#### Model Categories by Example

Lecture 3

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E, D model categories Quillen functor f: 6 20: U that interacted well with the model structures ~ F: Ho(B) → Ho(O): U Ho(E) = Ho(D) F+U FILL was a Queller Egul. 18 Sometimes we need Quiller Cymaline > Holb) ~ Ho(0)

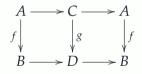
a Zig-Zag

sSet: = Set DOP Equipped with Kan model struct SSet Kan \* Fis dis. were then Kan Complexes \* Catib were the Monomorphisms Key fact: SSet Kan ~ a Top awllen (via geometric real) \* Used Ex: SSet -> sSet to construct "Ex "(-)" as a junctorial f.b. (eplacement

#### Being closed under retracts

MC3) The three classes of morphisms are closed under retracts.

The meaning of this is that they are closed under retracts in  $\mathbf{Mor}(\mathcal{C})$ . That is, f is a retract of g if and only if there is a commutative diagram



where the horizontal composites are the identity.

### An infinite family of model structures

in sset

Let fib<sub>n</sub> be the collection of morphisms f such that  $Ex^n(f)$  is a Kan fibration.

Proposition: (Bere] There is a model structure on sset such what \* Wear Egu ose He wear Egus in SSet Kon sSet, (sset, = sset kan)

\* Fibrations are fibr

· Cof = LLP( Decartic fib)

id: Set, -> 5 Set,+1

is a Quiller Eguiv.

### New models from old

Proving MCI-MCS in general is hard

- 1) Passing model structures over adjunctions
- 2) Adding weak Equidaces
- 3) Cotegories enriched in model categories.

  All of there regure proporties/structures on the model cat we stort with

## Cofibrant generation

Idea: If a model structure is cofibrantly generated then we can test various things against a small set of maps as opposed to all the (acyclic) cofibrations

# Examples







### Examples

Cat<sub>Nat</sub> 
$$T = \{ \phi \hookrightarrow \xi + 3, \{ \delta_0, 1 \} \hookrightarrow \{ \delta_0 \rightarrow 1 \}, \{ \delta_0 \Rightarrow 1 \} \}$$

### Strøm model structure

Prop There is a model structure on Top where
weak Equal = homotopy Equal III This is not

· fils (Hurcicz) Subrations

· cox = closed Hurersicz cystrations

Top strøm

All objects are bigiblent

id: Top Quillen -> Top strøm

This is <u>not</u> coxil.

Generated. [Raptis]

Ch(R)
weak egus are the
chan homotopies

### Fibrant generation

Coxib gen => Smallner Condition fib gen a cosmal new condition Only \$ 1 are cosmall in Set. is fibrally generated. s Pro (Set) [Quak]

### Right transferred model structures

Suppose that  ${\mathcal C}$  is a model category, and that we have an adjunction

$$U: \mathcal{D} \leftrightarrows \mathcal{C}: F$$

The *right transferred model structure* on  $\mathcal{D}$  (if it exists) has  $f \colon X \to Y$  a:

- weak equivalence if U(f) is a we in 6
- fibration if U(f) is a fib in 6
- · cofibration if LLP ( acrel & Jb)

Moreover the pair  $F \dashv U$  is a Quillen adjunction between these model structures.

### Existence of right transferred model structures

A map in  $\mathcal{D}$  is an *anodyne map* if it has the LLP with respect to all fibrations.

Proposition: Necessary and sufficient conditions for the right transferred model structure to exist are:

- · [factorization] Every morphism in ] justous as a Cogibration followed by an acy. f.b + as an analyse map followed by a fis.
- · [Acyclaity] Every another Map is a weak Equivalence.

#### **Factorizations**

Proposition: Suppose that

· F present small objects.

Then every morphism factors as a cofibration followed a trivial fibration, and as an anodyne map followed by a fibration. Moreover if the model structure exists then it is cofibrantly generated by F(I) and F(J).

