#### Model Categories by Example

Lecture 1

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#### Logistics

- Supported by the LMS in conjunction with the TTT (Transpennine Topology Triangle).
- Lectures will be recorded + available to watch online at a later date.
- There are notes to go along with the course at <u>bifibrant.com</u>.
- If you have a question please type it in chat.
- Keep yourself muted!

#### What to expect

- To come out with a working knowledge of model categories.
- A collection of tools for constructing model categories.
- Plenty of examples (many of which I hope to be non-standard)!
- No proofs (but references to be found in lecture notes).
- No  $(\infty, 1)$ -categories.
- Subscripts on categories indicate a model structure.

#### Motivation

\* Homotopy theory ws localizations of categories \* La category Wc Mor(b) \* would like to from & [wi] such that Maps in wave isos \* y: C -> C(w) initial among all fundors sending W to isos.

# Canonical examples

## A construction of $\mathcal{C}[W^{-1}]$

## A construction of $C[W^{-1}]$

Problem

(1) No way to compute Hom & [w-1] (X, Y)

(2) Zig-Zaap of additional length

## How to go on?

I dea: Limit the length of Zig-Zigs

3 Everythy is more tame

(1) Ask gor a "Calculus of Ceft fractions" [Gabriel-Zismen 167]

=> Zig-Zegp of length &

X

Problem There are super rare.

## How to go on?

[167 Quillen] Model corregories ~> Zig-Zago cy Centh 3

- \* Hovery
- \* Hirschhorn
- \* Dwyer-Spalinski
- \* (2021) BARCHIN.

## Model categories

Idea Category & Egupped with 3 classes of Maps to invert) X=7 \* Fibrations (Surjections") \* Cotbrations ("injections") +Axioms

my Allows Us to construct a category Ho(B) ~ B[w-1]

### Lifting properties

Suppose we have a commutative square



a *lift* in the diagram is a morphism  $h\colon B\to X$  such that hi=f and ph=g. A morphism  $i\colon A\to B$  is said to have the *left lifting property* (LLP) with respect to another morphism  $p\colon X\to Y$  if a lifting exists for any choice of f and g making the square commute.

#### Retracts

Let  $f: X \to Y$  be a morphism in a category  $\mathcal{C}$ , then a *retraction* of f is left-inverse. That is, there exists a morphism r such that  $r \circ f = \mathrm{id}_X$ .

Let 
$$\mathcal{E}$$
 have a terminal object  $*$ ,  $f:* \to X$   
then the unique map  $X \to *$  is a netract,

A *model category* is a category C with three distinguished classes of morphisms:

- Weak equivalences W<sub>C</sub>;
- Fibrations Fib<sub>C</sub>;
- Cofibrations  $Cof_{\mathcal{C}}$ ;

each of which is closed under composition.



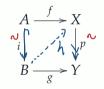
- A morphism which is both a fibration and a weak equivalence is an acyclic fibration
- A morphism which is both a cofibration and a weak equivalence is an acyclic cofibration.

MC1) 
$$C$$
 has all small limits and colimits.  $\Rightarrow \phi$ 

MC2) If f and g are morphism such that gf is defined and if two of f, g and gf are weak equivalences, then so is the third.

MC3) The three distinguished classes of morphisms are closed under retracts.

MC4) Given a commutative square of the form



a lift exists when either i is a cofibration and p is an acyclic fibration or when i is an acyclic cofibration and p is a fibration.

MC5) Each morphism f in C can be factored in two ways:

(1) 
$$f = p \circ i$$
, where  $p$  is an acyclic fibration and  $i$  is a cofibration.

(2) 
$$f = p \circ i$$
, where  $p$  is a fibration and  $i$  is an acyclic cofibration.

## Bifibrant objects

Let  $\mathcal C$  be a model category. An object  $X \in \mathcal C$  is said to be:

· bifibrant if fibrent + coxibrant

## Bifibrant replacements

Let C be a model category and  $X \in C$ .

