Chapter 2

Fractional Derivatives and Integrals

In this chapter several approaches to the generalization of the notion of differentiation and integration are considered. The choice has been reduced to those definitions which are related to applications.

2.1 The Name of the Game

Mathematics is the art of giving things misleading names. The beautiful – and at first look mysterious – name *the fractional calculus* is just one of the those misnomers which are the essence of mathematics.

For example, we know such names as *natural numbers* and *real numbers*. We use them very often; let us think for a moment about these names. The notion of a *natural number* is a natural abstraction, but is the *number* itself *natural*?

The notion of a *real number* is a generalization of the notion of a natural number. The word *real* emphasizes that we pretend that they reflect real quantities. The *real numbers* do reflect real quantities, but this cannot change the fact that they do not exist. Everything is in order in mathematical analysis, and the notion of a *real number* makes it easier, but if one wants to compute something, he immediately discovers for himself that there is no place for *real numbers* in the *real world*; nowadays, computations are performed mostly on digital computers, which can work only with finite sets of finite fractions, which serve as approximations to unreal *real numbers*.

Let us now return to the name of the *fractional calculus*. It does not mean the calculus of fractions. Neither does it mean a fraction of any calculus — differential, integral or calculus of variations. *The fractional calculus* is a name for the theory of integrals and derivatives of arbitrary order, which unify and generalize the notions of integer-order differentiation and n-fold integration.

Let us consider the infinite sequence of n-fold integrals and n-fold derivatives:

$$\ldots, \quad \int_a^t d\tau_2 \int_a^{\tau_2} f(\tau_1)d\tau_1, \quad \int_a^t f(\tau_1)d\tau_1, \quad f(t), \quad \frac{df(t)}{dt}, \quad \frac{d^2f(t)}{dt^2}, \quad \ldots$$

The derivative of arbitrary real order α can be considered as an interpolation of this sequence of operators; we will use for it the notation suggested and used by Davis [39], namely

$${}_aD_t^{\alpha} f(t).$$

The short name for derivatives of arbitrary order is *fractional derivatives*.

The subscripts a and t denote the two limits related to the operation of fractional differentiation; following Ross [227] we will call them the *terminals* of fractional differentiation. The appearance of the terminals in the symbol of fractional differentiation is essential. This helps to avoid ambiguities in applications of fractional derivatives to real problems.

The words *fractional integrals* mean in this book integrals of arbitrary order and correspond to negative values of α. We will not use a separate notation for fractional integrals; we will denote the fractional integral of order $\beta > 0$ by

$${}_aD_t^{-\beta} f(t).$$

A *fractional differential equation* is an equation which contains fractional derivatives; a *fractional integral equation* is an integral equation containing fractional integrals.

A *fractional-order system* means a system described by a fractional differential equation or a fractional integral equation or by a system of such equations.

2.2 Grünwald–Letnikov Fractional Derivatives

2.2.1 Unification of Integer-order Derivatives and Integrals

In this section we describe an approach to the unification of two notions, which are usually presented separately in classical analysis: derivative of integer order n and n-fold integrals. As will be shown below, these notions are closer to each other than one usually assumes.

Let us consider a continuous function $y = f(t)$. According to the well-known definition, the first-order derivative of the function $f(t)$ is defined by

$$f'(t) = \frac{df}{dt} = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}. \tag{2.1}$$

Applying this definition twice gives the second-order derivative:

$$\begin{aligned} f''(t) = \frac{d^2 f}{dt^2} &= \lim_{h \to 0} \frac{f'(t) - f'(t-h)}{h} \\ &= \lim_{h \to 0} \frac{1}{h} \left\{ \frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right\} \\ &= \lim_{h \to 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned} \tag{2.2}$$

Using (2.1) and (2.2) we obtain

$$f'''(t) = \frac{d^3 f}{dt^3} = \lim_{h \to 0} \frac{f(t) - 3f(t-h) + 3f(t-2h) - f(t-3h)}{h^3} \tag{2.3}$$

and, by induction,

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(t - rh), \tag{2.4}$$

where

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} \tag{2.5}$$

is the usual notation for the binomial coefficients.

Let us now consider the following expression generalizing the fractions in (2.1)–(2.4):

$$f_h^{(p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} (-1)^k \binom{p}{r} f(t - rh), \tag{2.6}$$

where p is an arbitrary integer number; n is also integer, as above.

Obviously, for $p \leq n$ we have

$$\lim_{h\to 0} f_h^{(p)}(t) = f^{(p)}(t) = \frac{d^p f}{dt^p}, \tag{2.7}$$

because in such a case, as follows from (2.5), all the coefficients in the numerator after $\binom{p}{p}$ are equal to 0.

Let us consider negative values of p. For convenience, let us denote

$$\begin{bmatrix} p \\ r \end{bmatrix} = \frac{p(p+1)\ldots(p+r-1)}{r!}. \tag{2.8}$$

Then we have

$$\binom{-p}{r} = \frac{-p(-p-1)\ldots(-p-r+1)}{r!} = (-1)^r \begin{bmatrix} p \\ r \end{bmatrix} \tag{2.9}$$

and replacing p in (2.6) with $-p$ we can write

$$f_h^{(-p)}(t) = \frac{1}{h^p} \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh), \tag{2.10}$$

where p is a positive integer number.

If n is fixed, then $f_h^{(-p)}(t)$ tends to the uninteresting limit 0 as $h \to 0$. To arrive at a non-zero limit, we have to suppose that $n \to \infty$ as $h \to 0$. We can take $h = \frac{t-a}{n}$, where a is a real constant, and consider the limit value, either finite or infinite, of $f_h^{(-p)}(t)$, which we will denote as

$$\lim_{\substack{h\to 0 \\ nh=t-a}} f_h^{(-p)}(t) = {}_aD_t^{-p} f(t). \tag{2.11}$$

Here ${}_aD_t^{-p} f(t)$ denotes, in fact, a certain operation performed on the function $f(t)$; a and t are the *terminals* — the limits relating to this operation.

Let us consider several particular cases.

For $p = 1$ we have:

$$f_h^{(-1)}(t) = h \sum_{r=0}^{n} f(t-rh). \tag{2.12}$$

Taking into account that $t - nh = a$ and that the function $f(t)$ is assumed to be continuous, we conclude that

$$\lim_{\substack{h\to 0 \\ nh=t-a}} f_h^{(-1)}(t) = {}_aD_t^{-1} f(t) = \int_0^{t-a} f(t-z)dz = \int_a^t f(\tau)d\tau. \tag{2.13}$$

Let us take $p = 2$. In this case

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{2 \cdot 3 \cdot \ldots \cdot (2 + r - 1)}{r!} = r + 1,$$

and we have:

$$f_h^{(-2)}(t) = h \sum_{r=0}^{n} (rh) f(t - rh). \tag{2.14}$$

Denoting $t + h = y$ we can write

$$f_h^{(-2)}(t) = h \sum_{r=1}^{n+1} (rh) f(t - rh), \tag{2.15}$$

and taking $h \to 0$ we obtain

$$\lim_{\substack{h \to 0 \\ nh = t-a}} f_h^{(-2)}(t) = {}_aD_t^{-2} f(t) = \int_0^{t-a} z f(t - z) dt = \int_a^t (t - \tau) f(\tau) d\tau, \tag{2.16}$$

because $y \to t$ as $h \to 0$.

The third particular case, namely $p = 3$, will show us the general expression for ${}_aD_t^{-p}$.

Taking into account that

$$\begin{bmatrix} 2 \\ r \end{bmatrix} = \frac{3 \cdot 4 \cdot \ldots \cdot (3 + r - 1)}{r!} = \frac{(r+1)(r+2)}{1 \cdot 2},$$

we have

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=0}^{n} (r+1)(r+2) h^2 f(t - rh). \tag{2.17}$$

Denoting, as above, $t + h = y$, we write

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=1}^{n+1} r(r+1) h^2 f(y - rh) \tag{2.18}$$

Expression (2.18) can be written as

$$f_h^{(-3)}(t) = \frac{h}{1 \cdot 2} \sum_{r=1}^{n+1} (rh)^2 f(y - rh) + \frac{h^2}{1 \cdot 2} \sum_{r=1}^{n+1} rh f(y - rh). \tag{2.19}$$

Taking now $h \to 0$, we obtain

$${}_aD_t^{-3} f(t) = \frac{1}{2!} \int_0^{t-a} z^2 f(t - z) dz = \int_a^t (t - \tau)^2 f(\tau) d\tau, \tag{2.20}$$

because $y \to t$ as $h \to 0$ and

$$\lim_{\substack{h\to 0\\ nh=t-a}} \frac{h^2}{1\cdot 2}\sum_{r=1}^{n+1} rhf(y-rh) = \lim_{\substack{h\to 0\\ nh=t-a}} h\int_a^t (t-\tau)f(\tau)d\tau = 0.$$

Relationships (2.13)–(2.20) suggest the following general expression:

$${}_aD_t^{-p}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{(p-1)!}\int_a^t (t-\tau)^{p-1}f(\tau)d\tau. \tag{2.21}$$

To prove the formula (2.21) by induction we have to show that if it holds for some p, then it holds also for $p+1$.

Let us introduce the function

$$f_1(t) = \int_a^t f(\tau)d\tau, \tag{2.22}$$

which has the obvious property $f_1(a) = 0$, and consider

$$\begin{aligned} {}_aD_t^{-p-1}f(t) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{p+1}\sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f(t-rh) \\ &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{p}\sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f_1(t-rh) \\ &\quad - \lim_{\substack{h\to 0\\ nh=t-a}} h^{p}\sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r \end{bmatrix} f_1(t-(r+1)h) \end{aligned} \tag{2.23}$$

Using (2.8) it is easy to verify that

$$\begin{bmatrix} p+1 \\ r \end{bmatrix} = \begin{bmatrix} p \\ r \end{bmatrix} + \begin{bmatrix} p+1 \\ r-1 \end{bmatrix}, \tag{2.24}$$

where we must put

$$\begin{bmatrix} p+1 \\ -1 \end{bmatrix} = 0.$$

Relationship (2.24) applied to the first sum in (2.23) and the replacement of r by $r-1$ in the second sum gives:

$${}_aD_t^{-p-1}f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{p}\sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f_1(t-rh)$$

$$
\begin{aligned}
&+ \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p+1 \\ r-1 \end{bmatrix} f_1(t-rh) \\
&- \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=1}^{n+1} \begin{bmatrix} p+1 \\ r-1 \end{bmatrix} f_1(t-rh) \\
= \quad & {}_aD_t^{-p} f_1(t) - \lim_{\substack{h\to 0\\ nh=t-a}} h^p \begin{bmatrix} p+1 \\ n \end{bmatrix} f_1(t-(n+1)h) \\
= \quad & {}_aD_t^{-p} f_1(t) - (t-a)^p \lim_{n\to\infty} \begin{bmatrix} p+1 \\ n \end{bmatrix} \frac{1}{n^p} f_1(a - \frac{t-a}{n}).
\end{aligned}
$$

It follows from the definition (2.22) of the function $f_1(t)$ that

$$
\lim_{n\to\infty} f_1(a - \frac{t-a}{n}) = 0.
$$

Taking into account the known limit (1.7)

$$
\lim_{n\to\infty} \begin{bmatrix} p+1 \\ n \end{bmatrix} \frac{1}{n^p} = \lim_{n\to\infty} \frac{(p+1)(p+2)\ldots(p+n)}{n^p n!} = \frac{1}{\Gamma(p+1)},
$$

we obtain

$$
\begin{aligned}
{}_aD_t^{-p-1} f(t) &= {}_aD_t^{-p} f_1(t) = \frac{1}{(p-1)!} \int\limits_a^t (t-\tau)^{p-1} f_1(\tau) d\tau \\
&= -\left. \frac{(t-\tau)^p f_1(\tau)}{p!} \right|_{\tau=a}^{\tau=t} + \frac{1}{p!} \int\limits_a^t (t-\tau)^p f(\tau) d\tau \\
&= \frac{1}{p!} \int\limits_a^t (t-\tau)^p f(\tau) d\tau, \qquad (2.25)
\end{aligned}
$$

which ends the proof of formula (2.21) by induction.

Now let us show that formula (2.21) is a representation of a p-fold integral.

Integrating the relationship

$$
\frac{d}{dt}\left({}_aD_t^{-p} f(t) \right) = \frac{1}{(p-2!)} \int\limits_a^t (t-\tau)^{p-2} f(\tau) d\tau = {}_aD_t^{-p+1} f(t)
$$

from a to t we obtain:

$$\begin{aligned}
{}_aD_t^{-p}f(t) &= \int_a^t \left({}_aD_t^{-p+1}f(t)\right) dt, \\
{}_aD_t^{-p+1}f(t) &= \int_a^t \left({}_aD_t^{-p+2}f(t)\right) dt, \text{etc.},
\end{aligned}$$

and therefore

$$\begin{aligned}
{}_aD_t^{-p}f(t) &= \int_a^t dt \int_a^t \left({}_aD_t^{-p+2}f(t)\right) dt \\
&= \int_a^t dt \int_a^t dt \int_a^t \left({}_aD_t^{-p+3}f(t)\right) dt \\
&= \underbrace{\int_a^t dt \int_a^t dt \ldots \int_a^t}_{p \text{ times}} f(t)dt. \qquad (2.26)
\end{aligned}$$

We see that the derivative of an integer order n (2.4) and the p-fold integral (2.21) of the continuous function $f(t)$ are particular cases of the general expression

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh), \qquad (2.27)$$

which represents the derivative of order m if $p = m$ and the m-fold integral if $p = -m$.

This observation naturally leads to the idea of a generalization of the notions of differentiation and integration by allowing p in (2.27) to be an arbitrary real or even complex number. We will restrict our attention to real values of p.

2.2.2 Integrals of Arbitrary Order

Let us consider the case of $p < 0$. For convenience let us replace p by $-p$ in the expression (2.27). Then (2.27) takes the form

$$ {}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} h^{p} \sum_{r=0}^{n} \left[{p \atop r} \right] f(t-rh), \qquad (2.28)$$

where, as above, the values of h and n relate as $nh = t - a$.

To prove the existence of the limit in (2.28) and to evaluate that limit we need the following theorem (A. V. Letnikov, [124]):

THEOREM 2.1 ∘ *Let us take a sequence β_k, $(k = 1, 2, \ldots)$ and suppose that*

$$\lim_{k\to\infty} \beta_k = 1, \tag{2.29}$$

$$\lim_{n\to\infty} \alpha_{n,k} = 0 \quad \textit{for all } k, \tag{2.30}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k} = A \quad \textit{for all } k, \tag{2.31}$$

$$\sum_{k=1}^{n} |\alpha_{n,k}| < K \quad \textit{for all } n. \tag{2.32}$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k = A. \qquad \bullet \tag{2.33}$$

Proof. The condition (2.29) allows us to put

$$\beta_k = 1 - \sigma_k, \qquad \text{where} \quad \lim_{k\to\infty} \sigma_k = 0. \tag{2.34}$$

It follows from the condition (2.30) that for every fixed r

$$\lim_{n\to\infty} \sum_{k=1}^{r-1} \alpha_{n,k}\beta_k = 0 \tag{2.35}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{r-1} \alpha_{n,k} = 0. \tag{2.36}$$

Using subsequently (2.35), (2.34), (2.31), and (2.36) we obtain

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k &= \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\beta_k \\
&= \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k} - \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k} - \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k \\
&= A - \lim_{n\to\infty} \sum_{k=r}^{n} \alpha_{n,k}\sigma_k.
\end{aligned}$$

Now, using (2.36) and (2.32), we can perform the following estimation:

$$\begin{aligned}\left| A - \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k \right| &< \lim_{n\to\infty} \sum_{k=r}^{n} |\alpha_{n,k}| \cdot |\sigma_k| \\ &< \sigma^* \lim_{n\to\infty} \sum_{k=r}^{n} |\alpha_{n,k}| = \sigma^* \lim_{n\to\infty} \sum_{k=1}^{n} |\alpha_{n,k}| \\ &< \sigma^* K\end{aligned}$$

where $\sigma^* = \max\limits_{k\geq r} |\sigma_k|$.

It follows from (2.34) that for each arbitrarily small $\epsilon > 0$ there exists r such that $\sigma^* < \epsilon/K$ and, therefore,

$$\left| A - \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k \right| < \epsilon,$$

and the statement (2.33) of the theorem holds.

Theorem 2.1 has a simple consequence. Namely, if we take

$$\lim_{k\to\infty} \beta_k = B,$$

then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\beta_k = AB. \tag{2.37}$$

Indeed, introducing the sequence

$$\tilde{\beta}_k = \frac{\beta_k}{B}, \qquad \lim_{k\to\infty} \tilde{\beta}_k = 1,$$

we can apply Theorem 2.1 to obtain

$$\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\tilde{\beta}_k = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_{n,k}\frac{\beta_k}{B} = A,$$

from which the statement (2.37) follows.

To apply Theorem 2.1 for the evaluation of the limit (2.28), we write

$$ {}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t - rh)$$

$$
\begin{aligned}
&= \lim_{\substack{h\to 0\\ nh=t-a}} \sum_{r=0}^{n} \frac{1}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix} h(rh)^{p-1} f(t-rh) \\
&= \frac{1}{\Gamma(p)} \lim_{\substack{h\to 0\\ nh=t-a}} \sum_{r=0}^{n} \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix} h(rh)^{p-1} f(t-rh) \\
&= \frac{1}{\Gamma(p)} \lim_{n\to\infty} \sum_{r=0}^{n} \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix} \frac{t-a}{n} \left(r\frac{t-a}{n} \right)^{p-1} f(t - r\frac{t-a}{n})
\end{aligned}
$$

and take

$$\beta_r = \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix},$$

$$\alpha_{n,r} = \frac{t-a}{n} \left(r\frac{t-a}{n} \right)^{p-1} f(t - r\frac{t-a}{n}).$$

Using (1.7) we have

$$\lim_{r\to\infty} \beta_r = \lim_{r\to\infty} \frac{\Gamma(p)}{r^{p-1}} \begin{bmatrix} p \\ r \end{bmatrix} = 1. \tag{2.38}$$

Obviously, if the function $f(t)$ is continuous in the closed interval $[a, t]$, then

$$
\begin{aligned}
\lim_{n\to\infty} \sum_{r=0}^{n} \alpha_{n,r} &= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{t-a}{n} \left(r\frac{t-a}{n} \right)^{p-1} f(t - r\frac{t-a}{n}) \\
&= \lim_{h\to 0} \sum_{r=0}^{n} h(rh)^{p-1} f(t-rh) \\
&= \int_a^t (t-\tau)^{p-1} f(\tau) d\tau.
\end{aligned} \tag{2.39}
$$

Taking into account (2.38) and (2.39) and applying Theorem 2.1 we conclude that

$${}_aD_t^{-p} f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^p \sum_{r=0}^{n} \begin{bmatrix} p \\ r \end{bmatrix} f(t-rh) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{2.40}$$

If the derivative $f'(t)$ is continuous in $[a, b]$, then integrating by parts we can write (2.40) in the form

$${}_aD_t^{-p} f(t) = \frac{f(a)(t-a)^p}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau, \tag{2.41}$$

and if the function $f(t)$ has $m+1$ continuous derivatives, then

$$ {}_aD_t^{-p}f(t)=\sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{p+k}}{\Gamma(p+k+1)}+\frac{1}{\Gamma(p+k+1)}\int_a^t(t-\tau)^{p+m}f^{(m+1)}(\tau)d\tau. \tag{2.42}$$

The formula (2.42) immediately provides us with the asymptotics of ${}_aD_t^{-p}f(t)$ at $t=a$.

2.2.3 Derivatives of Arbitrary Order

Let us now consider the case of $p>0$. Our aim is, as above, to evaluate the limit

$$ {}_aD_t^{p}f(t)=\lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}f(t-rh)=\lim_{\substack{h\to 0\\ nh=t-a}} f_h^{(p)}(t) \tag{2.43}$$

where

$$ f_h^{(p)}(t)=h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}f(t-rh). \tag{2.44}$$

To evaluate the limit (2.43), let us first transform the expression for $f_h^{(p)}(t)$ in the following way.