## Acyclicity

Proposition: If a seguential colin of poshouts of images under f

y generating acyclic could in a weak Equi in D

Acy. I

Proposition: If D has fisher (eplacements

2) B has path objects for fish objects

### Projective model on functor categories

Let  $\mathcal D$  be a small category and  $\mathcal C$  a cofibrantly generated model category. Then there is a *projective model structure* on the functor category  $\mathcal C^{\mathcal D}$  where a map  $f\colon X\to Y$  is a:

• weak equivalence if 
$$\chi(d) \rightarrow \gamma(d)$$
 is a we in  $\mathcal{E}$   $\forall d \in \mathcal{D}$ 

· fibration if 
$$\chi(a) \rightarrow \gamma(d)$$
 is a fib in & \forall \delta \ell

## Projective model on functor categories

Let Daise Idea of proof using right trasfer machinery. be the discrete category of ob. on D. Codec (0) = Tob (D) 6 This has a cog. gen model struck when weak Egu yib trey are determined levelse.

## Simplicial presheaves

#### Thomason model structure

Idea Transfer SSetkan to Cat such that they are Quillen Equivalent.

N: Cat SSet: Ti Problem Fibration objects and up being grapails

Set 
$$\underset{E\times}{\text{SSet}}$$
  $\underset{E\times}{\text{SSet}}$   $\underset{N}{\text{Cat}}$ 

### Thomason model structure

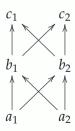
Prop There is a model structure on Cat such that a functor Fi & >D is in \* W if Ex N(f) is a weak gun Set Kan \*FB if Ex N(F) is a fileation in Set Kan · Cox if Up(age. fib) Cat thom

Catthon ~ @ sSet Kan

#### Thomason model structure

Proposition: If W is the category of weak equivlaneces of a model category, then W is fibrant in  $\mathbf{Cat}_{\mathrm{Thom}}$ .

Proposition: Every poset with five or less elements is cofibrant in  $Cat_{Thom}$ . The following poset is non-cofibrant in  $Cat_{Thom}$  and is the minimal such example in dimension and in cardinality.



## Diffeological spaces

Diss the cost, of developed spaces. (Kihara) Egup DEPJ with a dyeologish structe such that 1 Cp3 Co DCp3 smooth deg. retractract → Sort: Ditt = aSet: 1-1 Ditt

#### Left transfer

Suppose that  $\ensuremath{\mathcal{C}}$  is a model category, and that we have an adjunction

$$U:\mathcal{D}\rightleftarrows\mathcal{C}:G$$

The *left transferred model structure* on  $\mathcal{D}$  (if it exists) has  $f: X \to Y$  a:

- weak equivalence if U(f) 'is in 6
- fibration if RUP (acy. Cog)
- cofibration if U(f) is in 6,

Moreover the pair G, U is a Quillen adjunction between these model structures.

### Existence of left transferred model structures

Hard No way to get at generating cop + acyclic cop.

Lack of fibrantly generated things.

Women in Topology

#### Combinatorial model structures

A model category is said to be *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable.

### Injective model on functor categories

Let  $\mathcal{D}$  be a small category and  $\mathcal{C}$  a combinatorial model category. Then there is an *injective model structure* on the functor category  $\mathcal{C}^{\mathcal{D}}$  where a map  $f: X \to Y$  is a:

- weak equivalence if X(d) → y(d) is a were in 6 tode()
- · fibration if RLP (age, cyliations)
- cofibration if  $\chi(A) \Rightarrow \chi(A)$  is cog in  $\beta$   $\forall A \in D$

### Functoriality of functor categories

Let  $F: \mathcal{B} \rightleftarrows \mathcal{C}: G$  be a Quillen adjunction of combinatorial model categories, then composition determines Quillen adjunctions

$$\mathcal{B}_{\mathrm{proj}}^{\mathcal{D}} 
ightleftharpoons \mathcal{C}_{\mathrm{proj}}^{\mathcal{D}}$$

$$\mathcal{B}_{\mathsf{inj}}^{\mathcal{D}} 
ightleftharpoons \mathcal{C}_{\mathsf{inj}}^{\mathcal{D}}$$

If the original Quillen adjunction is a Quillen equivalence then so is the induced adjunction between the functor categories.

### Functoriality of functor categories

Let  $f \colon \mathcal{D} \to \mathcal{E}$  be a functor between small categories. Then the following are Quillen adjunctions for  $f^*$  the induced pullback:

(1) 
$$f_!: \mathcal{C}^{\mathcal{D}}_{\text{proj}} \rightleftarrows \mathcal{C}^{\mathcal{E}}_{\text{proj}}: f^*$$
.

(2) 
$$f^*: \mathcal{C}^{\mathcal{D}}_{\text{inj}} \rightleftarrows \mathcal{C}^{\mathcal{E}}_{\text{inj}}: f_*$$
.