*“THERE IS NO KNOWLEDGE THAT IS NOT POWER.”*

RALPH WALDO EMERSON, (1803-1882)

LEAH EDELSTEIN-KESHET

# CALCULUS FOR THE LIFE SCIENCES (II)

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*This book is dedicated to students who have a genuine desire to learn about the beauty and usefulness of mathematics and to many colleagues who have helped to shape my interest and philosophy of teaching life-science calculus.*

# 1

# Power functions as building blocks

Like tall architectural marvels that are made of simple units (beams, bricks, and tiles), many interesting functions can be constructed from simpler building blocks. In this chapter, we study a family of simple functions, the power functions - those of the form $f(x) = x^n$.

Our first task is to understand properties of the members of this "family". We will see that basic observations of power functions such as $x^2, x^3$ leads to insights into a biological problem of why the size of living cells is limited. Later, we use power functions as " building blocks" to construct polynomials, and rational functions . We then develop important approaches to sketch the shapes of the resulting graphs.

Mastered Material Check

1. Can you define **function**?
2. Give an example of a polynomial function; a rational function.

## 1.1 Power functions

**Section 1.1 Learning goals**

1. Interpret the shapes of power functions relative to one another.
2. Justify that power functions with low powers dominate near the origin, and power functions with high powers dominate far away from the origin.
3. Identify the points of intersection of two power functions.

Let us consider the power functions, that is functions of the form

$$y = f(x) = x^n,$$

where $n$ is a positive integer. Power functions are among the most elementary and "elegant" functions - we only need multiplications to compute their value at any point. They are thus easy to calculate, very predictable and smooth, and, from the point of view of calculus, very easy to handle.

Click on this link and then adjust the slider on this interactive **desmos** graph to see how the power $n$ affects the shape of a power function in the first quadrant.

From Figure 1.1, we see that the power functions ($y = x^n$ for powers $n = 2, \ldots, 5$) intersect at $x = 0$ and $x = 1$. This is true for all positive integer powers. The same figure also demonstrates another fact helpful for curve-sketching: the greater the power $n$, the *flatter* the graph near the origin and the *steeper* the graph beyond $x > 1$. This can be restated in terms of the relative size of the power functions. We say that *close to the origin, the functions with lower powers dominate, while far from the origin, the higher powers dominate*.

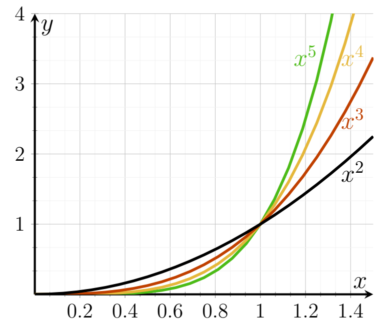



Figure 1.1: Graphs of a few power functions $y = x^n$. All intersect at $x = 0, 1$. As the power $n$ increases, the graphs become flatter close to the origin, $(0,0)$, and steeper at large $x$-values.

Mastered Material Check

3. Use Figure 1.1 to approximate when $x^5 = 2$.
4. What is the first quadrant?

More generally, a power function has the form

$$y = f(x) = K \cdot x^n$$

where $n$ is a positive integer and $K$, sometimes called the **coefficient**, is a constant. So far, we have compared power functions whose coefficient is $K = 1$. We can extend our discussion to a more general case as well.

**Example 1.1** *Find points of intersection and compare the sizes of the two power functions*

$$y_1 = ax^n, \quad \text{and} \quad y_2 = bx^m.$$

*where a and b are constants. You may assume that both a and b are positive.*

**Solution.** This comparison is a slight generalization of the previous discussion. First, we note that the coefficients $a$ and $b$ merely scale the vertical behaviour (i.e. stretch the graph along the $y$ axis). It is still true that the two functions intersect at $x = 0$; further, as before, the higher the power, the flatter the graph close to $x = 0$, and the steeper for large positive or negative values of $x$. However, now another point of intersection of the graphs occur when

$$ax^n = bx^m \quad \Rightarrow \quad x^{n-m} = (b/a).$$

We can solve this further to obtain a solution in the first quadrant

$$x = (b/a)^{1/(n-m)}. \tag{1.1}$$

This is shown in Figure 1.2 for the specific example of $y_1 = 5x^2, y_2 = 2x^3$. Close to the origin, the quadratic power function has a larger value, whereas for large $x$, the cubic function has larger values. The functions intersect when $5x^2 = 2x^3$, which holds for $x = 0$ or $x = \frac{5}{2} = 2.5$. $\Diamond$

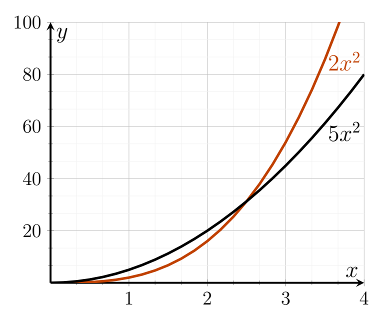



Figure 1.2: Graphs of two power functions, $y = 5x^2$ and $y = 2x^3$.

If $b/a$ is positive, then in general the value given in (1.1) is a real number.

**Example 1.2** *Determine points of intersection for the following pairs of functions:*

*(a)* $y_1 = 3x^4$ *and* $y_2 = 27x^2$,

*(b)* $y_1 = \left(\frac{4}{3}\right)\pi x^3$ *and* $y_2 = 4\pi x^2$.

**Solution.**

(a) Intersections occur at $x = 0$ and at $\pm(27/3)^{1/(4-2)} = \pm\sqrt{9} = \pm 3$.

(b) These functions intersect at $x = 0, 3$ but there are no other intersections at negative values of $x$. $\Diamond$

Note that in many cases, the points of intersection are irrational numbers whose decimal approximations can only be obtained by a scientific calculator or by some approximation method (such as **Newton's Method**, studied in Section **??**)

Mastered Material Check

5. What is an irrational number?

With only these observations we can examine a biological problem related to the size of cells. By applying these ideas, we can gain insight into why cells have a size limitation, as discussed in the next section.

## *1.2 How big can a cell be? A model for nutrient balance*

A summary of the cell size model. We discuss what cell size is consistent with a balance between nutrient absorption and consumption in a cell.

**Section 1.2 Learning goals**

1. Describe the derivation of a mathematical model for cell nutrient absorption and consumption.
2. Use parameters $(k_1, k_2)$ rather than specific numbers in mathematical expressions.
3. Demonstrate the link between power functions in Section 1.1 and cell nutrient balance in the model.
4. Interpret the results of the model.

Consider the following biologically motivated questions:

- What physical and biological constraints determine the size of a cell?
- Why do some size limitations exist?
- Why should animals be made of millions of tiny cells, instead of a just a few large ones?

We already have enough mathematical prowess to address these questions - particularly if we assume a cell is spherical. Of course, this is often not the case. The shapes of living cells uniquely suit their functions. Many have long appendages, cylindrical parts, or branch-like structures. But here, we neglect all these beautiful complexities and look at a simple spherical cell because it suffices to answer our questions. Such mathematical simplifications can be very illuminating: they allow us to form a **mathematical model**. A mathematical model is just a representation of a real situation which simplifies things by representing the most important aspects, and neglecting or idealizing complicating details.

In this section, we follow a reasonable set of assumptions and mathematical facts to explore how nutrient balance can affect and limit cell size.

### *Building the model*

In order to build the model we make some simplifying assumptions and then restate them mathematically. We base the model on the following

**assumptions**:

1. The cell is roughly spherical (See Figure 1.3).
2. The cell absorbs oxygen and nutrients through its surface. The larger the surface area, $S$, the faster the total rate of absorption. We assume that the rate at which nutrients (or oxygen) are absorbed is **proportional** to the surface area of the cell.
3. The rate at which nutrients are consumed (i.e., used up) in metabolism is proportional the volume, $V$, of the cell. The bigger the volume, the more nutrients are needed to keep the cell alive.

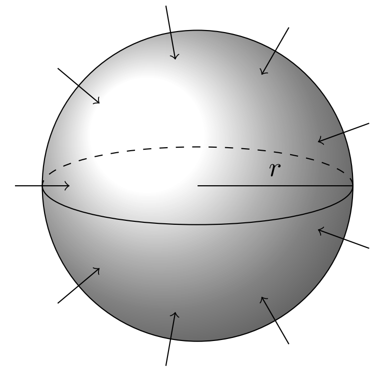



Figure 1.3: An assumed spherical cell absorbs nutrients at a rate proportional to its surface area $S$, but consumes nutrients at a rate proportional to its volume $V$.

Mastered Material Check

6. What does "A is proportional to B" mean?
7. What might the units be for quantities $A, C, V, S$ and $r$?
8. Given your choices for 7., what are the units associated with $k_1$, $k_2$?

We define the following quantities for our model of a single cell:

$$\begin{aligned}
A &= \text{net rate of absorption of nutrients per unit time,}\\
C &= \text{net rate of consumption of nutrients per unit time,}\\
V &= \text{cell volume,}\\
S &= \text{cell surface area,}\\
r &= \text{ radius of the cell.}
\end{aligned}$$

We now rephrase the assumptions mathematically. By Assumption 2, the absorption rate, $A$, is proportional to $S$: this means that

$$A = k_1 S,$$

where $k_1$ is a **constant of proportionality**. Since absorption and surface area are positive quantities, only positive values of the proportionality constant make sense, so $k_1$ must be positive. The value of this constant depends on properties of the cell membrane such as its permeability or how many pores it contains to permit passage of nutrients. *By using a generic constant - called a **parameter** - to represent this proportionality constant, we keep the model general enough to apply to many different cell types.*

By Assumption 3, the rate of nutrient consumption, $C$ is proportional to $V$, so that

$$C = k_2 V,$$

where $k_2 > 0$ is a second (positive) proportionality constant. The value of $k_2$ depend son the cell metabolism, i.e. how quickly it consumes nutrients in carrying out its activities.

By Assumption 1, the cell is spherical, thus its surface area, $S$, and volume, $V$, are:

$$S = 4\pi r^2, \quad V = \frac{4}{3}\pi r^3. \tag{1.2}$$

Putting these facts together leads to the following relationships between nutrient absorption $A$, consumption $C$, and cell radius $r$:

$$A = k_1(4\pi r^2) = (4\pi k_1)r^2,$$
$$C = k_2\left(\frac{4}{3}\pi r^3\right) = \left(\frac{4}{3}\pi k_2\right)r^3.$$

Rewriting this relationship as

$$A(r) = (4\pi k_1)r^2, \quad \text{and} \quad C(r) = \left(\frac{4}{3}\pi k_2\right)r^3. \tag{1.3}$$

we observe that $A, C$ are simply *power functions* of the cell radius, $r$, that is

$$A(r) = ar^2, \quad C(r) = cr^3.$$

*Note:* the powers are $n = 3$ for consumption and $n = 2$ for absorption.

The discussion of power functions in Section 1.1 now contributes to our analysis of how nutrient balance depends on cell size.

Mastered Material Check

9. What are constants $a$ and $c$ in terms of $k_1$ and $k_2$?
10. Why are we considering different values of $r$ in Example 1.3?

### *Nutrient balance depends on cell size*

In our discussion of cell size, we found two power functions that depend on the cell radius $r$, namely the nutrient absorption $A(r)$ and consumption $C(r)$ given in Eqns. (1.3). We first ask whether absorption or consumption of nutrients dominates for small, medium, or large cells.

**Example 1.3 (A fine balance)** *For what cell size is the consumption rate exactly balanced by the absorption rate? Which rate (consumption or absorption) dominates for small cells? For large cells?*

**Solution.**

The two rates "balance" (and their graphs intersect) when

$$A(r) = C(r) \quad \Rightarrow \quad \left(\frac{4}{3}\pi k_2\right)r^3 = (4\pi k_1)r^2.$$

A trivial solution to this equation is $r = 0$.

*Note:* this solution is not interesting biologically, but we should not forget it in mathematical analysis of such problems.

If $r \neq 0$, then, canceling a factor of $r^2$ from both sides gives:

$$r = 3\frac{k_1}{k_2}.$$

This means absorption and consumption rates are equal for cells of this size. For small $r$, the power function with the smaller power of $r$ (namely $A(r)$) dominates, but for very large values of $r$, the power function with the higher power of $r$ (namely $C(r)$) dominates. It follows that for smaller cells, absorption $A \approx r^2$ is the dominant process, while for larger cells, consumption rate $C \approx r^3$ dominates. *We conclude that cells larger than the critical size $r = 3k_1/k_2$ are unable to keep up with the nutrient demand, and cannot survive since consumption overtakes absorption of nutrients.* $\Diamond$

Using the above simple geometric argument, we deduced that cell size has strong implications on its ability to absorb nutrients or oxygen quickly enough to feed itself. For these reasons, cells larger than some maximal size (roughly 1mm in diameter) rarely occur.

A similar strategy also allows us to consider the energy balance and sustainability of life on Earth - as seen next, in Section 1.3.

## 1.3 Sustainability and energy balance on Earth

**Section 1.3 Learning goals**

1. Justify the given mathematical model that describes the energy input and output on Planet Earth.
2. Use the given model to determine the energy equlibrium of the planet.

The sustainability of life on Planet Earth depends on a fine balance between the temperature of its oceans and land masses and the ability of life forms to tolerate climate change. As a follow-up to our model for nutrient balance, we introduce a simple energy balance model to track incoming and outgoing energy and determine a rough estimate for the Earth's temperature. We use the following basic assumptions:

1. Energy input from the sun, given the Earth's radius $r$, can be approximated as
$$E_{in} = (1-a)S\pi r^2, \tag{1.4}$$
where $S$ is incoming radiation energy per unit area (also called the **solar constant**) and $0 \leq a \leq 1$ is the fraction of that energy reflected; $a$ is also called the **albedo**, and depends on cloud cover, and other planet characteristics (such as percent forest, snow, desert, and ocean).

2. Energy lost from Earth due to radiation into space depends on the current temperature of the Earth $T$, and is approximated as
$$E_{out} = 4\pi r^2 \varepsilon \sigma T^4, \tag{1.5}$$
where $\varepsilon$ is the **emissivity** of the Earth's atmosphere, which represents the Earth's tendency to emit radiation energy. This constant depends on cloud cover, water vapour, as well as on **greenhouse gas** concentration in the atmosphere; $\sigma$ is a physical constant (the Stephan-Bolzmann constant) which is fixed for the purpose of our discussion.

Mastered Material Check

11. Do you think $E_{in}$ is proportional to Earth's surface area or volume?

Notice there are several different symbols in Eqns. (1.4) and (1.5). Being clear about which are constants and which are variables is critical to using any mathematical model. As the next example points out, sometimes you have a choice to make.

**Example 1.4 (Energy expressions are power functions)** *Explain in what sense the two forms of energy above can be viewed as power functions, and what types of power functions they represent.*

**Solution.** Both $E_{in}$ and $E_{out}$ depend on Earth's radius as the power $\sim r^2$. However, since this radius is a constant, it is not fruitful to consider it as an interesting variable for this problem (unlike the cell size example in Section 1.2). However, we note that $E_{out}$ depends on temperatureas $\sim T^4$. (We might also select the albedo as a variable and in that case, we note that $E_{in}$ depends linearly on the albedo $a$.) $\Diamond$

**Example 1.5 (Energy equilibrium for the Earth)** *Explain how the assumptions above can be used to determine the equilibrium temperature of the Earth, that is, the temperature at which the incoming and outgoing radiation energies are balanced.*

**Solution.** The Earth is at equilibrium when

$$E_{in} = E_{out} \quad \Rightarrow \quad (1-a)S\pi r^2 = 4\pi r^2 \varepsilon\sigma T^4.$$

$\Diamond$

We observe that the factors $\pi r^2$ cancel, and we obtain an equation that can be solved for the temperature $T$. It is instructive to examine how this temperature depends on the constants in the problem, and how it is affected by cloud cover and greenhouse gas level. This is also explored in Exercise 21

## *1.4 Sketching simple functions*

**Section 1.4 Learning goals**

1. Sketch the graph of a simple polynomial of the form $y = ax^n + bx^m$.
2. Sketch a rational function such as $y = Ax^n/(b + x^m)$.

### *Even and odd power functions*

So far, we have considered power functions $y = x^n$ with $x > 0$. But in mathematical generality, there is no reason to restrict the independent variable $x$ to positive values. Thus we expand the discussion to consider all real values of $x$. We examine now some symmetry properties that arise.

In Figure 1.4 (a) we see that power functions with an even power, such as $y = x^2$, $y = x^4$, and $y = x^6$, are symmetric about the $y$-axis. In Figure 1.4(b)

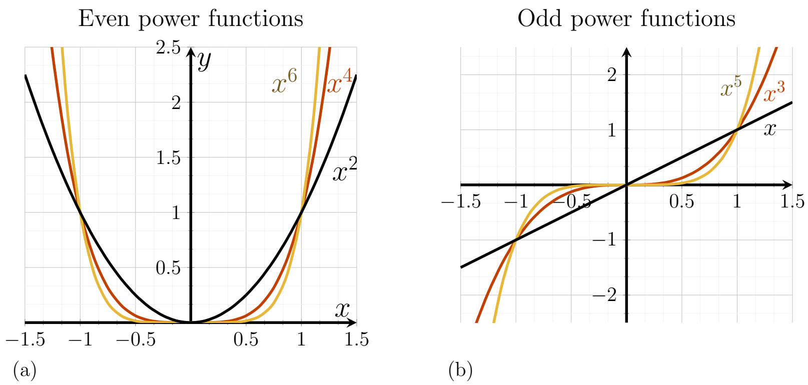



Figure 1.4: Graphs of power functions. (a) A few even power functions: $y = x^2$, $y = x^4$ and $y = x^6$. (b) Some odd power functions: $y = x$, $y = x^3$ and $y = x^5$. Note the symmetry properties.

we notice that power functions with an odd power, such as $y = x$, $y = x^3$ and $y = x^5$ are symmetric when rotated 180° about the origin. We adopt the term **even function** and **odd function** to describe such symmetry properties.

All power functions are continuous and **unbounded**: for $x \to \infty$ both even and odd power functions satisfy $y = x^n \to \infty$. For $x \to -\infty$, odd power functions tend to $-\infty$. Odd power functions are **one-to-one**: that is, each value of $y$ is obtained from a unique value of $x$ and vice versa. This is not true for even power functions. From Fig 1.4 we see that all power functions go through the point $(0,0)$. Even power functions have a **local minimum** at the origin whereas odd power functions do not.

Mastered Material Check

12. Highlight the $y$-axes and circle the origins in Fig 1.4.
13. Consider Figure 1.4: where do even power functions intersect? Odd?
14. Show that $f(a) = a^5 - 3a$ is an odd function.
15. Give an example of a function which is **bounded**.
16. Verify $y = x^2$ is **not** one-to-one.
17. What graphical property do one-to-one functions share?

### *Sketching a simple (two-term) polynomial*

Based on our familiarity with power functions, we now discuss functions made up of such components. In particular, we extend the discussion to **polynomials** (sums of power functions) and **rational functions** (ratios of such functions). We also develop skills in sketching graphs of these functions.

**Example 1.6 (Sketching a simple cubic polynomial)** *Sketch a graph of the polynomial*

$$y = p(x) = x^3 + ax. \tag{1.6}$$

*How would the sketch change if the constant a changes from positive to negative?*

Adjust the slider to see how positive and negative values of the coefficient $a$ affect the shape of the polynomial $y = x^3 + ax$.

**Solution.** The polynomial in Eqn. (1.6) has two terms, each one a power function. Let us consider their effects individually. Near the origin, for $x \approx 0$ the term $ax$ dominates so that, close to $x = 0$, the function behaves as

$$y \approx ax.$$

Mastered Material Check

17. Justify why the linear term dominates near the origin, while the cubic term dominates further out.
18. Sketch the graph of *any* function with horizontal asymptote $y = 2$.

This is a straight line with slope $a$. Hence, near the origin, if $a > 0$ we would see a line with positive slope, whereas if $a < 0$ the slope of the line should be negative. Far away from the origin, the cubic term dominates, so

$$y \approx x^3$$

at large (positive or negative) $x$ values. Figure 1.5 illustrates these ideas.

