2018-10-02 Autonomous ODEs (2.5)

From last lecture:

Keep in mind ideas 1, 2, 3 - Model satisfies:
Idea 1. If, $P > K$ then $r(P) < 0$ (overpopulation)
Idea 2. If, $S < P < K$ then $r(P) > 0$ (normal population)
Idea 3. If, $P < S$ then $r(P) < 0$ (endangered)

Note: The above is **not** the solution P(t). This is part of finding the ODE.

1. Give a function r(P) that will satisfy these conditions?

$r\left(P\right) = -c\left(P - S\right)\left(P - K\right)$

According to Malthusian Growth, $P' = rP$

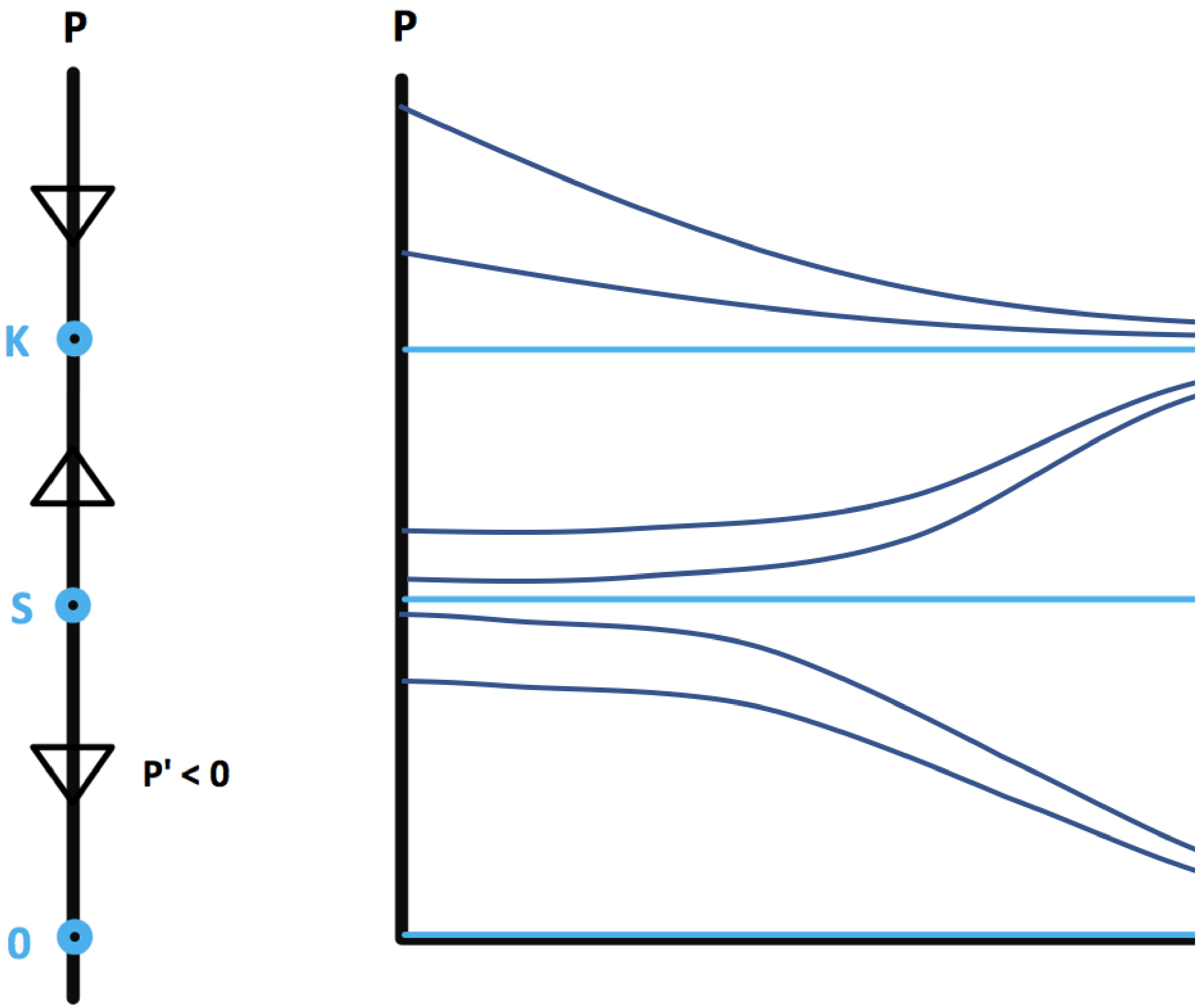
Since $\frac{dP}{dt} = rP$ and taking a constant **c**,

This population satisfies $\frac{dP}{dt} = -c\left(P - S\right)\left(P - K\right)P$ or $-c\left(1 - \frac{P}{S}\right)\left(1 - \frac{P}{K}\right)P$

where $c > 0$ and $0 < S < K$.
Note: The c in the first and the second equation is different; the negative sign is for the parabola opening downward.

