Chapter 4

The Laplace Transform Method

Differential equations of fractional order appear more and more frequently in various research areas and engineering applications. An effective and easy-to-use method for solving such equations is needed.

However, known methods have certain disadvantages. Methods, described in detail in [179, 153, 13] for fractional differential equations of rational order, do not work in the case of arbitrary real order. On the other hand, there is an iteration method described in [232], which allows solution of fractional differential equations of arbitrary real order but it works effectively only for relatively simple equations, as well as the series method [179, 70]. Other authors (e.g. [13, 29]) used in their investigations the one-parameter Mittag-Leffler function $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$. Still other authors [235, 80] prefer the Fox H-function [69], which seems to be too general to be frequently used in applications.

Instead of this variety of different methods, we introduce here a method which is free of these disadvantages and suitable for a wide class of initial value problems for fractional differential equations. The method uses the Laplace transform technique and is based on the formula of the Laplace transform of the Mittag-Leffler function in two parameters $E_{\alpha,\beta}(z)$. We hope that this method could be useful for obtaining solutions of different applied problems appearing in physics, chemistry, electrochemistry, engineering, etc.

This chapter deals with the solution of fractional linear differential equations with constant coefficients.

In Section 4.1 we give solutions to some initial-value problems for "standard" fractional differential equations. Some of them were solved by other authors earlier by other methods, and the comparison in such cases just underlines the simplicity and the power of the Laplace transform method.

In Section 4.2 we extend the proposed method for the case of so-called "sequential" fractional differential equations, i.e. equations in terms of the Miller–Ross sequential derivatives). For this purpose, we use the Laplace transform for the Miller–Ross sequential fractional derivative given by formula (2.259). The "sequential" analogues of the problems solved in Section 4.1 are considered. Naturally, we arrive at solutions which are different from those obtained in the Section 4.1.

The operational calculus, which can be applied to the fractional differential equations considered in this chapter, has been developed in the papers by Yu. F. Luchko and H. M. Srivastava [128], and by S. B. Hadid and Yu. Luchko [100]. R. Gorenflo and Yu. Luchko also developed an operational method for solving generalized Abel integral equations of the second kind [86].

4.1 Standard Fractional Differential Equations

The following examples illustrate the use of (1.80) for solving fractional-order differential equations with constant coefficients. In this chapter we use the classical formula for the Laplace transform of the fractional derivative, as given, e.g., in [179, p. 134] or [153, p. 123]:

$$\int_0^\infty e^{-st}\ {}_0D_t^\alpha f(t)\,dt = s^\alpha F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{\alpha-k-1} f(t)\right]_{t=0}, \tag{4.1}$$

$$(n-1 < \alpha \le n).$$

4.1.1 Ordinary Linear Fractional Differential Equations

In this section we give some examples of the solution of ordinary linear differential equations of fractional order.

Example 4.1. A slight generalization of an equation solved in [179, p. 157]:

$${}_0D_t^{1/2} f(t) + af(t) = 0, \quad (t>0); \qquad \left[{}_0D_t^{-1/2} f(t)\right]_{t=0} = C. \tag{4.2}$$

Applying the Laplace transform we obtain

$$F(s) = \frac{C}{s^{1/2} + a}, \qquad C = \left[{}_0D_t^{-1/2} f(t) \right]_{t=0}$$

and the inverse transform with the help of (1.81) gives the solution of (4.2):

$$f(t) = Ct^{-1/2} E_{\frac{1}{2},\frac{1}{2}}(-a\sqrt{t}). \tag{4.3}$$

Using the series expansion (1.56) of $E_{\alpha,\beta}(t)$, it is easy to check that for $a = 1$ the solution (4.3) is identical to the solution

$$f(t) = C(\frac{1}{\sqrt{\pi t}} - e^t \mathrm{erfc}(\sqrt{t})),$$

obtained in [179] in a more complex way.

Example 4.2. Let us consider the following equation:

$${}_0D_t^Q f(t) + {}_0D_t^q f(t) = h(t), \tag{4.4}$$

which "encounters very great difficulties except when the difference $q - Q$ is integer or half-integer" [179, p.156].

