Packing to fewer dimensions

(for space compression and query speedup)

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Speeding up *cosine* computation

- What if we could take our vectors and "pack" them into fewer dimensions (say 50,000→100) while preserving distances?
  - Now, O(nm) to compute cos(d,q) for all n docs
  - Then, O(km+kn) where k << n,m

- Two methods:
  - Latent semantic indexing
  - Random projection

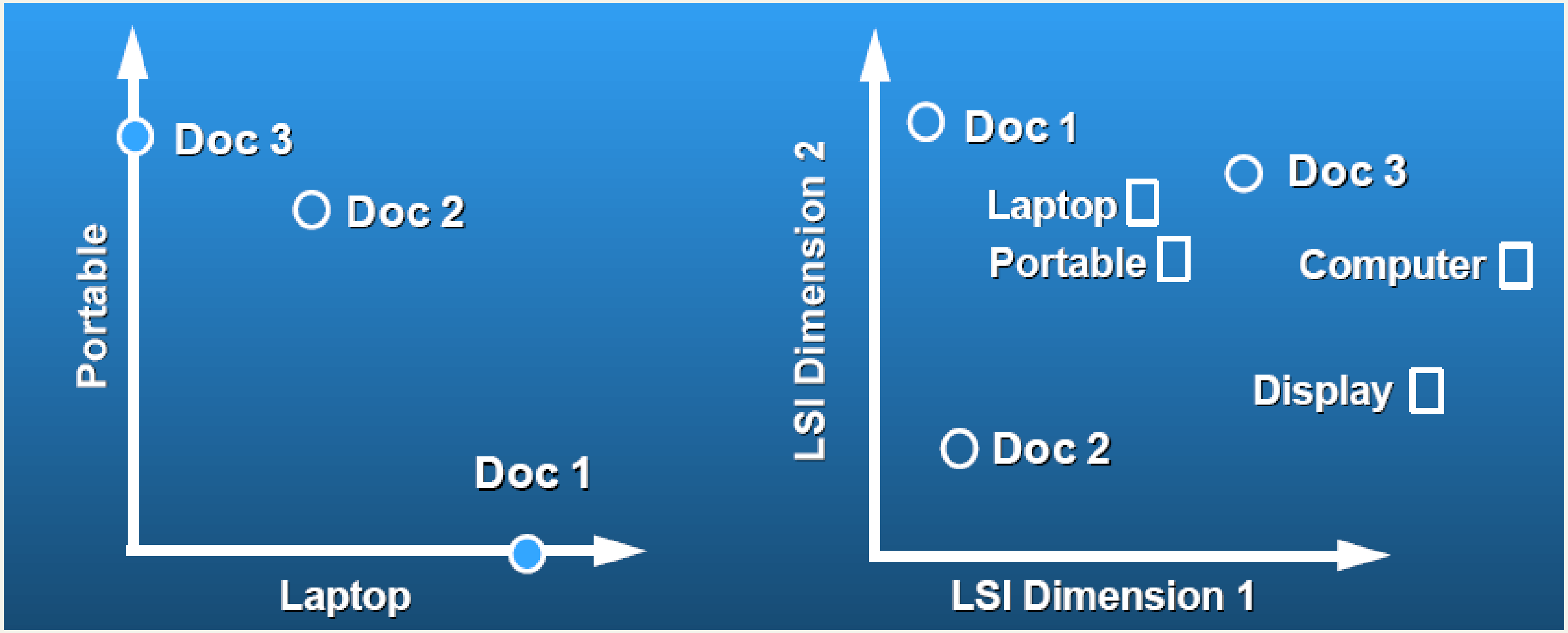
Briefly

- LSI is data-*dependent*
  - Create a k-dim subspace by eliminating redundant axes
  - Pull together “related” axes – hopefully
    - ***car*** and ***automobile***

What about polysemy ?

- Random projection is data-*independent*
  - Choose a k-dim subspace that guarantees good *stretching properties* with high probability between any pair of points.

Latent Semantic Indexing



courtesy of Susan Dumais

Notions from linear algebra

- Matrix A, vector v
- Matrix transpose (A^t)
- Matrix product
- Rank
- Eigenvalues λ and eigenvector v: $Av = \lambda v$

Example

$$\begin{pmatrix} 6 & -2 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Overview of LSI

- Pre-process docs using a technique from linear algebra called <u>Singular Value Decomposition</u>

