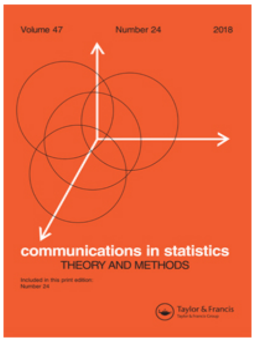

Taylor & Francis
Taylor & Francis Group

Communications in Statistics—Theory and Methods

ISSN: 0361-0926 (Print) 1532-415X (Online) Journal homepage: www.tandfonline.com/journals/lsta20

Gluing Copulas

Karl Friedrich Siburg & Pavel A. Stoimenov

To cite this article: Karl Friedrich Siburg & Pavel A. Stoimenov (2008) Gluing Copulas, Communications in Statistics—Theory and Methods, 37:19, 3124-3134, DOI: 10.1080/03610920802074844

To link to this article: https://doi.org/10.1080/03610920802074844

Communications in Statistics—Theory and Methods, 37: 3124–3134, 2008

ISSN: 0361-0926 print/1532-415X online
DOI: 10.1080/03610920802074844

Multivariate Analysis

Gluing Copulas

KARL FRIEDRICH SIBURG[1] AND
PAVEL A. STOIMENOV[2]

[1]Fakultät für Mathematik, Technische Universität Dortmund, Dortmund, Germany
[2]Fakultät für Statistik, Technische Universität Dortmund, Dortmund, Germany

We present a new way of constructing n-copulas, by scaling and gluing finitely many n-copulas. Gluing for bivariate copulas produces a copula that coincides with the independence copula on some grid of horizontal and vertical sections. Examples illustrate how gluing can be applied to build complicated copulas from simple ones. Finally, we investigate the analytical as well as statistical properties of the copulas obtained by gluing, in particular, the behavior of Spearman's ρ and Kendall's τ.

Keywords Construction of copulas; Copulas; Horizontal section; Kendall's tau; Spearman's rho; Vertical section.

Mathematics Subject Classification 60E05; 62H05.

1. Introduction

Let I be the closed unit interval [0,1].

Definition 1.1. For any integer $n \geq 2$, an n-dimensional copula (or n-copula) is a function $C : I^n \to I$ with the following properties:

(C1) $C(x_1, \ldots, x_k, \ldots, x_n) = 0$ when $x_k = 0$ for some $k = 1, \ldots, n$.
(C2) $C(1, \ldots, 1, x_k, 1, \ldots, 1) = x_k$ for all $k = 1, \ldots, n$ and for all $x_k \in I$.
(C3) C is n-increasing, i.e., for all n-boxes $B = \times_{k=1}^{n}[c_k, d_k] \subseteq I^n$ we have:

$$V_C(B) := \sum_{v \in B} \operatorname{sgn}(v)\, C(v) \geq 0$$

Received August 28, 2007; Accepted March 24, 2008
Address correspondence to Karl Friedrich Siburg, Fakultät für Mathematik, Technische Universität Dortmund, Vogelpothsweg 87, Dortmund 44227, Germany; E-mail: karl.f.siburg@math.tu-dortmund.de

where the sum is taken over all vertices $v = (v_1, \ldots, v_n)$ of B, with $v_k \in \{c_k, d_k\}$ for each $k = 1, \ldots, n$, and $\operatorname{sgn}(v)$ is defined to be 1 if $v_k = c_k$ for an even number of k's, and -1 otherwise.

It can be shown that for any copula C and for all $(x_1, \ldots, x_n) \in I^n$:

$$C^-(x_1, \ldots, x_n) \leq C(x_1, \ldots, x_n) \leq C^+(x_1, \ldots, x_n), \tag{1}$$

where C^- and C^+ are the so-called Fréchet–Hoeffding bounds given by:

$$C^-(x_1, \ldots, x_n) = \max(x_1 + \cdots + x_n - n + 1, 0),$$
$$C^+(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n).$$