Suppose that $0 < q < Q < 1$. The Laplace transform of equation (4.4) leads to

$$(s^Q + s^q)F(s) = C + H(s), \tag{4.5}$$

$$C = \left[{}_0D_t^{q-1} f(t) + {}_0D_t^{Q-1} f(t) \right]_{t=0},$$

and then

$$F(s) = \frac{C + H(s)}{s^Q + s^q} = \frac{C + H(s)}{s^q(s^{Q-q} + 1)} = \Big(C + H(s)\Big) \frac{s^{-q}}{s^{Q-q} + 1}. \tag{4.6}$$

After inversion with the help of (1.80) for $\alpha = Q - q$ and $\beta = Q$, we obtain the solution:

$$f(t) = C\,G(t) + \int_0^t G(t - \tau)h(\tau)d\tau, \tag{4.7}$$

$$C = \left[{}_0D_t^{q-1} f(t) + {}_0D_t^{Q-1} f(t) \right]_{t=0}, \quad G(t) = t^{Q-1} E_{Q-q,Q}(-t^{Q-q}).$$

The case $0 < q < Q < n$ (for example, the equation obtained in [184]) can be solved similarly.

Example 4.3. Let us consider the following initial value problem for a non-homogeneous fractional differential equation under non-zero initial conditions:

$$ {}_0D_t^{\alpha} y(t) - \lambda y(t) = h(t), \quad (t > 0); \tag{4.8}$$

$$ \left[{}_0D_t^{\alpha-k} y(t) \right]_{t=0} = b_k, \quad (k = 1, 2, \ldots, n), \tag{4.9}$$

where $n - 1 < \alpha < n$. The problem (4.8) was solved in [232] by the iteration method. With the help of the Laplace transform and the formula (1.80) we obtain the same solution directly and easily.

Indeed, taking into account the initial conditions (4.9), the Laplace transform of equation (4.8) yields

$$ s^{\alpha} Y(s) - \lambda Y(s) = H(s) + \sum_{k=1}^{n} b_k s^{k-1}, $$

from which

$$ Y(s) = \frac{H(s)}{s^{\alpha} - \lambda} + \sum_{k=1}^{n} b_k \frac{s^{k-1}}{s^{\alpha} - \lambda}, \tag{4.10}$$

and the inverse Laplace transform using (1.80) gives the solution:

$$ y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda t^{\alpha}) + \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda (t - \tau)^{\alpha}) h(\tau) d\tau. \tag{4.11}$$

4.1.2 Partial Linear Fractional Differential Equations

The proposed approach can be successfully used for solving partial linear differential equations of fractional order.

Example 4.4. Nigmatullin's fractional diffusion equation

Let us consider the following initial boundary value problem for the fractional diffusion equation in one space dimension:

$$ {}_0D_t^{\alpha} u(x, t) = \lambda^2 \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (t > 0, \quad -\infty < x < \infty); \tag{4.12}$$

$$ \lim_{x \to \pm\infty} u(x, t) = 0; \quad \left[{}_0D_t^{\alpha-1} u(x, t) \right]_{t=0} = \varphi(x). \tag{4.13}$$

We assume here $0 < \alpha < 1$. An equation of the type (4.12) was deduced by Nigmatullin [164] and by Westerlund [253] and studied by Mainardi [131]). We will give a simple solution of problem (4.12) demonstrating once again the advantage of using the Mittag-Leffler function in two parameters (1.56).

Taking into account the boundary conditions (4.13), the Fourier transform with respect to variable x gives:

$$ {}_0D_t^\alpha \overline{u}(\beta,t) + \lambda^2\beta^2\overline{u}(\beta,t) = 0 \tag{4.14} $$

$$ \left[{}_0D_t^{\alpha-1}\overline{u}(x,t)\right]_{t=0} = \overline{\varphi}(\beta), \tag{4.15} $$

where β is the Fourier transform parameter. Applying the Laplace transform to (4.14) and using the initial condition (4.15) we obtain

$$ \overline{U}(\beta,s) = \frac{\varphi(\beta)}{s^\alpha + \lambda^2\beta^2}. \tag{4.16} $$