2) Sketch the phase plane and solution graphs?

Phase Plane and Solution Graph:



Solutions will never touch each other, meaning they will never touch the equilibrium points at S, K, and 0 (for ex. when P(0) is between S and 0, the population will approach 0, but never be 0).

3) What are the critical points? Are they stable, unstable, or semi-stable?

From the ODE above, the critical points are S, K, and 0.

Note:

- Solutions may get very close to the equilibrium point (light blue lines) but will never touch them
- Solutions can never touch (uniqueness and existence theorem)

From the phase plane:

<u>Equilibrium points</u>

S = unstable(since the arrows move away from S)

K = stable (since the arrows move towards K)

0 = stable

Note: For this scenario, it does not matter what happens below 0 since it is impossible for the population to be negative. Regarding 0 not being semi-stable - anything around 0 will always tend to 0 and the population will never be negative, so 0 is stable.

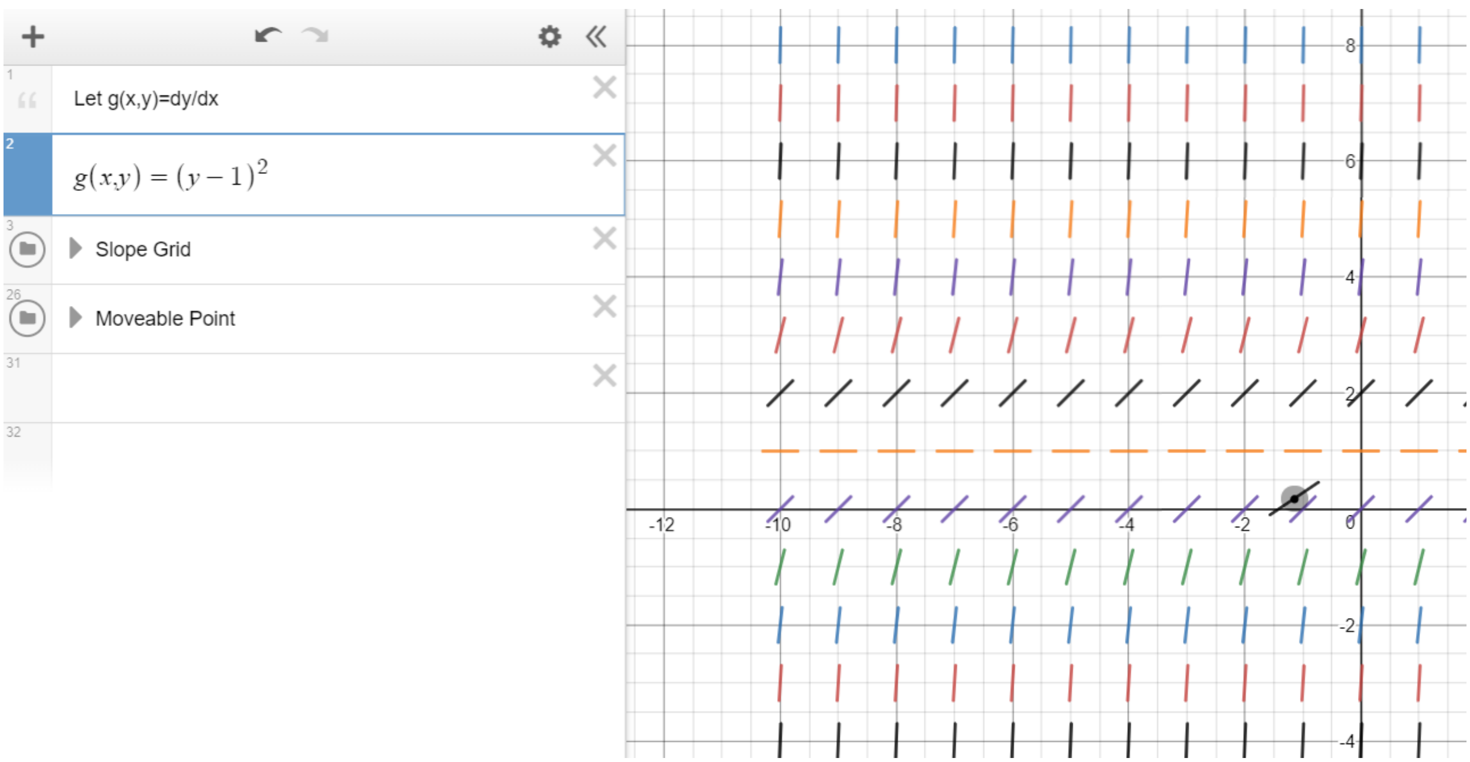
Note:

