Professor Gia Barboza-Salerno

College of Social Work

The Ohio State University

Week 5: Probability Theory and Sampling Distributions II

September 23, 2023

Hypothesis Testing in a Probabilistic Framework

Intro to probability theory

The magazine Discover once had a special issue on "Life at Risk." In an article, Jeffrey Kluger describes the risks of making it through one day:

> *Imagine my relief when I made it out of bed alive last Monday morning. It was touch and go there for a while, but I managed to scrape through. Getting up was not the only death-defying act I performed that day. There was shaving, for example; that was no walk in the park. Then there was showering, followed by leaving the house and walking to work and spending eight hours at the office. By the time I finished my day – a day that also included eating lunch, exercising, going out to dinner, and going home – I counted myself lucky to have survived in one piece.*

Is this writer unusually fearful? No, why?

Intro to probability theory

- "There is not a single thing you can do ... that isn't risky enough to be your last."
- Examples
  - 1 out of 2 million people will die from falling out of bed.
  - 1 out of 400 will be injured falling out of bed
  - 1 out of 77 adults over 35 will have a heart attack this year
  - Black Americans are twice as likely to be shot by the police compared to Whites (https://www.washingtonpost.com/graphics/investigations/police-shootings-database/)
  - The average American faces a 1 in 13 risk of suffering some kind of injury in home that necessitates medical attention
  - 1 out of 32 risk of being the victim of some violent crime, 1 out of 14 risk of having property stolen this year
- These are all probabilities that were calculated from counts of reported accidents

Probability

The point is that we talk loosely about probability all the time, but we need to be more specific in terms of defining and using probabilities because as I will demonstrate with a couple of examples, the consequences are deadly

The Sample Space

- A **Sample Space** is a list of all possible outcomes, or events, of an experiment
- *Example*: List the possible sample space for 1 toss of a fair coin.
- S = {*H*, *T*}
- *Example*: List the possible sample space for 2 tosses of a fair coin.

- Let *A* be the event that the outcome of one toss of a fair coin is heads. Then $P(H) = \frac{1}{2}$

Events & Sample Spaces

A random process is a phenomenon whose *outcome cannot be predicted with certainty*

A probability event, *A*, is a collection of outcomes of an experiment. More specifically an event is the subset of the sample space. The sample space is the entire possible set of outcomes of an experiment.

Random Process	Event
Flipping a coin	Obtaining heads
Rolling a die	Getting a "5"
Flipping a coin 10 times	Getting 3 heads in 10 tosses
Getting the covid vaccine	Getting Covid
100 children graduate high school	4 of the 100 will go to Yale

Probability: An assessment of risk

There is a modest amount of theory we can cover. I will introduce some basic concepts of probability theory and then present some applications. This is meant to give you some basic skills by which to challenge what you read.

A probability is a number that lies between 0 and 1 that measures the uncertainly of a particular event

Mathematically, the probability of an outcome is simply

$$P(outcome) = \frac{A}{S}$$

where A = the number of ways an event can occur and S = the total number of possible outcomes

The probability of an event occurring is the fraction of time that it would happen if the random process occurs ***repeatedly under the same conditions***

Note: A = S x P(outcome), i.e., the number of ways an event can occur = the total number of possible outcomes times the probability that the event will occur. This is also called the **expectation**.

Harder: The Sample Space

- List all possible outcomes of 1 toss of a fair die
- List all possible outcomes of 1 toss of two fair die and then compute the probability of observing a total score of 6? At least 6? At most 12?

Example

The probability of being dealt a full house in poker is .0014. If you were dealt 100,000 poker hands, how many full houses should you <u>expect</u>?

Let A = the number of full houses we might expect, the P(A) = .0014 and S = 100,000, then:

$$A = S \times P(A)$$

$$A = 100000 \times .0014 = 140$$

Note: this is interpreted as a long-run average… Clearly, anything can happen at anytime.