• A fibrant replacement of X is a fibrant object RX along with on a cyclic cofibrator X C RX

(X ->\*) mcs X ->> \*\*

• A cofibrant replacement of X is a cops diplet QX dury with on acyclic film a X >>> X

## The trivial model structure(s)

Let & be a (co) complete Structure on & where:

- · W= isomorphisms
- · Fib s any map
- · Coz arey map

The trivial model be triv

integory. Then ther is a model

Fibrant Objects X > \* a jbrotin

a All ob, are bisbrant,

## The trivial model structure(s)

we any map Fib = isos Car. any map 6 terminal

#### Too much data!

#### Proposition:

- The cofibrations are the morphisms with the LLP with respect to all acyclic fibrations.
- The fibrations are the morphisms with the RLP with respect to all acyclic cofibrations.

#### **Determination**

Proposition: The following incomplete data uniquely determines a model structure:

```
* Weak Egur + (con fibrations

* Fibrations + confidentians

* Fibrations + agibrant obj.
```

#### Determination

Proposition: The following incomplete data does not uniquely determine a model structure:

## Canonical examples (revisited)

\* X > # is duays a Serie

56" > all object fibrent · Coglib obj. ONE CW-amplxs

## Canonical examples (revisited)

Costs replacement is a prejectu resolution.

## Canonical examples (revisited)

 $\mathcal{C} = \mathbf{Cat}$ 

Wr Egwoderer of Cots

Fib = RLP(Wn Cog)

Coscinjectu on obj.

Catnat

fact This model is determined by its weak Equateries

(Canonical)

### The homotopy category

Goal: Show that for C a model category, we can construct  $C[W^{-1}]$  and that this is a category.

Strategy: Form a certain category  $\sim$  and show it is equivalent to  $\mathcal{C}[W^{-1}]$ .

Intuition: When we have a model category the zig-zags can be taken as

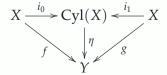
$$X \stackrel{\triangleright}{\longleftarrow} QX \longrightarrow RY \stackrel{\triangleleft}{\longleftarrow} Y$$

### Left homotopy

Let  $\mathcal{C}$  be a model category and  $X \in \mathcal{C}$ . A *cylinder object*  $\operatorname{Cyl}(X)$  for X is a factorization of the codiagonal  $\nabla_X \colon X \sqcup X \to X$  as

$$\nabla_X : \underbrace{X \sqcup X}^{c(i_0, i_1) \in \operatorname{Cof}_{\mathcal{C}}} \operatorname{Cyl}(X) \xrightarrow{p \in W_{\mathcal{C}}} X.$$

Let  $f,g\colon X\to Y$  be a pair of morphisms in a model category. Then a *left homotopy*  $\eta\colon f\sim_L g$  is a morphism  $\eta\colon\operatorname{Cyl}(X)\to Y$  which makes the following diagram commute:

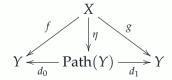


### Right homotopy

Let  $\mathcal{C}$  be a model category and  $X \in \mathcal{C}$ . A *path object*  $\operatorname{Path}(X)$  for X is a factorization of the diagonal  $\Delta_X \colon X \to X \times X$  as

$$\Delta_X \colon X \xrightarrow{\mathfrak{S} \in W_{\mathcal{C}}} \operatorname{Path}(X) \xrightarrow{(d_0, d_1) \in \operatorname{Fib}_{\mathcal{C}}} X \times X.$$

Let  $f,g\colon X\to Y$  be a pair of morphisms in a model category. Then a *right homotopy*  $\eta\colon f\sim_R g$  is a morphism  $\eta\colon X\to \operatorname{Path}(Y)$  which makes the following diagram commute:



#### Homotopy equivalences

- A pair of morphisms  $f,g:X\to Y$  in a model category are *homotopic*, written  $f\sim g$  if they are both left and right homotopic.
- A morphism  $f \colon X \to Y$  in a model category is a *homotopy equivalence* is there is a morphism  $h \colon Y \to X$  such that  $hf \sim \mathrm{id}_X$  and  $fh \sim \mathrm{id}_Y$ .

## Homotopy as an equivalence relation

Proposition:

Proposition: X.YEB, left homotopy is an equal rel on Hom(XiY)

(resp. righ) for X cylibrat,

+ they coincide.

A morphism fix=1'in byis a wear Ezw It it is a hoppy Exwalve

Ect = & of pitts of.

## The homotopy category

Des 6 a model cet. The homotypy cut tho (6)

· Ob; = ob; of by

· morphors one hope classon of maps under hotely

The There is an Equivalence of conteyons to (6) & ECW-)

Home (QRX, QRY) / = Homecon (X17)