Heaps

Data Structures & Algorithms

CONTENT

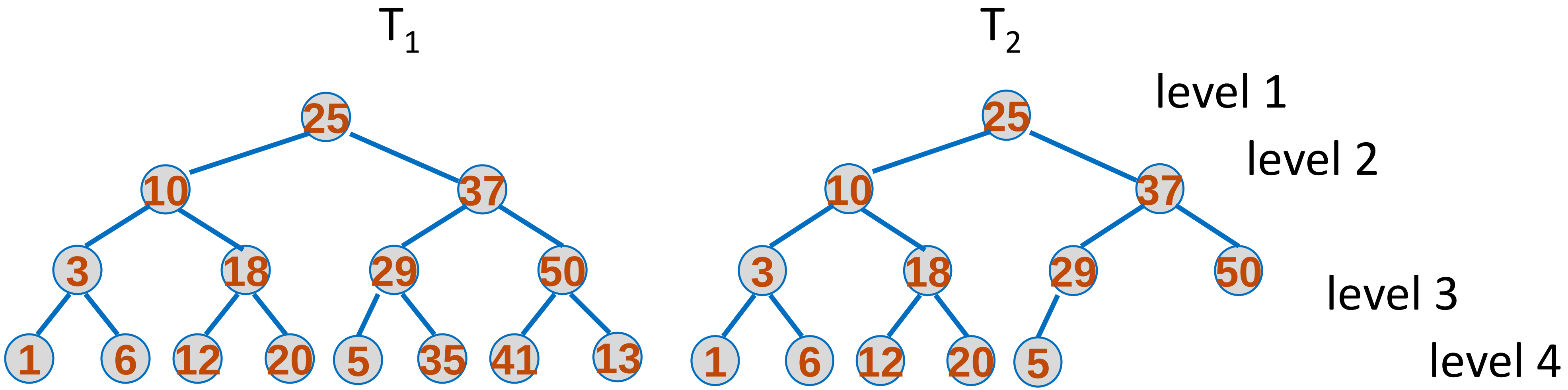
- Introduction
- Basic Operations
- Heap Sort

CONTENT

- **Introduction**
- Basic Operations
- Heap Sort

- **Heap**
  - is a specialized tree-based data structure
  - is an application of complete binary tree (also called **priority queue**)
- **Definition**
  - max/min tree
    - a tree in which the key value in each node is no smaller/greater than the key values in its children (if any)
  - max/min heap
    - a max/min complete binary tree

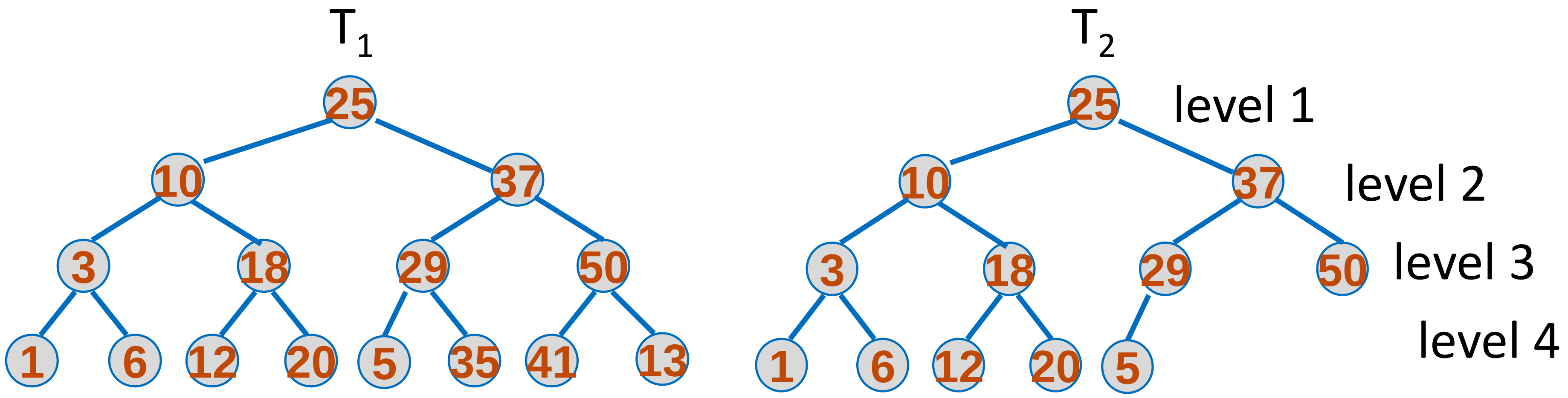
- **Complete binary tree**
  - A complete binary tree is a binary tree that satisfies two properties:
  - First, in a complete binary tree, every level, except possibly the last, is completely filled.
  - Second, all nodes appear as far left as possible.



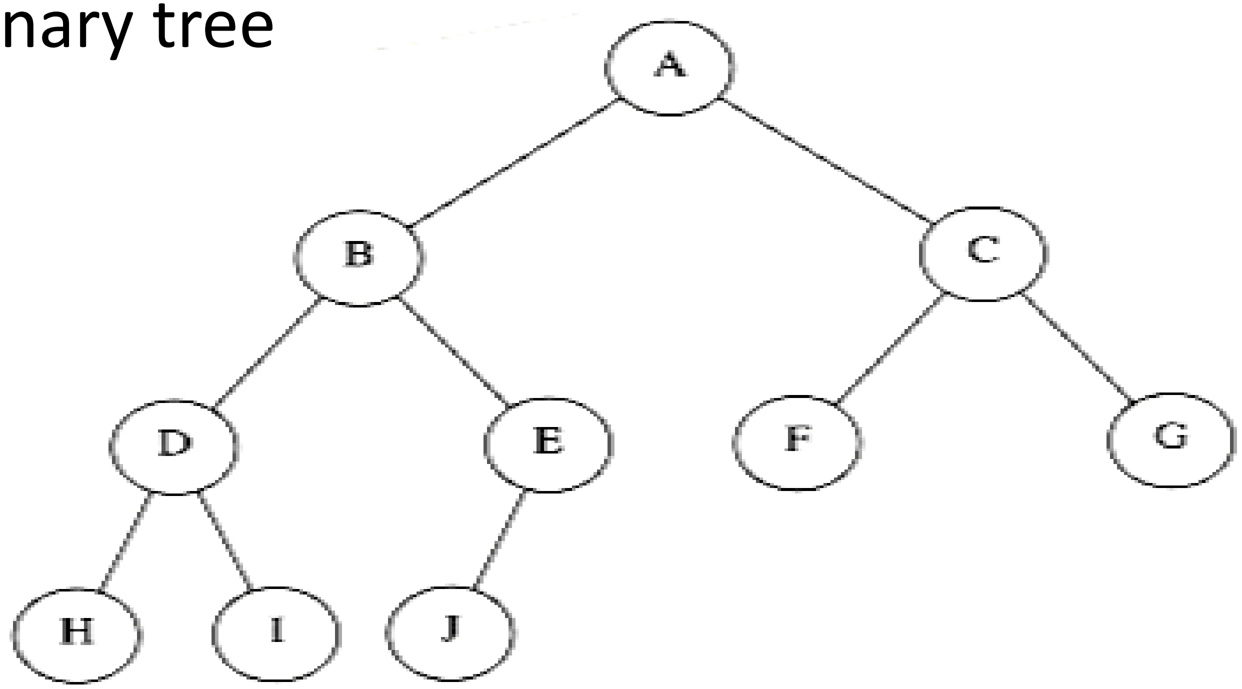
  - In a complete binary tree T, there are exactly n nodes and level r of T can have at most 2^{r-1} nodes.

