Model Categories by Example

Lecture 1

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Logistics

- Supported by the LMS in conjunction with the TTT (Transpennine Topology Triangle).
- Lectures will be recorded + available to watch online at a later date.
- There are notes to go along with the course at **bifibrant.com**.
- If you have a question please type it in chat.
- Keep yourself muted!

What to expect

- To come out with a working knowledge of model categories.
- A collection of tools for constructing model categories.
- Plenty of examples (many of which I hope to be non-standard)!
- No proofs (but references to be found in lecture notes).
- No $(\infty, 1)$-categories.
- Subscripts on categories indicate a model structure.

Motivation

* Homotopy theory $\rightsquigarrow$ localizations of categories
* $\mathcal{C}$ a category $W \subset \mathrm{Mor}(\mathcal{C})$
* would like to form $\mathcal{C}[W^{-1}]$ such that maps in W are isos
* $\gamma : \mathcal{C} \to \mathcal{C}[W^{-1}]$ initial among all functors sending W to isos.

Canonical examples

$\mathcal{C} = \mathrm{Top}$ $\quad W =$ weak homotopy equivalences

(homotopy)

$\mathcal{C} = \mathrm{Ch}_{\geq 0}(R)$ $\quad W =$ quasi-isomorphisms $\quad H_n(-)$ isos

(homology)

$\mathcal{C} = \mathrm{Cat}$ $\quad W =$ equivalence of categories

(fully faith. ess. surj)

A construction of $\mathcal{C}[W^{-1}]$

* $\mathcal{C}[W^{-1}]$ is the category w/ objects those of $\mathcal{C}$

* Morphisms are zig-zags $f : X \to Y$ in $\mathcal{C}[W^{-1}]$

w^{-1}

$$\begin{tikzcd}
 & Y \arrow[dl, "w_1"'] \arrow[dr] & & \cdot \arrow[dl, "w_2"'] & \cdots & Y_n \arrow[dr] & \\
X & & Y_1' & & & & Y
\end{tikzcd} \Big/ \sim \quad \text{(Contracting identities)}$$

A construction of $\mathcal{C}[W^{-1}]$

$\Rightarrow$ $\mathcal{C}[W^{-1}]$ always exists

Problem

① No way to compute $\mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)$

② Zig-zags of arbitrary length

$\Rightarrow$ $\mathcal{C}[W^{-1}]$ may not have hom sets

hom classes

How to go on?

Idea: Limit the length of Zig-zags

⇒ Everything is more tame

① Ask for a "Calculus of left fractions" [Gabriel-Zisman '67]

⇒ Zig-zags of length 2

$$\begin{tikzcd} & X' \arrow[dl, "w"'] \arrow[dr] & \\ X \arrow[rr, squiggly] & & Y \end{tikzcd}$$

<u>Problem</u> There are super rare.

How to go on?

['67 Quillen] Model categories $\rightsquigarrow$ Zig-Zags of length 3

* Hovey
* Hirschhorn
* Dwyer - Spaliński
* (2021) BALCHIN.

Model categories

<u>Idea</u> Category $\mathcal{C}$ equipped with 3 classes of maps

* Weak equivalences (maps we want to invert) $\xrightarrow{\sim}$ $X \simeq Y$

* Fibrations ("surjections") $\twoheadrightarrow$

* Cofibrations ("injections") $\hookrightarrow$

+ Axioms

$\rightsquigarrow$ Allows us to construct a <u>category</u> $Ho(\mathcal{C}) \approx \mathcal{C}[W^{-1}]$

Lifting properties

Suppose we have a commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

a *lift* in the diagram is a morphism $h\colon B \to X$ such that $hi = f$ and $ph = g$.
A morphism $i\colon A \to B$ is said to have the *left lifting property* (LLP) with respect to another morphism $p\colon X \to Y$ if a lifting exists for any choice of f and g making the square commute.

Retracts

Let $f\colon X \to Y$ be a morphism in a category $\mathcal{C}$, then a *retraction* of f is left-inverse. That is, there exists a morphism r such that $r \circ f = \mathrm{id}_X$.

let $\mathcal{C}$ have a terminal object $*$, $f: * \to X$

then the unique map $X \to *$ is a retract,

The axioms of a model category

A *model category* is a category $\mathcal{C}$ with three distinguished classes of morphisms:

- Weak equivalences – $\mathrm{W}_{\mathcal{C}}$;
- Fibrations – $\mathrm{Fib}_{\mathcal{C}}$;
- Cofibrations – $\mathrm{Cof}_{\mathcal{C}}$;

each of which is closed under composition.