Using the known property of the binomial coefficients

$$\binom{p}{r}=\binom{p-1}{r}+\binom{p-1}{r-1} \tag{2.45}$$

we can write

$$\begin{aligned} f_h^{(p)}(t) &= h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p-1}{r}f(t-rh)\\ &\quad + h^{-p}\sum_{r=1}^{n}(-1)^r\binom{p-1}{r-1}f(t-rh)\\ &= h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p-1}{r}f(t-rh)\\ &\quad + h^{-p}\sum_{r=0}^{n-1}(-1)^{r+1}\binom{p-1}{r}f(t-(r+1)h)\\ &= (-1)^n\binom{p-1}{n}h^{-p}f(a)\\ &\quad + h^{-p}\sum_{r=0}^{n-1}(-1)^r\binom{p-1}{r}\Delta f(t-rh), \end{aligned} \tag{2.46}$$

where we denote

$$\Delta f(t-rh) = f(t-rh) - f(t-(r+1)h).$$

Obviously, $\Delta f(t-rh)$ is a first-order backward difference of the function $f(\tau)$ at the point $\tau = t - rh$.

Applying the property (2.45) of the binomial coefficients repeatedly m times, we obtain starting from (2.46):

$$\begin{aligned}
f_h^{(p)}(t) &= (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1}\binom{p-2}{n-1} h^{-p}\Delta f(a+h) \\
&\quad + h^{-p}\sum_{r=0}^{n-2}(-1)^r \binom{p-2}{r}\Delta^2 f(t-rh) \\
&= (-1)^n \binom{p-1}{n} h^{-p} f(a) + (-1)^{n-1}\binom{p-2}{n-1} h^{-p}\Delta f(a+h) \\
&\quad + (-1)^{n-2}\binom{p-3}{n-3} h^{-p}\Delta^2 f(a+2h) \qquad (2.47)\\
&\quad + h^{-p}\sum_{r=0}^{n-3}(-1)^r \binom{p-3}{r}\Delta^3 f(t-rh) \\
&= \ldots \\
&= \sum_{k=0}^{m}(-1)^{n-k}\binom{p-k-1}{n-k} h^{-p}\Delta^k f(a+kh) \\
&\quad + h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1} f(t-rh). \qquad (2.48)
\end{aligned}$$

Let us evaluate the limit of the k-th term in the first sum in (2.48):

$$\begin{aligned}
&\lim_{\substack{h\to 0\\ nh=t-a}} (-1)^{n-k}\binom{p-k-1}{n-k} h^{-p}\Delta^k f(a+kh) \\
&\quad = \lim_{\substack{h\to 0\\ nh=t-a}} (-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k} \\
&\qquad\qquad \times \left(\frac{n}{n-k}\right)^{p-k} (nh)^{-p+k}\frac{\Delta^k f(a+kh)}{h^k} \\
&\quad = (t-a)^{-p+k}\lim_{n\to\infty}(-1)^{n-k}\binom{p-k-1}{n-k}(n-k)^{p-k} \\
&\qquad\qquad \times \lim_{n\to\infty}\left(\frac{n}{n-k}\right)^{p-k} \times \lim_{h\to 0}\frac{\Delta^k f(a+kh)}{h^k}
\end{aligned}$$

$$= \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)}, \tag{2.49}$$

because using (1.7) gives

$$\lim_{n\to\infty} (-1)^{n-k} \binom{p-k-1}{n-k} (n-k)^{p-k}$$
$$= \lim_{n\to\infty} \frac{(-p+k+1)(-p+k+2)\dots(-p+n)}{(n-k)^{-p+k}(n-k)!} = \frac{1}{\Gamma(-p+k+1)}$$

and

$$\lim_{n\to\infty} \left(\frac{n}{n-k}\right)^{p-k} = 1,$$
$$\lim_{h\to 0} \frac{\Delta^k f(a+kh)}{h^k} = f^{(k)}(a).$$

Knowing the limit (2.49) we can easily write the limit of the first sum in (2.48).

To evaluate the limit of the second sum in (2.48) let us write it in the form

$$\frac{1}{\Gamma(-p+m+1)} \sum_{r=0}^{n-m-1} (-1)^r \Gamma(-p+m+1) \binom{p-m-1}{r} r^{-m+p}$$
$$\times h(rh)^{m-p} \frac{\Delta^{m+1} f(t-rh)}{h^{m+1}}. \tag{2.50}$$

To apply Theorem 2.1 we take

$$\beta_r = (-1)^r \Gamma(-p+m+1) \binom{p-m-1}{r} r^{-m+p},$$

$$\alpha_{n,r} = h(rh)^{m-p} \frac{\Delta^{m+1} f(t-rh)}{h^{m+1}}, \quad h = \frac{t-a}{n}.$$

Using (1.7) we verify that

$$\lim_{r\to\infty} \beta_r = \lim_{r\to\infty} (-1)^r \Gamma(-p+m+1) \binom{p-m-1}{r} r^{-m+p} = 1. \tag{2.51}$$

In addition, if $m-p > -1$, then

$$\lim_{n\to\infty} \sum_{r=0}^{n-m-1} \alpha_{n,r} = \lim_{\substack{h\to 0\\ nh=t-a}} \sum_{r=0}^{n-m-1} h(rh)^{m-p} \frac{\Delta^{m+1} f(t-rh)}{h^{m+1}}$$
$$= \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)\,d\tau. \tag{2.52}$$

Taking into account (2.51) and (2.52) and applying Theorem 2.1 we conclude that

$$\lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n-m-1}(-1)^r\binom{p-m-1}{r}\Delta^{m+1}f(t-rh)$$
$$= \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau. \quad (2.53)$$

Using (2.49) and (2.53) we finally obtain the limit (2.43):

$$\begin{aligned} {}_aD_t^p f(t) &= \lim_{\substack{h\to\infty\\ nh=t-a}} f_h^{(p)}(t) \\ &= \sum_{k=0}^{m}\frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\ &\quad + \frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p}f^{(m+1)}(\tau)d\tau. \quad (2.54)\end{aligned}$$

The formula (2.54) has been obtained under the assumption that the derivatives $f^{(k)}(t)$, $(k = 1, 2, \ldots, m+1)$ are continuous in the closed interval $[a, t]$ and that m is an integer number satisfying the condition $m > p-1$. The smallest possible value for m is determined by the inequality

$$m < p < m+1.$$

2.2.4 Fractional Derivative of $(t-a)^\beta$

Let us evaluate the Grünwald–Letnikov fractional derivative ${}_aD_t^p f(t)$ of the power function

$$f(t) = (t-a)^\nu,$$

where ν is a real number.

We will start by considering negative values of p, which means that we will start with the evaluation of the fractional integral of order $-p$. Let us use the formula (2.40):

$${}_aD_t^p (t-a)^\nu = \frac{1}{\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}(\tau-a)^\nu d\tau, \qquad (2.55)$$

and suppose $\nu > -1$ for the convergence of the integral. Performing in (2.55) the substitution $\tau = a + \xi(t-a)$ and then using the definition of the beta function (1.20) we obtain:

$$
\begin{aligned}
{}_aD_t^p(t-a)^\nu &= \frac{1}{\Gamma(-p)}(t-a)^{\nu-p}\int\limits_0^1 \xi^\nu(1-\xi)^{-p-1}d\xi \\
&= \frac{1}{\Gamma(-p)}B(-p,\nu+1)(t-a)^{\nu-p} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(\nu-p+1)}(t-a)^{\nu-p}, \quad (p<0,\ \nu>-1). \quad (2.56)
\end{aligned}
$$

Now let us consider the case $0 \le m \le p < m+1$. To apply the formula (2.54), we must require $\nu > m$ for the convergence of the integral in (2.54). Then we have:

$$
{}_aD_t^p(t-a)^\nu = \frac{1}{\Gamma(-p+m+1)}\int\limits_a^t (t-\tau)^{m-p}\frac{d^{m+1}(\tau-a)^\nu}{d\tau^{m+1}}d\tau, \qquad (2.57)
$$

because all non-integral addends are equal to 0.

Taking into account that

$$
\frac{d^{m+1}(\tau-a)^\nu}{d\tau^{m+1}} = \nu(\nu-1)\dots(\nu-m)(\tau-a)^{\nu-m-1} = \frac{\Gamma(\nu+1)}{\nu-m}(\tau-a)^{\nu-m-1}
$$

and performing the substitution $\tau = a + \xi(t-a)$ we obtain:

$$
\begin{aligned}
{}_aD_t^p(t-a)^\nu &= \frac{\Gamma(\nu+1)}{\Gamma(\nu-m)\Gamma(-p+m+1)}\int\limits_a^t (t-\tau)^{m-p}(\tau-a)^{\nu-m-1}d\tau \\
&= \frac{\Gamma(\nu+1)B(-p+m+1,\nu-m)}{\Gamma(\nu-m)\Gamma(-p+m+1)}(t-a)^{\nu-p} \\
&= \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p}. \qquad (2.58)
\end{aligned}
$$

Noting that the expression (2.58) is formally identical to the expression (2.56) we can conclude that the Grünwald–Letnikov fractional derivative of the power function $f(t) = (t-a)^\nu$ is given by the formula

$$
{}_aD_t^p(t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-p+\nu+1)}(t-a)^{\nu-p}, \qquad (2.59)
$$

$$(p < 0, \quad \nu > -1) \quad \text{or} \quad (0 \le m \le p < m+1, \quad \nu > m).$$

We will return to formula (2.59) for the Grünwald–Letnikov fractional derivative of the power function later, when we consider some other approaches to fractional differentiation. The formula will be the same, but the conditions for its applicability will be different.

From the theoretical point of view, the class of functions for which the considered Grünwald–Letnikov definition of the fractional derivative is defined ($(m+1)$-times continuously differentiable functions) is very narrow. However, in most applied problems describing continuous physical, chemical and other processes we deal with such very smooth functions.

2.2.5 Composition with Integer-order Derivatives

Noting that we have only one restriction for m in the formula (2.54), namely the condition $m > p-1$, let us write s instead of m and rewrite (2.54) as

$$
\begin{aligned}
{}_aD_t^p f(t) &= \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\
&\quad + \frac{1}{\Gamma(-p+s+1)} \int_a^t (t-\tau)^{s-p} f^{(s+1)}(\tau)\,d\tau. \qquad (2.60)
\end{aligned}
$$

In what follows we assume that $m < p < m+1$.

Let us evaluate the derivative of integer order n of the fractional derivative of fractional order p in the form (2.60), where we take $s \geq m+n-1$. The result is:

$$
\begin{aligned}
\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) &= \sum_{k=0}^{s} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} \\
&\quad + \frac{1}{\Gamma(-p-n+s+1)} \int_a^t (t-\tau)^{s-p-n} f^{(s+1)}(\tau)\,d\tau \qquad (2.61) \\
&= {}_aD_t^{p+n} f(t). \qquad (2.62)
\end{aligned}
$$

Since $s \geq m+n-1$ is arbitrary, let us take $s = m+n-1$. This gives:

$$\frac{d^n}{dt^n}\left({}_aD_t^p f(t)\right) = {}_aD_t^{p+n} f(t)$$

$$= \sum_{k=0}^{m+n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau) d\tau \qquad (2.63)$$

Let us now consider the reverse order of operations and evaluate the fractional derivative of order p of an integer-order derivative $\frac{d^n f(t)}{dt^n}$. Using the formula (2.60) we obtain:

$${}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) = \sum_{k=0}^{s} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(-p+s+1)} \int_a^t (t-\tau)^{s-p} f^{(n+s+1)}(\tau) d\tau. \qquad (2.64)$$

Putting here $s = m-1$ we obtain:

$${}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) = \sum_{k=0}^{m-1} \frac{f^{(n+k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} + \frac{1}{\Gamma(m-p)} \int_a^t (t-\tau)^{m-p-1} f^{(m+n)}(\tau) d\tau, \qquad (2.65)$$

and comparing (2.63) and (2.65) we arrive at the conclusion that

$$\frac{d^n}{dt^n} \left({}_aD_t^p f(t) \right) = {}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-p-n+k}}{\Gamma(-p-n+k+1)}. \qquad (2.66)$$

The relationship (2.66) says that the operations $\frac{d^n}{dt^n}$ and ${}_aD_t^p$ are commutative, i.e., that

$$\frac{d^n}{dt^n} \left({}_aD_t^p f(t) \right) = {}_aD_t^p \left(\frac{d^n f(t)}{dt^n} \right) = {}_aD_t^{p+n} f(t), \qquad (2.67)$$

only if at the lower terminal $t = a$ of the fractional differentiation we have

$$f^{(k)}(a) = 0, \qquad (k = 0, 1, 2, \ldots, n-1). \qquad (2.68)$$

2.2.6 Composition with Fractional Derivatives

Let us now consider the fractional derivative of order q of a fractional derivative of order p:

$$ {}_aD_t^q\Big({}_aD_t^p f(t)\Big). $$

Two cases will be considered separately: $p < 0$ and $p > 0$. The first case means that — depending on the sign of q — differentiation of order $q > 0$ or integration of order $-q > 0$ is applied to the fractional integral of order $-p > 0$. In the second case, the object of the outer operation is the fractional derivative of order $p > 0$.

In both cases we will obtain an analogue of the well-known property of integer-order differentiation:

$$ \frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^{m+n} f(t)}{dt^{m+n}}. $$

Case $p < 0$

Let us first take $q < 0$. Then we have:

$$ \begin{aligned} {}_aD_t^q\Big({}_aD_t^p f(t)\Big) &= \frac{1}{\Gamma(-q)}\int_a^t (t-\tau)^{-q-1}\Big({}_aD_\tau^p f(\tau)\Big)d\tau \\ &= \frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t (t-\tau)^{-q-1}d\tau \int_a^\tau (\tau-\xi)^{-q-1} f(\xi)d\xi \\ &= \frac{1}{\Gamma(-q)\Gamma(-p)}\int_a^t f(\xi)d\xi \int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau \\ &= \frac{1}{\Gamma(-p-q)}\int_a^t (t-\xi)^{-p-q-1} f(\xi)d\xi \\ &= {}_aD_t^{p+q} f(t), \end{aligned} \tag{2.69} $$

where the integral

$$ \begin{aligned} \int_\xi^t (t-\tau)^{-q-1}(\tau-\xi)^{-p-1}d\tau &= (t-\xi)^{-p-q-1}\int_0^1 (1-z)^{-q-1}z^{-p-1}dz \\ &= \frac{\Gamma(-q)\Gamma(-p)}{\Gamma(-p-q)}(t-\xi)^{-p-q-1} \end{aligned} $$

is evaluated with the help of the substitution $\tau = \xi + z(t-\xi)$ and the definition of the beta function (1.20).

Let us now suppose that $0 < n < q < n+1$. Noting that $q = (n+1) + (q-n-1)$, where $q-n-1 < 0$, and using the formulas (2.62) and (2.69) we obtain:

$$
\begin{aligned}
{}_aD_t^q\Big({}_aD_t^p f(t)\Big) &= \frac{d^{n+1}}{dt^{n+1}}\Big\{{}_aD_t^{q-n-1}\big({}_aD_t^p f(t)\big)\Big\} \\
&= \frac{d^{n+1}}{dt^{n+1}}\Big\{{}_aD_t^{p+q-n-1} f(t)\Big\} \\
&= {}_aD_t^{p+q} f(t). \qquad (2.70)
\end{aligned}
$$

Combining (2.69) and (2.70) we conclude that if $p < 0$, then for any real q

$$
{}_aD_t^q\Big({}_aD_t^p f(t)\Big) = {}_aD_t^{p+q} f(t).
$$

Case $p > 0$

Let us assume that $0 \leq m < p < m+1$. Then, according to formula (2.54), we have

$$
\begin{aligned}
{}_aD_t^p f(t) &= \lim_{\substack{h\to\infty \\ nh=t-a}} f_h^{(p)}(t) \\
&= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \\
&\quad + \frac{1}{\Gamma(-p+m+1)} \int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau. \qquad (2.71)
\end{aligned}
$$

Let us take $q < 0$ and evaluate

$$
{}_aD_t^q\Big({}_aD_t^p f(t)\Big).
$$

Examining the right-hand side of (2.71) we see that the functions $(t-a)^{-p+k}$ have non-integrable singularities for $k = 0, 1, \ldots, m-1$. Therefore, the derivative of real order q of ${}_aD_t^p f(t)$ exists only if

$$
f^{(k)}(a) = 0, \qquad (k = 0, 1, \ldots, m-1). \qquad (2.72)
$$

The integral in the right-hand side of (2.71) is equal to ${}_aD_t^{p-m-1}f(t)$ (the fractional integral of order $-p+m+1$ of the function $f(t)$). Therefore, under the conditions (2.72) the representation (2.71) of the p-th derivative of $f(t)$ takes the following form:

$$ {}_aD_t^p f(t) = \frac{f^{(m)}(a)(t-a)^{-p+m}}{\Gamma(-p+m+1)} + {}_aD_t^{p-m-1} f^{(m+1)}(t). \tag{2.73} $$

Now we can find the derivative of order $q < 0$ (in other words, the integral of order $-q > 0$) of the derivative of order p given by (2.73):

$$ \begin{aligned} {}_aD_t^q\Big({}_aD_t^p f(t)\Big) &= \frac{f^{(m)}(a)(t-a)^{-p-q+m}}{\Gamma(-p-q+m+1)} \\ &\quad + \frac{1}{\Gamma(-p-q+m+1)} \int\limits_a^t \frac{f^{(m+1)}(\tau)d\tau}{(t-\tau)^{p+q-m}}, \end{aligned} \tag{2.74} $$

because

$$ \begin{aligned} {}_aD_t^q\Big({}_aD_t^{p-m-1} f^{(m+1)}(t)\Big) &= {}_aD_t^{p+q-m-1} f^{(m+1)}(t) \\ &= \frac{1}{\Gamma(-p-q+m+1)} \int\limits_a^t \frac{f^{(m+1)}(\tau)d\tau}{(t-\tau)^{p+q-m}}. \end{aligned} $$

Taking into account the conditions (2.72) and the formula (2.71) we arrive at

$$ {}_aD_t^q\Big({}_aD_t^p f(t)\Big) = {}_aD_t^{p+q} f(t). \tag{2.75} $$

Let us now take $0 \le n < q < n+1$. Assuming that $f(t)$ satisfies the conditions (2.72) and taking into account that $q-n-1 < 0$ and, therefore, the formula (2.75) can be used, we obtain:

$$ \begin{aligned} {}_aD_t^q\Big({}_aD_t^p f(t)\Big) &= \frac{d^{n+1}}{dt^{n+1}}\Big\{{}_aD_t^{q-n-1}\Big({}_aD_t^p f(t)\Big)\Big\} \\ &= \frac{d^{n+1}}{dt^{n+1}}\Big\{{}_aD_t^{p+1-n-1} f(t)\Big\} \\ &= {}_aD_t^{p+q} f(t), \end{aligned} \tag{2.76} $$

which is the same as (2.75).

Therefore, we can conclude that if $p < 0$, then the relationship (2.75) holds for arbitrary real q; if $0 \le m < p < m+1$, then the relationship

(2.75) holds also for arbitrary real q, if the function $f(t)$ satisfies the conditions (2.72).

Moreover, if $0 \leq m < p < m+1$ and $0 \leq n < q < n+1$ and the function $f(t)$ satisfies the conditions

$$f^{(k)}(a) = 0, \quad (k = 0, 1, \ldots, r-1), \tag{2.77}$$

where $r = \max(n, m)$, then the operators of fractional differentiation ${}_aD_t^p$ and ${}_aD_t^q$ commute:

$${}_aD_t^q\Big({}_aD_t^p f(t)\Big) = {}_aD_t^p\Big({}_aD_t^q f(t)\Big) = {}_aD_t^{p+q} f(t). \tag{2.78}$$

2.3 Riemann–Liouville Fractional Derivatives

Manipulation with the Grünwald–Letnikov fractional derivatives defined as a limit of a fractional-order backward difference is not convenient. The obtained expression (2.54) looks better because of the presense of the integral in it; but what about the non-integral terms? The answer is simple and elegant: to consider the expression (2.54) as a particular case of the integro-differential expression

$${}_a\mathbf{D}_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int\limits_a^t (t-\tau)^{m-p} f(\tau)d\tau, \qquad (m \leq p < m+1). \tag{2.79}$$

The expression (2.79) it is the most widely known definition of the fractional derivative; it is usually called the Riemann–Liouville definition.

Obviously, the expression (2.54), which has been obtained for the Grünwald–Letnikov fractional derivative under the assumption that the function $f(t)$ must be $m+1$ times continuously differentiable, can be obtained from (2.79) *under the same assumption* by performing repeatedly integration by parts and differentiation. This gives

$$\begin{aligned} {}_a\mathbf{D}_t^p f(t) &= \left(\frac{d}{dt}\right)^{m+1} \int\limits_a^t (t-\tau)^{m-p} f(\tau)d\tau \\ &= \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{-p+k}}{\Gamma(-p+k+1)} \end{aligned}$$

$$+\frac{1}{\Gamma(-p+m+1)}\int_a^t (t-\tau)^{m-p} f^{(m+1)}(\tau)d\tau$$
$$= {}_aD_t^p f(t), \qquad (m \le p < m+1). \tag{2.80}$$

Therefore, if we consider a class of functions $f(t)$ having $m+1$ continuous derivatives for $t \ge 0$, then the Grünwald–Letnikov definition (2.43) (or, what is in this case the same, its integral form (2.54)) is equivalent to the Riemann–Liouville definition (2.79).