- Create a new (*smaller*) vector space

- Queries handled (*faster*) in this new space

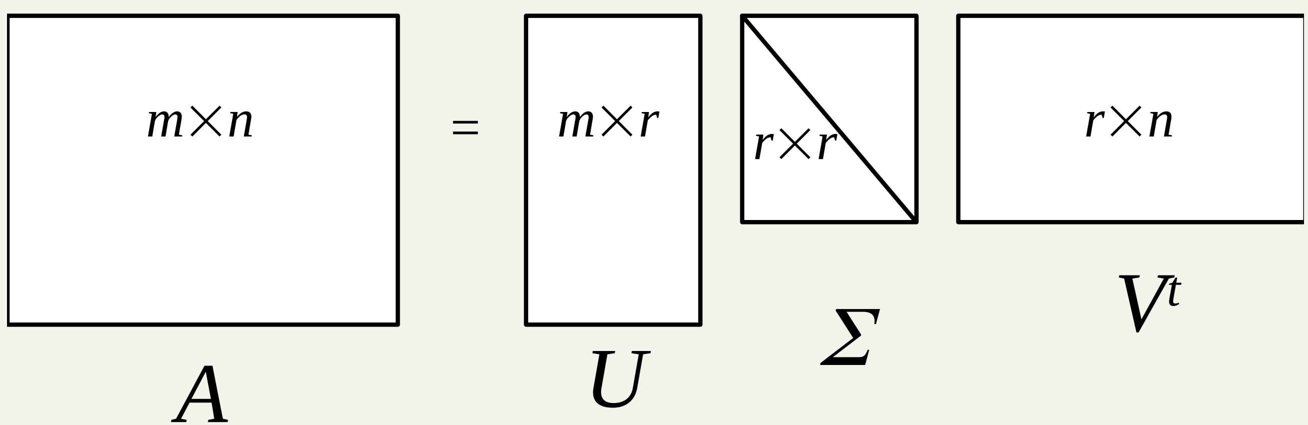
Singular-Value Decomposition

- Recall $m \times n$ matrix of terms $\times$ docs, A.
  - A has rank $r \leq m,n$
- Define term-term correlation matrix $T=AA^t$
  - T is a square, symmetric $m \times m$ matrix
  - Let U be $m \times r$ matrix of **r** eigenvectors of T
- Define doc-doc correlation matrix $D=A^tA$
  - D is a square, symmetric $n \times n$ matrix
  - Let V be $n \times r$ matrix of **r** eigenvectors of D

A's decomposition

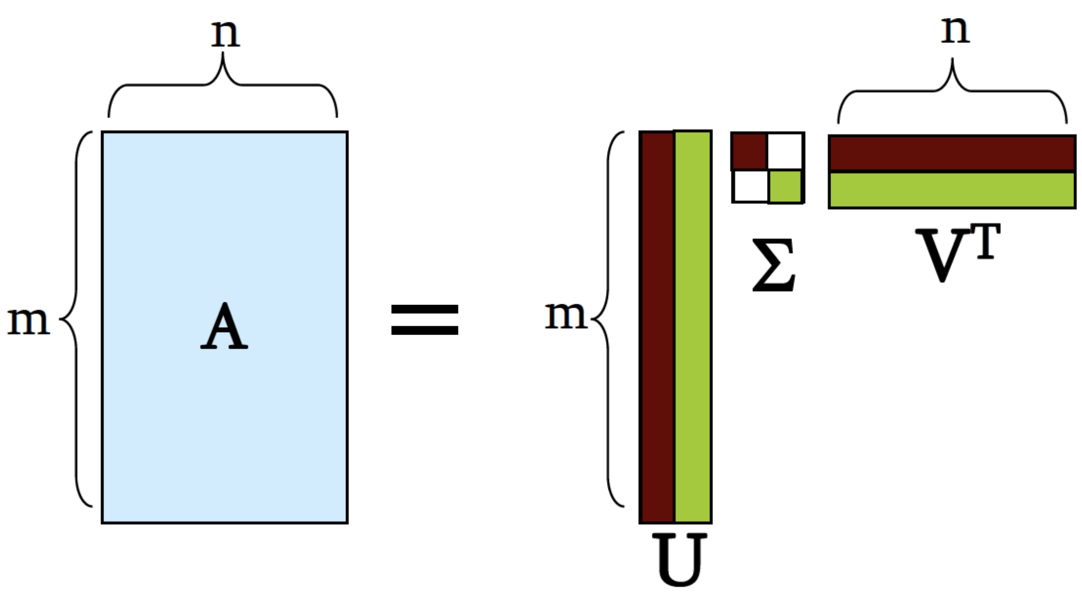
- Given U (for T, $m \times r$) and V (for D, $n \times r$) formed by orthonormal columns (unit dot-product)

- It turns out that $A = U \Sigma V^t$
  - Where Σ is a diagonal matrix with the **singular values** (=square root of the eigenvalues of $T=AA^t$) in **decreasing order**.



