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Continuous Random Variable

Probability density function (pdf)

$f(x)$

- $f(x) \geq 0$
- f is piecewise continuous.
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Aside:

Discrete r.v.

$\Downarrow$

pmf:

r.V	Prob.
1	1/3
2	1/3
3	1/3

$P(X=1) = \frac{1}{3}$

$P(X=2) = \frac{1}{3}$

$P(X=3) = \frac{1}{3}$

$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

Difference between pmf and pdf

pmf	pdf
① Discrete random variable	① Continuous random variable
② It gives a probability (direct way)	② It does not give probability directly

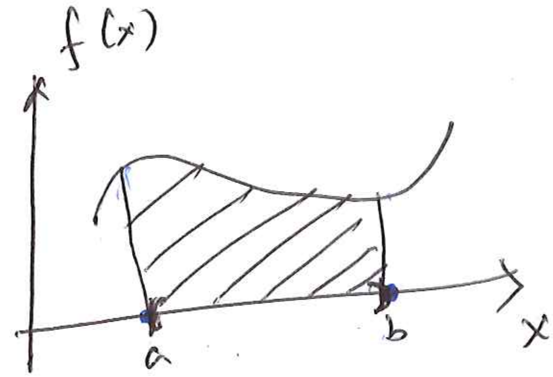
If X is a random variable, with density function f, then for any $a < b$,

$$\boxed{P(a < X < b) = \int_a^b f(x)\,dx}$$

$$\boxed{\begin{aligned} P(a < X < b) &= P(a \le X < b) \\ &= P(a < X \le b) \\ &= P(a \le X \le b). \end{aligned}}$$

continuous random variable

$$P(a < X < b) = \int_a^b f(x)\,dx$$



Caution: NOT true for a discrete random variable

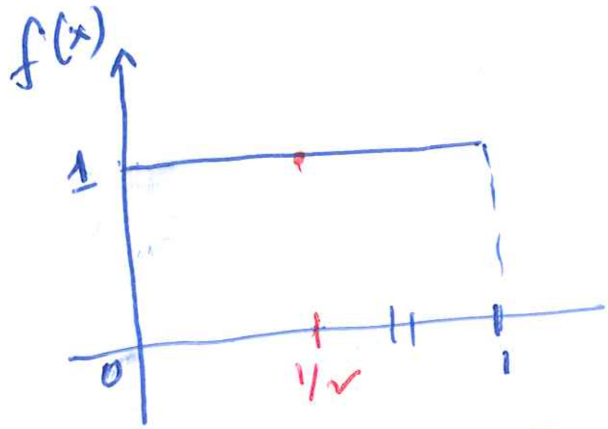
Question: $f(x)$: pdf

$$\boxed{P(X = a) = P(a \le X \le a) = \int_a^a f(x)\,dx = 0}$$

(Difference between pmf and pdf)

Example: Uniform random variable on [0, 1]

$[0,1]$ $0 \le x \le 1$
$(0,1)$ $0 < x < 1$



pdf:

$$f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x < 0 \quad \text{or} \quad x > 1 \end{cases}$$

Notation: $X \sim \text{Uni}([0,1])$

(**Note:** $P(X = \frac{1}{2}) = 0 = \int_{1/2}^{1/2} f(x)\,dx = 0$

$f(\frac{1}{2}) = 1$)

$X \sim \text{Uni}([0,1])$

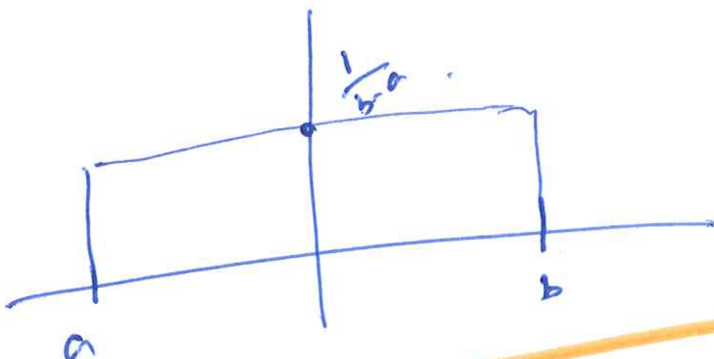
$$P\left(\frac{1}{2} < X \le 1\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} 1\,dx$$

$$= x \Big|_{1/2}^{1}$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

Uniform random variable on $[a, b]$ $\quad a < b$.