From the pure mathematical point of view such a class of functions is narrow; however, this class of functions is very important for applications, because the character of the majority of dynamical processes is smooth enough and does not allow discontinuities. Understanding this fact is important for the proper use of the methods of the fractional calculus in applications, especially because of the fact that the Riemann–Liouville definition (2.79) provides an excellent opportunity to weaken the conditions on the function $f(t)$. Namely, it is enough to require the integrability of $f(t)$; then the integral (2.79) exists for $t > a$ and can be differentiated $m+1$ times. The weak conditions on the function $f(t)$ in (2.79) are necessary, for example, for obtaining the solution of the Abel integral equation.

Let us look at how the Riemann–Liouville definition (2.79) appears as the result of the unification of the notions of integer-order integration and differentiation.

2.3.1 Unification of Integer-order Derivatives and Integrals

Let us suppose that the function $f(\tau)$ is continuous and integrable in every finite interval (a, t); the function $f(t)$ may have an integrable singularity of order $r < 1$ at the point $\tau = a$:

$$\lim_{\tau \to a} (\tau - a)^r f(t) = \text{const}\,(\neq 0).$$

Then the integral

$$f^{(-1)}(t) = \int_a^t f(\tau)d\tau \tag{2.81}$$

exists and has a finite value, namely equal to 0, as $t \to a$. Indeed, performing the substitution $\tau = a + y(t-a)$ and then denoting $\epsilon = t - a$, we obtain

$$\begin{aligned}
\lim_{t\to a} f^{(-1)}(t) &= \lim_{t\to a} \int_a^t f(\tau)d\tau \\
&= \lim_{t\to a} (t-a) \int_0^1 f(a+y(t-a))dy \\
&= \lim_{\epsilon\to 0} \epsilon^{1-r} \int_0^1 (\epsilon y)^r f(a+y\epsilon) y^{-r} dy = 0, \qquad (2.82)
\end{aligned}$$

because $r < 1$. Therefore, we can consider the two-fold integral

$$\begin{aligned}
f^{(-2)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} f(\tau)d\tau = \int_a^t f(\tau)d\tau \int_\tau^t d\tau_1 \\
&= \int_a^t (t-\tau)f(\tau)d\tau. \qquad (2.83)
\end{aligned}$$

Integration of (2.83) gives the three-fold integral of $f(\tau)$:

$$\begin{aligned}
f^{(-3)}(t) &= \int_a^t d\tau_1 \int_a^{\tau_1} d\tau_2 \int_a^{\tau_2} f(\tau_3)d\tau_3 \\
&= \int_a^t d\tau_1 \int_a^{\tau_1} (\tau_1 - \tau) f(\tau)d\tau \\
&= \frac{1}{2} \int_a^t (t-\tau)^2 f(\tau)d\tau, \qquad (2.84)
\end{aligned}$$

and by induction in the general case we have the Cauchy formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau. \qquad (2.85)$$

Let us now suppose that $n \geq 1$ is fixed and take integer $k \geq 0$. Obviously, we will obtain

$$f^{(-k-n)}(t) = \frac{1}{\Gamma(n)} D^{-k} \int_a^t (t-\tau)^{n-1} f(\tau)d\tau, \qquad (2.86)$$

where the symbol D^{-k} $(k \geq 0)$ denotes k iterated integrations.

On the other hand, for a fixed $n \geq 1$ and integer $k \geq n$ the $(k-n)$-th derivative of the function $f(t)$ can be written as

$$f^{(k-n)}(t) = \frac{1}{\Gamma(n)} D^k \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{2.87}$$

where the symbol D^k $(k \geq 0)$ denotes k iterated differentiations.

We see that the formulas (2.86) and (2.87) can be considered as particular cases of one of them, namely (2.87), in which n $(n \geq 1)$ is fixed and the symbol D^k means k integrations if $k \leq 0$ and k differentiations if $k > 0$. If $k = n-1$, $n-2$, $\ldots$, then the formula (2.87) gives iterated integrals of $f(t)$; for $k = n$ it gives the function $f(t)$; for $k = n+1$, $n+2$, $n+3$, $\ldots$ it gives derivatives of order $k - n = 1$, 2, 3, $\ldots$ of the function $f(t)$.

2.3.2 Integrals of Arbitrary Order

To extend the notion of n-fold integration to non-integer values of n, we can start with the Cauchy formula (2.85) and replace the integer n in it by a real $p > 0$:

$${}_a\mathbf{D}_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} f(\tau) d\tau. \tag{2.88}$$

In (2.85) the integer n must satisfy the condition $n \geq 1$; the corresponding condition for p is weaker: for the existence of the integral (2.88) we must have $p > 0$.

Moreover, under certain reasonable assumptions

$$\lim_{p \to 0} {}_a\mathbf{D}_t^{-p} f(t) = f(t), \tag{2.89}$$

so we can put

$${}_a\mathbf{D}_t^{0} f(t) = f(t). \tag{2.90}$$

The proof of the relationship (2.89) is very simple if $f(t)$ has continuous derivatives for $t \geq 0$. In such a case, integration by parts and the use of (1.3) gives

$${}_a\mathbf{D}_t^{-p} f(t) = \frac{(t-a)^p f(a)}{\Gamma(p+1)} + \frac{1}{\Gamma(p+1)} \int_a^t (t-\tau)^p f'(\tau) d\tau,$$

and we obtain

$$\lim_{p\to 0} {}_a\mathbf{D}_t^{-p} f(t) = f(a) + \int_a^t f'(\tau)d\tau = f(a) + \Big(f(t) - f(a)\Big) = f(t).$$

If $f(t)$ is only continuous for $t \geq a$, then the proof of (2.89) is somewhat longer. In such a case, let us write ${}_a\mathbf{D}_t^{-p} f(t)$ in the form:

$$\begin{aligned}
{}_a\mathbf{D}_t^{-p} f(t) &= \frac{1}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} \left(f(\tau) - f(t)\right) d\tau + \frac{f(t)}{\Gamma(p)} \int_a^t (t-\tau)^{p-1} d\tau \\
&= \frac{1}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} \left(f(\tau) - f(t)\right) d\tau \qquad (2.91) \\
&\quad + \frac{1}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} \left(f(\tau) - f(t)\right) d\tau \qquad (2.92) \\
&\quad + \frac{f(t)(t-a)^p}{\Gamma(p+1)}. \qquad (2.93)
\end{aligned}$$

Let us consider the integral (2.92). Since $f(t)$ is continuous, for every $\delta > 0$ there exists $\epsilon > 0$ such that

$$|f(\tau) - f(t)| < \epsilon.$$

Then we have the following estimate of the integral (2.92):

$$|I_2| < \frac{\epsilon}{\Gamma(p)} \int_{t-\delta}^{t} (t-\tau)^{p-1} d\tau < \frac{\epsilon \delta^p}{\Gamma(p+1)}, \qquad (2.94)$$

and taking into account that $\epsilon \to 0$ as $\delta \to 0$ we obtain that for all $p \geq 0$

$$\lim_{\delta \to 0} |I_2| = 0. \qquad (2.95)$$

Let us now take an arbitrary $\epsilon > 0$ and choose δ such that

$$|I_2| < \epsilon \qquad (2.96)$$

for all $p \geq 0$. For this fixed δ we obtain the following estimate of the integral (2.91):

$$|I_1| \leq \frac{M}{\Gamma(p)} \int_a^{t-\delta} (t-\tau)^{p-1} d\tau \leq \frac{M}{\Gamma(p+1)} \Big(\delta^p - (t-a)^p\Big), \qquad (2.97)$$

from which it follows that for fixed $\delta > 0$

$$\lim_{p \to 0} |I_1| = 0. \tag{2.98}$$

Considering

$$\left| {}_a\mathbf{D}_t^{-p} f(t) - f(t) \right| \leq |I_1| + |I_2| + |f(t)| \cdot \left| \frac{(t-a)^p}{\Gamma(p+1)} - 1 \right|$$

and taking into account the limits (2.95) and (2.95) and the estimate (2.96) we obtain

$$\limsup_{p \to 0} \left| {}_a\mathbf{D}_t^{-p} f(t) - f(t) \right| \leq \epsilon,$$

where ϵ can be chosen as small as we wish. Therefore,

$$\limsup_{p \to 0} \left| {}_a\mathbf{D}_t^{-p} f(t) - f(t) \right| = 0,$$

and (2.89) holds if $f(t)$ is continuous for $t \geq a$.

If $f(t)$ is continuous for $t \geq a$, then integration of arbitrary real order defined by (2.88) has the following important property:

$${}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{-q} f(t) \Big) = {}_a\mathbf{D}_t^{-p-q} f(t). \tag{2.99}$$

Indeed, we have

$$\begin{aligned}
{}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{-q} f(t) \Big) &= \frac{1}{\Gamma(q)} \int_a^t (t-\tau)^{q-1} \, {}_a\mathbf{D}_\tau^{-p} f(\tau) d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t (t-\tau)^{q-1} d\tau \int_a^\tau (\tau-\xi)^{p-1} f(\xi) d\xi \\
&= \frac{1}{\Gamma(p)\Gamma(q)} \int_a^t f(\xi) d\xi \int_\xi^t (t-\tau)^{q-1} (\tau-\xi)^{p-1} d\tau \\
&= \frac{1}{\Gamma(p+q)} \int_a^t (t-\xi)^{p+q-1} f(\xi) d\xi \\
&= {}_a\mathbf{D}_t^{-p-q} f(t).
\end{aligned}$$

(For the evaluation of the integral from ξ to t we used the substitution $\tau = \xi + \zeta(t - \xi)$ allowing us to express it in terms of the beta function (1.20).)

Obviously, we can interchange p and q, so we have

$$ {}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{-q} f(t)\Big) = {}_a\mathbf{D}_t^{-q}\Big({}_a\mathbf{D}_t^{-p} f(t)\Big) = {}_a\mathbf{D}_t^{-p-q} f(t). \qquad (2.100) $$

One may note that the rule (2.100) is similar to the well-known property of integer-order derivatives:

$$ \frac{d^m}{dt^m}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{d^n}{dt^n}\left(\frac{d^m f(t)}{dt^m}\right) = \frac{d^{m+n} f(t)}{dt^{m+n}}. \qquad (2.101) $$

2.3.3 Derivatives of Arbitrary Order

The representation (2.87) for the derivative of an integer order $k - n$ provides an opportunity for extending the notion of differentiation to non-integer order. Namely, we can leave integer k and replace integer n with a real α so that $k - \alpha > 0$. This gives

$$ {}_a\mathbf{D}_t^{k-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (0 < \alpha \le 1), \qquad (2.102) $$

where the only substantial restriction for α is $\alpha > 0$, which is necessary for the convergence of the integral in (2.102). This restriction, however, can be — without loss of generality — replaced with the narrower condition $0 < \alpha \le 1$; this can be easily shown with the help of the property (2.100) of the integrals of arbitrary real order and the definition (2.102).

Denoting $p = k - \alpha$ we can write (2.102) as

$$ {}_a\mathbf{D}_t^{p} f(t) = \frac{1}{\Gamma(k-p)} \frac{d^k}{dt^k} \int_a^t (t-\tau)^{k-p-1} f(\tau) d\tau, \quad (k-1 \le p < k) \qquad (2.103) $$

or

$$ {}_a\mathbf{D}_t^{p} f(t) = \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-(k-p)} f(t)\Big), \quad (k-1 \le p < k). \qquad (2.104) $$

If $p = k - 1$, then we obtain a conventional integer-order derivative of order $k - 1$:

$$ \begin{aligned} {}_a\mathbf{D}_t^{k-1} f(t) &= \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-(k-(k-1))} f(t)\Big) \\ &= \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^{-1} f(t)\Big) = f^{(k-1)}(t). \end{aligned} $$

Moreover, using (2.90) we see that for $p = k \geq 1$ and $t > a$

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d^k}{dt^k}\Big({}_a\mathbf{D}_t^0 f(t)\Big) = \frac{d^k f(t)}{dt^k} = f^{(k)}(t), \tag{2.105}$$

which means that for $t > a$ the Riemann–Liouville fractional derivative (2.103) of order $p = k > 1$ coincides with the conventional derivative of order k.

Let us now consider some properties of the Riemann–Liouville fractional derivatives.

The first — and maybe the most important — property of the Riemann–Liouville fractional derivative is that for $p > 0$ and $t > a$

$$ {}_a\mathbf{D}_t^p\Big({}_a\mathbf{D}_t^{-p} f(t)\Big) = f(t), \tag{2.106}$$

which means that the Riemann–Liouville fractional differentiation operator is a left inverse to the Riemann–Liouville fractional integration operator of the same order p.

To prove the property (2.106), let us consider the case of integer $p = n \geq 1$:

$$\begin{aligned} {}_a\mathbf{D}_t^n\Big({}_a\mathbf{D}_t^{-n} f(t)\Big) &= \frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau \\ &= \frac{d}{dt}\int_a^t f(\tau)d\tau = f(t). \end{aligned}$$

Taking now $k - 1 \leq p < k$ and using the composition rule (2.100) for the Riemann–Liouville fractional integrals, we can write

$$ {}_a\mathbf{D}_t^{-k} f(t) = {}_a\mathbf{D}_t^{-(k-p)}\Big({}_a\mathbf{D}_t^{-p} f(t)\Big), \tag{2.107}$$

and, therefore,

$$\begin{aligned} {}_a\mathbf{D}_t^p\Big({}_a\mathbf{D}_t^{-p} f(t)\Big) &= \frac{d^k}{dt^k}\Big\{ {}_a\mathbf{D}_t^{-(k-p)}\Big({}_a\mathbf{D}_t^{-p} f(t)\Big)\Big\} \\ &= \frac{d^k}{dt^k}\Big\{ {}_a\mathbf{D}_t^{-p} f(t)\Big\} = f(t), \end{aligned}$$

which ends the proof of the property (2.106).

As with conventional integer-order differentiation and integration, fractional differentiation and integration do not commute.

If the fractional derivative ${}_a\mathbf{D}_t^p f(t)$, $(k-1 \leq p < k)$, of a function $f(t)$ is integrable, then

$$ {}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^p f(t)\Big) = f(t) - \sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{p-j} f(t)\Big]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(p-j+1)}. \quad (2.108) $$

Indeed, on the one hand we have

$$ \begin{aligned} {}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^p f(t)\Big) &= \frac{1}{\Gamma(p)}\int_a^t (t-\tau)^{p-1}\, {}_a\mathbf{D}_\tau^p f(\tau)d\tau \\ &= \frac{d}{dt}\left\{\frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^{p}\, {}_a\mathbf{D}_\tau^p f(\tau)d\tau\right\}. \quad (2.109) \end{aligned} $$

On the other hand, repeatedly integrating by parts and then using (2.100) we obtain

$$ \begin{aligned} &\frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^{p}\, {}_a\mathbf{D}_\tau^p f(\tau)d\tau \\ &= \frac{1}{\Gamma(p+1)}\int_a^t (t-\tau)^{p}\frac{d^k}{d\tau^k}\left\{{}_a\mathbf{D}_\tau^{-(k-p)} f(\tau)\right\}d\tau \\ &= \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}\left\{{}_a\mathbf{D}_\tau^{-(k-p)} f(\tau)\right\}d\tau \\ &\qquad -\sum_{j=1}^{k}\left[\frac{d^{k-j}}{dt^{k-j}}\Big({}_a\mathbf{D}_t^{-(k-p)} f(t)\Big)\right]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \\ &= \frac{1}{\Gamma(p-k+1)}\int_a^t (t-\tau)^{p-k}\left\{{}_a\mathbf{D}_\tau^{-(k-p)} f(\tau)\right\}d\tau \\ &\qquad -\sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{p-j} f(t)\Big]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \quad (2.110) \\ &= {}_a\mathbf{D}_t^{-(p-k+1)}\Big({}_a\mathbf{D}_t^{-(k-p)} f(t)\Big) \\ &\qquad -\sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{p-j} f(t)\Big]_{t=a}\frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)} \quad (2.111) \end{aligned} $$

$$= {}_a\mathbf{D}_t^{-1} f(t) - \sum_{j=1}^{k} \Big[{}_a\mathbf{D}_t^{p-j} f(t) \Big]_{t=a} \frac{(t-a)^{p-j+1}}{\Gamma(2+p-j)}. \tag{2.112}$$

The existence of all terms in (2.110) follows from the integrability of ${}_a\mathbf{D}_t^p f(t)$, because due to this condition the fractional derivatives ${}_a\mathbf{D}_t^{p-j} f(t)$, $(j = 1, 2, \ldots, k)$, are all bounded at $t = a$.

Combining (2.109) and (2.112) ends the proof of the relationship (2.108).

An important particular case must be mentioned. If $0 < p < 1$, then

$${}_a\mathbf{D}_t^{-p} \Big({}_a\mathbf{D}_t^p f(t) \Big) = f(t) - \Big[{}_a\mathbf{D}_t^{p-1} f(t) \Big]_{t=a} \frac{(t-a)^{p-1}}{\Gamma(p)}. \tag{2.113}$$

The property (2.106) is a particular case of a more general property

$${}_a\mathbf{D}_t^p \Big({}_a\mathbf{D}_t^{-q} f(t) \Big) = {}_a\mathbf{D}_t^{p-q} f(t), \tag{2.114}$$

where we assume that $f(t)$ is continuous and, if $p \geq q \geq 0$, that the derivative ${}_a\mathbf{D}_t^{p-q} f(t)$ exists.

Two cases must be considered: $q \geq p \geq 0$ and $p > q \geq 0$.

If $q \geq p \geq 0$, then using the properties (2.100) and (2.106) we obtain

$$\begin{aligned} {}_a\mathbf{D}_t^p \Big({}_a\mathbf{D}_t^{-q} f(t) \Big) &= {}_a\mathbf{D}_t^p \Big({}_a\mathbf{D}_t^{-p} \, {}_a\mathbf{D}_t^{-(q-p)} \Big) \\ &= {}_a\mathbf{D}_t^{-(q-p)} = {}_a\mathbf{D}_t^{p-q} f(t). \end{aligned}$$

Now let us consider the case $p > q \geq 0$. Let us denote by m and n integers such that $0 \leq m-1 \leq p < m$ and $0 \leq n \leq p-q < n$. Obviously, $n \leq m$. Then, using the definition (2.103) and the property (2.100) we obtain

$$\begin{aligned} {}_a\mathbf{D}_t^p \Big({}_a\mathbf{D}_t^{-q} f(t) \Big) &= \frac{d^m}{dt^m} \Big\{ {}_a\mathbf{D}_t^{-(m-p)} \Big({}_a\mathbf{D}_t^{-q} f(t) \Big) \Big\} \\ &= \frac{d^m}{dt^m} \Big\{ {}_a\mathbf{D}_t^{p-q-m} f(t) \Big\} \\ &= \frac{d^n}{dt^n} \Big\{ {}_a\mathbf{D}_t^{p-q-n} f(t) \Big\} = {}_a\mathbf{D}_t^{p-q} f(t). \end{aligned}$$

The above mentioned property (2.108) is a particular case of the more general property

$${}_a\mathbf{D}_t^{-p} \Big({}_a\mathbf{D}_t^q f(t) \Big) = {}_a\mathbf{D}_t^{q-p} f(t) - \sum_{j=1}^{k} \Big[{}_a\mathbf{D}_t^{q-j} f(t) \Big]_{t=a} \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}, \tag{2.115}$$

$$(0 \le k - 1 \le q < k).$$

To prove the formula (2.115) we first use property (2.100) (if $q \le p$) or property (2.114) (if $q \ge p$) and then property (2.108). This gives:

$$\begin{aligned}
{}_a\mathbf{D}_t^{-p}\Big({}_a\mathbf{D}_t^{q} f(t)\Big) &= {}_a\mathbf{D}_t^{q-p}\Big\{{}_a\mathbf{D}_t^{-q}\Big({}_a\mathbf{D}_t^{q} f(t)\Big)\Big\} \\
&= {}_a\mathbf{D}_t^{q-p}\left\{ f(t) - \sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{q-j} f(t)\Big]_{t=a}\frac{(t-a)^{q-j}}{\Gamma(p-j+1)}\right\} \\
&= {}_a\mathbf{D}_t^{q-p} f(t) - \sum_{j=1}^{k}\Big[{}_a\mathbf{D}_t^{q-j} f(t)\Big]_{t=a}\frac{(t-a)^{p-j}}{\Gamma(1+p-j)},
\end{aligned}$$

where we used the known derivative of the power function (2.117):

$${}_a\mathbf{D}_t^{q-p}\left\{\frac{(t-a)^{q-j}}{\Gamma(1+q-j)}\right\} = \frac{(t-a)^{p-j}}{\Gamma(1+p-j)}.$$

2.3.4 Fractional Derivative of $(t-a)^{\beta}$

Let us now evaluate the Riemann–Liouville fractional derivative ${}_a\mathbf{D}_t^{p} f(t)$ of the power function

$$f(t) = (t-a)^{\nu},$$

where ν is a real number.