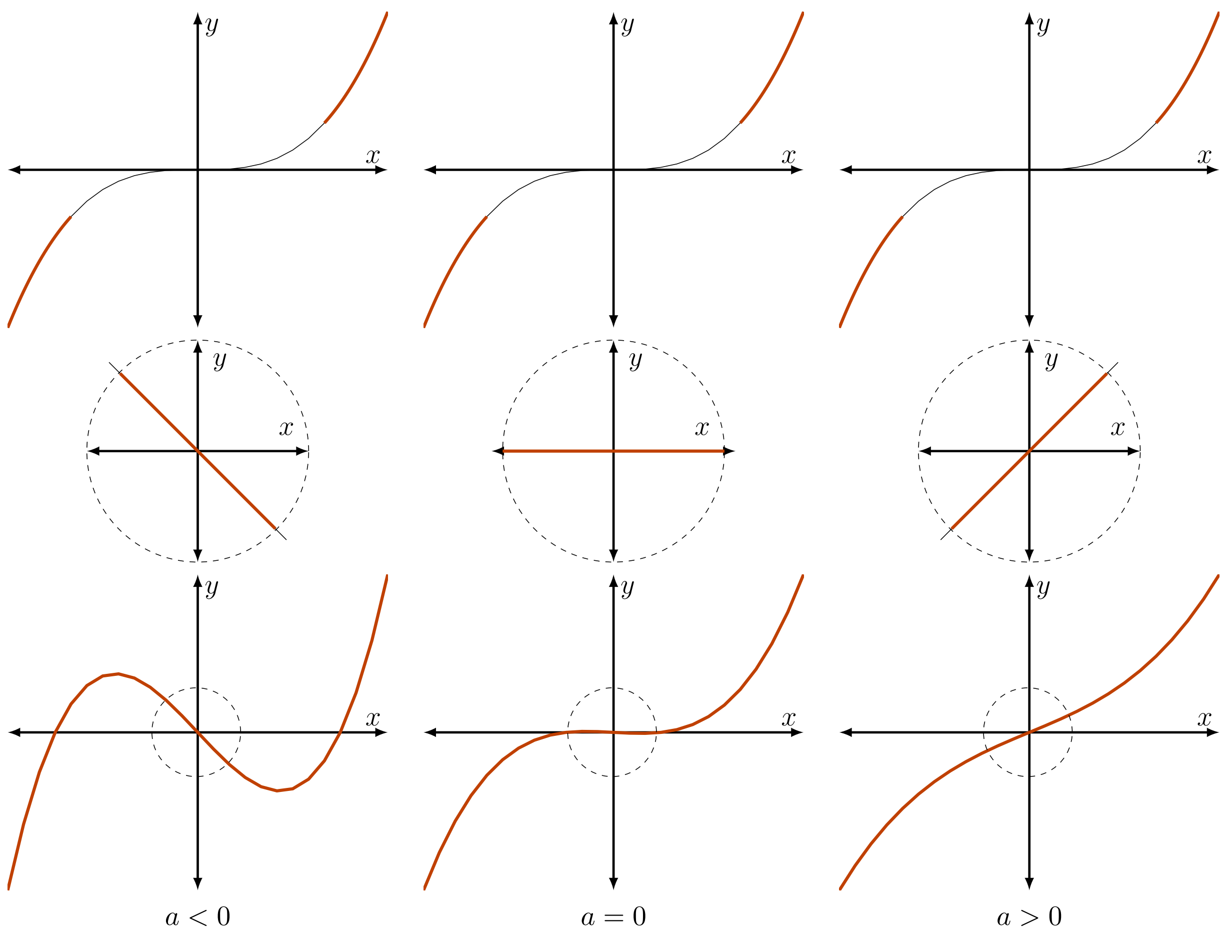



Figure 1.5: The graph of the polynomial $y = p(x) = x^3 + ax$ can be obtained by combining its two power function components. The cubic "arms" $y \approx x^3$ (top row) dominate for large $x$ (far from the origin), while the linear part $y \approx ax$ (middle row) dominates near the origin. When these are smoothly connected (bottom row) we obtain a sketch of the desired polynomial. Shown here are three possibilities, for $a < 0, a = 0, a > 0$, left to right. The value of $a$ determines the slope of the curve near $x = 0$ and thus also affects presence of a local maximum and minimum (for $a < 0$).

In the first row we see the behaviour of $y = p(x) = x^3 + ax$ for large $x$, in the second for small $x$. The last row shows the graph for an intermediate range. We might notice that for $a < 0$, the graph has a local minimum as well as a local maximum. Such an argument already leads to a fairly reasonable sketch of the function in Eqn. (1.6). We can add further details using algebra to find **zeros** - that is where $y = p(x) = 0$. ♢

**Example 1.7 (Zeros)** *Find the places at which the polynomial Eqn.* (1.6) *crosses the x axis, that is, find the* **zeros** *of the function* $y = x^3 + ax$.

**Solution.** The zeros of the polynomial can be found by setting

$$y = p(x) = 0 \quad \Rightarrow \quad x^3 + ax = 0 \quad \Rightarrow \quad x^3 = -ax.$$

The above equation always has a solution $x = 0$, but if $x \neq 0$, we can cancel and obtain

$$x^2 = -a.$$

This would have no solutions if $a$ is a positive number, so that in that case, the graph crosses the $x$ axis only once, at $x = 0$, as shown in Figure 1.5. If $a$ is negative, then the minus signs cancel, so the equation can be written in the form

$$x^2 = |a|$$

and we would have two new zeros at

$$x = \pm\sqrt{|a|}.$$

For example, if $a = -1$ then the function $y = x^3 - x$ has zeros at $x = 0, 1, -1$. ◊

Mastered Material Check

19. Find the zeros of $y = x^3 + 3x$.

**Example 1.8 (A more general case)** *Explain how you would use the ideas of Example 1.6 to sketch the polynomial $y = p(x) = ax^n + bx^m$. Without loss of generality, you may assume that $n > m \geq 1$ are integers.*

**Solution.** As in Example 1.6, this polynomial has two terms that dominate at different ranges of the independent variable. Close to the origin, $y \approx bx^m$ (since $m$ is the lower power) whereas for large $x$, $y \approx ax^n$. The full behaviour is obtained by smoothly connecting these pieces of the graph. Finding zeros can refine the graph. ◊

### *Sketching a simple rational function*

We apply similar reasoning to consider the graphs of simple rational functions. A **rational function** is a function that can be written as

$$y = \frac{p_1(x)}{p_2(x)}, \quad \text{where} \quad p_1(x) \text{ and } p_2(x) \text{ are polynomials.}$$

**Example 1.9 (A rational function)** *Sketch the graph of the rational function*

$$y = \frac{Ax^n}{a^n + x^n}, \quad x \geq 0. \tag{1.7}$$

*What properties of your sketch depend on the power $n$? What would the graph look like for $n = 1, 2, 3$?*

Adjust the sliders to see how the values of $n$, $A$, and $a$ affect the shape of the rational function in (1.7).

**Solution.** We can break up the process of sketching this function into the following steps:

- The graph of the function in Eqn. (1.7) goes through the origin (at $x = 0$, we see that $y = 0$).
- For very small $x$, (i.e., $x << a$) we can approximate the denominator by the constant term $a^n + x^n \approx a^n$, since $x^n$ is negligible by comparison, so that
$$y = \frac{Ax^n}{a^n + x^n} \approx \frac{Ax^n}{a^n} = \left(\frac{A}{a^n}\right)x^n \quad \text{for small } x.$$
This means that near the origin, the graph looks like a power function, $y \approx Cx^n$ (where $C = A/a^n$).
- For large $x$, i.e. $x >> a$, we have $a^n + x^n \approx x^n$ since $x$ overtakes and dominates over the constant $a$, so that
$$y = \frac{Ax^n}{a^n + x^n} \approx \frac{Ax^n}{x^n} = A \quad \text{for large } x.$$
This reveals that the graph has a horizontal asymptote $y = A$ at large values of $x$.
- Since the function behaves like a simple power function close to the origin, we conclude directly that the higher the value of $n$, the flatter is its graph near 0. Further, large $n$ means sharper rise to the eventual asymptote.

The results are displayed in Figure 1.6. ◊

Mastered Material Check

20. Why is $a^n$ a constant?
21. Sketch the graph of *any* function with horizontal asymptote $y = 2$.

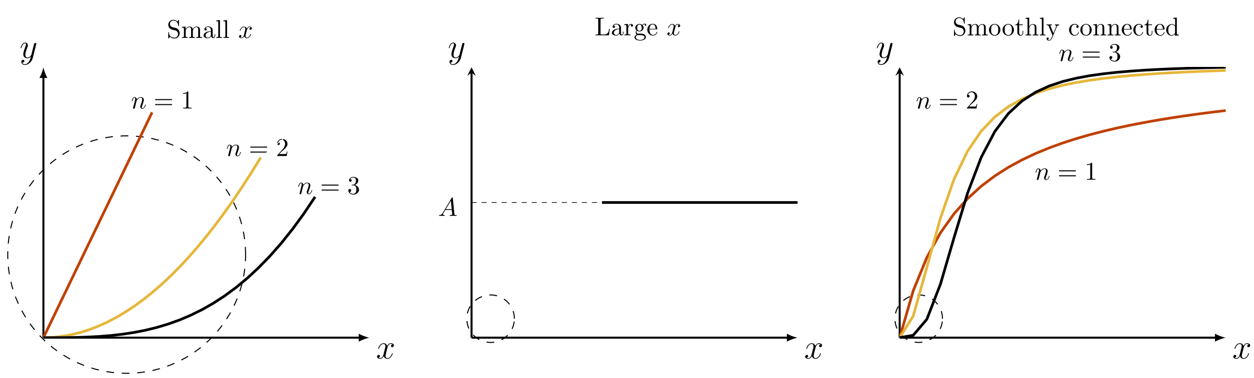



Figure 1.6: The rational functions Eqns.(1.7) with $n = 1, 2, 3$ are compared on this graph. Close to the origin, the function behaves like a power function, whereas for large $x$ there is a horizontal asymptote at $y = A$. As $n$ increases, the graph becomes flatter close to the origin, and steeper in its rise to the asymptote.

## 1.5 *Rate of an enzyme-catalyzed reaction*

**Section 1.5 Learning goals**

1. Describe the connection between Michaelis-Menten kinetics in biochemistry and rational functions described in Section 1.4.
2. Interpret properties of a graph such as Figure 1.8 in terms of properties of an enzyme-catalyzed reactions.

Rational functions introduced in Example 1.9 often play a role in biochemistry. Here we discuss two such examples and the contexts in which they appear. In both cases, we consider the initial rise of the function as well as its eventual saturation.

*Saturation and Michaelis-Menten kinetics*

Biochemical reactions are often based on the action of proteins known as **enzymes** that catalyze reactions in living cells. Fig. 1.7 depicts an enzyme E binding to its **substrate** S to form a **complex** C. The complex breaks apart into a **product**, P, and the original enzyme that can act once more. Substrate is usually plentiful relative to the enzyme.

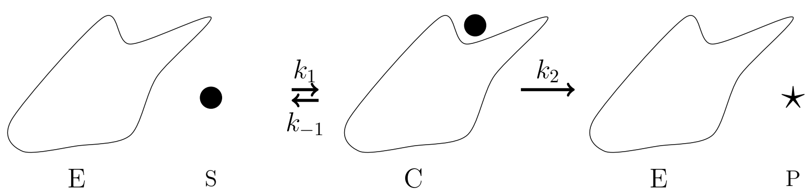



Figure 1.7: An enzyme (catalytic protein) is shown binding to a substrate molecule (circular dot) and then processing it into a product (star shaped molecule).

In the context of this example, $x$ represents the concentration of substrate in the reaction mixture. The speed of the reaction, $v$, (namely the rate at which product is formed) depends on $x$. When you actually graph the speed of the reaction as a function of the concentration, you see that it is not linear: Figure 1.8 is typical. This relationships, known as **Michaelis-Menten** kinetics, has the mathematical form

$$\text{speed of reaction } = v = \frac{Kx}{k_n + x}, \tag{1.8}$$

where $K, k_n > 0$ are constants specific to the enzyme and the experimental conditions.

Equation (1.8) is a rational function. Since $x$ is a concentration, it must be a positive quantity, so we restrict attention to $x \geq 0$. The expression in Eqn. (1.8) is a special case of the rational functions explored in Example 1.9, where $n = 1, A = K, a = k_n$. In Figure 1.8, we used plot this function for specific values of $K, k_n$. The following observations can be made

1. The graph of Eqn. (1.8) goes through the origin. Indeed, when $x = 0$ we have $v = 0$.

2. Close to the origin, the initial rise of the graph "looks like" a straight line. We can see this by considering values of $x$ that are much smaller than $k_n$. Then the denominator $(k_n + x)$ is well approximated by the constant $k_n$. Thus, for small $x$, $v \approx (K/k_n)x$, so that the graph resembles a straight line through the origin with slope $(K/k_n)$.

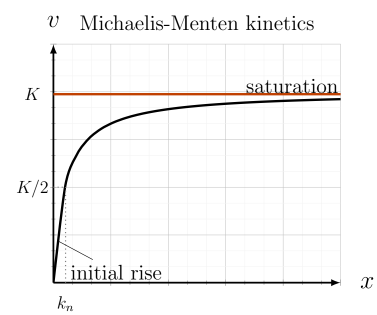



Figure 1.8: The graph of reaction speed, $v$, versus substrate concentration, $x$ in an enzyme-catalyzed reaction, as in Eqn. 1.8. This behaviour is called Michaelis-Menten kinetics. Note that the graph at first rises almost like a straight line, but then it curves and approaches a horizontal asymptote. This graph tells us that the speed of the enzyme cannot exceed some fixed level, i.e. it cannot be faster than $K$.

3. For large $x$, there is a horizontal asymptote. A similar argument for $x \gg k_n$, verifies that $v$ is approximately constant at large enough $x$.

Michaelis-Menten kinetics represents one relationship in which **saturation** occurs: the speed of the reaction at first increases as substrate concentration $x$ is raised, but the enzymes saturate and operate at a fixed constant speed $K$ as more and more substrate is added.

| | units | example |
|---|---|---|
| $x$ | concentration | "nano Molar" , $nM \equiv 10^{-9}$ Moles per litre |
| $v$ | concentration over time | nM min$^{-1}$ |
| $k_n$ | | |
| $K$ | | |

Table 1.1: Units for Michaelis-Menten kinetics, $v = \frac{Kx}{k_n + x}$.

Mastered Material Check

22. Complete Table 1.1.

**Units.** It is worth considering the units in Eqn. (1.8). Given that only quantities with identical units can be added or compared, and that the units of the two sides of the relationship *must balance*, fill Table 1.1.

**Featured Problem 1.1 (Fish population growth 1)** *The Beverton-Holt model relates the number of salmon in a population this year $N_1$ to the number of salmon that were present last year $N_0$, according to the relationship*

$$N_1 = k_1 \frac{N_0}{(1 + k_2 N_0)}, \quad k_1, k_2 > 0 \tag{1.9}$$

*Sketch $N_1$ as a function of $N_0$ and explain how the constants $k_1$ and $k_2$ affect the shape of the graph you obtain. Is there a population level $N_0$ that would be exactly the same from one year to the next? Are there any restrictions on $k_1$ or $k_2$ for this kind of static ("steady state") population to be possible?*

### *Hill functions*

The Michaelis-Menten kinetics we discussed above fit into a broader class of **Hill functions**, which are rational functions of the form shown in Eqn. (1.7) with $n > 1$ and $A, a > 0$. This function is often referred to in the life sciences as *a Hill function with coefficient n*, (although the "coefficient" is actually a power in the terminology used in this chapter). Hill functions occur when an enzyme-catalyzed reaction benefits from **cooperativity** of a multi-step process. For example, the binding of the first substrate molecule may enhance the binding of a second.

Michaelis-Menten kinetics coincides with a Hill function for $n = 1$. In biochemistry, expressions of the form of Eqn. (1.7) with $n > 1$ are often denoted "sigmoidal" kinetics. Several such functions are plotted in Figure 1.9. We examined the shapes of these functions in Example 1.9.

All Hill functions have a horizontal asymptote $y = A$ at large values of $x$. If $y$ is the speed of a chemical reaction (analogous to the variable we called

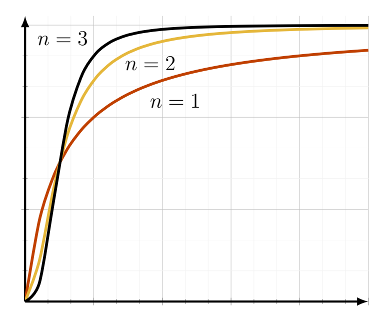



Figure 1.9: Hill function kinetics, from Eqn. (1.7), with $A = 3, a = 1$ and Hill coefficient $n = 1, 2, 3$. See also Fig 1.6 for an analysis of the shape of this graph.

$v$), then $A$ is the "maximal rate" or "maximal speed" of the reaction. Since the Hill function behaves like a simple power function close to the origin, the higher the value of $n$, the flatter is its graph near 0, and the sharper the rise to the eventual asymptote. Hill functions with large $n$ are often used to represent "switch-like" behaviour in genetic networks or biochemical signal transduction pathways.

The constant $a$ is sometimes called the "half-maximal activation level" for the following reason: when $x = a$ then

$$v = \frac{Aa^n}{a^n + a^n} = \frac{Aa^2}{2a^2} = \frac{A}{2}.$$

This shows that the level $x = a$ leads to a reaction speed of $A/2$ which is half of the maximal possible rate.

**Featured Problem 1.2** **Lineweaver-Burk plots.** *Hill functions can be transformed to a linear relationship through a change of variables. Consider the Hill function*

$$y = \frac{Ax^3}{a^3 + x^3}.$$

*define $y = 1/Y$, $X = 1/x^3$. Show that $Y$ and $X$ satisfy a linear relationship. Because we take the reciprocals of $x$ and $y$, $X$ and $Y$ are sometimes called* reciprocal coordinates.

## 1.6 Predator Response

Interactions of predators and prey are often studied in ecology. Professor C.S. ("Buzz") Holling, (a former Director of the Institute of Animal Resource Ecology at the University of British Columbia) described three types of predators, termed "Type I", "Type II" and "Type III", according to their ability to consume prey as the prey density increases. The three Holling "predator functional responses" are shown in Fig. 1.10.

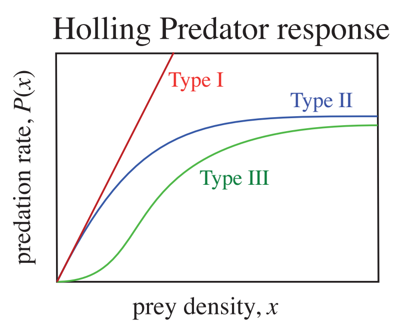



Figure 1.10: Holling's Type I, II, and III predator response. The predation rate $P(x)$ is the number of prey eaten by a predator per unit time. Note that the predation rate depends on the prey density $x$.

**Quick Concept Checks**

1. Match the predator responses shown in Fig. 1.10 with the descriptions given below

   1. As a predator, I get satiated and cannot keep eating more and more prey.
   2. I can hardly find the prey when the prey density is low, but I also get satiated at high prey density.
   3. The more prey there is, the more I can eat.

Based on Fig. 1.10, match the predator responses to functions shown below.

Hint: One of the curves "looks like a straight line" (so which function here is linear?). One of the choices is a power function. (Will it fit any of the other curves? why or why not?). Now consider the saturating curves and use our description of rational functions in Section 1.5 to select appropriate formulae for these functions.

$$P_1(x) = kx,$$
$$P_2(x) = K\frac{x}{a+x},$$
$$P_3(x) = Kx^n, \quad n \geq 2$$
$$P_4(x) = K\frac{x^n}{a^n+x^n}, \quad n \geq 2$$

The generality of mathematics allows us to adapt concepts we studied in one setting (enzyme biochemistry) to an apparently new topic (behaviour of predators).

### 1.6.1 A ladybug eating aphids

Here we use ideas developed so far to address a problem in population growth and biological control.

See this short video explanation of the ladybug Type III predator response to its aphid prey.

**Featured Problem 1.3 (A balance of predation and aphid population growth)** *Ladybugs are predators that love to eat aphids (their prey). Fig. 1.11 provides data* [1] *that supports the idea that ladybugs are type 3 predators.*

*Let x= the number of aphids in some unit area (i.e., the density of the prey). Then the number of aphids eaten by a ladybug per unit time in that unit area will be called the* **predation rate** *and denoted $P(x)$. The predation rate usually depends on the prey density, and we approximate that dependence by*

$$P(x) = K\frac{x^n}{a^n+x^n}, \quad \text{where} \quad K,a > 0. \tag{1.10}$$

*Here we consider the case that $n = 2$. The aphids reproduce at a rate proportional to their number, so that the growth rate of the aphid population G (number of new aphids per hour) is*

$$G(x) = rx \quad \text{where} \quad r > 0. \tag{1.11}$$

*(a) For what aphid population density x does the predation rate exactly balance the aphid population growth rate?*

*(b) Are there situations where the predation rate cannot match the growth rate? Explain your results in terms of the constants $K,a,r$.*

1

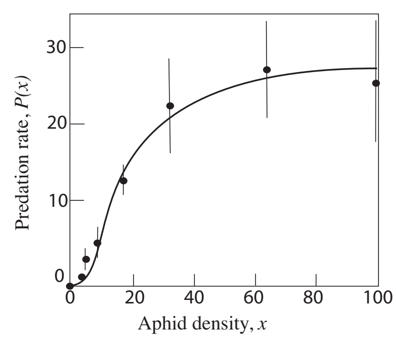



Figure 1.11: The predation rate of a ladybug depends on its aphid (prey) density.

