More on Hypothesis Testing Part - II

Dr. Moutushi Chatterjee
Indian Statistical Institute, Bangalore

Likelihood Ratio Test (LRT)

Suppose, a continuous random variable X has p.d.f. $f(x, \theta)$, where 'θ' is the unknown population parameter. Suppose, we want to test $H_0: \theta = \theta_0$ Ag. $H_1: \theta \neq \theta_0$.
Let a random sample of size 'n', say, $x_1, x_2, \ldots, x_n$ be drawn from the above popl^n. Then, $L_{\underline{x}}(\theta) = \prod_{i=1}^{n} f_{x_i}(\theta)$ is the likelihood function of θ for given $x_1, x_2, \ldots, x_n$.

Then, we reject our null hypothesis $H_0: \theta = \theta_0$ if $L_{\underline{x}}(\theta_0)$ is very small & accept H_0 otherwise.

Hence, a testing problem can be performed in this regard on the basis of the likelihood ratio criterion $\lambda(x_1, x_2, \ldots, x_n) < \lambda_0$, where, the likelihood ratio ~~criterion~~ is, $\lambda(x_1, x_2, \ldots, x_n) = \dfrac{L(\omega_0)}{\underset{\Theta \in \Theta}{\text{Max}} L(\Theta)}$, ... -(*)

Θ being the parametric space for ω regarding the given hypothesis - & λ_0 is a positive fraction.

Here, the critical region will be, $\omega = \{\underline{x} : \lambda(x_1, x_2, \ldots, x_n) < \lambda_0\}$, where λ_0 is $\ni$ $P_{H_0}(\omega) = \alpha$ [Size condition]. This type of testing procedure is called the likelihood ratio testing.

Since, here, ~~In other~~ the ~~al~~ alternative null hypothesis under consideration is composite, so this is also called the generalised LRT.

Decision Rule

Let us now consider the case of simple L.R.T.

Let, $X_1, X_2, \dots, X_n$ be a random sample from either $f_0(\cdot)$ or $f_1(\cdot)$. Then, a test 'γ' of $H_0: X_i \sim f_0(\cdot)$ $[\equiv \Theta = \omega_0]$ vs. $H_1: X_i \sim f_1(\cdot)$ $[\equiv \Theta = \omega_1 (\neq \omega_0)]$ is defined to be a simple likelihood-ratio test if 'γ' is defined by

Reject H_0 if $\lambda(\underline{x}) < \lambda_0$; Accept H_0 if $\lambda(\underline{x}) > \lambda_0$ & either accept or reject H_0 or randomise, if $\lambda(\underline{x}) = \lambda_0$, where,

$$\lambda(\underline{x}) = \frac{\prod_{i=1}^{n} f_0(x_i)}{\prod_{i=1}^{n} f_1(x_i)} = \frac{L_0(\underline{x})}{L_1(\underline{x})} = \frac{L_0}{L_1} = \frac{L_0(\omega_0)}{L_0(\omega_1)}$$

$$= \frac{L_0(\underline{x})}{L_1(\underline{x})} = \frac{L_0}{L_1},$$

where, λ_0 is a positive constant

& $L_j = L_j(\underline{x})$ is the likelihood function for sampling from the density $f_j(\cdot)$, $\forall j = 0, 1$.

density $f(\ldots)$

Note: (1) Although we used the same notation $\lambda(\underline{x})$ to denote the simple & composite L.R.T, but the generalized likelihood-ratio does not reduce to the simple likelihood ratio for $\Theta = \{\theta_0, \theta_1\}$.

(2) $\lambda(\underline{x})$ in (*) necessarily satisfies $0 \le \lambda \le 1$; $\lambda(\underline{x}) = \lambda \geq 0$, since we have a ratio of non-negative quantities & $\lambda \le 1$ since denominator is maximized for $\hat{\theta} \in \Theta$.

Merit of LRT: The generalised LRT makes good intuitive sense, since $\lambda(\underline{x})$ will tend to be small when H_0 is not true, as then, the dinominator of $\lambda(\underline{x})$ tends to be larger than the numerator.