For this purpose let us assume that $n-1 \le p < n$ and recall that by the definition of the Riemann–Liouville derivative

$${}_a\mathbf{D}_t^{p} f(t) = \frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{-(n-p)} f(t)\Big), \qquad (n-1 \le p < n). \tag{2.116}$$

Substituting into the formula (2.116) the fractional integral of order $\alpha = n - p$ of this function, which we have evaluated earlier (see formula (2.56), p. 56), i.e.

$${}_a\mathbf{D}_t^{-\alpha}\Big((t-a)^{\nu}\Big) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+\alpha)}(t-a)^{\nu+\alpha},$$

we obtain:

$${}_a\mathbf{D}_t^{p}\Big((t-a)^{\nu}\Big) = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)}(t-a)^{\nu-p}, \tag{2.117}$$

and the only restriction for $f(t) = (t-a)^{\nu}$ is its integrability, namely $\nu > -1$.

2.3.5 Composition with Integer-order Derivatives

In many applied problems the composition of the Riemann–Liouville fractional derivative with integer-order derivatives appears.

Let us consider the n-th derivative of the Riemann–Liouville fractional derivative of real order p.

Using the definition (2.102) of the Riemann–Liouville derivative we obtain:

$$\frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{k-\alpha} f(t)\Big) = \frac{1}{\Gamma(\alpha)} \frac{d^{n+k}}{dt^{n+k}} \int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau = {}_a\mathbf{D}_t^{n+k-\alpha} f(t), \tag{2.118}$$

$$(0 < \alpha \leq 1),$$

and denoting $p = k - \alpha$ we have

$$\frac{d^n}{dt^n}\Big({}_a\mathbf{D}_t^{p} f(t)\Big) = {}_a\mathbf{D}_t^{n+p} f(t). \tag{2.119}$$

To consider the reversed order of operations, we must take into account that

$$\begin{aligned} {}_a\mathbf{D}_t^{-n} f^{(n)}(t) &= \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau \\ &= f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)} \end{aligned} \tag{2.120}$$

and that

$${}_a\mathbf{D}_t^{p} g(t) = {}_a\mathbf{D}_t^{p+n}\Big({}_a\mathbf{D}_t^{-n} g(t)\Big). \tag{2.121}$$

Using (2.120), (2.121) and (2.117) we obtain:

$$\begin{aligned} {}_a\mathbf{D}_t^{p}\Big(\frac{d^n f(t)}{dt^n}\Big) &= {}_a\mathbf{D}_t^{p+n}\Big({}_a\mathbf{D}_t^{-n} f^{(n)}(t)\Big) \\ &= {}_a\mathbf{D}_t^{p+n}\Big(f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^j}{\Gamma(j+1)}\Big) \\ &= {}_a\mathbf{D}_t^{p+n} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)(t-a)^{j-p-n}}{\Gamma(1+j-p-n)}, \end{aligned} \tag{2.122}$$

which is the same as the relationship (2.66).

Therefore, as in the case of the Grünwald–Letnikov derivatives, we see that the Riemann–Liouville fractional derivative operator ${}_a\mathbf{D}_t^p$ commutes with $\frac{d^n}{dt^n}$, i.e., that

$$\frac{d^n}{dt^n}\left({}_a\mathbf{D}_t^p f(t)\right) = {}_a\mathbf{D}_t^p\left(\frac{d^n f(t)}{dt^n}\right) = {}_a\mathbf{D}_t^{p+n} f(t), \tag{2.123}$$

only if at the lower terminal $t = a$ of the fractional differentiation the function $f(t)$ satisfies the conditions

$$f^{(k)}(a) = 0, \qquad (k = 0, 1, 2, \ldots, n-1). \tag{2.124}$$

2.3.6 Composition with Fractional Derivatives

Let us now turn our attention to the composition of two fractional Riemann–Liouville derivative operators: ${}_a\mathbf{D}_t^p$, $(m-1 \le p < m)$, and ${}_a\mathbf{D}_t^q$, $(n-1 \le q < n)$.

Using subsequently the definition of the Riemann–Liouville fractional derivative (2.104), the formula (2.108) and the composition with integer-order derivatives (2.119) we obtain:

$$\begin{aligned}
{}_a\mathbf{D}_t^p\Big({}_a\mathbf{D}_t^q f(t)\Big) &= \frac{d^m}{dt^m}\Big\{{}_a\mathbf{D}_t^{-(m-p)}\Big({}_a\mathbf{D}_t^q f(t)\Big)\Big\} \\
&= \frac{d^m}{dt^m}\Big\{{}_a\mathbf{D}_t^{p+q-m} f(t) \\
&\qquad - \sum_{j=1}^{n}\Big[{}_a\mathbf{D}_t^{q-j} f(t)\Big]_{t=a}\frac{(t-a)^{m-p-j}}{\Gamma(1+m-p-j)}\Big\} \\
&= {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{n}\Big[{}_a\mathbf{D}_t^{q-j} f(t)\Big]_{t=a}\frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}.
\end{aligned} \tag{2.125}$$

Interchanging p and q (and therefore m and n), we can write:

$${}_a\mathbf{D}_t^q\Big({}_a\mathbf{D}_t^p f(t)\Big) = {}_a\mathbf{D}_t^{p+q} f(t) - \sum_{j=1}^{m}\Big[{}_a\mathbf{D}_t^{p-j} f(t)\Big]_{t=a}\frac{(t-a)^{-q-j}}{\Gamma(1-q-j)}. \tag{2.126}$$

The comparison of the relationships (2.125) and (2.126) says that in the general case the Riemann–Liouville fractional derivative operators ${}_a\mathbf{D}_t^p$ and ${}_a\mathbf{D}_t^p$ do not commute, with only one exception (besides the trivial case $p = q$): namely, for $p \neq q$ we have

$${}_a\mathbf{D}_t^p\Big({}_a\mathbf{D}_t^q f(t)\Big) = {}_a\mathbf{D}_t^q\Big({}_a\mathbf{D}_t^p f(t)\Big) = {}_a\mathbf{D}_t^{p+q} f(t), \tag{2.127}$$

only if both sums in the right-hand sides of (2.125) and (2.126) vanish. For this we have to require the simultaneous fulfillment of the conditions

$$\left[{}_a\mathbf{D}_t^{p-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, m), \tag{2.128}$$

and the conditions

$$\left[{}_a\mathbf{D}_t^{q-j} f(t)\right]_{t=a} = 0, \quad (j = 1, 2, \ldots, n). \tag{2.129}$$

As will be shown below in Section 2.3.7, if $f(t)$ has a sufficient number of continous derivatives, then the conditions (2.128) are equivalent to

$$f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, m-1) \tag{2.130}$$

and the conditions (2.129) are equivalent to

$$f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, n-1), \tag{2.131}$$

and the relationship (2.127) holds (i.e. the p-th and q-th derivatives commute) if

$$f^{(j)}(a) = 0, \quad (j = 0, 1, 2, \ldots, r-1), \tag{2.132}$$

where $r = \max(n, m)$.

2.3.7 Link to the Grünwald–Letnikov Approach

As we mentioned above, see p. 63, there exists a link between the Riemann–Liouville and the Grünwald–Letnikov approaches to differentiation of arbitrary real order. The exact conditions of the equivalence of these two approaches are the following.

Let us suppose that the function $f(t)$ is $(n-1)$-times continuously differentiable in the interval $[a, T]$ and that $f^{(n)}(t)$ is integrable in $[a, T]$. Then for every p $(0 < p < n)$ the Riemann–Liouville derivative ${}_a\mathbf{D}_t^p f(t)$ exists and coincides with the Grünwald–Letnikov derivative ${}_aD_t^p f(t)$, and if $0 \leq m-1 \leq p < m \leq n$, then for $a < t < T$ the following holds:

$${}_a\mathbf{D}_t^p f(t) = {}_aD_t^p f(t) = \sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^{j-p}}{\Gamma(1+j-p)} + \frac{1}{\Gamma(m-p)} \int_a^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{p-m+1}}. \tag{2.133}$$

Indeed, on the one hand the right-hand side of formula (2.133) is equal to the Grünwald–Letnikov derivative ${}_aD_t^p f(t)$. On the other hand, it can be written as

$$\frac{d^m}{dt^m}\left\{\sum_{j=0}^{m-1} \frac{f^{(j)}(a)(t-a)^{m+j-p}}{\Gamma(1+m+j-p)} + \frac{1}{\Gamma(2m-p)} \int_a^t (t-\tau)^{2m-p-1} f^{(m)}(\tau)d\tau\right\},$$

which after m integrations by parts takes the form of the Riemann–Liouville derivative ${}_a\mathbf{D}_t^p f(t)$

$$\begin{aligned}\frac{d^m}{dt^m}\left\{\frac{1}{\Gamma(m-p)}\int_a^t (t-\tau)^{m-p-1} f(\tau)d\tau\right\} &= \frac{d^m}{dt^m}\left\{{}_a\mathbf{D}_t^{-(m-p)} f(t)\right\} \\ &= {}_a\mathbf{D}_t^p f(t).\end{aligned}$$

The following particular case of the relationship (2.133) is important from the viewpoint of numerous applied problems.

If $f(t)$ is continuous and $f'(t)$ is integrable in the interval $[a, T]$, then for every p $(0 < p < 1)$ both Riemann–Liouville and Grünwald–Letnikov derivatives exist and can be written in the form

$$ {}_a\mathbf{D}_t^p f(t) = {}_a D_t^p f(t) = \frac{f(a)(t-a)^{-p}}{\Gamma(1-p)} + \frac{1}{\Gamma(1-p)}\int_a^t (t-\tau)^{-p} f'(\tau)d\tau. \tag{2.134}$$

Obviously, the derivative given by the expression (2.134) is integrable.

Another important property following from (2.133) is that the existence of the derivative of order $p > 0$ implies the existence of the derivative of order q for all q such that $0 < q < p$.

More precisely, if for a given continuous function $f(t)$ having integrable derivative the Riemann–Liouville (Grünwald–Letnikov) derivative ${}_a\mathbf{D}_t^p f(t)$ exists and is integrable, then for every q such that $(0 < q < p)$ the derivative ${}_a\mathbf{D}_t^q f(t)$ also exists and is integrable.

Indeed, if we denote $g(t) = {}_a\mathbf{D}_t^{-(1-p)} f(t)$, then we can write

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d}{dt}\left({}_a\mathbf{D}_t^{-(1-p)} f(t)\right) = g'(t).$$

Noting that $g'(t)$ is integrable and taking into account the formula (2.134) and the inequality $0 < 1 + q - p < 1$ we conclude that the derivative ${}_a\mathbf{D}_t^{1+q-p} g(t)$ exists and is integrable. Then, using the property (2.114), we obtain:

$$ {}_a\mathbf{D}_t^{1+q-p} g(t) = {}_a\mathbf{D}_t^{1+q-p}\left({}_a\mathbf{D}_t^{-(1-p)} f(t)\right) = {}_a\mathbf{D}_t^q f(t).$$

The relationship (2.133) between the Grünwald–Letnikov and the Riemann–Liouville definitions also has another consequence which is very important for the formulation of applied problems, manipulation with fractional derivatives and the formulation of physically meaningful initial-value problems for fractional-order differential equations.

Under the same assumptions on the function $f(t)$ ($f(t)$ is $(m-1)$-times continuously differentiable and its m-th derivative is integrable in $[a,T]$) and on p ($m-1 \le p < m$) the condition

$$\Big[{}_a\mathbf{D}_t^p f(t) \Big]_{t=a} = 0 \tag{2.135}$$

is equivalent to the conditions

$$f^{(j)}(a) = 0, \quad (j = 0,\, 1,\, 2,\, \ldots,\, m-1). \tag{2.136}$$

Indeed, if the conditions (2.136) are fulfilled, then putting $t \to a$ in (2.133) we immediately obtain (2.135).

On the other hand, if the condition (2.135) is fulfilled, then multiplying both sides of (2.133) subsequently by $(t-a)^{p-j}$, ($j = m-1$, $m-2$, $m-3$, ..., 2, 1, 0) and taking the limits as $t \to a$ we obtain $f^{(m-1)}(a) = 0$, $f^{(m-2)}(a) = 0$, ..., $f''(a) = 0$, $f'(a) = 0$, $f(a) = 0$ — i.e., the conditions (2.136).

Therefore, (2.135) holds if and only if (2.136) holds.

From the equivalence of the conditions (2.135) and (2.136) it immediately follows that if for some $p > 0$ the p-th derivative of $f(t)$ is equal to zero at the terminal $t = a$, then all derivatives of order q ($0 < q < p$) are also equal to zero at $t = a$:

$$\Big[{}_a\mathbf{D}_t^q f(t) \Big]_{t=a} = 0.$$

2.4 Some Other Approaches

Among other approaches to the generalization of the notion of differentiation and integration we decided to pay attention to the approach suggested by M. Caputo and to the approach based on generalized functions (distributions), because of its possible usefulness for the formulation and solution of applied problems and their transparency.

The approach developed by M. Caputo allows the formulation of initial conditions for initial-value problems for fractional-order differential equations in a form involving only the limit values of integer-order derivatives at the lower terminal (initial time) $t = a$, such as $y'(a)$, $y''(a)$ etc.

The generalized functions approach allows consideration and utilization of the Dirac delta function $\delta(t)$ and the Heaviside (unit-step) function $H(t)$; both functions are frequently used as models (or parts of models) for test signals and loading.

2.4.1 Caputo's Fractional Derivative

The definition (2.103) of the fractional differentiation of the Riemann–Liouville type played an important role in the development of the theory of fractional derivatives and integrals and for its applications in pure mathematics (solution of integer-order differential equations, definitions of new function classes, summation of series, etc.).

However, the demands of modern technology require a certain revision of the well-established pure mathematical approach. There have appeared a number of works, especially in the theory of viscoelasticity and in hereditary solid mechanics, where fractional derivatives are used for a better description of material properties. Mathematical modelling based on enhanced rheological models naturally leads to differential equations of fractional order — and to the necessity of the formulation of initial conditions to such equations.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $f(a)$, $f'(a)$, etc.

Unfortunately, the Riemann–Liouville approach leads to initial conditions containing the limit values of the Riemann–Liouville fractional derivatives at the lower terminal $t = a$, for example

$$\begin{aligned} &\lim_{t\to a} {}_a\mathbf{D}_t^{\alpha-1} f(t) = b_1, \\ &\lim_{t\to a} {}_a\mathbf{D}_t^{\alpha-2} f(t) = b_2, \\ &\qquad\qquad \dots, \\ &\lim_{t\to a} {}_a\mathbf{D}_t^{\alpha-n} f(t) = b_n, \end{aligned} \tag{2.137}$$

where b_k, $k = 1, 2, \dots, n$ are given constants.

In spite of the fact that initial value problems with such initial conditions can be successfully solved mathematically (see, for example, solutions given in [232] and in this book), their solutions are practically useless, because there is no known physical interpretation for such types of initial conditions.

Here we observe a conflict between the well-established and polished mathematical theory and practical needs.

A certain solution to this conflict was proposed by M. Caputo first in his paper [23] and two years later in his book [24], and recently (in Banach spaces) by El-Sayed [55, 56]). Caputo's definition can be written as

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \quad (n-1<\alpha<n). \qquad (2.138)$$

Under natural conditions on the function $f(t)$, for $\alpha \to n$ the Caputo derivative becomes a conventional n-th derivative of the function $f(t)$. Indeed, let us assume that $0 \le n-1 < \alpha < n$ and that the function $f(t)$ has $n+1$ continuous bounded derivatives in $[a,T]$ for every $T > a$. Then

$$\begin{aligned} \lim_{\alpha\to n} {}_a^C D_t^\alpha f(t) &= \lim_{\alpha\to n} \Bigg(\frac{f^{(n)}(a)(t-a)^{n-\alpha}}{\Gamma(n-\alpha+1)} \\ &\qquad + \frac{1}{\Gamma(n-\alpha+1)} \int_a^t (t-\tau)^{n-\alpha} f^{(n+1)}(\tau)d\tau \Bigg) \\ &= f^n(a) + \int_a^t f^{(n+1)}(\tau)d\tau = f^{(n)}(t), \quad n = 1, 2, \ldots \end{aligned}$$

This says that, similarly to the Grünwald–Letnikov and the Riemann–Liouville approaches, the Caputo approach also provides an interpolation between integer-order derivatives.

The main advantage of Caputo's approach is that the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential equations, i.e. contain the limit values of integer-order derivatives of unknown functions at the lower terminal $t = a$.

To underline the difference in the form of the initial conditions which must accompany fractional differential equations in terms of the Riemann–Liouville and the Caputo derivatives, let us recall the corresponding Laplace transform formulas for the case $a = 0$.

The formula for the Laplace transform of the Riemann–Liouville fractional derivative is

$$\int_0^\infty e^{-pt} \{ {}_0\mathbf{D}_t^\alpha f(t)\} \; dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \; {}_0\mathbf{D}_t^{\alpha-k-1} f(t)\Big|_{t=0}, \quad (2.139)$$

$$(n-1 \le \alpha < n),$$

whereas Caputo's formula, first obtained in [23], for the Laplace transform of the Caputo derivative is (see Section 2.8.3)

$$\int_0^\infty e^{-pt} \left\{ {}_0^C D_t^\alpha f(t) \right\} \, dt = p^\alpha F(p) - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0), \qquad (2.140)$$

$$(n-1 < \alpha \leq n).$$

We see that the Laplace transform of the Riemann–Liouville fractional derivative allows utilization of initial conditions of the type (2.137), which can cause problems with their physical interpretation. On the contrary, the Laplace transform of the Caputo derivative allows utilization of initial values of classical integer-order derivatives with known physical interpretations.

The Laplace transform method is frequently used for solving applied problems. To choose the appropriate Laplace transform formula, it is very important to understand which type of definition of fractional derivative (in other words, which type of initial conditions) must be used.

Another difference between the Riemann–Liouville definition (2.103) and the Caputo definition (2.138) is that the Caputo derivative of a constant is 0, whereas in the cases of a finite value of the lower terminal a the Riemann–Liouville fractional derivative of a constant C is not equal to 0, but

$${}_0\mathbf{D}_t^\alpha C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)}. \qquad (2.141)$$

This fact led, for example, Ochmann and Makarov [174] to using the Riemann–Liouville definition with $a = -\infty$, because, on the one hand, from the physical point of view they need the fractional derivative of a constant equal to zero and on the other hand formula (2.141) gives 0 if $a \to -\infty$. The physical meaning of this step is that the starting time of the physical process is set to $-\infty$. In such a case transient effects cannot be studied. However, taking $a = -\infty$ is the necessary abstraction for the consideration of the steady-state processes, for example for studying the response of the fractional-order dynamic system to the periodic input signal, wave propagation in viscoelastic materials, etc.

Putting $a = -\infty$ in both definitions and requiring reasonable behaviour of $f(t)$ and its derivatives for $t \to -\infty$, we arrive at the same formula

$${}_{-\infty}\mathbf{D}_t^\alpha f(t) = {}_{-\infty}^{\ C} D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (2.142)$$

$$(n-1<\alpha<n),$$

which shows that for the study of steady-state dynamical processes the Riemann–Liouville definition and the Caputo definition must give the same results.