- **Complete binary tree**
  - Example:

- In trees T_1 and T_2: level 1: 2^{1-1}=1 node, level 2: 2^{2-1}=2 nodes, level 3: 2^{3-1}=4 nodes
- In tree T_1: level 4: 2^{4-1}=8 nodes
- In tree T_2: level 4: 5 nodes (have at most 2^{4-1}=8 nodes)



- **Examples**
  - A complete binary tree



  - Array implementation of the tree

	A	B	C	D	E	F	G	H	I	J			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

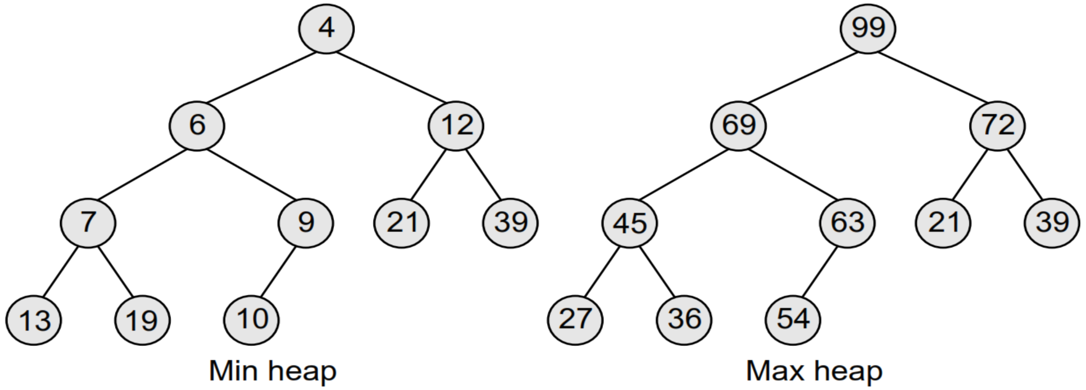
- Binary heap is a **complete binary tree**
- Every node satisfies the heap property:

If B is a child of A, then key(A) ≥ key(B) or key(B) ≥ key(A)

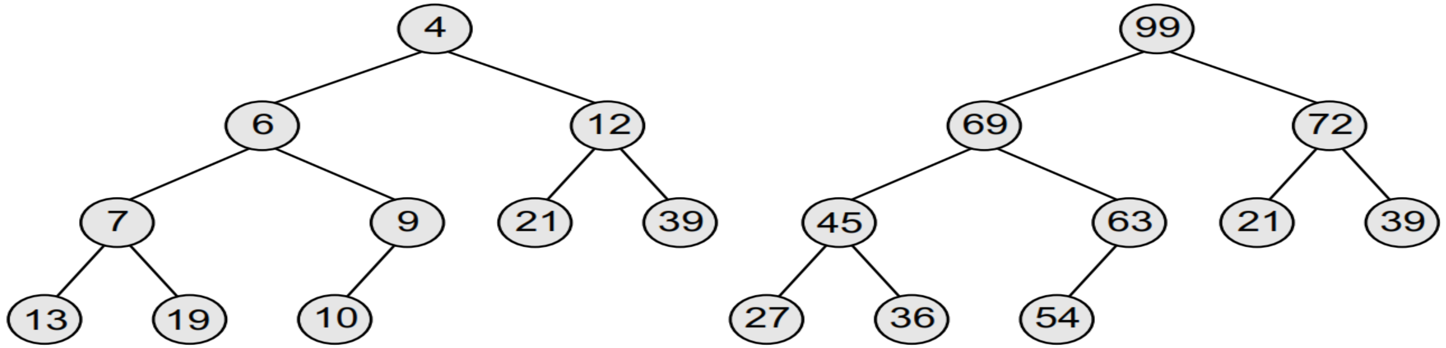
- Alternatively, elements at every node will be either less than or equal to the element at its left and right child.

⇨Thus, the root node has the highest key value in the heap. Such a heap is commonly known as a **max heap** (**key(A) ≥ key(B)**)

⇨Thus, the root node has the lowest key value in the heap. Such a heap is called a **min heap** (**key(B) ≥ key(A)**)



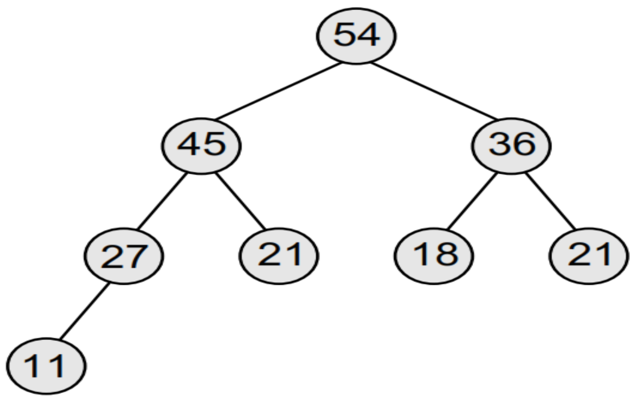
- Based on this criteria, a heap can be of two types
  - **Max Heap**: Where the value of the root node is greater than or equal to either of its children.
  - **Min Heap**: Where the value of the root node is less than or equal to either of its children.



Min heap Max heap

- **Data structure of Binary heap**
  - Using array (the same rules as that of a complete binary tree)
  - Element is at position i in the array (i=1,2,3,…):
    - Left child is stored at position 2i
    - Right child is stored at position 2i+1.
    - Parent is stored at position i/2.
  - Parent A[i] (for array A[1..n], A[1] is the root)
    - Left child: A[2i]
    - Right child: A[2i + 1]

1	54
2	45
3	36
4	27
5	29
6	18
7	21
8	11
9	
10	
11	
12	
13	
14	
15	



- **Data structure of Binary heap**
  - All the levels of the tree except the last level are completely filled.
  - The height of a binary tree is given as $\log_2 n$, (n: number of elements)

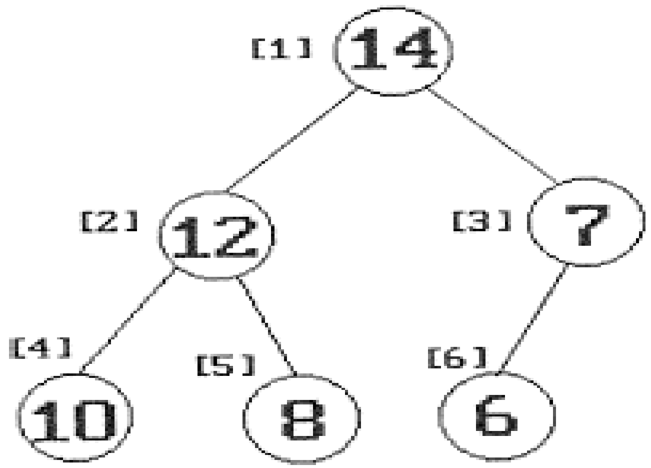
⇨ Heaps are a very popular data structure for implementing priority queues.

⇨ A binary heap is a useful data structure in which elements can be added randomly.

⇨ But only the element with the highest value is removed in case of max heap and lowest value in case of min heap.

- **Heap Representation**
  - Since heaps are complete trees, we may use an array representation

```
#define MAX_ELEMENTS 100
typedef struct {
   int key;
   /* other fields */
} element;
element heap[MAX_ELEMENTS];
int n = 0;
```

