Model Categories by Example

Lecture 3

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Recap

$\mathcal{C}, \mathcal{D}$ model categories

Quillen functor $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ that interacted well with the model structures

$\rightsquigarrow \mathbb{F} : Ho(\mathcal{C}) \rightleftarrows Ho(\mathcal{D}) : \mathbb{U}$

$F \dashv U$ was a Quillen equiv. if $Ho(\mathcal{C}) \simeq Ho(\mathcal{D})$ $\mathbb{F} \dashv \mathbb{U}$

Quillen equivalence $\Rightarrow Ho(\mathcal{C}) \simeq Ho(\mathcal{D})$

Sometimes we need a zig-zag

Recap

Introduce $sSet := Set^{\Delta^{op}}$

Equipped with Kan model struct $sSet_{Kan}$

* Fib obj. are the Kan complexes
* Cofib are the monomorphisms.

Key fact: $sSet_{Kan} \simeq_Q Top$ quillen (via geometric real + singular set)

* Used $Ex : sSet \longrightarrow sSet$ to construct "$Ex^\infty(-)$" as a functorial f.b. replacement.

Being closed under retracts

MC3) The three classes of morphisms are closed under retracts.

The meaning of this is that they are closed under retracts in **Mor**$(\mathcal{C})$. That is, f is a retract of g if and only if there is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & B \end{tikzcd}$$

where the horizontal composites are the identity.

An infinite family of model structures

in sSet.

Let fib$_n$ be the collection of morphisms f such that $\mathrm{Ex}^n(f)$ is a Kan fibration.

Proposition: [Beke] There is a model structure on sSet such that

* Weak equ are the weak equiv in sSet Kan
* Fibrations are fib$_n$
* Cof = LLP(acyclic fib)

sSet$_n$ (sSet$_0$ = sSet Kan)

id: sSet$_n$ $\longrightarrow$ sSet$_{n+1}$ is a Quillen Equiv.

New models from old

Proving MC1-MC5 in general is <u>hard</u>.

1) Passing model structures over adjunctions

2) Adding weak equivalences

3) Categories enriched in model categories.

All of these require properties/structures on the model cat we start with

Cofibrant generation

Idea: If a model structure is cofibrantly generated then we can test various things against a small set of maps as opposed to all the (acyclic) cofibrations

A model cat. is "cofib. gen" if there exists sets I, J such that 1) $LLP(RLP(I)) = $ Cof of $\mathcal{C}$.

2) $LLP(RLP(J)) = $ Acyclic cof of $\mathcal{C}$.

I, J satisfy a smallness condition on their domains

$$\ni \operatorname{colim}_\beta (A, X_\beta) \cong \operatorname{Hom}(A, \operatorname{colim}_\beta X_\beta)$$

Examples

Top$_{\text{Quillen}}$

$$I = \{ S^{n-1} \hookrightarrow D^n \mid n \geq 0 \}$$

$$J = \{ D^n \hookrightarrow D^n \times [0,1] \mid n \geq 0 \}$$

sSet$_{\text{Kan}}$

$$I = \{ \partial\Delta[n] \to \Delta[n] \mid n \geq 0 \}$$

$$J = \{ \Lambda^k[n] \to \Delta[n] \mid n \geq 0,\ 0 \leq k \leq n \}.$$

Examples

Cat$_{\mathrm{Nat}}$

$$I = \{\emptyset \hookrightarrow \{*\},\ \{0,1\} \hookrightarrow \{0 \to 1\},\ \{0 \rightrightarrows 1\} \to \{0 \to 1\}\}$$

$$J = \{* \to \{0 \rightleftarrows 1\}\}$$

Mod$(R)_{\mathrm{stab}}$

$$I = \{i \longrightarrow R \mid i \text{ a left ideal}\}$$

$$J = \{0 \to R\},$$

Strøm model structure

<u>Prop</u> There is a model structure on Top where

- weak equiv = homotopy equivs
- fib = (Hurewicz) fibrations
- cof = closed Hurewicz cofibrations

!!! This is <u>not</u> cofib. generated. [Raptis]

$Top_{Strøm}$

All objects are bifibrant

$id: Top_{Quillen} \to Top_{Strøm}$

$Ch(R)$

weak equivs are the chain homotopies

Fibrant generation

Cofib gen $\Rightarrow$ smallness condition

fib gen $\Rightarrow$ cosmallness condition

Only ϕ, $*$ are cosmall in Set.