The case of r = 2

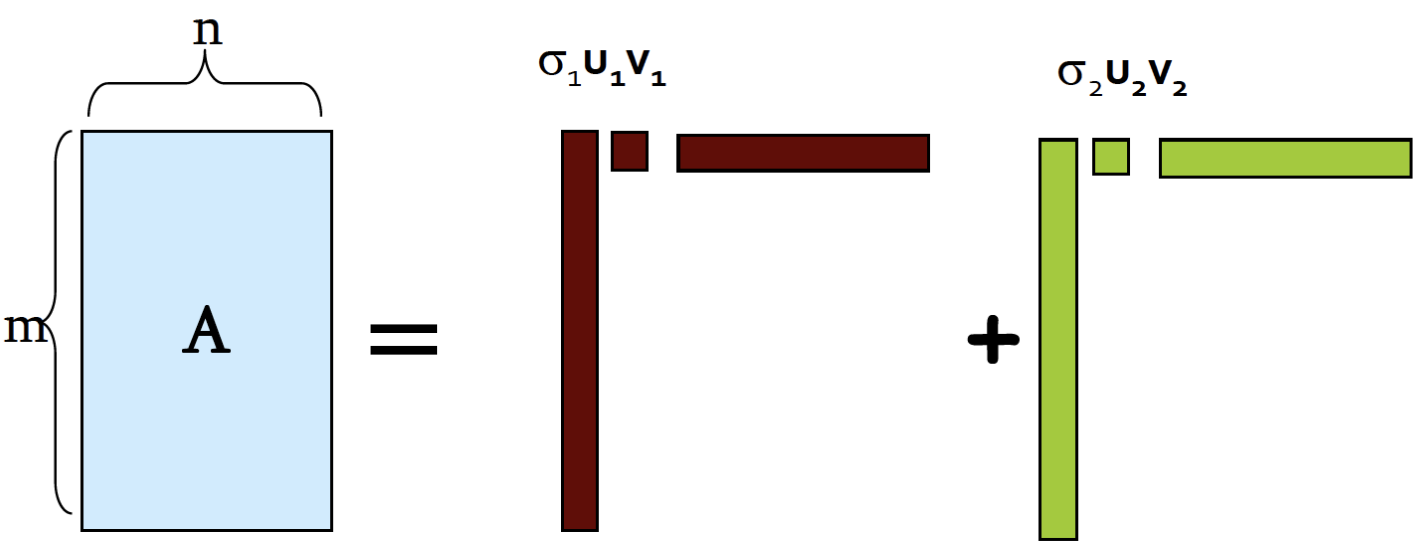
$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathsf{T}}$$



From Jure Leskovec's slides (Stanford), this and next ones

The case of r=2

$$\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathsf{T}}$$



σ_i … scalar

u_i … vector

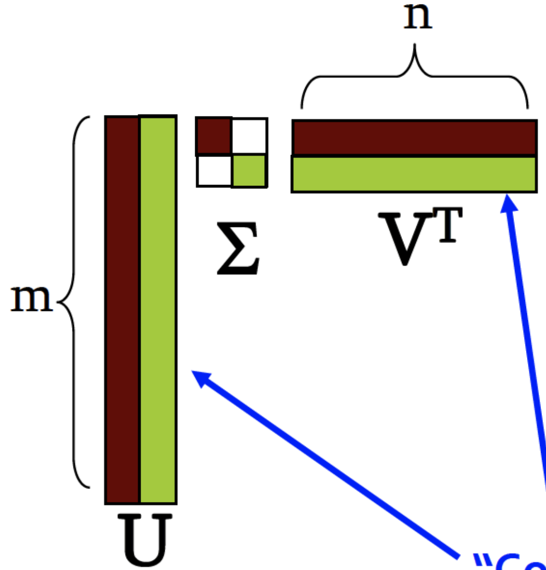
v_i … vector

SVD – Example: Users-to-Movies

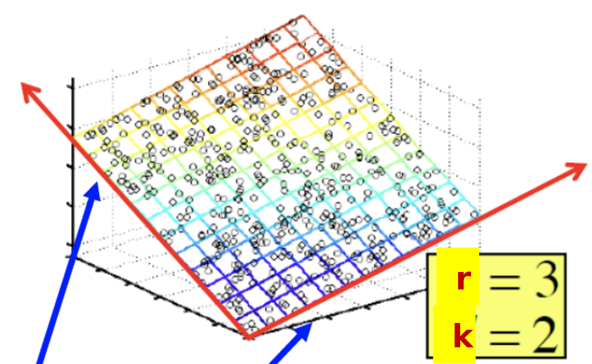
- $A = U \Sigma V^T$ - example: **Users to Movies**

7 5

$$\begin{array}{cc} & \begin{array}{ccccc}\text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie}\end{array} \\ \begin{array}{c}\uparrow\\ \text{SciFi}\\ \downarrow\\ \uparrow\\ \text{Romnce}\\ \downarrow\end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \end{array} \approx$$



k = projected space size



"Concepts"
AKA Latent dimensions
AKA Latent factors

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example: **Users to Movies**

$$
\begin{array}{cc}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

r = 3

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example: **Users to Movies**

$$
\begin{array}{c}
 \\
\text{SciFi} \\
 \\
 \\
\text{Romnce} \\
 \\
 \\
\end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{ccc} \text{SciFi-concept} & \text{Romance-concept} & \end{array} \\
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
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0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
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\end{bmatrix}
\end{array}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
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\begin{bmatrix}
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0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example:

SciFi-concept

“strength” of the SciFi-concept

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{array}{c} \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \\ \end{array} &
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
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\mathbf{0.55} & 0.09 & -0.04 \\
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\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example:

V is "movie-to-concept" similarity matrix

SciFi-concept

$$
\begin{array}{c} \text{SciFi} \\ \\ \\ \text{Romnce} \end{array}
\begin{array}{c}
\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
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\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
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0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SciFi-concept