The upper bound C^+ is a copula itself for all $n \geq 2$, whereas the lower bound C^- is a copula only for $n = 2$. Another distinguished copula is the product copula

$$P(x_1, \ldots, x_n) = x_1 \ldots x_n.$$

The importance of copulas to the theory of statistics stems from the well-known Sklar's theorem (see Schweizer and Sklar, 2005; Sklar, 1996, 1959), which states that for all real-valued random variables $X_1, \ldots, X_n$ with joint distribution function H and univariate margins $F_1, \ldots, F_n$ there exists an n-copula C such that:

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{2}$$

Conversely, given an n-copula C and univariate distribution functions $F_1, \ldots, F_n$, then the function H defined by (2) is an n-dimensional distribution function with univariate margins $F_1, \ldots, F_n$. Moreover, if $F_1, \ldots, F_n$ are all continuous, then C is unique and is called the copula of $X_1, \ldots, X_n$; otherwise, C is uniquely determined on $\operatorname{Range} F_1 \times \cdots \times \operatorname{Range} F_n$. In view of Sklar's theorem, a rich collection of n-copulas yields a rich collection of n-dimensional joint distribution functions with arbitrary margins, which proves useful in statistical modeling and simulation. Furthermore, the decomposition of the joint distribution function in (2) implies that copulas represent the dependence structure of random variables, irrespective of their distributions. Thus, as noted in Nelsen (2006), the study of concepts and measures of nonparametric dependence is equivalent to the study of properties of copulas and, therefore, it is useful to have a variety of copulas at our disposal.

Several general methods of constructing copulas exist. Among them are the inversion method, geometric methods (e.g., ordinal sums, shuffles of min, and copulas with prescribed horizontal, vertical, or diagonal sections), algebraic methods (e.g., a copula transformation), and methods based on generators, leading to the large class of Archimedean copulas; for details we refer to Nelsen (2006). For more recent constructions see, e.g., Alfonsi and Brigo (2005), De Baets and De Meyer (2007), Durante et al. (2007), Erdely and González-Barrios (2006), Klement et al. (2007), and the references therein. Note, however, that most of the existing construction methods apply to the bivariate case only and have not been generalized to the n-dimensional situation. To quote from Nelsen (2006), "constructing n-copulas is difficult."

In this article, we present a new method for constructing copulas in any dimension, the so-called gluing construction. In its simplest form, gluing with respect to the first variable x_1 proceeds as follows. Given two n-copulas C_1 and C_2 and a number $\theta \in (0, 1)$, the graphs of C_1 and C_2 are scaled and pasted into the boxes $[0, \theta] \times I^{n-1}$ and $[\theta, 1] \times I^{n-1}$, respectively, i.e., they are glued together along the hyperplane $\{x_1 = \theta\}$. Of course, this construction can be carried out with respect to any other variable, and with finitely many copulas. Finally, successive gluing with respect to different variables leads to the gluing method in its most general form. The copulas obtained by gluing exhibit several interesting statistical properties. For instance, the gluing of two bivariate copulas produces a copula which has the same section as the independence copula. Furthermore, we deduce formulas for the behavior of Spearman's ρ and Kendall's τ under gluing.

The article is organized as follows. Section 2 introduces the general gluing construction for N-copulas and illustrates the method by examples. In Sec. 3, we discuss analytical and statistical properties of copulas obtained by gluing.

2. The Gluing Construction

2.1. *Gluing Two Copulas*

For the sake of clarity, we illustrate the gluing construction in its most basic form. Consider two copulas C_1, C_2 on I^n with $n \geq 2$. Fix any index $i \in \{1, \ldots, n\}$ and real number $\theta \in (0, 1)$, and partition the unit cube as:

$$I^n = (I \times \cdots \times [0, \theta] \times \cdots \times I) \cup (I \times \cdots \times [\theta, 1] \times \cdots \times I).$$