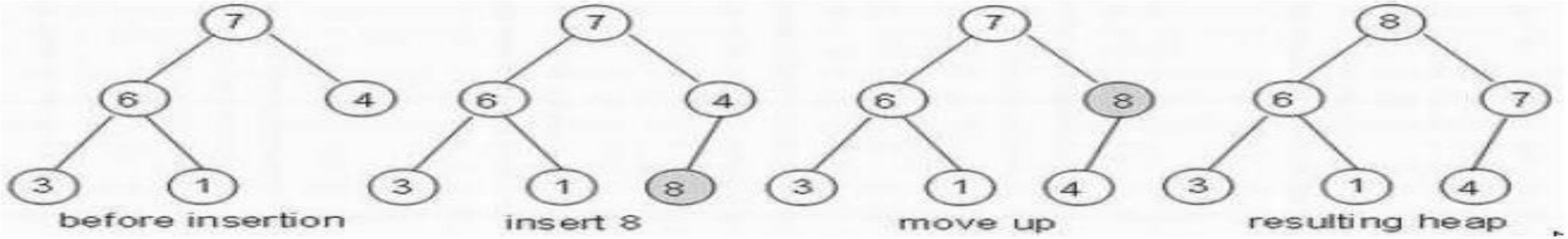


	14	12	7	10	8	6
0	1	2	3	4	5	6

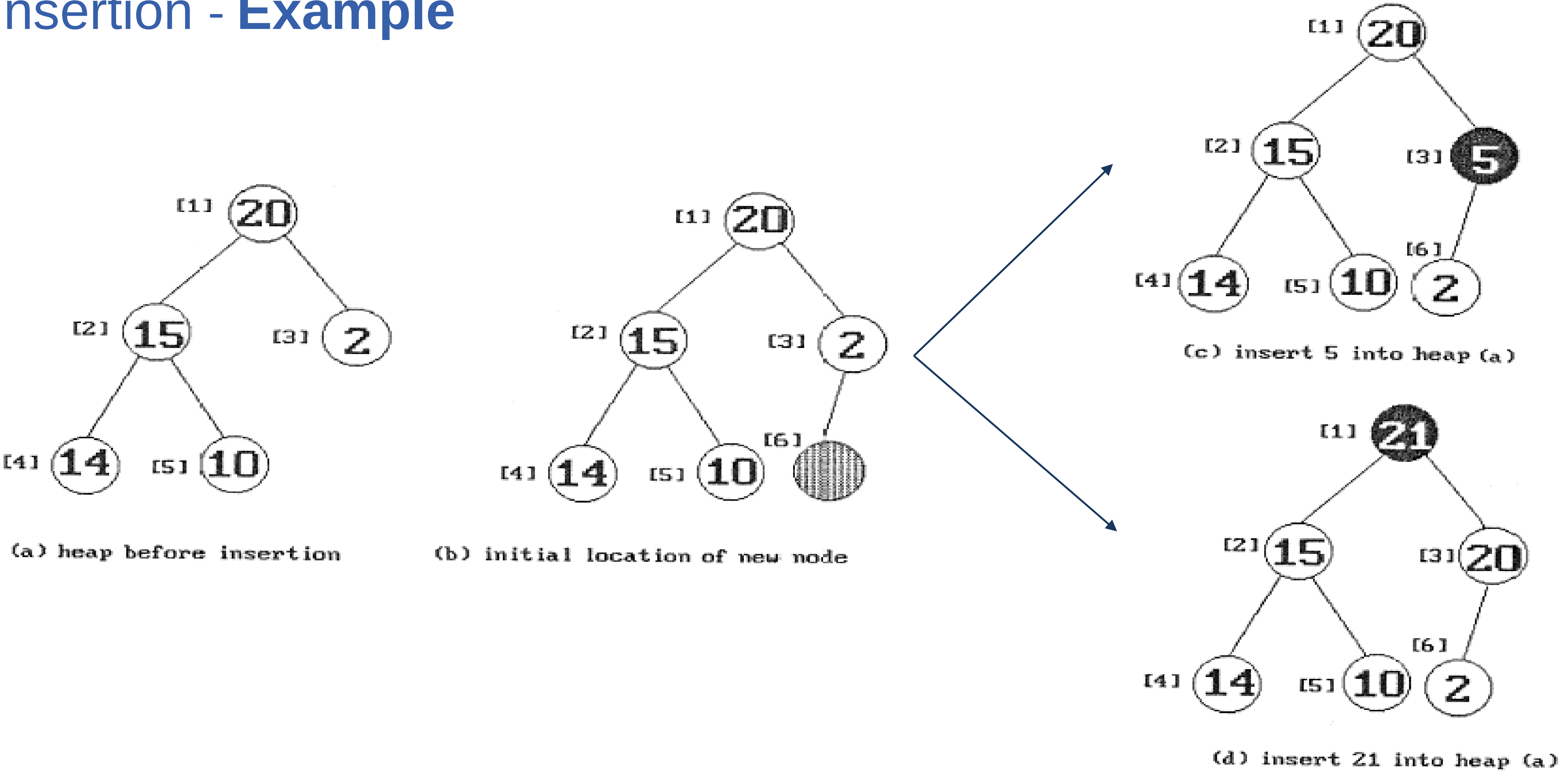
CONTENT

- Introduction
- **Basic Operations**
- Heap Sort

- **Insertion**
  - Consider a max heap **H** with **n** elements
  - Find a proper place for the new element in the array implementation
  - The parent of node **i** is located at **i/2**
    - Step 1: Put the new element at the last entry of the array
    - Step 2: Exchange the new element with its parent, if the new element is greater
    - Step 3: Repeat Step 2 until no more exchange is necessary



- Insertion - **Example**



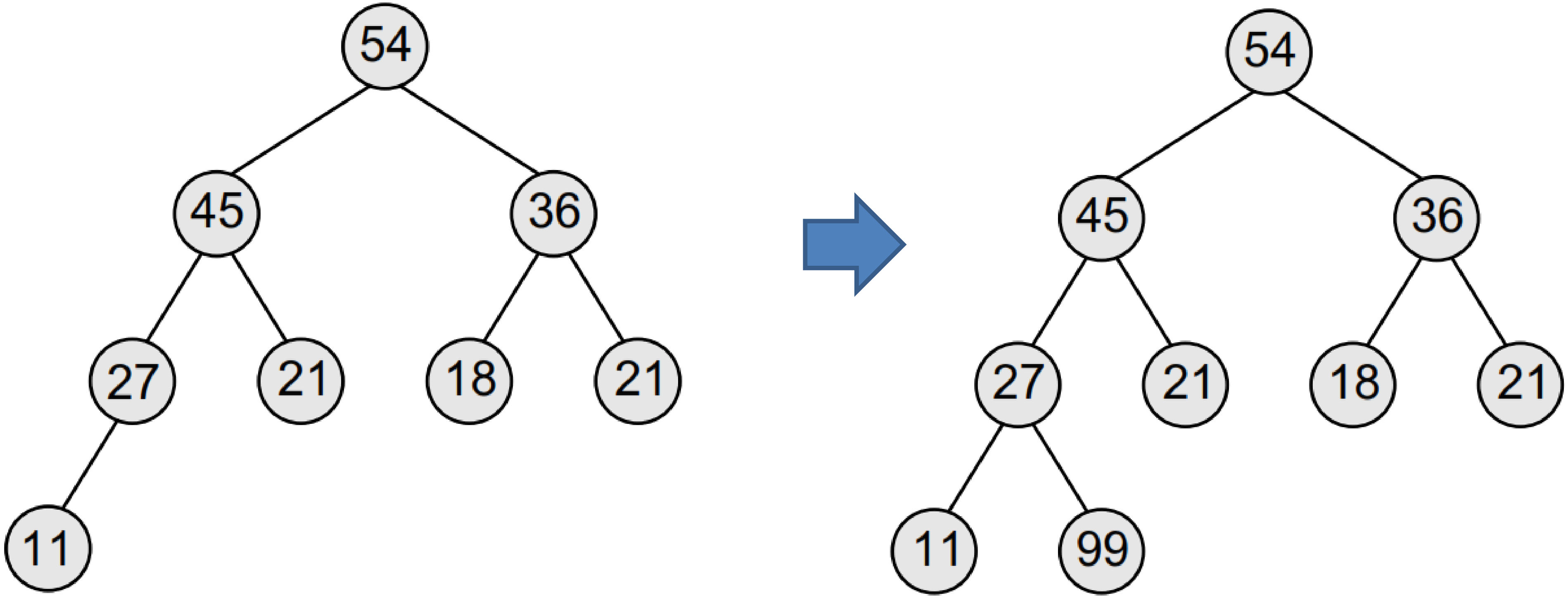
(a) heap before insertion

(b) initial location of new node

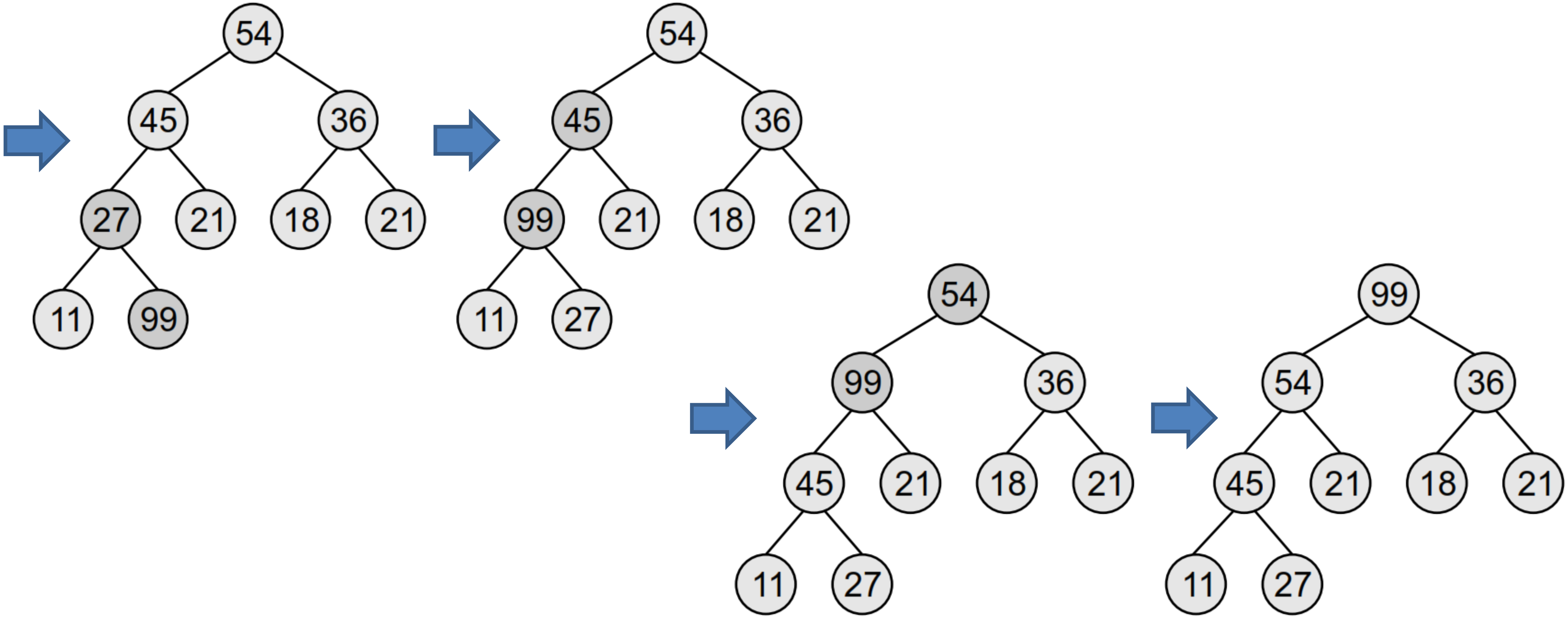
(c) insert 5 into heap (a)