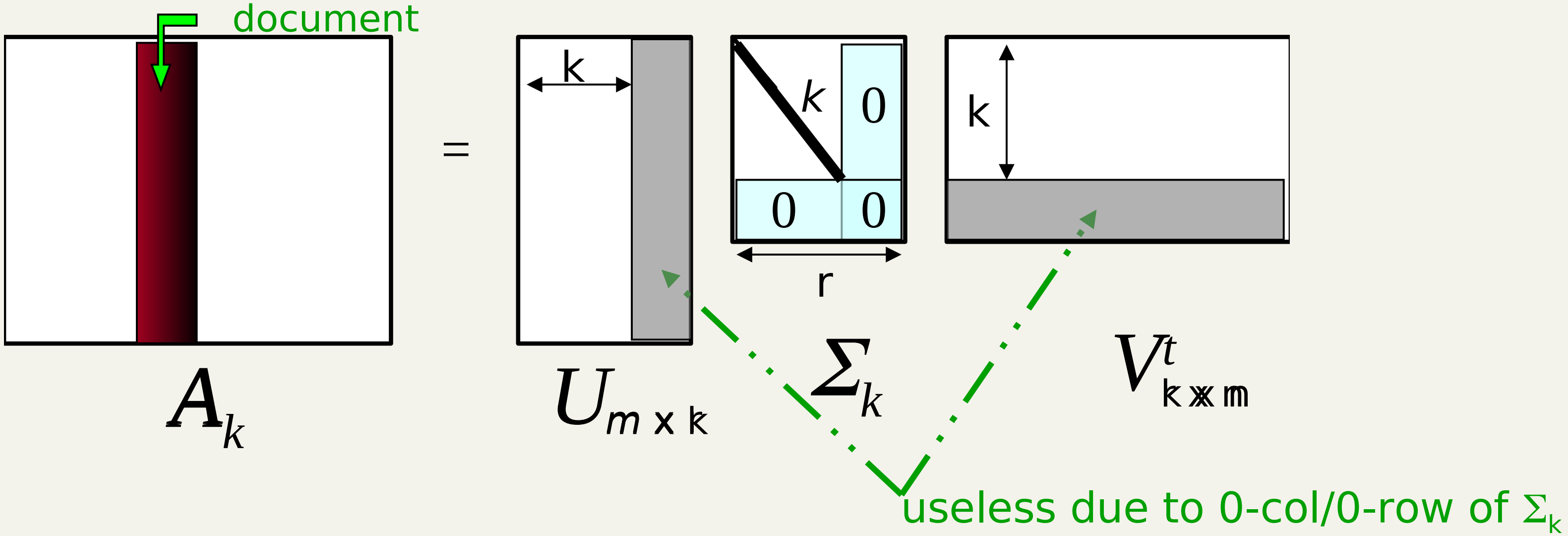
SVD - Interpretation #1

'**movies**', '**users**' and '**concepts**':

- ***U***: user-to-concept similarity matrix
- ***V***: movie-to-concept similarity matrix
- **Σ**: its diagonal elements: 'strength' of each concept

Dimensionality reduction

- Fix some $k \ll r$, *zero out* all but the k biggest eigenvalues in Σ *[choice of k is crucial]*
  - Denote by Σ_k this new version of Σ, having rank k
  - Typically k is about 100, while r (*A's rank*) is > 10,000



