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Short Communication

A note on the lifted Miller–Tucker–Zemlin subtour elimination constraints for the capacitated vehicle routing problem

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Abstract

Corrected Miller–Tucker–Zemlin type subtour elimination constraints for the capacitated vehicle routing problem are presented.

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The *capacitated vehicle routing problem* (CVRP) is defined on a graph $G=(V,A)$, where $V=\{v_1,\dots,v_n\}$ is a vertex set, v_1 is a depot, and the remaining vertices are customers; the set $A=\{(v_i,v_j):v_i,v_j\in V,\ i\neq j\}$ is an arc set. With each vertex $v_i\in V\setminus\{v_1\}$ is associated a positive integer demand q_i and with each arc (v_i,v_j) is associated a travel cost c_{ij}. The CVRP consists of determining a set of m vehicle routes satisfying the following conditions: (i) each route starts and ends at the depot, (ii) each customer belongs to exactly one route, (iii) the total demand of each route does not exceed the vehicle capacity Q, (iv) the total cost of all routes is minimized. There exists an extensive literature on the CVRP. For a recent overview, see [1].

There are several ways to formulate the CVRP. In the following formulation, x_{ij} is a binary variable equal to 1 if and only if arc (v_i,v_j) belongs to the solution, and an integer u_i variable is associated with each customer v_i. Variables x_{ij} and x_{ji} are only defined if $q_i+q_j\leqslant Q$. The CVRP is then:

$$\text{minimize} \quad \sum_{i\neq j} c_{ij}x_{ij} \tag{1}$$

$$\text{subject to} \quad \sum_{j=2}^{n} x_{1j}=m, \tag{2}$$

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$$\sum_{i=2}^{n} x_{i1} = m, \tag{3}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij} = 1 \quad (i = 2, \ldots, n), \tag{4}$$

$$\sum_{\substack{i=1\\ i\neq j}}^{n} x_{ij} = 1 \quad (i = 2, \ldots, n), \tag{5}$$

$$u_i - u_j + Qx_{ij} \leqslant Q - q_j \quad (i, j = 2, \ldots, n;\ i \neq j), \tag{6}$$

$$q_i \leqslant u_i \leqslant Q \quad (i = 2, \ldots, n), \tag{7}$$

$$x_{ij} = 0 \text{ or } 1 \quad (i, j = 1, \ldots, n;\ i \neq j), \tag{8}$$

$$m \geqslant 1 \text{ and integer.} \tag{9}$$

In this formulation, constraints (2)–(5) are degree constraints whereas constraints (6) ensure that the solution contains no subtours disconnected from the depot. Together with (7), they also present the formation of vehicle routes containing the depot but having a total demand exceeding Q. Constraints (6) and (7), proposed by Kulkarni and Bhave [2], extend the Miller–Tucker–Zemlin [3] subtour elimination constraints for the *traveling salesman problem*. Note that the right-hand side of (6) is mistakenly written as $Q - q_i$ in [2], which forbids a number of feasible solutions. For example, consider the legal route (v_1, v_i, v_j, v_1) where $Q = 3$, $q_i = 2$ and $q_j = 1$. Constraints (7) imply $u_i \geqslant 2$, and constraints (6) with a right-hand side of $Q - q_i$ imply $u_j \geqslant u_i + 2 \geqslant 4$, which violates (7).

Proposition 1. *The constraints*

$$u_i - u_j + Qx_{ij} \leqslant Q - q_j \quad (i, j = 2, \ldots, n;\ i \neq j) \tag{6}$$

are valid inequalities for the CVRP.