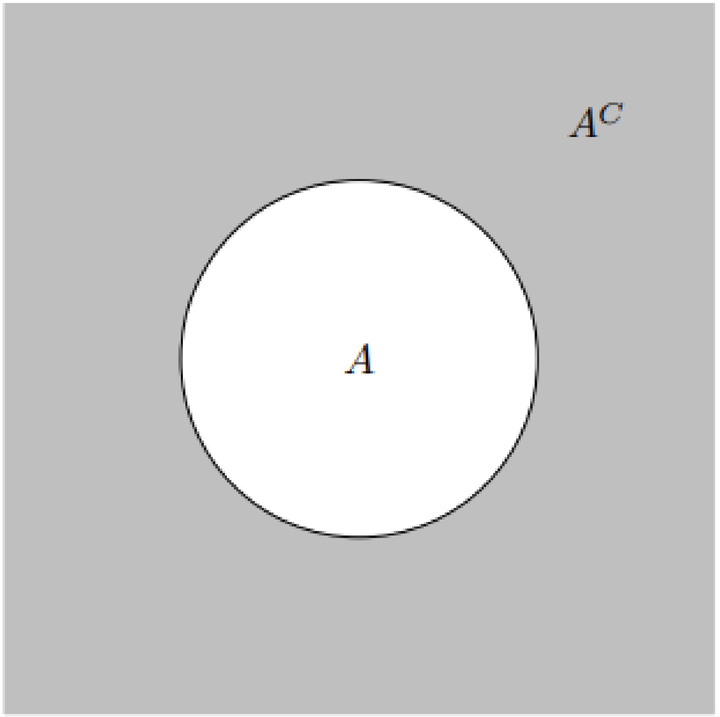
Assigning Probabilities

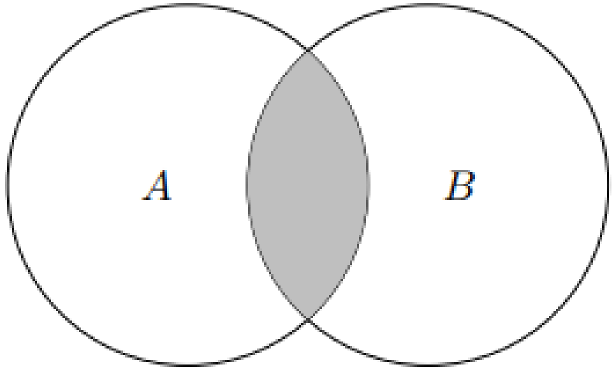
- Any probability that is assigned must fall between 0 and 1
- The sum of the probabilities across all outcomes must be equal to 1
- An outcome will be assigned a probability of 0 if one is sure that that outcome will never occur
- Likewise, if one assigns a probability of 1 to an event, then that event must occur all the time.

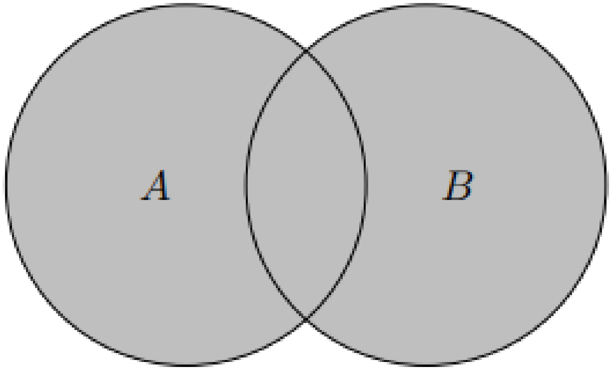
Formal Rules of Probability

- Numbers called probabilities are assigned to outcomes in the sample space such that the sum of the numbers over all outcomes is equal to one. **Example**.
  - Suppose that the sample space for our random experiment is S
  - We denote an event by a letter, say A, where A is a subset of S
- We are often interested in events that are derived from other events
  - Rolling a 2 OR a 3
  - Patient who receives therapy for trauma is relieved of symptoms AND has better communication skills
- Events derived from other events can be described as follows
  - $A \cap B$ is the event that both A and B occur, i.e. the intersection of two events
  - $A \cup B$ is the event that either A or B occur, i.e. the union of two events
  - A^c is the event that A does not occur (called the complement of A)

Venn Diagrams







Illustration

Suppose I choose a student at random from class and record the month he, she or they were born. The student could be born during any of the 12 months.

- List the sample space of the experiment
- Let A = the student was born during the last half of the year and B = the student is born during a month that is four letters long.
- Describe the following events in words and find each
  - $A \cap B$
  - $A \cup B$
  - A^c and B^c

Union of Events

Let A = rolling a 2, and B = rolling a 6 in one toss of a fair die

What is the probability of rolling a 2 or a 6?

i.e., $\mathrm{P}(\mathrm{A} \cup B)$ =??

It seems that P(A∪B)= P(A) + P(B) however this is generally not true

The Addition Rule

To determine the probability of $\mathrm{A} \cup B$:

- $\mathrm{P}(A)$
- $\mathrm{P}(B)$
- $\mathrm{P}(A \cap B)$

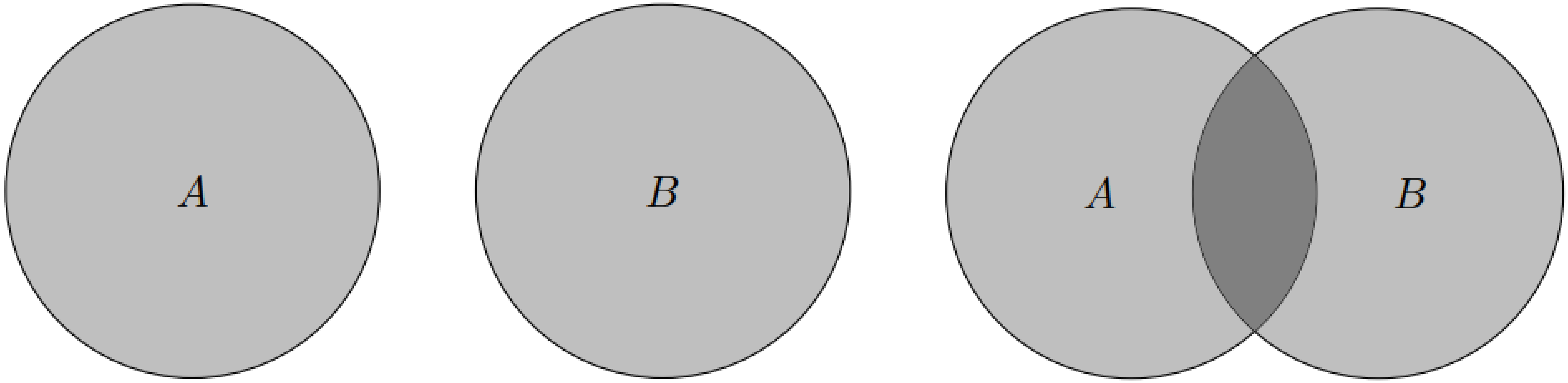
$$\mathrm{P}(\mathrm{A} \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

In the previous example, why does $\mathrm{P}(\mathrm{A} \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

Example

Let A = the event of rolling a 1,2, or 3 and B = the event of getting an odd number. What is the probability of A OR B?