There is also another difference between the Riemann–Liouville and the Caputo approaches, which we would like to mention here and which seems to be important for applications. Namely, for the Caputo derivative we have

$$ {}_a^C D_t^\alpha \left({}_a^C D_t^m f(t) \right) = {}_a^C D_t^{\alpha+m} f(t), \qquad (m=0,1,2,\ldots; \quad n-1<\alpha<n) \tag{2.143}$$

while for the Riemann–Liouville derivative

$$ {}_a\mathbf{D}_t^m \left({}_a\mathbf{D}_t^\alpha f(t) \right) = {}_a\mathbf{D}_t^{\alpha+m} f(t), \qquad (m=0,1,2,\ldots; \quad n-1<\alpha<n) \tag{2.144}$$

The interchange of the differentiation operators in formulas (2.143) and (2.144) is allowed under different conditions:

$$ {}_a^C D_t^\alpha \left({}_a^C D_t^m f(t) \right) = {}_a^C D_t^m \left({}_a^C D_t^\alpha f(t) \right) = {}_a^C D_t^{\alpha+m} f(t), \tag{2.145}$$

$$f^{(s)}(0)=0, \quad s=n,n+1,\ldots,m$$

$$(m=0,1,2,\ldots;n-1<\alpha<n)$$

$$ {}_a\mathbf{D}_t^m \left({}_a\mathbf{D}_t^\alpha f(t) \right) = {}_a\mathbf{D}_t^\alpha \left({}_a\mathbf{D}_t^m f(t) \right) = {}_a\mathbf{D}_t^{\alpha+m} f(t), \tag{2.146}$$

$$f^{(s)}(0)=0, \quad s=0,1,2,\ldots,m$$

$$(m=0,1,2,\ldots;n-1<\alpha<n).$$

We see that contrary to the Riemann–Liouville approach, in the case of the Caputo derivative there are no restrictions on the values $f^{(s)}(0)$, $(s=0,1,\ldots,n-1)$.

2.4.2 Generalized Functions Approach

This approach is based on the observation that the Cauchy formula (2.85), see page 64,

$$f^{(-n)}(t)=\frac{1}{\Gamma(n)}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau,$$

which allows replacement of the n-fold integral of the function $f(t)$ with a single integration, can be written as a convolution of the function $f(t)$ and the power function t^{n-1}:

$$f^{(-n)}(t) = f(t) * \frac{t^{n-1}}{\Gamma(n)}, \tag{2.147}$$

where both functions, $f(t)$ and t^{n-1}, are replaced with zero for $t < a$ and $t < 0$ correspondingly; the asterisk means the convolution:

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau.$$

Let us consider the function $\Phi_p(t)$ defined by [76]

$$\Phi_p(t) = \begin{cases} \dfrac{t^{p-1}}{\Gamma(\gamma)}, & t > 0 \\ 0, & t \leq 0. \end{cases} \tag{2.148}$$

Using the function $\Phi_p(t)$ the formula (2.147) can be considered as a particular case of the more general convolution of the function $f(t)$ and the function $\Phi_p(t)$:

$$f^{(-p)}(t) = f(t) * \Phi_p(t). \tag{2.149}$$

To handle both positive and negative values of p in the same way, it is convenient to consider the function $\Phi_p(t)$ as a generalized function. Its properties are known [76]; for our purposes it is essential that

$$\lim_{p \to -k} \Phi_p(t) = \Phi_{-k}(t) \delta^{(k)}(t), \qquad (k = 0, 1, 2, \ldots), \tag{2.150}$$

where $\delta(t)$ is the Dirac delta function [76]. The Dirac delta function is often used in applied problems for the description of impulse loading (impulse forces). The convolution of the k-th derivative of the delta function and $f(t)$ is given by

$$\int_{-\infty}^{\infty} f(\tau) \delta^{(k)}(t-\tau) d\tau = f^{(k)}(t). \tag{2.151}$$

Obviously, if p is a positive integer ($p = n$), then the formula (2.149) reduces to (2.147). On the other hand, it follows from the relationship

(2.150) and the properties of the delta function that for negative integer values of p ($p = -n$, $n > 0$)

$$f^{(0)}(t) = f(t) * \Phi_0(t) = f(t) * \delta(t) = f(t),$$

$$f^{(1)}(t) = f(t) * \Phi_{-1}(t) = f(t) * \delta'(t) = f'(t),$$

$$\dots \qquad \dots \qquad \dots$$

$$f^{(k)}(t) = f(t) * \Phi_{-k}(t) = f(t) * \delta^{(k)}(t) = f^{(k)}(t).$$

Therefore, both integer-order integrals and derivatives of a generalized function $f(t)$ can be obtained as particular cases of the convolution (2.149), which is also meaningful for non-integer values of p. This means that the formula (2.149) provides a unification of n-fold integrals and n-th order derivatives of a generalized function and an extention of these notions to real order p and that we can define the derivative of real order p of a generalized function $f(t)$, which is equal to zero for $t < a$, as

$$_a\tilde{D}_t^p f(t) = f(t) * \Phi_p(t). \tag{2.152}$$

Another property of the function $\Phi_p(t)$, which leads to important consequences, is

$$\Phi_p(t-a) * \Phi_q(t) = \Phi_{p+q}(t-a). \tag{2.153}$$

To prove (2.153), let us first suppose that $p > 0$ and $q > 0$. Then using the substitution $\tau = a + \zeta(t-a)$ and the definition of the beta function (1.20) we obtain

$$\begin{aligned}
\Phi_p(t-a) * \Phi_q(t) &= \int_a^t \frac{(\tau-a)^{p-1}}{\Gamma(p)} \frac{(t-\tau)^{q-1}}{\Gamma(q)} d\tau \\
&= \frac{1}{\Gamma(p)\Gamma(q)} \int_0^t (\tau-a)^{p-1}(t-\tau)^{q-1} d\tau \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p)\Gamma(q)} \int_0^1 \zeta^{p-1}(1-\zeta)^{q-1} d\zeta \\
&= \frac{(t-a)^{p+q-1}}{\Gamma(p+q)},
\end{aligned} \tag{2.154}$$

and analytic continuation with respect to p and q gives (2.153).

It follows from (2.153) that if the function $f(t)$ is zero for $t < a$, then

$$\Big(f(t) * \Phi_p(t)\Big) * \Phi_q(t) = f(t) * \Big(\Phi_p(t) * \Phi_q(t)\Big) = f(t) * \Phi_{p+q}(t), \quad (2.155)$$

from which immediately follows the composition law

$${}_a\tilde{D}_t^p\Big({}_a\tilde{D}_t^q f(t)\Big) = {}_a\tilde{D}_t^q\Big({}_a\tilde{D}_t^p f(t)\Big) = {}_a\tilde{D}_t^{p+q} f(t). \quad (2.156)$$

for all p and q. The simplicity of the composition law (2.156) is, of course, a great advantage of the use of generalized functions.

From formula (2.153) we directly obtain the derivative of real order p of the generalized function

$$\Phi_{q+1}(t) = \frac{t_+^q}{\Gamma(q+1)} = \begin{cases} \frac{t^q}{\Gamma(q+1)}, & (t > 0) \\ 0, & (t \leq 0) \end{cases}$$

in the form

$${}_a\tilde{D}_t^p\Big(\frac{(t-a)^q}{\Gamma(q+1)}\Big) = \frac{(t-a)^{p-q}}{\Gamma(1+q-p)}, \quad (t > a). \quad (2.157)$$

In the particular case $q = 0$ we obtain the fractional derivative of the Heaviside unit-step function $H(t)$:

$${}_a\tilde{D}_t^p H(t-a) = \frac{(t-a)^{-p}}{\Gamma(1-p)}, \quad (t > a), \quad (2.158)$$

and, in general, for all $b < a$

$${}_b\tilde{D}_t^p H(t-a) = \begin{cases} \frac{(t-a)^{-p}}{\Gamma(1-p)} & (t > a) \\ 0, & (b \leq t \leq a). \end{cases} \quad (2.159)$$

Putting $q = -n-1$ $(n \geq 0)$ in (2.157), we obtain the fractional derivative of order p of the n-th derivative of the Dirac delta function:

$${}_a\tilde{D}_t^p \delta^{(n)}(t-a) = \frac{(t-a)^{-n-p-1}}{\Gamma(-n-p)}, \quad (t > a), \quad (2.160)$$

and, in general, for $b < a$ we have

$${}_b\tilde{D}_t^p \delta^{(n)}(t-a) = \begin{cases} \frac{(t-a)^{-n-p-1}}{\Gamma(-n-p)}, & (t > a) \\ 0, & (b \leq t \leq a). \end{cases} \quad (2.161)$$

Finally, if $q - p + 1 = -n$ $(n \geq 0)$ then from (2.157) it follows that

$$ {}_a\tilde{D}_t^p \left(\frac{(t-a)^{p-n-1}}{\Gamma(p-n)} \right) = \delta^{(n)}(t-a), \quad (t > a). \tag{2.162} $$

Relationships (2.158), (2.160) and (2.162) represent an interesting and useful link between the power function, the Heaviside unit-step function and the Dirac delta function.

The generalized function approach allows the establishment of an interesting link between the Riemann–Liouville and the Caputo approaches and their relationship to conventional and generalized integer-order derivatives.

Using the function $\Phi_p(t)$, the Riemann–Liouville definition (2.103) can be written as

$$ {}_a\mathbf{D}_t^p f(t) = \frac{d^n}{dt^n} \Big(f(t) * \Phi_{n-p}(t) \Big), \tag{2.163} $$

the Caputo definition can be written as

$$ {}_a^C D_t^p f(t) = \left(\frac{d^n f(t)}{dt^n} * \Phi_{n-p}(t) \right) \tag{2.164} $$

and the relationship (2.133) takes the form

$$ {}_a\mathbf{D}_t^p f(t) = {}_a^C D_t^p f(t) + \sum_{k=0}^{n-1} \Phi_{k-p+1}(t-a) f^{(k)}(a). \tag{2.165} $$

Taking $p \longrightarrow n$, where n is a positive integer number and using (2.150), we obtain from (2.165) the following relationship:

$$ {}_a^L D_t^n f(t) = {}_a^C D_t^n f(t) + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a). \tag{2.166} $$

Comparing relationship (2.166) with the well known relationship between the classical derivative $f_C^{(n)}(t)$ and the generalized derivative $\tilde{f}^{(n)}(t)$

$$ \tilde{f}^{(n)}(t) = f_C^{(n)}(t) + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a), \tag{2.167} $$

where $\tilde{f}(t) = f(t)$ for $t \geq a$ and $\tilde{f}(t) \equiv 0$ for $t < a$, we conclude that *the Riemann–Liouville definition (2.79) serves as a generalization of the notion of the generalized (in the sense of generalized functions) derivative,* while *the Caputo derivative (2.138) is a generalization of differentiation in the classical sense.*

Similar results can be found in D. Matignon's work [143], where a relationship between the fractional derivative in the sense of distributions and the "smooth fractional derivative" (which coincides with Caputo's derivative) has been given, and in F. Mainardi's paper [135], where the relationship between the Riemann–Liouville and the Caputo definitions of fractional differentiation is also discussed.

2.5 Sequential Fractional Derivatives

The main idea of differentiation and integration of arbitrary order is the generalization of iterated integration and differentiation.

In all these approaches the general aim is the same: to "replace" the integer-valued parameter n of an operation denoted, for example, by the symbols

$$\frac{d^n}{dt^n}$$

with a non-integer parameter p. Other details vary (function classes, methods of "replacement" of n with p, some properties for non-integer values of p), but it is obvious that all efforts are made for the direct intermediate replacement of an integer n with a non-integer p.

However, there is also another way which is less well known but can be of great importance for many applications. This approach is based on the observation that, in fact, n-th order differentiation is simply a series of first-order differentiations:

$$\frac{d^n f(t)}{dt^n} = \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} f(t). \tag{2.168}$$

If there is a suitable method for "replacing" the derivative of first order $\frac{d}{dt}$ with the derivative of non-integer order D^α, where $0 \leq \alpha \leq 1$, then it is possible to consider the following analogue of (2.168):

$$D^{n\alpha} f(t) = \underbrace{D^\alpha D^\alpha \ldots D^\alpha}_{n} f(t). \tag{2.169}$$

K. S. Miller and B. Ross called the generalized differentiation defined by (2.169), where D^α is the Riemann–Liouville fractional derivative, *sequential differentiation* and considered differential equations with sequential fractional derivatives of type (2.169) in their book [153, Chapter VI, section 4].

Other mutations of sequential fractional derivatives can be obtained by interpreting D^α as the Grünwald–Letnikov derivative, the Caputo derivative or any other type of fractional derivative not considered here.

Instead of (2.169) it is possible to replace each first-order derivative in (2.168) by fractional derivatives of orders which are not necessarily equal, and to consider the more general expression:

$$\mathcal{D}^\alpha f(t) = D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(t), \tag{2.170}$$

$$\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

which we will also call the *sequential fractional derivative.* Depending on the problem, the symbol $\mathcal{D}^\alpha$ in (2.170) can mean the Riemann–Liouville, the Grünwald–Letnikov, the Caputo or any other mutation of the operator of generalized differentiation. Moreover, from this point of view, the Riemann–Liouville fractional derivative and the Caputo fractional derivative are also just particular cases of the sequential derivative (2.170).

Indeed, the Riemann–Liouville fractional derivative can be written as

$${}_a\mathbf{D}_t^p f(t) = \underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n}\,{}_aD_t^{-(n-p)} f(t), \qquad (n-1 \le p < n), \tag{2.171}$$

while the Caputo fractional differential operator can be written as

$${}_a^C D_t^p f(t) = {}_a\mathbf{D}_t^{-(n-p)}\underbrace{\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}}_{n} f(t), \qquad (n-1 < p \le n). \tag{2.172}$$

The properties of the Riemann–Liouville derivatives and the Caputo derivatives of the same cumulative order p are different due to the different *sequence* of differential operators $\frac{d}{dt}$ and ${}_a\mathbf{D}_t^{-(n-p)}$.

In the case of the Grünwald–Letnikov approach (p. 59) and the Riemann–Liouville approach (p. 68) we saw that for the fractional integrals it always holds that

$$D^p D^q f(t) = D^q D^p f(t) = D^{p+q} f(t), \qquad (p < 0, \quad q < 0). \qquad (2.173)$$

Because of this, we do not see a reason for considering sequential *integral* operators.

However, in the general case, the property (2.173) does not hold for $p > 0$ and/or $q > 0$ (this explains the difference between the Riemann–Liouville and the Caputo fractional derivatives). Therefore, only consideration of sequential fractional *derivative* operators can be of interest and can give new results.

On the other hand, sequential fractional derivatives can appear in a natural way in the formulation of various applied problems in physics and applied science. Indeed, differential equations modelling processes or objects arise usually as a result of a substitution of one relationship involving derivatives into another one. If the derivatives in both relationships are fractional derivatives, then the resulting expression (equation) will contain — in the general case — sequential fractional derivatives.

It is worth mentioning that the sequential fractional integro-differential operators of the form (2.170), with $\alpha_1 < 0, \alpha_2 > 0, \ldots, \alpha_n > 0$ were first considered and used for various purposes by M. M. Dzhrbashyan and A. B. Nersesyan at least since 1958 [46, 47, 49, 45, 50]. However, in this book we call sequential fractional derivatives also *Miller–Ross fractional derivatives*, because they clearly outlined the difference between the (single) Riemann–Liouville differentiation and sequential fractional differentiation [153, Chapter VI].

2.6 Left and Right Fractional Derivatives

Until now, we considered the fractional derivatives ${}_aD_t^p f(t)$ with fixed lower terminal a and moving upper terminal t. Moreover, we supposed that $a < t$. However, it is also possible to consider fractional derivatives with moving lower terminal t and fixed upper terminal b.

Let us suppose that the function $f(t)$ is defined in the interval $[a, b]$, where a and b can even be infinite.

The fractional derivative with the lower terminal at the left end of the interval $[a, b]$, ${}_aD_t^p f(t)$, is called the *left fractional derivative*. The

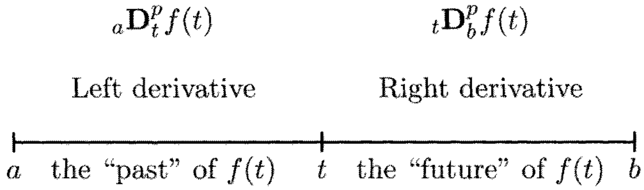

Figure 2.1: *The left and right derivatives as operations on the "past" and the "future" of $f(t)$.*

fractional derivative with the upper terminal at the right end of the interval $[a, b]$ is called the *right fractional derivative.* Obviously, the notions of left and right fractional derivatives can be introduced for any mutation of fractional differentiation — Riemann–Liouville, Grünwald–Letnikov, Caputo and others, which are not considered in this book.

For example, if $k - 1 \leq p < k$, then the left Riemann–Liouville fractional derivative is, as we know, defined by

$$ {}_a\mathbf{D}_t^p f(t) = \frac{1}{\Gamma(k-p)} \left(\frac{d}{dt}\right)^k \int_a^t (t-\tau)^{k-p-1} f(\tau)d\tau. \tag{2.174}$$

The corresponding right Riemann–Liouville derivative is defined by [232, §2.3]

$$ {}_t\mathbf{D}_b^p f(t) = \frac{1}{\Gamma(k-p)} \left(-\frac{d}{dt}\right)^k \int_t^b (\tau-t)^{k-p-1} f(\tau)d\tau. \tag{2.175}$$

The right Caputo and Grünwald–Letnikov derivatives can be defined in a similar manner.

The notions of left and right fractional derivatives can be considered from the physical and the mathematical viewpoints.

Sometimes the following physical interpretation of the left and right derivative can be helpful.

Let us suppose that t is time and the function $f(t)$ describes a certain dynamical process developing in time. If we take $\tau < t$, where t is the present moment, then the state $f(\tau)$ of the process f belongs to the past of this process; if we take $\tau > t$, then $f(\tau)$ belongs to the future of the process f.

From such a point of view, the left derivative (2.174) is an operation performed on the past states of the process f and the right derivative is an operation performed on the future states of the process f.

The physical causality principle means that the present state of the process started at the instant $\tau = a$, i.e. the current value of $f(t)$, depends on all its previous (past) states $f(\tau)$ $(a \leq \tau < t)$. Since we are not aware of the dependence of the present state of any process on the results of its development in the future, only left derivatives are considered in this book. Perhaps once the right derivatives will also get a certain physical interpretation in terms of dynamical processes.

On the other hand, from the viewpoint of mathematics the right derivatives remind us of the operators conjugate to the operators of left differentiation. This means that the complete theory of fractional differential equations, especially the theory of boundary value problems for fractional differential equations, can be developed only with the use of both left and right derivatives.

At present, the above interpretation of fractional derivatives and integrals, related to dynamical processes, seems to be the most transparent and usable. There was an attempt undertaken by R. R. Nigmatullin [165] to derive a relationship between a static fractal structure and fractional integration, but it follows from R. S. Rutman's [231] critics that a suitable practically useful relationship between static fractals and fractional integration or differentiation still has not been established.

2.7 Properties of Fractional Derivatives

Let us turn our attention to the properties of fractional-order integration and differentiation, which are most frequently used in applications.

2.7.1 Linearity

Similarly to integer-order differentiation, fractional differentiation is a linear operation:

$$D^p\Big(\lambda f(t) + \mu g(t)\Big) = \lambda\, D^p f(t) + \mu\, D^p g(t), \qquad (2.176)$$

where D^p denotes any mutation of the fractional differentiation considered in this book.

The linearity of fractional differentiation follows directly from the corresponding definition. For example, for the Grünwald–Letnikov fractional derivatives defined by (2.43) we have:

$$
\begin{aligned}
{}_aD_t^p\Big(\lambda f(t)+\mu g(t)\Big) &= \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p}{r}\Big(\lambda f(t-rh)+\mu g(t-rh)\Big)\\
&= \lambda \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p}{r} f(t-rh)\\
&\qquad +\mu \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p}\sum_{r=0}^{n}(-1)^r\binom{p}{r} g(t-rh)\\
&= \lambda\, {}_aD_t^p f(t) + \mu\, {}_aD_t^p g(t).
\end{aligned}
$$

Similarly, for Riemann–Liouville fractional derivatives of order p ($k-1 \le p < k$) defined by (2.103) we have

$$
\begin{aligned}
{}_a\mathbf{D}_t^p\Big(\lambda f(t)+\mu g(t)\Big) &= \frac{1}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-p-1}\Big(\lambda f(\tau)+\mu g(\tau)\Big)d\tau\\
&= \frac{\lambda}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-p-1} f(\tau)d\tau\\
&\qquad + \frac{\mu}{\Gamma(k-p)}\frac{d^k}{dt^k}\int_a^t (t-\tau)^{k-p-1} g(\tau)d\tau\\
&= \lambda\, {}_a\mathbf{D}_t^p f(t) + \mu\, {}_a\mathbf{D}_t^p g(t).
\end{aligned}
$$

2.7.2 The Leibniz Rule for Fractional Derivatives

Let us take two functions, $\varphi(t)$ and $f(t)$, and start with the known Leibniz rule for evaluating the n-th derivative of the product $\varphi(t)f(t)$:

$$
\frac{d^n}{dt^n}\Big(\varphi(t)f(t)\Big) = \sum_{k=0}^{n}\binom{n}{k}\varphi^{(k)}(t)f^{(n-k)}(t). \tag{2.177}
$$

Let us now take the right-hand side of formula (2.177) and replace the integer parameter n with the real-valued parameter p. This means that the integer-order derivative $f^{(n-k)}(t)$ will be replaced with the Grünwald–Letnikov fractional-order derivative ${}_aD_t^{p-k}f(t)$. Denoting

$$
\Omega_n^p(t) = \sum_{k=0}^{n}\binom{p}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t), \tag{2.178}
$$

let us evaluate the sum (2.178).