"trivial fibration"

- A morphism which is both a fibration and a weak equivalence is an *acyclic fibration*
- A morphism which is both a cofibration and a weak equivalence is an *acyclic cofibration*.

The axioms of a model category

Quillen only asks for finite

MC1) $\mathcal{C}$ has all small limits and colimits. $\Rightarrow \emptyset$ $*$

MC2) If f and g are morphism such that gf is defined and if two of f, g and gf are weak equivalences, then so is the third. "2-out-of-3"

MC3) The three distinguished classes of morphisms are closed under retracts.

The axioms of a model category

MC4) Given a commutative square of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "i"', "\sim"' near start] & X \arrow[d, two heads, "p", "\sim" near start] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "h"'] & Y \end{tikzcd}$$

a lift exists when either i is a cofibration and p is an acyclic fibration or when i is an acyclic cofibration and p is a fibration.

The axioms of a model category

MC5) Each morphism f in $\mathcal{C}$ can be factored in two ways:

(1) $f = p \circ i$, where p is an acyclic fibration and i is a cofibration.

(2) $f = p \circ i$, where p is a fibration and i is an acyclic cofibration.

$$f : X \to Y \quad \overset{\text{mc5}}{\rightsquigarrow} \quad X \underset{\sim}{\hookrightarrow} \bullet \underset{\sim}{\twoheadrightarrow} Y$$

Bifibrant objects

Let $\mathcal{C}$ be a model category. An object $X \in \mathcal{C}$ is said to be:

- *fibrant* if the map $X \longrightarrow *$ is a fibration
- *cofibrant* if the map $\phi \longrightarrow X$ is a cofibration
- *bifibrant* if fibrant + cofibrant

Bifibrant replacements

Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$.

- A *fibrant replacement* of X is a fibrant object RX along with an acyclic cofibration $X \overset{\sim}{\hookrightarrow} RX$

$$(X \to *) \overset{\text{mcs}}{\rightsquigarrow} X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$$

- A *cofibrant replacement* of X is a cofib object QX along with an acyclic fibration $QX \overset{\sim}{\twoheadrightarrow} X$

$$(\emptyset \to X) \overset{\text{mcs}}{\rightsquigarrow} \emptyset \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$$

$RQX \simeq QRX \Rightarrow$ bifibrant

The trivial model structure(s)

Let $\mathcal{C}$ be a (co)complete category. Then there is a model structure on $\mathcal{C}$ where:

- W = isomorphisms
- Fib = any map
- Cof = any map

The trivial model $\mathcal{C}_{triv}$

Fibrant objects $X \to *$ a fibration

$\Rightarrow$ all obj. fibrant.

$\Rightarrow$ All obj are bifibrant.

The trivial model structure(s)

W = any map

Fib = any map

Cof = isomorphisms

$\mathcal{C}$ initial

$\emptyset \to X$ cofibration

$\Leftrightarrow \emptyset \cong X$

W = any map

Fib = isos

Cof. any map

$\mathcal{C}$ terminal

Too much data!

Proposition:

- The cofibrations are the morphisms with the LLP with respect to all acyclic fibrations.
- The fibrations are the morphisms with the RLP with respect to all acyclic cofibrations.

Determination

Proposition: The following incomplete data uniquely determines a model structure:

* weak equiv + (co)fibrations
* Fibrations + cofibrations
* Fibrations + cofibrant obj.

Determination

Proposition: The following incomplete data does not uniquely determine a model structure:

* weak equivalences
* fibrat obj + weak equiv
* fibrat obj + cof. obj

Canonical examples (revisited)

$\mathcal{C} = \mathbf{Top}$

W = weak hpty equivalences

Fib = Serre fibrations

Cof = retracts of relative cell complexes

$\mathrm{Top}_{\mathrm{Quillen}}$

- $X \to *$ is always a Serre fib^n ⇒ all object fibrant
- Cof^{ib} obj. are CW-cmplxs

Canonical examples (revisited)

$\mathcal{C} = \mathbf{Ch}_{\geq 0}(R)$

W: quasi-iso

Fib = degreewise epimorphisms in positive degree

$\mathrm{Cof} = LLP(W \cap \mathrm{Fib})$

$\mathrm{Ch}_{\geq 0}(R)_{proj}$

Cofib replacement is a projective resolution.