- should be careful doing measurements around S because it is an unstable population. If measurements must be taken, it should be around K.
- In this context, unstable means if a point is taken a little bit above or under S, it will tend to diverge from S. Stable means if a point is taken a little bit above or under K, it will tend to converge to K.

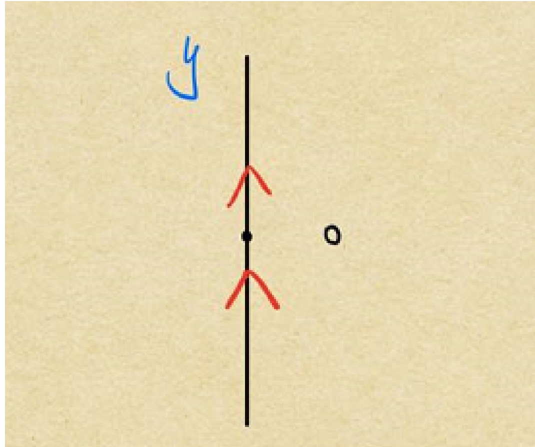
4) Write an autonomous ODE with a semi stable equilibrium solution

$y' = (y-1)^2$

Graph:

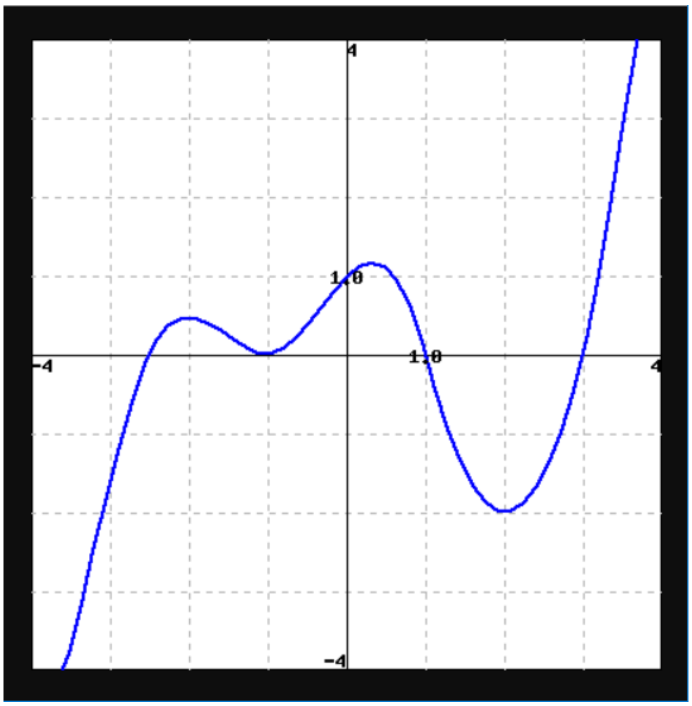


The slope of the graph is positive for R except for 1. Based on the graph, the slopes before and after the critical point y = 1 are all positive. In which matches the semi-stable requirements mentioned in the video. The phase diagram below gives a clear demonstration:



There is also another way to identify the stability of critical points:

Here is a graph of function f(x) (Adopted from WebWork)



When the graph:
- goes from negative to positive through a critical point, the critical point is unstable
- goes from positive to negative through a critical point, the critical point is stable
- approaches 0 but remains positive/negative, in which the graph looks like a smiling/sad face curve, the critical point is semi-stable