Power and MDE

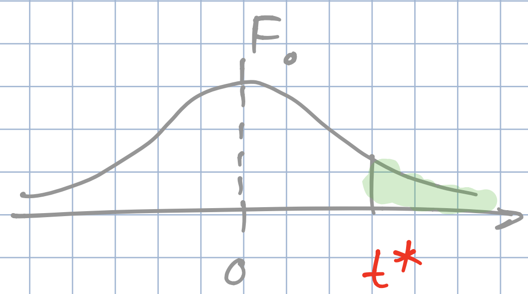
$$X_1^t, X_2^t, \dots, X_n^t \sim_{iid} \mathcal{N}(\mu^t, \sigma^{t2})$$
$$X_1^c, X_2^c, \dots, X_m^c \sim_{iid} \mathcal{N}(\mu^c, \sigma^{c2})$$

$H_0: \quad \mu^t = \mu^c$

$H_1: \quad \mu^t \neq \mu^c$

Statistics: $$T(x^t, x^c) = \frac{\bar{X}^t - \bar{X}^c}{\sqrt{\frac{\operatorname{Var} X_i^t}{n} + \frac{\operatorname{Var} X_i^c}{m}}}$$

$$F_0 = St(n+m-1) \sim \mathcal{N}(0,1)$$

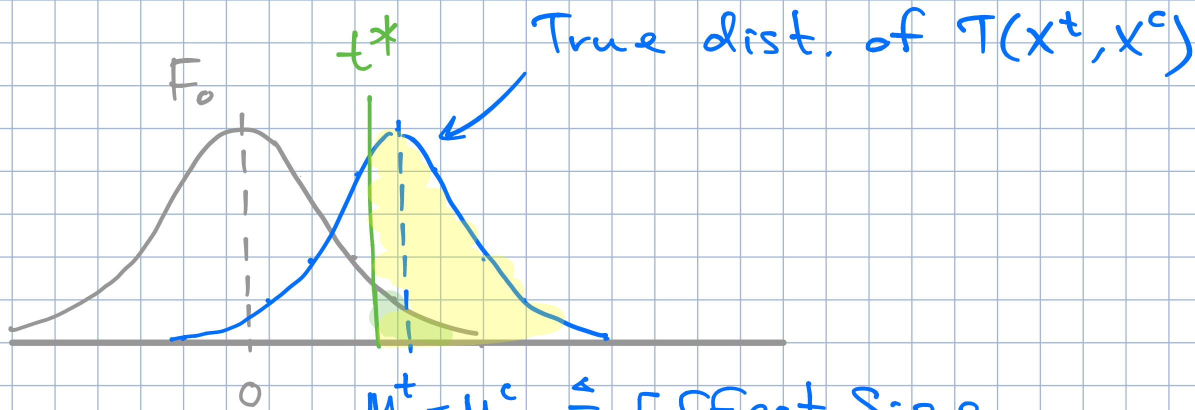


$$t^* = z_{1-\alpha/2} \quad \text{where} \quad z_\alpha : \overbrace{\mathbb{P}\{x \leq z_\alpha\}}^{\Phi(z_\alpha)} = \alpha$$

Usually $\alpha = 0.05$ (Type I Error)
aka $\mathbb{P}\{\text{reject } H_0 \mid H_0\}$