#### Hints and partial solution

Use the sliders to manipulate the predation constants $K,a$ and the aphid growth rate parameter $r$. How many solutions are there to $P(x) = G(x)$? Show that for some parameter values, there is only a trivial solution at $x = 0$. Make a connection between this observation and part (b) of Example 1.3.

(a) The wording "the predation rate exactly balances the reproduction rate" means that the two functions $P(x)$ and $G(x)$ are exactly equal. Hence, to solve this problem, equate $P(x) = G(x)$ and determine the value of $x$ (i.e., the number of aphids) at which this equality holds. You will find that one solution to this equation is $x = 0$. But if $x \neq 0$, you can cancel one factor of $x$ from both sides and rearrange the equation to obtain a quadratic equation whose solution can be written down (in terms of the positive constants $K,r,a$).

Hint: Recall that a quadratic equation $ax^2+bx+c = 0$ has roots $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. These roots are real provided $b^2-4ac \geq 0$.

(b) The solution you find in (a) is only a real number (i.e. a real solution exists) if the **discriminant** (quantity inside the square-root) is positive. Determine when this situation can occur and interpret your answer in terms of the aphid and ladybugs.

The solution to this problem is based on solving a quadratic equation, and so, relies on the fact that we chose the value $n = 2$ in the predation rate. What happens if $n > 2$? How do we solve the same kind of problem if $n = 3, 4$ etc? We return to this issue, and develop an approximate technique (Newton's method) in a later chapter.

## *1.7 Summary*

1. Functions of the form $f(x) = K \cdot x^n$ ($n$ a positive integer) are called *power functions* with coefficient $K$.
2. Power functions with larger powers of $n$ form graphs that are flatter near the origin and steeper for $x > 1$.
3. An even function satisfies $f(-x) = f(x)$; an odd function satisfies $f(-x) = -f(x)$. Identifying even and odd functions can aid in graph sketching.
4. The zeros of a function $f(x)$ are roots of the equation $f(x) = 0$. Identifying the root(s) of a function helps in sketching its graph.
5. Polynomials are sums of power functions. Rational functions are ratios of polynomials. By examining the behaviour of terms that dominate near and far from the origin, we can obtain a rough sketch of such functions.
6. Mathematical models can be used to describe scientific phenomenon. Making reasonable assumptions and observations are necessary for building a successful model. Translating these assumptions and observations into mathematics is the key.
7. Hill functions can be transformed into a linear relationship using a change of variables; the plots that result are called Lineweaver-Burk plots.
8. The mathematical models explored in this chapter concerned:

   (a) cell size, based on nutrient balance;
   (b) energy balance on Earth;
   (c) biochemical reactions and Michaelis-Menten kinetics; and
   (d) enzyme-catalyzed reactions and Hill functions.

9. Units, while often suppressed in math texts, can be immensely useful in solving application problems. Only quantities with identical units can be added, or compared. Two sides of an equation must have identical units.

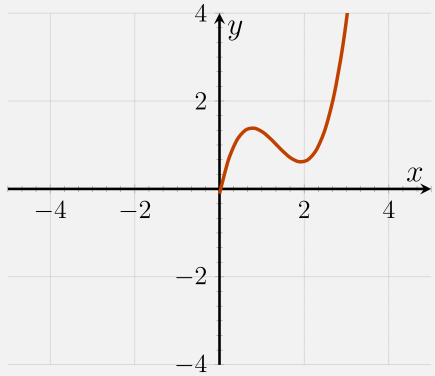
**Quick Concept Checks**

1. When is $x^2 > x^{10}$?

   (a) never (b) always (c) for small $x$ (d) for large $x$

2. Why do we make assumptions when we build mathematical models?

3. Complete the sketch of the following graph, given that it is




   (a) an even function
   (b) an odd function

4. What is the relationship between Michaelis-Menten kinetics and Hill functions?

## *Exercises*

1.1. **Power functions.** Consider the power function

$$y = ax^n, \quad -\infty < x < \infty.$$

Explain, possibly using a sketch, how the shape of the function changes when the coefficient $a$ increases or decreases (for fixed $n$). How is this change in shape different from the shape change that results from changing the power $n$?

1.2. **Transformations.** Consider the graphs of the simple functions $y = x$, $y = x^2$, and $y = x^3$. Describe what happens to each of these graphs when the functions are *transformed* as follows:

(a) $y = Ax$, $y = Ax^2$, and $y = Ax^3$ where $A > 1$ is some constant.

(b) $y = x + a$, $y = x^2 + a$, and $y = x^3 + a$ where $a > 0$ is some constant.

(c) $y = (x - b)^2$, and $y = (x - b)^3$ where $b > 0$ is some constant.

1.3. **Sketching transformations.** Sketch the graphs of the following functions:

(a) $y = x^2$,

(b) $y = (x + 4)^2$,

(c) $y = a(x - b)^2 + c$ for the case $a > 0$, $b > 0$, $c > 0$.

(d) Comment on the effects of the constants $a$, $b$, $c$ on the properties of the graph of $y = a(x - b)^2 + c$.

1.4. **Sketching polynomials.** Use arguments from Section 1.4 to sketch graphs of the following polynomials:

(a) $y = 2x^5 - 3x^2$,

(b) $y = x^3 - 4x^5$.

1.5. **Finding points of intersection.**

(a) Consider the two functions $f(x) = 3x^2$ and $g(x) = 2x^5$. Find all points of intersection of these functions.

(b) Repeat for functions $f(x) = x^3$ and $g(x) = 4x^5$.

*Note:* finding these points of intersection is equivalent to calculating the **zeros** of the functions in Exercise 4.

1.6. **Qualitative sketching skills.**

(a) Sketch the graph of the function $y = \rho x - x^5$ for positive and negative values of the constant $\rho$. Comment on behaviour close to zero and far away from zero.

(b) What are the zeros of this function and how does this depend on $\rho$?

(c) For what values of $\rho$ would you expect that this function would have a local maximum ("peak") and a local minimum ("valley")?

1.7. **Finding points of intersection.** Consider functions $f(x) = Ax^n$ and $g(x) = Bx^m$. Suppose $m > n > 1$ are integers, and $A, B > 0$. Determine the values of $x$ at which the the functions are the same - i.e. they intersect. Are there two places of intersection or three? How does this depend on the integer $m - n$?

*Note:* The point $(0,0)$ is always an intersection point. Thus, we are asking: when is there only *one* more and when there are *two* more intersection points? See Exercise 5 for an example of both types.

1.8. **More intersection points.** Find the intersection of each pair of functions.

(a) $y = \sqrt{x}$, $y = x^2$,

(b) $y = -\sqrt{x}$, $y = x^2$,

(c) $y = x^2 - 1$, $\frac{x^2}{4} + y^2 = 1$.

1.9. **Crossing the $x$-axis.** Answer the following by solving for $x$ in each case. Find all values of $x$ for which the following functions cross the $x$-axis (equivalently: the **zeros** of the function, or **roots** of the equation $f(x) = 0$.)

(a) $f(x) = I - \gamma x$, where $I, \gamma$ are positive constants.

(b) $f(x) = I - \gamma x + \varepsilon x^2$, where $I, \gamma, \varepsilon$ are positive constants. Are there cases where this function does not cross the $x$ axis?

(c) In the case where the root(s) exist in part (b), are they positive, negative or of mixed signs?

1.10. **Crossing the $x$-axis.** Answer Exercise 9 by sketching a rough graph of each of the functions in parts (a-b) and using these sketches to determine how many real roots there are and where they are located (positive vs. negative $x$-axis).

*Note:* this exercise provides qualitative analysis skills that are helpful in later applications.

1.11. **Power functions.** Consider the functions $y = x^n$, $y = x^{1/n}$, $y = x^{-n}$, where $n$ is an integer $n = 1, 2, \ldots$).

(a) Which of these functions increases most steeply for values of $x$ greater than 1?

(b) Which decreases for large values of $x$?

(c) Which functions are not defined for negative $x$ values?

(d) Compare the values of these functions for $0 < x < 1$.

(e) Which of these functions are not defined at $x = 0$?

1.12. **Roots of a quadratic.** Find the range $m$ such that the equation $x^2 - 2x - m = 0$ has two unequal roots.

1.13. **Rational Functions.** Describe the shape of the graph of the function $y = Ax^n/(b+x^m)$ in two cases:

(a) $n > m$ and

(b) $m > n$.

1.14. **Power functions with negative powers.** Consider the function

$$f(x) = \frac{A}{x^a}$$

where $A > 0, a > 1$, with $a$ an integer. This is the same as the function $f(x) = Ax^{-a}$, which is a power function with a negative power.

(a) Sketch a rough graph of this function for $x > 0$.

(b) How does the function change if $A$ is increased?

(c) How does the function change if $a$ is increased?

1.15. **Intersections of functions with negative powers.** Consider two functions of the form

$$f(x) = \frac{A}{x^a}, \quad g(x) = \frac{B}{x^b}.$$

Suppose that $A, B > 0,\ a, b > 1$ and that $A > B$. Determine where these functions intersect for positive $x$ values.

1.16. **Zeros of polynomials.** Find all real zeros of the following polynomials:

(a) $x^3 - 2x^2 - 3x$,

(b) $x^5 - 1$,

(c) $3x^2 + 5x - 2$.

(d) Find the points of intersection of the functions $y = x^3 + x^2 - 2x + 1$ and $y = x^3$.

1.17. **Inverse functions.** The functions $y = x^3$ and $y = x^{1/3}$ are *inverse functions* (see Section 2.3 for a discussion of inverse functions).

(a) Sketch both functions on the same graph for $-2 < x < 2$ showing clearly where they intersect.

(b) The tangent line to the curve $y = x^3$ at the point $(1,1)$ has slope $m = 3$, whereas the tangent line to $y = x^{1/3}$ at the point $(1,1)$ has slope $m = 1/3$. Explain the relationship of the two slopes.

1.18. **Properties of a cube.** The volume $V$ and surface area $S$ of a cube whose sides have length $a$ are given by the formulae

$$V = a^3, \quad S = 6a^2.$$

Note that these relationships are expressed in terms of power functions. The independent variable is $a$, not $x$. We say that "$V$ is a function of $a$" (and also "$S$ is a function of $a$").

(a) Sketch $V$ as a function of $a$ and $S$ as a function of $a$ on the same set of axes. Which one grows faster as $a$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $a$? Sketch a graph of $\frac{V}{S}$ as a function of $a$.

(c) The formulae above tell us the volume and the area of a cube of a given side length. Suppose we are given either the volume or the surface area and asked to find the side.

(i) Find the length of the side as a function of the volume (i.e. express $a$ in terms of $V$).

(ii) Find the side as a function of the surface area.

(iii) Use your results to find the side of a cubic tank whose volume is 1 litre.

(iv) Find the side of a cubic tank whose surface area is 10 $\text{cm}^2$.

**Units.**
Note that 1 litre = $10^3$ $\text{cm}^3$.

1.19. **Properties of a sphere.** The volume $V$ and surface area $S$ of a sphere of radius $r$ are given by the formulae

$$V = \frac{4\pi}{3} r^3, \quad S = 4\pi r^2.$$

Note that these relationships are expressed in terms of power functions with constant multiples such as $4\pi$. The independent variable is $r$, not $x$. We say that "$V$ is a function of $r$" (and also "$S$ is a function of $r$").

(a) Sketch $V$ as a function of $r$ and $S$ as a function of $r$ on the same set of axes. Which one grows faster as $r$ increases?

(b) What is the ratio of the volume to the surface area; that is, what is $\frac{V}{S}$ in terms of $r$? Sketch a graph of $\frac{V}{S}$ as a function of $r$.

(c) The formulae above tell us the volume and the area of a sphere of a given radius. But suppose we are given either the volume or the surface area and asked to find the radius.

(i) Find the radius as a function of the volume (i.e. express $r$ in terms of $V$).

(i) Find the radius as a function of the surface area.

(i) Use your results to find the radius of a balloon whose volume is 1 litre.

(i) Find the radius of a balloon whose surface area is 10 $\text{cm}^2$

1.20. **The size of cell.** Consider a cell in the shape of a thin cylinder (length $L$ and radius $r$). Assume that the cell absorbs nutrient through its surface at rate $k_1 S$ and consumes nutrients at rate $k_2 V$ where $S, V$ are the surface area and volume of the cylinder. Here we assume that $k_1 = 12\mu\text{M}\ \mu\text{m}^{-2}$ per min and $k_2 = 2\mu\text{M}\ \mu\text{m}^{-3}$ per min.

**Units.**
Note that $\mu$M is $10^{-6}$ moles and $\mu$m is $10^{-6}$ meters.

(a) Use the fact that a cylinder (without end-caps) has surface area $S = 2\pi rL$ and volume $V = \pi r^2 L$ to determine the cell radius such that the rate of consumption exactly balances the rate of absorption.

(b) What do you expect happens to cells with a bigger or smaller radius?

(c) How does the length of the cylinder affect this nutrient balance?

1.21. **Energy equilibrium for Earth.** This exercise focuses on Earth's temperature, climate change, and sustainability.

(a) Complete the calculation for Example 1.5 by solving for the temperature $T$ of the Earth at which incoming and outgoing radiation energies balance.

(b) Assume that greenhouse gases decrease the emissivity $\varepsilon$ of the Earth's atmosphere. Explain how this would affect the Earth's temperature.

(c) Explain how the size of the Earth affects its energy balance according to the model.

(d) Explain how the albedo $a$ affects the Earth's temperature.

1.22. **Allometric relationship.** Properties of animals are often related to their physical size or mass. For example, the metabolic rate of the animal ($R$), and its pulse rate ($P$) may be related to its body mass $m$ by the approximate formulae $R = Am^b$ and $P = Cm^d$, where $A, C, b, d$ are positive constants. Such relationships are known as *allometric* relationships.

(a) Use these formulae to derive a relationship between the metabolic rate and the pulse rate (*hint:* eliminate $m$).

(b) A similar process can be used to relate the Volume $V = (4/3)\pi r^3$ and surface area $S = 4\pi r^2$ of a sphere to one another. Eliminate $r$ to find the corresponding relationship between volume and surface area for a sphere.

1.23. **Rate of a very simple chemical reaction.** We consider a chemical reaction that does not saturate, and a simple linear relationship between reaction speed and reactant concentration.

A chemical is being added to a mixture and is used up in a reaction. The rate of change of the chemical, (also called "the rate of the reaction") $v$ M/sec is observed to follow a relationship

$$v = a - bc$$

where $c$ is the reactant concentration (in units of M) and $a, b$ are positive constants.
*Note:* $v$ is considered to be a function of $c$, and moreover, the relationship between $v$ and $c$ is assumed to be linear.

**Units.**
Note that M stands for Molar, which is the number of moles per litre.

(a) What units should $a$ and $b$ have to make this equation consistent?

*Note:* in an equation such as $v = a - bc$, each of the three terms *must have* the same units. Otherwise, the equation would not make sense.

(b) Use the information in the graph shown in Figure 1.12 (and assume that the intercept on the $c$ axis is at 0.001M) to find the values of $a$ and $b$ (*hint:* find the equation of the line in the figure, and compare it to the relationship $v = a - bc$).

(c) What is the rate of the reaction when $c = 0.005$ M?

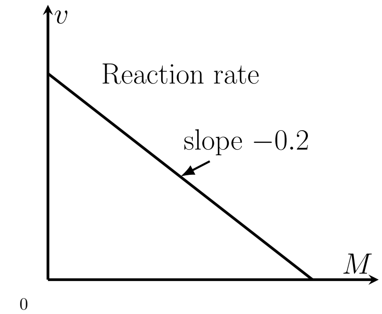



Figure 1.12: Figure for Exercise 23; rate of a chemical reaction. Assume that the intercept of the line on the $c$ axis is at 0.01M

1.24. **Michaelis-Menten kinetics.** Consider the Michaelis-Menten kinetics where the speed of an enzyme-catalyzed reaction is given by $v = Kx/(k_n + x)$.

(a) Explain the statement that "when $x$ is large there is a horizontal asymptote" and find the value of $v$ to which that asymptote approaches.

(b) Determine the reaction speed when $x = k_n$ and explain why the constant $k_n$ is sometimes called the "half-max" concentration.

1.25. **A polymerization reaction.** Consider the speed of a polymerization reaction shown in Figure 1.13. Here the rate of the reaction is plotted as a function of the substrate concentration; this experiment concerned the polymerization of actin, an important structural component of cells; data from [**?**]. The experimental points are shown as dots, and a Michaelis-Menten curve has been drawn to best fit these points. Use the data in the figure to determine approximate values of $K$ and $k_n$ in the two treatments shown.

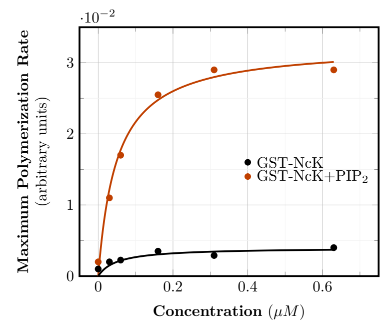



Figure 1.13: Figure for Exercise. 25; speed of polymerization.

1.26. **Hill functions.** Hill functions are sometimes used to represent a biochemical "switch," that is a rapid transition from one state to another. Consider the Hill functions

$$y_1 = \frac{x^2}{1+x^2}, \quad y_2 = \frac{x^5}{1+x^5},$$

(a) Where do these functions intersect?

(b) What are the asymptotes of these functions?

(c) Which of these functions increases fastest near the origin?

(d) Which is the sharpest "switch" and why?

1.27. **Transforming a Hill function to a linear relationship.** A Hill function is a nonlinear function - but if we redefine variables, we can transform it into a linear relationship. The process is analogous to transforming Michaelis-Menten kinetics into a Lineweaver-Burk plot, as discussed in Appendix **??**.

(a) Determine how to define appropriate variables $X$ and $Y$ (in terms of the original variables $x$ and $y$) so that the Hill function $y = Ax^3/(a^3+x^3)$ is turned into a linear relationship between $X$ and $Y$.

(b) Indicate how the slope and intercept of the line are related to the original constants $A, a$ in the Hill function.

1.28. **Hill function and sigmoidal chemical kinetics.** It is known that the *rate* $v$ at which a certain chemical reaction proceeds depends on the *concentration* of the reactant $c$ according to the formula

$$v = \frac{Kc^2}{a^2+c^2},$$

where $K$, $a$ are some constants. When the chemist plots the values of the quantity $1/v$ (on the "$y$" axis) versus the values of $1/c^2$ (on the "$x$ axis"), she finds that the points are best described by a straight line with $y$-intercept 2 and slope 8. Use this result to find the values of the constants $K$ and $a$.

1.29. **Lineweaver-Burk plots.** Shown in the Figure 1.14(a) and (b) are two Lineweaver-Burk plots (see Appendix **??**). By noting properties of these figures comment on the comparison between the following two enzymes:

(a) Enzyme (1) and (2).

(b) Enzyme (1) and (3).

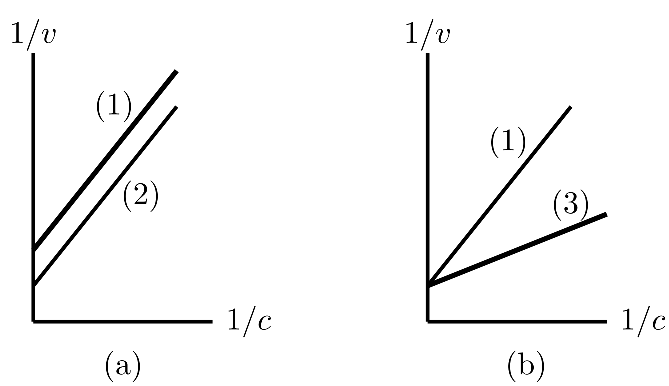



Figure 1.14: Figure for Exercise 29: Lineweaver-Burk plots.

