Chapter 1

Special Functions of the Fractional Calculus

In this chapter some basic theory of the special functions which are used in the other chapters is given. We give here some information on the gamma and beta functions, the Mittag-Leffler functions, and the Wright function; these functions play the most important role in the theory of differentiation of arbitrary order and in the theory of fractional differential equations.

1.1 Gamma Function

Undoubtedly, one of the basic functions of the fractional calculus is Euler's gamma function $\Gamma(z)$, which generalizes the factorial $n!$ and allows n to take also non-integer and even complex values.

We will recall in this section some results on the gamma function which are important for other parts of this work.

1.1.1 Definition of the Gamma Function

The gamma function $\Gamma(z)$ is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.1}$$

which converges in the right half of the complex plane $Re(z) > 0$. Indeed, we have

$$\Gamma(x+iy) = \int_0^\infty e^{-t} t^{x-1+iy} dt = \int_0^\infty e^{-t} t^{x-1} e^{iy\log(t)} dt$$
$$= \int_0^\infty e^{-t} t^{x-1} [\cos(y\log(t)) + i\sin(y\log(t))] dt. \qquad (1.2)$$

The expression in the square brackets in (1.2) is bounded for all t; convergence at infinity is provided by e^{-t}, and for the convergence at $t = 0$ we must have $x = Re(z) > 1$.

1.1.2 Some Properties of the Gamma Function

One of the basic properties of the gamma function is that it satisfies the following functional equation:

$$\Gamma(z+1) = z\Gamma(z), \qquad (1.3)$$

which can be easily proved by integrating by parts:

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z dt = \left[-e^{-t}t^z\right]_{t=0}^{t=\infty} + z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z).$$

Obviously, $\Gamma(1) = 1$, and using (1.3) we obtain for $z = 1, 2, 3, \ldots$:

$$\begin{aligned}
\Gamma(2) &= 1 \cdot \Gamma(1) = 1 = 1!, \\
\Gamma(3) &= 2 \cdot \Gamma(2) = 2 \cdot 1! = 2!, \\
\Gamma(4) &= 3 \cdot \Gamma(3) = 3 \cdot 2! = 3!, \\
\cdots & \quad \cdots \quad \cdots \quad \cdots \\
\Gamma(n+1) &= n \cdot \Gamma(n) = n \cdot (n-1)! = n!
\end{aligned}$$

Another important property of the gamma function is that it has simple poles at the points $z = -n$, $(n = 0, 1, 2, \ldots)$. To demonstrate this, let us write the definition (1.1) in the form:

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt. \qquad (1.4)$$

The first integral in (1.4) can be evaluated by using the series expansion for the exponential function. If $Re(z) = x > 0$ (i.e. z is in the right half-plane), then $Re(z+k) = x + n > 0$ and $t^{z+k}\Big|_{t=0} = 0$. Therefore,

$$\begin{aligned}\int_0^1 e^{-t}t^{z-1}dt &= \int_0^1 \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} t^{z-1}dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 t^{k+z-1}dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+z)}.\end{aligned}$$

The second integral defines an entire function of the complex variable z. Indeed, let us write

$$\varphi(z) = \int_1^{\infty} e^{-t}t^{z-1}dt = \int_1^{\infty} e^{(z-1)\log(t)-t}dt. \tag{1.5}$$

The function $e^{(z-1)\log(t)-t}$ is a continuous function of z and t for arbitrary z and $t \geq 1$. Moreover, if $t \geq 1$ (and therefore $\log(t) \geq 0$), then it is an entire function of z. Let us consider an arbitrary bounded closed domain D in the complex plane ($z = x + iy$) and denote $x_0 = \max_{z \in D} Re(z)$. Then we have:

$$\begin{aligned}\left|e^{-t}t^{z-1}\right| &= \left|e^{(z-1)\log(t)-t}\right| = \left|e^{(x-1)\log(t)-t}\right| \cdot \left|e^{iy\log(t)}\right| \\ &= \left|e^{(x-1)\log(t)-t}\right| \leq e^{(x_0-1)\log(t)-t} = e^{-t}t^{x_0-1}.\end{aligned}$$

This means that the integral (1.5) converges uniformly in D and, therefore, the function $\varphi(z)$ is regular in D and differentiation under the integral in (1.5) is allowed. Because the domain D has been chosen arbitrarily, we conclude that the function $\varphi(z)$ has the above properties in the whole complex plane. Therefore, $\varphi(z)$ is an entire function allowing differentiation under the integral.

Bringing together the above considerations, we see that

$$\begin{aligned}\Gamma(z) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \int_1^{\infty} e^{-t}t^{z-1}dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{k+z} + \text{entire function},\end{aligned} \tag{1.6}$$

and, indeed, $\Gamma(z)$ has only simple poles at the points $z = -n$, $n = 0, 1, 2, \ldots$.

1.1.3 Limit Representation of the Gamma Function

The gamma function can be represented also by the limit

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)\dots(z+n)}, \tag{1.7}$$

where we initially suppose $Re(z) > 0$.

To prove (1.7), let us introduce an auxiliary function

$$f_n(z) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt. \tag{1.8}$$

Performing the substitution $\tau = \frac{t}{n}$ and then repeating integration by parts we obtain

$$\begin{aligned} f_n(z) &= n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau \\ &= \frac{n^z}{z} n \int_0^1 (1-\tau)^{n-1} \tau^z d\tau \\ &= \frac{n^z\, n!}{z(z+1)\dots(z+n-1)} \int_0^1 \tau^{z+n-1} d\tau \\ &= \frac{n^z\, n!}{z(z+1)\dots(z+n-1)(z+n)}. \end{aligned} \tag{1.9}$$

Taking into account the well-known limit

$$\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n = e^{-t}$$

we may expect that

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt = \int_0^\infty e^{-t} t^{z-1} dt, \tag{1.10}$$

which ends the proof of the limit representation (1.7) of the gamma function, if the interchange of the limit and the integral in (1.10) is justified. To do this, let us estimate the difference

$$\begin{aligned}\Delta &= \int_0^\infty e^{-t} t^{z-1} dt - f_n(z) \\ &= \int_0^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt. \qquad (1.11)\end{aligned}$$

Let us take an arbitrary $\epsilon > 0$. Because of the convergence of the integral (1.1) there exists an N such that for $n \geq N$ we have

$$\left| \int_n^\infty e^{-t} t^{z-1} dt \right| \leq \int_n^\infty e^{-t} t^{x-1} dt < \frac{\epsilon}{3}, \qquad (x = Re(z)). \qquad (1.12)$$

Fixing now N and considering $n > N$ we can write Δ as a sum of three integrals:

$$\Delta = \left(\int_0^N + \int_N^n \right) \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt + \int_n^\infty e^{-t} t^{z-1} dt. \qquad (1.13)$$

The last term is less then $\frac{\epsilon}{3}$. For the second integral we have:

$$\begin{aligned}\left| \int_N^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt \right| &\leq \int_N^n \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{x-1} dt \\ &< \int_N^\infty e^{-t} t^{x-1} dt < \frac{\epsilon}{3}, \qquad (1.14)\end{aligned}$$

where, as above, $x = Re(z)$.

For the estimation of the first integral in (1.13) we need the following auxiliary inequality:

$$0 < e^{-t} - \left(1 - \frac{t}{n}\right)^n < \frac{t^2}{2n}, \quad (0 < t < n), \qquad (1.15)$$

which follows from the relationships

$$1 - e^t \left(1 - \frac{t}{n}\right)^n = \int_0^t e^\tau \left(1 - \frac{\tau}{n}\right)^n \frac{\tau}{n} d\tau \qquad (1.16)$$

and

$$0 < \int_0^t e^\tau \left(1 - \frac{\tau}{n}\right)^n \frac{\tau}{n} d\tau < \int_0^t e^\tau \frac{\tau}{n} d\tau = e^t \frac{t^2}{2n}. \qquad (1.17)$$

(Relationship (1.16) can be verified by differentiating both sides.)

Using the auxiliary inequality (1.15) we obtain for large n and fixed N:

$$\left| \int_0^N \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n \right] t^{z-1} dt \right| < \frac{1}{2n} \int_0^N t^{x+1} dt < \frac{\epsilon}{3}. \qquad (1.18)$$

Taking into account inequalities (1.12), (1.14) and (1.18) and the arbitrariness of ϵ we conclude that the interchange of the limit and the integral in (1.10) is justified.

This definitely completes the proof of the formula (1.7) for the limit representation of the gamma function for $Re(z) > 0$.

With the help of (1.3) the condition $Re(z) > 0$ can be weakened to $z \neq 0, -1, -2, \ldots$ in the following manner.

If $-m < Re(z) \leq -m + 1$, where m is a positive integer, then

$$\begin{aligned}
\Gamma(z) &= \frac{\Gamma(z+m)}{z(z+1)\ldots(z+m-1)} \\
&= \frac{1}{z(z+1)\ldots(z+m-1)} \lim_{n\to\infty} \frac{n^{z+m} n!}{(z+m)\ldots(z+m+n)} \\
&= \frac{1}{z(z+1)\ldots(z+m-1)} \lim_{n\to\infty} \frac{(n-m)^{z+m}(n-m)!}{(z+m)(z+m+1)\ldots(z+n)} \\
&= \lim_{n\to\infty} \frac{n^z n!}{z(z+1)\ldots(z+n)}. \qquad (1.19)
\end{aligned}$$

Therefore, the limit representation (1.7) holds for all z excluding $z \neq 0, -1, -2, \ldots$

1.1.4 Beta Function

In many cases it is more convenient to use the so-called beta function instead of a certain combination of values of the gamma function.

The beta function is usually defined by

$$B(z, w) = \int_0^1 \tau^{z-1}(1-\tau)^{w-1} d\tau, \quad (Re(z) > 0, \quad Re(w) > 0). \qquad (1.20)$$

To establish the relationship between the gamma function defined by (1.1) and the beta function (1.20) we will use the Laplace transform.

Let us consider the following integral

$$h_{z,w}(t) = \int_0^t \tau^{z-1}(1-\tau)^{w-1}d\tau. \tag{1.21}$$

Obviously, $h_{z,w}(t)$ is a convolution of the functions t^{z-1} and t^{w-1} and $h_{z,w}(1) = B(z,w)$.