Demerit of L.R.T.:

Although, in general, generalised LRT will be a good test, but, there are examples where it makes a poor showing compared to other tests.

i) One possible drawback of the test is that ~~it becomes~~ it is, sometimes difficult to find $\max_{\theta \in \Theta} L(\theta; x_1, \ldots, x_n)$.

ii) Another drawback of LRT is, that it may be difficult to find the dist^n of Λ which is required to evaluate the power of the test, where, Λ ~~is the~~ $= \lambda(X_1, \ldots, X_n)$ is a function of the random variables $X_1, X_2, \ldots, X_n$ & is itself a r.v. & as well as a statistic as it does not depend on any unknown parameter.

Properties of Likelihood ratio criterion

(i) $0 \le \lambda(\underset{\sim}{x}) \le 1$; under $H_0 \cup H_1$ it approaches to zero & when $\theta \in \omega_0$, then, $\lambda(\underset{\sim}{x}) \to 1$.

(ii) If α is the size of the ~~test~~ CR, then, the C.R. is defined by the relation $\lambda \le \lambda_\alpha$, where, λ_α is a constant $\ni P(\lambda \le \lambda_\alpha | H_0) = \alpha$. Hence, the distn of $\lambda(\underset{\sim}{x})$ is indispensible for the determination of the C.R. of the L.R. criterion.

(P.T.O)

(iii) Under certain regularity conditions, $-2\ln\lambda \overset{a}{\sim} \chi^2_{k-1}$.

(iv) The L.R. test is also consistent under certain very general assumptions.

(v) LR test may be biased.

(vi) If the UMP test exists, then it coincides with the LR test.

Example ①: Let, $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known. Suppose, we want a testing procedure to test $H_0: \mu = \mu_0$ vs. $H_1: \mu \neq \mu_0$. Perform on the basis of a likelihood ratio [L.R] criterion.

Solution: Since, $X_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall\, i = 1(1)n$, so, the likelihood function of μ is

$$L(\mu) = \begin{cases} \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}}, & -\infty < x_i < \infty,\ i = 1(1)n, \\ & -\infty < \mu < \infty \\ 0 & \text{o.w.} \end{cases}$$

$$\therefore L(\mu_0) = \begin{cases} \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu_0)^2}{\sigma^2}}, & -\infty < x_i < \infty, \forall i=1(1)n, \\ & -\infty < \mu < \infty \\ 0, & o.w. \end{cases}$$

Now, $\max\limits_{\mu \in H_0 \cup H_1} L(\mu) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2}}$, $\therefore \hat{\mu}_{MLE} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$,

where, $\mu \in H_0 \cup H_1 = \{\mu : -\infty < \mu < \infty\}$.

$\therefore$ According to LR principle, the LR criterion, will be,

$$\lambda(\underline{x}) < \lambda_0 \quad \text{ie,} \quad \frac{L(\mu_0)}{\max\limits_{\mu \in H_0 \cup H_1} L(\mu)} < \lambda_0$$

$$\Leftrightarrow \frac{\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu_0)^2}{\sigma^2}}}{\frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\bar{x})^2}{\sigma^2}}} < \lambda_0.$$

$$\Leftrightarrow e^{+\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n}(x_i-\bar{x})^2-\sum_{i=1}^{n}(x_i-\mu_0)^2\right\}} < \lambda_0.$$

$$\Leftrightarrow \sum_{i=1}^{n}(x_i-\bar{x})^2-\sum_{i=1}^{n}(x_i-\mu_0)^2 < \lambda_1, \quad \lambda_1 = 2\sigma^2 \ln \lambda_0.$$

$$\Leftrightarrow \sum_{i=1}^{n}x_i^2 - n\bar{x}^2 - \sum_{i=1}^{n}x_i^2 - n\mu_0^2 + 2n\mu_0\bar{x} < \lambda_1$$

$$\Leftrightarrow -n(\bar{x}^2 - 2\bar{x}\mu_0 + \mu_0^2) < \lambda_1$$

$$\Leftrightarrow n(\bar{x}-\mu_0)^2 > \lambda_2, \quad \text{where, } \lambda_2 = -\lambda_1, \text{ say.}$$

$$\Leftrightarrow \frac{n(\bar{x}-\mu_0)^2}{\sigma^2} > \lambda_3 \Leftrightarrow \frac{(\bar{x}-\mu_0)^2}{\sigma^2/n} > \lambda_3 \Leftrightarrow \left(\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right)^2 > \lambda_3.$$

$$\Leftrightarrow \left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > \lambda^*.$$

$\therefore$ Our size α UMPCR will be, $\omega = \{\underline{x} : \lambda(\underline{x}) < \lambda_0\} = \{\underline{x} : |\tau_{H_0}| > \lambda^*\}$.

where, $\tau_{H_0} = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \overset{H_0}{\sim} N(0, 1)$,

& λ^* is to be obtained from the size condition, ie $P_{H_0}(\omega) = \alpha$.