If $\underbrace{T(x^t, x^c)}_{t_{obs}} \geq t^*$ then we reject H_0

$$\mathbb{P}\{T \geq t_{obs}\} \triangleq \text{p-value}$$



$\frac{\mu^t - \mu^c}{\sqrt{\dots}} \triangleq$ Effect Size

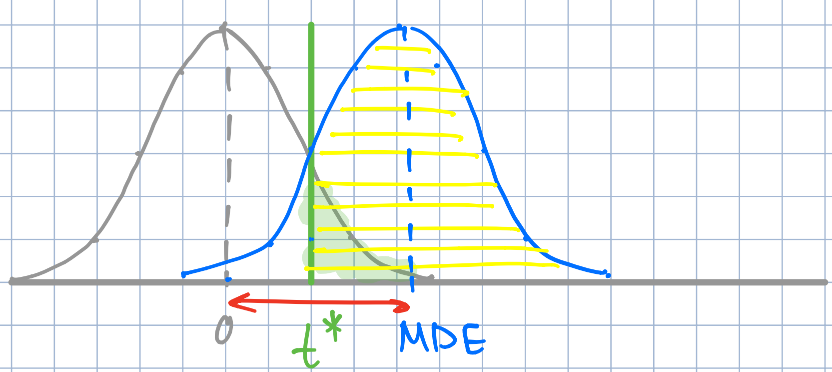
Def

Power $\triangleq$ Fraction of True distribution which lies to the right of t^*

Therefore, in order to compute Power for a fixed n, m, we need:

* Estimate of $\mu^t - \mu^c$
* Estimate of $\operatorname{Var} X^t + \operatorname{Var} X^c$

Now we are going to assume that we already have X^t and X^c, and we would like to know Minimum Effect Size that can be detected with given Power (80%)



<u>Def</u>: MDE is a point such that for a $\mathcal{N}(MDE, \sigma^2)$ $\mathbb{P}\{x \geq t^*\} = \gamma$ where $t^* = z_{1-\alpha}$ and σ^2 - is known, $\gamma = 1 - \beta$ - power (i.e. 80%)

How to compute MDE

$$\gamma = \frac{1}{\sqrt{2\pi}} \int_{t^* - MDE}^{+\infty} e^{-t^2/2}\, dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{MDE - t^*} e^{-t^2/2}\, dt$$

$$= \Phi(MDE - t^*)$$

Therefore $MDE - z_{1-\alpha} = z_\gamma \Rightarrow$

$$MDE = z_\gamma + z_{1-\alpha}$$

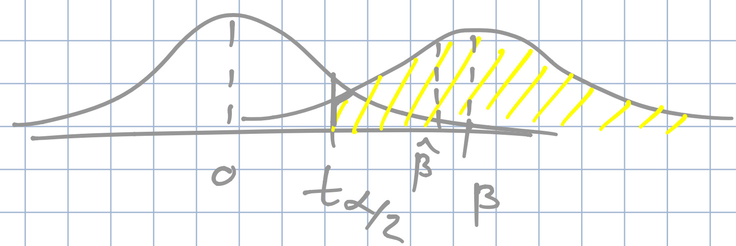
$$MDE \triangleq \frac{\mu^t - \mu^c}{\sqrt{Var}}$$

$$(\mu^t - \mu^c) = (z_\gamma + z_{1-\alpha}) \sqrt{Var}$$

$$\hat{\beta} = \bar{X}^t - \bar{X}^c$$

$$\beta = \mu^t - \mu^c$$

$$\mathbb{P}\left\{ \frac{\hat{\beta}}{\sigma_{\hat{\beta}}} > t_{\alpha/2} \,\middle|\, \beta \right\} = \gamma$$



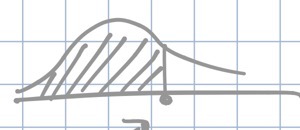
$$\mathbb{P}\left\{ \frac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}} > t_{\alpha/2} - \frac{\beta}{\sigma_{\hat{\beta}}} \,\middle|\, \beta \right\} = \gamma$$

since $\dfrac{\hat{\beta} - \beta}{\sigma_{\hat{\beta}}} \longrightarrow \mathcal{N}(0,1)$

and $\mathbb{P}\{X > y\} = 1 - \underbrace{\mathbb{P}\{X \leq y\}}_{\Phi(y)}$ and

$$1 - \Phi(y) = \Phi(-y)$$

we have $\gamma = \Phi\Big(\underbrace{\dfrac{\beta}{\sigma_{\hat{\beta}}} - t_{\alpha/2}}\Big)$



$$\frac{\beta}{\sigma_{\hat{\beta}}} - t_{\alpha/2} = t_{1-\gamma}$$

$$\beta_{MDE} = \underbrace{(t_{1-\gamma} + t_{\alpha/2})\,\sigma_{\hat{\beta}}}_{\text{Minimum Detectable Effect}}$$