(d) insert 21 into heap (a)

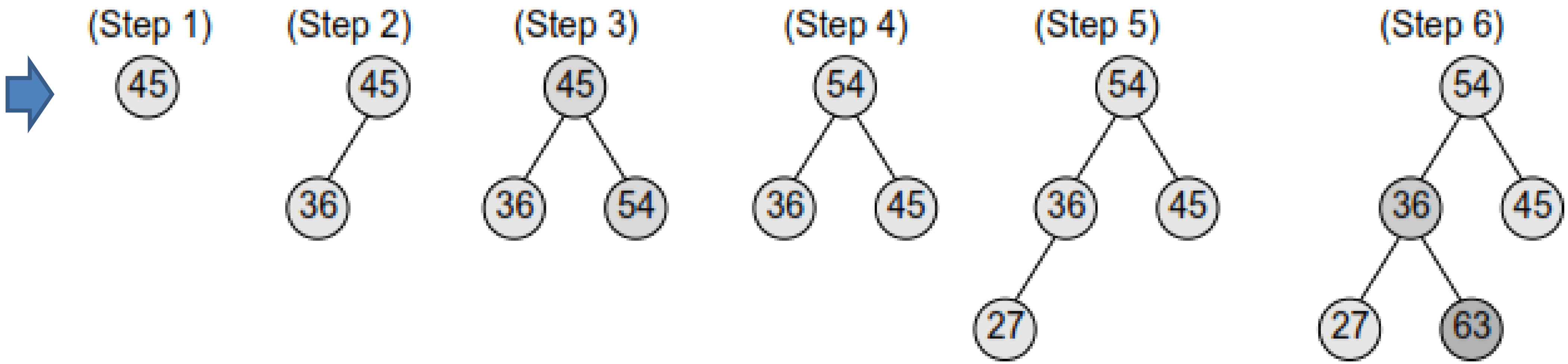
- **Insertion**
  - Example: insert 99 in the max heap
    - **Step 1**: Put the new element at the last entry of the array



- **Insertion**
  - **Step 2, 3**:



- **Insertion**
    - Example: Build a max heap H from the given set of numbers 45, 36, 54, 27, 63, 72, 61, 18.



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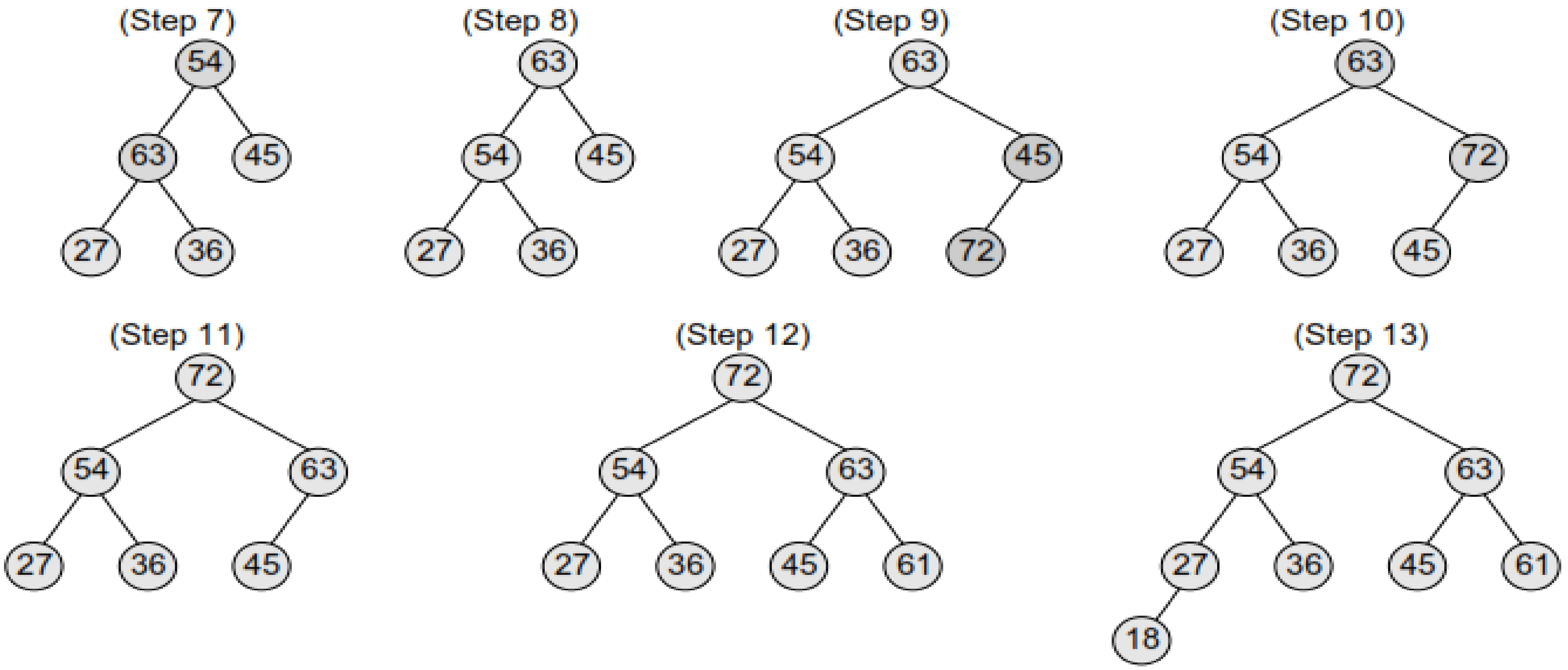
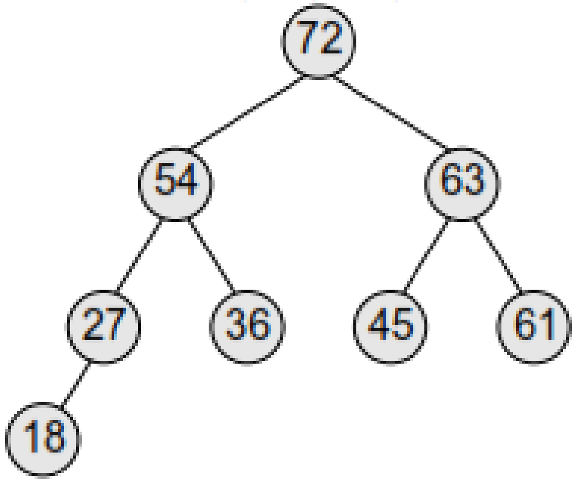
  - Example: Build a max heap H from the given set of numbers 45, 36, 54, 27, 63, 72, 61, 18



- **Insertion**
  - Example: Build a max heap H from the given set of numbers 45, 36, 54, 27, 63, 72, 61, 18.

(Step 13)