Proof. If $x_{ij} = 0$, then the left-hand side attains a maximum of $Q - q_j$ when u_i takes its maximal value Q and u_j takes its minimal value q_j. To show that (6) is valid when $x_{ij} = 1$, consider any legal vehicle route $(v_{i_1} = v_1, v_{i_2}, \ldots, v_{i_r} = v_1)$ induced by the x_{ij} variables equal to 1. Then setting $u_{i_1} = 0$ and $u_{i_t} = u_{i_{t-1}} + q_{i_t}$ $(t = 2, \ldots, r)$ means that (6) and (7) hold for these values of $u_{i_t}, x_{i_t i_{t+1}} = 1$ $(t = 1, \ldots, r-1)$, and $x_{i_r,1} = 1$. □

Desrochers and Laporte [4] also use constraints (6) but their lifting of these constraints is incorrectly written as

$$u_i - u_j + Qx_{ij} + (Q - q_i - q_j)x_{ji} \leqslant Q - q_i \quad (i, j = 2, \ldots, n;\ i \neq j), \tag{10}$$

which can also be shown to eliminate some feasible solutions by using the above example. This can be corrected as follows.

Proposition 2. *The constraints*

$$u_i - u_j + Qx_{ij} + (Q - q_i - q_j)x_{ji} \leqslant Q - q_j \quad (i, j = 2, \ldots, n;\ i \neq j) \tag{11}$$

are valid inequalities for the CVRP.

Proof. Introducing an extra term $\alpha_{ji}x_{ji}$ in (6) yields

$$u_i - u_j + Qx_{ij} + \alpha_{ji}x_{ji} \leqslant Q - q_j \quad (i, j = 2, \ldots, n;\ i \neq j). \tag{12}$$

We seek the largest value of α_{ji} so that (12) is valid. If $x_{ji} = 0$, then α_{ji} can take any value. If $x_{ji} = 1$, then $x_{ij} = 0$ and we obtain

$$\alpha_{ji} \leqslant Q + (u_j - u_i) - q_j \leqslant Q - q_i - q_j$$

since constraints (6) for $x_{ji} = 1$ imply $u_j - u_i \leqslant -q_i$. □

Finally, Desrochers and Laporte [4] propose two liftings for the u_i variables:

$$u_i \geqslant q_i + \sum_{\substack{j=2\\ j\neq i}}^{n} q_j x_{ji} \quad (i = 2, \ldots, n) \tag{13}$$

and

$$u_i \leqslant Q - \left(Q - \max_{j\neq i}\{q_j\} \quad - q_i\right)x_{1i} - \sum_{\substack{j=2\\ j\neq i}}^{n} q_j x_{ji} \quad (i = 2, \ldots, n). \tag{14}$$

Using again the same example as above, it can be shown that (14) applied to u_j eliminates some feasible solutions. The correct constraints must be written with x_{ij} in the right-hand side summation.

Proposition 3. *The constraints*

$$u_i \leqslant Q - \left(Q - \max_{j \neq i}\{q_j\} - q_i\right)x_{1i} - \sum_{\substack{j=2 \\ j \neq i}}^{n} q_j x_{ij} \quad (i = 2, \ldots, n) \tag{15}$$

are valid inequalities for the CVRP.

Proof. We first show that the constraints

$$u_i \leqslant Q - \sum_{\substack{j=2 \\ j \neq i}}^{n} q_j x_{ij} \quad (i = 2, \ldots, n) \tag{16}$$

are valid inequalities. If $x_{i1} = 1$, then (16) holds trivially since the summation is zero. If $x_{i1} = 0$, then $x_{ij} = 1$ for a unique index $j \neq 1$ and (16) follows from (6). Lifting (16) into (15) is done by subtracting a term $\alpha_{1i}x_{1i}$ from the right-hand side. The maximal value of α_{1i} is constrained by the case $x_{1i} = 1$. There are two subcases. If $x_{i1} = 1$, then the summation is equal to zero and

$$\alpha_{1i} \leqslant Q - u_i \leqslant Q - q_i.$$

If $x_{i1} = 0$, then $x_{ij} = 1$ for a unique index $j \neq 1$; the summation is then equal to q_j and

$$\alpha_{1i} \leqslant Q - q_j - u_i \leqslant Q - q_j - q_i \quad (j = 2, \ldots, n;\ j \neq i).$$

Combining these two cases, it follows that $\alpha_{1i} = Q - \max_{j \neq i}\{q_j\} - q_i$ is a valid lifting coefficient. □

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