Then we define the function

$$C_1 \underset{x_i=\theta}{\circledast} C_2 : I^n \to I$$

by setting

$$\left(C_1 \underset{x_i=\theta}{\circledast} C_2\right)(x_1, \ldots, x_i, \ldots, x_n) = \theta C_1\left(x_1, \ldots, \frac{x_i}{\theta}, \ldots, x_n\right) \tag{3}$$

if $0 \leq x_i \leq \theta$, and

$$\begin{aligned}\left(C_1 \underset{x_i=\theta}{\circledast} C_2\right)&(x_1, \ldots, x_i, \ldots, x_n)\\ &= (1-\theta)C_2\left(x_1, \ldots, \frac{x_i - \theta}{1-\theta}, \ldots, x_n\right) + \theta C_1(x_1, \ldots, 1, \ldots, x_n)\end{aligned} \tag{4}$$

if $\theta \leq x_i \leq 1$. Thus, $C_1 \circledast_{x_i=\theta} C_2$ can be seen as the result of gluing C_1 and C_2 along the section $\{x_i = \theta\}$. We claim that it is indeed a copula.

Theorem 2.1. *For any two n-copulas C_1, C_2, any index $i \in \{1, \ldots, n\}$, and any number $\theta \in (0, 1)$, the function $C_1 \circledast_{x_i=\theta} C_2$ is an n-copula.*

Proof. We show that $C = C_1 \circledast_{x_i=\theta} C_2$ satisfies the axioms (C1)–(C3) from Definition 1.1. Indeed, since C_1 and C_2 satisfy (C1), the same follows for C.

Similarly, it is easy to check that, if all but one variable x_k are equal to 1, then $C(1, \ldots, x_k, \ldots, 1) = x_k$; one just has to distinguish the cases $k = i$ and $k \neq i$.

Hence, the only condition to be checked is (C3), i.e., that $V_C(B) \geq 0$ for all n-boxes $B \subseteq I^n$. Since the volume V_f is additive for every function f, i.e., $V_f(B) = V_f(B_1) + V_f(B_2)$ if $B = B_1 \cup B_2$ where B_1 and B_2 have disjoint interior, we may restrict ourselves to the case where B is contained in either $I^n \cap \{x_i \leq \theta\}$ or $I^n \cap \{x_i \geq \theta\}$. Now the claim follows from the observation that the right sides of (3) and (4) are n-increasing. This finishes the proof of the theorem. □

Let us illustrate the gluing construction in the simplest case of bivariate copulas. A bivariate copula C is called singular if $\partial^2 C/\partial x_1 \partial x_2$ vanishes almost everywhere in I^2; in this case, the support of C has Lebesgue measure zero in I^2. We refer the reader to Nelsen (2006) for more details.

Example 2.1. Let $\theta \in (0, 1)$, and suppose that probability mass θ is uniformly distributed along the line segment joining $(0, 0)$ and $(\theta, 1)$, and probability mass $1 - \theta$ is uniformly distributed along the segment between $(\theta, 1)$ and $(1, 0)$. Consider the resulting singular copula C whose support consists of these two line segments; see Fig. 1. It follows (see Nelsen, 2006, Example 3.3) that:

$$C(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 \leq \theta x_2 \\ \theta x_2 & \text{if } \theta x_2 < x_1 < 1 - (1-\theta)x_2 \\ x_1 + x_2 - 1 & \text{if } 1 - (1-\theta)x_2 \leq x_1. \end{cases} \tag{5}$$

This copula is a standard example of a singular copula. In terms of gluing, C can be written as:

$$C = C^+ \underset{x_1=\theta}{\circledast} C^-$$

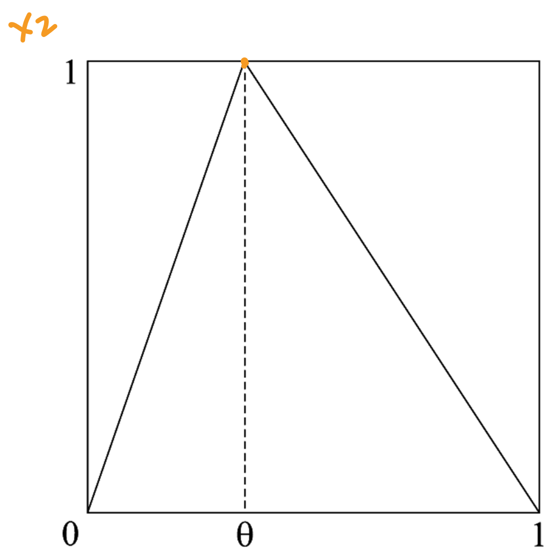

Figure 1. The support of the singular copula C in Example 2.1.

where $C^+(x_1, x_2) = \min(x_1, x_2)$ and $C^-(x_1, x_2) = \max(x_1 + x_2 - 1, 0)$ is the Fréchet–Hoeffding upper and lower bound, respectively.