Because the Laplace transform of a convolution of two functions is equal to the product of their Laplace transforms, we obtain:

$$H_{z,w}(s) = \frac{\Gamma(z)}{s^z} \cdot \frac{\Gamma(w)}{s^w} = \frac{\Gamma(z)\Gamma(w)}{s^{z+w}}, \tag{1.22}$$

where $H_{z,w}(s)$ is the Laplace transform of the function $h_{z,w}(t)$.

On the other hand, since $\Gamma(z)\Gamma(w)$ is a constant, it is possible to restore the original function $h_{z,w}(t)$ by the inverse Laplace transform of the right-hand side of (1.22). Due to the uniqueness of the Laplace transform, we therefore obtain:

$$h_{z,w}(t) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} t^{z+w-1}, \tag{1.23}$$

and taking $t = 1$ we obtain the following expression for the beta function:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \tag{1.24}$$

from which it follows that

$$B(z,w) = B(w,z). \tag{1.25}$$

The definition of the beta function (1.20) is valid only for $Re(z) > 0$, $Re(w) > 0$. The relationship (1.24) provides the analytical continuation of the beta function for the entire complex plane, if we have the analytically continued gamma function.

With the help of the beta function we can establish the following two important relationships for the gamma function.

The first one is

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{1.26}$$

We will obtain the formula (1.26) under the condition $0 < Re(z) < 1$ and then show that it holds for $z \neq 0, \pm 1, \pm 2, \ldots$

Using (1.24) and (1.20) we can write

$$\Gamma(z)\Gamma(1-z) = B(z, 1-z) = \int\limits_0^1 \left(\frac{t}{1-t}\right)^{z-1} \frac{dt}{1-t}, \tag{1.27}$$

where the integral converges if $0 < Re(z) < 1$. Performing the substitution $\tau = t/(1-t)$ we obtain

$$\Gamma(z)\Gamma(1-z) = \int\limits_0^\infty \frac{\tau^{z-1}}{1+\tau} d\tau. \tag{1.28}$$

Let us now consider the integral

$$\int\limits_L f(s)ds, \quad f(s) = \frac{s^{z-1}}{1+s}, \tag{1.29}$$

along the contour shown in Fig. 1.1. The complex plane is cut along the real positive semi-axis.

The function $f(\tau)$ has a simple pole at $s = e^{\pi i}$. Therefore, for $R > 1$ we have

$$\int\limits_L f(s)ds = 2\pi i \Big[\mathrm{Res} f(s)\Big]_{s=e^{\pi i}} = -2\pi i e^{i\pi z}. \tag{1.30}$$

On the other hand, the integrals along the circumferences $|s| = \epsilon$ and $|s| = R$ vanish as $\epsilon \to 0$ and $R \to \infty$, and the integral along the lower cut edge differs from the integral along the upper cut edge by the factor $-e^{2\pi i z}$. Because of this, for $\epsilon \to 0$ and $R \to \infty$ we obtain:

$$\int\limits_L f(s)ds = 2\pi i \Big[\mathrm{Res} f(s)\Big]_{s=e^{\pi i}} = -2\pi i e^{i\pi z} = \Gamma(z)\Gamma(1-z)(1 - e^{2\pi i z}), \tag{1.31}$$

and

$$\Gamma(z)\Gamma(1-z) = \frac{2\pi i e^{i\pi z}}{e^{2\pi i z} - 1} = \frac{\pi}{\sin(\pi z)}, \quad (0 < Re(z) < 1). \tag{1.32}$$

If $m < Re(z) < m + 1$, then we can put $z = \alpha + m$, where $0 < Re(\alpha) < 1$. Using (1.3) we obtain

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= (-1)^m \Gamma(\alpha)\Gamma(1-\alpha) \\ &= \frac{(-1)^m \pi}{\sin(\pi\alpha)} = \frac{\pi}{\sin(\pi(\alpha+m))} = \frac{\pi}{\sin(\pi z)}, \end{aligned} \tag{1.33}$$

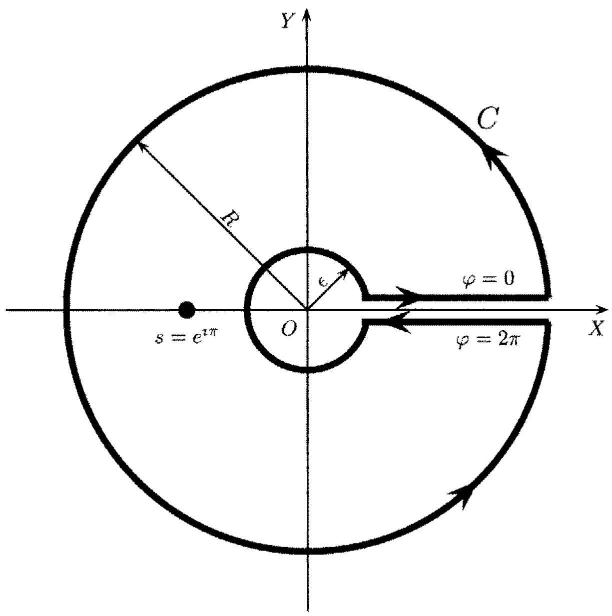

Figure 1.1: *Contour L.*

which shows that the relationship (1.26) holds for $z \neq 0, \pm 1, \pm 2, \ldots$

Taking $z = 1/2$ we obtain from (1.26) a useful particular value of the gamma function:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{1.34}$$

The second important relationship for the gamma function, easily obtained with the help of the beta function, is the Legendre formula

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = \sqrt{\pi} 2^{2z-1}\Gamma(2z), \quad (2z \neq 0, -1, -2, \ldots). \tag{1.35}$$

To prove the relationship (1.35) let us consider

$$B(z, z) = \int_0^1 [\tau(1-\tau)]^{z-1} d\tau, \quad (Re(z) > 0). \tag{1.36}$$

Taking into account the symmetry of the function $y(\tau) = \tau(1 - \tau)$

and performing the substitution $s = 4\tau(1-\tau)$ we obtain

$$B(z,z) = 2\int_0^{1/2} [\tau(1-\tau)]^{z-1} d\tau$$
$$= \frac{1}{2^{2z-1}} \int_0^1 s^{z-1}(1-s)^{-1/2} ds = 2^{1-2z} B(z, \frac{1}{2}), \qquad (1.37)$$

and using the relationship (1.24) we obtain from (1.37) the Legendre formula (1.35).

Taking $z = n + \frac{1}{2}$ in (1.35) we obtain a set of particular values of the gamma function

$$\Gamma(n + \frac{1}{2}) = \frac{\sqrt{\pi}\Gamma(2n+1)}{2^{2n}\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!} \qquad (1.38)$$

containing also (1.34).

1.1.5 Contour Integral Representation

The integration variable t in the definition of the gamma function (1.1) is real. If t is complex, then the function $e^{(z-1)\log(t)-t}$ has a branch point $t = 0$. Cutting the complex plane (t) along the real semi-axis from $t = 0$ to $t = +\infty$ makes this function single-valued. Therefore, according to Cauchy's theorem, the integral

$$\int_C e^{-t} t^{z-1} dt = \int_C e^{(z-1)\log(t)-t} dt$$

has the same value for any contour C running around the point $t = 0$ with both ends at $+\infty$.

Let us consider the contour C (see Fig. 1.2) consisting of the part of the upper edge $(+\infty, \epsilon)$ of the cut, the circle C_ϵ of radius ϵ with the centre at $t = 0$ and the part of the lower cut edge $(\epsilon, +\infty)$.

Taking $\log(t)$ to be real on the upper cut edge, we have

$$t^{z-1} = e^{(z-1)\log(t)}.$$

On the lower cut edge we must replace $\log(t)$ by $\log(t) + 2\pi i$:

$$t^{z-1} = e^{(z-1)(\log(t)+2\pi i)} = e^{(z-1)\log(t)} e^{(z-1)2\pi i} = t^{z-1} e^{2(z-1)\pi i}.$$

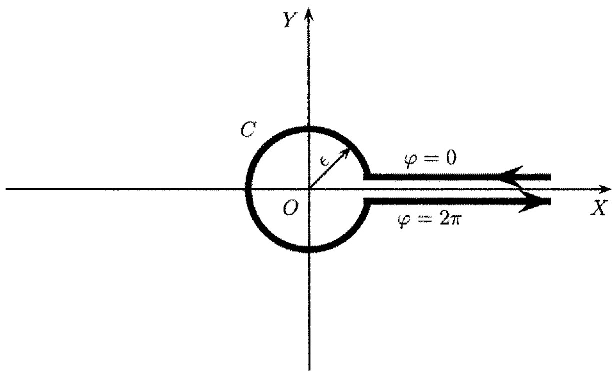

Figure 1.2: *Contour C.*

Therefore,

$$\int_C e^{-t}t^{z-1}dt = \int_{+\infty}^{\epsilon} e^{-t}t^{z-1}dt + \int_{C_\epsilon} e^{-t}t^{z-1}dt + e^{2(z-1)\pi\imath} \int_{\epsilon}^{+\infty} e^{-t}t^{z-1}dt. \tag{1.39}$$

Let us show that the integral along C_ϵ tends to zero as $\epsilon \to 0$. Indeed, taking into account that $|t| = \epsilon$ on C_ϵ and denoting

$$M = \max_{t \in C_\epsilon} \left| e^{-y\arg(t)-t} \right|, \qquad (y = Im(z)),$$

where M is independent of t, we obtain ($z = x + iy$):

$$\begin{aligned} \left| \int_{C_\epsilon} e^{-t}t^{z-1}dt \right| &\leq \int_{C_\epsilon} \left| e^{-t}t^{z-1} \right| dt = \int_{C_\epsilon} \left| t^{x-1} \right| \cdot \left| e^{-y\arg(t)-t} \right| dt \\ &\leq M\epsilon^{x-1} \int_{C_\epsilon} dt = M\epsilon^{x-1} \cdot 2\pi\epsilon = 2\pi M \epsilon^x, \end{aligned}$$

and therefore

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} e^{-t}t^{z-1}dt = 0 \tag{1.40}$$

and

$$\int_C e^{-t}t^{z-1}dt = \int_{+\infty}^{0} e^{-t}t^{z-1}dt + e^{2(z-1)\pi\imath} \int_{0}^{+\infty} e^{-t}t^{z-1}dt. \tag{1.41}$$

Using (1.1) we obtain:

$$\Gamma(z) = \frac{1}{e^{2\pi i z} - 1} \int_C e^{-t} t^{z-1} dt. \tag{1.42}$$

The function $e^{2\pi i z} - 1$ has its zeros at the points $z = 0, \pm 1, \pm 2, \ldots$. The points $z = 1, 2, \ldots$ are not the poles of $\Gamma(z)$, because in this case the function $e^{-t}t^{z-1}$ is single-valued and regular in the complex plane (t) and according to Cauchy's theorem

$$\int_C e^{-t} t^{z-1} dt = 0.$$

If $z = 0, -1, -2, \ldots$, then the function $e^{-t}t^{z-1}$ is not an entire function of t and the integral of it along the contour C is not equal to zero. Therefore, the points $z = 0, -1, -2, \ldots$ are the poles of $\Gamma(z)$. According to the principle of analytic continuation, the integral representation (1.42) holds not only for $Re(z) > 0$, as assumed at the beginning, but in the whole complex plane (z).