List the sample space…



The Complement

An event can either occur (A) or not occur, i.e., *not* A. The latter is denoted A^c

If we know the probability that an event did occur we can compute the probability it will not occur as follows:

$$P(A^c) = 1 - P(A)$$

Summary

The probability of an event is the fraction of time that it happens (under identical repeated conditions)

Know the meaning of complements (A^c), intersections (A ∩ B), and unions (A ∪ B)

Addition rule: P(A ∪ B) = P(A) + P(B) − P(A ∩ B)

If (and only if!) A and B are mutually exclusive, we can ignore P(A ∩ B) in the addition rule

Complement rule: P(A^c) = 1 − P(A)

Independence: Balls in Urns

Imagine a random process in which balls are placed into an urn and picked at random

If the urn contained 1 red ball and 2 black balls, and we let *A* = the event of drawing a red ball, what is P(A)?

What if we draw a ball, put it back into the urn, draw a second ball. Let *A* = the event of drawing a red ball on the first draw, and B = the event of drawing a red ball on the second draw. What is $\mathrm{P}(A \cap B)$? → (1/3)(1/3)=1/9

On the surface it seems like

$\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$ **BUT** $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \cdot \mathrm{P}(B)$

Note carefully, if $\mathrm{P}(A \cap B) \neq \mathrm{P}(A) \cdot \mathrm{P}(B)$ then A and B are NOT independent

Balls in Urns

Suppose we draw the first ball but don't put the ball back in the urn. After the first draw, there are no red balls in the urn. The probability of a red ball in the second draw is 0. Why doesn't the rule of multiplying probabilities work?

The outcome of the first event changed the distribution, now *conditional on drawing a red ball on the first draw,* the P(B) = 0 $\neq \frac{1}{3}$

When we draw *without replacement, what happens on the second draw depends on what happened on the first draw.* The notion that the probability of an event may change depending on other events is called conditional probability, denoted $P(A|B)$

In this example $P(A) = \frac{1}{3}; P(B|A) = 0; P(B|A^c) = \frac{1}{2}$

The multiplication rule

To determine $P(A \cap B)$ we use the multiplication rule, which is always true for any two events:

$$P(A \cap B) = P(A) \cdot P(B|A)$$

$$P(A \cap B) = P(B) \cdot P(A|B)$$

The multiplication rule allows us to compute conditional probabilities as well, by re-arranging the formula

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Note: this is the idea behind dependence

Here it is

1 red balls, 2 black balls

A = the event of drawing a red ball on the first toss,
B = the event of drawing a red ball on the second toss

$$P(A) = \frac{\text{1 red ball}}{\text{3 total balls}}$$

$P(B|A)$ = the probability of getting a red ball on the second draw conditional on getting a red ball on the first draw which is clearly 0. There are no red balls left

P$(B|A^c) =$
the probability of getting red on the second draw conditional on NOT getting red on the first draw $\rightarrow$
the probability of getting red on the first draw times the probability of getting red on the second draw given you got black on the first draw
1/3 x 1/2 = 1/6

Independence

As we have seen before, there are times when one event is completely unaffected by another event

If one event, B, is not affected by another event, A, then the probability of B given A is equal to the probability of B, i.e.,

$$P(B|A) = P(B)$$

Then we say that A and B are independent. If the probability of A depends on B (or vice versa) then A and B are dependent

Dependence and Independence

For each, state P(B|A)

Event A	Event B
Patient recovers	Patient got treatment
Student is admitted to Yale	Students father went to Yale
Person is arrested while bird watching	Bird watcher is Black

Independence and the Multiplication Rule

If two events, A and B, are independent then $P(A \cap B) = P(A) \cdot P(B)$

Example: What is the probability of two heads in two tosses of a fair coin?

If two events, A and B, are dependent we need to use the multiplicative rule

Example: An urn contains 5 balls, 3 red and 2 black, and we sample without replacement two balls. What is the probability of drawing two black balls?

Review

If two events, A and B, are independent then

$$P(A \cap B) = P(A) \cdot P(B)$$

If two events, A and B, are NOT independent then

$$P(A \cap B) = P(A) \cdot P(B|A)$$
$$P(A \cap B) = P(B) \cdot P(A|B)$$

Example

An article in the American Journal of Public Health reported that in a certain population, the probability that a child's gestational age is less than 37 weeks is 0.142

- The probability that his or her birth weight is less than 2500 grams is 0.051
- The probability of both is 0.031

Let A = P(gestational age < 37 weeks) and
Let B = P(bw < 2500 g)

1) Are A and B independent?
2) For a randomly selected child, what is the probability that A or B will occur?

Example

Assume the probability that a child's gestational age is less than 37 weeks is .142, the probability that the child's birth weight is less than 2500 grams is .051, and the probability of both is .031.

What is the probability that a child's birth weight will be less than 2500 g given that the child's gestational age is less than 37 weeks?

The Case of Sally Clark

- Clark's first son died in December 1996 within a few weeks of his birth.
- Her second son died in similar circumstances in January 1998.
- A month later, Clark was arrested and tried for both deaths.
- Both sides agreed that neither baby died of SIDS
- Sir Roy Meadow, a pediatrician testified as to the probability of observing two deaths by SIDS in the same household

A gross miscarriage of justice….

"… it's the chance of backing that long odds outsider at the Grand National, you know; let's say it's a 80 to 1 chance, you back the winner last year, then the next year there's another horse at 80 to 1 and it is still 80 to 1 and you back it again and it wins. Now here we're in a situation that, you know, to get to these odds of 73 million you've got to back that 1 in 80 chance four years running, so yes, you might be very, very lucky because each time it's just been a 1 in 80 chance and you know, you've happened to have won it, but the chance of it happening four years running we all know is extraordinarily unlikely. So it's the same with these deaths. You have to say two unlikely events have happened and together it's very, very, very unlikely."