$s\mathrm{Pro}(\mathrm{Set})$ [Quick] is fibrantly generated.

Right transferred model structures

Suppose that $\mathcal{C}$ is a model category, and that we have an adjunction

$$U : \mathcal{D} \leftrightarrows \mathcal{C} : F$$

The *right transferred model structure* on $\mathcal{D}$ (if it exists) has $f : X \to Y$ a:

- weak equivalence if $U(f)$ is a w.e in $\mathcal{C}$
- fibration if $U(f)$ is a fib in $\mathcal{C}$
- cofibration if LLP (acyclic fib)

Moreover the pair $F \dashv U$ is a Quillen adjunction between these model structures.

Existence of right transferred model structures

A map in $\mathcal{D}$ is an *anodyne map* if it has the LLP with respect to all fibrations.

Proposition: Necessary and sufficient conditions for the right transferred model structure to exist are:

- [Factorization] Every morphism in D factors as a cofibration followed by an acy. fib + as an anodyne map followed by a fib.
- [Acyclicity] Every anodyne Map is a weak equivalence

Factorizations

Proposition: Suppose that

- $\mathcal{C}$ is cofibrantly gen.
- F preserve small objects.

Then every morphism factors as a cofibration followed a trivial fibration, and as an anodyne map followed by a fibration. Moreover if the model structure exists then it is cofibrantly generated by $F(I)$ and $F(J)$.

Acyclicity

Proposition: If a sequential colim of pushouts of images under F of $\underbrace{\text{generating acyclic cofib}}_{J}$ in a weak equiv in D

$\Rightarrow$ Acy.

Proposition: If 1) D has fibrant replacements

2) D has path objects for fib. obj

$\Rightarrow$ Acy.

Projective model on functor categories

Let $\mathcal{D}$ be a small category and $\mathcal{C}$ a cofibrantly generated model category. Then there is a *projective model structure* on the functor category $\mathcal{C}^{\mathcal{D}}$ where a map $f\colon X \to Y$ is a:

- weak equivalence if $X(d) \to Y(d)$ is a we in $\mathcal{C}$ $\forall d \in \mathcal{D}$
- fibration if $X(d) \to Y(d)$ is a fib in $\mathcal{C}$ $\forall d \in \mathcal{D}$
- cofibration if LLP(acyclic fib)

$\mathcal{C}^{\mathcal{D}}_{proj}$

Projective model on functor categories

Idea of proof using right transfer machinery. Let D^{disc} be the discrete category of ob. on D.

$$\mathcal{C}^{disc(D)} = \prod_{ob(D)} \mathcal{C}$$

This has a cof. gen model structure where weak equiv + fib + cof are determined levelwise.

$$U : \mathcal{C}^{D} \rightleftarrows \mathcal{C}^{disc(D)} : F$$

Simplicial presheaves

Ex D a small category.

$$sPr(D) = Pr(D)^{\Delta^{op}} = sSet^{D^{op}}$$

Equip sSet with the Kan model structure.

$\Rightarrow$ $sPr(D)_{proj}$.

$D = \Delta$. $\rightsquigarrow$ Bisimplicial sets.

Thomason model structure

Idea: Transfer $sSet_{Kan}$ to Cat such that they are Quillen equivalent.

$$N : Cat \rightleftarrows sSet : \tau_1$$

Problem: fibrant objects end up being groupoids

$$\begin{tikzcd} sSet \arrow[r, bend left, "sd"] & sSet \arrow[l, bend left, "Ex"] \arrow[r, bend left, "sd"] & sSet \arrow[l, bend left, "Ex"] \arrow[r, bend left, "\tau_1"] & Cat \arrow[l, bend left, "N"] \end{tikzcd}$$

Thomason model structure

<u>Prop</u> There is a model structure on Cat such that a functor $F: \mathcal{C} \to \mathcal{D}$ is in

- W if $Ex^2 N(f)$ is a weak equiv in $sSet_{Kan}$
- Fib if $Ex^2 N(f)$ is a fibration in $sSet_{Kan}$
- Cof if LLP(acyc. fib)

Cat_{Thom}

$$Cat_{Thom} \simeq_Q sSet_{Kan}$$

Thomason model structure

Proposition: If W is the category of weak equivlaneces of a model category, then W is fibrant in $\mathbf{Cat}_{\mathrm{Thom}}$.