1.30. **Michaelis-Menten enzyme kinetics.** The rate of an enzymatic reaction according to the *Michaelis-Menten kinetics* assumption is

$$v = \frac{Kc}{k_n+c},$$

where $c$ is concentration of substrate (shown on the $x$-axis) and $v$ is the reaction speed (given on the $y$-axis). Consider the data points given in Table 1.2.

| Substrate concentration | nM | $c$ | 5 | 10 | 20 | 40 | 50 | 100 |
|---|---|---|---|---|---|---|---|---|
| **Reaction speed** | nM/min | $v$ | 0.068 | 0.126 | 0.218 | 0.345 | 0.39 | 0.529 |

Table 1.2: Chemical reaction speed data.

(a) Convert this data to a Lineweaver-Burk (linear) relationship.

(b) Plot the transformed data values on a graph or spreadsheet, and estimate the slope and $y$-intercept of the line you get.

(c) Use these results to find the best estimates for $K$ and $k_n$.

1.31. **Spacing in a school of fish** According to the biologist Breder [**?**], two fish in a school prefer to stay some specific distance apart. Breder suggested that the fish that are a distance $x$ apart are attracted to one another by a force $F_A(x) = A/x^a$ and repelled by a second force $F_R(x) = R/x^r$, to keep from getting too close. He found the preferred spacing distance (also called the *individual distance*) by determining the value of $x$ at which the repulsion and the attraction exactly balance.

Find the *individual distance* in terms of the quantities $A, R, a, r$ (all assumed to be positive constants.)

# 2

# Exponential functions

*“The mathematics of uncontrolled growth are frightening. A single cell of the bacterium E. coli would, under ideal circumstances, divide every twenty minutes. That is not particularly disturbing until you think about it, but the fact is that bacteria multiply geometrically: one becomes two, two become four, four become eight, and so on. In this way it can be shown that in a single day, one cell of E. coli could produce a super-colony equal in size and weight to the entire planet Earth."*

Michael Crichton, The Andromeda Strain, p. 247

In this chapter, we introduce the exponential functions. We first describe the discrete process of **population doubling**, represented by powers of 2, namely, $2^n$, where $n$ is some integer. We generalize to a continuous function $2^x$ where $x$ is any real number. We can then attach meaning to the notion of the derivative of an exponential function. In doing so, we encounter a specially convenient base denote $e$, leading to the most useful member of this class of functions, $y = e^x$. We discuss applications to unlimited growth in a population.

Mastered Material Check

1. If a population has size $P$, what do we mean by a doubled poulation size??
2. How large would the population be it it doubled twice?

## 2.1 Unlimited growth and doubling

**Section 2.1 Learning goals**

1. Explain the link between population doubling and integer powers of the base 2.
2. Given information about the doubling time of a population and its initial size, determine the size of that population at some later time.
3. Appreciate the connection between $2^n$ for integer values of $n$ and $2^x$ for a real number $x$.

*The Andromeda Strain*

The Andromeda Strain scenario (described by Crichton in the opening quotation) motivates our investigation of population doubling and uncontrolled growth. Consider $2^n$ where $n = 1, 2\dots$ is an integer. We will study values of this discrete function as the "variable", $n$ in the exponent changes. We list some values and display a graph of $2^n$ versus $n$ in Figure 2.1. Notice that an initially "gentle" growth becomes extremely steep in just a few steps, as shown in the accompanying graph.

A screencast summary of population doubling and the Andromeda Strain. Edu.Cr.

Mastered Material Check

3. Compare the function $f(n) = 2^n$ and $g(n) = n^2$ for $n = 1, 2, \dots, 5$. How do these differ?

*Note:* properties of $2^n$ (and related expressions) are reviewed in Appendix **??** where common manipulations are illustrated. We assume the reader is familiar with this material.

| **n** | $\mathbf{2^n}$ |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |
| 6 | 64 |
| 7 | 128 |
| 8 | 256 |
| 9 | 512 |
| 10 | 1024 |

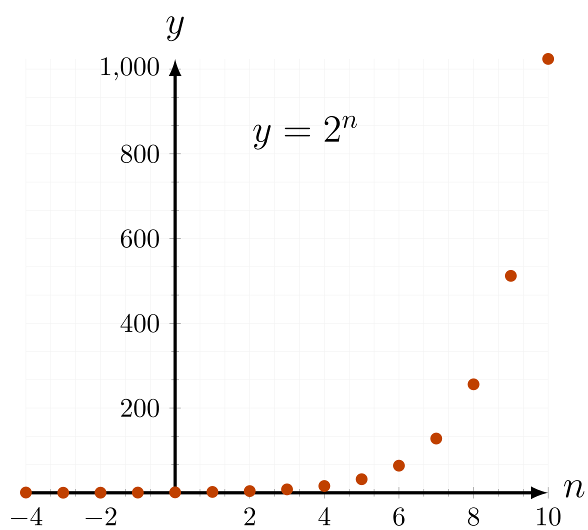



Figure 2.1: Powers of 2 including both negative and positive integers: here we show $2^n$ for $-4 < n < 10$.

The function $2^n$ first grows slowly, but then grows faster and faster as $n$ increases. As a side remark, the fact that $2^{10} \approx 1000 = 10^3$, will prove useful for simple approximations. With this preparation, we can now check the accuracy of Crichton's statement about bacterial growth.

Mastered Material Check

4. Why would the approximation $2^{10} \approx 10^3$ be helpful?

**Example 2.1 (Growth of E. coli)** *Use the following facts to check the assertion made by Crichton's statement at the beginning of this chapter.*

- *Mass of* 1 *E. coli cell :* 1 *nanogram* = $10^{-9}$*gm* = $10^{-12}$*kg.*
- *Mass of Planet Earth :* $6 \cdot 10^{24}$ *kg.*

**Solution.** Based on the above two facts, we surmise that the size of an E. coli colony (number of cells, $m$) that together form a mass equal to Planet Earth would be

$$m = \frac{6 \cdot 10^{24}\ \text{kg}}{10^{-12}\ \text{kg}} = 6 \cdot 10^{36}.$$

Mastered Material Check

5. How many cells of E. coli would there be after 20 minutes? 1 hour? 2 hours?

Each hour corresponds to 3 twenty-minute generations. In a period of 24 hours, there are $24 \times 3 = 72$ generations, each doubling the colony size. After 1 day of uncontrolled growth, the number of cells would be $2^{72}$. We can find a decimal approximation using the observation that $2^{10} \approx 10^3$:

$$2^{72} = 2^2 \cdot 2^{70} = 4 \cdot (2^{10})^7 \approx 4 \cdot (10^3)^7 = 4 \cdot 10^{21}.$$

Using a scientific calculator, the value is found to be $4.7 \cdot 10^{21}$, so the approximation is relatively good. $\Diamond$

Apparently, the estimate made by Crichton is not quite accurate. However it can be shown that it takes less than 2 days to produce a number far in excess of the "size of Planet Earth". The exact number of generations is left as an exercise for the reader and is discussed in Example 2.12.

Mastered Material Check

6. Verify that it takes less than 2 days to produce a number far in excess of the size of Planet Earth.

*The function $2^x$ and its "relatives"*

We would like to generalize the function $2^n$ to a continuous function, so that the tools of calculus - such as derivatives - can be used. To this end, we start with values that can be calculated based on previous mathematical experience, and then "fill in gaps".

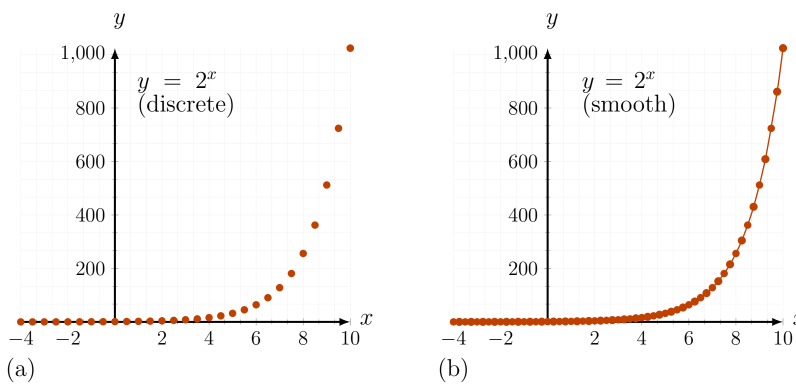



Figure 2.2: (a) Values of the function $2^x$ for discrete value of $x$. We can compute many values (e.g. for $x = 0, \pm 1, \pm 2$, by simple arithmetical operations, and for $x = \pm 1/2, \pm 3/2$ by evaluating square roots). (b) The function $2^x$ is connected smoothly to form a continuous curve.

From previous familiarity with power functions such as $y = x^2$ (not to be confused with $2^x$), we know the value of

$$2^{1/2} = \sqrt{2} \approx 1.41421\ldots$$

We can use this value to compute

$$2^{3/2} = (\sqrt{2})^3, \quad 2^{5/2} = (\sqrt{2})^5,$$

and all other fractional exponents that are multiples of $1/2$. We can add these to the graph of our previous powers of 2 to fill in additional points. This is shown on Figure 2.2(a).

Similarly, we could also calculate exponents that are multiples of $1/4$ since

$$2^{1/4} = \sqrt{\sqrt{2}}$$

is a value that we can obtain. Adding these values leads to an even finer set of points. By continuing in the same way, we "fill in" the graph of the emerging function. Connecting the dots smoothly allows us to define a value for any real $x$, of a new continuous function,

$$y = f(x) = 2^x.$$

Here $x$ is no longer restricted to an integer, as shown by the smooth curve in Figure 2.2(b).

Mastered Material Check

7. Given $2^{1/2} \approx 1.41421$, find $2^{3/2}$ and $2^{5/2}$ without using fractional powers.
8. What method might you use to determine a decimal approximation of $2^{1/4}$ without computing fractional powers?
9. Why do we need to assume that $a > 0$ for the exponential function $y = a^x$?

**Example 2.2 (Generalization to other bases)** *Plot "relatives" of $2^x$ that have other bases, such as $y = 3^x$, $y = 4^x$ and $y = 10^x$ and comment about the function $y = a^x$ where $a > 0$ is a constant (called the* **base***).*

**Solution.** We first form the discrete function $a^n$ for integer values of $n$, simply by multiplying $a$ by itself $n$ times. This is analogous to Figure 2.1. So long as $a$ is positive, we can "fill in" values of $a^x$ when $x$ is rational (in the same way as we did for $2^x$), and we can smoothly connect the points to lead to the continuous function $a^x$ for any real $x$. Given some positive constant $a$, we define the new function $f(x) = a^x$ as the exponential function with base $a$. Shown in Figure 2.3 are the functions $y = 2^x$, $y = 3^x$, $y = 4^x$ and $y = 10^x$. ◊

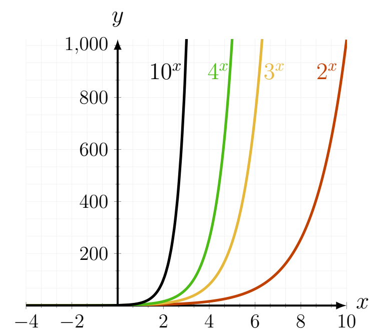



Figure 2.3: The function $y = f(x) = a^x$ is shown here for a variety of bases, $a = 2, 3, 4$, and 10.

## 2.2 *Derivatives of exponential functions and the function $e^x$*

**Section 2.2 Learning goals**

1. Using the definition of the derivative, calculate the derivative of the function $y = a^x$ for an arbitrary base $a > 0$.
2. Describe the significance of the special base $e$.
3. Summarize the properties of the function $e^x$, its derivatives, and how to manipulate it algebraically.
4. Recall the fact that the function $y = e^{kx}$ has a derivative that is proportional to the same function ($y = e^{kx}$).

### *Calculating the derivative of $a^x$*

A screencast with the calculations for this section and motivation for the natural base $e$. Edu.Cr.

In this section we show how to compute the derivative of the exponential function. Rather then restricting attention to the special case $y = 2^x$, we consider an arbitrary positive constant $a$ as the base. Note that the base has to be positive to ensure that the function is defined for all real $x$. For $a > 0$ let

$$y = f(x) = a^x.$$

Then, using the definition of the derivative,

$$\begin{aligned}
\frac{da^x}{dx} &= \lim_{h\to 0} \frac{\left(a^{x+h} - a^x\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(a^x a^h - a^x\right)}{h} \\
&= \lim_{h\to 0} a^x \frac{\left(a^h - 1\right)}{h} \\
&= a^x \left[\lim_{h\to 0} \frac{a^h - 1}{h}\right].
\end{aligned}$$

The variable $x$ appears only in the common factor $a^x$ that can be factored out. The limit applies to $h$, not $x$. The terms inside square brackets depend only on the base $a$ and on $h$, but once the limit is evaluated, that term is some constant (independent of $x$) that we denote by $C_a$. To summarize, we have found that

The **derivative of an exponential function** $a^x$ is of the form $C_a a^x$ where $C_a$ is a constant that depends only on the base $a$.

We now examine this in more detail with bases 2 and 10.

**Example 2.3 (Derivative of $2^x$)** *Write down the derivative of $y = 2^x$ using the above result.*

**Solution.** For base $a = 2$, we have

$$\frac{d2^x}{dx} = C_2 \cdot 2^x,$$

where

$$C_2(h) = \lim_{h\to 0} \frac{2^h - 1}{h} \approx \frac{2^h - 1}{h} \quad \text{for small } h.$$

The decimal expansion value of $C_2$ is determined in the next example. ♢

**Example 2.4 (The value of $C_2$)** *Find an approximation for the value of the constant $C_2$ in Example 2.3 by calculating the value of the ratio $(2^h - 1)/h$ for small (finite) values of $h$, e.g., $h = 0.1, 0.01$, etc. Do these successive approximations for $C_2$ value approach a fixed real number?*

**Solution.** We take these successively smaller values of $h$ and compute the value of $C_2 = (2^h - 1)/h$ on a spreadsheet.

The results are shown in Table 2.1, where we find that $C_2 \approx 0.6931$. (The actual value has an infinitely long decimal expansion that we here represent by its first few digits.) Thus, the derivative of $2^x$ is

$$\frac{d2^x}{dx} = C_2 \cdot 2^x \approx (0.6931) \cdot 2^x.$$

♢

Mastered Material Check

10. Describe geometrically the derivative of $a^x$.

Link to Google Sheets. The constant $C_a$ in the derivative of $a^x$ is calculated on this spreadsheet for $a = 2$. You can copy and paste this to our own spreadsheet and experiment with the value of the base $a$. Try to find a value of $a$ between 2 and 3 for which $C_a$ is close to 1.0.

| $h$ | $C_2$ |
|---|---|
| 0.1 | 0.717735 |
| 0.01 | 0.695555 |
| 0.001 | 0.693387 |
| 0.0001 | 0.693171 |
| 0.00001 | 0.693150 |
| 0.000001 | 0.693147 |
| 0.0000001 | 0.693147 |

Table 2.1: The constant $C_2$ in Example 2.4 is found by letting $h$ get smaller and smaller. The value converges to $C_2 = 0.693147$ .

**Example 2.5 (The base 10 and the derivative of $10^x$)** *Determine the derivative of $y = f(x) = 10^x$.*

**Solution.** For base 10 we have

$$C_{10}(h) \approx \frac{10^h - 1}{h} \quad \text{for small } h.$$

We find, by similar approximation (Table 2.2), that $C_{10} \approx 2.3026$, so that

$$\frac{d10^x}{dx} = C_{10} \cdot 10^x \approx (2.3026) \cdot 10^x.$$

◊

| $h$ | $C_{10}$ |
|---|---|
| 0.1 | 2.589254 |
| 0.01 | 2.329299 |
| 0.001 | 2.305238 |
| 0.0001 | 2.302850 |
| 0.00001 | 2.302612 |
| 0.000001 | 2.302588 |
| 0.0000001 | 2.302585 |

Table 2.2: As in Table 2.1 but for the constant $C_{10}$ in Example 2.5. (The advantage of using a spreadsheet is that we only need to change one cell to obtain this new set of values.)

Mastered Material Check

11. What does it mean for a function $f(x)$ to be proportional to itself?

Thus, the derivative of $y = a^x$ is proportional to itself, but the constant of proportionality ($C_a$) depends on the base.

### *The natural base e is convenient for calculus*

In Examples 2.3-2.5, we found that the derivative of $a^x$ is $C_a a^x$, where the constant $C_a$ depends on the base. These constants are somewhat inconvenient, but unavoidable if we use an arbitrary base. Here we ask:

Does there exists a convenient base (to be called "$e$") for which the constant is particularly simple, namely such that $C_e = 1$?

This is the property of the **natural base** that we next identify.

We can determine such a hypothetical base using only the property that

$$C_e = \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

This means that for small $h$

$$\frac{e^h - 1}{h} \approx 1,$$

so that

$$e^h - 1 \approx h \quad \Rightarrow \quad e^h \approx h + 1 \quad \Rightarrow \quad e \approx (1+h)^{1/h}.$$

More formally,

$$e = \lim_{h \to 0} (1+h)^{1/h}. \tag{2.1}$$

We can find an approximate decimal expansion for $e$ by calculating the ratio in Eqn. (2.1) for some very small (but finite value) of $h$ on a spreadsheet. Results are shown in Table 2.3. We find (e.g. for $h = 0.00001$) that

$$e \approx (1.00001)^{100000} \approx 2.71826.$$

To summarize, we have found that for the special base, $e$, we have the following property:

**The derivative of the function $e^x$ is $e^x$.**

Link to Google Sheets. The calculation of a decimal approximation to base $e$ as shown in Table 2.3 .

| $h$ | approximation to $e$ |
|---|---|
| 0.1 | 2.5937425 |
| 0.01 | 2.7048138 |
| 0.001 | 2.7169239 |
| 0.0001 | 2.7181459 |
| 0.00001 | 2.7182682 |

Table 2.3: We can use a spreadsheet to find a decimal approximation to the natural base $e$ using Eqn. (2.1) and letting $h$ approach zero.

Mastered Material Check

12. Why can't we simply plug in $h = 0$ into Eqn. (2.1) evaluate the limit?
13. Let $h = \frac{1}{n}$ and rewrite Eqn (2.1).
14. Explain why each of Properties 1. → 8. hold for the function $e^x$.

The value of base $e$ is obtained from the limit in Eqn. (2.1). This can be written in either of two equivalent forms.

The base of the natural exponential function is the real number defined as follows:

$$e = \lim_{h\to 0}(1+h)^{1/h} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

### *Properties of the function $e^x$*

We list below some of the key features of the function $y = e^x$. Note that all stem from basic manipulations of exponents as reviewed in Appendix **??**.

Use the slider to adjust the value of the base $a$ in the function $y = a^x$; Compare your result with the function $y = e^x$. Explain what you see for $a > 1$, $a = 1$, $0 < a < 1$ and $a = 0$.

1. $e^a e^b = e^{a+b}$ as with all similar exponent manipulations.
2. $(e^a)^b = e^{ab}$ also stems from simple rules for manipulating exponents.
3. $e^x$ is a function that is defined, continuous, and differentiable for all real numbers $x$.
4. $e^x > 0$ for all values of $x$.
5. $e^0 = 1$, and $e^1 = e$.
6. $e^x \to 0$ for increasing negative values of $x$.
7. $e^x \to \infty$ for increasing positive values of $x$.
8. The derivative of $e^x$ is $e^x$ (shown in this chapter).

**Example 2.6** **a)** *Find the derivative of $e^x$ at $x = 0$.*

**b)** *Show that the tangent line at that point is the line $y = x + 1$.*

Review: On this graph of $f(x) = e^x$ add a generic tangent line at any point $x_0$. (See Sections **??**-**??**). Adjust a slider for $x_0$ to get the configuration shown in Fig. 2.4.

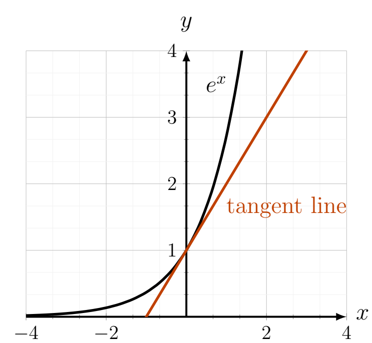



Figure 2.4: The function $y = e^x$ has the property that its tangent line at $x = 0$ has slope 1.

**Solution.**

**a)** The derivative of $e^x$ is $e^x$. At $x = 0$, $e^0 = 1$.

**b)** The slope of the tangent line at $x = 0$ is therefore 1. The tangent line goes through $(0, e^0) = (0, 1)$ so it has a $y$-intercept of 1. Thus the tangent line at $x = 0$ with slope 1 is $y = x + 1$. This is shown in Figure 2.4. ♢

### *Composite derivatives involving exponentials*

Using the derivative of $e^x$ and the chain rule, we can now differentiate composite functions in which the exponential function appears.