$\theta \min\left(\frac{x_1}{\theta}, x_2\right)$

2.2. *Gluing Many Copulas*

In the following, we introduce the gluing method for the general case of finitely many copulas. Note that this can also be realized by sequentially gluing two copulas as described in the previous section. Fix any $i \in \{1, \ldots, n\}$ and numbers θ_k such that $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$, and let $C_1, \ldots, C_N$ be n-copulas. Then we consider the partition

$$I^n = \bigcup_{k=1}^{N} I \times \cdots \times [\theta_{k-1}, \theta_k] \times \cdots \times I$$

and define the function $\circledast_{x_i=\theta_k} C_k : I^n \to I$ by:

$$\Big(\underset{x_i=\theta_k}{\circledast} C_k\Big)(x_1, \ldots, x_i, \ldots, x_n)$$
$$= (\theta_k - \theta_{k-1})C_k\left(x_1, \ldots, \frac{x_i - \theta_{k-1}}{\theta_k - \theta_{k-1}}, \ldots, x_n\right) + \theta_{k-1}C_{k-1}(x_1, \ldots, 1, \ldots, x_n) \quad (6)$$

if $x_i \in [\theta_{k-1}, \theta_k]$ with $1 \leq k \leq N$; note that the formal term involving C_0 is irrelevant since $\theta_0 = 0$. Then the same arguments as in the proof of Theorem 2.1 show the following result.

Theorem 2.2. *The function* $\circledast_{x_i=\theta_k} C_k : I^n \to I$ *is an* n*-copula.*

Finally, we may combine gluings in different variables, resulting in the most general gluing construction of gluing finitely many copulas in each variable x_i, $1 \leq i \leq n$. We illustrate this with the simplest nontrivial example, the gluing of four bivariate copulas.

Example 2.2. Given four bivariate copulas $C_1, \ldots, C_4$ and three numbers $\theta_1, \theta_2, \theta_3 \in (0, 1)$, let us first glue C_1 and C_2 along $\{x_1 = \theta_1\}$, and C_3 and C_4 along $\{x_1 = \theta_2\}$. Then we glue these two copulas along $\{x_2 = \theta_3\}$. This results in the copula

$$C = \Big(C_3 \underset{x_1=\theta_2}{\circledast} C_4\Big) \underset{x_2=\theta_3}{\circledast} \Big(C_1 \underset{x_1=\theta_1}{\circledast} C_2\Big) \quad (7)$$

which is represented by the partition of I^2 outlined in Fig. 2. If we consider the new copula C on the rectangle $[0, \theta_2] \times [0, \theta_3]$, for instance, we obtain:

$$C(x_1, x_2) = \theta_2\theta_3\, C_3\left(\frac{x_1}{\theta_2}, \frac{x_2}{\theta_3}\right). \quad (8)$$

As an example, the reader might want to visualize the copulas

$$\Big(C^- \underset{x_1=1/2}{\circledast} C^+\Big) \underset{x_2=1/2}{\circledast} \Big(C^+ \underset{x_1=1/2}{\circledast} C^-\Big)$$

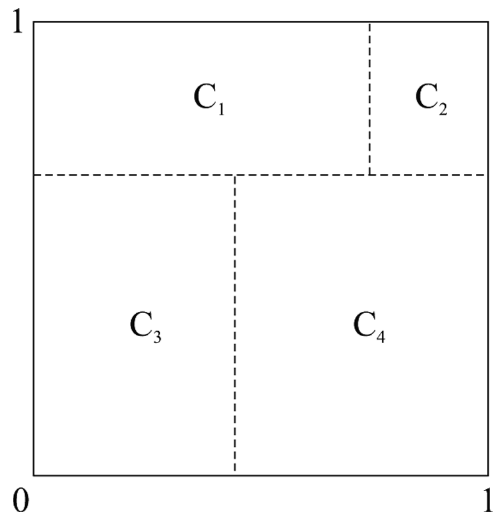

Figure 2. The copula C from Example 2.2.

and

$$\left(C^{+} \underset{x_1=1/2}{\circledast} C^{-}\right) \underset{x_2=1/2}{\circledast} \left(C^{-} \underset{x_1=1/2}{\circledast} C^{+}\right)$$

by drawing their supports.