First, let us suppose that $p = q < 0$. Then we have also $p - k = q - k < 0$ for all k, and according to (2.40)

$$ {}_aD_t^{p-k} f(t) = \frac{1}{\Gamma(-q+k)} \int_a^t (t-\tau)^{-q+k-1} f(\tau) d\tau, \tag{2.179} $$

which leads to

$$ \Omega_n^q(t) = \sum_{k=0}^{n} \binom{q}{k} \frac{1}{\Gamma(-q+k)} \int_a^t (t-\tau)^{-q+k-1} \varphi^{(k)}(t) f(\tau) d\tau \tag{2.180} $$

$$ = \int_a^t \left\{ \sum_{k=0}^{k=n} \binom{q}{k} \frac{1}{\Gamma(-q+k)} \varphi^{(k)}(t)(t-\tau)^k \right\} \frac{f(\tau)}{(t-\tau)^{q+1}} d\tau. \tag{2.181} $$

Taking into account the reflection formula (1.26) for the gamma function, we have

$$ \binom{q}{k} \frac{1}{\Gamma(-q+k)} = \frac{\Gamma(q+1)}{k!\,\Gamma(q-k+1)} \cdot \frac{1}{\Gamma(-q+k)} \tag{2.182} $$

$$ = \frac{\Gamma(q+1)}{k!} \cdot \frac{\sin(k-q)\pi}{\pi} \tag{2.183} $$

$$ = (-1)^{k+1} \frac{\Gamma(q+1)}{k!} \frac{\sin(q\pi)}{\pi}, \tag{2.184} $$

and, therefore, the expression (2.181) takes the form:

$$ \Omega_n^q(t) = -\frac{\sin(q\pi)}{\pi} \Gamma(q+1) \int_a^t \left\{ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \varphi^{(k)}(t)(t-\tau)^k \right\} \frac{f(\tau)}{(t-\tau)^{q+1}} f(\tau) d\tau. \tag{2.185} $$

Using the Taylor theorem we can write

$$ \sum_{k=0}^{n} \frac{(-1)^k}{k!} \varphi^{(k)}(t)(t-\tau)^k = \varphi(t) + \varphi'(t)(t-\tau) + \ldots + \frac{\varphi^{(n)}(t)}{n!}(t-\tau)^n $$

$$ = \varphi(\tau) + \frac{1}{n!} \int_a^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi, $$

and therefore we obtain

$$ \Omega_n^q(t) = -\frac{\sin(q\pi)\Gamma(q+1)}{\pi} \int_a^t (t-\tau)^{-q-1} \varphi(\tau) f(\tau) d\tau $$

$$
\begin{aligned}
&-\frac{\sin(q\pi)\Gamma(q+1)}{\pi\, n!}\int\limits_a^t (t-\tau)^{-q-1} f(\tau)d\tau \int\limits_a^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi \\
&= \frac{1}{\Gamma(-q)}\int\limits_a^t (t-\tau)^{-q-1}\varphi(\tau)f(\tau)d\tau \\
&\quad + \frac{1}{n!\,\Gamma(-q)}\int\limits_a^t (t-\tau)^{-q-1} f(\tau)d\tau \int\limits_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi \\
&= {}_aD_t^q\Big(\varphi(t)f(t)\Big) + R_n^q(t), \qquad (2.186)
\end{aligned}
$$

where

$$
R_n^q(t) = \frac{1}{n!\,\Gamma(-q)}\int\limits_a^t (t-\tau)^{-q-1} f(\tau)d\tau \int\limits_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi. \qquad (2.187)
$$

Let us now consider the case of $p > 0$. Our first step is to show that the evaluation of $\Omega_n^p(t)$ can be reduced to the evaluation of Ω_n^q for a certain negative q.

Taking into account that $\Gamma(0) = \infty$ we have to put

$$
\binom{p-1}{-1} = 0,
$$

and using the known property of the binomial coefficients

$$
\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}
$$

we can write

$$
\Omega_n^p(t) = \sum_{k=0}^{n}\binom{p-1}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t) + \sum_{k=1}^{n}\binom{p-1}{k-1}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t). \qquad (2.188)
$$

Replacing k with $k+1$ in the second sum gives

$$
\begin{aligned}
\Omega_n^p(t) &= \sum_{k=0}^{n}\binom{p-1}{k}\varphi^{(k)}(t)\frac{d}{dt}\Big({}_aD_t^{p-k-1}f(t)\Big) \\
&\quad + \sum_{k=0}^{n-1}\binom{p-1}{k}\frac{d\varphi^{(k)}(t)}{dt}\cdot {}_aD_t^{p-k-1}f(t), \qquad (2.189)
\end{aligned}
$$

which can be written as

$$\Omega_n^p(t) = \binom{p-1}{n} \varphi^{(n)}(t)\, {}_aD_t^{p-n} f(t) + \frac{d}{dt} \sum_{k=0}^{n-1} \binom{p-1}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k-1} f(t). \tag{2.190}$$

Adding and subtracting the expression

$$\frac{d}{dt}\left\{ \binom{p-1}{n} \varphi^{(n)}(t)\, {}_aD_t^{p-n-1} f(t) \right\}$$

we obtain

$$\Omega_n^p(t) = \frac{d}{dt} \sum_{k=0}^{n} \binom{p-1}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k-1} f(t) \tag{2.191}$$

$$- \binom{p-1}{n} \varphi^{(n+1)}\, {}_aD_t^{p-n-1} f(t) \tag{2.192}$$

or

$$\Omega_n^p(t) = \frac{d}{dt} \Omega_n^{p-1}(t) - \binom{p-1}{n} \varphi^{(n+1)}(t)\, {}_aD_t^{p-k-1} f(t). \tag{2.193}$$

The relationship (2.193) says that the evaluation of $\Omega_n^p(t)$ can be reduced to the evaluation of $\Omega_n^{p-1}(t)$. Repeating this procedure we can reduce the evaluation of $\Omega_n^p(t)$ $(p > 0)$ to the evaluation of $\Omega_n^q(t)$ $(q < 0)$.

Let us suppose that $0 < p < 1$. Then $p - 1 < 0$, and according to (2.186) we have

$$\Omega_n^{p-1}(t) = {}_aD_t^{p-1}\Big(\varphi(t) f(t)\Big) + R_n^{p-1}(t). \tag{2.194}$$

To combine (2.194) and (2.193), we have to differentiate (2.194) with respect to t. Taking into account that

$$\frac{d}{dt} R_n^{p-1}(t) = \frac{-p}{n!\,\Gamma(-p+1)} \int_a^t (t-\tau)^{-p-1} f(\tau) d\tau \int_\tau^t \varphi^{n+1}(\xi)(\tau - \xi)^n d\xi$$

$$+ \frac{(-1)^n \varphi^{n+1}(t)}{n!\,\Gamma(-p+1)} \int_a^t (t-\tau)^{-p+n} f(\tau) d\tau \tag{2.195}$$

and that

$$\int_a^t (t-\tau)^{-p+n} f(\tau) d\tau = \Gamma(-p+n+1)\, {}_aD_t^{p-n-1} f(t) \tag{2.196}$$

(since $n - p > 0$), we obtain:

$$\begin{aligned}
\frac{d}{dt}\Omega_n^{p-1}(t) &= {}_aD_t^p\Big(\varphi(t)f(t)\Big) \\
&\quad + \frac{(-1)^n\Gamma(-p+n+1)\varphi^{(n+1)}(t)}{n!\,\Gamma(-p+1)} \cdot {}_aD_t^{p-n-1}f(t) + R_n^p(t) \\
&= {}_aD_t^p\Big(\varphi(t)f(t)\Big) \\
&\quad + \binom{p-1}{n}\varphi^{(n+1)}(t)\,{}_aD_t^{p-n-1}f(t) + R_n^p(t),
\end{aligned} \tag{2.197}$$

and the substitution of this expression into (2.193) gives

$$\Omega_n^p(t) = {}_aD_t^p\Big(\varphi(t)f(t)\Big) + R_n^p(t), \tag{2.198}$$

which has the same form as (2.186).

Using mathematical induction we can prove that the relationship (2.198) holds for all p such that $p+1 < n$.

Obviously, the relationship (2.198) gives, in fact, the rule for the fractional differentiation of the product of two functions. This rule is a generalization of the Leibniz rule for integer-order differentiation, so it is convenient to preserve Leibniz's name also in the case of fractional differentiation.

The Leibniz rule for fractional differentiation is the following. If $f(\tau)$ is continuous in $[a, t]$ and $\varphi(\tau)$ has $n+1$ continuous derivatives in $[a, t]$, then the fractional derivative of the product $\varphi(t)f(t)$ is given by

$${}_aD_t^p\Big(\varphi(t)f(t)\Big) = \sum_{k=0}^{n}\binom{p}{k}\varphi^{(k)}(t)\,{}_aD_t^{p-k}f(t) - R_n^p(t) \tag{2.199}$$

where $n \geq p+1$ and

$$R_n^p(t) = \frac{1}{n!\,\Gamma(-p)}\int_a^t (t-\tau)^{-p-1}f(\tau)d\tau \int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi. \tag{2.200}$$

The sum in (2.199) can be considered as a partial sum of an infinite series and $R_n^p(t)$ as a remainder of that series.

Performing two subsequent changes of integration variables, first $\xi = \tau + \zeta(t-\tau)$ and then $\tau = a + \eta(t-a)$ we obtain the following expression for $R_n^p(t)$:

$$R_n^p(t) = \frac{(-1)^n}{n!\,\Gamma(-p)} \int_a^t (t-\tau)^{n-p} f(\tau) d\tau \int_0^1 \varphi^{(n+1)}(\tau + \zeta(t-\tau))\zeta^n d\zeta$$

$$= \frac{(-1)^n (t-a)^{n-p+1}}{n!\,\Gamma(-p)} \int_0^1 \int_0^1 F_a(t,\zeta,\eta) d\eta d\zeta, \qquad (2.201)$$

$$F_a(t,\zeta,\eta) = f(a+\eta(t-a))\varphi^{(n+1)}(a+(t-a)(\zeta+\eta-\zeta\eta)),$$

from which it directly follows that

$$\lim_{n\to\infty} R_n^p(t) = 0$$

if $f(\tau)$ and $\varphi(\tau)$ along with all its derivatives are continuous in $[a, t]$. Under this condition the Leibniz rule for fractional differentiation takes the form:

$${}_aD_t^p\Big(\varphi(t)f(t)\Big) = \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} f(t). \qquad (2.202)$$

The Leibniz rule (2.202) is especially useful for the evaluation of fractional derivatives of a function which is a product of a polynomial and a function with known fractional derivative.

To justify the above operations on $R_n^p(t)$ we have to show that $R_n^p(t)$ has a finite value for $p > 0$.

The function

$$\frac{f(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi}{(t-\tau)^{p+1}} \qquad (2.203)$$

gives an indefinite expression $\frac{0}{0}$ for $\tau = t$. To find the limit we can use the l'Hospital rule. Differentiating the numerator and the denominator with respect to τ we obtain

$$\frac{f'(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi + nf(\tau)\int_\tau^t \varphi^{(n+1)}(\xi)(\tau-\xi)^{n-1} d\xi}{-(p+1)(t-\tau)^p}, \qquad (2.204)$$

which again gives an indefinite expression $\frac{0}{0}$ for $\tau = t$. However, if $m < p \le m+1$, then applying the l'Hospital rule $m+2$ times we will obtain $(t-\tau)^{p-m-1}$ in the denominator (giving infinity for $\tau = t$), while the numerator will consist of the terms containing the multipliers of the form

$$\int_{\tau}^{t} \varphi^{(n+1)}(\xi)(\tau-\xi)^{n-k} d\xi \qquad (2.205)$$

which vanish as $\tau \to t$ if $n > k$. Obviously, k cannot be greater than $m+2$, so we can take $n \geq m+2$ and the function (2.203) will tend to 0 for $\tau \to t$. This means that the integral in (2.200) exists in the classical sense even for $p > -1$.

Taking into account the link between the Grünwald–Letnikov fractional derivatives and the Riemann–Liouville ones we see that under the above conditions on $f(t)$ and $\varphi(t)$ the Leibniz rule (2.202) holds also for the Riemann–Liouville derivatives.

2.7.3 Fractional Derivative of a Composite Function

One of the useful consequences of the Leibniz rule for the fractional derivative of a product is a rule for evaluating the fractional derivative of a composite function.

Let us take an analytic function $\varphi(t)$ and $f(t) = H(t-a)$, where $H(t)$ is the Heaviside function. Using the Leibniz rule (2.202) and the formula for the fractional differentiation of the Heaviside function (2.158) we can write:

$$\begin{aligned} {}_aD_t^p \varphi(t) &= \sum_{k=0}^{\infty} \binom{p}{k} \varphi^{(k)}(t)\, {}_aD_t^{p-k} H(t-a) \\ &= \frac{(t-a)^{-p}}{\Gamma(1-p)} \varphi(t) + \sum_{k=1}^{\infty} \binom{p}{k} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)} \varphi^{(k)}(t). \qquad (2.206) \end{aligned}$$

Now let us suppose that $\varphi(t)$ is a composite function:

$$\varphi(t) = F(h(t)). \qquad (2.207)$$

The k-th order derivative of $\varphi(t)$ is evaluated with the help of the Faà di Bruno formula [2, Chapter 24, §24.1.2]:

$$\frac{d^k}{dt^k} F(h(t)) = k! \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r}, \qquad (2.208)$$

where the sum $\sum$ extends over all combinations of non-negative integer values of a_1, a_2, ..., a_k such that

$$\sum_{r=1}^{k} r a_r = k \quad \text{and} \quad \sum_{r}^{k} a_r = m.$$

Substituting (2.207) and (2.208) into (2.206) we obtain the formula for the evaluation of the fractional derivative of a composite function:

$$\begin{aligned} {}_aD_t^p F(h(t)) &= \frac{(t-a)^{-p}}{\Gamma(1-p)}\varphi(t) \\ &+ \sum_{k=1}^{\infty} \binom{p}{k} \frac{k!(t-a)^{k-p}}{\Gamma(k-p+1)} \sum_{m=1}^{k} F^{(m)}(h(t)) \sum \prod_{r=1}^{k} \frac{1}{a_r!} \left(\frac{h^{(r)}(t)}{r!} \right)^{a_r}, \end{aligned} \tag{2.209}$$

where the sum $\sum$ and coefficients a_r have the meaning explained above.

2.7.4 Riemann–Liouville Fractional Differentiation of an Integral Depending on a Parameter

The well-known rule for the differentiation of an integral depending on a parameter with the upper limit depending on the same parameter, namely [68]

$$\frac{d}{dt} \int_0^t F(t,\tau)d\tau = \int_0^t \frac{\partial F(t,\tau)}{\partial t} d\tau + F(t, t-0), \tag{2.210}$$

has its analogue for fractional-order differentiation.

The rule for Riemann–Liouville fractional differentiation of an integral depending on a parameter, when the upper limit also depends on the parameter, is the following:

$$ {}_0D_t^\alpha \int_0^t K(t,\tau)d\tau = \int_0^t {}_\tau D_t^\alpha K(t,\tau)d\tau + \lim_{\tau \to t-0} {}_\tau D_t^{\alpha-1} K(t,\tau), \tag{2.211}$$

$$(0 < \alpha < 1).$$

Indeed, using (2.210) we have

$$
\begin{aligned}
{}_0D_t^{\alpha} \int\limits_0^t K(t,\tau)d\tau &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t \frac{d\eta}{(t-\eta)^{\alpha}} \int\limits_0^{\eta} K(\eta,\tau)d\tau \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t d\tau \int\limits_{\tau}^t \frac{K(\eta,\tau)d\eta}{(t-\eta)^{\alpha}} \\
&= \frac{d}{dt} \int\limits_0^t \widetilde{K}(t,\tau)d\tau \\
&= \int\limits_0^t \frac{\partial}{\partial t} \widetilde{K}(t,\tau)d\tau + \lim_{\tau \to t-0} \widetilde{K}(t,\tau) \\
&= \int\limits_0^t {}_{\tau}D_t^{\alpha} K(t,\tau)d\tau + \lim_{\tau \to t-0} {}_{\tau}D_t^{\alpha-1} K(t,\tau), \qquad (2.212)
\end{aligned}
$$

where

$$
\widetilde{K}(t,\xi) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{\xi}^t \frac{K(\eta,\xi)d\eta}{(t-\eta)^{\alpha}}.
$$

The following important particular case must be mentioned. If we have $K(t-\tau)f(\tau)$ instead of $K(t,\tau)$, then relationship (2.211) takes the form:

$$
{}_0D_t^{\alpha} \int\limits_0^t K(t-\tau)f(\tau)d\tau = \int\limits_0^t {}_0D_{\tau}^{\alpha} K(\tau)f(t-\tau)d\tau + \lim_{\tau \to +0} f(t-\tau)\, {}_0D_{\tau}^{\alpha-1} K(\tau). \qquad (2.213)
$$

It is worth noting that while in the right-hand side of the general formula (2.211) we have fractional derivatives with moving lower terminal τ, all fractional derivatives in (2.213) have the same lower terminal, namely 0. This significant simplification can be very useful in solving applied problems where the fractional differentiation of a convolution integral must be performed.

2.7.5 Behaviour near the Lower Terminal

We have shown in Section 2.3.7 that the Grünwald–Letnikov derivative ${}_aD_t^p f(t)$ and the Riemann–Liouville derivative ${}_a\mathbf{D}_t^p f(t)$ coincide if $f(t)$ is continuous and has a sufficient number of continuous derivatives in the closed interval $[a, t]$.

To study the behaviour of the fractional derivatives at the lower terminal, i.e. for $t \to a+0$, let us suppose that the function $f(t)$ is analytic at least in the interval $[a, \epsilon]$ for some small positive ϵ and, therefore, can be represented by the Taylor series

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(t-a)^k \tag{2.214}$$

in this interval.