1.1.6 Contour Integral Representation of $1/\Gamma(z)$

In this section we give formulas for the integral representation of the reciprocal gamma function.

To obtain the simplest integral representation formula for $1/\Gamma(z)$ let us replace z by $1 - z$ in the formula (1.42), which leads to

$$\int_C e^{-t} t^{-z} dt = (e^{-2z\pi i} - 1)\, \Gamma(1-z), \tag{1.43}$$

and then perform the substitution $t = \tau e^{\pi i} = -\tau$. This substitution will transform (namely, turn it counterclockwise) the complex plane (t) with the cut along the real positive semi-axis into the complex plane (τ) with the cut along the real negative semi-axis. The lower cut edge $\arg(\tau) = -\pi$ in the (τ)-plane will correspond to the upper cut edge $t = 0$ in the (t)-plane. The contour C will be transformed to Hankel's contour Ha shown in Fig. 1.3. Then we have:

$$\int_C e^{-t} t^{-z} dt = -\int_{Ha} e^{\tau} (e^{\pi i}\tau)^{-z} d\tau = -e^{-z\pi i} \int_{Ha} e^{\tau} \tau^{-z} d\tau. \tag{1.44}$$

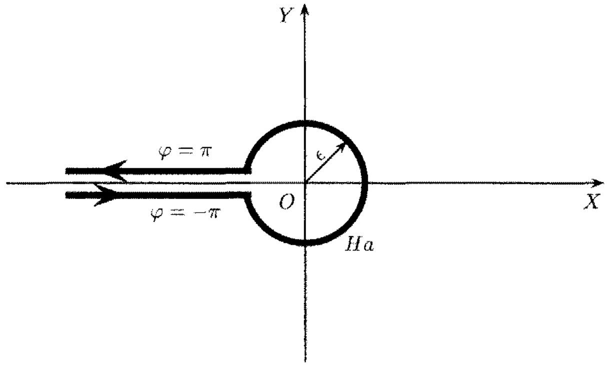

Figure 1.3: *The Hankel contour Ha.*

Taking into account the relationships (1.43) and (1.26) we obtain

$$\int\limits_{Ha} e^{\tau}\tau^{-z}d\tau = (e^{z\pi i} - e^{-z\pi i})\Gamma(1-z) = 2i\sin(\pi z)\Gamma(1-z) = \frac{2\pi i}{\Gamma(z)}. \quad (1.45)$$

Therefore, we have the following integral representation for the reciprocal gamma function:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int\limits_{Ha} e^{\tau}t^{-z}d\tau. \quad (1.46)$$

Let us now denote by $\gamma(\epsilon, \varphi)$ ($\epsilon > 0$, $0 < \varphi \leq \pi$) the contour consisting of the following three parts:

1) $\arg\tau = -\varphi, \qquad |\tau| \geq \epsilon$;
2) $-\varphi \leq \arg\tau \leq \varphi$, $|\tau| = \epsilon$;
3) $\arg\tau = -\varphi, \qquad |\tau| \geq \epsilon$.

The contour is traced so that $\arg\tau$ is non-decreasing. It is shown in Fig. 1.4.

The contour $\gamma(\epsilon, \varphi)$ divides the complex plane τ into two domains, which we denote by $G^-(\epsilon, \varphi)$ and $G^+(\epsilon, \varphi)$, lying correspondingly on the left and on the right side of the contour $\gamma(\epsilon, \varphi)$ (Fig. 1.4).

If $0 < \varphi < \pi$, then both $G^-(\epsilon, \varphi)$ and $G^+(\epsilon, \varphi)$ are infinite domains.

If $\varphi = \pi$, then $G^-(\epsilon, \varphi)$ becomes a circle $|\tau| < \epsilon$ and $G^+(\epsilon, \varphi)$ becomes a complex plane excluding the circle $|\tau| < \epsilon$ and the line $|\arg\varphi| = \pi$.

Let us show that instead of integrating along Hankel's contour Ha in (1.46) we can integrate along the contour $\gamma(\epsilon, \varphi)$, where $\frac{\pi}{2} < \varphi < \pi$,

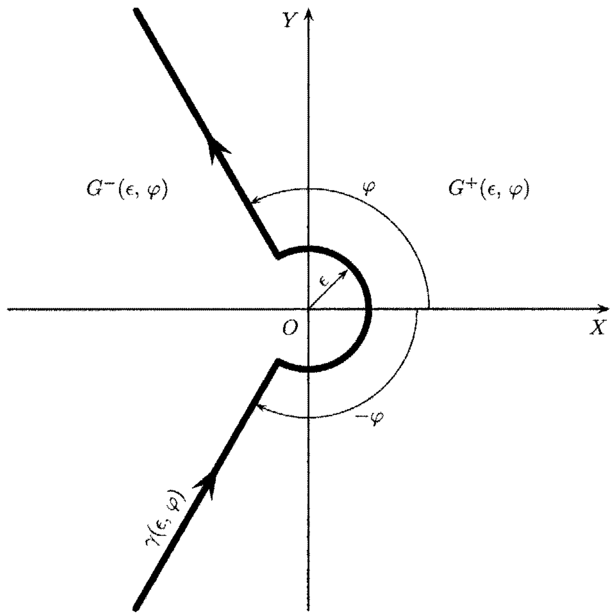

Figure 1.4: *Contour* $\gamma(\epsilon, \varphi)$.

i.e. that

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int\limits_{\gamma(\epsilon,\varphi)} e^{\tau} \tau^{-z} d\tau, \qquad (\epsilon > 0, \quad \frac{\pi}{2} < \varphi \leq \pi). \tag{1.47}$$

Let us consider the contour $(A^+B^+C^+D^+)$ shown in Fig. 1.5. Using the Cauchy theorem for the contour gives:

$$0 = \int\limits_{(A^+B^+C^+D^+)} e^{\tau} \tau^{-s} d\tau = \int\limits_{A^+}^{B^+} + \int\limits_{B^+}^{C^+} + \int\limits_{C^+}^{D^+} + \int\limits_{D^+}^{A^+} . \tag{1.48}$$

On the arc (A^+B^+) we have $|\tau| = R$ and

$$\begin{aligned} |e^{\tau}\tau^{-z}| &= e^{R\cos(\arg\tau) - x\log R + y\arg\tau} \\ &\leq e^{-R\cos(\pi-\varphi) - x\log R + 2\pi y}, \end{aligned}$$

from which it follows that

$$\lim_{R\to\infty} \int\limits_{A^+}^{B^+} = 0. \tag{1.49}$$

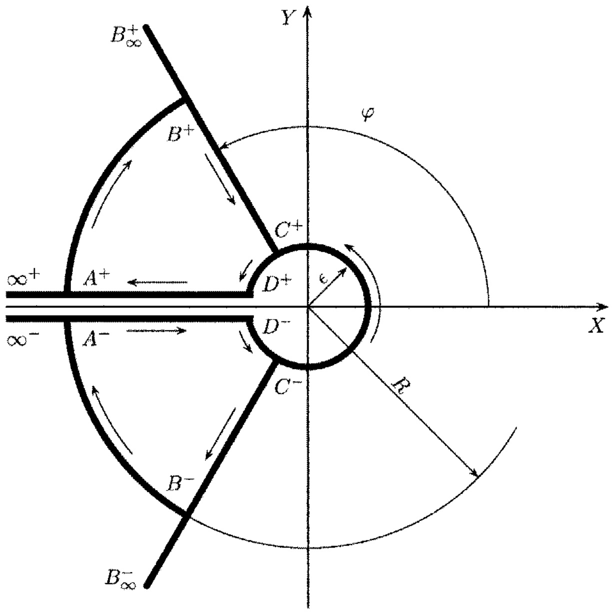

Figure 1.5: *Transformation of the contour Ha to the contour $\gamma(\epsilon, \varphi)$.*

Taking $R \to \infty$ in (1.48) and using (1.49) we obtain:

$$\int_{C^+}^{D^+} + \int_{D^+}^{\infty^+} + \int_{B_\infty^+}^{C^+} = 0$$

or

$$\int_{C^+}^{D^+} + \int_{D^+}^{\infty^+} = \int_{C^+}^{B_\infty^+} . \tag{1.50}$$

Similarly, consideration of the contour $(A^- D^- C^- B^-)$ leads to

$$\int_{\infty^-}^{D^-} + \int_{D^-}^{C^-} = \int_{B_\infty^-}^{C^-} . \tag{1.51}$$

Using (1.50) and (1.51) we see that

$$\int_{Ha} e^\tau t^{-z} d\tau = \left(\int_{B_\infty^-}^{C^-} + \int_{C^-}^{C^+} + \int_{C^+}^{B_\infty^+} \right) e^\tau t^{-z} d\tau = \int_{\gamma(\epsilon,\varphi)} e^\tau t^{-z} d\tau$$

and, indeed, the integral representation (1.47) for the reciprocal gamma function holds for all z.

Now we can obtain the following two integral representations for the reciprocal gamma function.