**Example 2.7** *Find the derivative of $y = e^{kx}$.*

**Solution.** Letting $u = kx$ gives $y = e^u$. Applying the simple chain rule leads to,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

but

$$\frac{du}{dx} = k \quad \text{so} \quad \frac{dy}{dx} = e^u k = ke^{kx}.$$

◊

We highlight this result for future use:

The derivative of $y = e^{kx}$ is

$$\frac{dy}{dx} = ke^{kx}.$$

Mastered Material Check

15. Let $y = e^{5x}$. What is $\frac{dy}{dx}$?
16. Let $y = e^{\pi x}$. What is $\frac{dy}{dx}$?
17. List all constants in Example 2.8.
18. List all variables in Example 2.8.

**Example 2.8 (Chemical reactions)** *According to the collision theory of bimolecular gas reactions, a reaction between two molecules occurs when the molecules collide with energy greater than some activation energy, $E_a$, referred to as the Arrhenius activation energy. $E_a > 0$ is constant for the given substance. The fraction of bimolecular reactions in which this collision energy is achieved is*

$$F = e^{-(E_a/RT)},$$

*where $T$ is temperature (in degrees Kelvin) and $R > 0$ is the gas constant. Suppose that the temperature $T$ increases at some constant rate, $C$, per unit time.*

*Determine the rate of change of the fraction $F$ of collisions that result in a successful reaction.*

**Solution.** This is a related rates problem involving an exponential function that depends on the temperature, which depends on time, $F = e^{-(E_a/RT(t))}$. We are asked to find the derivative of $F$ with respect to time when the temperature increases.

We are given that $dT/dt = C$. Let $u = -E_a/RT$. Then $F = e^u$. Using the chain rule,

$$\frac{dF}{dt} = \frac{dF}{du}\frac{du}{dT}\frac{dT}{dt}.$$

Further, we have $E_a, R, C$ are all constants, so

$$\frac{dF}{du} = e^u \quad \text{and} \quad \frac{du}{dT} = \frac{E_a}{RT^2}.$$

Assembling these parts, we have

$$\frac{dF}{dt} = e^u \frac{E_a}{RT^2} C = C\frac{E_a}{R} T^{-2} e^{-(E_a/RT)} = \frac{CE_a}{RT^2} e^{-(E_a/RT)}.$$

Thus, the rate of change of the fraction $F$ of collisions that result in a successful reaction is given by the expression above. ◊

**Featured Problem 2.1 (Ricker model for fish population growth)** *Salmon are fish with non-overlapping generations. The adults lay eggs that are fertilized by males before the entire population dies. The eggs hatch to form a new generation. In Featured Problem 1.1, we considered one model for fish populations. Here we discuss a second model, the Ricker Equation, wherein the fish population this year, $N_1$, is related to the population last year, $N_0$, by the rule*

$$N_1 = N_0 e^{r\left(1-\frac{N_0}{K}\right)}, \quad r, K > 0. \tag{2.2}$$

*Here $r$ is called an intrinsic growth rate, and $K$ is the carrying capacity of the population. We investigate the following questions.*

*(a) Is there a population level $N_0$ that would stay constant from one year to the next?*

*(b) Simplify the notation by setting $x = N_0, y = N_1$. Compute the derivative $dy/dx$ and interpret its meaning.*

*(c) What population level this year would result in the greatest possible population next year?*

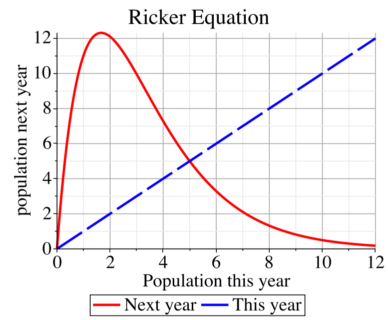



Figure 2.5: The functional form of the Ricker equation.

Adjust the sliders to observe how the parameters $K$ and $r$ affect the Ricker equation 2.2. What is special about the intersection of the two curves shown?

## *The function $e^x$ satisfies a new kind of equation*

We divert our attention momentarily to an interesting observation. We have seen that the function

$$y = f(x) = e^x$$

satisfies the relationship

$$\frac{dy}{dx} = f'(x) = f(x) = y.$$

In other words, when differentiating, we get the same function back again. We summarize this observation:

The function $y = f(x) = e^x$ is equal to its own derivative. It hence satisfies the equation

$$\frac{dy}{dx} = y.$$

An equation linking a function and its derivative(s) is called a **differential equation**.

This is a new type of equation, unlike others previously seen in this course. In later Chapters, we show that these differential equations have many applications to biology, physics, chemistry, and science in general.

## 2.3 *Inverse functions and logarithms*

**Section 2.3 Learning goals**

1. Explain the concept of inverse function from both algebraic and geometric points of view: given a function, determine whether (and for what restricted domain) an inverse function can be defined and sketch that inverse function.
2. Describe the relationship between the domain and range of a function and the range and domain of its inverse function. (Review Appendix **??**).
3. Apply these ideas to the logarithm, which is the inverse of an exponential function.
4. Reproduce the calculation of the derivative of $\ln(x)$ using implicit differentiation.

In this chapter we defined the new function $e^x$ and computed its derivative. Paired with this newcomer is an inverse function, the natural logarithm, $\ln(x)$. Recall the following key ideas:

- Given a function $y = f(x)$, its inverse function, denoted $f^{-1}(x)$ satisfies

$$f(f^{-1}(x)) = x, \quad \text{and} \quad f^{-1}(f(x)) = x.$$

- The range of $f(x)$ is the domain of $f^{-1}(x)$ (and vice versa), which implies that in many cases, the relationship holds only on some subset of the original domains of the functions.
- The functions $f(x) = x^n$ and $g(x) = x^{1/n}$ are inverses of one another for all $x$ when $n$ is odd.
- The domain of a function (such as $y = x^2$ or other even powers) must be restricted (e.g. to $x \geq 0$) so that its inverse function ($y = \sqrt{x}$) is defined.
- On that restricted domain, the graphs of $f$ and $f^{-1}$ are mirror images of one another about the line $y = x$. Essentially, this stems from the fact that the roles of $x$ and $y$ are interchanged.

Mastered Material Check

19. Are $f(x) = x^n$ and $g(x) = x^{1/n}$ also inverses of one another for even integer $n$? Is this true for all $x$?
20. What is the inverse function for $y = x$? Over what range of values is the inverse defined?
21. What is the inverse function to $y = x^{2/3}$ and over what domain are the two functions inverses of one another?

Note symmetry about the line $y = x$ for this graph of $f(x) = x^n$ and $g(x) = x^{1/n}$. Adjust the slider for $n$ to see how even and odd powers behave. What do you notice about the domain over which $g(x)$ is defined? Adjust the slider for $a$ to observe "corresponding points" on the two graphs.

### *The natural logarithm is an inverse function for $e^x$*

For $y = f(x) = e^x$ we define an inverse function, shown on Figure 2.6. We call this function the logarithm (base $e$), and write it as

$$y = f^{-1}(x) = \ln(x).$$

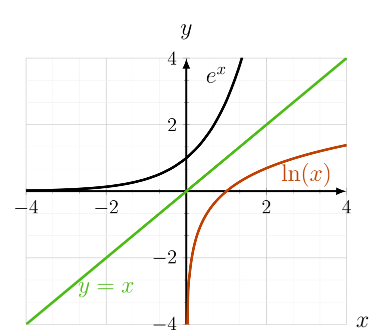



Figure 2.6: The function $y = e^x$ is shown with its inverse, $y = \ln x$.

We have the following connection: $y = e^x$ implies $x = \ln(y)$. The fact that the functions are inverses also implies that

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^x) = x.$$

The domain of $e^x$ is $-\infty < x < \infty$, and its range is $x > 0$. For the inverse function, this domain and range are interchanged, meaning that $\ln(x)$ is only defined for $x > 0$ (its domain) and returns values in $-\infty < x < \infty$ (its range). As shown in Figure 2.6, the functions $e^x$ and $\ln(x)$ are reflections of one another about the line $y = x$.

Properties of the logarithm stem directly from properties of the exponential function. A review of these is provided in Appendix **??**. Briefly,

1. $\ln(ab) = \ln(a) + \ln(b)$,
2. $\ln(a^b) = b\ln(a)$,
3. $\ln(1/a) = \ln(a^{-1}) = -\ln(a)$.

Mastered Material Check

22. Give algebraic justification of the three properties of logarithms.

**Featured Problem 2.2 (Agroforestry)** *In agroforestry, the farming of crops is integrated with growing of trees to benefit productivity and maintain the health of an ecosystem. A tree can provide advantage to nearby plants by creating better soil permeability, higher water retention, and more stable temperatures. At the same time, trees produce shade and increased competition for nutrients. Both the advantage $A(x)$ and the shading $S(x)$ depend on distance from the tree, with shading a dominant negative effect right under the tree. Suppose that at a distance $x$ from a given tree species, the net benefit $B$ to a crop plant can be expressed as the difference*

$$B(x) = A(x) - S(x), \quad \textit{where} \;\; A(x) = \alpha e^{-x^2/a^2}, \quad S(x) = \beta e^{-x^2/b^2}, \alpha\beta, a, b > 0$$

*(a) How far away from the tree will the two influences break even? (b) Find the optimal distance to plan crops so that they derive maximal benefit from the nearby tree.*

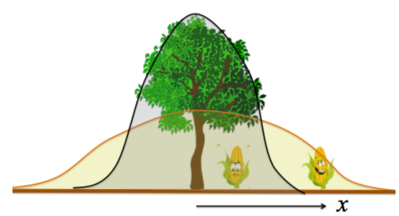



Figure 2.7: Too close to a tree, shading (grey) $S(x)$ interferes with crop growth. Just beyond this region, the advantage $A(x)$ to crop growth outweighs any disadvantage due to shading. We seek to find the optimal distance $x$ for planting the crops.

The Advantage $A(x)$, the shading effect $S(x)$, and the net benefit $B(x)$ for a crop as functions of distance $x$ from a tree are shown here. Move the sliders to see how the spatial range $a$ and the magnitude $\beta$ affect the graphs.

## *Derivative of $\ln(x)$ by implicit differentiation*

Implicit differentiation is helpful whenever an inverse function appears. Knowing the derivative of the original function allows us to compute the derivative of its inverse by using their relationship. We use implicit differentiation to find the derivative of $y = \ln(x)$.

First, restate the relationship in the inverse form, but consider $y$ as the dependent variable - that is think of $y$ as a quantity that depends on $x$:

$$y = \ln(x) \quad \Rightarrow \quad e^y = x \quad \Rightarrow \quad \frac{d}{dx}e^{y(x)} = \frac{d}{dx}x.$$

Applying the chain rule to the left hand side,

$$\frac{de^y}{dy}\frac{dy}{dx} = 1 \quad \Rightarrow \quad e^y\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

We have thus shown the following:

The derivative of $\ln(x)$ is $1/x$:

$$\frac{d\ln(x)}{dx} = \frac{1}{x}.$$

Tangent lines to the graphs of $y = e^x$ and $y = \ln(x)$ at **corresponding points** are mirror images about the line $y = x$. Adjust the slider to see the tangent lines at various points along the curves. What do we mean by "corresponding points"?

Inverse functions are mirror images of one another about the line $y = x$, since the role of independent and dependent variables are switched. Their tangent lines are also mirror images about the same line.

## 2.4 *Applications of the logarithm*

**Section 2.4 Learning goals**

1. Describe the relationships between properties of $e^x$ and properties of its inverse $\ln(x)$, and master manipulations of expressions involving both.
2. Use logarithms for base conversions.
3. Use logarithms to solve equations involving the exponential function, i.e. solve $A = e^{bt}$ for $t$.)
4. Given a relationship such as $y = ax^b$, show that $\ln(y)$ is related linearly to $\ln(x)$, and use data points for $(x, y)$ to determine the values of $a$ and $b$.

### *Using the logarithm for base conversion*

The logarithm is helpful in changing an exponential function from one base to another. We give some examples here.

Mastered Material Check

23. Why might one base be preferred over another?

**Example 2.9** *Rewrite $y = 2^x$ in terms of base $e$.*

**Solution.** We apply ln and then exponentiate the result. Manipulations of exponents and logarithms lead to the desired results as follows:

$$y = 2^x \quad \Rightarrow \quad \ln(y) = \ln(2^x) = x\ln(2).$$

$$e^{\ln(y)} = e^{x\ln(2)} \quad \Rightarrow \quad y = e^{x\ln(2)}.$$

We find (using a calculator) that $\ln(2) = 0.6931\ldots$ This coincides with the value we computed earlier for $C_2$ in Example 2.4, so we have

$$y = e^{kx} \quad \text{where} \quad k = \ln(2) = 0.6931\ldots$$

◊

**Example 2.10** *Find the derivative of $y = 2^x$.*

**Solution.** In Example 2.9 we expressed this function in the alternate form

$$y = 2^x = e^{kx} \quad \text{with} \quad k = \ln(2).$$

From Example 2.7 we have

$$\frac{dy}{dx} = ke^{kx} = \ln(2)e^{\ln(2)x} = \ln(2)2^x.$$

Through the above base conversion and chain rule, we relate the constant $C_2$ in Example 2.4 to the natural logarithm of 2: $C_2 = \ln(2)$. ◊

*The logarithm helps to solve exponential equations*

Equations involving the exponential function can sometimes be simplified and solved using the logarithm. We provide a few examples of this kind.

**Example 2.11** *Find zeros of the function $y = f(x) = e^{2x} - e^{5x^2}$.*

**Solution.** We seek values of $x$ for which $f(x) = e^{2x} - e^{5x^2} = 0$. We write

$$e^{2x} - e^{5x^2} = 0 \quad \Rightarrow \quad e^{2x} = e^{5x^2} \quad \Rightarrow \quad \frac{e^{5x^2}}{e^{2x}} = 1 \quad \Rightarrow \quad e^{5x^2-2x} = 1.$$

Taking logarithm of both sides, and using the facts that $\ln(e^{5x^2-2x}) = 5x^2 - 2x$ and $\ln(1) = 0$, we obtain

$$e^{5x^2-2x} = 1 \quad \Rightarrow \quad 5x^2 - 2x = 0 \quad \Rightarrow \quad x = 0, \frac{5}{2}.$$

We see that the logarithm is useful in the last step of isolating $x$, after simplifying the exponential expressions appearing in the equation. ◊

**Andromeda Strain, revisited.** In Section 2.1 we posed the question: how long does it take for the Andromeda strain population to attain a size of $6 \cdot 10^{36}$ cells, i.e. to grow to an Earth-sized colony? We now solve this problem using the continuous exponential function and the logarithm.

Recall that the bacterial doubling time is 20 min. If time is measured in minutes, the number, $B(t)$ of bacteria at time $t$ could be described by the smooth function:

$$B(t) = 2^{t/20}.$$

Mastered Material Check

24. Verify that $B(t)$ agrees with Figure 2.1 and give powers of 2 at $t = 20, 40, 60, 80, \ldots$ minutes.
25. When, in general, will $B(t)$ give a power of 2?

**Example 2.12 (The Andromeda strain)** *Starting from a single cell, how long does it take for an E. coli colony to reach size of $6 \cdot 10^{36}$ cells by doubling every 20 minutes?*

**Solution.** We compute the time it takes by solving for $t$ in $B(t) = 6 \cdot 10^{36}$, as shown below.

$$6 \cdot 10^{36} = 2^{t/20} \quad \Rightarrow \quad \ln(6 \cdot 10^{36}) = \ln(2^{t/20})$$
$$\ln(6) + 36\ln(10) = \frac{t}{20}\ln(2).$$

Solving for $t$,

$$t = 20\frac{\ln(6) + 36\ln(10)}{\ln(2)} = 20\frac{1.79 + 36(2.3)}{0.693} = 2441.27 \text{ min} = \frac{2441.27}{60} \text{ hr}.$$

Hence, it takes nearly 41 hours (but less than 2 days) for the colony to "grow to the size of planet Earth" (assuming the implausible scenario of unlimited growth). ♢

**Example 2.13 (Using base** $e$**)** *Express the number of bacteria in terms of base e (for practice with base conversions).*

**Solution.** Given $B(t) = 2^{t/20}$ is the number of bacteria at time $t$, we proceed as follows:

$$B(t) = 2^{t/20} \quad \Rightarrow \quad \ln(B(t)) = \frac{t}{20}\ln(2),$$
$$e^{\ln(B(t))} = e^{\frac{t}{20}\ln(2)} \quad \Rightarrow \quad B(t) = e^{kt} \quad \text{where} \quad k = \frac{\ln(2)}{20} \text{ per min}.$$

♢

The constant $k$ has units of 1/time. We refer to $k$ as the growth rate of the bacteria. We observe that this constant can be written as:

$$k = \frac{\ln(2)}{\text{doubling time}}.$$

As we see next, this approach is helpful in scientific applications.

### *Logarithms help plot data that varies on large scale*

Living organisms come in a variety of sizes, from the tiniest cells to the largest whales. Comparing attributes across species of vastly different sizes poses a challenge, as visualizing such data on a simple graph obscures both extremes.

Suppose we wish to compare the physiology of organisms of various sizes, from that of a mouse to that of an elephant. An example of such data for metabolic rate versus mass of the animal is shown in Table 2.4.

| **animal** | **body weight** $M$ (gm) | **basal metabolic rate** (BMR) |
|---|---|---|
| mouse | 25 | 1580 |
| rat | 226 | 873 |
| rabbit | 2200 | 466 |
| dog | 11700 | 318 |
| man | 70000 | 202 |
| horse | 700000 | 106 |

Table 2.4: Animals of various sizes (mass $M$ in gm) have widely different basal metabolic rates (BMR, generally measured in terms of oxygen consumption rate, i.e. ml $O_2$ consumed per hr). A log-log plot of this data is shown in Figure 2.8.

It would be hard to see all data points clearly on a regular graph. For this reason, it is helpful to use logarithmic scaling for either or both variables. We show an example of this kind of **log-log plot**, where both axes use logarithmic scales, in Figure 2.8.

In allometry, it is conjectured that such data fits some power function of the form

$$y \approx ax^b, \text{ where } a, b > 0. \tag{2.3}$$

*Note:* this is not an exponential function, but a power function with power $b$ and coefficient $a$.

Finding the **allometric constants** $a$ and $b$ using the graph in Fig 2.8 is now explained.

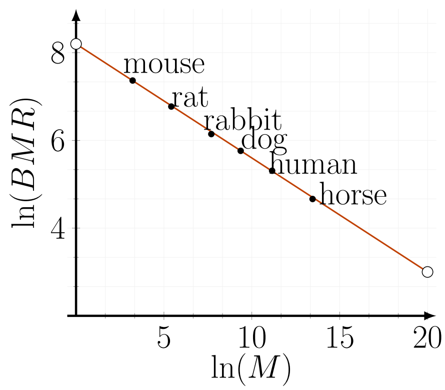



Figure 2.8: A log-log plot of the data in Table 2.4, showing ln(BMR) versus $\ln(M)$.

**Mastered Material Check**

26. Use software to plot the data given in Table 2.4. Why is it so hard to plot on a regular graph?

**Example 2.14 (Log transformation)** *Define* $Y = \ln(y)$ *and* $X = \ln(x)$. *Show that* (2.3) *can be rewritten as a linear relationship between* $Y$ *and* $X$.

**Solution.** We have

$$Y = \ln(y) = \ln(ax^b) = \ln(a) + \ln(x^b) = \ln(a) + b\ln(x) = A + bX,$$

where $A = \ln(a)$. Thus, we have shown that $X$ and $Y$ are related linearly:

$$Y = A + bX, \quad \text{where} \quad A = \ln(a).$$

This is the equation of a straight line with slope $b$ and $Y$ intercept $A$. $\diamond$

**Example 2.15 (Finding the constants)** *Use the straight line superimposed on the data in Figure* 2.8 *to estimate the values of the constants* $a$ *and* $b$.

**Solution.** We use the straight line that has been fitted to the data in Figure 2.8. The $Y$ intercept is roughly 8.2. The line goes approximately through $(20, 3)$ and $(0, 8.2)$ (open dots on plot) so its slope is $\approx (3 - 8.2)/20 = -0.26$. According to the relationship we found in Example 2.14,

$$8.2 = A = \ln(a) \quad \Rightarrow \quad a = e^{8.2} = 3640, \quad \text{and} \quad b = -0.26.$$

Thus, reverting to the original allometric relationship leads to

$$y = ax^b = 3640x^{-0.26} = \frac{3640}{x^{0.26}}.$$

From this we see that the metabolic rate $y$ decreases with the size $x$ of the animal, as indicated by the data in Table 2.4.