Note that in the special case when $\theta_1 = \theta_2 = \theta$ there are two different ways of representing the copula C from (7) as a gluing, namely:

$$C = \left(C_3 \underset{x_1=\theta}{\circledast} C_4\right) \underset{x_2=\theta_3}{\circledast} \left(C_1 \underset{x_1=\theta}{\circledast} C_2\right) = \left(C_3 \underset{x_2=\theta_3}{\circledast} C_1\right) \underset{x_1=\theta}{\circledast} \left(C_4 \underset{x_2=\theta_3}{\circledast} C_2\right). \tag{9}$$

Remark 2.1 (Bivariate Orthogonal Grid Construction)**.** It turns out that, in two dimensions, the gluing construction coincides with the so-called orthogonal grid construction with P as background copula, which has been developed in De Baets and De Meyer (2007). Indeed, the copula Q in De Baets and De Meyer (2007, Prop. 12) agrees with the bivariate gluing $C_1 \circledast_{x_1=\theta} C_2$ in the special case when $(u_1, u_1') = (0, \theta)$, $(u_2, u_2') = (\theta, 1)$, and $(v_1, v_1') = (0, 1)$.

3. Properties of Copulas Obtained by Gluing

3.1. *Analytical Properties*

It follows from the definition that the rescaling in the gluing construction is made in such a way that the relative volume is preserved. Suppose, for instance, that C is obtained by gluing a copula $\widetilde{C}$ into the rectangle $[0, \theta_1] \times [0, \theta_2]$. Then (8) implies

$$C(x_1, x_2) = \theta_1 \theta_2 \, \widetilde{C}\left(\frac{x_1}{\theta_1}, \frac{x_2}{\theta_2}\right)$$

on $[0, \theta_1] \times [0, \theta_2]$ so that we have:

$$V_C(\theta_1 I_1 \times \theta_2 I_2) = \theta_1 \theta_2 V_{\tilde{C}}(I_1 \times I_2) \tag{10}$$

for any closed intervals $I_1, I_2 \subseteq I$. Note that this quadratic scaling distinguishes the gluing construction from the well known ordinal sum construction (see, e.g., Nelsen, 2006) which has linear scaling.

A special role in the gluing construction is played by the independence copula $P(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n$. Note that $P \circledast_{x_i=\theta} C = P$ for $x_i \in [0, \theta]$, and $C \circledast_{x_i=\theta} P = P$ for $x_i \in [\theta, 1]$, respectively. In particular, gluing P with itself yields P again:

$$P \underset{x_i=\theta}{\circledast} P = P. \tag{11}$$

In fact, this is the only possibility to obtain P as a gluing in view of the following result.

Proposition 3.1. *For any two copulas C_1, C_2, any index $i \in \{1, \ldots, n\}$, and any number $\theta \in (0, 1)$ the following holds*:

$$C_1 \underset{x_i=\theta}{\circledast} C_2 = P \Leftrightarrow C_1 = C_2 = P.$$

Proof. In view of (11), we only need to show "$\Rightarrow$". But $C_1 \circledast_{x_i=\theta} C_2 = P$ implies $\theta C_1(x_1/\theta, x_2) = x_1 x_2$ for all $(x_1, x_2) \in [0, \theta] \times I$, which is equivalent to $C_1 = P$. Analogously, we see that $C_2 = P$. □

A bivariate copula C is called absolutely continuous if, considered as a joint distribution function, it has a joint density which is given by $\partial^2 C/\partial x_1 \partial x_2$. C is singular if $\partial^2 C/\partial x_1 \partial x_2 = 0$ almost everywhere on I^2. It is clear from the construction that both properties are preserved under arbitrary gluings.