Comment on the expert's testimony.

The Mistake

The doctor calculated the probability of one of the children' dying from SIDS as being 1 in 8543, and that therefore the probability that *both* would die as

$$\left(\frac{1}{8543}\right)^2 = \frac{1}{73{,}000{,}000}$$

This meant that the probability of guilt was

$$1 - \frac{1}{73{,}000{,}000} \sim .9999999$$

The Response

The calculation leading to 1 in 73 million is invalid. It would only be valid if SIDS cases arose independently within families, an assumption that would need to be justified empirically. Not only was no such empirical justification provided in the case, but there are very strong reasons for supposing that the assumption is false. There may well be unknown genetic or environmental factors that predispose families to SIDS, so that a second case within the family becomes much more likely than would be a case in another, apparently similar, family.

Also, IF a child is murdered the likelihood that a parent did it is quite high. However, the likelihood of a child being killed by a parent is thankfully very small…

“A Spectacular Miscarriage of Justice”: The Sudden Infant Death Study at Trial

Rate for groups with different factors	SIDS INCIDENCE IN THIS GROUP*
Anybody smokes in the household (SIDS more likely) *Nobody smokes in the household*	in 737 in 5041
No waged income in the household *At least one waged income in the household*	in 486 in 2088
Mother <27 years and parity *Mother > 26 years and parity*	in 567 in 1882
None of these factors One of these factors Two of these factors All three of these factors	in 8543 in 1616 in 596 in 214

People v. Collins (1968) – another miscarriage of justice

Juanita Brooks was robbed in an alley and her purse was stolen. A witness testified at trial that she saw a blonde woman run out of the alley and enter a partly yellow car driven by a Black male with a mustache and beard. After testimony from a local math professor about the multiplication rule in probability (i.e., the independence rule), the jury was told to consider the probability that the suspects, who fit the description of the witnesses, were not the culprits. The following probabilities were offered to the jury:

The Evidence

Yellow car 1/10
Man with moustache 1/4
Girl with ponytail 1/10
Girl with blond hair 1/3
Black man with beard 1/10
Interracial couple in car 1/1000

The probability of guilt is about 1 because the probability of these things occurring is 1/12,000,000…

Challenges…

Evidentiary Foundations?
Independent Events?
Reporting Bias? – is this what was really observed?
The 1 in 12 million figure does not relate to the probability of guilt

So far, we have discussed the probability of single events
In research, however, the data we collect consists of many events (for each subject, does he/she contract polio?)
We then summarize those events with a number (out of the 200,000 people who got the vaccine, how many contracted polio?)
Such a number is an example of a random variable

Distributions

In our sample, we observe a certain value of a random variable
In order to assess the variability of that value, we need to know the chances that our random variable could have taken on different values depending on the true values of the population parameters

- This is called a distribution
- A distribution describes the probability that a random variable will take on a specific value or fall within a specific range of values

Random Variable	Possible Outcomes
# of women who will be sexually assaulted	0,1,2,…
# of veterans diagnosed with PTSD	0,1,2,…
# of ACEs	0,1,2,…

Back to the Coin Toss

Suppose we flip a coin three times. What is the probability that exactly one of the flips will be heads?

Possible outcomes

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

P(HHH) = 3/8

How did I count the number of ways that I could get three heads? If there are a large number of “things” this quickly becomes infeasible…

Binomial Coefficients

When there are two possible outcomes that occur, or don't occur, *n* times, the number of ways that one event can occur *k* times is

$$\frac{n!}{k!\,(n-k)!}$$

where n! means to multiple *n* by all of the numbers that come before it

Examples: 3! = 3x2x1 and 100! = 100x99x98x97….x1

Note that 0!=1 *why?*

Revisit the coin example, how many ways can we get 3 heads (HHH) in 3 tosses of a fair coin?

In words, out of three things (the tosses) we need three things, or from “3” choose “3”

$$\binom{3}{3} = \frac{3!}{3!\,(3-3)!} = \frac{1}{1x1} = 1$$

How many ways can 2 things be chosen from 10?

```
> choose(3,1)
[1] 3
> choose(10,2)
[1] 45
```

Example

- Suppose one has a bowl with 4 black balls and 2 white balls. Draw two balls at random. What are the possibilities? What is the probability of getting 2 white balls?
- Suppose now the bowl has 100 balls, 25 balls are black and 75 balls are white. Select 12 balls at random. What is the probability that you get 12 white balls?
- How many black balls would you expect to see in your sample of $N = 12$?

The Binomial Sampling Distribution

To calculate the probability of observing *X* results in *n* trials use the Binomial formula

The binomial coefficient is defined by the next expression:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

The binomial formula is defined by :

$$p(K = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where *X* = the number of successes, *n* = the number of trials, *p* = the probability of success on each trial

According to the CDC, 22% of adults in the United States smoke. Suppose we sample 10 people, what is the probability that 5 of them will smoke?

We can use the binomial formula, with

$$\frac{10!}{5!\,(10-5)!}.22^5\,(1-.22)^{10-5} = .037$$

```
> dbinom(5, size=10, prob=.22)
[1] 0.03749617
```

What is the probability that our sample will contain two or fewer smokers?