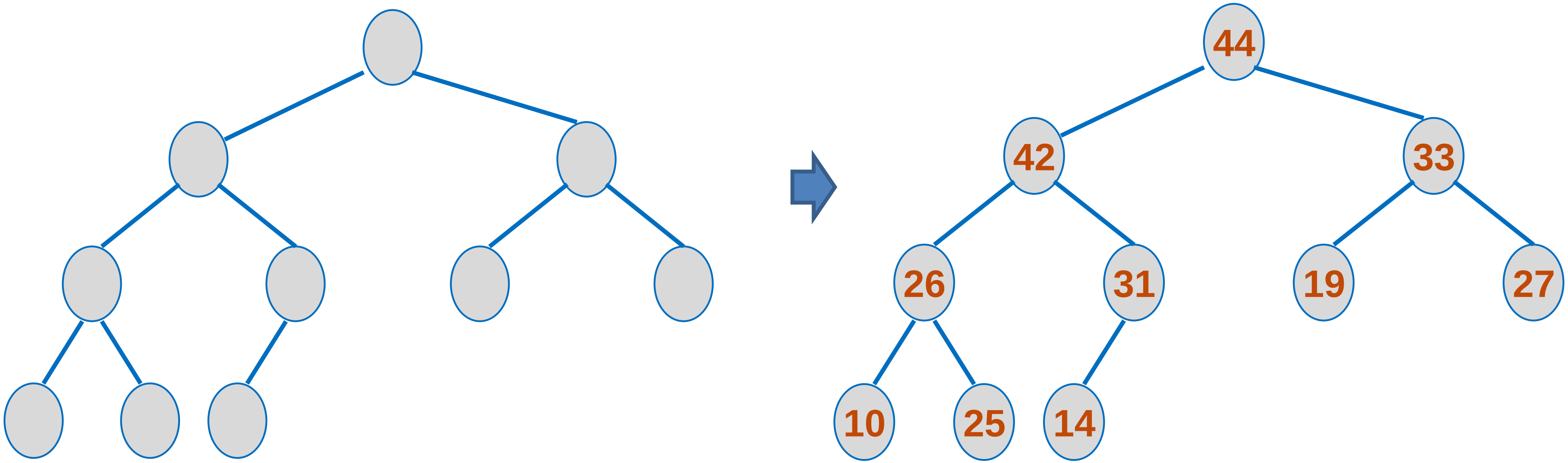


HEAP[1]	HEAP[2]	HEAP[3]	HEAP[4]	HEAP[5]	HEAP[6]	HEAP[7]	HEAP[8]	HEAP[9]	HEAP[10]
72	54	63	27	36	45	61	18		

Memory representation of binary heap H

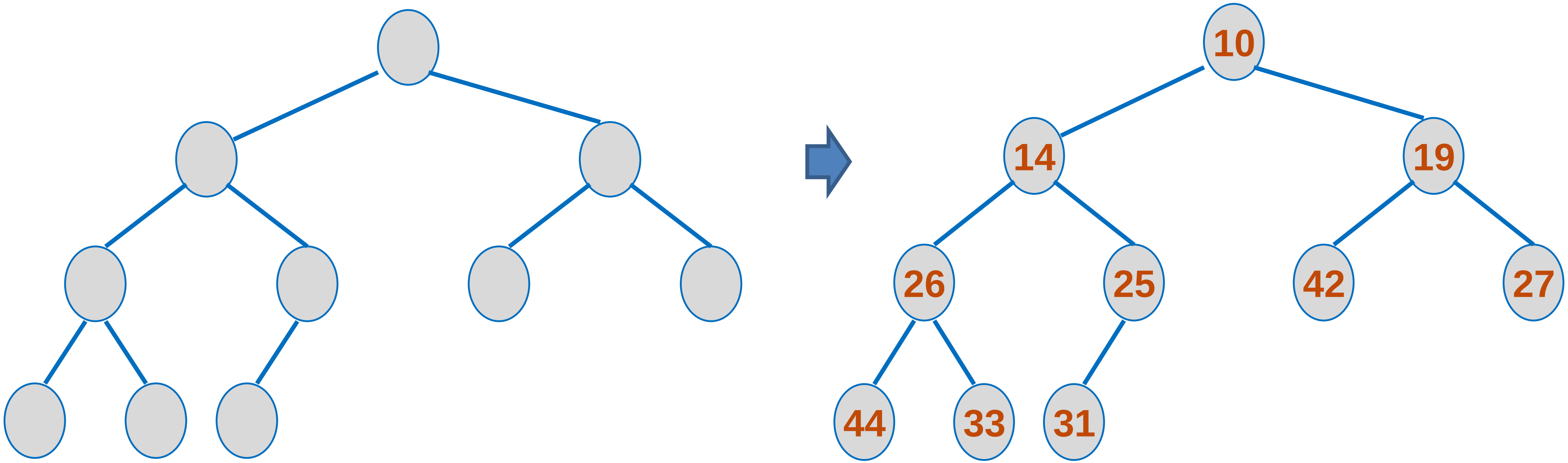
- **Max Heap Construction Algorithm**

25	33	42	10	14	19	27	44	26	31



- **Min Heap Construction Algorithm**

25	33	42	10	14	19	27	44	26	31



- **Max Heap Construction Algorithm**

55	43	40	65	84	20	37	4	22	35	31	25	42

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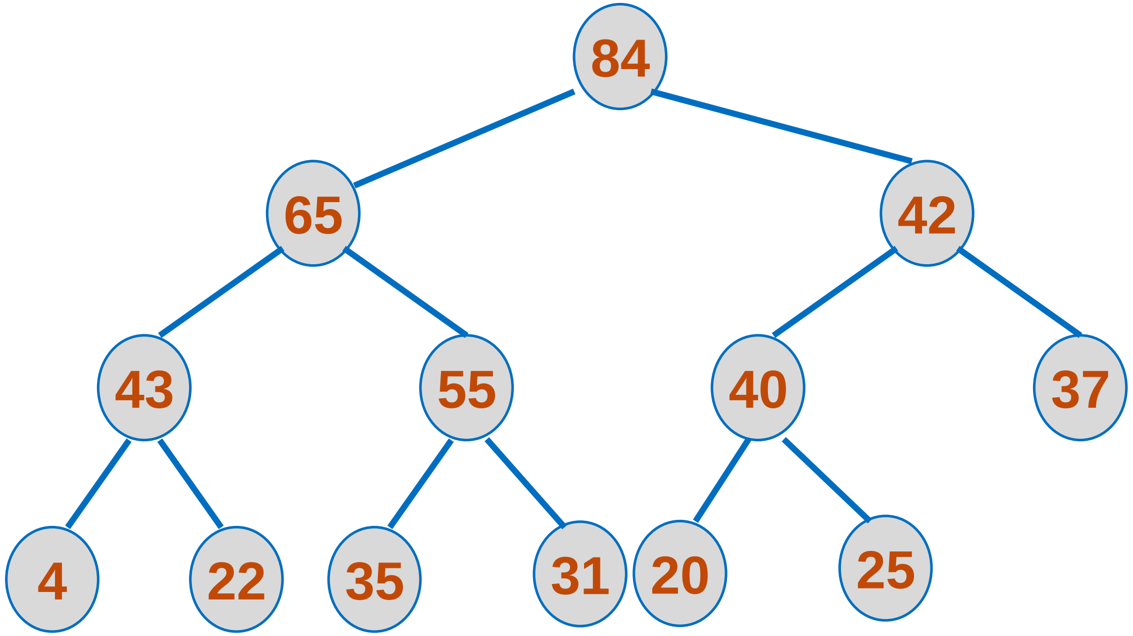
55	43	40	65	84	20	37	4	22	35	31	25	42



- **Min Heap Construction Algorithm**

55	43	40	65	84	20	37	4	22	35	31	25	42

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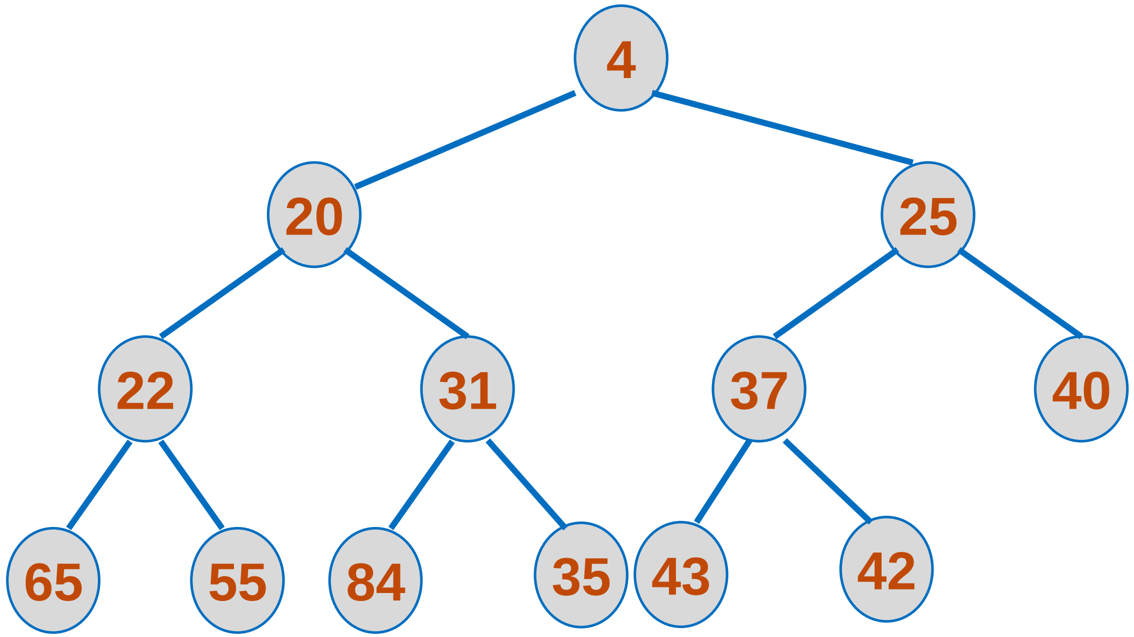
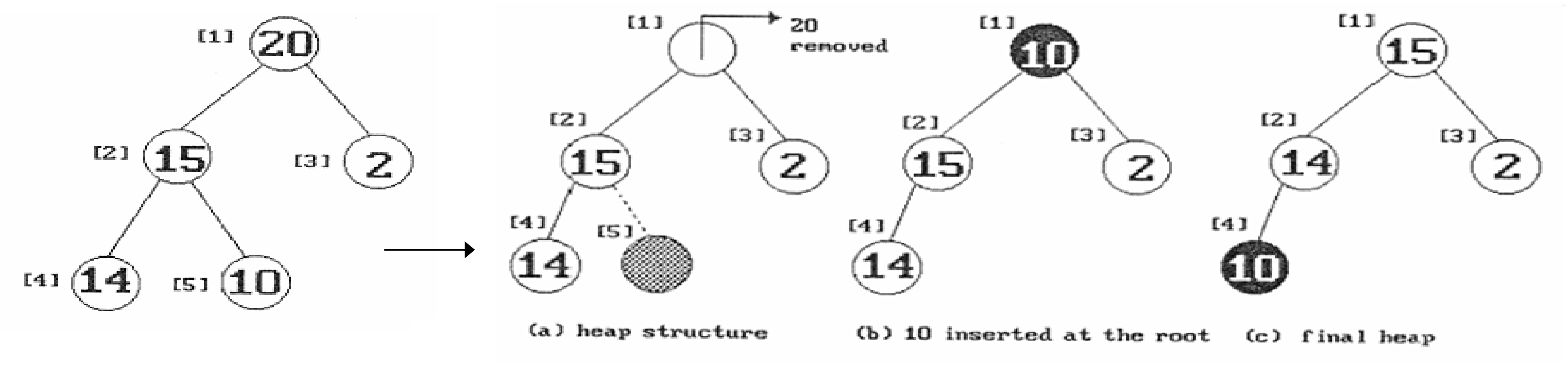


