

# Formula Sheet for Math 200, Fall 2019

$$\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \right) \mathbf{a}.$$

$$\text{Det}(\mathbf{a}, \mathbf{b}) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 = - \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} = -\text{Det}(\mathbf{b}, \mathbf{a})$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \mathbf{a} \cdot \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

$$\mathbf{b} \times \mathbf{c} = \left\langle \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}, - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right\rangle$$

$$|\mathbf{b} \times \mathbf{c}| = |\mathbf{b}| |\mathbf{c}| |\sin \theta|$$

$$\text{Det}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = -\text{Det}(\mathbf{b}, \mathbf{a}, \mathbf{c}) \quad \text{so} \quad \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct, \quad \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\mathbf{r}_0 + t\mathbf{v}, \quad (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0, \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\text{dist}(ax + by + cz + d_1 = 0, ax + by + cz + d_2 = 0) = \frac{|d_1 - d_2|}{|\langle a, b, c \rangle|}$$

Distance from a point  $P(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$  is

$$D = |ax_1 + by_1 + cz_1 + d| / \sqrt{a^2 + b^2 + c^2}.$$

$$\begin{aligned} 1/2 &= \sin(30^\circ) = \sin(\pi/6) = \cos(60^\circ) = \cos(\pi/3) \\ 1/\sqrt{2} &= \sin(45^\circ) = \sin(\pi/4) = \cos(45^\circ) = \cos(\pi/4) \\ \sqrt{3}/2 &= \cos(30^\circ) = \cos(\pi/6) = \sin(60^\circ) = \sin(\pi/3) \end{aligned}$$

$$\cos(2\alpha) = 1 - 2\sin^2 \alpha = 2\cos^2 \alpha - 1$$

$$\frac{d}{d\theta} \tan(\theta) = \sec^2(\theta) = 1/(\cos^2(\theta))$$

$$\begin{aligned} L(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ f(x, y) - f(x_0, y_0) &= z - z_0 \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ \Delta f &= \Delta z \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y \\ df &= dz = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy \end{aligned}$$

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

Directional derivative ( $\mathbf{u}$  a unit vector):

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \langle f_x, f_y, f_z \rangle \cdot \mathbf{u}$$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$f_{xx}, \quad D = f_{xx}f_{yy} - (f_{xy})^2$$

Local min:  $f_{xx} > 0, D > 0$ ; local max:  $f_{xx} < 0, D > 0$ ; saddle:  $D < 0$ ; degenerate/indeterminate:  $D = 0$ .

Lagrange multipliers to maximize/minimize  $f = f(x, y)$  or  $f = f(x, y, z)$  subject to  $g = C = \text{Constant}$  where  $g = g(x, y)$  or  $g = g(x, y, z)$ :

$$\nabla f = \lambda \nabla g; \quad g = C$$

Mass and centre of mass, density  $\rho$  for two variables

$$m = \int \int_D \rho(x, y) dA, \quad \bar{x} = \frac{1}{m} \int \int_D x \rho(x, y) dA, \quad \bar{y} = \frac{1}{m} \int \int_D y \rho(x, y) dA$$

Mass and centre of mass, density  $\rho$  for three variables

$$m = \int \int \int_E \rho(x, y, z) dV, \\ \bar{x} = \frac{1}{m} \int \int \int_E x \rho(x, y, z) dV, \quad \bar{y} = \frac{1}{m} \int \int \int_E y \rho(x, y, z) dV, \quad \bar{z} = \frac{1}{m} \int \int \int_E z \rho(x, y, z) dV$$

Polar/cylindrical coordinates:  $x = r \cos \theta$ ,  $y = r \sin \theta$ ,

$$dA = dx dy = r dr d\theta, \quad dV = dx dy dz = r dz dr d\theta$$

Spherical (like polar in  $x, y$  with radius  $r = \rho \sin \phi$ ,  $z = \rho \cos \phi$ , so  $\phi$  measures angle with positive  $z$ -axis,  $0 \leq \phi \leq \pi = 180^\circ$ ):

$$x = (\rho \sin \phi) \cos \theta, \quad y = (\rho \sin \phi) \sin \theta, \quad z = \rho \cos \phi, \quad dV = \rho^2 \sin \phi d\rho d\phi d\theta$$