Proposition: Every poset with five or less elements is cofibrant in $\mathbf{Cat}_{\mathrm{Thom}}$. The following poset is non-cofibrant in $\mathbf{Cat}_{\mathrm{Thom}}$ and is the minimal such example in dimension and in cardinality.

$$\begin{tikzcd}
c_1 & & c_2 \\
b_1 \arrow[u] \arrow[urr] & & b_2 \arrow[u] \arrow[ull] \\
a_1 \arrow[u] \arrow[urr] & & a_2 \arrow[u] \arrow[ull]
\end{tikzcd}$$

Diffeological spaces

Diff the cat. of diffeological spaces. [Kihara]

<u>Plan</u> Equip $\Delta[p]$ with a diffeological structure such that

$\Lambda^k[p] \hookrightarrow \Delta[p]$ smooth def. retract.

$\Rightarrow$ $S^{\mathrm{Diff}} : \mathrm{Diff} \rightleftarrows \mathrm{sSet} : |-|_{\mathrm{Diff}}$

Left transfer

Suppose that $\mathcal{C}$ is a model category, and that we have an adjunction

$$U : \mathcal{D} \rightleftarrows \mathcal{C} : G$$

The *left transferred model structure* on $\mathcal{D}$ (if it exists) has $f : X \to Y$ a:

- weak equivalence if $U(f)$ is in $\mathcal{E}$
- fibration if $RLP($asy. cof$)$
- cofibration if $U(f)$ is in $\mathcal{C}$.

Moreover the pair G, U is a Quillen adjunction between these model structures.

Existence of left transferred model structures

<u>Hard</u> No way to get at generating cof + acycl. c cof.

Lack of fibrantly generated things.

Women in Topology

Combinatorial model structures

A model category is said to be *combinatorial* if it is cofibrantly generated and the underlying category is locally presentable.

Ex $sSet_{Kan}$ is combinatorial

$Top_{Quillen}$ is not combinatorial

Injective model on functor categories

Let $\mathcal{D}$ be a small category and $\mathcal{C}$ a combinatorial model category. Then there is an *injective model structure* on the functor category $\mathcal{C}^{\mathcal{D}}$ where a map $f : X \to Y$ is a:

- weak equivalence if $X(d) \to Y(d)$ is a weak in $\mathcal{C}$ $\forall d \in D$
- fibration if RLP(ayc. cofibrations)
- cofibration if $X(d) \to Y(d)$ is cof in $\mathcal{C}$ $\forall d \in D$

$sP_1(D)_{inj}$

Functoriality of functor categories

Let $F : \mathcal{B} \rightleftarrows \mathcal{C} : G$ be a Quillen adjunction of combinatorial model categories, then composition determines Quillen adjunctions

$$\mathcal{B}^{\mathcal{D}}_{\text{proj}} \rightleftarrows \mathcal{C}^{\mathcal{D}}_{\text{proj}}$$

$$\mathcal{B}^{\mathcal{D}}_{\text{inj}} \rightleftarrows \mathcal{C}^{\mathcal{D}}_{\text{inj}}$$

If the original Quillen adjunction is a Quillen equivalence then so is the induced adjunction between the functor categories.

Functoriality of functor categories

Let $f : \mathcal{D} \to \mathcal{E}$ be a functor between small categories. Then the following are Quillen adjunctions for f^* the induced pullback:

(1) $f_! : \mathcal{C}^{\mathcal{D}}_{\mathrm{proj}} \rightleftarrows \mathcal{C}^{\mathcal{E}}_{\mathrm{proj}} : f^*$.

(2) $f^* : \mathcal{C}^{\mathcal{D}}_{\mathrm{inj}} \rightleftarrows \mathcal{C}^{\mathcal{E}}_{\mathrm{inj}} : f_*$.