The inverse Laplace transform of (4.16) using (1.80) gives

$$ \overline{u}(\beta,t) = \varphi(\beta)t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2\beta^2 t^\alpha), \tag{4.17} $$

and then the inverse Fourier transform produces the solution of the initial-value problem (4.12)–(4.13):

$$ u(x,t) = \int_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi, \tag{4.18} $$

$$ G(x,t) = \frac{1}{\pi}\int_0^\infty t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2\beta^2t^\alpha)\cos\beta x d\beta. \tag{4.19} $$

Let us evaluate integral (4.19). The Laplace transform of (4.19) and formula 1.2(11) from [62] produce

$$ g(x,s) = \frac{1}{\pi}\int_0^\infty \frac{\cos(\beta x)\,d\beta}{\lambda^2\beta^2 + s^\alpha} = \frac{1}{2\lambda}s^{-\alpha/2}e^{-|x|\lambda^{-1}s^{\alpha/2}}, \tag{4.20} $$

and the inverse Laplace transform gives

$$ G(x,t) = \frac{1}{4\lambda\pi i}\int_{Br} e^{st}s^{-\frac{\alpha}{2}}\exp(-|x|\lambda^{-1}s^{\alpha/2})ds. \tag{4.21} $$

Performing the substitutions $\sigma = st$ and $z = |x|\lambda^{-1}t^{-\rho}$ $(\rho = \alpha/2)$ and transforming the Bromwich contour Br to the Hankel contour Ha (see Fig. 2.2), as was done in a similar case by Mainardi [131], we obtain

$$G(x,t) = \frac{t^{1-\rho}}{2\lambda}\frac{1}{2\pi i}\int\limits_{Ha} e^{\sigma - z\sigma^{\rho}}\frac{d\sigma}{\sigma^{\rho}} = \frac{1}{2\lambda}t^{\rho-1}W(-z,-\rho,\rho), \qquad z = \frac{|x|}{\lambda t^{\rho}} \tag{4.22}$$

where $W(z,\lambda,\mu)$ is the Wright function (1.156). We would like to note that, in fact, we have just evaluated the Fourier cosine-transform of the function $u_1(\beta) = t^{\alpha-1}E_{\alpha,\alpha}(-\lambda^2\beta^2 t^{\alpha})$.

It is easy to check that for $\alpha = 1$ (the traditional diffusion equation) the fractional Green function (4.22) reduces to the classical expression

$$G(x,t) = \frac{1}{2\lambda\sqrt{\pi t}}\exp(-\frac{x^2}{4\lambda^2 t}). \tag{4.23}$$

Example 4.5. The Schneider–Wyss fractional diffusion equation

The following example shows that the proposed method can be effectively applied also to fractional integral equations. Let us consider the Schneider–Wyss type formulation of the diffusion equation [235] (for simplicity and comparison with the previous example — in one spatial dimension):

$$u(x,t) = \varphi(x) + \lambda^2\, {}_0D_t^{-\alpha}\frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (-\infty < x < \infty, \quad t > 0); \tag{4.24}$$

$$\lim_{x\to\pm\infty} u(x,t) = 0, \quad u(x,0) = \varphi(x). \tag{4.25}$$

Applying the Fourier transform with respect to the spatial variable x and the Laplace transform with respect to time t, we obtain:

$$\overline{U}(\beta,s) = \frac{\varphi(\beta)s^{\alpha-1}}{s^{\alpha}+\lambda^2\beta^2}, \tag{4.26}$$

where $\overline{U}(\beta,p)$ is the Fourier–Laplace transform of $u(x,t)$, β is the Fourier transfrom parameter and p is the Laplace transform parameter.

Inverting Laplace and Fourier transforms as was done in the previous Example 4.4, we obtain the solution of problem (4.24):

$$u(x,t) = \int\limits_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi, \tag{4.27}$$

$$G(x,t) = \frac{1}{\pi} \int_0^\infty E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha) \cos \beta x \, d\beta. \tag{4.28}$$

Let us evaluate integral (4.28). The Laplace transform of (4.28) and formula 1.2(11) from [62] produce

$$g(x,s) = \frac{s^{\alpha-1}}{\pi} \int_0^\infty \frac{\cos(\beta x) d\beta}{s^\alpha + \lambda^2 \beta^2} = \frac{1}{2\lambda} s^{\frac{\alpha}{2}-1} \exp(-|x| \lambda^{-1} s^{\alpha/2}), \tag{4.29}$$

and the inverse Laplace transform gives:

$$G(x,t) = \frac{1}{4\lambda \pi i} \int_{Br} e^{st} s^{\frac{\alpha}{2}-1} \exp(-|x|\lambda^{-1} s^{\alpha/2}) ds. \tag{4.30}$$