The Binomial Formula

This formula works for any random variable that counts the number of times an event occurs out of n trials, provided that the following assumptions are met:

- The number of trials n must be fixed in advance
- The probability that the event occurs, p, must be the same from trial to trial
- The trials must be independent

If these assumptions are met, the random variable is said to follow a binomial distribution, or to be binomially distributed

Real Juror Data: The Fifth circuit is ~40% black

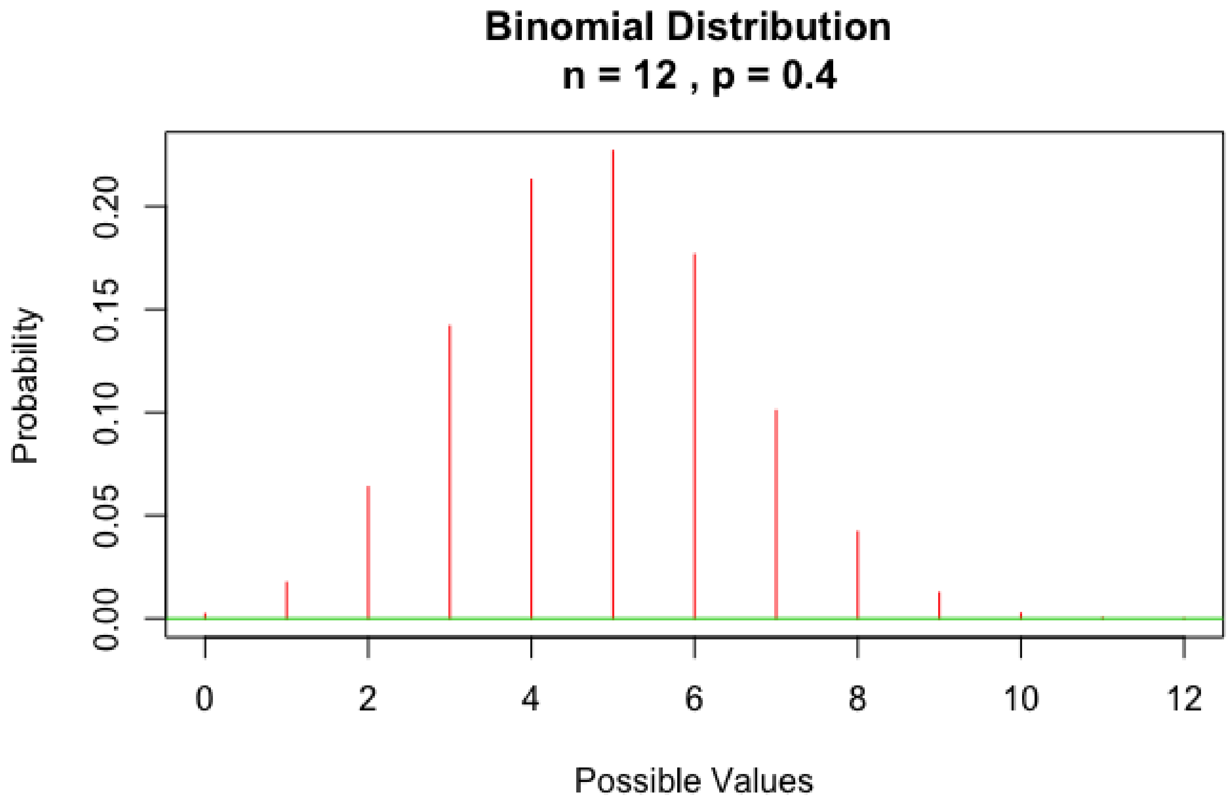
How many black jurors would you **expect if the trial is fair?**

Ratio of Black Jurors to White Jurors	Number of trials (literally)
0:12	14
1:11	19
2:10	27
3:9	28
4:8	27
5:7	24
6:6	22
7:5	13
8:4	4
9:3	2
10:2	1
11:1	0
12:0	0
Total	181

Question: How to test whether jury selection is a random process (i.e., the null hypothesis)

If the composition of the jury pool is 40% black how many black people would you expect to be on the jury?

We would expect .4*12 = ~4.8 blacks on the jury, or 5…



Does this cast doubt on whether the sampling was truly random?

What assumptions?

2 MS Co

https://github.com/APM-Reports/jury-data

To be eli[...] the trial is conducte[...] an 'infamous crime' or [...]er or habitual drunkard [...] An intere[...]