$\therefore P_{H_0}[|\tau_{H_0}| > \lambda^*] = \alpha \Rightarrow 1 - P_{H_0}[|\tau_{H_0}| < \lambda^*] = \alpha$

$\Rightarrow 1 - P_{H_0}[-\lambda^* < \tau_{H_0} < \lambda^*] = \alpha \Rightarrow 1 - \{P_{H_0}[\tau_{H_0} < \lambda^*] - P_{H_0}(\tau_{H_0} < -\lambda^*)\} = \alpha$.

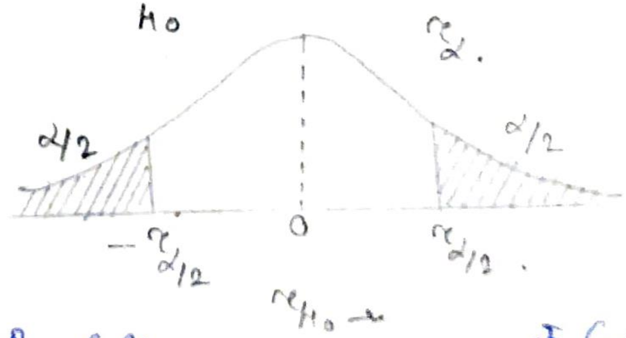
$\Rightarrow 1 - \{\Phi(\lambda^*) - 1 + \Phi(\lambda^*)\} = \alpha$.

$\therefore 2[1 - \Phi(\lambda^*)] = \alpha$

$\Rightarrow 1 - \Phi(\lambda^*) = \frac{\alpha}{2}$.

$\Rightarrow \Phi(-\lambda^*) = \alpha/2 = \Phi(-\tau_{\alpha/2})$.

~~Hence~~, $\Rightarrow -\lambda^* = -\tau_{\alpha/2} \Rightarrow \lambda^* = \tau_{\alpha/2}$.



$P_{H_0}(\tau_{H_0} < -\tau_{\alpha/2}) = \alpha/2 \Rightarrow \Phi(-\tau_{\alpha/2}) = \alpha/2$.

Hence, the size α UMPCR for testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ is given by, $\omega = \{\underline{x} : |\tau_{H_0}| > \tau_{\alpha/2}\} = \{\underline{x} : \left|\dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}\right| > \tau_{\alpha/2}\}$.

Problem : 4 Let, $x_1, x_2, \ldots, x_n \overset{iid}{\sim} N(\mu, \sigma^2)$ μ, σ^2 are both unknown. Then perform the LRT for testing $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$.

Solution: Since $x_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall\, i = 1(1)n$, so the where, μ & σ^2 are both unknown, so, the likelihood function of μ & σ^2 is,

$$L_{\underline{x}}(\mu, \sigma^2) = \frac{1}{(\sigma\sqrt{2\pi})^n}\, e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\sigma^2}}, \quad \forall\, i = 1(1)n.$$

$-\infty < x_i < \infty$, $-\infty < \mu < \infty$, $0 < \sigma^2 < \infty$.

Now, we know that, if $x_i \overset{iid}{\sim} N(\mu, \sigma^2)$, $\forall\, i = 1(1)n$ with μ & σ^2 both unknown, then, $\hat{\mu}_{MLE} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ & $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = s^2$,

[Also, we know that, if a distⁿ involves 'k' unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$ (say), & we want to test $H_0: \theta_1 = \theta_0$
vs. $H_1: \theta_1 \neq \theta_0$,

then, $H_0 = \{\theta_1, \ldots, \theta_k : \theta_1 = \theta_0, -\infty < \theta_i < \infty, \forall i = 2(1)k\}$.

& $H_0 \cup H_1 = \{\theta_1, \ldots, \theta_k : -\infty < \theta_i < \infty, \forall i = 1(1)k\}$.