Performing the substitutions $\sigma = st$ and $z = |x|\lambda^{-1} t^{-\rho}$ $(\rho = \alpha/2)$ and transforming the Bromwich contour Br to the Hankel contour Ha (see Fig. 2.2), as was done in a similar case by Mainardi [131], we obtain

$$G(x,t) = \frac{t^{-\rho}}{2\lambda} \frac{1}{2\pi i} \int_{Ha} e^{\sigma - z\sigma^\rho} \frac{d\sigma}{\sigma^{1-\rho}} = \frac{1}{2\lambda} t^{-\rho} M(z,\rho), \quad z = \frac{|x|}{\lambda t^\rho} \tag{4.31}$$

where $M(z,\rho) = W(-z, -\rho, 1-\rho)$ is the Mainardi function (1.160).

The last expression is identical to the expression which was obtained by Mainardi [131] in another way.

We would like to note at this point, as in the previous example, that we have just evaluated the Fourier cosine-transform of the function $u_2(\beta) = E_{\alpha,1}(-\lambda^2 \beta^2 t^\alpha)$.

For $\alpha = 1$ the fractional Green's function (4.31) also reduces to the classical expression (4.23). The case of an arbitrary number of space dimensions can be solved similarly.

For $\alpha = 1$ both generalizations (Nigmatullin's as well as that by Schneider and Wyss) of the diffusion problem give the standard diffusion problem, and the solutions reduce to the classical solution. However, it is obvious that the asymptotic behaviour of (4.18) and (4.27) for $t \to 0$, and $t \to \infty$ is different (see also the discussion in [80] on two different generalizations of the standard relaxation equation and the discussion in [72] on two fractional models — one based on fractional derivatives and the other based on fractional integrals — for mechanical stress relaxation).

This difference was caused by initial conditions of different types. The class of solutions is determined by the number and the type of initial conditions.

4.2 Sequential Fractional Differential Equations

Let us consider initial value problems of the form:

$$ {}_0\mathcal{L}_t y(t) = f(t); \qquad {}_0\mathcal{D}_t^{\sigma_k - 1} y(t)\Big|_{t=0} = b_k, \quad (k = 1, \ldots, n), \tag{4.32} $$

$$ {}_a\mathcal{L}_t y(t) \equiv {}_a\mathcal{D}_t^{\sigma_n} y(t) + \sum_{k=1}^{n-1} p_k(t)\, {}_a\mathcal{D}_t^{\sigma_{n-k}} y(t) + p_n(t) y(t), \tag{4.33} $$

where the following notation is used for the Miller–Ross sequential derivatives:

$$ {}_a\mathcal{D}_t^{\sigma_k} \equiv {}_aD_t^{\alpha_k}\, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1}; $$

$$ {}_a\mathcal{D}_t^{\sigma_k - 1} \equiv {}_aD_t^{\alpha_k - 1}\, {}_aD_t^{\alpha_{k-1}} \ldots {}_aD_t^{\alpha_1}; $$

$$ \sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \ldots, n); $$

$$ 0 < \alpha_j \leq 1, \quad (j = 1, 2, \ldots, n). $$

The fractional differential equation (4.32) is a sequential fractional differential equation, according to the terminology used by Miller and Ross [153]. To extend the Laplace transform method using the advantages of (1.80) for such equations with constant coefficients, the formula (2.259) can be used.

4.2.1 Ordinary Linear Fractional Differential Equations

In this section we give solutions of the "sequential" analogues of "standard" linear ordinary fractional differential equations with constant coefficients. Of course, we must take appropriate initial conditions, also in terms of sequential fractional derivatives.

