Time Series

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Outline I

- ACF and PACF of Stationary Time Series
 - ACF of Stationary Time Series
 - PACF of Stationary Time Series



ACF of Stationary Time Series I

• The autocorrelation function (ACF) of a stationary process, X_n , denoted as $\rho(h)$, for h = 0, 1, 2, ..., is defined as follows

$$\rho(h) = cor(X_{n+h}, X_n)$$

$$= \frac{E(X_{n+h}X_n)}{\sqrt{E[X_{n+h}^2]E[X_n^2]}}$$

- Remarks
 - The autocorrelation matrix R_n is positive definite for all n, where

$$R_n = \begin{bmatrix} 1 & \rho(1) & \cdots & \rho(n-1) \\ \rho(1) & 1 & \cdots & \rho(n-2) \\ \vdots & \vdots & \vdots & \vdots \\ \rho(n-1) & \rho(n-2) & \cdots & 1 \end{bmatrix}$$

ACF of MA(q) process I

q-order moving average or MA(q) process:

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \dots, \theta_q$ are real valued constants

ACF

$$\rho(h) = \begin{cases} \frac{1}{(1+\theta_1^2+\cdots+\theta_q^2)} \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & \text{if } |h| \leq q, \\ 0, & \text{if } |h| > q. \end{cases}$$

- where θ_0 is defined to be 1
- ACF of MA(q) process is **ZERO** for lags greater than q.
 - Cut-off to zero after lag q



ACF of AR(1) process I

• 1st order autoregressive or AR(1) process:

$$X_t = \phi X_{t-1} + Z_t, \ t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim \mathit{WN}(0,\sigma^2)$ and $|\phi| < 1$

• The ACF of an AR(1) process

$$\rho(h) = \phi^{|h|}$$

Tails off to zero

ACF of ARMA(1) process I

1st order ARMA or ARMA(1,1) process:

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2), |\phi| < 1, Z_t$ is uncorrelated with X_s for each s < t and $\phi + \theta \neq 0$

The ACF of an ARMA(1,1) process

$$\rho(h) = \begin{cases} 1, & \text{if } h = 0, \\ \frac{(\theta + \phi)(1 - \phi^2) + (\theta + \phi)^2 \phi}{(1 - \phi^2) + (\theta + \phi)^2}, & \text{if } h = \pm 1 \\ \phi^{|h| - 1} \rho(1), & \text{if } |h| \ge 2. \end{cases}$$

Tails off to zero

PACF of Stationary Time Series I

• The partial autocorrelation function (PACF) of a stationary process, X_n , denoted as $\alpha(h)$, for h = 0, 1, 2, ... is defined as follows

$$\alpha(\mathbf{0}) = \mathbf{1}, \alpha(\mathbf{1}) = \rho(\mathbf{1})$$

and

$$\alpha(h) = cor(X_{n+h} - X_{n+h}^{n+1,n+h-1}, X_n - X_n^{n+1,n+h-1}), h \ge 2$$

PACF of Stationary Time Series II

Remarks

- The PACF, $\alpha(h)$, is the correlation between X_{n+h} and X_n with the linear dependence of $\{X_{n+1}, \dots, X_{n+h-1}\}$ on each, removed.
- Both $(X_{n+h} X_{n+h}^{n+1,n+h-1})$ and $(X_n X_n^{n+1,n+h-1})$ are uncorrelated with $\{X_{n+1}, \dots, X_{n+h-1}\}$.
- If the process X_n is Gaussian, then

$$\alpha(h) = cor(X_{n+h}, X_n | X_{n+1}, \dots, X_{n+h-1}).$$

• That is, $\alpha(h)$ is the correlation coefficient between X_{n+h} and X_n in the bivariate distribution of (X_{n+h}, X_n) conditional on $\{X_{n+1}, \dots, X_{n+h-1}\}$.

PACF of Stationary Time Series III

- Theorem: $\alpha(h) = \phi_{hh}$
 - Recall, ϕ_{hh} is the last element of the vector ϕ_h and $\Gamma_h\phi_h=\gamma_h$
- Proof:-
 - Forward MSE:

$$E\left[\left(X_{n+h}-\sum_{i=1}^{h-1}a_iX_{n+h-i}\right)^2\right]$$

Normal Equations

$$E\left[\left(X_{n+h} - \sum_{i=1}^{h-1} a_i X_{n+h-i}\right) X_{n+h-j}\right] = 0, \text{ for } j = 1, \dots, h-1$$