- **Insertion**

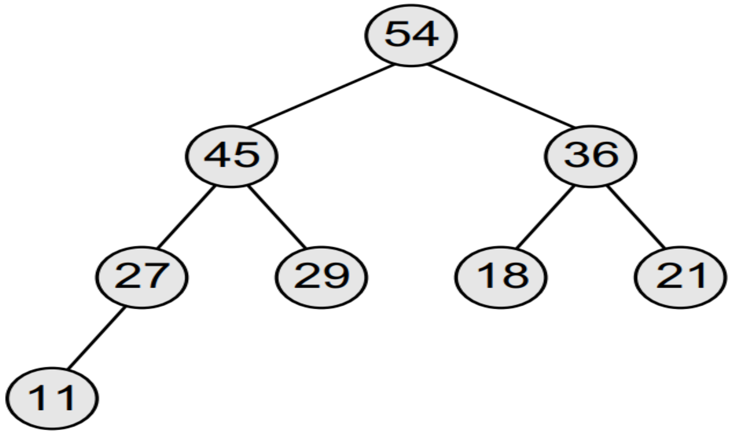
```
void insertMaxHeap(element item, int *n){
        int i;
        if (HEAP_FULL(*n))
                fprintf(stderr, “the heap is full.\n); exit(1);
        i = ++(*n);
        while ((i!=1) && (item.key>heap[i/2].key))
                { heap[i] = heap[i/2];
                  i /= 2;
                }
        heap[i] = item;
}
```

The height of *n* node heap = $\log_2(n+1)$
Time complexity = O (height) = O ($\log_2 n$)

- **Delete** - Delete the max (root) from a max heap
- Consider a max heap **H** with **n** elements (an element is always deleted from the root of heap H)
  - Step 1: Remove the root
  - Step 2: Replace the last element to the root (and delete the last element)
  - Step 3: Reestablish the heap (go down from root to leaf, exchange 2 elements as necessary)

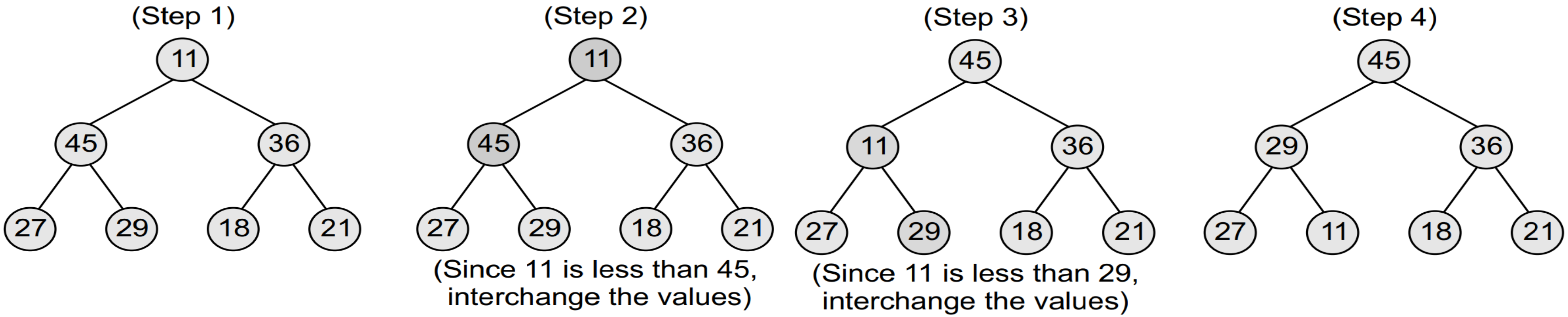


- **Delete**
  - Example: delete the root node's value from the max heap H

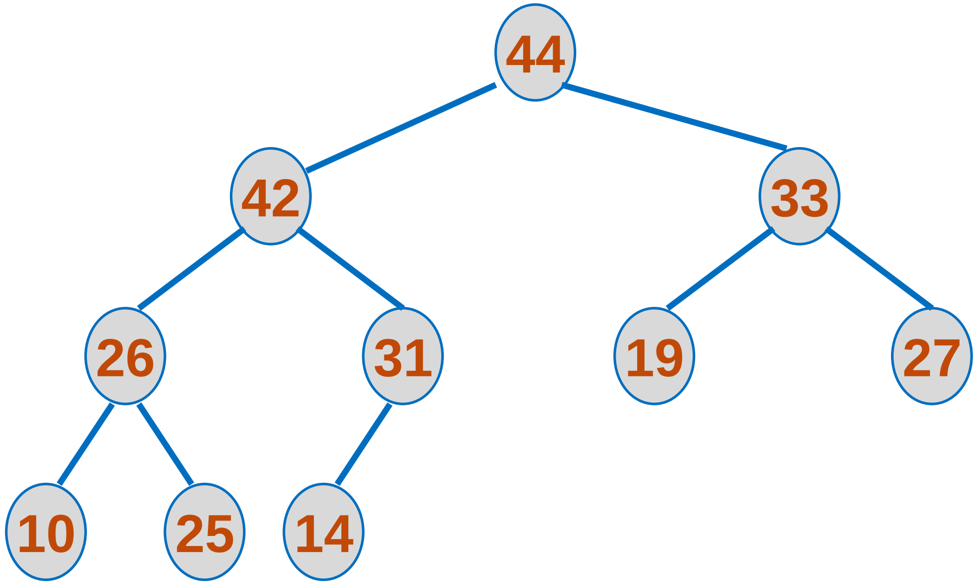


- the value of root node = 54
- the value of last node = 11

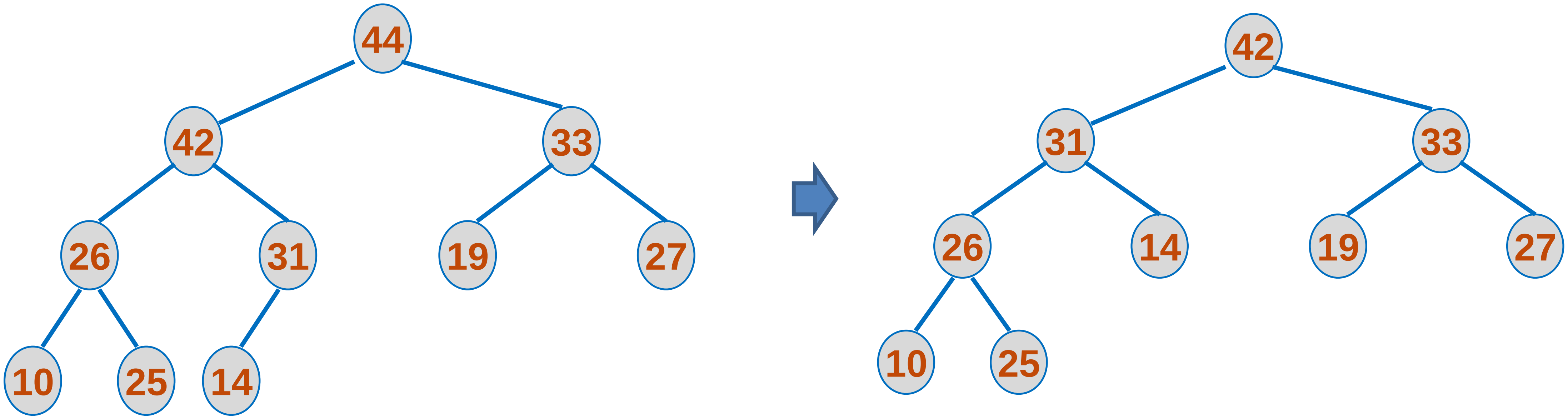
⇨ replace 54 with 11, delete the last node