Example 4.6. Let us consider the sequential analogue of Example 4.1:

$$ {}_0D_t^{\alpha}\left({}_0D_t^{\beta} y(t)\right) + a y(t) = 0 \tag{4.34} $$

$$ \left[{}_0D_t^{\alpha - 1}\left({}_0D_t^{\beta} y(t)\right)\right]_{t=0} = b_1, \qquad \left[{}_0D_t^{\beta - 1} y(t)\right]_{t=0} = b_2, \tag{4.35} $$

where $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta = 1/2$.

The formula (2.259) of the Laplace transform of the sequential fractional derivative allows us to utilize the initial conditions (4.35). To use (2.259), we take $\alpha_1 = \beta$, $\alpha_2 = \alpha$, and $m = 2$. Therefore, $\sigma_1 = \beta$, $\sigma_2 = \alpha + \beta$. Then the Laplace transform (2.259) of equation (4.34) gives:

$$(s^{\alpha+\beta} + a)Y(s) = s^{\alpha} b_2 + b_1, \tag{4.36}$$

from which it follows that

$$Y(s) = b_2 \frac{s^{\alpha}}{s^{\alpha+\beta} + a} + b_1 \frac{1}{s^{\alpha+\beta} + a}, \tag{4.37}$$

and after the Laplace inversion with the help of (1.80) we find the solution to the problem (4.34)-(4.35):

$$y(t) = b_2 t^{\beta-1} E_{\alpha+\beta,\beta}(-a t^{\alpha+\beta}) + b_1 t^{\alpha+\beta-1} E_{\alpha+\beta,\alpha+\beta}(-a t^{\alpha+\beta}). \tag{4.38}$$

For $\beta = 0$ and $\alpha = 1/2$ (and assuming, of course, $b_2 = 0$), we can obtain from (4.38) the solution of Example 4.1.

Example 4.7. Let us now consider the following sequential analogue for the equation considered in Example 4.2:

$${}_0D_t^{\alpha} \left({}_0D_t^{\beta} y(t) \right) + {}_0D_t^{q} y(t) = h(t), \tag{4.39}$$

where $0 < \alpha < 1$, $0 < \beta < 1$, $0 < q < 1$, $\alpha + \beta = Q > q$.

The Laplace transform (2.259) of equation (4.39) gives:

$$(s^{\alpha+\beta} + s^q)Y(s) = H(s) + s^{\alpha} b_2 + b_1, \tag{4.40}$$

$$b_1 = \left[{}_0D_t^{\alpha-1} \left({}_0D_t^{\beta} y(t) \right) \right]_{t=0} + \left[{}_0D_t^{q-1} y(t) \right]_{t=0},$$

$$b_2 = \left[{}_0D_t^{\beta-1} y(t) \right]_{t=0}.$$

Writing $Y(s)$ in the form

$$Y(s) = \frac{s^{-q} H(s)}{s^{\alpha+\beta-q} + 1} + b_2 \frac{s^{\alpha-q}}{s^{\alpha+\beta-q} + 1} + b_1 \frac{s^{-q}}{s^{\alpha+\beta-q} + 1} \tag{4.41}$$

and finding the inverse Laplace transform with the help of (1.80), we obtain the solution:

$$\begin{aligned} y(t) &= b_2 t^{\beta-1} E_{\alpha+\beta-q,\beta}(-t^{\alpha+\beta-q}) + b_1 t^{\alpha+\beta-q} E_{\alpha+\beta-q,\alpha+\beta}(-t^{\alpha+\beta-q}) \\ &+ \int_0^t (t-\tau)^{\alpha+\beta-1} E_{\alpha+\beta-q,\alpha+\beta} \left(-(t-\tau)^{\alpha+\beta-q} \right) h(\tau) d\tau. \end{aligned} \tag{4.42}$$

It is easy to see that this solution contains the solution of Example 4.2 as a special case.

