_{T_2}\right)(s) = \int_{-\infty}^{\infty} f_{T_1}(x) f_{T_2}(s-x)\, dx$$

$$= \int_{x=0}^{s} \underbrace{\left(\lambda e^{-\lambda x}\right)}_{f_{T_1}(x)} \underbrace{\left(\lambda e^{-\lambda(s-x)}\right)}_{f_{T_2}(s-x)} dx$$

$$= \lambda^2 \int_{x=0}^{s} e^{-\lambda s}\, dx$$

$$= \boxed{\lambda^2 s e^{-\lambda s}}$$

So, $\boxed{S \sim Gamma(2, \lambda)}$

$$f_{T_1}(x) = \begin{cases} \lambda e^{-\lambda x}, & \boxed{x > 0} \\ 0, & \text{otherwise} \end{cases}$$

$$f_{T_2}(s-x) = \begin{cases} \lambda e^{-\lambda(s-x)}, & \boxed{s > x} \\ 0, & \text{otherwise} \end{cases}$$

10) $E\left(\frac{1}{X+1}\right) = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{e^{-\lambda}\lambda^{k}}{k!}$

$$= \frac{1}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{k+1}}{(k+1)!}$$

$$= \frac{1}{\lambda}\left[\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^{k}}{k!} - \frac{e^{-\lambda}\lambda^{0}}{0!}\right]$$

$$= \frac{1}{\lambda}\left[1 - e^{-\lambda}\right]$$

So, $\boxed{E\left(\frac{1}{X+1}\right) = \frac{1}{\lambda}\left(1 - e^{-\lambda}\right)}$

11)

X = number of heads in the 2nd stage of the process.

In the first stage, the number of heads is binomially distributed with parameters (n, p)

So, $E(N) = np$ ---- (1)

[If N is fixed i.e. $\rightarrow E(X \mid N) = Np$]

Now,

$$E(N + X) = E\big(E(N + X \mid N)\big)$$

X: out of N (random) trials

$$= E\big(E(N \mid N) + E(X \mid N)\big)$$

$$= E(N + Np)$$

$$= E\big((p+1)N\big)$$

$$= (p+1)\,E(N)$$

by (1)

$$= (p+1)\,np$$

So, $\boxed{E(N + X) = np(p+1)}$

12) For continuous case: Let X has pdf $f(x)$

$$\sigma^2 = \int_{-\infty}^{\infty} \overset{\geq 0}{(x-\mu)^2}\, \overset{\geq 0}{f(x)}\, dx$$

$$\geq \int_A (x-\mu)^2 f(x)\, dx$$

Let $A = \{|X-\mu|>t\}$
$= \{x : |x-\mu| > t\}$

$$\geq \int_A t^2 f(x)\, dx$$

$$= t^2 \int_A f(x)\, dx$$

But $P(A) = \int_A f(x)\, dx$

Hence, $\sigma^2 \geq t^2 P(A)$

So, $P(A) \leq \dfrac{\sigma^2}{t^2}$

$\Rightarrow P(|X-\mu|>t) \leq \dfrac{\sigma^2}{t^2}$ //