Term-by-term fractional differentiation of (2.214) using the formula for the fractional differentiation of the power function (2.117) gives

$${}_aD_t^p f(t) = {}_a\mathbf{D}_t^p f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(k-p+1)}(t-a)^{k-p}, \tag{2.215}$$

from which it follows that if $f(t)$ has the form (2.214) then

$${}_aD_t^p f(t) = {}_a\mathbf{D}_t^p f(t) \sim \frac{f(a)}{\Gamma(1-p)}(t-a)^{-p}, \quad (t \to a+0), \tag{2.216}$$

and

$$\lim_{t \to a+0} {}_aD_t^p f(t) = \lim_{t \to a+0} {}_a\mathbf{D}_t^p f(t) = \begin{cases} 0, & (p < 0) \\ f(a), & (p = 0) \\ \infty, & (p > 0). \end{cases} \tag{2.217}$$

If we allow $f(t)$ to have an integrable singularity at $t = a$, then it can be written in the form $f(t) = (t-a)^q f_*(t)$, where $f_*(a) \neq 0$ and $q > -1$. Supposing that $f_*(t)$ can be represented by its Taylor series, we can write

$$f(t) = (t-a)^q f_*(t) = (t-a)^q \sum_{k=0}^{\infty} \frac{f_*^{(k)}(a)}{k!}(t-a)^k \tag{2.218}$$

$$= \sum_{k=0}^{\infty} \frac{f_*^{(k)}(a)}{k!}(t-a)^{q+k} \tag{2.219}$$

Performing the term-by-term Riemann–Liouville fractional differentiation of the series (2.219), we obtain

$${}_a\mathbf{D}_t^p f(t) = \sum_{k=0}^{\infty} \frac{f_*^{(k)}(a)}{k!} \frac{\Gamma(q+k+1)}{\Gamma(q+k-p+1)}(t-a)^{q+k-p}, \tag{2.220}$$

from which it follows that

$$ {}_a\mathbf{D}_t^p f(t) \approx \frac{f_*(a)\Gamma(q+1)}{\Gamma(q-p+1)}(t-a)^{q-p}, \quad (t \to a+0), \tag{2.221} $$

and

$$ \lim_{t\to a+0} {}_a\mathbf{D}_t^p f(t) = \begin{cases} 0, & (p<q) \\ \dfrac{f_*(a)\Gamma(q+1)}{\Gamma(q-p+1)}, & (p=q) \\ \infty, & (p>q). \end{cases} \tag{2.222} $$

2.7.6 Behaviour far from the Lower Terminal

To study the behaviour of the fractional derivative far from the lower terminal, i.e. for $t \to \infty$, let us start with the formula obtained for an analytic function $\varphi(t)$ in Section 2.7.3:

$$ {}_aD_t^p \varphi(t) = \sum_{k=0}^{\infty} \binom{p}{k} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)} \varphi^{(k)}(t). \tag{2.223} $$

Using the definition of the binomial coefficients and the reflection formula for the gamma function (1.26) we can write the relationship (2.223) as

$$ \begin{aligned} {}_aD_t^p \varphi(t) &= \sum_{k=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(k+1)\Gamma(p-k+1)} \frac{(t-a)^{k-p}}{\Gamma(k-p+1)} \varphi^{(k)}(t) \\ &= \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k (t-a)^{k-p}}{(p-k)\,k!} \varphi^{(k)}(t). \end{aligned} \tag{2.224} $$

Now let us suppose that t is far from the lower terminal a, i.e. that $|t| \gg |a|$. Then we can write

$$ (t-a)^{k-p} = t^{k-p}\left(1-\frac{a}{t}\right)^{k-p} = t^{k-p}\left(1 - \frac{(k-p)a}{t} + O\left(\frac{a^2}{t^2}\right)\right) \tag{2.225} $$

and therefore

$$ (t-a)^{k-p} \approx t^{k-p} + \frac{(p-k)at^k}{t^{p+1}}, \quad (|t| \gg |a|). \tag{2.226} $$

Substituting (2.226) into (2.224) we obtain

$$ \begin{aligned} {}_aD_t^p \varphi(t) \quad \approx \quad & \frac{\Gamma(p+1)\sin(p\pi)}{\pi} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k t^{k-p}}{(p-k)\,k!} \varphi^{(k)}(t) \right. \\ & \left. + \frac{a}{t^{p+1}} \sum_{k=0}^{\infty} \frac{(-1)^k t^k \varphi^{(k)}(t)}{k!} \right\} \end{aligned} \tag{2.227} $$

and using (2.223) gives

$$ {}_aD_t^p\varphi(t) \approx {}_0D_t^p\varphi(t) + \frac{a\Gamma(p+1)\sin(p\pi)\varphi(0)}{\pi t^{p+1}}, \quad (|t| \gg |a|). \tag{2.228} $$

Taking $t \to \infty$ we conclude that for large t

$$ {}_aD_t^p\varphi(t) \approx {}_0D_t^p\varphi(t). \tag{2.229} $$

This means that the impact of the instant at which the dynamical process $\varphi(t)$ started (and therefore the impact of the transient effects) vanishes as $t \to \infty$, and therefore for large t the fractional derivative with the lower terminal $t = a$ can be replaced, for example, with the fractional derivative with the lower terminal $t = 0$.

Another way of making the interval between the lower terminal and the upper terminal larger is considering $a \to -\infty$ for a fixed value of t. In this case we have $|a| \gg |t|$ and therefore

$$ (t-a)^{k-p} = a^{k-p}\left(1 - \frac{t}{a}\right)^{k-p} = a^{k-p}\left(1 - \frac{(k-p)t}{a} + O\left(\frac{t^2}{a^2}\right)\right), \tag{2.230} $$

from which it follows that

$$ (t-a)^{k-p} \approx a^{k-p} + \frac{(p-k)ta^k}{a^{p+1}}, \quad (|t| \gg |a|) \tag{2.231} $$

Substitution of (2.231) into (2.224) gives

$$ \begin{aligned} {}_aD_t^p\varphi(t) \approx \frac{\Gamma(p+1)\sin(p\pi)}{\pi} & \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k (t-(t-a))^{k-p}}{(p-k)\,k!}\varphi^{(k)}(t) \right. \\ & \left. + \frac{t}{a^{p+1}} \sum_{k=0}^{\infty} \frac{(-1)^k (t-(t-a))^k \varphi^{(k)}(t)}{k!} \right\} \end{aligned} \tag{2.232} $$

and using (2.223) we obtain

$$ {}_aD_t^p\varphi(t) \approx {}_{t-a}D_t^p\varphi(t) + \frac{t\Gamma(p+1)\sin(p\pi)\varphi(t-a)}{\pi a^{p+1}}, \quad (|a| \gg |t|). \tag{2.233} $$

Therefore, we may conclude that, under certain conditions on $\varphi(t)$, for large negative values of a the fractional derivative with a fixed lower terminal can be replaced with the fractional derivative with a moving lower terminal:

$$ {}_aD_t^p\varphi(t) \approx {}_{t-a}D_t^p\varphi(t). \tag{2.234} $$

2.8 Laplace Transforms of Fractional Derivatives

2.8.1 Basic Facts on the Laplace Transform

Let us recall some basic facts about the Laplace transform.

The function $F(s)$ of the complex variable s defined by

$$F(s) = L\{f(t);\, s\} = \int_0^\infty e^{-st} f(t)dt \tag{2.235}$$

is called the Laplace transform of the function $f(t)$, which is called the original. For the existence of the integral (2.235) the function $f(t)$ must be of exponential order α, which means that there exist positive constants M and T such that

$$e^{-\alpha t}|f(t)| \le M \quad \text{for all} \quad t > T.$$

In other words, the function $f(t)$ must not grow faster then a certain exponential function when $t \to \infty$.

We will denote the Laplace transforms by uppercase letters and the originals by lowercase letters.

The original $f(t)$ can be restored from the Laplace tranform $F(s)$ with the help of the inverse Laplace transform

$$f(t) = L^{-1}\{F(s);\, t\} = \int_{c-i\infty}^{c+i\infty} e^{st} F(s)ds, \quad c = Re(s) > c_0, \tag{2.236}$$

where c_0 lies in the right half plane of the absolute convergence of the Laplace integral (2.235).

The direct evaluation of the inverse Laplace transform using the formula (2.236) is often complicated; however, sometime it gives useful information on the behaviour of the unknown original $f(t)$ which we look for.

The Laplace transform of the convolution

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau \tag{2.237}$$

of the two functions $f(t)$ and $g(t)$, which are equal to zero for $t < 0$, is equal to the product of the Laplace transform of those function:

$$L\{f(t) * g(t);\, s\} = F(s)\, G(s) \tag{2.238}$$

under the assumption that both $F(s)$ and $G(s)$ exist. We will use the property (2.238) for the evaluation of the Laplace transform of the Riemann–Liouville fractional integral.

Another useful property which we need is the formula for the Laplace transform of the derivative of an integer order n of the function $f(t)$:

$$L\{f^{n}(t);\, s\} = s^{n}F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^{n}F(s) - \sum_{k=0}^{n-1} s^{k} f^{(n-k-1)}(0), \tag{2.239}$$

which can be obtained from the definition (2.235) by integrating by parts under the assumption that the corresponding integrals exist.

In the following sections on the Laplace transforms of fractional derivatives we consider the lower terminal $a = 0$.

2.8.2 Laplace Transform of the Riemann–Liouville Fractional Derivative

We will start with the Laplace transform of the Riemann–Liouville and Grünwald–Letnikov fractional integral of order $p > 0$ defined by (2.88), which we can write as a convolution of the functions $g(t) = t^{p-1}$ and $f(t)$:

$${}_0\mathbf{D}_t^{-p} f(t) = {}_0 D_t^{-p} f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} f(\tau) d\tau = t^{p-1} * f(t). \tag{2.240}$$

The Laplace transform of the function t^{p-1} is [62]

$$G(s) = L\{t^{p-1};\, s\} = \Gamma(p) s^{-p}. \tag{2.241}$$

Therefore, using the formula for the Laplace transform of the convolution (2.238) we obtain the Laplace transform of the Riemann–Liouville and the Grünwald–Letnikov fractional integral:

$$L\{{}_0\mathbf{D}_t^{-p} f(t);\, s\} = L\{{}_0 D_t^{-p} f(t);\, s\} = s^{-p} F(s). \tag{2.242}$$

Now let us turn to the evaluation of the Laplace transform of the Riemann–Liouville fractional derivative, which for this purpose we write in the form:

$$ {}_0\mathbf{D}_t^p f(t) = g^{(n)}(t), \tag{2.243}$$

$$ g(t) = {}_0\mathbf{D}_t^{-(n-p)} f(t) \frac{1}{\Gamma(k-p)} \int_0^t (t-\tau)^{n-p-1} f(\tau) d\tau, \tag{2.244}$$

$$ (n-1 \le p < n). $$

The use of the formula for the Laplace transform of an integer-order derivative (2.239) leads to

$$ L\{{}_0\mathbf{D}_t^p f(t);\, s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0). \tag{2.245}$$

The Laplace transform of the function $g(t)$ is evaluated by (2.242):

$$ G(s) = s^{-(n-p)} F(s). \tag{2.246}$$

Additionally, from the definition of the Riemann–Liouville fractional derivative (2.103) it follows that

$$ g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}}\, {}_0\mathbf{D}_t^{-(n-p)} f(t) = {}_0\mathbf{D}_t^{p-k-1} f(t). \tag{2.247}$$

Substituting (2.246) and (2.247) into (2.245) we obtain the following final expression for the Laplace transform of the Riemann–Liouville fractional derivative of order $p > 0$:

$$ L\{{}_0\mathbf{D}_t^p f(t);\, s\} = s^p F(s) - \sum_{k=0}^{n-1} s^k \left[{}_0\mathbf{D}_t^{p-k-1} f(t) \right]_{t=0}. \tag{2.248}$$

$$ (n-1 \le p < n). $$

This Laplace transform of the Riemann–Liouville fractional derivative is well known (see, for example, [179] or [153]). However, its practical applicability is limited by the absense of the physical interpretation of the limit values of fractional derivatives at the lower terminal $t = 0$. At the time of writing, such an interpretation is not known.

2.8.3 Laplace Transform of the Caputo Derivative

To establish the Laplace transform formula for the Caputo fractional derivative let us write the Caputo derivative (2.138) in the form:

$$ {}_{0}^{C}D_t^p f(t) = {}_0\mathbf{D}_t^{-(n-p)} g(t), \qquad g(t) = f^{(n)}(t), \tag{2.249}$$

$$(n-1 < p \le n). \tag{2.250}$$

Using the formula (2.242) for the Laplace transform of the Riemann–Liouville fractional integral gives

$$L\{ {}_{0}^{C}D_t^p f(t);\, s\} = s^{-(n-p)} G(s), \tag{2.251}$$

where, according to (2.239),

$$G(s) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0). \tag{2.252}$$

Introducing (2.252) into (2.251) we arrive at the Laplace transform formula for the Caputo fractional derivative:

$$L\{ {}_{0}^{C}D_t^p f(t)\} = s^p F(s) - \sum_{k=0}^{n-1} s^{p-k-1} f^{(k)}(0), \tag{2.253}$$

$$(n-1 < p \le n).$$

Since this formula for the Laplace transform of the Caputo derivative involves the values of the function $f(t)$ and its derivatives at the lower terminal $t = 0$, for which a certain physical interpretation exists (for example, $f(0)$ is the initial position, $f'(0)$ is the initial velocity, $f''(0)$ is the initial acceleration), we can expect that it can be useful for solving applied problems leading to linear fractional differential equations with constant coefficients with accompanying initial conditions in traditional form.

2.8.4 Laplace Transform of the Grünwald–Letnikov Fractional Derivative

First let us consider the case of $0 \le p < 1$, when the Grünwald–Letnikov fractional derivative (2.54) with the lower terminal $a = 0$ of the function

$f(t)$, which is bounded at $t = 0$, can be written in the following form:

$$ {}_0D_t^p f(t) = \frac{f(0)t^{-p}}{\Gamma(1-p)} + \frac{1}{\Gamma(1-p)} \int_0^t (t-\tau)^{-p} f'(\tau) d\tau. \qquad (2.254) $$

Using the Laplace transform of the power function (2.241), the formula for the Laplace transform of the convolution (2.238) and the Laplace transform of the integer-order derivative (2.239) we obtain:

$$ L\{ {}_0D_t^p f(t); \, s\} = \frac{f(0)}{s^{1-p}} + \frac{1}{s^{1-p}} \Big(sF(s) - f(0) \Big) = s^p F(s). \qquad (2.255) $$

An example of an application of the formula (2.255) is given in [75].

The Laplace transform of the Grünwald–Letnikov fractional derivative of order $p > 1$ does not exist in the classical sense, because in such a case we have non-integrable functions in the sum in the formula (2.54). The Laplace transforms of such functions are given by divergent integrals. However, the Laplace tranform of the power function (2.241) allows analytic continuation with respect to the parameter p. This approach is equivalent to the generalized functions (distributions) approach [76]. Divergent integrals in such a sense are called finite-part integrals. In this way, assuming that $m < p < m+1$, and using the Laplace transform of the power function (2.241), the formula for the Laplace transform of the convolution (2.238) and the Laplace transform of the integer-order derivative (2.239), we obtain:

$$ \begin{aligned} L\{ {}_0D_t^p f(t); \, s\} &= \sum_{k=0}^{m} f^{(k)}(0) L\{ \frac{t^{-p+k}}{\Gamma(-p+m+1)}; \, s\} \\ &\qquad + L\{ \frac{t^{m-p}}{\Gamma(-p+m+1)} * f^{(m+1)}(t); \, s\} \\ &= \sum_{k=0}^{m} f^{(k)}(0) s^{p-k-1} \\ &\qquad + s^{p-m-1} \Big(s^{m+1} F(s) - \sum_{k=0}^{m} f^{(k)}(0) s^{m-k} \Big) \\ &= s^p F(s). \end{aligned} \qquad (2.256) $$

We arrived at the same formula as (2.255).

In applications it is necessary to keep in mind that the formula (2.256) holds in the classical sense only for $0 < p < 1$; for $p > 1$ it holds in the sense of generalized functions (distributions) and, therefore, the formulation of an applied problem must also be done using the language of generalized functions, as well as interpretation of the results obtained in this way.

2.8.5 Laplace Transform of the Miller–Ross Sequential Fractional Derivative

Let us introduce the following notation for the Miller–Ross sequential derivative:

$$ {}_a\mathcal{D}_t^{\sigma_m} \equiv {}_aD_t^{\alpha_m}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1}; \tag{2.257} $$

$$ {}_a\mathcal{D}_t^{\sigma_m - 1} \equiv {}_aD_t^{\alpha_m - 1}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1}; \tag{2.258} $$

$$ \sigma_m = \sum_{j=1}^{m} \alpha_j, \qquad 0 < \alpha_j \le 1, \quad (j = 1, 2, \ldots, m). $$

We can establish the following formula for the Laplace transform of the sequential derivative (2.257):

$$ L\Big\{ {}_0\mathcal{D}_t^{\sigma_m} f(t);\, s \Big\} = s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m - \sigma_{m-k}} \Big[{}_0\mathcal{D}_t^{\sigma_{m-k}-1} f(t) \Big]_{t=0}, \tag{2.259} $$

$$ {}_a\mathcal{D}_t^{\sigma_{m-k}-1} \equiv {}_aD_t^{\alpha_{m-k}-1}\, {}_aD_t^{\alpha_{m-k-1}} \ldots {}_aD_t^{\alpha_1}, $$

$$ (k = 0, 1, \ldots, m-1). $$

The particular case of (2.259) for $f(t)$ m-times differentiable, $\alpha_m = \mu$, $\alpha_k = 1$, $(k = 1, 2, \ldots, m-1)$ was obtained by Caputo [24, p. 41] much earlier. Taking $\alpha_1 = \mu$, $\alpha_k = 1$, $(k = 2, 3, \ldots, m)$ leads under obvious assumptions to the classical formula (2.248).

To prove the formula (2.259) let us first recall the Laplace transform formula for the Riemann–Liouville fractional derivative (2.248), which in the case of $0 < \alpha \le 1$ takes the form:

$$ L\{ {}_0\mathbf{D}_t^{\alpha} f(t);\, s \} = s^{\alpha} F(s) - \Big[{}_0\mathbf{D}_t^{\alpha-1} f(t) \Big]_{t=0}, \tag{2.260} $$

and then use the formula (2.260) subsequently m times:

$$ L\Big\{ {}_0\mathcal{D}_t^{\sigma_m} f(t);\, s \Big\} = L\Big\{ {}_0D_t^{\alpha_m}\, {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s \Big\} $$

$$
\begin{aligned}
&= p^{\alpha_m} L\Big\{ {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s\Big\} \\
&\qquad\qquad - \Big[{}_0D_t^{\alpha_m - 1}\, {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t) \Big]_{t=0} \\
&= p^{\alpha_m} L\Big\{ {}_0\mathcal{D}_t^{\sigma_{m-1}} f(t);\, s\Big\} - \Big[{}_0\mathcal{D}_t^{\sigma_m - 1} f(t) \Big]_{t=0} \\
&= p^{\alpha_m + \alpha_{m-1}} L\Big\{ {}_0\mathcal{D}_t^{\sigma_{m-2}} f(t);\, s\Big\} \\
&\qquad\qquad - p^{\alpha_m} \Big[{}_0\mathcal{D}_t^{\sigma_{m-1} - 1} f(t) \Big]_{t=0} \\
&\qquad\qquad\qquad - \Big[{}_0\mathcal{D}_t^{\sigma_m - 1} f(t); \Big]_{t=0} \\
&\ldots \qquad \ldots \qquad \ldots \\
&= s^{\sigma_m} F(s) - \sum_{k=0}^{m-1} s^{\sigma_m - \sigma_{m-k}} \Big[{}_0\mathcal{D}_t^{\sigma_{m-k} - 1} f(t) \Big]_{t=0}.
\end{aligned}
$$

2.9 Fourier Transforms of Fractional Derivatives

2.9.1 Basic Facts on the Fourier Transform

The exponential Fourier transform of a continuous function $h(t)$ absolutely integrable in $(-\infty, \infty)$ is defined by

$$
F_e\{h(t);\, \omega\} = \int_{-\infty}^{\infty} e^{i\omega t} h(t)dt, \tag{2.261}
$$

and the original $h(t)$ can be restored from its Fourier transform $H_e(t)$ with the help of the inverse Fourier transform:

$$
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_e(\omega) e^{-i\omega t} d\omega. \tag{2.262}
$$

As above, we will denote originals by lowercase letters, and their transforms by uppercase letters.

The Fourier transform of the convolution

$$
h(t) * g(t) = \int_{-\infty}^{\infty} h(t-\tau) g(\tau) d\tau = \int_{-\infty}^{\infty} h(\tau) g(t-\tau) d\tau \tag{2.263}
$$

of the two functions $h(t)$ and $g(t)$, which are defined in $(-\infty, \infty)$, is equal to the product of their Fourier transforms:

$$F_e\{h(t) * g(t);\, \omega\} = H_e(\omega)\, G_e(\omega) \qquad (2.264)$$

under the assumption that both $H_e(\omega)$ and $G_e(\omega)$ exist. We will use the property (2.264) for the evaluation of the Fourier transforms of the Riemann–Liouville fractional integral and Fourier transforms of fractional derivatives.

Another useful property of the Fourier transform, which is frequently used in solving applied problems, is the Fourier transform of derivatives of $h(t)$. Namely, if $h(t)$, $h'(t)$, ..., $h^{(n-1)}(t)$ vanish for $t \to \pm\infty$, then the Fourier transform of the n-th derivative of $h(t)$ is

$$F_e\{h^{(n)}(t);\, \omega\} = (-i\omega)^n H_e(\omega). \qquad (2.265)$$

The Fourier transform is a powerful tool for frequency domain analysis of linear dynamical systems.

2.9.2 Fourier Transform of Fractional Integrals

First we will evaluate the Fourier transform of the Riemann–Liouville fractional integral with the lower terminal $a = -\infty$, i.e. of

$$_{-\infty}\mathbf{D}_t^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-\tau)^{\alpha-1} g(\tau)\, d\tau, \qquad (2.266)$$

where we assume $0 < \alpha < 1$.