Then, the LR criterion will be,

$$\lambda(\underline{x}) = \frac{\max\limits_{\theta_1, \ldots, \theta_k \in H_0} L(\underline{\theta})}{\max\limits_{\theta_1, \ldots, \theta_k \in H_0 \cup H_1} L(\underline{\theta})} \;]$$

Now, in the present case, we have, the parametric space of μ & σ^2 as,

$H_0 = \{\mu, \sigma^2 : \mu = \mu_0, 0 < \sigma^2 < \infty\}$.

$H_0 \cup H_1 = \{\mu, \sigma^2 : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$

$\therefore \max\limits_{\mu,\sigma^2 \in H_0} L(\mu,\sigma^2) = \frac{1}{(s_0\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum\limits_{i=1}^{n} \frac{(x_i-\mu_0)^2}{s_0^2}}$, where $s_0^2 = \frac{1}{n}\sum\limits_{i=1}^{n}(x_i-\mu_0)^2$.

Also, $\max\limits_{\mu,\sigma^2 \in H_0 \cup H_1} L(\mu,\sigma^2) = \frac{1}{(s\sqrt{2\pi})^n} e^{-\frac{1}{2}\sum\limits_{i=1}^{n} \frac{(x_i-\bar{x})^2}{s^2}} = \frac{1}{(s\sqrt{2\pi})^n} e^{-\frac{n}{2}}$.

& $\max\limits_{\mu,\sigma^2 \in H_0} L(\mu,\sigma^2) = \frac{1}{(s_0\sqrt{2\pi})^n} \cdot e^{-\frac{n}{2}}$.

Now, according to LR principle, the LR criterion will be

$\lambda(\underline{x}) < \lambda_0 \Rightarrow \frac{\max\limits_{\mu,\sigma^2 \in H_0} L(\mu,\sigma^2)}{\max\limits_{\mu,\sigma^2 \in H_0 \cup H_1} L(\mu,\sigma^2)} < \lambda_0$, where λ_0 is $\ni P_{H_0}(\omega_0) = \alpha$.

Now $\lambda(\underset{\sim}{x}) < \lambda_0 \iff \left(\frac{s}{s_0}\right)^n < \lambda_0 \iff \frac{s^2}{s_0^2} < \lambda_1 \iff \frac{s_0^2}{s^2} > \lambda_2$

$\iff \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2}{s^2} > \lambda_2 \Rightarrow \frac{\sum_{i=1}^{n}\{(x_i-\bar{x})+(\bar{x}-\mu_0)\}^2}{s^2} > \lambda_3$

$\iff \frac{n s^2 + n(\bar{x}-\mu_0)^2}{s^2/n} > \lambda_3 \iff \frac{n(\bar{x}-\mu_0)^2}{s^2} > \lambda_4 \iff \left|\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right| > \lambda^*$

$\therefore$ In this case, our size α UMPCR will be,

$\omega = \{ \underset{\sim}{x} : \lambda(\underset{\sim}{x}) < \lambda_0 \} = \{ \underset{\sim}{x} : |t_{H_0}| > \lambda^* \}$,

where, $t_{H_0} \overset{H_0}{\sim} t_{n-1}$ dist^n,

& λ^* is a constant to be obtained from the size cond^n $P_{H_0}(\omega_0) = \alpha$

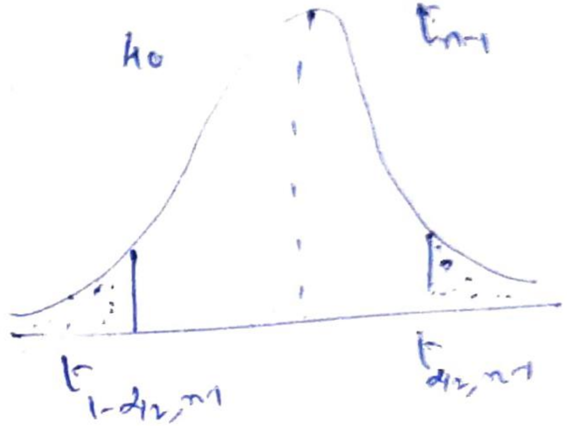
$\therefore P_{H_0}[\,|t_{H_0}| > \lambda^*] = \alpha \Rightarrow 1 - P_{H_0}[\,|t_{H_0}| < \lambda^*] = \alpha$.