Finally, a bivariate copula C is called symmetric if $C(x_1, x_2) = C(x_2, x_1)$ for all $(x_1, x_2) \in I^2$. In general, the gluing of two symmetric copulas is not symmetric anymore, as the simple copula (5) from Example 2.1 shows. On the other hand, gluing two non-symmetric copulas may yield a symmetric copula; this is illustrated by the following example.

Example 3.1. We come back to Example 2.2 and consider the two non symmetric copulas $C_1 = C^- \circledast_{x_2=1/2} C^+$ and $C_2 = C^+ \circledast_{x_2=1/2} C^-$. Then it follows from (9) that:

$$C = C_1 \underset{x_1=1/2}{\circledast} C_2 = \left(C^- \underset{x_1=1/2}{\circledast} C^+\right) \underset{x_2=1/2}{\circledast} \left(C^+ \underset{x_1=1/2}{\circledast} C^-\right),$$

and this copula is symmetric since

$$C(x_1, x_2) = \begin{cases} \max((x_1 + x_2)/2 - 1/4, 0) & \text{if } (x_1, x_2) \in [0, 1/2]^2 \\ \min((x_1 + x_2)/2 - 1/4, x_1) & \text{if } (x_1, x_2) \in [0, 1/2] \times [1/2, 1] \\ \min((x_1 + x_2)/2 - 1/4, x_2) & \text{if } (x_1, x_2) \in [1/2, 1] \times [0, 1/2] \\ \max((x_1 + x_2)/2 - 1/4, x_1 + x_2 - 1) & \text{if } (x_1, x_2) \in [1/2, 1]^2 \end{cases}.$$

The gluing of two bivariate copulas C_1 and C_2 along $\{x_1 = \theta\}$ produces a copula $C = C_1 \circledast_{x_1=\theta} C_2$ with the property that:

$$C(\theta, x_2) = \theta x_2 = P(\theta, x_2). \tag{12}$$

This means that C coincides on the vertical $\{x_1 = \theta\}$ with the independence copula P, which one might express by saying that C has a "vertical section of independence". Since a copula can be viewed as the joint distribution function of random variables with uniform distributions on I, the vertical and horizontal sections of a copula have the following statistical interpretation. If X_1 and X_2 are random variables uniformly distributed on I, the sections are proportional to conditional distribution functions; see Nelsen (2006). If C coincides with P on $\{x_1 = \theta\}$, then for $x_1 = \theta$ the conditional probability agrees with the unconditional one, i.e.,

$$\mathbb{P}(X_2 \leq x_2 \mid X_1 \leq \theta) = \frac{C(\theta, x_2)}{\theta} = x_2 = \mathbb{P}(X_2 \leq x_2).$$

The question arises if, conversely, any copula C satisfying (12) can be obtained by gluing. The affirmative answer is given in the following representation theorem.

Theorem 3.1. *Let C be a copula on I^2. Then the following are equivalent*:

1. *There is $\theta \in (0, 1)$ such that $C(\theta, x_2) = \theta x_2$ for all $x_2 \in I$*;
2. *There are two copulas C_1 and C_2 such that $C = C_1 \circledast_{x_1=\theta} C_2$*.

Proof. If $C = C_1 \circledast_{x_1=\theta} C_2$ then $C(\theta, x_2) = \theta x_2$. Conversely, we define two functions $C_1, C_2 : I^2 \to I$ by setting

$$C_1(x_1, x_2) = \frac{1}{\theta} C(\theta x_1, x_2), \quad C_2(x_1, x_2) = \frac{1}{1-\theta}(C(\theta + (1-\theta)x_1, x_2) - \theta x_2).$$

Then, since C is a copula with $C(\theta, x_2) = \theta x_2$, it is straightforward to check that also C_1 and C_2 are copulas, i.e., they satisfy the axioms (C1)–(C3) from Definition 1.1. Moreover, in view of (3) and (4), we have $C = C_1 \circledast_{x_1=\theta} C_2$. □

Note that this theorem yields *all* copulas with the given vertical section θx_2 at the point $x_1 = \theta$. In particular, we immediately obtain the precise upper and lower bounds for those copulas.