Guarantee

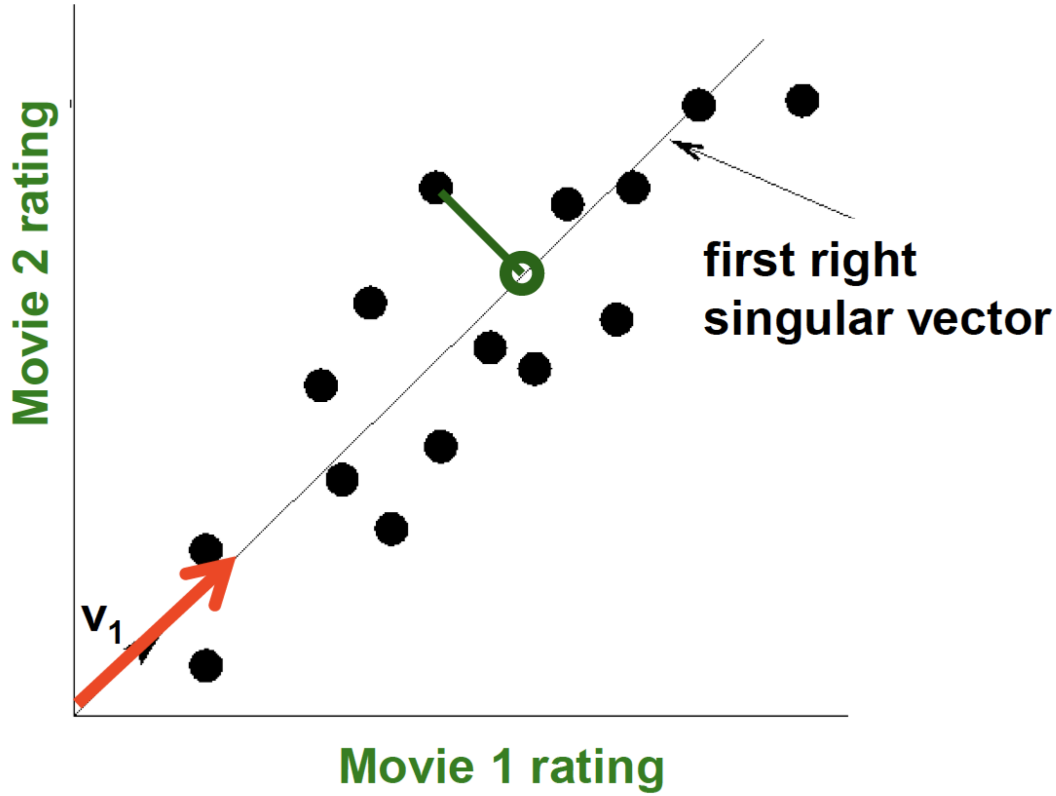
- A_k is a pretty good approximation to *A:*
  - Relative distances are (approximately) preserved
  - Of all $m \times n$ matrices of rank k, A_k is the best approximation to *A wrt* the following measures:
    - $\min_{B,\ rank(B)=k} ||A-B||_2 = ||A-A_k||_2 = \sigma_{k+1}$
    - $\min_{B,\ rank(B)=k} ||A-B||_F^2 = ||A-A_k||_F^2 = \sigma_{k+1}^2 + \sigma_{k+2}^2 + \ldots + \sigma_r^2$
- Frobenius norm $||A||_F^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_r^2$

Q: How many σs to keep?
A: Rule-of-a thumb:
keep 80-90% of 'energy' $= \sum_i \sigma_i^2$

SVD - Complexity

- To compute SVD:
  - **$O(nm^2)$** or **$O(n^2m)$** (whichever is less)
- But:
  - Less work, if we just want singular values
  - or if we want first *k* singular vectors
  - or if the matrix is sparse
- Implemented in linear algebra packages like
  - LINPACK, Matlab, SPlus, Mathematica ...

SVD – Dimensionality Reduction



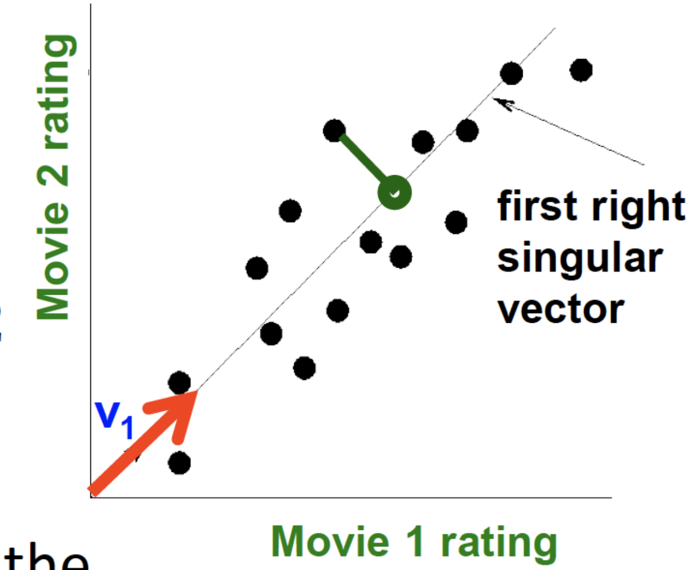
- Instead of using two coordinates (x, y) to describe point locations, let's use only one coordinate
- Point's position is its location along vector v_1
- **How to choose v_1? Minimize reconstruction error**

SVD – Dimensionality Reduction

- **Goal: Minimize the sum of reconstruction errors:**

$$\sum_{i=1}^{N}\sum_{j=1}^{D}\left\|x_{ij}-z_{ij}\right\|^2$$

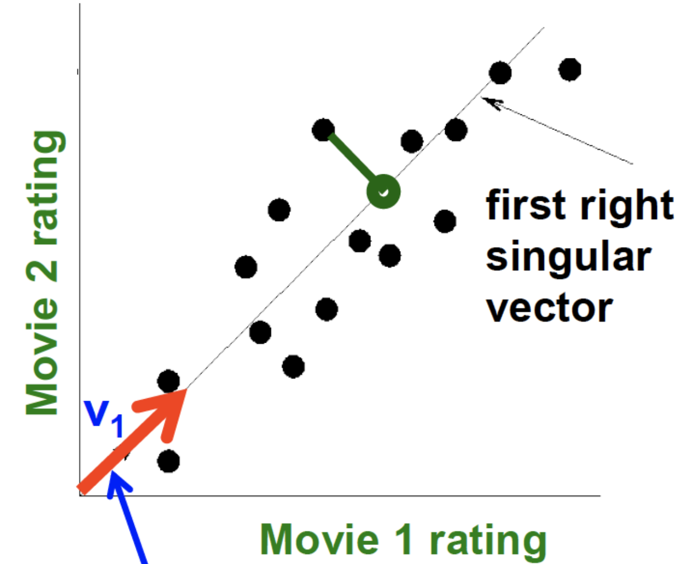
  - where x_{ij} are the "old" and z_{ij} are the "new" coordinates