$\Rightarrow 1 - P_{H_0}[-\lambda^* < t_{H_0} < \lambda^*] = \alpha$. $\therefore \lambda^* = t_{\alpha/2, n-1}$.

$\therefore$ The size α UMPCR for the test $H_0 : \mu = \mu_0$ vs. $H_1 : \mu \neq \mu_0$

is $\omega = \{ \underset{\sim}{x} : |t_{H_0}| > t_{\alpha/2, n-1} \}$.

$= \{ \underset{\sim}{x} : \left| \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \right| > t_{\alpha/2, n-1} \}$.



$= -t_{\alpha/2, n-1}$ $\therefore P(|t_{H_0}| < t_{\alpha/2, n-1}) = \alpha$.

Power function: Let, $\beta(\mu)$ denote the power function of the above UMP test. Then,

$$\beta(\mu) = P_H(\omega) = P_H\left(|t_{H_0}| > t_{\alpha/2, n-1}\right) = P_H\left(\left|\frac{\bar{x}-\mu_0}{s/\sqrt{n}}\right| > t_{\alpha/2, n-1}\right)$$

$$= 1 - P_H\left(-t_{\alpha/2, n-1} < \frac{\bar{x}-\mu_0}{s/\sqrt{n}} < t_{\alpha/2, n-1}\right)$$

$$= 1 - P_H\left[\frac{(\mu_0-\mu)}{s/\sqrt{n}} - t_{\alpha/2, n-1} < \frac{\bar{x}-\mu}{s/\sqrt{n}} < \frac{\mu_0-\mu}{s/\sqrt{n}} + t_{\alpha/2, n-1}\right]$$

$$= 1 - P\left(\frac{\bar{x}-\mu}{s/\sqrt{n}} < \frac{\mu_0-\mu}{s/\sqrt{n}} + t_{\alpha/2, n-1}\right) + P\left(\frac{\bar{x}-\mu}{s/\sqrt{n}} < \frac{\mu_0-\mu}{s/\sqrt{n}} - t_{\alpha/2, n-1}\right)$$

Ideal test

Definition: A test is said to be ideal if for this test, both the types of errors equal to 0, i.e. the # of all & consequently the power of the test is 1, so that the test is the most powerful.

Example: (1) Suppose, $X \sim R(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ & we want to test $H_0: \theta = 2.5$ vs. $H_1: \theta = 4.5$.

with the critical region ~~for~~ $\omega = \{4 \le x \le 5\}$ & acceptance region $\omega^c = \{2 \le x \le 3\}$.

$P(\text{type-I error}) = P_{H_0}(\omega) = 0$, $\because X \overset{H_0}{\sim} R(2,3)$, so, when $4 \le x \le 5$, ~~the probability~~ $f_{H_0}(x) = 0$

$P(\text{type-II error}) = P_{H_1}(\omega^c) = \int_2^3 \cancel{dx} = 0$, as $X \overset{H_1}{\sim} R(4,5)$, so, $\forall\, 2 \le x \le 3$ $f_{H_1}(x) = 0$.

$\therefore$ Power of the test $= \beta = 1 - P(\text{type-II error}) = 1 - 0 = 1. = P_{H_1}(\omega)$.

Example: (2) Suppose, $X \overset{H_0}{\sim} N(0,1)$ we want to test $H_0: X \sim N(0,1)$

vs. $H_1: X \sim P(1)$.

& our CR is $\omega = \{0,1,2,3 \cdots\}$ & acceptance region is $\omega^c = \{x : x \notin \omega\}$.

$\therefore P(\text{Type-I error}) = P_{H_0}(\omega) = 0$, $\because$ In CR as x assumes only discrete values, but a cont. variable cannot assume discrete values.

$$\text{Power} = \beta = P_{H_1}(\omega) = \sum_{x=0}^{\infty} \frac{e^{-1}}{x!} = 1, \quad \because \text{ sum is taken over the full range of } f_{H_1}(x).$$

$\therefore P(\text{Type-II error}) = 1 - \text{Power} = 1 - 1 = 0$

Alternatively $\because X \overset{H_1}{\sim} P(1)$, so, it can assume values $x = 0, 1, 2, \ldots$

But the acceptance region ω^c excludes all these values, so

so, $P(\text{type 2 error}) = P_{H_1}(\omega^c) = 0$. & as such $\beta = 1 - 0 = 1$.