Corollary 3.1. *Let C be a copula with $C(\theta, x_2) = \theta x_2$ for all $x_2 \in I$. Then*

$$\left(C^- \underset{x_1=\theta}{\circledast} C^-\right) \leq C \leq \left(C^+ \underset{x_1=\theta}{\circledast} C^+\right),$$

and all three copulas coincide on $\{x_1 = \theta\}$.

Note that an analogous result has been proved in Klement et al. (2007). Finally, we point out that these results can be generalized, by gluing finitely many copulas in each variable, to copulas coinciding with P on a given grid in I^2. Note that this grid may be more general than just a Cartesian product of line segments, as demonstrated in Fig. 2.

3.2. Statistical Properties

In this final section we investigate statistical properties of bivariate copulas obtained by gluing. In particular, we are interested in how classical measures of concordance, e.g., Spearman's ρ and Kendall's τ, behave under gluing. Also, we consider the effect of gluing on tail dependence.

If C is copula on I^2 then Spearman's ρ is given by:

$$\rho_C = 12\int_0^1\int_0^1 C(x_1, x_2)dx_1\,dx_2 - 3 \tag{13}$$

and Kendall's τ by:

$$\tau_C = 1 - 4\int_0^1\int_0^1 \partial_1 C(x_1, x_2)\partial_2 C(x_1, x_2)dx_1\,dx_2, \tag{14}$$

where $\partial_k C$ for $k = 1, 2$ denotes the partial derivative of C with respect to the kth variable. We refer to Nelsen (2006) for the original definitions and more details.

Theorem 3.2. *Let $C = C_1 \circledast_{x_1=\theta} C_2$ be a copula obtained by gluing two copulas C_1 and C_2 along $\{x_1 = \theta\}$. Then Spearman's ρ and Kendall's τ can be written as:*

$$\rho_C = F(\rho_{C_1}, \rho_{C_2}) \quad \text{and} \quad \tau_C = F(\tau_{C_1}, \tau_{C_2})$$

where $F(x, y) = \theta^2 x + (1-\theta)^2 y$.

Proof. In order to prove the formula for Spearman's ρ, we recall the gluing formulae (3) and (4) and calculate:

$$\begin{aligned}
&\int_0^1\int_0^1 C(x_1, x_2)dx_1\,dx_2 \\
&\quad = \theta\int_0^1\int_0^\theta C_1\left(\frac{x_1}{\theta}, x_2\right)dx_1\,dx_2 + (1-\theta)\int_0^1\int_\theta^1 C_2\left(\frac{x_1-\theta}{1-\theta}, x_2\right)dx_1\,dx_2 \\
&\quad\quad + \theta\int_0^1\int_\theta^1 x_2 dx_1\,dx_2 \\
&\quad = \theta^2\int_0^1\int_0^1 C_1(z, x_2)dz\,dx_2 + (1-\theta)^2\int_0^1\int_0^1 C_2(z, x_2)dz\,dx_2 + \frac{1}{2}\theta(1-\theta).
\end{aligned}$$

Using (13), it is easy to see that the constants involving θ cancel each other, which finally yields the result for ρ_C.

A completely analogous calculation proves the statement for τ_C. □

We immediately obtain the possible ranges for Spearman's ρ and Kendall's τ.

Corollary 3.2. *If $C = C_1 \circledast_{x_1=\theta} C_2$ then:*

$$\rho_C, \tau_C \in [-1 + 2\theta(1-\theta), 1 - 2\theta(1-\theta)].$$

Proof. As measures of concordance, Spearman's ρ and Kendall's τ respect the concordance ordering on the set of bivariate copulas; see Nelsen (2006). Therefore, the claim follows from Theorem 3.2 and Corollary 3.2, together with the fact that $\rho_{C^-} = \tau_{C^-} = -1$ and $\rho_{C^+} = \tau_{C^+} = 1$. $\square$

Example 3.2. By gluing two copulas one can obtain any value for ρ and τ, respectively. Indeed, consider the singular copula

$$C = C^+ \underset{x_1=\theta}{\circledast} C^-$$

from Example 2.1. Theorem 3.2 shows immediately that

$$\rho_C = \tau_C = 2\theta - 1,$$

in accordance with the calculations suggested in Nelsen (2006, Exer. 5.6).