- **SVD gives 'best' axis to project on:**
  - **'best'** = minimizing the sum of reconstruction errors
- **In other words, minimum reconstruction error**

SVD - Interpretation #2

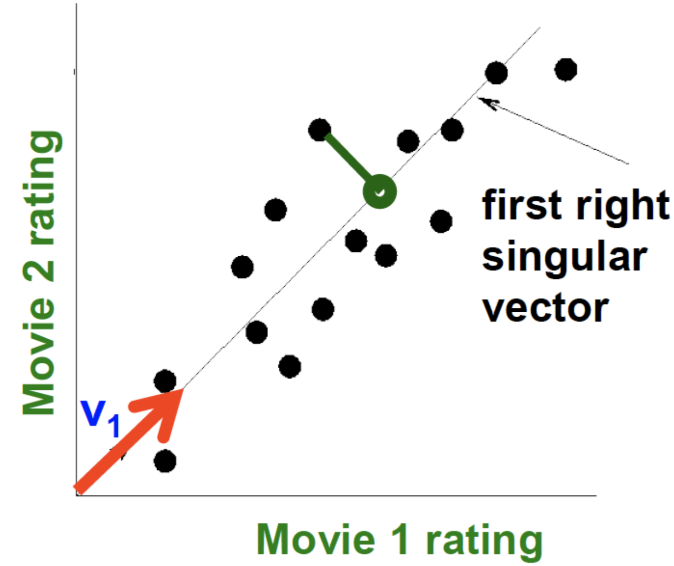
- **$A = U \Sigma V^T$ - example:**
  - **V**: “movie-to-concept” matrix
  - **U**: “user-to-concept” matrix



$$
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}
=
\begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix}
\times
\begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix}
\times
\begin{bmatrix} \boxed{\mathbf{0.56} \quad \mathbf{0.59} \quad \mathbf{0.56} \quad 0.09 \quad 0.09} \\ 0.12 \quad -0.02 \quad 0.12 \quad \mathbf{-0.69} \quad \mathbf{-0.69} \\ 0.40 \quad \mathbf{-0.80} \quad 0.40 \quad 0.09 \quad 0.09 \end{bmatrix}
$$

SVD - Interpretation #2

- $A = U \Sigma V^T$ - example:



variance ('spread') on the v_1 axis

$$
\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}
=
\begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix}
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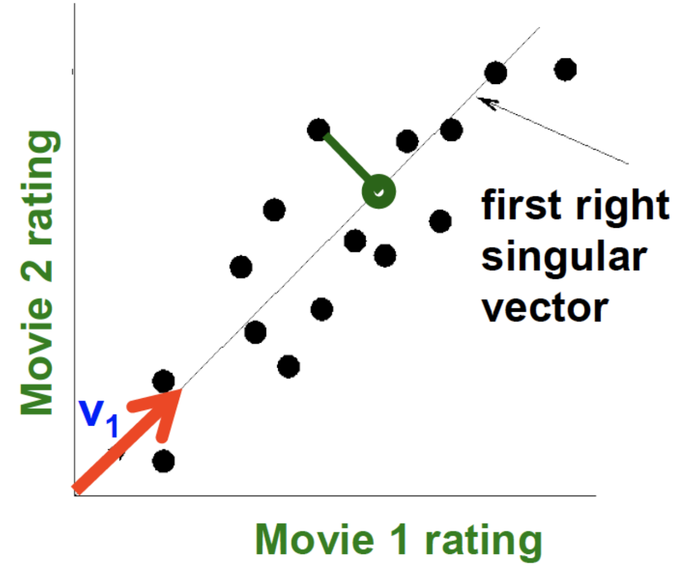
$$

SVD - Interpretation #2

$A = U \Sigma V^T$ - example:

- $U \Sigma$: Gives the coordinates of the points in the projection axis



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix}$$

Projection of users on the "Sci-Fi" axis $(U \Sigma)^T$:

$$\begin{bmatrix} 1.61 & 0.19 & -0.01 \\ 5.08 & 0.66 & -0.03 \\ 6.82 & 0.85 & -0.05 \\ 8.43 & 1.04 & -0.06 \\ 1.86 & -5.60 & 0.84 \\ 0.86 & -6.93 & -0.87 \\ 0.86 & -2.75 & 0.41 \end{bmatrix}$$

SVD - Interpretation #2

More details

- **Q:** How exactly is dim. reduction done?

