2018-11-05 Second-Order ODEs - Linear Homogeneous (4.3)

4.3 Linear Homogeneous

Pre-lecture:

Solving non-homogenous DEs: ay''(t) + by'(t) + cy(t) = g(t) for some function g(t)

The method of undetermined coefficients:

E.g. Given y' - 2y = x + 2

y' - 2y = 0 -> y' = 2y -> $y = c \cdot e^{2x}$

Let y = ax + b, y' = a -> y' -2y = a - 2(ax+b) ->(-2a)x + (a-2b) = x+2 -> -2a = 1, a = -1/2, a - 2b = 2, b = -5/4

$y = ce^{2x} + \left(-\frac{1}{2}x - \frac{5}{4}\right)$

Continuation:

Q. Model the position y(t) of a keypress of one laptop key:

For key being released,

my'' = -ky - γy'

y(0) = 0.5

y'(0) = 0

Idea to find solution: try y = e^{rt}

2. Find a formula for r.

$mr^2 = -k - \gamma r$

$mr^2 + \gamma r + k = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

3. What kind of number can r be?

- 2 distinct real numbers
- 1 repeated real number
- 2 distinct complex numbers

4. What happens to the key when γ is large? Do we want this?

y'' = -13y - 14y'

y(0) = 0.5

y'(0) = 0

Solve the system: $r^2 + 14r + 13 = 0$, r = -13 or r = -1, so y(t) = $c_1 e^{-13t} + c_2 e^{-t}$.

Plug in y(0) = 0.5 and y'(0) = 0, we get y(t) = $-\frac{1}{24} e^{-13t} + \frac{13}{24} e^{-t}$.

- γ large means dampening will be larger, which results in the key being harder to push down due to being stiff

5. What happens to the key when γ is small? Do we want this?

y'' = -13y - 4y'

y(0) = 0.5

y'(0) = 0

Solve the system: $r^2 + 4r + 13 = 0$, r = -2$\pm$3i, so y(t) = $e^{-2t} \left(a_1 \cos(3t) + a_2 \sin(3t)\right)$ or y(t) = $c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}$.

y(t) = $c_1 e^{-2t} e^{3it} + c_2 e^{-2t} e^{-3it}$

(NOTE: Euelers Formula: e^{it} = sin(t) + cos(t))

- γ small means dampening will be smaller, which results in the key being easier to push down
- as t goes to infinity, this solution oscillates

6. We want a laptop key that doesn't oscillate, but we also don't want too much damping. What is the minimum amount of damping necessary for the key not to oscillate?

$$y'' = -\beta y - \gamma y'$$

What is γ* such that:

γ < γ*, oscillation,discriminant < 0

γ > γ*, no oscillation, discriminant > 0

discriminant = 0, γ* = $\sqrt{52}$ which is between 7 and 8

Note: We want the discriminant to be 0 because we don't want over and under damping.

7. What happens to the key that is critically damped?

(NOTE: this is when we have one value of r that is repeated!)

y'' = -9y - 6y'

- **Find one solution $y_1(t)$.**
  - From y'' = -9y - 6y' we get $r^2 + 6r + 9 = 0$, r = -3
  - Therefore, $y_1(t) = c_1 e^{-3t}$
- **How do we find a second solution?**
  - Look for solutions of the form $y(t) = y_1(t)\, v(t) = c_1 e^{-3t} v(t)$
- **Which ODE does v(t) satisy?**
- **Find v(t). Find y(t).**