Finally, we turn to a different dependence concept, the so-called tail dependence, which measures the dependence in the tails of a joint distribution. More precisely, given a copula C, its lower-tail dependence parameter is defined as:

$$\lambda_C = \lim_{t \searrow 0} \frac{C(t,t)}{t} \tag{15}$$

whenever this limit exists; see Nelsen (2006). It does not make sense to study the tail dependence of a copula $C = C_1 \circledast_{x_1=\theta} C_2$ in the general case. However, if we construct C by gluing copulas in such a way that the copula C_1, say, is fit into the square $[0, \theta]^2$ then:

$$\lambda_C = \theta^2 \lim_{t \searrow 0} \frac{C_1\left(\frac{t}{\theta}, \frac{t}{\theta}\right)}{t} = \theta \lim_{u \searrow 0} \frac{C_1(u,u)}{u} = \theta \lambda_{C_1}. \tag{16}$$

Consequently, the lower-tail dependence parameter scales with θ and tends to zero if the square $[0, \theta]^2$ is made smaller and smaller. This phenomenon is due to the fact that gluing scales quadratically, and not linearly, as described in (10).

Example 3.3. Let $\alpha, \beta \in I$ with $\alpha + \beta \le 1$, and consider the Fréchet copula $C_{\alpha,\beta} = \alpha C^+ + (1 - \alpha - \beta)P + \beta C^-$; then $\lambda_{C_{\alpha,\beta}} = \alpha$. It follows that the copula $C = (C_{\alpha,\beta} \circledast_{x_1=\theta} P) \circledast_{x_2=\theta} P$ satisfies

$$C(x_1, x_2) = \begin{cases} \theta^2 C_{\alpha,\beta}\left(\dfrac{x_1}{\theta}, \dfrac{x_2}{\theta}\right) & \text{if } (x_1, x_2) \in [0, \theta]^2 \\ x_1 x_2 & \text{otherwise} \end{cases}. \tag{17}$$

In view of (16), the copula C has lower-tail dependence parameter $\lambda_C = \theta\alpha$. For instance, if $\alpha = 1/2$, $\beta = 1/3$, and $\theta = 1/10$, then $\lambda_C = 1/20$.

Acknowledgments

We are grateful to Peter Becker-Kern for interesting discussions, and to one of the anonymous referees for useful comments and remarks. The second author gratefully acknowledges financial support from the German Research Council (SFB 475, Reduction of Complexity in Multivariate Data Structures).

References

Alfonsi, A., Brigo, D. (2005). New families of copulas based on periodic functions. *Commun. Statist. Theor. Meth.* 34(7):1437–1447.

De Baets, B., De Meyer, H. (2007). Orthogonal grid construction of copulas. *IEEE Trans. Fuzzy Syst.* 15(6):1053–1062.

Durante, F., Kolesárová, A., Mesiar, R., Sempi, C. (2007). Copulas with given values on a horizontal and a vertical section. *Kybernetika* 43(2):209–220.

Erdely, A., González-Barrios, J. M. (2006). On the construction of families of absolutely continuous copulas with given restrictions. *Commun. Statist. Theor. Meth.* 35(4–6):649–659.

Klement, E. P., Kolesárová, A., Mesiar, R., Sempi, C. (2007). Copulas constructed from horizontal sections. *Commun. Statist. Theor. Meth.* 36(16):2901–2911.

Nelsen, R. B. (2006). *An Introduction to Copulas*. 2nd ed. New York: Springer.

Schweizer, B., Sklar, A. (2005). *Probabilistic Metric Spaces*. New York: Dover Publications.

Sklar, A. (1996). Random variables, distribution functions, and copulas—a personal look backward and forward. In: Rüschendorf, L., Schweizer, B., Taylor, M. D., eds. *Distributions with Fixed Marginals and Related Topics*. Hayward, CA: Institute of Mathematical Statistics, pp. 1–14.

Sklar, M. (1959). Fonctions de répartition à *n* dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* 8:229–231.