Canonical examples (revisited)

$\mathcal{C} = \mathbf{Cat}$

W = equivalence of cats

Fib = RLP(W ∩ Cof)

Cof = injective on obj.

$\mathcal{C}at_{Nat}$

Fact This model is determined by its weak equivalences

(Canonical)

The homotopy category

Goal: Show that for $\mathcal{C}$ a model category, we can construct $\mathcal{C}[W^{-1}]$ and that this is a category.

Strategy: Form a certain category $\mathcal{C}_{cf}/\sim$ and show it is equivalent to $\mathcal{C}[W^{-1}]$.

Intuition: When we have a model category the zig-zags can be taken as

$$X \xleftarrow{\sim} QX \longrightarrow RY \xleftarrow{\sim} Y$$

Left homotopy

Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$. A *cylinder object* $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal $\nabla_X\colon X \sqcup X \to X$ as

$$\nabla_X\colon \underline{X \sqcup X} \xhookrightarrow{(i_0,i_1)\in\mathrm{Cof}_{\mathcal{C}}} \mathrm{Cyl}(X) \xrightarrow[\sim]{p\in \mathrm{W}_{\mathcal{C}}} X.$$

"$X \times [0,1]$"

Let $f, g\colon X \to Y$ be a pair of morphisms in a model category. Then a *left homotopy* $\eta\colon f \sim_L g$ is a morphism $\eta\colon \mathrm{Cyl}(X) \to Y$ which makes the following diagram commute:

$$\begin{tikzcd}
X \arrow[r, "i_0"] \arrow[dr, "f"'] & \mathrm{Cyl}(X) \arrow[d, "\eta"] & X \arrow[l, "i_1"'] \arrow[dl, "g"] \\
 & Y &
\end{tikzcd}$$

Right homotopy

Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$. A *path object* $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\Delta_X \colon X \to X \times X$ as

$$\Delta_X \colon X \underset{\sim}{\overset{s \in W_{\mathcal{C}}}{\hookrightarrow}} \mathrm{Path}(X) \overset{(d_0,d_1) \in \mathrm{Fib}_{\mathcal{C}}}{\twoheadrightarrow} X \times X.$$

"$X^{[0,1]}$"

Let $f, g \colon X \to Y$ be a pair of morphisms in a model category. Then a *right homotopy* $\eta \colon f \sim_R g$ is a morphism $\eta \colon X \to \mathrm{Path}(Y)$ which makes the following diagram commute:

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, "\eta"] \arrow[dr, "g"] & \\
Y & \mathrm{Path}(Y) \arrow[l, "d_0"] \arrow[r, "d_1"'] & Y
\end{tikzcd}$$

Homotopy equivalences

- A pair of morphisms $f, g\colon X \to Y$ in a model category are *homotopic*, written $f \sim g$ if they are both left and right homotopic.
- A morphism $f\colon X \to Y$ in a model category is a *homotopy equivalence* is there is a morphism $h\colon Y \to X$ such that $hf \sim \mathrm{id}_X$ and $fh \sim \mathrm{id}_Y$.

Homotopy as an equivalence relation

Proposition: $X, Y \in \mathcal{C}$. left homotopy (resp. right) is an equivalence rel^n on $\mathrm{Hom}(X,Y)$ for X cofibrant, Y fibrant.

+ they coincide.

Proposition: $\mathcal{C}_{cf} \subseteq \mathcal{C}$ of bifib. obj.

A morphism $f: X \to Y$ in $\mathcal{C}_{cf}$ is a weak equiv iff it is a htpy equivalence.

The homotopy category

<u>Def</u>

Def^n $\mathcal{C}$ a model cat. The homotopy cat $Ho(\mathcal{C})$

- obj = obj of $\mathcal{C}_{cf}$
- morphisms are hpty classes of maps under hpty.

<u>Thm</u> There is an equivalence of categories $Ho(\mathcal{C}) \simeq \mathcal{C}[W^{-1}]$

$$\mathrm{Hom}_{\mathcal{C}}(QRX, QRY)/\sim \;\cong\; \mathrm{Hom}_{\mathcal{C}[W^{-1}]}(X, Y)$$