Example 4.8. Let us consider the following initial value problem for the sequential fractional differential equation:

$$ {}_0D_t^{\alpha_2}({}_0D_t^{\alpha_1}y(t)) - \lambda y(t) = h(t); \tag{4.43}$$

$$ \left[{}_0D_t^{\alpha_2-1}\left({}_0D_t^{\alpha_1}y(t)\right)\right]_{t=0} = b_1, \quad \left[{}_0D_t^{\alpha_1-1}y(t)\right]_{t=0} = b_2. \tag{4.44}$$

Let us consider $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$. The Laplace transform (2.259) of equation (4.43) gives

$$ (s^{\alpha_1+\alpha_2} - \lambda)Y(s) = s^{\alpha_2}b_2 + b_1, $$

and after inversion using (1.80) we obtain the solution:

$$ y(t) = b_2 t^{\alpha_1-1}E_{\alpha,\alpha_1}(\lambda t^{\alpha}) + b_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-\tau)^{\alpha})h(\tau)d\tau, \tag{4.45}$$

$$ (\alpha = \alpha_1 + \alpha_2). $$

Let us take α the same as in Example 4.3. Using (1.56), (1.82) and (2.213), it is easy to verify that (4.45) is the solution of (4.43). It is also worthwhile to note that if $b_1 \neq 0$, $b_2 \neq 0$, then (4.45) is *not* a solution of the equation ${}_0D_t^{\alpha}y(t) - \lambda y(t) = h(t)$ from Example 4.3; also (4.11) is *not* a solution of equation (4.43). On the other hand, equations (4.8) and (4.43) are very close to one another: the *fractional Green's function* in both cases is $G(t) = (t)^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$. We will return to this observation in Chapter 5.

4.2.2 Partial Linear Fractional Differential Equations

Example 4.9. Let us consider Mainardi's [131] initial value problem for the fractional diffusion–wave equation:

$$ {}_0D_t^{\alpha}u(x,t) = \lambda^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad (|x| < \infty, t > 0) \tag{4.46}$$

$$ u(x,0) = f(x), \qquad (|x| < \infty) \tag{4.47}$$

$$\lim_{x \pm \infty} u(x,t) = 0, \qquad (t > 0) \tag{4.48}$$

where $0 < \alpha < 1$.

The type of the initial condition (4.47) suggests that the fractional derivative in equation (4.46) must be interpreted as a properly chosen sequential fractional derivative ${}_0\mathcal{D}_t^\alpha = {}_0D_t^{\alpha_2}\, {}_0D_t^{\alpha_1}$. The Laplace transform formula (2.259) for $\alpha_2 = \alpha - 1$, $\alpha_1 = 1$, and $k = 2$ (this gives Caputo's formula [24]), i.e.,

$$\mathcal{L}\{\, {}_0\mathcal{D}_t^\alpha y(t);\, s\} = s^\alpha Y(s) - s^{\alpha-1} y(0), \tag{4.49}$$

applied to the problem (4.46)–(4.48), yields:

$$s^\alpha \overline{u}(x,s) - s^{\alpha-1} f(x) = \lambda^2 \frac{\partial^2 \overline{u}(x,s)}{\partial x^2}, \qquad (|x| < \infty) \tag{4.50}$$

$$\lim_{x \pm \infty} \overline{u}(x,s) = 0, \qquad (t > 0). \tag{4.51}$$

Applying now the Fourier exponential transform to equation (4.50) and utilizing the boundary conditions (4.51), we obtain:

$$U(\beta, s) = \frac{s^{\alpha-1}}{s^\alpha + \lambda^2 \beta^2} F(\beta), \tag{4.52}$$

where $U(\beta, p)$ and $F(\beta)$ are the Fourier transforms of $\overline{u}(x,s)$ and $f(x)$.

The inverse Laplace transform of the fraction $s^{\alpha-1}/(s^\alpha + \lambda^2\beta^2)$ is $E_{\alpha,1}(-\lambda^2\beta^2 t^\alpha)$ (where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function in two parameters). Therefore, the inversion of the Fourier and the Laplace transform gives the solution in the following form:

$$u(x,t) = \int_{-\infty}^{\infty} G(x - \xi, t) f(\xi) d\xi, \tag{4.53}$$

$$\begin{aligned} G(x,t) &= \frac{1}{\pi} \int_0^\infty E_{\alpha,1}(-\lambda^2\beta^2 t^\alpha) \cos(\beta x) d\beta \\ &= \frac{1}{2\lambda} t^{-\rho} W(-z, -\rho, 1-\rho), \end{aligned} \tag{4.54}$$

where $W(z, \lambda, \mu)$ is the Wright function (1.156). This solution is identical to the solution of the Schneider–Wyss fractional (integro-differential) diffusion equation (4.27).