The first integral representation is obtained for arbitrary complex z. Let us perform the substitution $\tau = \zeta^{1/\alpha}$, $(\alpha < 2)$ in (1.47) and in the case of $1 \leq \alpha < 2$ consider only such contours $\gamma(\epsilon, \varphi)$ for which $\frac{\pi}{2} < \varphi < \frac{\pi}{\alpha}$. Due to this, since $\epsilon > 0$ is arbitrary, we arrive at the following integral representation

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha i} \int\limits_{\gamma(\epsilon,\mu)} \exp(\zeta^{1/\alpha})\zeta^{(1-z-\alpha)/\alpha} d\zeta, \tag{1.52}$$

$$\left(\alpha < 2, \quad \frac{\pi\alpha}{2} < \mu < \min\{\pi, \ \pi\alpha\}\right).$$

Another integral representation for $1/\Gamma(z)$ can be obtained if we note that in the case of $Re(z) > 0$ the formula (1.47) holds also for $\alpha = \pi/2$:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int\limits_{\gamma(\epsilon,\frac{\pi}{2})} e^u u^{-z} du, \qquad (\epsilon > 0, \quad Re(z) > 0). \tag{1.53}$$

Performing the substitution $u = \sqrt{\zeta}$ in (1.53), we obtain the integral representation

$$\frac{1}{\Gamma(z)} = \frac{1}{4\pi i} \int\limits_{\gamma(\epsilon,\pi)} \exp(\zeta^{1/2})\zeta^{-(z+1)/2} d\zeta, \qquad (\epsilon > 0, \quad Re(z) > 0). \tag{1.54}$$

We would like to emphasize that the integral representation (1.52) is valid for arbitrary z, whereas the integral representation (1.54) holds only if $Re(z) > 0$.

1.2 Mittag-Leffler Function

The exponential function, e^z, plays a very important role in the theory of integer-order differential equations. Its one-parameter generalization, the function which is now denoted by [65]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{1.55}$$

was introduced by G. M. Mittag-Leffler [155, 156, 157] and studied also by A. Wiman [256, 257].

The two-parameter function of the Mittag-Leffler type, which plays a very important role in the fractional calculus, was in fact introduced by Agarwal [3]. A number of relationships for this function were obtained by Humbert and Agarwal [107] using the Laplace transform technique. This function could have been called the Agarwal function. However, Humbert and Agarwal generously left the same notation as for the one-parameter Mittag-Leffler function, and that is the reason that now the two-parameter function is called the Mittag-Leffler function. We will use the name and the notation used in the fundamental handbook on special functions [65]. In spite of using the same notation as Agarwal, the definition given there differs from Agarwal's definition by a non-constant factor. Some parts of this section are based on results by M. M. Dzhrbashyan [45, Chapter III].

Regarding the distribution of zeros, the papers by A. Wiman [257], A. M. Sedletskii [240], R. Gorenflo, Yu. Luchko, and S. Rogosin [87], and the book by M. M. Dzhrbashyan [45, pp. 139–146] must be mentioned; we will not discuss them here.

1.2.1 Definition and Relation to Some Other Functions

A two-parameter function of the Mittag-Leffler type is defined by the series expansion [65]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad (\alpha > 0, \quad \beta > 0). \tag{1.56}$$

It follows from the definition (1.56) that

$$E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z, \tag{1.57}$$

$$E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)} = \frac{e^z - 1}{z}, \tag{1.58}$$

$$E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \sum_{k=0}^{\infty} \frac{z^k}{(k+2)!} = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)} = \frac{e^z - 1 - z}{z^2}, \tag{1.59}$$

and in general

$$E_{1,m}(z) = \frac{1}{z^{m-1}} \left\{ e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!} \right\}. \tag{1.60}$$

The hyperbolic sine and cosine are also particular cases of the Mittag-Leffler function (1.56):

$$E_{2,1}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} = \cosh(z), \tag{1.61}$$

$$E_{2,2}(z^2) = \sum_{k=0}^{\infty} \frac{z^{2k}}{\Gamma(2k+2)} = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} = \frac{\sinh(z)}{z}. \tag{1.62}$$

The hyperbolic functions of order n [65], which are generalizations of the hyperbolic sine and cosine, can also be expressed in terms of the Mittag-Leffler function:

$$h_r(z,n) = \sum_{k=0}^{\infty} \frac{z^{nk+r-1}}{(nk+r-1)!} = z^{r-1} E_{n,r}(z^n), \qquad (r = 1, 2, \ldots, n), \tag{1.63}$$

as well as the trigonometric functions of order n, which are generalizations of the sine and cosine functions:

$$k_r(z,n) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{nj+r-1}}{(nj+r-1)!} = z^{r-1} E_{n,r}(-z^n), \qquad (r = 1, 2, \ldots, n). \tag{1.64}$$

Using [2, formulas 7.1.3 and 7.1.8] we obtain

$$E_{1/2,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\frac{k}{2}+1)} = e^{z^2} \operatorname{erfc}(-z), \tag{1.65}$$

where $\operatorname{erfc}(z)$ is the error function complement defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-t^2} dt.$$

For $\beta = 1$ we obtain the Mittag-Leffler function in one parameter:

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} \equiv E_\alpha(z). \tag{1.66}$$

The function $\mathcal{E}_t(\nu, a)$, introduced in [153] for solving differential equations of rational order, is a particular case of the Mittag-Leffler function (1.56):

$$\mathcal{E}_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + k + 1)} = t^\nu E_{1,\nu+1}(at). \tag{1.67}$$

Yu. N. Rabotnov's [218] function $\ni_\alpha(\beta, t)$ is a particular case of the Mittag-Leffler function (1.56) too:

$$\ni_\alpha(\beta, t) = t^\alpha \sum_{k=0}^{\infty} \frac{\beta^k t^{k(\alpha+1)}}{\Gamma((k+1)(1+\alpha))} = t^\alpha E_{\alpha+1,\alpha+1}(\beta t^{\alpha+1}). \tag{1.68}$$

It follows from the relationships (1.67) and (1.68) that the properties of the Miller–Ross function and Rabotnov's function can be deduced from the properties of the Mittag-Leffler function in two parameters (1.56).

Plotnikov [190, cf. [250]] and Tseytlin [250] used in their investigations two functions $Sc_\alpha(z)$ and $Cs_\alpha(z)$, which they call the fractional sine and cosine. Those functions are also just particular cases of the Mittag-Leffler function in two parameters:

$$Sc_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n+1}}{\Gamma((2-\alpha)n + 2)} = z E_{2-\alpha,2}(-z^{2-\alpha}), \tag{1.69}$$

$$Cs_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{(2-\alpha)n}}{\Gamma((2-\alpha)n + 1)} = E_{2-\alpha,1}(-z^{2-\alpha}). \tag{1.70}$$

Another "fractionalization" of the sine and cosine functions, which can also be expressed in terms of the Mittag-Leffler function (1.56), has been suggested by Luchko and Srivastava [128]:

$$\sin_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{\Gamma(2\mu k + 2\mu - \lambda + 1)} = z E_{2\mu,2\mu-\lambda+1}(-z^2), \tag{1.71}$$

$$\cos_{\lambda,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\Gamma(2\mu k + \mu - \lambda + 1)} = E_{2\mu,\mu-\lambda+1}(-z^2). \tag{1.72}$$

Of course, the properties of both versions of the fractional sine and cosine follow from the properties of the Mittag-Leffler function (1.56).

Generalizations of the Mittag-Leffler function (1.56) to two variables, suggested by P. Humbert and P. Delerue [108] and by A. M. Chak [36], were further extended by H. M. Srivastava [243] to the following symmetric form:

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^{m+\frac{\beta(\nu n+1)-1}{\alpha}} y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma\Big(m\alpha + (\nu n+1)\beta\Big)\,\Gamma\Big(n\lambda + (\sigma m+1)\mu\Big)}. \tag{1.73}$$

An interesting generalization of the Mittag-Leffler function to several variables has been suggested by S. B. Hadid and Yu. Luchko [100], who used it for solving linear fractional differential equations with constant coefficients by the operational method:

$$E_{(\alpha_1,\ldots,\alpha_m),\beta}(z_1,\ldots,z_m) \tag{1.74}$$

$$= \sum_{k=0}^{\infty} \sum_{\substack{l_1+\ldots+l_m=k \\ l_1>0,\ldots,l_m>0}} \frac{(k;l_1,\ldots,l_m)\prod\limits_{i=1}^{m} z_i^{l_i}}{\Gamma(\beta+\sum\limits_{i=1}^{m}\alpha_i l_i)},$$

where $(k;l_1,\ldots,l_m)$ are multinomial coefficients [2].

1.2.2 The Laplace Transform of the Mittag-Leffler Function in Two Parameters

As follows from relationship (1.57), the Mittag-Leffler function $E_{\alpha,\beta}(z)$ is a generalization of the exponential function e^z and, therefore, the exponential function is a particular case of the Mittag-Leffler function.

We will outline here the way to obtain the Laplace transform of the Mittag-Leffler function with the help of the analogy between this function and the function e^z. For this purpose, let us obtain the Laplace transform of the function $t^k e^{at}$ in an untraditional way.

First, let us prove that

$$\int_0^{\infty} e^{-t} e^{\pm zt}\,dt = \frac{1}{1\mp z}, \qquad |z|<1. \tag{1.75}$$

Indeed, using the series expansion for e^z, we obtain

$$\int_0^{\infty} e^{-t}e^{zt}\,dt = \frac{1}{1-z} = \sum_{k=0}^{\infty}\frac{(\pm z)^k}{k!}\int_0^{\infty} e^{-t}t^k\,dt = \sum_{k=0}^{\infty}(\pm z)^k = \frac{1}{1\mp z}. \tag{1.76}$$

Second, we differentiate both sides of equation (1.75) with respect to z. The result is

$$\int_0^\infty e^{-t} t^k e^{\pm zt} dt = \frac{k!}{(1-z)^{k+1}}, \quad (|z| < 1), \tag{1.77}$$

and after obvious substitutions we obtain the well-known pair of Laplace transforms of the function $t^k e^{\pm at}$:

$$\int_0^\infty e^{-pt} t^k e^{\pm at} dt = \frac{k!}{(p \mp a)^{k+1}}, \quad (Re(p) > |a|). \tag{1.78}$$

Let us now consider the Mittag-Leffler function (1.56). Substitution of (1.56) in the integral below leads to

$$\int_0^\infty e^{-t} t^{\beta-1} E_{\alpha,\beta}(zt^\alpha) dt = \frac{1}{1-z}, \quad (|z| < 1), \tag{1.79}$$

and we obtain from (1.79) a pair of Laplace transforms of the function $t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm z t^\alpha)$, $(E^{(k)}_{\alpha,\beta}(y) \equiv \frac{d^k}{dy^k} E_{\alpha,\beta}(y))$:

$$\int_0^\infty e^{-pt} t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\pm a t^\alpha) dt = \frac{k!\, p^{\alpha-\beta}}{(p^\alpha \mp a)^{k+1}}, \quad (Re(p) > |a|^{1/\alpha}). \tag{1.80}$$

The particular case of (1.80) for $\alpha = \beta = \frac{1}{2}$

$$\int_0^\infty e^{-pt} t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}(\pm a\sqrt{t}) dt = \frac{k!}{(\sqrt{p} \mp a)^{k+1}}, \quad (Re(p) > a^2). \tag{1.81}$$

is useful for solving the semidifferential equations considered in [179, 153].