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
=
\begin{bmatrix}
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\times
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\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD - Interpretation #2

More details

- Q: How exactly is dim. reduction done?
- A: Set smallest singular values to zero

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
=
\begin{bmatrix}
0.13 & 0.02 & -0.01 \\
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0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\
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$$

SVD - Interpretation #2

More details

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- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
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\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
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\times
\begin{bmatrix}
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0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD - Interpretation #2

This is Rank 2 approximation to A. We could also do Rank 1 approx. The larger the rank the more accurate the approximation.

More details

- Q: **How exactly is dim. reduction done?**
- A: Set smallest singular values to zero

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} 0.13 & 0.02 & \cancel{-0.01} \\ 0.41 & 0.07 & \cancel{-0.03} \\ 0.55 & 0.09 & \cancel{-0.04} \\ 0.68 & 0.11 & \cancel{-0.05} \\ 0.15 & -0.59 & \cancel{0.65} \\ 0.07 & -0.73 & \cancel{-0.67} \\ 0.07 & -0.29 & \cancel{0.32} \end{bmatrix} \times \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & \cancel{1.3} \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & -0.69 & -0.69 \\ \cancel{0.40} & \cancel{-0.80} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09} \end{bmatrix}$$

SVD - Interpretation #2

This is Rank 2 approximation to A. We could also do Rank 1 approx. The larger the rank the more accurate the approximation.

More details

- Q: **How exactly is dim. reduction done?**
- A: **Set smallest singular values to zero**

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 \\
\mathbf{0.41} & 0.07 \\
\mathbf{0.55} & 0.09 \\
\mathbf{0.68} & 0.11 \\
0.15 & \mathbf{-0.59} \\
0.07 & \mathbf{-0.73} \\
0.07 & \mathbf{-0.29}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 \\
0 & \mathbf{9.5}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69}
\end{bmatrix}
$$

SVD - Interpretation #2

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- Q: How exactly is dim. reduction done?
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$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.92} & \mathbf{0.95} & \mathbf{0.92} & 0.01 & 0.01 \\ \mathbf{2.91} & \mathbf{3.01} & \mathbf{2.91} & -0.01 & -0.01 \\ \mathbf{3.90} & \mathbf{4.04} & \mathbf{3.90} & 0.01 & 0.01 \\ \mathbf{4.82} & \mathbf{5.00} & \mathbf{4.82} & 0.03 & 0.03 \\ 0.70 & \mathbf{0.53} & 0.70 & \mathbf{4.11} & \mathbf{4.11} \\ -0.69 & 1.34 & -0.69 & \mathbf{4.78} & \mathbf{4.78} \\ 0.32 & \mathbf{0.23} & 0.32 & \mathbf{2.01} & \mathbf{2.01} \end{bmatrix}$$

Frobenius norm:

$\|M\|_F = \sqrt{\Sigma_{ij} M_{ij}^2}$

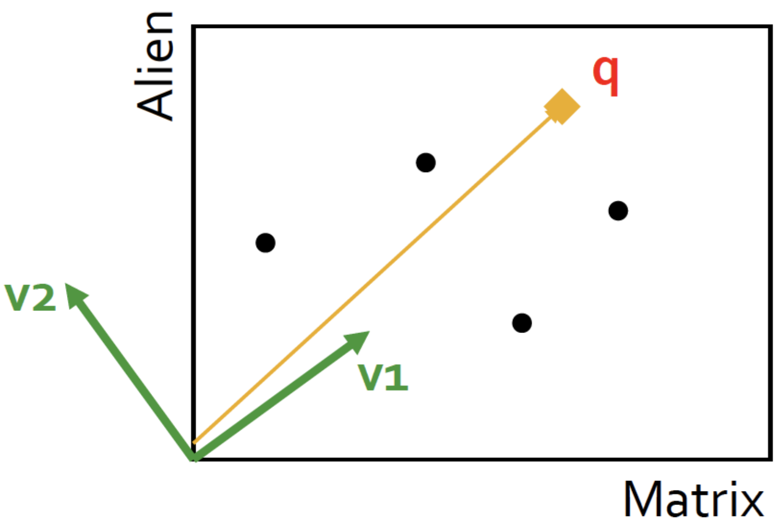
$\|A-B\|_F = \sqrt{\Sigma_{ij} (A_{ij}-B_{ij})^2}$
is "small"

Case study: How to query?

- **Q: Find users that like ‘Matrix’**
- **A: Map query into a ‘concept space’ – how?**

$$\mathbf{q} = \begin{array}{c} \begin{array}{ccccc}\text{Matrix}&\text{Alien}&\text{Serenity}&\text{Casablanca}&\text{Amelie}\end{array} \\ \begin{bmatrix}5&0&0&0&0\end{bmatrix}\end{array}$$

Project into concept space:
Inner product with each ‘concept’ vector $\mathbf{v}_i$



Interpret as another row in the matrix

Case study: How to query?