Illiteracy

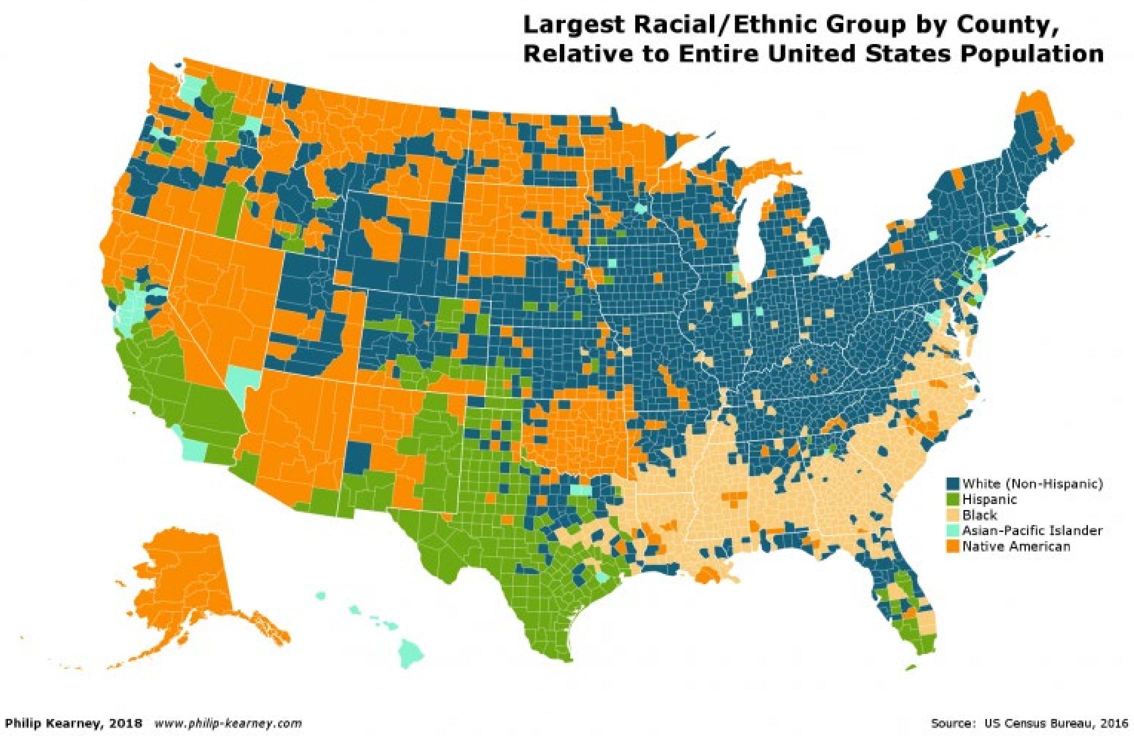
Percentage of population age 16 and older that lacks basic prose literacy skills.

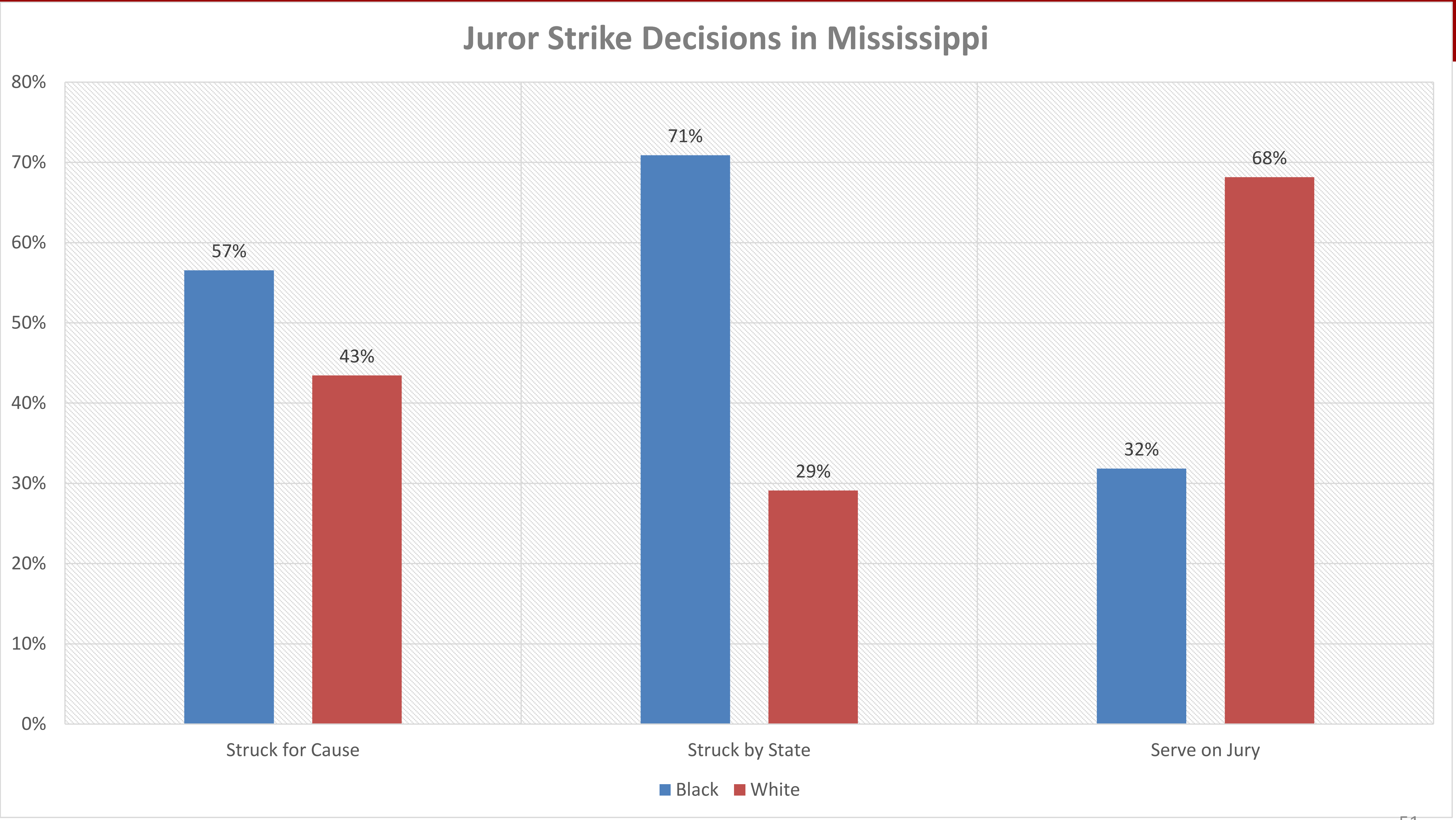
The 2011 County Health Rankings used data from 2003 for this measure.