$a, b \in \mathbb{R}$



$$\boxed{\begin{aligned} f(x) &= \frac{1}{b-a}, \quad a \le x \le b \\ &= 0, \quad \text{otherwise} \end{aligned}}$$

$$\cancel{\int_a^b f(x)\,dx = \int_a^b \frac{1}{b-a}\,dx}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\cancel{\int_{-\infty}^{a} f(x)\,dx}}_{0} + \int_a^b f(x)\,dx + \underbrace{\cancel{\int_b^{\infty} f(x)\,dx}}_{0}$$

$$= \int_a^b \frac{1}{b-a}\,dx = 1$$

So, $f(x)$ is a pdf //.

Cumulative distribution function (cdf)

$$\boxed{F(x)} = P(X \le x)$$

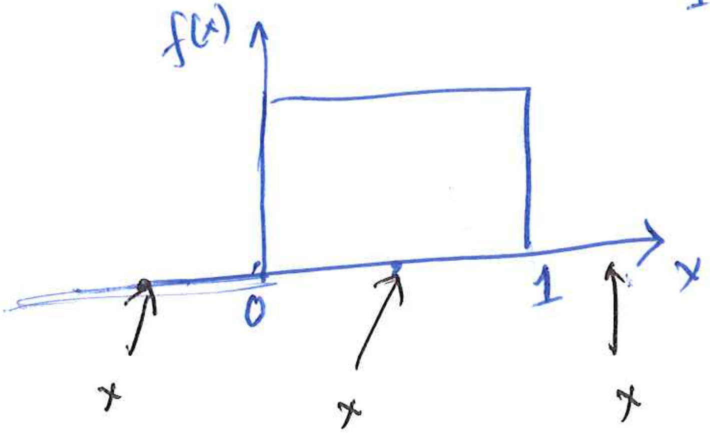
$$= P(-\infty < X \le x)$$

$$= \boxed{\int_{-\infty}^{x} f(x)\,dx}$$

$$\boxed{F'(x) = f(x)}$$

Unif. distribution $X \sim \mathrm{Unif}([0,1])$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$



If $\boxed{x \le 0}$:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= 0$$

If $\boxed{0 \le x \le 1}$:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \cancel{\int_{-\infty}^{0} f(x)\,dx}^{\,0} + \int_{0}^{x} f(x)\,dx$$

$$= x$$

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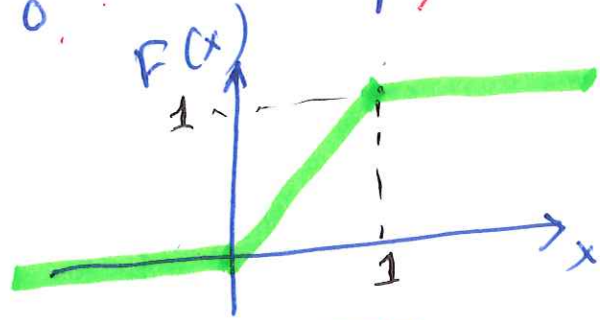
If $\boxed{x \geq 1}$:

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{0} \cancelto{0}{f(x)}\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} \cancelto{0}{f(x)}\,dx$$

$$= \boxed{1}$$



So, the cdf is:

$$\boxed{F(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x \geq 1 \end{cases}}$$

Suppose that F is the cdf of a continuous random variable $\underline{X}$ and is $\underline{\text{STRICTLY INCREASING}}$ on some interval I, and $\left.\begin{matrix} F = 0 \text{ on the left of } I \\ F = 1 \text{ on the right of } I \end{matrix}\right\}$

(I: may be unbounded).

On I: F^{-1} is well-defined

$x = F^{-1}(y)$ [when $y = F(x)$]