## 2.5 Summary

1. We reviewed exponential functions of the form $y = a^x$, where $a > 0$, the base, is constant.
2. The function $y = e^x$ is its own derivative, that is $\frac{dy}{dx} = e^x$. This function satisfies $\frac{dy}{dx} = y$, which is an example of a **differential equation**.
3. If $y = f(x)$, its inverse function is denoted $f^{-1}(x)$ and satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The graph of $f^{-1}$ is the same as the graph of $f$ reflected across the line $y = x$. The domain of a function may have to be restricted so that its inverse function exists.
4. Let $f(x) = e^x$. The inverse of this function is $f^{-1}(x) = \ln(x)$. The derivative of $\ln(x)$ is $\frac{1}{x}$.
5. We can transform exponential relationships into linear relationships using logarithms. Such transformations allow for more meaningful plots, and can aid us in finding unknown constants in exponential relationships.
6. The applications in this chapter included:

   (a) the Andromeda strain of E. coli (a bacterium) and its doubling;
   (b) the Ricker equation for fish population growth from one year to the next;
   (c) chemical reactions: the fraction which result in a successful reaction;
   (d) how the advantage and disadvantage of plants growing near a tree depend on distance from the tree; and
   (e) allometry: the relationship between body weight and basal metaboloci rate.

**Quick Concept Checks**

1. Instead of 1 E. coli cell, suppose we began with 2 which also doubled every 20 min. How long would it take for the population to grow to the size of the earth?

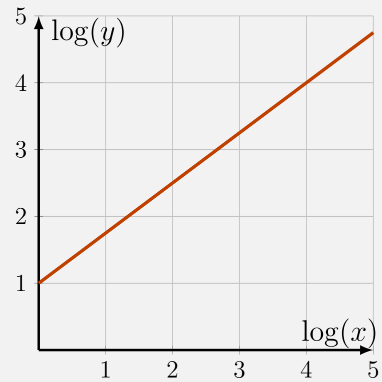
2. Given $\sqrt{3} \approx 1.74205$, compute without taking square roots:

   (a) $3^{3/2}$,

   (b) $3^{5/2}$,

3. Let $x = e^{\rho a}$. Determine $\frac{dx}{da}$.

4. Consider the following log-log plot




   (a) Let $Y = \log(y)$ and $X = \log(x)$. Find constants $A$ and $B$ such that $Y = AX + B$.

   (b) Determine constants $a$ and $b$ such that $y = ax^b$.

## *Exercises*

2.1. **Polymerase Chain reaction (PCR).** The polymerase chain reaction (PCR) was invented by Mullis in 1983 to amplify DNA. It is based on the fact that each strand of (double-stranded) DNA can act as a template for the synthesis of the second ("complementary") strand. The method consists of repeated cycles of heating (which separates the DNA strands) and cooling (allowing for new DNA to be assembled on each strand). The reaction mixture includes the original DNA to be amplified, plus enzymes and nucleotides, the components needed to form the new DNA). Each cycle doubles the amount of DNA.

A particular PCR experiment consisted of 35 cycles.

(a) By what factor was the original DNA amplified? Give your answer both in terms of powers of 2 and in approximate decimal (powers of ten) notations.

(b) Use the approximation in the caption of Table 2.1 (rather than a scientific calculator) to find the decimal approximation.

2.2. **Invention of the game of chess.** According to some legends, the inventor of the game of chess (who lived in India thousands of years ago) so pleased his ruler, that he was asked to chose his reward.

*"I would be content with grains of wheat. Let one grain be placed on the first square of my chess board, and double that number on the second, double that on the third, and so on,"* said the inventor. The ruler gladly agreed.

A chessboard has $8 \times 8$ squares. How many grains of wheat would be required for the last square on that board? Give your answer in decimal notation.

*Note:* in the original wheat and chessboard problem, we are asked to find the total number of wheat grains on all squares. This requires summing a geometric series, and is a problem ideal for early 2nd term calculus.

2.3. **Computing powers of 2.** In order to produce the graph of the continuous function $2^x$ in Figure 2.2, it was desirable to generate many points on that graph using simple calculations. Suppose you have an ordinary calculator with the operations $+, -, \times, /$. You also know that $\sqrt{2} \approx 1.414$.

How would you compute $2^x$ for the values $x = 7/2, x = -1/2$, and $x = -5$?

2.4. **Exponential base requirement.** Explain the requirement that $a$ must be positive in the exponential function $y = a^x$. What could go wrong if $a$ was a negative base?

2.5. **Derivative of $3^x$.** Find the derivative of $y = 3^x$. What is the value of the multiplicative constant $C_3$ that shows up in your calculation?

2.6. **Graphing functions.** Graph the following functions:

(a) $f(x) = x^2 e^{-x}$,

(b) $f(x) = \ln(e^{2x})$.

2.7. **Changing bases.** Express the following in terms of base $e$:

(a) $y = 3^x$,

(b) $y = \frac{1}{7^x}$,

(c) $y = 15^{x^2+2}$.

Express the following in terms of base 2:

(d) $y = 9^x$,

(e) $y = 8^x$,

(f) $y = -e^{x^2+3}$.

Express the following in terms of base 10:

(g) $y = 21^x$,

(h) $y = 1000^{-10x}$,

(i) $y = 50^{x^2-1}$.

2.8. **Comparing numbers expressed using exponents.** Compare the values of each pair of numbers (i.e. indicate which is larger):

(a) $5^{0.75}, 5^{0.65}$

(b) $0.4^{-0.2}, 0.4^{0.2}$

(c) $1.001^2, 1.001^3$

(d) $0.999^{1.5}, 0.999^{2.3}$

2.9. **Logarithms.** Rewrite each of the following equations in logarithmic form:

(a) $3^4 = 81$,

(b) $3^{-2} = \frac{1}{9}$,

(c) $27^{-\frac{1}{3}} = \dfrac{1}{3}$.

2.10. **Equations with logarithms.** Solve the following equations for $x$:

(a) $\ln x = 2\ln a + 3\ln b$,

(b) $\log_a x = \log_a b - \frac{2}{3}\log_a c$.

2.11. **Reflections and transformations.** What is the relationship between the graph of $y = 3^x$ and the graph of each of the following functions?

(a) $y = -3^x$ ,

(b) $y = 3^{-x}$,

(c) $y = 3^{1-x}$,

(d) $y = 3^{|x|}$,

(e) $y = 2 \cdot 3^x$,

(f) $y = \log_3 x$.

2.12. **Equations with exponents and logarithms.** Solve the following equations for $x$:

(a) $e^{3-2x} = 5$,

(b) $\ln(3x-1) = 4$,

(c) $\ln(\ln(x)) = 2$,

(d) $e^{ax} = Ce^{bx}$, where $a \neq b$ and $C > 0$.

2.13. **Derivative of exponential and logarithmic functions.** Find the first derivative for each of the following functions:

**Formula.**
$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$

(a) $y = \ln(2x+3)^3$,

(b) $y = \ln^3(2x+3)$,

(c) $y = \ln(\cos \frac{1}{2}x)$,

(d) $y = \log_a(x^3 - 2x)$

(e) $y = e^{3x^2}$,

(f) $y = a^{-\frac{1}{2}x}$,

(g) $y = x^3 \cdot 2^x$,

(h) $y = e^{e^x}$,

(i) $y = \dfrac{e^t - e^{-t}}{e^t + e^{-t}}$.

2.14. **Maximum, minimum and inflection points.** Find the maximum and minimum points as well as all inflection points of the following functions:

(a) $f(x) = x(x^2 - 4)$,

(b) $f(x) = x^3 - \ln(x), x > 0$,

(c) $f(x) = xe^{-x}$,

(d) $f(x) = \frac{1}{1-x} + \frac{1}{1+x}, -1 < x < 1$,

(e) $f(x) = x - 3\sqrt[3]{x}$,

(f) $f(x) = e^{-2x} - e^{-x}$.

2.15. **Using graph information.** Shown in Figure 15 is the graph of $y = Ce^{kt}$ for some constants $C, k$, and a tangent line. Use data from the graph to determine $C$ and $k$.

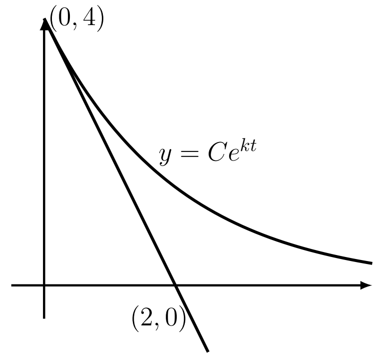



Figure 2.9: Figure for Exercise 15; $y = Ce^{kt}$ and a tangent line.

2.16. **Comparing exponential functions.** Consider the two functions

1. $y_1(t) = 10e^{-0.1t}$,
2. $y_2(t) = 10e^{0.1t}$.

Answer the following:

(a) Which one is decreasing and which one is increasing?

(b) In each case, find the value of the function at $t = 0$.

(c) Find the time at which the increasing function has doubled from this initial value.

(d) Find the time at which the decreasing function has fallen to half of its initial value.

*Note:* these values of $t$ are called the doubling time, and half-life, respectively

2.17. **Invasive species.** An ecosystem with mature trees has a relatively constant population of beetles (species 1) - around $10^9$. At $t = 0$, a single reproducing invasive beetle (species 2) is introduced accidentally.

If this population grows at the exponential rate

$$N_2(t) = e^{rt}, \quad \text{where } r = 0.5 \text{ per month},$$

how long does it take for species 2 to overtake the population of the resident species 1? Assume exponential growth for the entire duration.

2.18. **Human population growth.** It is sometimes said that the population of humans on Earth is growing exponentially. This means

$$P(t) = Ce^{rt}, \quad \text{where } r > 0.$$

We investigate this claim. To this end, we consider the human population beginning in year 1800 ($t = 0$). Hence, we ask whether the data in Table 2.5 fits the relationship

$$P(t) = Ce^{r(t-1800)}, \quad \text{where } t \text{ is time in years and } r > 0?$$

(a) Show that the above relationship implies that $\ln(P)$ is a linear function of time, and that $r$ is the slope of the linear relationship (*hint:* take the natural logarithm of both sides of the relationship and simplify).

(b) Use the data from Table 2.5 for the years 1800 to 2020 to investigate whether $P(t)$ fits an exponential relationship (*hint:* plot $\ln(P)$, where $P$ is human population (in billions) against time $t$ in years - we refer to this process as "transforming the data".

(c) A spreadsheet can be used to fit a straight line through the transformed data you produced in (b).

(i) Find the best fit for the growth rate parameter $r$ using that option.

(ii) What are the units of $r$?

(iii) What is the best fit value of $C$?

(d) Based on your plot of $\ln(P)$ versus $t$ and the best fit values of $r$ and $C$, over what time interval was the population growing more slowly than the overall trend, and when was it growing more rapidly than this same overall trend?

(e) Under what circumstances could an exponentially growing population be **sustainable**?

| **year** | **human population** (billions) |
|---|---|
| 1 | 0.2 |
| 1000 | 0.275 |
| 1500 | 0.45 |
| 1650 | 0.5 |
| 1750 | 0.7 |
| 1804 | 1 |
| 1850 | 1.2 |
| 1900 | 1.6 |
| 1927 | 2 |
| 1950 | 2.55 |
| 1960 | 3 |
| 1980 | 4.5 |
| 1987 | 5 |
| 1999 | 6 |
| 2011 | 7 |
| 2020 | 7.7 |

Table 2.5: The human population (billions) over the years AD 1 to AD 2020.

2.19. **A sum of exponentials.** Researchers that investigated the molecular motor dynein found that the number of motors $N(t)$ remaining attached to their microtubule tracks at time $t$ (in sec) after a pulse of activation was well described by a double exponential of the form

$$N(t) = C_1e^{-r_1t} + C_2e^{-r_2t}, \quad t \geq 0.$$

They found that $r_1 = 0.1, r_2 = 0.01$ per second, and $C_1 = 75, C_2 = 25$ percent.

(a) Plot this relationship for $0 < t < 8$ min. Which of the two exponential terms governs the behaviour over the first minute? Which dominates in the later phase?

(b) Now consider a plot of $\ln(N(t))$ versus $t$. Explain what you see and what the slopes and other aspects of the graph represent.

2.20. **Exponential Peeling.** The data in Table 2.6 is claimed to have been generated by a double exponential function of the form

$$N(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}, \quad t \geq 0.$$

Use the data to determine the values of the constants $r_1$, $r_2$, $C_1$, and $C_2$.

| time | $N(t)$ |
|---|---|
| 0.0000 | 100.0000 |
| 0.1000 | 57.6926 |
| 0.2000 | 42.5766 |
| 0.3000 | 35.8549 |
| 0.4000 | 31.8481 |
| 0.5000 | 28.8296 |
| 2.5000 | 4.7430 |
| 4.5000 | 0.7840 |
| 6.0000 | 0.2032 |
| 8.0000 | 0.0336 |

Table 2.6: Table for Exercise 20; data to be fit to a function of the form $N(t) = C_1 e^{-r_1 t} + C_2 e^{-r_2 t}, \quad t \geq 0.$

2.21. **Shannon Entropy.** In a recent application of information theory to the field of genomics, a function called the Shannon entropy, $H$, was considered. In it, a given gene is represented as a binary device: it can be either "on" or "off" (i.e. being expressed or not).

If $x$ is the probability that the gene is "on" and $y$ is the probability that it is "off", the Shannon entropy function for the gene is defined as

$$H = -x\log(x) - y\log(y)$$

Note that

- $x$ and $y$ being probabilities just means that they satisfy $0 < x \leq 1$, and $0 < y \leq 1$ and
- the gene can only be in one of these two states, so $x + y = 1$.

Use these facts to show that the Shannon entropy for the gene is greatest when the two states are equally probable, i.e. for $x = y = 0.5$.

2.22. **A threshold function.** The response of a regulatory gene to inputs that affect it is not simply linear. Often, the following so-called "squashing function" or "threshold function" is used to link the input $x$ to the output $y$ of the gene:

$$y = f(x) = \frac{1}{1 + e^{(ax+b)}},$$

where $a$, $b$ are constants.

(a) Show that $0 < y < 1$.

(b) For $b = 0$ and‘$a = 1$ sketch the shape of this function.

(c) How does the shape of the graph change as $a$ increases?

2.23. **Graph sketching.** Sketch the graph of the function $y = e^{-t}\sin \pi t$.

2.24. **The Mexican Hat.** Consider the function

$$y = f(x) = 2e^{-x^2} - e^{-x^2/3}$$

(a) Find the critical points of $f$.

(b) Determine the value of $f$ at those critical points.

(c) Use these results and the fact that for very large $x$, $f \to 0$ to draw a rough sketch of the graph of this function.

(d) Comment on why this function might be called "a Mexican Hat".

*Note:* The second derivative is not very informative here, and we do not ask you to use it for determining concavity in this example. However, you may wish to calculate it for practice with the chain rule.

2.25. **The Ricker Equation.** In studying salmon populations, a model often used is the Ricker equation which relates the size of a fish population this year, $x$ to the expected size next year $y$. The Ricker equation is

$$y = \alpha x e^{-\beta x}$$

where $\alpha, \beta > 0$.

(a) Find the value of the current population which maximizes the salmon population next year according to this model.

(b) Find the value of the current population which would be exactly maintained in the next generation.

(c) Explain why a very large population is not sustainable.

*Note:* these populations do not actually change continuously, since all the parents die before the eggs are hatched.

2.26. **Spacing in a fish school.** Life in a social group has advantages and disadvantages: protection from predators is one advantage. Disadvantages include competition for food or resources. Spacing of individuals in a school of fish or a flock of birds is determined by the mutual attraction and repulsion of neighbours from one another: each individual does not want to stray too far from others, nor get too close.

Suppose that when two fish are at distance $x > 0$ from one another, they are attracted with "force" $F_a$ and repelled with "force" $F_r$ given by:

$$F_a = Ae^{-x/a}$$

$$F_r = Re^{-x/r}$$

where $A, R, a, r$ are positive constants.

*Note:* $A, R$ are related to the magnitudes of the forces, while $a, r$ to the spatial range of these effects.

(a) Show that at distance $x = a$, the first function has fallen to $(1/e)$ times its value at the origin. (Recall $e \approx 2.7$.)

(b) For what value of $x$ does the second function fall to $(1/e)$ times its value at the origin? Note that this is the reason why $a, r$ are called spatial ranges of the forces.

(c) It is generally assumed that $R > A$ and $r < a$. Interpret what this mean about the comparative effects of the forces.

(d) Sketch a graph showing the two functions on the same set of axes.

(e) Find the distance at which the forces exactly balance. This is called the comfortable distance for the two individuals.

(f) If either $A$ or $R$ changes so that the ratio $R/A$ decreases, does the comfortable distance increase or decrease? Justify your response.

(g) Similarly comment on what happens to the comfortable distance if $a$ increases or $r$ decreases.

2.27. **Seed distribution.** The density of seeds at a distance $x$ from a parent tree is observed to be

$$D(x) = D_0 e^{-x^2/a^2},$$

where $a > 0, D_0 > 0$ are positive constants. Insects that eat these seeds tend to congregate near the tree so that the fraction of seeds that get eaten is

$$F(x) = e^{-x^2/b^2}$$

where $b > 0$.

*Note:* These functions are called Gaussian or Normal distributions. The parameters $a, b$ are related to the “width” of these bell-shaped curves.

The number of seeds that survive (i.e. are produced and not eaten by insects) is

$$S(x) = D(x)(1 - F(x))$$

Determine the distance $x$ from the tree at which the greatest number of seeds survive.

2.28. **Euler’s ‘*e*’.** In 1748, Euler wrote a classic book on calculus, “Introductio in Analysin Infinitorum” [**?**]in which he showed that the function $e^x$ could be written in an expanded form similar to an (infinitely long) polynomial:

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + ...$$

Use as many terms as necessary to find an approximate value for the number $e$ and for $1/e$ to 5 decimal places.

*Note:* in other mathematics courses we see that such expansions, called power series, are central to approximations of many functions.

# 3

# Differential equations for exponential growth and decay

In Section 2.2 we made an observation about exponential functions and a new kind of equation - a **differential equation** - that such functions satisfy. In this chapter we explore this observation in more detail. At first, this link is based on the simple relationship between an exponential function and its derivatives. Later, this expands into a more encompassing discussion of

1. how differential equations arise in scientific problems,
2. how we study their predictions, and
3. what their solutions can tell us about the natural world.

We begin by reintroducing these equations.

## 3.1 *Introducing a new kind of equation*

A screencast summary of the introduction: differential equations for exponential growth and decay.

**Section 3.1 Learning goals**

1. Explain that the exponential function and its derivative are proportional to one another, and thereby satisfy a relationship of the form $dy/dx = ky$.
2. Give the definitions of a differential equation and of a solution to a differential equation.
3. Explain that $y = e^{kt}$ is a solution to the differential equation $dy/dt = ky$.

### *Observations about the exponential function*

In Chapter 2, we introduced the exponential function $y = f(x) = e^x$, and noted that it satisfies the relationship

$$\frac{de^x}{dx} = e^x, \quad \Rightarrow \quad \frac{dy}{dx} = y.$$

The equation on the right (linking a function to its own derivative) is a new kind of equation called a **differential equation** (abbreviated DE). We say that

$f(x) = e^x$ is a function that "satisfies" the equation, and we call this a **solution to the differential equation**.

**Note:** The solution to an algebraic equation is a number, whereas the solution to a differential equation is a function.

We call this a **differential equation** because it connects (one or more) derivatives of a function with the function itself.

Mastered Material Check

1. For whar constant $C$ does $y = Ce^x$ satisfy the differential equation $dy/dx = y$?
2. What function satisfies the DE $dy/dz = y$?

**Definition 3.1 (Differential equation)** *A differential equation is a mathematical equation that relates one or more derivatives of some function to the function itself. Solving the differential equation is the process of identifying the function(s) that satisfies the given relationship.*

We will be interested in applications in which a system or process varies over time. For this reason, we will henceforth use the independent variable $t$, for **time** in place of the former generic "$x$".

### Observations.

1. Consider the function of time: $y = f(t) = e^t$. Show that this function satisfies the differential equation

$$\frac{dy}{dt} = y.$$

Hint: Notice that we merely changed the notation very slightly. Now the derivative is "with respect to" $t$ rather than $x$.