Data | Description | Data Source

Place	Population	% Illiterate	Error Margin
Jefferson	6,718	30.1%	15.2-49.2%
Wilkinson	7,254	29.2%	15.2-47.3%
Holmes	14,891	28.6%	14.2-47.1%
Noxubee	8,866	28.4%	14.7-46.0%
Humphreys	7,594	27.5%	13.9-45.4%
Quitman	6,890	27.0%	13.8-44.5%
Issaquena	1,422	26.8%	13.8-43.9%
Sharkey	4,414	26.4%	13.4-43.7%
Tunica	7,085	25.7%	12.8-43.0%
Sunflower	21,763	25.6%	13.0-42.2%

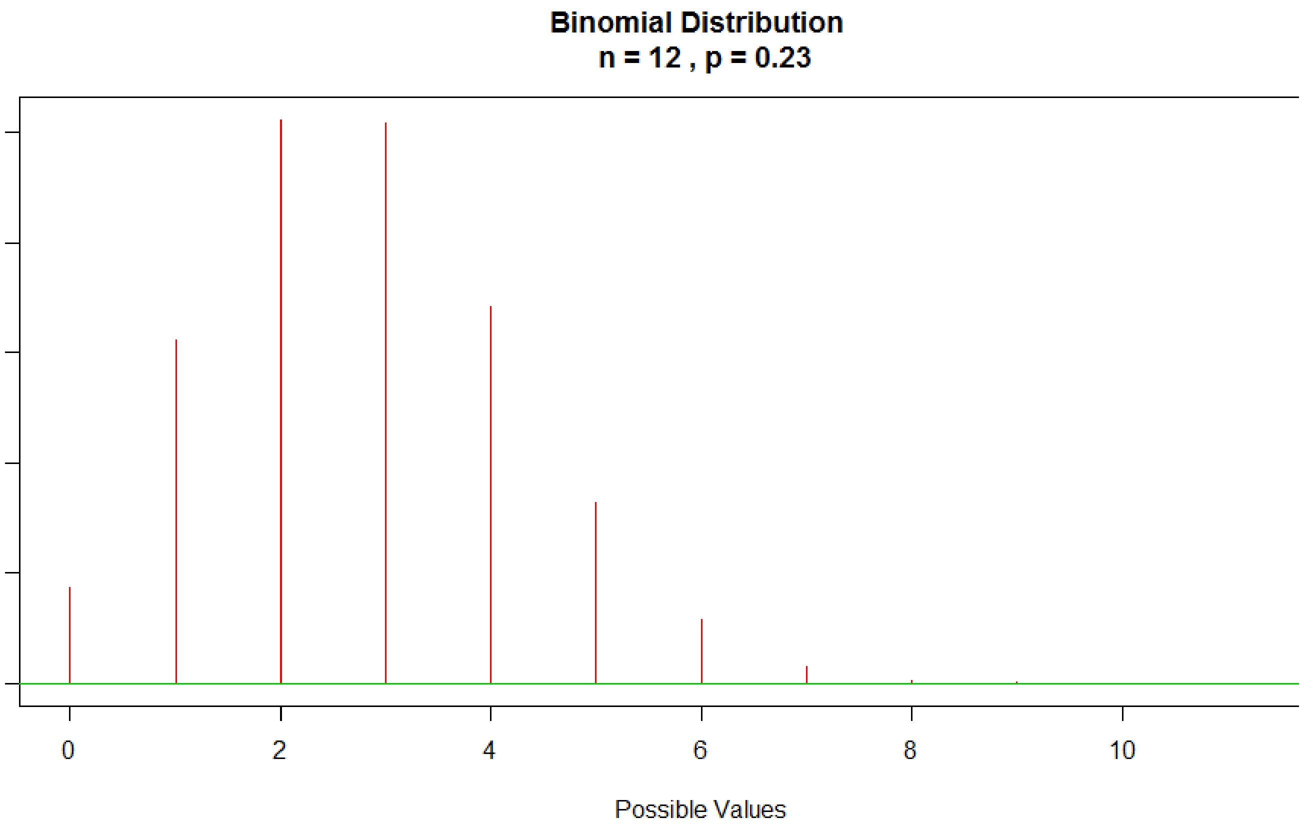
A jury of someone else's peers




Juror Strike Decisions in Mississippi
80%
70%
60%
50%
40%
30%
20%
10%
0%
57%
43%
71%
29%
32%
68%
Struck for Cause
Struck by State
Serve on Jury
Black
White

Oddity in Picking Jurors Opens Door to Racial Bias

Allen Snyder, a black man, is on death row in Louisiana. An all-white jury in Jefferson Parish, in the New Orleans suburbs, sentenced him to death in 1996 for the fatal stabbing of a man his wife was seeing. (How) is it possible to get an all-white jury in Parish? In the 20 years following the lifting of the moratorium on the death penaltiy, there were 20 murder trials in Jefferson Parish that ended in death sentences. Information about the race of the jurors is available in 18 of them. Parish is 23% black, so how many jurors might we expect on each 12-member panel? of the 18 juries, 10 had no black members. Seven had one. One had two. None had three.