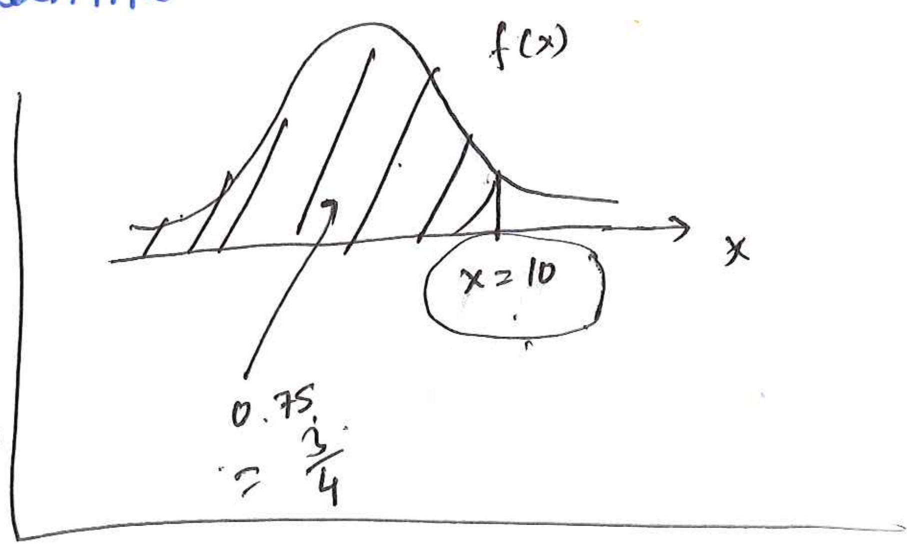
The p^{th} quantile of F is defined as the value x_p such that

$$F(x_p) = p$$

(i.e., $P(\underline{X} \leq x_p) = p$)

[Example: Suppose $x_p = 10$, for $p = 0.75$.
Then $P(\underline{X} \leq 10) = 0.75$]

75% quantile is $= 10$.



Specific case:

$p = \frac{1}{2}$: x_p : Median

$p = \frac{1}{4}$ or $p = \frac{3}{4}$

x_p : lower quartile

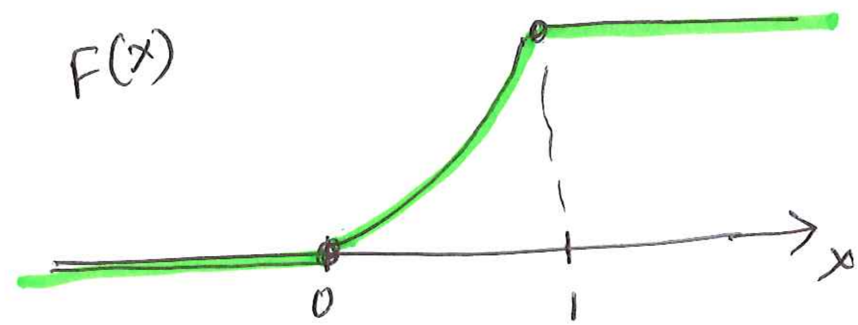
x_p : upper quartile

Example: Suppose

$F(x) = x^2$ for $0 \le x \le 1$

(for some continuous r.v. X)

$(f(x) = F'(x) = 2x, \quad 0 \le x \le 1)$



Goal: find x_p such that

(x_p: Median) $F(x_p) = 0.5$

$x_p = F^{-1}(0.5)$

$= \sqrt{0.5}$

$= 0.707$

$y = F(x) = x^2$

$x = \sqrt{y}$

$y = \sqrt{x}$

$\uparrow$

$F^{-1}(x)$

$\boxed{\text{Median} = 0.707}$

Upper quartile $= F^{-1}\left(\frac{3}{4}\right)$

$= \sqrt{\frac{3}{4}} = 0.866$

Lower quartile $= F^{-1}\left(\frac{1}{4}\right) = \sqrt{\frac{1}{4}} = 0.5$

Exponential density

If X has the pdf

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases} \quad \lambda > 0}$$

X is said to have an exponential distribution.

$\boxed{x \geq 0}$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = \cancel{\int_{-\infty}^{x} \lambda e^{-\lambda u}\,du}$$

$$= \int_{-\infty}^{0} 0\,du + \int_{0}^{x} \lambda e^{-\lambda u}\,du$$

$$= \left(1 - e^{-\lambda x}\right)$$

$\boxed{x < 0,}$

$$F(x) = \int_{-\infty}^{x} f(u)\,du = 0$$

$$\boxed{F(x) = \begin{cases} 1 - e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}}$$

Median of X: (Want to find x_p)

$$F(x_p) = \frac{1}{2}$$

$$1 - e^{-\lambda x_p} = \frac{1}{2}$$

$$e^{-\lambda x_p} = \frac{1}{2}$$

$$\cancel{-}\lambda x_p = \ln\left(\frac{1}{2}\right) = \cancel{-}\ln 2$$

$$\boxed{x_p = \frac{\ln 2}{\lambda}}$$