Solution:

$$\gamma_{h-1} = \Gamma_{h-1} \mathbf{a}_{h-1}$$

PACF of Stationary Time Series IV

Backward MSE:

$$E\left[\left(X_n - \sum_{i=1}^{h-1} b_i X_{n+i}\right)^2\right]$$

Normal Equations

$$E\left[\left(X_{n}-\sum_{i=1}^{h-1}b_{i}X_{n+i}\right)X_{n+j}\right]=0, \text{ for } j=1,\ldots,h-1$$

Solution

$$\gamma_{h-1} = \Gamma_{h-1} b_{h-1}$$

Therefore,

$$\mathbf{a_{h-1}} = \mathbf{b_{h-1}} = \phi_{h-1}$$

PACF of Stationary Time Series V

As a result,

$$\alpha(h) = cor(X_{n+h} - X_{n+h}^{n+h-1,n+1}, X_n - X_n^{n+1,n+h-1})$$

$$= \frac{E\left[\left(X_{n+h} - \phi'_{h-1}X_{n+h-1,n+1}\right)\left(X_n - \phi'_{h-1}X_{n+1,n+h-1}\right)\right]}{\sqrt{E\left[\left(X_{n+h} - \phi'_{h-1}X_{n+h-1,n+1}\right)^2\right]E\left[\left(X_n - \phi'_{h-1}X_{n+1,n+h-1}\right)^2\right]}}$$

$$= \frac{E\left[\left(X_{n+h} - \phi'_{h-1}X_{n+h-1,n+1}\right)\left(X_n - \phi'^{(r)}_{h-1}X_{n+h-1,n+1}\right)\right]}{\sqrt{E\left[\left(X_{n+h} - \phi'_{h-1}X_{n+h-1,n+1}\right)^2\right]E\left[\left(X_n - \phi'^{(r)}_{h-1}X_{n+h-1,n+1}\right)^2\right]}}$$

$$= \frac{\gamma(h) - \phi'_{h-1}\gamma^{(r)}_{h-1} - \phi'^{(r)}_{h-1}\gamma_{h-1} + \phi'_{h-1}\Gamma_{h-1}\phi^{(r)}_{h-1}}{\sqrt{\left[\gamma(0) - \phi'_{h-1}\Gamma_{h-1}\phi_{h-1}\right]\left[\gamma(0) - \phi'^{(r)}_{h-1}\Gamma_{h-1}\phi^{(r)}_{h-1}\right]}}$$

$$= \frac{\gamma(h) - \phi'_{h-1}\gamma^{(r)}_{h-1} - \phi'_{h-1}\gamma^{(r)}_{h-1} + \phi'_{h-1}\gamma^{(r)}_{h-1}}{\gamma(0) - \phi'_{h-1}\Gamma_{h-1}\phi_{h-1}}$$

$$= \frac{\gamma(h) - \phi'_{h-1}\gamma^{(r)}_{h-1}}{V_{h-1}} = \phi_{hh}$$

PACF of AR(p) process I

• p-order autoregressive or AR(p) process:

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t, t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$, Z_t is uncorrelated with X_s for each s < t and all roots of the polynomial $(1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p)$ lie outside the unit circle.

PACF of AR(p) process II

- PACF of causal AR(p)
 - For $h \ge p$ the best linear predictor of X_{h+1} in terms of $1, X_1, \dots, X_h$ is

$$X_{h+1} = \phi_1 X_h + \phi_2 X_{h-1} + \cdots + \phi_p X_{h+1-p}.$$

Since the coefficient ϕ_{hh} of X_1 is ϕ_p if h = p and 0 if h > p, we conclude that the

$$\alpha(h) = \phi_p \text{ for } h = p$$

and

$$\alpha(h) = 0$$
 for $h > p$

- PACF of a causal AR(p) process is **ZERO** for lags greater than p.
 - Cut-off to zero after lag p

PACF of MA(1) process I

1st order moving average or MA(1) process:

$$X_t = Z_t + \theta Z_{t-1}, \ t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and θ is a real constant.

The PACF of an MA(1) process

$$\alpha(h) = \phi_{hh} = -(-\theta)^h/(1+\theta^2+\cdots+\theta^{2h}).$$

Tails off to zero

ACF & PACF of Stationary Time Series I

 Behavior of the ACF and PACF for Causal and Invertible ARMA Models

| | AR(p) | MA(q) | ARMA(p, q) |
|------|----------------------|----------------------|------------|
| ACF | Tails off | Cuts off after lag q | Tails off |
| PACF | Cuts off after lag p | Tails off | Tails off |