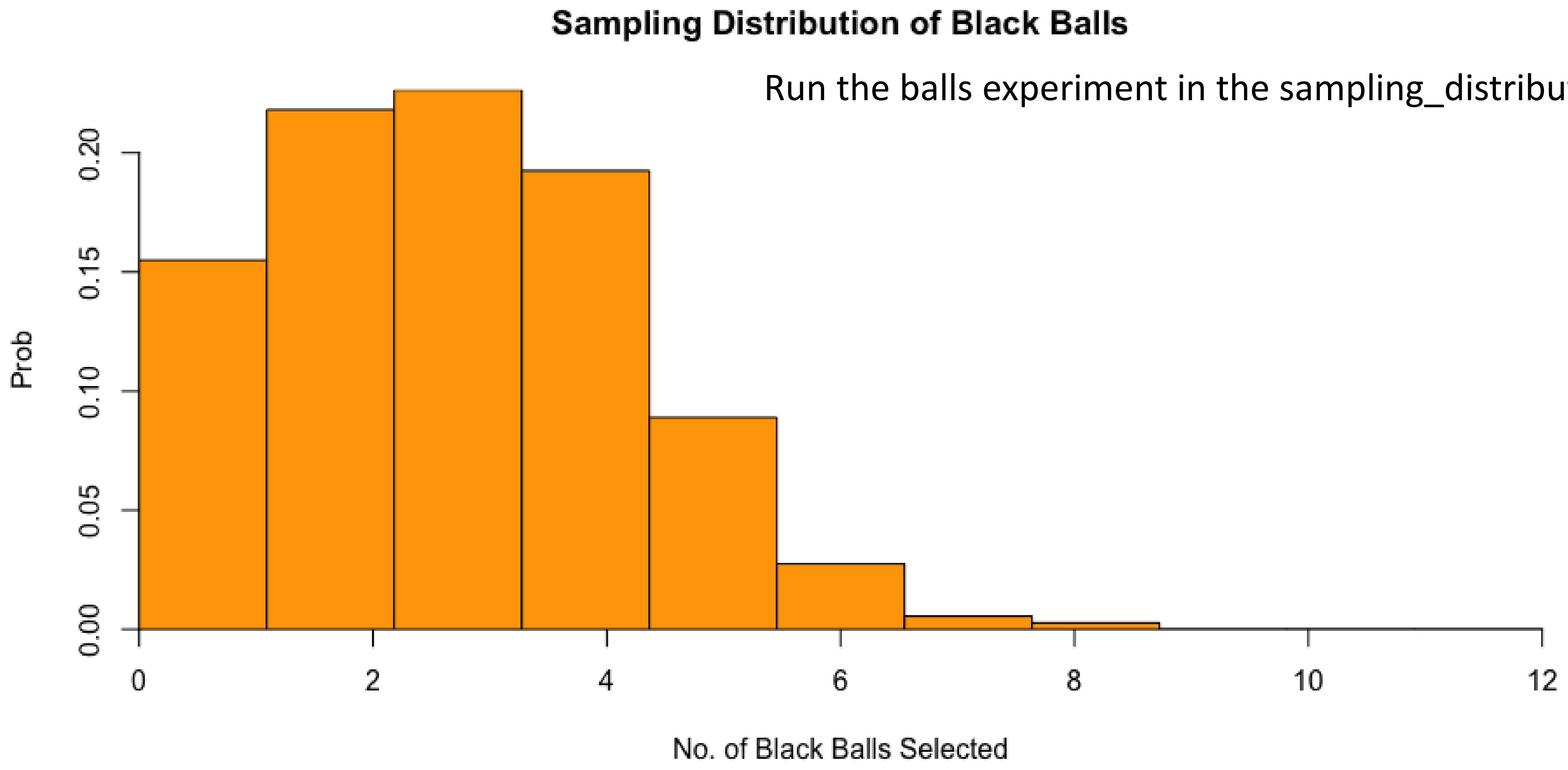
We should expect about 3 black jurors on each 12-member panel: 12 X .23 = 2.76.

The probability of observing no black jurors in this county is about .04 as follows:

$$\binom{n}{x} p^x (1-p)^{n-x}$$

$$\binom{12}{0} .23^0 \,(1-.23)^{12-0} \approx 0.04$$

Results of Experiment



Run the balls experiment in the sampling_distributions.Rmd

No matter how many times you run this, you will never see a high probability of getting 12 black balls. Think about the implications for this... JURY SELECTION IS NOT RANDOM...

Summary

A random variable is a number that can equal different values depending on the outcome of a random process

The distribution of a random variable describes the probability that the random variable will take on those different values

The number of ways to choose k things out of n possibilities is:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial distribution: the probability that an event will occur *k* times out of *n* is:

$$p(K=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

where *p* is the probability that the event will occur on any particular trial

Examples

- Calculate the probability of each of the following:
    1. Two tails in two tosses of a fair coin.
    2. Two heads in three tosses of a fair coin
    3. Four heads in four tosses of an unfair coin where the probability of a head is 0.75
    4. One six in three rolls of a fair die
    5. Five fours in five rolls of an unfair die where the probability of a four is 0.25.

The Fair Coin

- Example: In football, a coin toss$\Rightarrow$ at what point do you become suspicious that the coin is unfair?
- Statistical inference provides a ***systematic way*** of determining risk
- The coin toss can be thought of as a test statistic
  - What is H_0
  - What is H_1
  - What is the Type I error of the test statistic?
  - What is the Type II error?
  - What percentage of the time are you willing to be wrong?

The Fair Coin

- How can we calculate the risk of a Type I error associated with a specific outcome in a test of statistical significance also called the observed significance level?
  - One way is to determine how often a fair coin would be that the Patriots do not always get the ball
- Ultimately you are interested in determining the likelihood of getting X heads in a very large number of ***samples or trials*** of a fair coin (i.e., tosses in 16 football games for 10 years)
  - The resulting distribution is the sampling distribution
  - The shape of the distribution of the number of heads in many trials is called a binomial distribution
- Let's consider what thousands of samples of X = 10 tosses of a fair coin looks like...