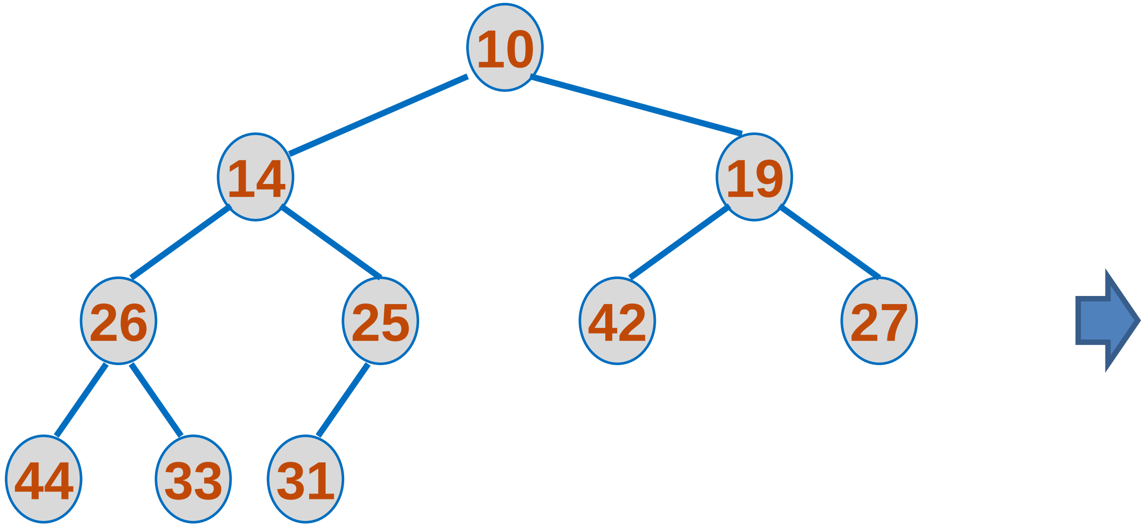
- Example: **Max Heap**



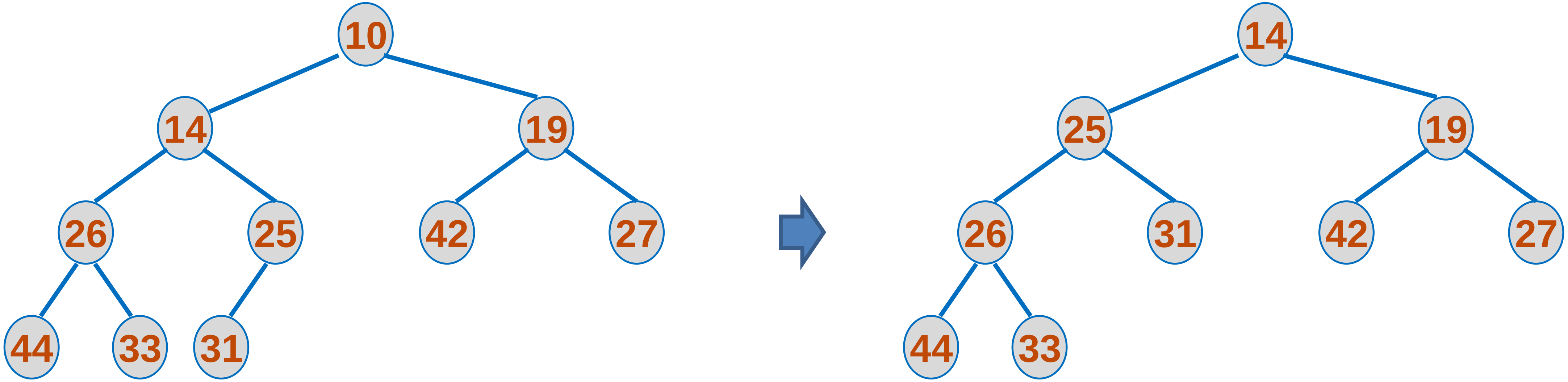
- Example: **Max Heap**



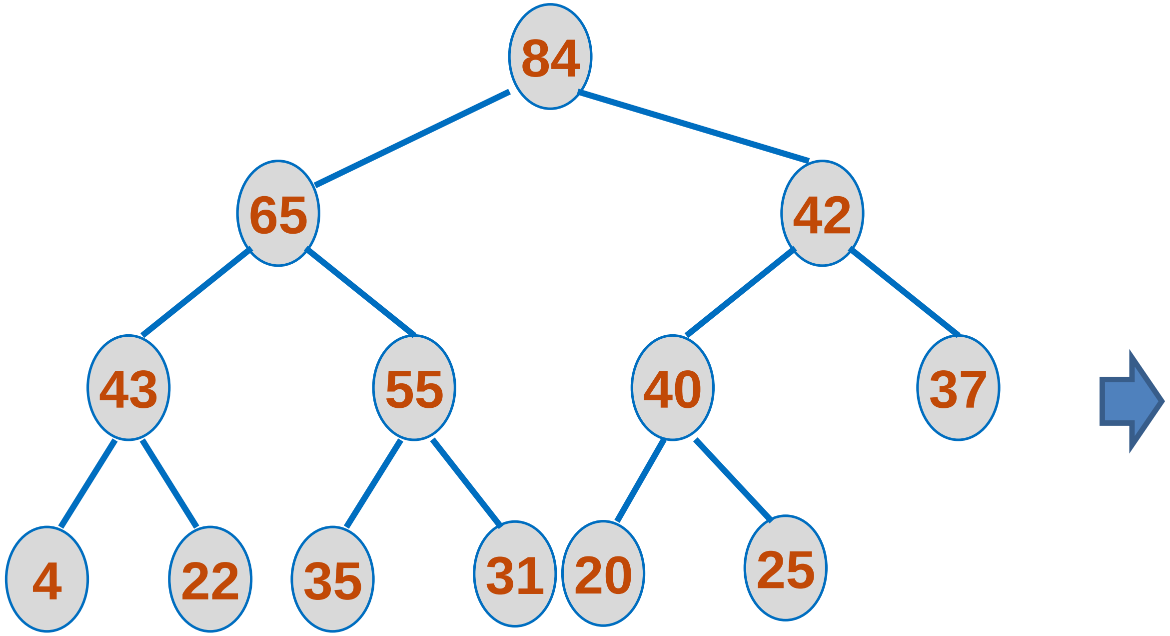
- Example: **Min Heap**



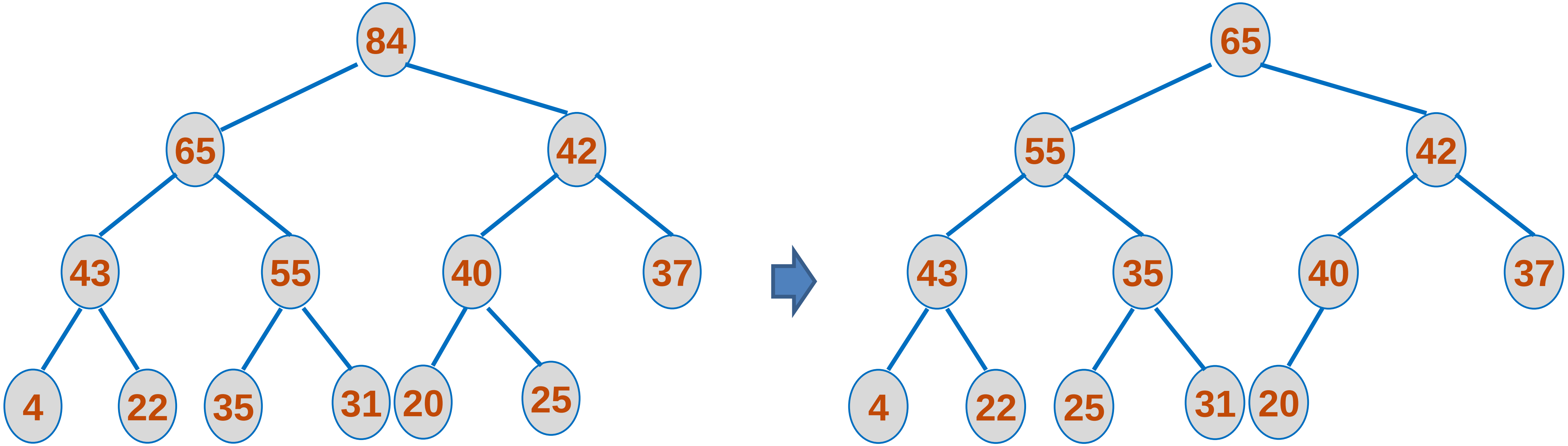
- Example: **Min Heap**



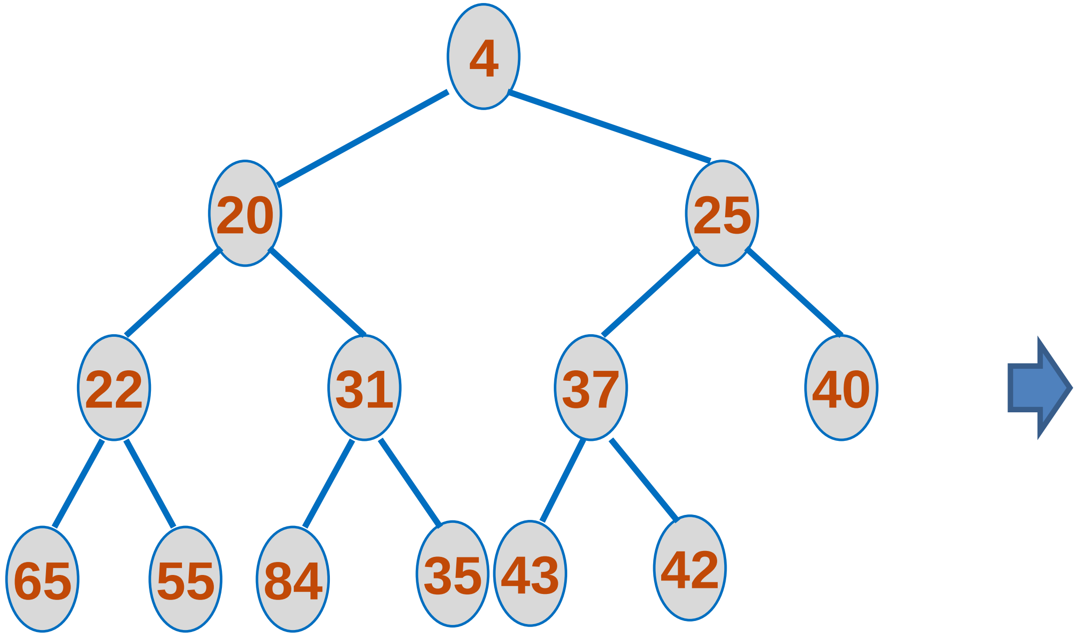
- Example: **Max Heap**



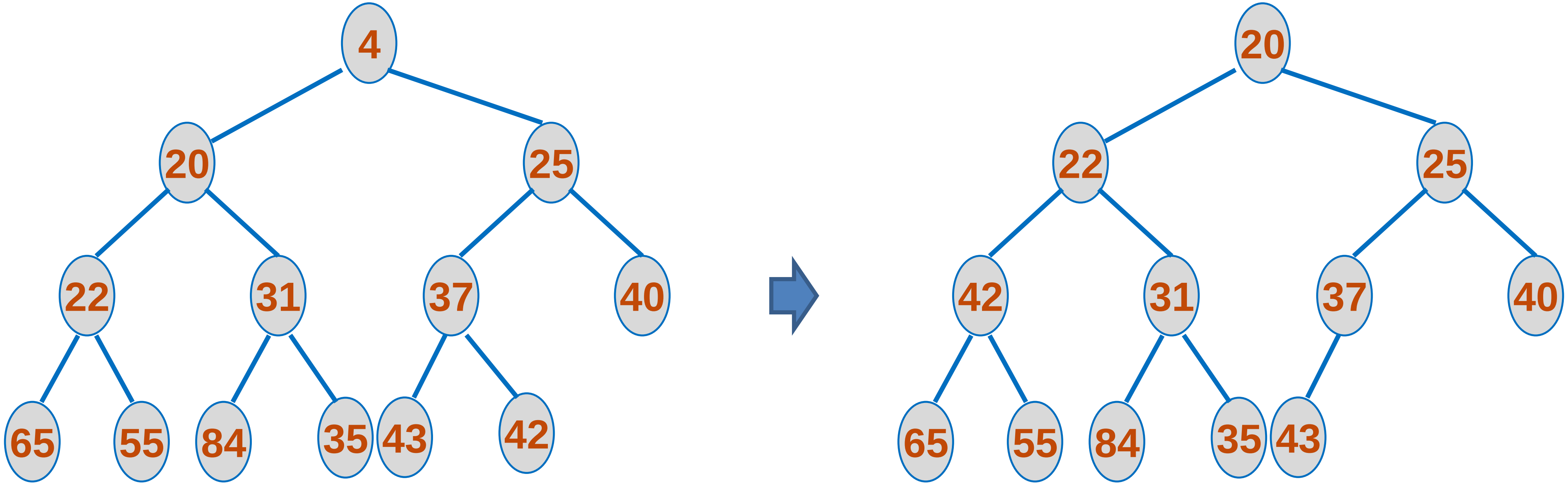
- Example: **Max Heap**



- Example: **Min Heap**



- Example: **Min Heap**



- **Delete** - Delete the max (root) from a max heap

```
element   deleteMaxHeap(int *n){
   int parent, child;  element item, temp;
   if (HEAP_EMPTY(*n)) {
                        fprintf(stderr, "The heap is empty\n");        exit(1);
   }
   item = heap[1];                         /* save value of the element with the highest key */
   temp = heap[(*n)--];                    /* use last element in heap to adjust heap */
   parent = 1;  child = 2;
   while (child <= *n) {                   /* find the larger child of the current parent */
            if ((child < *n) && (heap[child].key<heap[child+1].key))          child++;
            if (temp.key >= heap[child].key)       break;
            heap[parent] = heap[child];            /* move to the next lower level */
            parent= child; child *= 2;
   }
   heap[parent] = temp;
   return  item;
}
```

CONTENT

- Introduction
- Basic Operations
- **Heap Sort**

- **Heap Sort**
  - Given *n* elements (in an array A[1..n]) to be sorted
  - Recall: max heap
    - An array is represented by a complete binary tree, in which the key value in each node is no smaller than the key values in its children (if any)
    - A[1] is the root (suppose the first element of the array is A[1])
    - A[i] is parent, so A[2i] is the left child and A[2i+1] is the right child (if A[0] is the root, so A[2i+1] and A[2i+2]) respectively
  - O(*n* log *n*) time

(1). Build a max heap

- Use function *adjust*(A, i, n)
  - both the left and the right sub-trees of A[i] are already max heaps