- **Q: Find users that like 'Matrix'**
- **A: Map query into a 'concept space' – how?**

This product U * Σ gives the projection of users to the 2 new dims, corresponding to the vectors v1, v2 of V matrix below

$$
\begin{array}{c|c}
 & \text{Matrix} \quad \text{Alien} \quad \text{Serenity} \quad \text{Casablanca} \quad \text{Amelie} \\
\begin{array}{c} \\ \text{SciFi} \\ \\ \\ \\ \text{Romnce} \\ \\ \end{array} &
\left[\begin{array}{ccccc}
1 & 1 & 1 & 0 & 0 \\
3 & 3 & 3 & 0 & 0 \\
4 & 4 & 4 & 0 & 0 \\
5 & 5 & 5 & 0 & 0 \\
0 & 2 & 0 & 4 & 4 \\
0 & 0 & 0 & 5 & 5 \\
0 & 1 & 0 & 2 & 2
\end{array}\right]
\end{array}
=
\left[\begin{array}{cc}
\mathbf{0.13} & 0.02 \\
\mathbf{0.41} & 0.07 \\
\mathbf{0.55} & 0.09 \\
\mathbf{0.68} & 0.11 \\
0.15 & \mathbf{-0.59} \\
0.07 & \mathbf{-0.73} \\
0.07 & \mathbf{-0.29}
\end{array}\right]
\times
\left[\begin{array}{ccc}
12.4 & 0 & 0 \\
0 & 9.5 & 0 \\
0 & 0 &
\end{array}\right]
\times
\left[\begin{array}{ccccc}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69}
\end{array}\right]
$$

Case study: How to query?

Compactly, we have:

$$q_{concept} = q\ V$$

E.g.:

$$q = \begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\ \end{array}$$

$$q = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.12 \\ 0.59 & -0.02 \\ 0.56 & 0.12 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix} = \begin{array}{cc} \text{SciFi-concept} & \\ \end{array}$$

$$= \begin{bmatrix} 2.8 & 0.6 \end{bmatrix}$$

movie-to-concept similarities (V)

Case study: How to query?

- **How would the user *d* that rated ('Alien', 'Serenity') be handled?**

$d_{concept}$ = d V

This is a new user that expresses preferences on-the-fly at query time

E.g.:

$$\underbrace{\begin{array}{ccccc} \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\ \hline 0 & 4 & 5 & 0 & 0 \end{array}}_{\mathbf{d}} \times \begin{bmatrix} 0.56 & 0.12 \\ 0.59 & -0.02 \\ 0.56 & 0.12 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix} = \begin{array}{cc} \text{SciFi-concept} & \\ \downarrow & \\ \left[\,5.2\right. & \left.0.4\,\right] \end{array}$$

movie-to-concept similarities (V)

Case study: How to query?

- **Observation:** User ***d*** that rated (‘*Alien*’, ‘*Serenity*’) will be **similar** to user **q** that rated (‘*Matrix*’), although ***d*** and **q** have **zero ratings in common**!

$$\begin{array}{cccccc} & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \end{array} \qquad \begin{array}{cc} \text{SciFi-concept} & \end{array}$$

$$\mathbf{d} = \begin{bmatrix} 0 & 4 & 5 & 0 & 0 \end{bmatrix} \dashrightarrow \begin{bmatrix} 5.2 & 0.4 \end{bmatrix}$$

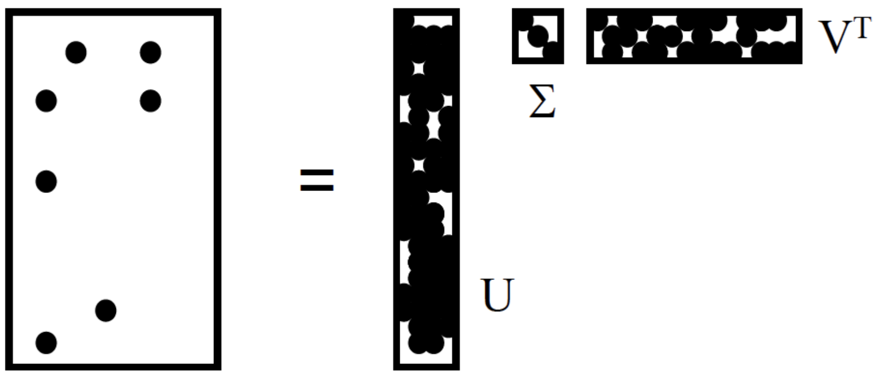
$$\mathbf{q} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \end{bmatrix} \dashrightarrow \begin{bmatrix} 2.8 & 0.6 \end{bmatrix}$$

Zero ratings in common

Similarity > 0

SVD: Drawbacks

- + **Optimal low-rank approximation** in terms of Frobenius norm
- − **Interpretability problem:**
  - A singular vector specifies a linear combination of all input columns or rows
- − **Lack of sparsity:**
  - Singular vectors are **dense!**



Recap

- c-th concept = c-th col of U_k *(which is m x k)*
  - $U_k[i][c]$ = strength of association between c-th concept and i-th row (term/user/…)
  - $V^t_k[c][j]$ = strength of association between c-th concept and j-th column (document/movie/…)
- Projected query: $q' = q\, V_k$
  - $q'[c]$ = strenght of concept *c* in q
- Projected column (doc/movie): $d'_j = d_j\, V_k$
  - $d'_j[c]$ = strenght of concept *c* in d_j