Let us start with the Laplace transform of the function

$$h(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$$

(see formula (2.241)), which can be written as

$$\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1} e^{-st}\, dt = s^{-\alpha}. \qquad (2.267)$$

Let us take $s = -i\omega$, where ω is real. It follows from the Dirichlet theorem [68, p. 564] that in such a case the integral (2.267) converges if $0 < \alpha < 1$. Therefore, we immediately obtain the Fourier transform of the function

$$h_+(t) = \begin{cases} \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & (t > 0) \\ 0, & (t \leq 0) \end{cases}$$

in the form

$$F_e\{h_+(t);\, \omega\} = (-i\omega)^{-\alpha}. \tag{2.268}$$

Now we can find the Fourier transform of the Riemann–Liouville fractional integral (2.266), which can be written as a convolution (2.263) of the functions $h_+(t)$ and $g(t)$:

$${}_{-\infty}\mathbf{D}_t^{-\alpha} f(t) = h_+(t) * g(t). \tag{2.269}$$

Using the rule (2.264) we obtain:

$$F_e\left\{{}_{-\infty}\mathbf{D}_t^{-\alpha} g(t);\, \omega\right\} = (i\omega)^{-\alpha} G(\omega), \tag{2.270}$$

where $G(\omega)$ is the Fourier transform of the function $g(t)$.

The formula (2.270) gives also the Fourier transform of the Grünwald–Letnikov fractional integral ${}_{-\infty}D_t^{-\alpha} g(t)$ and the Caputo fractional integral ${}_{-\infty}^{\ C}D_t^{-\alpha} g(t)$, because in this case they coincide with the Riemann–Liouville fractional integral.

2.9.3 Fourier Transform of Fractional Derivatives

Let us now evaluate the Fourier transform of fractional derivatives.

Considering the lower terminal $a = -\infty$ and requiring the resonable behaviour of $g(t)$ and its derivatives for $t \to -\infty$ we can perform integration by parts and write the Riemann–Liouville, the Grünwald–Letnikov and the Caputo definition in the same form:

$$\left.\begin{matrix} {}_{-\infty}\mathbf{D}_t^{\alpha} g(t) \\ {}_{-\infty}D_t^{\alpha} g(t) \\ {}_{-\infty}^{\ C}D_t^{\alpha} g(t) \end{matrix}\right\} = \frac{1}{\Gamma(n-\alpha)} \int\limits_{-\infty}^{t} \frac{g^{(n)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-n}} = {}_{-\infty}\mathbf{D}_t^{\alpha-n} g^{(n)}(t), \tag{2.271}$$

$$(n-1 < \alpha < n).$$

The Fourier transform of (2.271) with the use of the Fourier transform of the Riemann–Liouville fractional integral (2.270) and then the

Fourier transform of an integer-order derivative (2.265) gives the following formula for the exponential Fourier transform of the Riemann–Liouville, Grünwald–Letnikov and Caputo fractional derivatives with the lower terminal $a = -\infty$:

$$\begin{aligned} F_e\left\{D^\alpha g(t);\, \omega\right\} &= (-i\omega)^{\alpha-n} F_e\left\{g^{(n)}(t);\, \omega\right\} \\ &= (-i\omega)^{\alpha-n}(-i\omega)^n G(\omega) \\ &= (-i\omega)^\alpha G(\omega), \end{aligned} \tag{2.272}$$

where the symbol D^α denotes any of the mentioned fractional differentiations (Riemann–Liouville ${}_{-\infty}\mathbf{D}_t^\alpha$, Grünwald–Letnikov ${}_{-\infty}D_t^\alpha g(t)$ or Caputo ${}_{-\infty}^{\ C}D_t^\alpha g(t)$).

The Fourier transform of fractional derivatives has been used, for example, by H. Beyer and S. Kempfle [19] for analysing the oscillation equation with a fractional-order damping term:

$$y''(t) + a\ {}_{-\infty}D_t^\alpha y(t) + by(t) = f(t), \tag{2.273}$$

by S. Kempfle and L. Gaul [115] for constructing global solutions of linear fractional differential equations with constant coefficients, and implicitly by R. R. Nigmatullin and Ya. E. Ryabov [166] for studying relaxation processes in insulators.

2.10 Mellin Transforms of Fractional Derivatives

2.10.1 Basic Facts on the Mellin Transform

The Mellin integral transform $F(s)$ of a function $f(t)$, which is defined in the interval $(0, \infty)$ is

$$F(s) = \mathcal{M}\left\{f(t);\, s\right\} = \int_0^\infty f(t) t^{s-1} dt, \tag{2.274}$$

where s is complex, such as

$$\gamma_1 < Re(s) < \gamma_2.$$

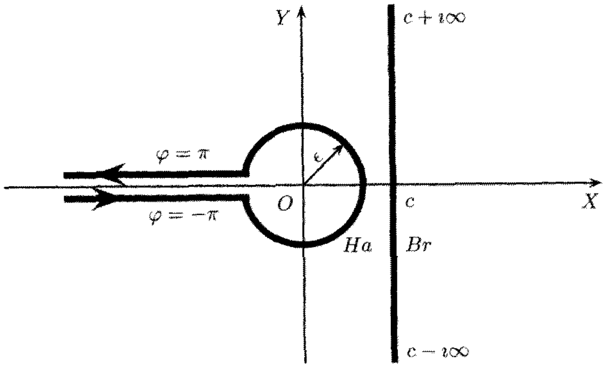

Figure 2.2: *The Bromwich (Br) and the Hankel (Ha) contours.*

The Mellin transform (2.274) exists if the function $f(t)$ is piecewise continuous in every closed interval $[a, b] \subset (0, \infty)$ and

$$\int_0^1 |f(t)|t^{\gamma_1 - 1}dt < \infty, \qquad \int_1^\infty |f(t)|t^{\gamma_2 - 1}dt < \infty. \tag{2.275}$$

If the function $f(t)$ also satisfies the Dirichlet conditions in every closed interval $[a, b] \subset (0, \infty)$, then the function $f(t)$ can be restored using the inverse Mellin transform formula

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s)t^{-s}ds, \quad (0 < t < \infty), \tag{2.276}$$

in which $\gamma_1 < \gamma < \gamma_2$. The integration contour in (2.276) is the Bromwich contour (contour Br in Fig. 2.2).

It follows from the definition (2.274) that

$$\mathcal{M}\Big\{t^\alpha f(t);\ s\Big\} = \mathcal{M}\Big\{f(t);\ s + \alpha\Big\} = F(s + \alpha). \tag{2.277}$$

The Mellin transform of the Mellin convolution

$$f(t) * g(t) = \int_0^\infty f(t\tau)g(\tau)d\tau \tag{2.278}$$

of functions $f(t)$ and $g(t)$, the Mellin transforms of which are $F(s)$ and $G(s)$, is given by the formula (see, for example, [249]):

$$\mathcal{M}\Big\{\int_0^\infty f(t\tau)g(\tau)d\tau;\ s\Big\} = F(s)\,G(1-s), \tag{2.279}$$

and combining (2.277) and (2.279) gives

$$\mathcal{M}\Big\{t^\lambda\int_0^\infty \tau^\mu f(t\tau)g(\tau)d\tau;\ s\Big\} = F(s+\lambda)\,G(1-s-\lambda+\mu). \tag{2.280}$$

Integrating repeatedly by parts, we have the following relationship for the Mellin transform of an integer-order derivative:

$$\begin{aligned}
\mathcal{M}\Big\{f^{(n)}(t);\ s\Big\} &= \int_0^\infty f^{(n)}(t)\,t^{s-1}dt \\
&= \Big[f^{(n-1)}(t)\,t^{s-1}\Big]_0^\infty - (s-1)\int_0^\infty f^{(n-1)}(t)\,t^{s-2}dt \\
&= \Big[f^{(n-1)}(t)\,t^{s-1}\Big]_0^\infty - (s-1)\mathcal{M}\Big\{f^{(n-1)}(t);\ s-1\Big\} \\
&= \ldots \\
&= \sum_{k=0}^{n-1}(-1)^k\frac{\Gamma(s)}{\Gamma(s-k)}\Big[f^{(n-k-1)}(t)\,t^{s-k-1}\Big]_0^\infty \\
&\qquad\qquad + (-1)^n\frac{\Gamma(s)}{\Gamma(s-n)}F(s-n) \\
&= \sum_{k=0}^{n-1}\frac{\Gamma(1-s+k)}{\Gamma(1-s)}\Big[f^{(n-k-1)}(t)\,t^{s-k-1}\Big]_0^\infty \\
&\qquad\qquad + \frac{\Gamma(1-s+n)}{\Gamma(1-s)}F(s-n)
\end{aligned} \tag{2.281}$$

where $F(s)$ is the Mellin transform of $f(t)$.

If $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t=0$ and $t=\infty$ give zero, then the formula (2.281) takes the simplest form:

$$\mathcal{M}\Big\{f^{(n)}(t);\ s\Big\} = \frac{\Gamma(1-s+n)}{\Gamma(1-s)}F(s-n). \tag{2.282}$$

2.10.2 Mellin Transform of the Riemann–Liouville Fractional Integrals

Let us evaluate the Mellin transform of the Riemann–Liouville fractional integral ${}_0D_t^{-\alpha}f(t)$, $(\alpha > 0)$. Using the substitution $\tau = t\xi$ we can write

$$\begin{aligned} {}_0D_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau \\ &= \frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\xi)^{\alpha-1}f(t\xi)d\xi \\ &= \frac{t^\alpha}{\Gamma(\alpha)}\int_0^\infty f(t\xi)g(\xi)d\xi \end{aligned} \tag{2.283}$$

where

$$g(t) = \begin{cases} (1-t)^{\alpha-1}, & (0 \le t < 1) \\ 0, & (t \ge 1). \end{cases}$$

The Mellin transform of the function $g(t)$ gives simply the Euler beta function (1.20),

$$\mathcal{M}\Big\{g(t);\ s\Big\} = B(\alpha,\ s) = \frac{\Gamma(\alpha)\,\Gamma(s)}{\Gamma(\alpha+s)}. \tag{2.284}$$

Then using the formulas (2.280), (2.283), and (2.284), we obtain:

$$\mathcal{M}\Big\{\,{}_0D_t^{-\alpha}f(t);\ s\Big\} = \frac{1}{\Gamma(\alpha)}\,F(s+\alpha)\,B(\alpha,\ 1-s-\alpha),$$

or

$$\mathcal{M}\Big\{\,{}_0D_t^{-\alpha}f(t);\ s\Big\} = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)}\,F(s+\alpha), \tag{2.285}$$

where $F(s)$ is the Mellin transform of the function $f(t)$.

The obtained formula (2.285) reminds us of the particular case of the Mellin transform of the n-th derivative of $f(t)$ (2.282), which can be formally obtained from (2.285) by putting $\alpha = -n$.

2.10.3 Mellin Transform of the Riemann–Liouville Fractional Derivative

Let us take $0 \le n-1 < \alpha < n$. According to the definition of the Riemann–Liouville fractional derivative, we can write

$${}_0D_t^{\alpha}f(t) = \frac{d^n}{dt^n}\,{}_0D_t^{-(n-\alpha)}f(t).$$

Temporarily denoting $g(t) = {}_0D_t^{-(n-\alpha)} f(t)$, and using formulas (2.281) and (2.285), we have:

$$\begin{aligned}
\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} &= \mathcal{M}\Big\{ \frac{d^n}{dt^n}\, {}_0D_t^{\alpha-n} f(t);\ s\Big\} = \mathcal{M}\Big\{ g^{(n)};\ s\Big\} \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[g^{(n-k-1)}(t)\, t^{s-k-1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1-s+n)}{\Gamma(1-s)} G(s-n) \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \left[\frac{d^{n-k-1}}{dt^{n-k-1}}\, {}_0D_t^{\alpha-n} f(t)\, t^{s-k-1} \right]_0^{\infty} \\
&\quad + \frac{\Gamma(1-s+n)}{\Gamma(1-s)}\, \frac{\Gamma(1-(s-n)-(n-\alpha))}{\Gamma(1-(s-n))} \\
&\quad \times F((s-n)+(n-\alpha)),
\end{aligned} \tag{2.286}$$

or

$$\begin{aligned}
\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} &= \sum_{k=0}^{n-1} \frac{\Gamma(1-s+k)}{\Gamma(1-s)} \Big[{}_0D_t^{\alpha-k-1} f(t)\, t^{s-k-1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\, F(s-\alpha).
\end{aligned} \tag{2.287}$$

If $0 < \alpha < 1$, then (2.287) takes on the form:

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \Big[{}_0D_t^{\alpha-1} f(t)\, t^{s-1} \Big]_0^{\infty} + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\, F(s-\alpha). \tag{2.288}$$

If the function $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t=0$ and $t=\infty$ in the formula (2.287) give zero, then it takes on the simplest form:

$$\mathcal{M}\Big\{ {}_0D_t^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)}\, F(s-\alpha). \tag{2.289}$$

2.10.4 Mellin Transform of the Caputo Fractional Derivative

Let us take $0 \le n-1 \le \alpha < n$. Temporarily denoting $h(t) = f^{(n)}(t)$ and using the formulas (2.285) and (2.281), we have:

$$\mathcal{M}\Big\{ {}_0^C D_t^{\alpha} f(t);\ s\Big\} = \mathcal{M}\Big\{ {}_0D_t^{-(n-\alpha)} f^{(n)}(t);\ s\Big\}$$

$$
\begin{aligned}
&= \mathcal{M}\Big\{ {}_0D_t^{-(n-\alpha)} h(t);\ s \Big\} \\
&= \frac{1-s-(n-\alpha)}{\Gamma(1-s)} H(s+(n-\alpha)) \\
&= \frac{\Gamma(1-s-n+\alpha)}{\Gamma(1-s)} \\
&\quad \times \left\{ \sum_{k=0}^{n-1} \frac{\Gamma(1-(s+n-\alpha)+k)}{\Gamma(1-(s+n-\alpha))} \Big[f^{(n-k-1)}(t)\, t^{(s+n-\alpha)-k-1} \Big]_0^\infty \right. \\
&\qquad \left. + \frac{\Gamma(1-(s+n-\alpha)+n)}{\Gamma(s+n-\alpha-n)} F((s+n-\alpha)-n) \right\} \\
&= \sum_{k=0}^{n-1} \frac{\Gamma(1-s-n+\alpha+k)}{\Gamma(1-s)} \Big[f^{(n-k-1)}(t)\, t^{s+n-\alpha-k-1} \Big]_0^\infty \\
&\qquad + \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} F(s-\alpha)
\end{aligned}
\tag{2.290}
$$

or

$$
\begin{aligned}
\mathcal{M}\Big\{ {}_0^C D_t^\alpha f(t);\ s \Big\} = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha-m-s)}{\Gamma(1-s)} \Big[f^{(k)}\, t^{s-\alpha+k} \Big]_0^\infty \\
+ \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} F(s-\alpha).
\end{aligned}
\tag{2.291}
$$

For $0 < \alpha < 1$ the formula (2.291) takes on the form:

$$
\mathcal{M}\Big\{ {}_0^C D_t^\alpha f(t);\ s \Big\} = \frac{\Gamma(\alpha-s)}{\Gamma(1-s)} \Big[f(t)\, t^{s-\alpha} \Big]_0^\infty + \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha).
\tag{2.292}
$$

If the function $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t = 0$ and $t = \infty$ in the formula (2.291) give zero, then it takes on the simplest form:

$$
\mathcal{M}\Big\{ {}_0^C D_t^\alpha f(t);\ s \Big\} = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} F(s-\alpha).
\tag{2.293}
$$

2.10.5 Mellin Transform of the Miller–Ross Fractional Derivative

Let us recall the following notation for the Miller–Ross sequential fractional derivative defined by (2.257):

$$
{}_a\mathcal{D}_t^{\sigma_m} \equiv {}_aD_t^{\alpha_m}\, {}_aD_t^{\alpha_{m-1}} \dots {}_aD_t^{\alpha_1};
$$

$$
{}_a\mathcal{D}_t^{\sigma_m - 1} \equiv {}_aD_t^{\alpha_m - 1}\, {}_aD_t^{\alpha_{m-1}} \ldots {}_aD_t^{\alpha_1};
$$
$$
\sigma_m = \sum_{j=1}^{m} \alpha_j, \qquad 0 < \alpha_j \leq 1, \quad (j = 1, 2, \ldots, m).
$$

Let us start with $m = 2$. Temporarily denoting $g(t) = {}_0D_t^{\beta}(t)$ and using the formula (2.287), we have:

$$
\begin{aligned}
\mathcal{M}\Big\{ {}_0\mathcal{D}_t^{\sigma_2} f(t);\ s\Big\} &= \mathcal{M}\Big\{ {}_0D_t^{\alpha_2} g(t);\ s\Big\} \\
&= \Big[{}_0D_t^{\alpha_2 - 1} g(t)\, t^{s-1} \Big]_0^{\infty} + \frac{\Gamma(1 - s + \alpha_2)}{\Gamma(1 - s)}\, G(s - \alpha_2) \\
&= \Big[{}_0D_t^{\sigma_2 - 1} f(t)\, t^{s-1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1 - s + \alpha_2)}{\Gamma(1 - s)} \Big\{ \Big[{}_0D_t^{\alpha_1 - 1} f(t)\, t^{s - \alpha_2 - 1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1 - (s - \alpha_2) + \alpha_1)}{\Gamma(1 - (s - \alpha_2))}\, F((s - \alpha_2) - \alpha_1) \Big\} \\
&= \Big[{}_0\mathcal{D}_t^{\sigma_2 - 1} f(t)\, t^{s-1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1 - s + \alpha_2)}{\Gamma(1 - s)} \Big[{}_0\mathcal{D}_t^{\sigma_1 - 1} f(t)\, t^{s - \alpha_2 - 1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1 - s + \sigma_2)}{\Gamma(1 - s)} F(s - \sigma_2).
\end{aligned} \tag{2.294}
$$

It can be shown by induction that in the general case the Mellin transform of the Miller–Ross sequential fractional derivative is given by the following expression:

$$
\begin{aligned}
\mathcal{M}\Big\{ {}_0\mathcal{D}_t^{\sigma_n} f(t);\ s\Big\} &= \sum_{k=1}^{n} \frac{\Gamma(1 - s + \sigma_n - \sigma_k)}{\Gamma(1 - s)} \Big[{}_0\mathcal{D}_t^{\sigma_k - 1} f(t)\, t^{s - \sigma_n + \sigma_k - 1} \Big]_0^{\infty} \\
&\quad + \frac{\Gamma(1 - s + \sigma_n)}{\Gamma(1 - s)}\, F(s - \sigma_n).
\end{aligned} \tag{2.295}
$$

If the function $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t = 0$ and $t = \infty$ in the formula (2.295) give zero, then it takes on the simplest form:

$$
\mathcal{M}\Big\{ {}_0\mathcal{D}_t^{\sigma_n} f(t);\ s\Big\} = \frac{\Gamma(1 - s + \sigma_n)}{\Gamma(1 - s)}\, F(s - \sigma_n), \tag{2.296}
$$

which is the same as expressions (2.287) and (2.291) for the Riemann–Liouville derivative and the Caputo derivative. Therefore, for functions with suitable behaviour for $t \to 0$ and $t \to \infty$ the Mellin transform of the Riemann–Liouville, Caputo, and Miller–Ross fractional derivative may coincide. This is similar to what we also observe in the case of the Laplace and Fourier transforms.

Under the conditions of coincidence, using of (2.277) gives

$$\mathcal{M}\Big\{t^{\alpha} D^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s)}{\Gamma(1-s-\alpha)} F(s), \tag{2.297}$$

and

$$\begin{aligned}\mathcal{M}\Big\{\sum_{k=0}^{n} a_k t^{\alpha+k} D^{\alpha+k} f(t);\ s\Big\} &= F(s)\Gamma(1-s) \sum_{k=0}^{n} \frac{a_k}{\Gamma(1-s-\alpha-k)} \\ &= \frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-s-\alpha)} \sum_{k=0}^{n} (-1)^k a_k \prod_{j=0}^{k-1} (s+\alpha+1),\end{aligned} \tag{2.298}$$

where D^{α} denotes the Riemann–Liouville, or Caputo, or Miller–Ross fractional derivative.

In particular, we have

$$\mathcal{M}\Big\{t^{\alpha+1} D^{\alpha+1} f(t) + t^{\alpha} D^{\alpha} f(t);\ s\Big\} = \frac{\Gamma(1-s)(1-s-\alpha)}{\Gamma(1-s-\alpha)} F(s), \tag{2.299}$$

and putting $\alpha = 1$ gives the well-known property of the Mellin transform:

$$\mathcal{M}\Big\{t^2 f''(t) + t f'(t);\ s\Big\} = s^2 F(s), \tag{2.300}$$

which is often used in applied problems.