  - the element A[i] will be moved to one of its descendant so that the sub-tree rooted at A[i] becomes a max heap
- Function *adjust*() is invoked for the sub-trees rooted at A[n/2],A[n/2-1], …, A[1] in that order (i.e. all the non-leaf nodes)

(2). Sort by using the heap

(a). A[1..n] is a heap, exchange A[1] & A[n] -> A[n] is rightly position

(b). Rebuild a max heap for A[1..n-1]. Repeat steps (a) & (b) until array has only one element

- **Example**

6 5 3 1 8 7 2 4

- **(1).Build a max heap - Use function *adjust*(A, i, n)**
  - both the left and the right sub-trees of A[i] are already max heaps
  - the element A[i] will be moved to one of its descendant so that the sub-tree rooted at A[i] becomes a max heap

```
void adjust(int list[], int root, int n) {
        int child, rootkey; int temp;
        temp = list[root]; rootkey = list[root].key; child = 2*root;
        while (child <= n) {
                if ((child<n) && (list[child].key<list[child+1].key))     child++;
                if (rootkey > list[child].key)     break;
                else {   list[child/2] = list[child];       child *= 2; }
        }
        list[child/2] = temp;
}
```

- **(2).Sort by using heap**

```
void heap_sort(int list[], int n) {
     /* Initially data is in list[1.. n] */
    int i, j;
    /* build a max heap */
    for (i = n/2; i > 0; i--)   adjust(list, i, n);
    /* at this point we have a max heap */
    for (i = n-1; i > 0; i--) {
            SWAP(list[1], list[i+1]);  /* swap the root & element at pos. i+1*/
            adjust(list, 1, i);          /* rebuild list from element 1 to i */
    }
}
```

- Example

25	33	42	10	14	19	27	44	26	31

- Example

55	43	40	65	84	20	37	4	22	35	31	25	42

- How to sort the list in descending order?

- Example

25	33	42	10	14	19	27	44	26	31

- Example

55	43	40	65	84	20	37	4	22	35	31	25	42

SUMMARY

- Introduction
- Basic Operations
- Heap Sort

Nhân bản – Phụng sự – Khai phóng

Enjoy the Course…!