2. The functions $y = e^{kt}$ (for $k$ constant) satisfy the differential equation

$$\frac{dy}{dt} = ky. \tag{3.1}$$

We can verify by differentiating $y = e^{kt}$, using the chair rule. Setting $u = kt$, and $y = e^u$, we have

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = e^u \cdot k = ke^{kt} = ky \quad \Rightarrow \quad \frac{dy}{dt} = ky$$

Hence, we have established that $y = e^{kt}$ satisfies the DE (3.1).

It is interesting to ask: *I*s this is the only function that satisfies the differential equation 3.1? Are there other possible solutions? What about a function such as $y = 2e^{kt}$ or $y = 400e^{kt}$?

The reader should show that for any constant $C$, the function $y = Ce^{kt}$ is a solution to the DE (3.1). To do so, differentiate the function and plug into (3.1). Verifying that the two sides of the equation are then the same establishes the result. While we do not prove it here, it turns out that $y = Ce^{kt}$ are the *only* functions that satisfy Eqn. (3.1).

Hint: Notice that the constant $C$ in front will appear in both the derivative and the function, and so will not change the equation.

Let us summarize what we have found out so far:

Solutions to the differential equation

$$\frac{dy}{dt} = ky \tag{3.2}$$

are the functions

$$y = Ce^{kt} \tag{3.3}$$

for $C$ an arbitrary constant.

A few comments are in order. First, unlike *algebraic* equations - whose solutions are numbers - **differential equations** have solutions that are *functions*. Second, the constant $k$ that appears in Eqn. (3.2), is the same as the constant $k$ in $e^{kt}$. Depending on the sign of $k$, we get either

**a)** *exponential growth* for $k > 0$, as illustrated in Figure 3.1(a), or

**b)** *exponential decay* for $k < 0$, as illustrated in Figure 3.1(b).

Third, since $e^{kt}$ is always positive, the constant $C$ determines the sign of the function as a whole - whether its graph lies above or below the $t$ axis.

A few curves of each type ($C > 0, C < 0$) are shown in each panel of Figure 3.1. The collection of curves in a panel is called a **family** of solution curves. The family shares the same value of $k$, but each member has a distinct value of $C$. Next, we ask how to specify a particular member of the family as *the* solution.

Mastered Material Check

3. Give an example of an algebraic equation and its solution.
4. Verify that $y = 3e^{-t}$ satisfies differential equation $\frac{dy}{dt} = -y$.
5. Why is $e^{kt}$ always positive?
6. Plot, using software, $y = Ce^{t}$ for each of $C = -4, -2, 2$ and 4.
7. Plot, using software, $y = Ce^{-t}$ for each of $C = -4, -2, 2$, and 4.

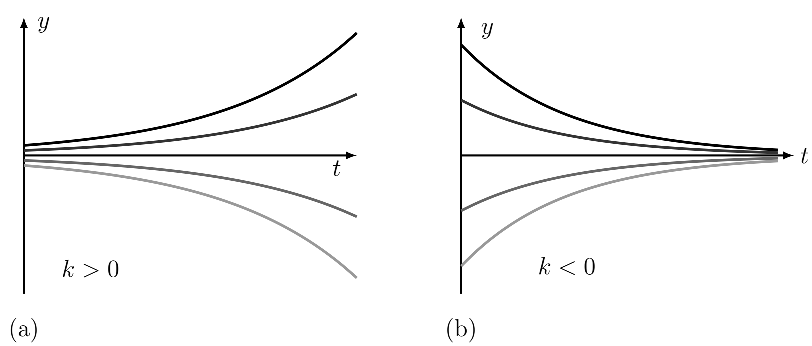



Figure 3.1: (a) A family of solutions to the differential equation (DE) (3.2). These are functions of the form $y = Ce^{kt}$ for $k > 0$ and arbitrary constant $C$. (b) Ampther family of solutions of a DE of the form (3.2), but for $k < 0$.

## *The solution to a differential equation*

**Definition 3.2 (Solution to a differential equation)** *By a* **solution** *to a differential equation, we mean a function that satisfies that equation.*

We often refer to “solution curves” - the graphs of the family of solutions of a differential equation, as shown, for example in the panels of Figure 3.1.

So far, we found that “many” functions can be valid solutions of the differential equation (3.2), since we can chose the constant $C$ arbitrarily in the family of solutions $y = Ce^{kt}$. Hence, in order to distinguish one specific solution of interest, we need additional information. This additional information is called an **initial value**, or **initial condition**, and it specifies one point belonging to the solution curve of interest. A common way to set an initial value is to specify a fixed value of the function (say $y = y_0$) at time $t = 0$.

**Definition 3.3 (Initial value)** *An* **initial value** *for a differential equation is a specified, known value of the solution at some specific time point (usually at time $t = 0$).*

**Example 3.1** *Given the differential Eqn. (3.2) and the initial value*

$$y(0) = y_0,$$

*find the value of $C$ for the solution in Eqn. (3.3).*

**Solution.** We proceed as follows:

$$y(t) = Ce^{kt}, \quad \text{so} \quad y(0) = Ce^{k \cdot 0} = Ce^0 = C \cdot 1 = C.$$

But, by the initial condition, $y(0) = y_0$. So,

$$C = y_0$$

and we have established that

$$y(t) = y_0 e^{kt}, \quad \text{where } y_0 \text{ is the initial value.}$$

◊

For example, in Figure 3.1, the initial value specifies that the solution we want passes through a specific point in the $ty$-plane - namely, the point $(0, y_0)$. Only one curve in the family of curves has that property. Hence, the initial value picks out a unique solution.

**Example 3.2** *Find the solution to the differential equation*

$$\frac{dy}{dt} = -0.5y$$

*that satisfies the initial condition $y(0) = 3$. Describe the behaviour of the solution you have found.*

**Solution.** The DE indicates that $k = -0.5$, so solutions are exponential functions $y = Ce^{-0.5t}$. The initial condition sets the value of $C$. From previous discussion, we know that $C = y(0) = 3$. Hence, the solution is $y = 3e^{-0.5t}$. This is a decaying exponential. ◊

Adjust the sliders in this interactive graph to see how the values of $k$ and $C$ affect the shape of the graph of the function $y = Ce^{kt}$ as well as its initial value $y(0) = y_0$.
Note the transitions that take place when $k$ changes from positive to negative.

Mastered Material Check

8. Given differential Eqn. (3.2) and the initial value $y(0) = 1$, find $C$ for the solution in Eqn (3.3).
9. Repeat the above but for the initial value $y(0) = 10$.
10. Draw the $ty$-plane with the points $(0, y_0)$ for $y_0 = 1, 10$.
11. Use differentiation to verify that the unction $y = 3e^{-0.5t}$ in Example 3.2 is a solution to $dy/dt = -0.5y$ with initial condition $y(0) = 3$.

## 3.2 *Differential equation for unlimited population growth*

**Section 3.2 Learning goals**

1. Recall the derivation of a model for human population growth and describe how it leads to a differential equation.
2. Identify that the solution to that equation is an exponential function.
3. Define per capita birth rates and rates of mortality, and explain the process of estimating their values from assumptions about the population.
4. Compute the doubling time of a population from its growth rate and vice versa.

Differential equations are important because they turn up in the study of many natural processes that vary continuously. In this section we examine the way that a simple differential equation arises when we study continuous uncontrolled population growth.

A screencast summary of the model for (unlimited) human population growth.

Here we set up a mathematical model for population growth. Let $N(t)$ be the number of individuals in a population at time $t$. The population changes with time due to births and mortality. (Here we ignore migration). Consider the changes that take place in the population size between time $t$ and $t+h$, where $\Delta t = h$ is a small time increment. Then

Mastered Material Check

12. What is the dependent variable in this model? The independent variable?
13. What are the units associated with each variable in this model?
14. What does "$x$ is proportional to $y$" mean?

$$N(t+h) - N(t) = \begin{bmatrix} \text{Change} \\ \text{in } N \end{bmatrix} = \begin{bmatrix} \text{Number of} \\ \text{births} \end{bmatrix} - \begin{bmatrix} \text{Number of} \\ \text{deaths} \end{bmatrix} \tag{3.4}$$

Eqn. (3.4) is just a "book-keeping" equation that keeps track of people entering and leaving the population. It is sometimes called a **balance equation**. We use it to derive a differential equation linking the *derivative* of $N$ to the *value* of $N$ at the given time.

Notice that dividing each term by the time interval $h$, we obtain

$$\frac{N(t+h)-N(t)}{h} = \left[\frac{\text{Number of births}}{h}\right] - \left[\frac{\text{Number of deaths}}{h}\right].$$

The term on the left "looks familiar". If we shrink the time interval, $h \to 0$, this term is a derivative $dN/dt$, so

$$\frac{dN}{dt} = \begin{bmatrix} \text{Rate of} \\ \text{change of N} \\ \text{per unit time} \end{bmatrix} = \begin{bmatrix} \text{Number of} \\ \text{births per} \\ \text{unit time} \end{bmatrix} - \begin{bmatrix} \text{Number of} \\ \text{deaths per} \\ \text{unit time} \end{bmatrix}$$

For simplicity, we assume that all individuals are identical and that the number of births per unit time is proportional to the population size. Denote by $r$ the constant of proportionality. Similarly, we assume that the number of deaths per unit time is proportional to population size with $m$ the constant of proportionality.

Both $r$ and $m$ have meanings: $r$ is the average **per capita birth rate**, and $m$ is the average **per capita mortality rate** . Here, both are assumed to be fixed positive constants that carry units of $1/\text{time}$. This is required to make the units match for every term in Eqn. (3.4). Then

$$r = \text{per capita birth rate} = \frac{\text{number births per unit time}}{\text{population size}},$$

$$m = \text{per capita mortality rate} = \frac{\text{number deaths per unit time}}{\text{population size}}.$$

Consequently, we have

$$\text{Number of births per unit time} = rN,$$
$$\text{Number of deaths per unit time} = mN.$$

We refer to constants such as $r, m$ as **parameters**. In general, for a given population, these would have specific numerical values that could be found through experiment, by collecting data, or by making simple assumptions. In Section 3.2, we show how some elementary assumptions about birth and mortality could help to estimate approximate values of $r$ and $m$.

Taking the assumptions and the form of the balance equation (3.4) together we have arrived at:

$$\frac{dN}{dt} = rN - mN = (r-m)N. \tag{3.5}$$

This is a differential equation: it links the derivative of $N(t)$ to the function $N(t)$. By solving the equation (i.e. identifying its solution), we are be able to make a projection about how fast a population is growing.

Define the constant $k = r - m$. Then $k$ is the **net growth rate**, of the population, so

$$\frac{dN}{dt} = kN, \quad \text{for } k = (r-m).$$

Suppose we also know that at time $t = 0$, the population size is $N_0$. Then:

- The function that describes population over time is (by previous results),

$$N(t) = N_0 e^{kt} = N_0 e^{(r-m)t}. \tag{3.6}$$

  (The result is identical to what we saw previously, but with $N$ rather than $y$ as the time-dependent function. We can easily check by differentiation that this function satisfies Eqn. (3.5).)

- Since $N(t)$ represents a population size, it has to be non-negative to have biological relevance. This is true so long as $N_0 \geq 0$.

- The initial condition $N(0) = N_0$, allows us to specify the (otherwise arbitrary) constant multiplying the exponential function.

Mastered Material Check

15. If there are 10 births/year in a population of size 1000, what is the birth rate $r$? Give units.
16. If there are 11 deaths/year in a population of size 1000, what is the mortality rate $m$? Give units.
17. Given the above conditions, what is the net growth rate $k$ for such a population? Give units. Is the population growing or shrinking?

- The population grows provided $k > 0$ which happens when $r - m > 0$ i.e. when birth rate exceeds mortality rate.
- If $k < 0$, or equivalently, $r < m$ then more people die on average than are born, so that the population shrinks and (eventually) go extinct.

### *A simple model for human population growth*

The differential equation (3.5) and its initial condition led us to predict that a population grows or decays exponentially in time, according to Eqn. (3.6). We can make this prediction quantitative by estimating the values of parameters $r$ and $m$. To this end, let us consider the example of a human population and make further simplifying assumptions. We measure time in years.

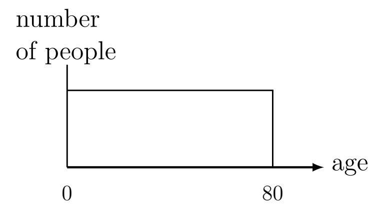



Figure 3.2: We assume a uniform age distribution to to determine the fraction of people who are fertile (and can give birth) or who are old (and likely to die). While slightly silly, this simplification helps estimate the desired parameters.

**Assumptions.**

- The age distribution of the population is "flat", i.e. there are as many 10 year-olds as 70 year olds. Of course, this is quite inaccurate, but a good place to start since it is easy to estimate some of the quantities we need. Figure 3.2 shows such a **uniform age distribution**.
- The sex ratio is roughly 50%. This means that half of the population is female and half male.
- Women are fertile and can have babies only during part of their lives: we assume that the fertile years are between age 15 and age 55, as shown in Figure 3.3.
- A lifetime lasts 80 years. This means that for half of that time a given woman can contribute to the birth rate, or that $\frac{(55-15)}{80} = 50\%$ of women alive at any time are able to give birth.
- During a woman's fertile years, we assume that on average, she has one baby every 10 years.
- We assume that deaths occur only from old age (i.e. we ignore disease, war, famine, and child mortality.)
- We assume that everyone lives precisely to age 80, and then dies instantly.

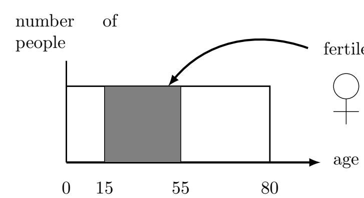



Figure 3.3: We assume that only women between the ages of 15 and 55 years old are fertile and can give birth. Then, according to our uniform age distribution assumption, half of all women are between these ages and hence fertile.

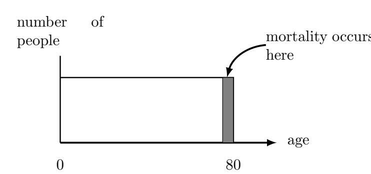



Figure 3.4: We assume that the people in the age bracket 79-80 years old all die each year, and that those are the only deaths. This, too, is a silly assumption, but makes it easy to estimate mortality in the population.

Based on the above assumptions, we can estimate the birthrate parameter $r$ as follows:

$$r = \frac{\text{number women}}{\text{population}} \cdot \frac{\text{years fertile}}{\text{years of life}} \cdot \frac{\text{number babies per woman}}{\text{number of years}}$$

Thus we compute that

$$r = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{10} = 0.025 \text{ births per person per year.}$$

Note that this value is now a rate per person per year, averaged over the entire population (male and female, of all ages). We need such an average rate since our model of Eqn. (3.5) assumes that individuals "are identical". We now have an approximate value for the average human per capita birth rate, $r \approx 0.025$ per year.

Next, using our assumptions, we estimate the mortality parameter, $m$. With the flat age distribution shown in Figure 3.2, there would be a fraction of $1/80$ of the population who are precisely removed by mortality every year (i.e. only those in their $80^{th}$ year.) In this case, we can estimate that the per capita mortality is:

$$m = \frac{1}{80} = 0.0125 \text{ deaths per person per year.}$$

The net per capita growth rate is $k = r - m = 0.025 - 0.0125 = 0.0125$ per person per year. We often refer to the constant $k$ as a **growth rate constant** and we also say that the population grows at the rate of 1.25% per year.

Mastered Material Check

18. Under these assumptions, for a population size of 800, how many male 35 year-olds would you expect? Women in their 60's?
19. Is the fertility assumption reasonable? Why or why not?
20. Explain the units attached to the birthrate parameter $r$.

**Example 3.3** *Using the results of this section, find a prediction for the population size $N(t)$ as a function of time $t$.*

**Solution.** We have found that our population satisfies the equation

$$\frac{dN}{dt} = (r - m)N = kN = 0.0125N,$$

so that

$$N(t) = N_0 e^{0.0125t}, \tag{3.7}$$

where $N_0$ is the starting population size. Figure 3.5 illustrates how this function behaves, using a starting value of $N(0) = N_0 = 7$ billion. $\Diamond$

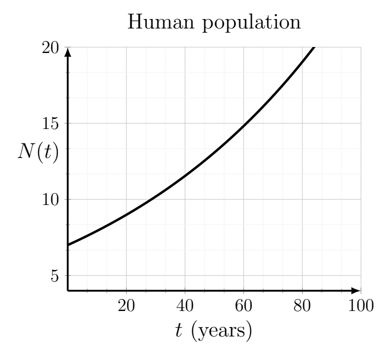



Figure 3.5: Projected world population (in billions) over 100 years, based on the model in Eqn. (3.7) and assuming that the initial population is $\approx 7$ billion.

**Example 3.4 (Human population in 100 years)** *Given the initial condition $N(0) = 7$ billion, determine the size of the human population at $t = 100$ years predicted by the model.*

**Solution.** At time $t = 0$, the population is $N(0) = N_0 = 7$ billion. Then in billions,

$$N(t) = 7e^{0.0125t}$$

so that when $t = 100$ we would have

$$N(100) = 7e^{0.0125 \cdot 100} = 7e^{1.25} = 7 \cdot 3.49 = 24.43.$$

Thus, with a starting population of 7 billion, there would be about 24.4 billion after 100 years based on the uncontrolled continuous growth model. $\Diamond$

Mastered Material Check

21. Based on Figure 3.5, when would we expect the human population to reach 15 billion?

**A critique.** Before leaving our population model, we should remember that our projections hold only so long as some rather restrictive assumptions are made. We have made many simplifications, and ignored many features that would seriously affect these results. These include (among others),

- variations in birth and mortality rates that stem from competition for resources and,
- epidemics that take hold when crowding occurs, and
- uneven distributions of resources or space.

We have also assumed that the age distribution is uniform (flat), but that is not accurate: the population grows only by adding new infants, and this would skew the distribution even if it is initially uniform. All these factors suggest that some "healthy skepticism" should be applied to any model predictions. Predictions may cease to be valid if model assumptions are not satisfied. This caveat will lead us to think about more realistic models for population growth. Certainly, the uncontrolled exponential growth would not be sustainable in the long run. That said, such a model is a good starting point for a first description of population growth, later to be adjusted.

### *Growth and doubling*

In Chapter 2, we used base 2 to launch our discussion of exponential growth and population doublings. We later discovered that base $e$ is more convenient for calculus, having a more elegant derivative. We also saw in Chapter 2, that bases of exponents can be inter-converted. These skills are helpful in our discussion of doubling times below.

**The doubling time.** How long would it take a population to double, given that it is growing exponentially with growth rate $k$? We seek a time $t$ such that $N(t) = 2N_0$. Then

$$N(t) = 2N_0 \quad \text{and} \quad N(t) = N_0 e^{kt},$$

implies that the population has doubled when $t$ satisfies

$$2N_0 = N_0 e^{kt}, \quad \Rightarrow \quad 2 = e^{kt} \quad \Rightarrow \quad \ln(2) = \ln(e^{kt}) = kt.$$

We solve for $t$. Thus, the **doubling time**, denoted $\tau$ is:

$$\tau = \frac{\ln(2)}{k}.$$

Mastered Material Check

22. What are the units associated with $\tau$?
23. The human population hit 3 billion in 1959. How does this fit with our (imperfect) model?

**Example 3.5 (Human population doubling time)** *Determine the doubling time for the human population based on the results of our approximate growth model.*

**Solution.** We have found a growth rate of roughly $k = 0.0125$ per year for the human population. Based on this, it would take

$$\tau = \frac{\ln(2)}{0.0125} = 55.45 \text{ years}$$

for the population to double. Compare this with the graph of Fig 3.5, and note that over this time span, the population increases from 6 to 12 billion. ♢

*Note:* the observant student may notice that we are simply converting back from base $e$ to base 2 when we compute the doubling time.

We summarize an important observation:

In general, an equation of the form

$$\frac{dy}{dt} = ky$$

that represents an exponential growth has a **doubling time** of

$$\tau = \frac{\ln(2)}{k}.$$

This is shown in Figure 3.6. We have discovered that based on the uncontrolled growth model, the population doubles *every 55 years*! After 110 years, for example, there have been two doublings, or a quadrupling of the population.

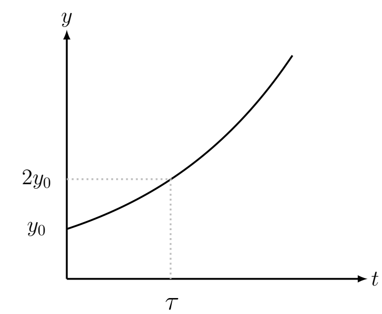


Figure 3.6: Doubling time for exponential growth.