1.2.3 Derivatives of the Mittag-Leffler Function

By the Riemann–Liouville fractional-order differentiation ${}_0D_t^\gamma$ (γ is an arbitrary real number) of series representation (1.56) we obtain

$${}_0D_t^\gamma \left(t^{\alpha k+\beta-1} E^{(k)}_{\alpha,\beta}(\lambda t^\alpha)\right) = t^{\alpha k+\beta-\gamma-1} E^{(k)}_{\alpha,\beta-\gamma}(\lambda t^\alpha). \tag{1.82}$$

The particular case of relationship (1.82) for $k=0$, $\lambda=1$ and integer γ is given in [65], equation 18.1(25) and has the form

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{\alpha,\beta}(t^\alpha)\right) = t^{\beta-m-1}E_{\alpha,\beta-m}(t^\alpha), \quad (m=1,2,3,\ldots). \tag{1.83}$$

Formula (1.83) has some interesting consequences. Taking $\alpha = \frac{m}{n}$, where m and n are natural numbers, we obtain

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{m/n,\beta}(t^{m/n})\right) = t^{\beta-1}E_{m/n,\beta}(t^{m/n}) + t^{\beta-1}\sum_{k=1}^{n}\frac{t^{-\frac{m}{n}k}}{\Gamma\left(\beta-\frac{m}{n}k\right)}, \tag{1.84}$$

$$(m,n=1,2,3,\ldots).$$

Setting $n=1$ and taking into account the well-known property of the gamma function

$$\frac{1}{\Gamma(-\nu)} = 0 \quad (\nu=0,1,2,\ldots),$$

we obtain from (1.84) that

$$\left(\frac{d}{dt}\right)^m \left(t^{\beta-1}E_{m,\beta}(t^m)\right) = t^{\beta-1}E_{m,\beta}(t^m), \tag{1.85}$$

$$(m=1,2,3,\ldots; \quad \beta=0,1,2,\ldots,m).$$

Performing the substitution $t=z^{n/m}$ in (1.84) we obtain

$$\left(\frac{m}{n}z^{1-\frac{n}{m}}\frac{d}{dz}\right)^m \left(z^{(\beta-1)n/m}E_{m/n,\beta}(z)\right)$$

$$= z^{(\beta-1)n/m}E_{m/n,\beta}(z) + t^{(\beta-1)n/m}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{m}{n}k\right)} \tag{1.86}$$

$$(m,n=1,2,3,\ldots).$$

Taking $m=1$ in (1.86), we obtain the following expression:

$$\frac{1}{n}\frac{d}{dz}\left(z^{(\beta-1)n}E_{1/n,\beta}(z)\right) = z^{\beta n-1}E_{1/n,\beta}(z) + z^{\beta n-1}\sum_{k=1}^{n}\frac{z^{-k}}{\Gamma\left(\beta-\frac{k}{n}\right)}, \tag{1.87}$$

$$(n=1,2,3,\ldots).$$

1.2.4 Differential Equations for the Mittag-Leffler Function

It is worthwhile noting that relationships (1.84)–(1.87) can also be interpreted as differential equations for the Mittag-Leffler function; namely, if we denote

$$
\begin{aligned}
y_1(t) &= t^{\beta-1} E_{m/n,\beta}(t^{m/n}), \\
y_2(t) &= t^{\beta-1} E_{m,\beta}(t^m), \\
y_3(t) &= t^{(\beta-1)n/m} E_{m/n,\beta}(t), \\
y_4(t) &= t^{(\beta-1)n} E_{1/n,\beta}(t),
\end{aligned}
$$

then these functions satisfy the following differential equations respectively:

$$
\frac{d^m y_1(t)}{dt^m} - y_1(t) = t^{\beta-1} \sum_{k=1}^{n} \frac{t^{-\frac{m}{n}k}}{\Gamma\left(\beta - \frac{m}{n}k\right)}, \tag{1.88}
$$

$$
(m, n = 1, 2, 3, \ldots)
$$

$$
\frac{d^m y_2(t)}{dt^m} - y_2(t) = 0, \tag{1.89}
$$

$$
(m = 1, 2, 3, \ldots; \quad \beta = 0, 1, 2, \ldots, m)
$$

$$
\left(\frac{m}{n} t^{1-\frac{n}{m}} \frac{d}{dt}\right)^m y_3(t) - y_3(t) = t^{(\beta-1)n/m} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma\left(\beta - \frac{m}{n}k\right)} \tag{1.90}
$$

$$
(m, n = 1, 2, 3, \ldots)
$$

$$
\frac{1}{n} \frac{dy_4(t)}{dt} y_4(t) - t^{n-1} y_4(t) = t^{\beta n-1} \sum_{k=1}^{n} \frac{t^{-k}}{\Gamma\left(\beta - \frac{k}{n}\right)}, \tag{1.91}
$$

$$
(n = 1, 2, 3, \ldots).
$$

1.2.5 Summation Formulas

Let us start with the obvious relationship

$$
\sum_{\nu=0}^{m-1} e^{i2\pi\nu k/m} = \begin{cases} m, & \text{if } k \equiv 0 \pmod{m} \\ 0, & \text{if } k \not\equiv 0 \pmod{m} \end{cases} \tag{1.92}
$$

where the notation $k \equiv p \pmod{m}$ means that the remainder of the division of $k - p$ by m is zero (k, p and m are integer numbers).

Combining (1.92) and the definition (1.56) of the Mittag-Leffler function, we obtain

$$\sum_{\nu=0}^{m-1} E_{\alpha,\beta}(z e^{\imath 2\pi\nu/m}) = m E_{m\alpha,\beta}(z^m), \quad (m \geq 1). \tag{1.93}$$

Replacing α with $\frac{\alpha}{m}$ and z with $z^{1/m}$ in (1.93), we arrive at

$$E_{\alpha,\beta}(z) = \frac{1}{m} \sum_{\nu=0}^{m-1} E_{\alpha/m,\beta}(z^{1/m} e^{\imath 2\pi\nu/m}), \quad (m \geq 1). \tag{1.94}$$

The following particular case of formula (1.94) must be mentioned. Taking $m = 2$ and $z = t^2$, we obtain

$$E_{\alpha,\beta}(z) + E_{\alpha,\beta}(-z) = 2E_{\alpha,\beta}(z^2). \tag{1.95}$$

Similarly, starting with the obvious formula

$$\sum_{\nu=-m}^{m} e^{\imath 2\pi\nu k/(2m+1)} = \begin{cases} 2m+1, & \text{if } k \equiv 0 \pmod{2m+1} \\ 0, & \text{if } k \not\equiv 0 \pmod{2m+1}, \end{cases} \tag{1.96}$$

we obtain

$$E_{\alpha,\beta}(z) = \frac{1}{2m+1} \sum_{\nu=-m}^{m-1} E_{\alpha/(2m+1),\beta}(z^{1/(2m+1)} e^{\imath 2\pi\nu/(2m+1)}), \quad (m \geq 0). \tag{1.97}$$

A generalization of the summation formula (1.93) has been obtained by H. M. Srivastava [244]:

$$\sum_{\nu=0}^{m-1} e^{i 2\pi\nu(m-n)/m} E_{\alpha,\,\beta}(z e^{\imath 2\pi\nu}/m) = m z^n E_{m\alpha,\,\beta+n\alpha}(z^m). \tag{1.98}$$

Obviously, for $n = 0$ the relationship (1.98) gives the summation formula (1.93).

1.2.6 Integration of the Mittag-Leffler Function

Integrating (1.56) term-by-term, we obtain

$$\int_0^z E_{\alpha,\beta}(\lambda t^\alpha) t^{\beta-1} dt = z^\beta E_{\alpha,\beta+1}(\lambda z^\alpha), \quad (\beta > 0). \tag{1.99}$$

Relationship (1.99) is a particular case of the following more general relationship obtained by the fractional-order term-by-term integration of the series (1.56):

$$\frac{1}{\Gamma(\nu)}\int_0^z (z-t)^{\nu-1}E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt = z^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda z^\alpha), \qquad (1.100)$$

$$(\beta > 0, \quad \nu > 0).$$

From (1.100) and formulas (1.57), (1.61) and (1.62) we obtain:

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}e^{\lambda t}dt = z^\alpha E_{1,\alpha+1}(\lambda z), \quad (\alpha > 0), \qquad (1.101)$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\cosh(\sqrt{\lambda}t)dt = z^\alpha E_{2,\alpha+1}(\lambda z^2), \quad (\alpha > 0), \qquad (1.102)$$

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}\frac{\sinh(\sqrt{\lambda}t)}{\sqrt{\lambda}t}dt = z^{\alpha+1}E_{2,\alpha+2}(\lambda z^2), \quad (\alpha > 0). \qquad (1.103)$$

Let us also prove the following formula for the fractional integration of the Mittag-Leffler function:

$$\frac{1}{\Gamma(\alpha)}\int_0^z (z-t)^{\alpha-1}E_{2\alpha,\beta}(t^{2\alpha})t^{\beta-1}dt = -z^{\beta-1}E_{2\alpha,\beta}(z^{2\alpha}) + z^{\beta-1}E_{\alpha,\beta}(z^\alpha). \qquad (1.104)$$

To prove (1.104), let us consider the integral

$$\begin{aligned}
&\int_0^z E_{2\alpha,\beta}(t^{2\alpha})t^{\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt \\
&= \sum_{k=0}^\infty \frac{1}{\Gamma(2k\alpha+\beta)}\int_0^z t^{2k\alpha+\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt \\
&= z^\beta\sum_{k=0}^\infty \frac{z^{2k\alpha}}{\Gamma(2k\alpha+\beta+1)} + z^\beta\sum_{k=0}^\infty \frac{z^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+\beta+1)} \\
&= z^\beta\sum_{k=0}^\infty \frac{z^{k\alpha}}{\Gamma(k\alpha+\beta+1)} = z^\beta E_{\alpha,\beta+1}(z^\alpha). \qquad (1.105)
\end{aligned}$$

Comparing (1.105) and (1.99) we have

$$\int_0^z E_{2\alpha,\beta}(t^{2\alpha})t^{\beta-1}\left\{1+\frac{(z-t)^\alpha}{\Gamma(1+\alpha)}\right\}dt = \int_0^z E_{\alpha,\beta}(\lambda t^\alpha)t^{\beta-1}dt, \quad (\beta>0). \tag{1.106}$$

Differentiating (1.106) with respect to z, we obtain (1.104).

