Ramakrishna Mission Vivekananda Educational and Research Institute
Problems - Random walk

1. Let S_n be a simple random walk (i.e., $S_n = \sum_{i=1}^n X_i$, for $X_i = \pm 1$, with (each) probability 1/2 and X_i's are independent.). Fix a number σ and define the process

$$P_n = e^{\sigma S_n}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^n. \quad , \sigma > 0 .$$

Show that $E(P_{n+1}|S_n) = P_n$. (i.e., when you know the random walk upto epoch n, the expected value of P_{n+1} is simply P_n.)

2. Fix an integer m. Let τ_m denote the first time the random walk reaches level m. That means, $\tau_m = \min\{n : S_n = m\}$. The random variable τ_m is called the *first passage time* of the random walk to level m. From the last problem it is clear that

$$1 = P_0 = E(P_{\min\{n,\tau_m\}}),$$

for every n, τ_m. Take the limit $n \to \infty$ to show that when $\tau_m < \infty$,

$$E\left[e^{\sigma m}\left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^{\tau_m}\right] = 1.$$

3. From the result in last problem show that for non-zero integer m,

$$E(\alpha^{\tau_m}) = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^{|m|},$$

for all $\alpha \in (0,1)$.

Solution

1) $$P_n = e^{\sigma S_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^n, \quad \sigma > 0.$$

$$P_{n+1} = e^{\sigma S_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}$$

$$\boxed{S_{n+1} = \sum_{i=1}^{n+1} X_i = \sum_{i=1}^{n} X_i + X_{n+1} = S_n + X_{n+1}}$$

$$\Rightarrow \quad P_{n+1} = e^{\sigma(S_n + X_{n+1})}\cdot\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{n+1}$$

$$= e^{\sigma S_n}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^n \cdot e^{\sigma X_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)$$

$$= P_n \, e^{\sigma X_{n+1}}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)$$

So, $$E\left(P_{n+1} \mid S_n\right) = P_n\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right) E\left(e^{\sigma X_{n+1}} \mid S_n\right)$$

$$= P_n\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)\left(\frac{1}{2}e^{\sigma(+1)} + \frac{1}{2}e^{\sigma(-1)}\right)$$

($\because X_{n+1} = \pm 1$ with prob $= \frac{1}{2}$)

$$= P_n \quad //$$ (done)

2) ~~$T_m = \min\{n : S_n = m\}$~~

By the last problem.

$$E\left(P_{n+1} \mid S_n\right) = P_n \quad \text{, for every } n. \quad \text{---: (1)}$$

So,

$$E\left(P_{n+2} \mid S_n\right) = E\left(E\left(P_{n+2} \mid S_{n+1}\right) \mid S_n\right)$$

by iterated condition expectation

by (1)

$$= E\left(P_{n+1} \mid S_n\right)$$

by (1)

$$= P_n.$$

So,

$$\boxed{E\left(P_m \mid S_n\right) = P_n \quad \text{for } 0 \le n \le m \quad \text{--- (2)}}$$

Now, if $\tau_m = \min\{n : S_n = m\}$

then

$$E\left(P_{\min\{n,\tau_m\}}\right) = E\left(P_{\min\{n,\tau_m\}} \mid S_0\right)$$

$\boxed{\text{by (2)}}$

$$= P_0 = e^{\sigma S_0}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^0$$

$$= 1 \quad (\because S_0 = 0)$$

for every n, τ_m. //

When $\tau_m < \infty$

$$\lim_{n\to\infty} \min\{n, \tau_m\} = \tau_m$$

So, $E\left(P_{\min\{n,\tau_m\}}\right) = 1$ gives

(by taking $n \to \infty$)

$$E\left(P_{\tau_m}\right) = 1$$

$$\Rightarrow E\left(e^{\sigma S_{\tau_m}} \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{\tau_m}\right) = 1$$

But $S_{\tau_m} = m$ (by definition of τ_m)!

So, $$E\left(e^{\sigma m} \left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{\tau_m}\right) = 1$$

(done)

3) At first, ~~WLOG~~ assume $m > 0$.

We know our discussion on the simple random walk that $S_n = m$ is recurrent.

So, $P(\tau_m < \infty) = 1$.

Now from problem #2 we have

$$E\left(e^{\sigma m}\left(\frac{2}{e^{\sigma}+e^{-\sigma}}\right)^{\tau_m}\right) = 1 \quad \cdots (*)$$

(for every $\sigma > 0$).

Let $\alpha \in (0,1)$ be given and solve for $\sigma > 0$ that satisfies

$$\alpha = \frac{2}{e^{\sigma}+e^{-\sigma}}.$$

$$\Rightarrow \alpha e^{\sigma} + \alpha e^{-\sigma} - 2 = 0.$$

(Multiply by e^{σ})

$$\Rightarrow \alpha e^{2\sigma} + \alpha - 2e^{\sigma} = 0$$

(Multiply by $e^{-2\sigma}$)

$$\Rightarrow \alpha e^{-2\sigma} - 2e^{-\sigma} + \alpha = 0$$

$$\Rightarrow \alpha (e^{-\sigma})^2 - 2(e^{-\sigma}) + \alpha = 0.$$

$$\Rightarrow \quad e^{-\sigma} = \frac{2 \pm \sqrt{4-4\alpha^2}}{2\alpha} = \frac{1 \pm \sqrt{1-\alpha^2}}{\alpha}$$

We want $\sigma > 0$. $\Rightarrow e^{-\sigma} < 1$.

$$\Rightarrow \quad e^{-\sigma} = \frac{1-\sqrt{1-\alpha^2}}{\alpha}$$

(the $\ominus$ sign is admissible)

Aside:

Verification that this means $\sigma > 0$

Note that we have chosen

$$0 < \alpha < 1$$

$$\Rightarrow \quad 0 < (1-\alpha)^2 = (1-\alpha) < 1-\alpha^2$$

Taking positive sq. root.

$$1-\alpha < \sqrt{1-\alpha^2}$$

$$\Rightarrow \quad 1-\sqrt{1-\alpha^2} < \alpha.$$

$$\Rightarrow \frac{1-\sqrt{1-\alpha^2}}{\alpha} < 1$$

$$\Rightarrow \quad e^{-\sigma} < 1 \Rightarrow \sigma > 0$$

With α and σ as defined.

(*) (on pg-4) gives.

$$E\left(\left(e^{\sigma}\right)^{m} . \alpha^{\tau_m}\right) = 1$$

$$\Rightarrow E\left(\left(\frac{\alpha}{1-\sqrt{1-\alpha^2}}\right)^{m} \alpha^{\tau_m}\right) = 1$$

$$\cancel{\Rightarrow E\left(\alpha^{\tau_m}\right) \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)}$$

$$\Rightarrow \left(\frac{\alpha}{1-\sqrt{1-\alpha^2}}\right)^{m} E\left(\alpha^{\tau_m}\right) = 1$$

$$\Rightarrow E\left(\alpha^{\tau_m}\right) = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^{m}.$$

This proves the result for $m>0$.

Now, τ_m and τ_{-m} are (due to symmetry) having the same distribution ~~for m>0~~

So, $E\left(\alpha^{\tau_m}\right) = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^{|m|}$

Done

Similarly, it can be shown that when $m < 0$,

$$E\left(\alpha^{T_m}\right) = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^{-m}$$

Hence, in general

$$E\left(\alpha^{T_m}\right) = \left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^{|m|}$$

(Done).