There is also an interesting relationship for the Mittag-Leffler function, which is similar to the Cristoffel–Darboux formula for orthogonal polynomials; namely,

$$\int_0^t \tau^{\gamma-1}E_{\alpha,\gamma}(y\tau^\alpha)(t-\tau)^{\beta-1}E_{\alpha,\beta}(z(t-\tau)^\alpha)d\tau$$

$$= \frac{yE_{\alpha,\gamma+\beta}(yt^\alpha)-zE_{\alpha,\gamma+\beta}(zt^\alpha)}{y-z}t^{\gamma+\beta-1}, \quad (\gamma>0, \quad \beta>0), \tag{1.107}$$

where y and z ($y \neq z$) are arbitrary complex numbers.

Indeed, using the definition of the Mittag-Leffler function (1.56) we have:

$$\int_0^t \tau^{\gamma-1}E_{\alpha,\gamma}(y\tau^\alpha)(t-\tau)^{\beta-1}E_{\alpha,\beta}(z(t-\tau)^\alpha)d\tau$$

$$\begin{aligned}
&= \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{y^n z^m}{\Gamma(\alpha n+\gamma)\Gamma(\alpha m+\beta)}\int_0^t \tau^{\alpha n+\gamma-1}(t-\tau)^{\alpha m+\beta-1}d\tau \\
&= \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{y^n z^m t^{\alpha(n+m)+\beta+\gamma-1}}{\Gamma(\alpha(n+m)+\beta+\gamma)} \\
&= t^{\beta+\gamma-1}\sum_{n=0}^\infty\sum_{k=n}^\infty \frac{y^n z^{k-n}t^{\alpha k}}{\Gamma(\alpha k+\beta+\gamma)} \\
&= t^{\beta+\gamma-1}\sum_{k=0}^\infty \frac{z^k t^{\alpha k}}{\Gamma(\alpha k+\beta+\gamma)}\sum_{n=0}^k\left(\frac{y}{z}\right)^n \\
&= \frac{t^{\beta+\gamma-1}}{y-z}\sum_{k=0}^\infty \frac{t^{\alpha k}(y^{k+1}-z^{k+1})}{\Gamma(\alpha k+\beta+\gamma)},
\end{aligned} \tag{1.108}$$

and utilizing the definition (1.56) we obtain (1.107).

Another interesting formula establishes the relationship between the Mittag-Leffler function and the function $e^{-x^2/4t}$. This relationship plays an important role in the solution of the diffusion (heat conduction, mass transfer) equation:

$$\int_0^\infty e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} dx = \sqrt{\pi} t^{\beta/2} E_{\alpha/2,(\beta+1)/2}(t^{\alpha/2}), \qquad (1.109)$$

$$(\beta > 0, \quad t > 0).$$

To prove formula (1.109) we note that for every fixed value of t the series

$$e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} = \sum_{k=0}^\infty \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} e^{-x^2/4t}, \quad (\beta > 0) \qquad (1.110)$$

can be term-by-term integrated from 0 to ∞. Performing the integration we obtain:

$$\begin{aligned} \int_0^\infty e^{-x^2/4t} E_{\alpha,\beta}(x^\alpha) x^{\beta-1} dt &= \int_0^\infty \left(\sum_{k=0}^\infty \frac{x^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)} e^{-x^2/4t} \right) dx \\ &= \sum_{k=0}^\infty \frac{1}{\Gamma(\alpha k+\beta)} \int_0^\infty x^{\alpha k+\beta-1} e^{-x^2/4t} dx \\ &= \sum_{k=0}^\infty \frac{\Gamma\left(\frac{\alpha k+\beta}{2}\right)}{2\Gamma(\alpha k+\beta)} (2\sqrt{t})^{\alpha k+\beta}, \qquad (1.111) \end{aligned}$$

and the use of the Legendre formula

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)$$

gives formula (1.109).

The use of the Laplace transform of the Mittag-Leffler function (1.80) is also a convenient way for obtaining various useful relationships for the Mittag-Leffler function.

For example, it follows from the identity (s denotes the Laplace transform parameter)

$$\frac{1}{s^2} = \frac{s^{\alpha-\beta}}{s^\alpha - 1} \left[s^{\beta-2} - s^{\beta-\alpha-2} \right] \qquad (1.112)$$

and from the known Laplace transform of the function t^{ν} [62, formula 4.3(1)]

$$L\left\{t^{\nu};s\right\}=\Gamma(\nu+1)s^{-\nu-1}, \qquad (Re(s)>0) \tag{1.113}$$

that

$$\int_0^t \tau^{\beta-1}E_{\alpha,\beta}(\tau^{\alpha})\left[\frac{(t-\tau)^{1-\beta}}{\Gamma(2-\beta)}-\frac{(t-\tau)^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+2)}\right]d\tau=t, \tag{1.114}$$

$$(0<\beta<2, \quad \alpha>0).$$

The formula for the fractional integration of the Mittag-Leffler function (1.104) can also be obtained immediately by the inverse Laplace transform of the identity

$$\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1}\cdot s^{-\alpha}=-\frac{s^{2\alpha-\beta}}{s^{2\alpha}-1}+\frac{s^{\alpha-\beta}}{s^{\alpha}-1}. \tag{1.115}$$

The formula (1.109) can also be obtained with the help of the Laplace transform technique. Indeed, if $F(s)$ denotes the Laplace transform of a function $f(t)$, i.e.

$$F(s)=L\left\{f(t);s\right\}=\int_0^{\infty}e^{-st}f(t)dt,$$

then [62, formula 4.1(33)]

$$L\left\{\frac{1}{\sqrt{\pi t}}\int_0^{\infty}e^{-x^2/4t}f(x)dx;s\right\}=s^{-1/2}F(s^{1/2}). \tag{1.116}$$

Let us take in (1.116)

$$f(x)=x^{\beta-1}E_{\alpha,\beta}(x^{\alpha}) \tag{1.117}$$

According to (1.80) we have

$$F(s)=\frac{s^{\alpha-\beta}}{s^{\alpha}-1}$$

and, therefore,

$$s^{-1/2}F(s^{1/2})=\frac{s^{\alpha/2-(\beta+1)/2}}{s^{\alpha/2}-1}=L\left\{t^{\frac{\beta+1}{2}-1}E_{\alpha,\beta}(t^{\alpha/2});s\right\}. \tag{1.118}$$

Comparing (1.116) and (1.118) we arrive at the relationship (1.109).

Similarly, using the Laplace transform of the Mittag-Leffler function (1.80), starting with the identity

$$\frac{s^{\alpha-\beta}}{s^{\alpha}-a} \cdot \frac{s^{\alpha-\gamma}}{s^{\alpha}+a} = \frac{s^{2\alpha-(\beta+\gamma)}}{s^{2\alpha}-a^2} \tag{1.119}$$

we obtain the convolution of two Mittag-Leffler functions:

$$\int_0^t \tau^{\beta-1} E_{\alpha,\beta}(a\tau^{\alpha})(t-\tau)^{\gamma-1} E_{\alpha,\gamma}(-a(t-\tau)^{\alpha})d\tau = t^{\beta+\gamma-1} E_{2\alpha,\beta+\gamma}(a^2 t^{2\alpha}) \tag{1.120}$$

$$(\beta > 0, \quad \gamma > 0).$$

The relationship (1.120) can also be obtained from (1.107), where we can take $z = -y$ and then utilize the relationship (1.95).

1.2.7 Asymptotic Expansions

Integration of the relationship (1.87) gives

$$\begin{aligned} E_{1/n,\beta}(z) = z^{(1-\beta)n} e^{z^n} \Big\{ & z_0^{(1-\beta)n} e^{-z_0^n} E_{1/n,\beta}(z_0) \\ & + n \int_{z_0}^{z} e^{-\tau^n} \left(\sum_{k=1}^{n} \frac{\tau^{-k}}{\Gamma\left(\beta - \frac{k}{n}\right)} \tau^{\beta n-1} \right) d\tau \Big\}, \quad (n \geq 1) \end{aligned} \tag{1.121}$$

which is valid for arbitrary $z_0 \neq 0$.

If $\beta = 1$, then $z_0 = 0$ can be taken in (1.121). This gives:

$$E_{1/n,1}(z) = e^{z^n} \left\{ 1 + n \int_0^z e^{-\tau^n} \left(\sum_{k=1}^{n-1} \frac{\tau^{k-1}}{\Gamma\left(\frac{k}{n}\right)} \right) \right\}, \quad (n \geq 2). \tag{1.122}$$

Taking $n = 2$ in (1.122), we obtain the formula

$$E_{1/2,1}(z) = e^{z^2} \left\{ 1 + \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2} d\tau \right\} \tag{1.123}$$

from which the following asymptotic formula follows:

$$E_{1/2,1} \sim 2e^{z^2}, \quad |\arg(z)| < \frac{\pi}{4}, \quad |z| \to \infty. \tag{1.124}$$

General asymptotic formulas for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ are given below in the form of theorems. The contour $\gamma(\epsilon, \varphi)$ and the domains $G^-(\epsilon, \varphi)$ $G^+(\epsilon, \varphi)$ used below have been defined in Section 1.1.6. The cases $\alpha < 2$, $\alpha = 2$ and $\alpha > 2$ are considered separately.

First let us obtain the corresponding integral representation formulas, which are necessary for obtaining the asymptotic formulas.

THEOREM 1.1 ∘ *Let $0 < \alpha < 2$ and let β be an arbitrary complex number. Then for an arbitrary $\epsilon > 0$ and μ such that*

$$\pi\alpha/2 < \mu \leq \min\{\pi, \pi\alpha\} \tag{1.125}$$

we have

$$E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon,\mu)} \frac{\exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta, \qquad z \in G^-(\epsilon, \mu), \tag{1.126}$$

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon,\mu)} \frac{\exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta,$$

$$z \in G^+(\epsilon, \mu) \quad \bullet \tag{1.127}$$

Let us prove this statement.

If $|z| < \epsilon$, then

$$\left|\frac{z}{\zeta}\right| < 1, \qquad \zeta \in \gamma(\epsilon, \mu). \tag{1.128}$$

Using the definition of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ (1.56) and the integral representation for the function $1/\Gamma(s)$ (1.52) and taking into account the inequality (1.128), we obtain for $\alpha < 2$ and $|z| < \epsilon$ that

$$\begin{aligned} E_{\alpha,\beta}(z) &= \sum_{k=0}^{\infty} \frac{1}{2\alpha\pi i} \left\{ \int\limits_{\gamma(\epsilon,\mu)} \exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha-k-1} d\zeta \right\} z^k \\ &= \frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon,\mu)} \exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha-1} \left\{ \sum_{k=0}^{\infty} \left(\frac{z}{\zeta}\right)^k \right\} d\zeta \\ &= \frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon,\mu)} \frac{\exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta. \end{aligned} \tag{1.129}$$

It follows from the condition (1.125) that this integral is absolutely convergent and defines a function of z, which is analytic in $G^-(\epsilon, \mu)$ and in $G^+(\epsilon, \mu)$. On the other hand, for every $\mu \in (\pi\alpha/2, \min\{\pi, \pi\alpha\})$ the circle $|z| < \epsilon$ lies in $G^-(\epsilon, \mu)$. Therefore, in accordance with the principle of analytic continuation, the integral (1.129) is equal to $E_{\alpha,\beta}(z)$ not only in the circle $|z| < \epsilon$, but in the entire domain $G^-(\epsilon, \mu)$, and we have proved formula (1.126).

Now let us take $z \in G^+(\epsilon, \mu)$. Then for an arbitrary $\epsilon_1 > |z|$ we have $z \in G^-(\epsilon_1, \mu)$, and using the formula (1.126) gives, on the one hand,

$$E_{\alpha,\beta}(z) = \frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon_1,\mu)} \frac{\exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta. \tag{1.130}$$

On the other hand, if $\epsilon < |z| < \epsilon_1$ and $-\mu < \arg(z) < \mu$, then the use of the Cauchy theorem gives

$$\frac{1}{2\alpha\pi i} \int\limits_{\gamma(\epsilon_1,\mu)-\gamma(\epsilon,\mu)} \frac{\exp(\zeta^{1/\alpha})\zeta^{(1-\beta)/\alpha}}{\zeta - z} d\zeta = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}), \tag{1.131}$$

and combining (1.130) and (1.131) we obtain the integral representation formula (1.127).

THEOREM 1.2 $\circ$ *If $Re(\beta) > 0$, then for arbitrary $\epsilon > 0$*

$$E_{2,\beta}(z) = \frac{1}{4\pi i} \int\limits_{\gamma(\epsilon,\pi)} \frac{\exp(\zeta^{1/2})\zeta^{(1-\beta)/2}}{\zeta - z} d\zeta, \qquad z \in G^-(\epsilon, \pi), \tag{1.132}$$

$$E_{2,\beta}(z) = \frac{1}{2} z^{(1-\beta)/2} \exp(z^{1/2}) + \frac{1}{4\pi i} \int\limits_{\gamma(\epsilon,\pi)} \frac{\exp(\zeta^{1/2})\zeta^{(1-\beta)/2}}{\zeta - z} d\zeta, \tag{1.133}$$

$$z \in G^-(\epsilon, \pi). \quad \bullet$$

The proof of this theorem is similar to the previous one. However, instead of the integral representation (1.52) of the function $1/\Gamma(s)$ we must use the formula (1.54) leading to the relationship (1.132). The integral on the right-hand side of equation (1.132) converges for $Re(\beta) > 0$ and converges absolutely for $Re(\beta) > 1$. Taking into account that formula (1.131) holds also for $\alpha = 2$ and $\mu = \pi$, we obtain (1.133).

Now let us use Theorem 1.1 for establishing the following asymptotic formulas.

THEOREM 1.3 ∘ *If* $0 < \alpha < 2$, β *is an arbitrary complex number and* μ *is an arbitrary real number such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{1.134}$$

then for an arbitrary integer $p \geq 1$ *the following expansion holds:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right), \tag{1.135}$$

$$|z| \to \infty, \qquad |\arg(z)| \leq \mu. \quad \bullet$$

Let us start the proof of formula (1.135) by taking φ satisfying the condition

$$\frac{\pi\alpha}{2} < \mu < \varphi \leq \min\{\pi, \pi\alpha\}. \tag{1.136}$$

Taking now $\epsilon = 1$ and substituting the representation

$$\frac{1}{\zeta - z} = -\sum_{k=1}^{p} \frac{\zeta^{k-1}}{z^k} + \frac{\zeta^p}{z^p(\zeta - z)} \tag{1.137}$$

into equation (1.127) of Theorem 1.1, we obtain the following expression for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ in the domain $G^{+}(1, \varphi)$ (i.e., on the right side of the contour $\gamma(1, \varphi)$):

$$\begin{aligned} E_{\alpha,\beta}(z) &= \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) \\ &\quad - \sum_{k=1}^{p} \left(\frac{1}{2\pi\alpha i} \int\limits_{\gamma(1,\varphi)} \exp(\zeta^{1/\alpha}) \zeta^{(1-\beta)/\alpha + k - 1} d\zeta \right) z^{-k} \\ &\quad + \frac{1}{2\pi\alpha i z^p} \int\limits_{\gamma(1,\varphi)} \exp(\zeta^{1/\alpha}) \zeta^{(1-\beta)/\alpha + p} d\zeta. \end{aligned} \tag{1.138}$$

The first integral can be evaluated with the help of formula (1.52):

$$\frac{1}{2\pi\alpha i} \int\limits_{\gamma(1,\varphi)} \exp(\zeta^{1/\alpha}) \zeta^{(1-\beta)/\alpha + k - 1} d\zeta = \frac{1}{\Gamma(\beta - \alpha k)}, \qquad (K \geq 1). \tag{1.139}$$

Substituting this expression into equation (1.138) and taking into account the condition (1.136), we obtain:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)}$$
$$+ \frac{1}{2\pi\alpha i z^p} \int\limits_{\gamma(1,\varphi)} \exp(\zeta^{1/\alpha}) \zeta^{(1-\beta)/\alpha+p} d\zeta, \qquad (1.140)$$

$$(|\arg(z)| \leq \mu, |z| > 1).$$

Let us estimate the integral

$$I_p(z) = \frac{1}{2\pi\alpha i z^p} \int\limits_{\gamma(1,\varphi)} \exp(\zeta^{1/\alpha}) \zeta^{(1-\beta)/\alpha+p} d\zeta,$$

for large $|z|$ and $|\arg(z)| \leq \mu$.

For large $|z|$ and $|\arg(z)| \leq \mu$ we have

$$\min_{\zeta \in \gamma(1,\varphi)} |\zeta - z| = |z| \sin(\varphi - \mu),$$

and therefore for large $|z|$ and $|\arg(z)| \leq \mu$ we have

$$|I_p(z)| \leq \frac{|z|^{-1-p}}{2\pi\alpha \sin(\varphi - \mu)} \int\limits_{\gamma(1,\varphi)} \left|\exp(\zeta^{1/\alpha})\right| \left|\zeta^{(1-\beta)+p}\right| d\zeta. \qquad (1.141)$$

The integral on the right-hand side converges, because for ζ such that $\arg(\zeta) = \pm\varphi$ and $|\zeta| \geq 1$ the following holds:

$$\left|\exp(\zeta^{1/\alpha})\right| = \exp\left(|\zeta|^{1/\alpha} \cos\left(\frac{\varphi}{\alpha}\right)\right),$$

where $\cos(\varphi/\alpha) < 0$ due to condition (1.136).

Combining equation (1.140) and the estimate (1.141) we obtain the asymptotic formula (1.135).

THEOREM 1.4 ∘ *If $0 < \alpha < 2$, β is an arbitrary complex number and μ is an arbitrary real number such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \qquad (1.142)$$

then for an arbitrary integer $p \geq 1$ *the following expansion holds:*

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-1-p}\right), \tag{1.143}$$

$$|z| \to \infty, \qquad \mu \leq |\arg(z)| \leq \pi. \quad \bullet$$

To prove Theorem 1.4, let us take

$$\frac{\pi\alpha}{2} < \varphi < \mu < \min\{\pi, \pi\alpha\} \tag{1.144}$$

Taking $\epsilon = 1$ in equation (1.126) of Theorem 1.1 and using formula (1.137), we obtain

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + I_p(z), \qquad z \in G^{-}(1, \varphi), \tag{1.145}$$

where $I_p(z)$ is the same as above.

For large $|z|$, such that $\mu \leq |\arg(z)| \leq \pi$, the following holds:

$$\min_{\zeta \in \gamma(1,\varphi)} |\zeta - z| = |z| \sin(\varphi - \mu).$$

Additionally, the domain $\mu \leq |\arg(z)| \leq \pi$ lies in the domain $G^{-}(1, \varphi)$, for which equation (1.145) holds. Therefore, for large $|z|$ we have the estimate

$$|I_p(z)| \leq \frac{|z|^{-1-p}}{2\pi\alpha \sin(\varphi - \mu)} \int_{\gamma(1,\varphi)} \left|\exp(\zeta^{1/\alpha})\right| \left|\zeta^{(1-\beta)+p}\right| d\zeta, \tag{1.146}$$

$$(\mu \leq |\arg(z)| \leq \pi).$$

Combining equation (1.145) and the estimate (1.146), we obtain the asymptotic formula (1.143).

The following two theorems, which give estimates of the behaviour of the Mittag-Leffler function $E_{\alpha,\beta}(z)$ in different parts of the complex plane, are obvious consequences of Theorems 1.3 and 1.4:

THEOREM 1.5 ∘ *If $\alpha < 2$, β is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ and C_1 and C_2 are real constants, then*

$$|E_{\alpha,\beta}(z)| \le C_1(1+|z|)^{(1-\beta)/\alpha} \exp\left(Re(z^{1/\alpha})\right) + \frac{C_2}{1+|z|}, \tag{1.147}$$

$$(|\arg(z)| \le \mu), \quad |z| \ge 0. \quad \bullet$$

THEOREM 1.6 ∘ *If $\alpha < 2$, β is an arbitrary real number, μ is such that $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$ and C is a real constant, then*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \tag{1.148}$$

$$(\mu \le |\arg(z)| \le \pi), \quad |z| \ge 0. \quad \bullet$$

Let us now turn our attention to the case of $\alpha \ge 2$.

THEOREM 1.7 ∘ *If $\alpha \ge 2$ and β is arbitrary, then for an arbitrary integer number $p \ge 1$ the following asymptotic formula holds:*

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}\sum_{n}\left(z^{1/\alpha}\exp\left(\frac{2\pi n i}{\alpha}\right)\right)^{1-\beta}\exp\left\{\exp\left(\frac{2\pi n i}{\alpha}\right)z^{1/\alpha}\right\}$$
$$- \sum_{k=1}^{p}\frac{z^{-k}}{\Gamma(\beta-\alpha k)} + O\left(|z|^{-1-p}\right), \tag{1.149}$$

where the sum is taken for integer n satisfying the condition

$$|\arg(z) + 2\pi n| \le \frac{\pi\alpha}{2}. \quad \bullet$$

Let us start the proof by recalling formula (1.97)

$$E_{\alpha,\beta}(z) = \frac{1}{2m+1}\sum_{\nu=-m}^{m-1} E_{\alpha/(2m+1),\beta}(z^{1/(2m+1)}e^{i2\pi\nu/(2m+1)}), \quad (m \ge 0),$$

where $\alpha > 0$. Taking into account that under the conditions of the theorem $\alpha \ge 2$, let us take integer $m \ge 1$ such that $\alpha_1 = \alpha/(2m+1) < 2$.

In such a case we can apply Theorems 1.3 and 1.4 to all terms of the above sum (1.97).

Let us take an arbitrary μ satisfying the inequality

$$\frac{\pi\alpha_1}{2} < \mu < \min\{\pi, \pi\alpha_1\}, \qquad \left(\alpha_1 = \frac{\alpha}{2m+1}\right).$$

Taking an arbitrary integer $q \geq 1$ and using the asymptotic formula (1.135) of Theorem 1.3 and (1.143) of Theorem 1.4, we obtain

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum \left(z^{1/\alpha} \exp\left(\frac{2\pi n i}{\alpha}\right)\right)^{1-\beta} \exp\left\{\exp\left(\frac{2\pi n i}{\alpha}\right) z^{1/\alpha}\right\}$$
$$-\frac{1}{2m+1} \sum_{n=-m}^{m} \left\{ \sum_{k=1}^{q} \frac{z^{-k/(2m+1)} \exp\left(\frac{-2\pi k n i}{2m+1}\right)}{\Gamma(\beta - \frac{k\alpha}{2m+1})} + O\left(|z|^{-(q+1)/(2m+1)}\right) \right\}. \tag{1.150}$$

The first sum in (1.150) is taken for integer values of n satisfying the condition

$$\left|\arg\left(z^{1/(2m+1)} \exp\left(\frac{2\pi n i}{2m+1}\right)\right)\right| \leq \mu. \tag{1.151}$$

Obviously, the condition (1.151) is equivalent to the condition

$$|\arg(z) + 2\pi n| \leq (2m+1)\mu. \tag{1.152}$$

Now let us suppose that z is fixed. If we take $\mu_* > \pi\alpha/2$ and μ_* is close enough to $\pi\alpha/2$, then the inequalities

$$|\arg(z) + 2\pi n| \leq \frac{\pi\alpha}{2} \tag{1.153}$$

and

$$|\arg(z) + 2\pi n| \leq \mu_*$$

are satisfied for the same set of values of n.

The number $(2m+1)\mu > \frac{\pi\alpha}{2}$ can be chosen close enough to $\frac{\pi\alpha}{2}$; therefore, the expression (1.150) can be written as

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} \sum \left(z^{1/\alpha} \exp\left(\frac{2\pi n i}{\alpha}\right)\right)^{1-\beta} \exp\left\{\exp\left(\frac{2\pi n i}{\alpha}\right) z^{1/\alpha}\right\}$$
$$- \frac{1}{2m+1} \sum_{k=1}^{q} \frac{z^{-k/(2m+1)}}{\Gamma\left(\beta - \frac{k\alpha}{2m+1}\right)} \left\{ \sum_{n=-m}^{m} \exp\left(-\frac{2\pi k n i}{2m+1}\right)\right\}$$
$$+ O\left(|z|^{-(q+1)/(2m+1)}\right), \tag{1.154}$$

where the first sum is taken for n satisfying the condition (1.153).

Until now, q was an arbitrary natural number. Now for a given p let us take

$$q = (2m+1)(p+1) - 1.$$

Then, taking into account that

$$\sum_{n=-m}^{m} \exp\left(-\frac{2\pi kni}{2m+1}\right) = \begin{cases} 2m+1 & k \equiv 0(\mathrm{mod}(2m+1)) \\ 0 & k \not\equiv 0(\mathrm{mod}(2m+1)), \end{cases} \tag{1.155}$$

the asymptotic expansion (1.149) follows from (1.154). The proof of Theorem 1.7 is complete.

1.3 Wright Function

The Wright function plays an important role in the solution of linear partial fractional differential equations, e.g. the fractional diffusion–wave equation.

This function, related to the Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$, was introduced by Wright [258, cf. [65, 107]]. A number of useful relationships were obtained by Humbert and Agarwal [107] with the help of the Laplace transform.

For convenience we adopt here Mainardi's notation for the Wright function $W(z;\alpha,\beta)$.

1.3.1 Definition

The Wright function is defined as [65, formula 18.1(27)]

$$W(z;\alpha,\beta) = \sum_{k=0}^{\infty} \frac{z^k}{k!\,\Gamma(\alpha k+\beta)}. \tag{1.156}$$

1.3.2 Integral Representation

This function can be represented by the following integral [65, formula 18.1(29)]:

$$W(z;\alpha,\beta) = \frac{1}{2\pi i}\int_{Ha} \tau^{-\beta} e^{\tau + z\tau^{-\alpha}}\, d\tau \tag{1.157}$$

where Ha denotes Hankel's contour.

To prove (1.157), let us write the integrated function in the form of a power series in z and perform term-by-term integration using the integral representation formula for the reciprocal gamma function (1.46)

$$\frac{1}{\Gamma(z)} = \int_{Ha} e^{\tau}\tau^{-z}d\tau.$$

1.3.3 Relation to Other Functions

It follows from the definition (1.156) that

$$W(z;0,1) = e^{z} \tag{1.158}$$

$$\left(\frac{z}{2}\right)^{\nu} W(\mp\frac{z^2}{4};1,\nu+1) = \begin{cases} J_{\nu}(z) \\ I_{\nu}(z). \end{cases} \tag{1.159}$$

Taking $\beta = 1-\alpha$, we obtain Mainardi's function $M(z;\alpha)$:

$$W(-z;-\alpha,1-\alpha) = M(z;\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{k!\,\Gamma(-\alpha(k+1)+1)}. \tag{1.160}$$

The following particular case of the Wright function was considered by Mainardi [131]:

$$W(-z;-\frac{1}{2},-\frac{1}{2}) = M(z;\frac{1}{2}) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{z^2}{4}\right). \tag{1.161}$$

We see that the Wright function is a generalization of the exponential function and the Bessel functions. For $\alpha > 0$ and $\beta > 0$ it is an entire function in z [65].

Recently Mainardi [131] pointed out that $W(z;\alpha,\beta)$ is an entire function in z also for $-1 < \alpha < 0$.

Let us prove this statement. Using the well-known relationship (1.26)

$$\Gamma(y)\Gamma(1-y) = \frac{\pi}{\sin(\pi y)},$$

we can write the Wright function in the form

$$W(z;\alpha,\beta) = \frac{1}{\pi}\sum_{k=0}^{\infty} \frac{z^k\Gamma(1-\alpha k-\beta)\sin\pi(\alpha k+\beta)}{k!}. \tag{1.162}$$

Let us introduce an auxiliary majorizing series

$$S = \frac{1}{\pi} \sum_{k=0}^{\infty} \left| \frac{\Gamma(1 - \alpha k - \beta)}{k!} \right| |z|^k. \tag{1.163}$$

The convergence radius of series (1.163) for $-1 < \alpha < 0$ is infinite:

$$R = \lim_{k \to \infty} \left| \frac{\Gamma(1 - \alpha k - \beta)}{k!} \frac{(k+1)!}{\Gamma(1 - \alpha k - \alpha - \beta)} \right| = \lim_{k \to \infty} \frac{k+1}{|\alpha|^{\alpha} k^{-\alpha}} = \infty. \tag{1.164}$$

(We use here relationship [63, formula 1.18(4)].)

It follows from the comparison of the series (1.156) and (1.163) that for $\alpha > -1$ and arbitrary β the convergence radius of the series representation of the Wright function $W(z; \alpha, \beta)$ is infinite, and the Wright function is an entire function.

There is an interesting link between the Wright function and the Mittag-Leffler function. Namely, the Laplace transform of the Wright function is expressed with the help of the Mittag-Leffler function:

$$\begin{aligned} L\left\{W(t; \alpha, \beta);\ s\right\} &= L\left\{ \sum_{k=0}^{\infty} \frac{t^k}{k!\ \Gamma(\alpha k + \beta)};\ s \right\} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} \cdot \frac{1}{s^{k+1}} \\ &= s^{-1} E_{\alpha,\beta}(s^{-1}). \end{aligned} \tag{1.165}$$