**Example 3.6 (A ten year doubling time)** *Suppose we are told that some animal population doubles every* 10 *years. What growth rate would lead to such a trend?*

**Solution.** In this case, $\tau = 10$ years. Rearranging

$$\tau = \frac{\ln(2)}{k},$$

we obtain

$$k = \frac{\ln(2)}{\tau} = \frac{0.6931}{10} \approx 0.07 \text{ per year.}$$

Thus, a growth rate of 7% leads to doubling roughly every 10 years. ♢

## 3.3 *Radioactive decay*

**Section 3.3 Learning goals**

1. Describe the model for the number of radioactive atoms and explain how this leads to a differential equation.
2. Determine the solution of the resulting differential equation.
3. Given the initial amount, determine the amount of radioactivity remaining at a future time.
4. Describe the link between half-life of the radioactive material and its decay rate; given the value of one, be able to find the value of the other.

A radioactive material consists of atoms that undergo a spontaneous change. Every so often, some radioactive atom emits a particle, and decays into an inert form. We call this a process of **radioactive decay**. For any one atom, it is impossible to predict when this event would occur exactly, but based on the behaviour of a large number of atoms decaying spontaneously, we can assign a **probability** $k$ of decay per unit time.

In this section, we use the same kind of book-keeping (keeping track of the number of radioactive atoms remaining) as in the population growth example, to arrive at a differential equation that describes the process. Once we have the equation, we determine its solution and make a long-term prediction about the amount of radioactivity remaining at a future time.

### *Deriving the model*

We start by letting $N(t)$ be the number of radioactive atoms at time $t$. Generally, we would know $N(0)$, the number present initially. Our goal is to make simple assumptions about the process of decay that allows us to arrive at a mathematical model to predict values of $N(t)$ at any later time $t > 0$.

**Assumptions.**

**(1)** The process of radioactive decay is random, but on average, the probability of decay for a given radioactive atom is $k$ per unit time where $k > 0$ is some constant.

**(2)** During each (small) time interval of length $\Delta t = h$, a radioactive atom has probability $kh$ of decaying. This is merely a restatement of **(1)**.

Suppose that at some time $t$, there are $N(t)$ radioactive atoms. Then, according to our assumptions, during the time period $t \leq t \leq t+h$, on average $khN(t_0)$ atoms would decay. How many are there at time $t+h$? We can write the following balance-equation:

$$\begin{bmatrix} \text{Amount left} \\ \text{at time} \\ t+h \end{bmatrix} = \begin{bmatrix} \text{Amount present} \\ \text{at time} \\ t \end{bmatrix} - \begin{bmatrix} \text{Amount decayed} \\ \text{during time interval} \\ t \leq t \leq t+h \end{bmatrix}$$

or, restated:

$$N(t+h) = N(t) - khN(t). \tag{3.8}$$

Here we have assumed that $h$ is a small time period. Rearranging Eqn. (3.8) leads to

$$\frac{N(t+h) - N(t)}{h} = -kN(t).$$

Considering the left hand side of this equation, we let $h$ get smaller and smaller ($h \to 0$) and recall that

$$\lim_{h \to 0} \frac{N(t+h) - N(t)}{h} = \frac{dN}{dt} = N'(t)$$

Mastered Material Check

24. Suppose a given atom has a 1% chance of decay per 24 hours. What is this atom's probability of decay per week? Per hour?

where we have used the notation for a derivative of $N$ with respect to $t$. We have thus shown that a description of the population of radioactive atoms reduces to

$$\frac{dN}{dt} = -kN. \tag{3.9}$$

We have, once more, arrived at a differential equation that provides a link between a function of time $N(t)$ and its own rate of change $dN/dt$. Indeed, this equation specifies that $dN/dt$ is proportional to $N$, but with a negative constant of proportionality which implies decay.

Above we formulated the entire model in terms of the **number** of radioactive atoms. However, as shown below, the same equation holds regardless of the system of units used measure the amount of radioactivity

**Example 3.7** *Define the number of moles of radioactive material by $y(t) = N(t)/A$ where $A$ is* **Avogadro's number** *(the number of molecules in* 1 *mole:* $\approx 6.022 \times 10^{23}$ *- a dimensionless quantity, i.e. just a number with no associated units). Determine the differential equation satisfied by $y(t)$.*

**Solution.** We write $y(t) = N(t)/A$ in the form $N(t) = Ay(t)$ and substitute this expression for $N(t)$ in Eqn. (3.9). We use the fact that $A$ is a constant to simplify the derivative. Then

$$\frac{dN}{dt} = -kN \quad \Rightarrow \quad \frac{Ady(t)}{dt} = -k(Ay(t)) \quad \Rightarrow \quad A\frac{dy(t)}{dt} = A(-ky(t))$$

cancelling the constant $A$ from both sides of the equations leads to

$$\frac{dy(t)}{dt} = -ky(t), \quad \text{or simply} \quad \frac{dy}{dt} = -ky. \tag{3.10}$$

Thus $y(t)$ satisfies the same kind of differential equation (with the same negative proportionality constant) between the derivative and the original function. We will refer to (3.10) as the **decay equation**. ♢

*Solution to the decay equation* (3.10)

Suppose that initially, there was an amount $y_0$. Then, together, the differential equation and initial condition are

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0. \tag{3.11}$$

We often refer to this pairing between a differential equation and an initial condition as an **initial value problem**. Next, we show that an exponential function is an appropriate solution to this problem

**Example 3.8 (Checking a solution)** *Show that the function*

$$y(t) = y_0 e^{-kt}. \tag{3.12}$$

*is a solution to initial value problem* (3.11).

**Solution.** Ee compute the derivative of the candidate function (3.12), and rearrange, obtaining

$$\frac{dy(t)}{dt} = \frac{d}{dt}[y_0 e^{-kt}] = y_0 \frac{de^{-kt}}{dt} = -ky_0 e^{-kt} = -ky(t).$$

This verifies that for the derivative of the function is $-k$ times the original function, so satisfies the DE in (3.11). We can also check that the initial condition is satisfied:

$$y(0) = y_0 e^{-k \cdot 0} = y_0 e^0 = y_0 \cdot 1 = y_0.$$

Hence, Eqn. (3.12) is the solution to the initial value problem for radioactive decay. For $k > 0$ a constant, this is a decreasing function of time that we refer to as **exponential decay**. ◊

*The half life*

Given a process of exponential decay, how long would it take for half of the original amount to remain? Let us recall that the "original amount" (at time $t = 0$) is $y_0$. Then we are looking for the time $t$ such that $y_0/2$ remains. We must solve for $t$ in

$$y(t) = \frac{y_0}{2}.$$

We refer to the value of $t$ that satisfies this as the **half life**.

**Example 3.9 (Half life)** *Determine the half life in the exponential decay described by Eqn.* (3.12).

**Solution.** We compute:

$$\frac{y_0}{2} = y_0 e^{-kt} \quad \Rightarrow \quad \frac{1}{2} = e^{-kt}.$$

Now taking reciprocals:

$$2 = \frac{1}{e^{-kt}} = e^{kt}.$$

Thus we find the same result as in our calculation for doubling times, namely,

$$\ln(2) = \ln(e^{kt}) = kt,$$

so that the half life is

$$\tau = \frac{\ln(2)}{k}.$$

This is shown in Figure 3.7.

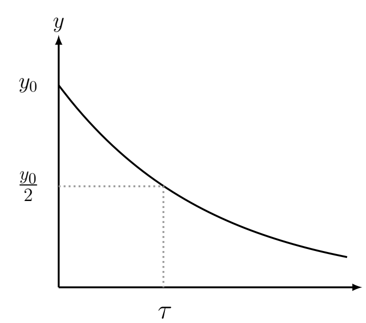



Figure 3.7: Half-life in an exponentially decreasing process.

**Example 3.10 (Chernobyl: April 1986)** *In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. The radioactive element iodine-131 ($I^{131}$) has half-life of 8 days whereas cesium-137 ($Cs^{137}$) has half life of 30 years. Use the model for radioactive decay to predict how much of this material would remain over time.*

**Solution.** We first determine the decay constants for each of these two elements, by noting that

$$k = \frac{\ln(2)}{\tau},$$

and recalling that $\ln(2) \approx 0.693$. Then for $I^{131}$ we have

$$k = \frac{\ln(2)}{\tau} = \frac{\ln(2)}{8} = 0.0866 \text{ per day.}$$

Then the amount of $I^{131}$ left at time $t$ (in days) would be

$$y_I(t) = y_0 e^{-0.0866t}.$$

For $Cs^{137}$

$$k = \frac{\ln(2)}{30} = 0.023 \text{ per year.}$$

so that for $T$ in years,

$$y_C(T) = y_0 e^{-0.023T}.$$

*Note:* we have used $T$ rather than $t$ to emphasize that units are different in the two calculations done in this example.

**Example 3.11 (Decay to** $0.1\%$ **of the initial level)** *How long it would take for $I^{131}$ to decay to 0.1 % of its initial level? Assume that the initial level occurred just after the explosion at Chernobyl.*

**Solution.** We must calculate the time $t$ such that $y_I = 0.001 y_0$:

$$0.001 y_0 = y_0 e^{-0.0866t} \quad \Rightarrow \quad 0.001 = e^{-0.0866t} \quad \Rightarrow \quad \ln(0.001) = -0.0866t.$$

Therefore,

$$t = \frac{\ln(0.001)}{-0.0866} = \frac{-6.9}{-0.0866} = 79.7 \text{ days.}$$

Thus it would take about 80 days for the level of Iodine-131 to decay to $0.1\%$ of its initial level. $\Diamond$

Mastered Material Check

25. Repeat the calculation in Example 3.11 for Cesium.
26. Convert the Cesium decay time units to days and repeat the calculation of Example 3.10 with the new time units.
27. If the decay rate of a substance is 10% per day, what is its half-life?

## *3.4 Deriving a differential equation for the growth of cell mass*

Derivation of a differential equation that describes cell growth resulting from absorption and consumption of nutrients.

In Section 1.2, we asked how the size of a living cell influences the balance between the rates of nutrient absorption (called $A$) and consumption (denoted $C$). But what if the two processes do not balance? What happens to the cell if the rates are unequal?

If a cell absorb nutrients faster than nutrients are consumed ($A > C$), some of the excess nutrients accumulate, and this buildup of nutrient mass can be converted into cell mass. This can result in growth (increase of cell mass). Conversely, if the consumption rate exceeds the rate of absorption of nutrients ($C > A$), the cell has a shortage of metabolic "fuel", and needs to convert some of its own mass into energy reserves that can power its metabolism - this would lead to loss in cell mass.

We can keep track of such changes in cell mass by using a simple "balance equation". The balance equation states that "the rate of change of cell mass is the difference between the rate of nutrient (mass) coming in ($A$) and the rate of nutrient (mass) being consumed ($C$), i.e.

$$\frac{dm}{dt} = A - C. \tag{3.13}$$

Each term in this equation must have the same units, mass of nutrient per unit time. $A$ contributes positively to mass increase, whereas $C$ is a rate of depletion that makes a negative contribution (hence the signs associated with terms in the equation). It also makes sense to adopt the assumptions previously made in Section 1.2 (and Featured Problem **??**) that

$$A = k_1 S, \quad C = k_2 V, \quad m = \rho V,$$

where $S, V, \rho$ are the surface area, volume, and density of the cell, and $k_1, k_2, \rho$ are positive constants. Then Eqn. (3.13) becomes

Mastered Material Check What are the units of $k_1, k_2, \rho$?

$$\frac{dm}{dt} = A - C \quad \Rightarrow \quad \frac{d(\rho V)}{dt} = k_1 S - k_2 V. \tag{3.14}$$

The above equation is rather general, and does not depend on cell shape.

Now consider the special case of a spherical cell for which $V = (4/3)\pi r^3$, $S = 4\pi r^2$. This simplification will permit us to convert the balance equation into a differential equation that describes changes in cell radius over time. Now Eqn. (3.14) can be rewritten as

$$\frac{d[\rho \cdot (4/3)\pi r^3]}{dt} = k_1(4\pi r^2) - k_2(4/3)\pi r^3. \tag{3.15}$$

We can simplify the derivative on the right hand side using the chain rule, as done in Featured Problem **??**, obtaining

$$\rho \frac{4\pi}{3}\pi(3r^2)\frac{dr}{dt} = k_1(4\pi r^2) - k_2(4/3)\pi r^3. \tag{3.16}$$

What does this tell us about cell radius?

One way to satisfy Eqn. (3.16) is to set $r = 0$ in each term. While this is a "solution" to the equation, it is not biologically interesting. (It merely describes a "cell" of zero radius that never changes.) Suppose $r \neq 0$. In that case, we can cancel out a factor of $r^2$ from both sides of the equation. (We can also cancel out $4\pi$.) After some simplification, we arrive at

$$\rho \frac{dr}{dt} = k_1 - \frac{k_2}{3} r, \quad \Rightarrow \quad \frac{dr}{dt} = \frac{1}{\rho}\left(k_1 - \frac{k_2}{3} r\right).$$

With appropriate units and taking into account typical cell size and density, this equation might look something like

$$\frac{dr}{dt} = (1 - 0.1\, r). \tag{3.17}$$

Hint: If we use units of $\mu$m (=$10^{-6}$m) for cell radius, pg (=$10^{-12}$gm) for mass, and measure time in hours, then approximate values of the constants are
$\rho = 1$pg $\mu$m$^{-3}$,
$k_1 = 1$pg $\mu$m$^{-2}$ hr$^{-1}$, and
$k_2 = 0.3$pg $\mu$m$^{-3}$ hr$^{-1}$.
In that case, the equation for cell radius is $dr/dt = (1 - 0.1 \cdot r)$.

From a statement about how cell mass changes, we have arrived at a resultant prediction about the rate of change of the cell radius. The equation we obtained is a differential equation that tells us something about a growing cell. In an upcoming chapter, we will build tools to be able to understand what this equation says, how to solve it for the cell radius $r(t)$ as a function of time $t$, and what such analysis predicts about the dynamics of cells with different initial sizes.

## 3.5 Summary

1. A differential equation is a statement linking the rate of change of some state variable with current values of that variable. An example is the simplest population growth model: if $N(t)$ is population size at time $t$:
$$\frac{dN}{dt} = kN.$$
2. A solution to a differential equation is a function that satisfies the equation. For instance, the function $N(t) = Ce^{kt}$ (for any constant $C$) is a solution to the unlimited population growth model (we check this by the appropriate differentiation). Graphs of such solutions (e.g. N versus t) are called solution curves.
3. To select a specific solution, more information (an initial condition) is needed. Given this information, e.g. $N(0) = N_0$, we can fully characterize the desired solution.
4. The **decay equation** is one representative of the same class of problems, and has an exponentially decaying solution.
$$\frac{dy}{dt} = -ky, \quad y(0) = y_0 \quad \Rightarrow \quad \text{Solution: } y(t) = y_0 e^{-kt}. \tag{3.18}$$
5. So far, we have seen simple differential equations with simple (exponential) functions for their solutions. In general, it may be quite challenging to make the connection between the differential equation (stemming from some application or model) with the solution (which we want in order to understand and predict the behaviour of the system.)

In this chapter, we saw examples in which a natural phenomenon (population growth, radioactive decay, cell growth) motivated a mathematical model that led to a differential equation. In both cases, that equation was derived by making a statement that tracked the amount or number or mass of a system over time. Numerous simplifications were made to derive each differential equation. For example, we assumed that the birth and mortality rates stay fixed even as the population grows to huge sizes.

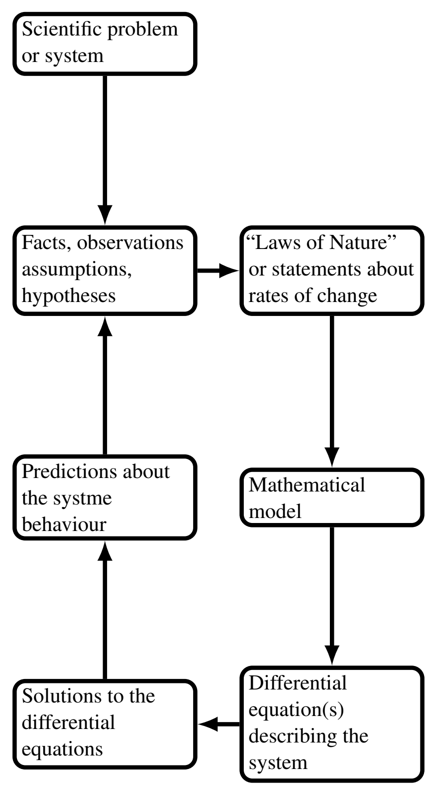


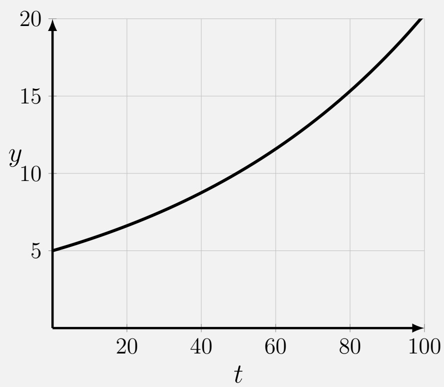

Figure 3.8: A "flow chart" showing how differential equations originate from scientific problems.

**With regard to a larger context.**

- Our purpose was to illustrate how a simple model is created, and what such models can predict.
- In general, differential equation models are often based on physical laws ("$F = ma$") or conservation statements ("rate in minus rate out equals net rate of change", or "total energy = constant").
- In biology, where the laws governing biochemical events are less formal, the models are often based on some mix of speculation and reasonable assumptions.
- In Figure 3.8 we illustrate how the scientific method leads to a cycle between the mathematical models and their test and validation using observations about the natural world.

**Quick Concept Checks**

1. Identify each of the following with either exponential growth or exponential decay:
   (a) $y = 20e^{3t}$;
   (b) $y = 5e^{-3t}$;
   (c) $\frac{dy}{dt} = 3t$;
   (d) $\frac{dy}{dx} = -5x$.
2. Determine the doubling time of the exponential growth function $N(t) = 500e^{2t}$.
3. Determine the half life of the of the exponential decay function $N(t) = 500e^{-2t}$.
4. Consider the following figure depicting exponential growth:




   What is the doubling time of this function?

## *Exercises*

3.1. **Checking solutions of differential equations.** A differential equation is an equation in which some function is related to its own derivative(s).

For each of the following functions, calculate the appropriate derivative, and show that the function satisfies the indicated *differential equation*

(a) $f(x) = 2e^{-3x}, \quad f'(x) = -3f(x)$

(b) $f(t) = Ce^{kt}, \quad f'(t) = kf(t)$

(c) $f(t) = 1 - e^{-t}, \quad f'(t) = 1 - f(t)$

3.2. **Linear differential equations.** Consider the function $y = f(t) = Ce^{kt}$ where $C$ and $k$ are constants. For what value(s) of these constants does this function satisfy the equation

(a) $\frac{dy}{dt} = -5y$,

(b) $\frac{dy}{dt} = 3y$.

*Note:* an equation which involves a function and its derivative is called a differential equation.

3.3. **Checking initial value solution to a differential equation.** Check that the function (3.6) satisfies the differential equation (3.2) and the initial condition $N(0) = N_0$.

3.4. **Solving linear differential equations.** Find a function that satisfies each of the following *differential equations*.

*Note:* all your answers should be exponential functions, but they may have different dependent and independent variables.

(a) $\dfrac{dy}{dt} = -y$,

(b) $\dfrac{dc}{dx} = -0.1c$ and $c(0) = 20$,

(c) $\dfrac{dz}{dt} = 3z$ and $z(0) = 5$.

3.5. **Andromeda strain, revisited.** In Chapter 2 we discussed the growth of bacteria, starting from a single cell. The doubling time of the bacteria was given as 20 min.

Find the appropriate differential equation that describes this growth, the appropriate initial condition, and the exponential function (with base $e$) that is the solution to that differential equation. Use units of hours for time $t$.

3.6. **Population growth in developed and developing countries.** In Canada, women have only about 2 children during their 40 years of fertility, and people live to age 80. In underdeveloped countries, people on average live to age 60 and women have a child roughly every 4 years between ages 13 and 45.

Compare the per capita birth and mortality rates and the predicted population growth or decay in each of these scenarios, using arguments analogous to those of Section 3.2.

Find the growth rate $k$ in percent per year and the doubling time for the growing population.

3.7. **Population growth and doubling.** A population of animals has a per-capita birth rate of $b = 0.08$ per year and a per-capita death rate of $m = 0.01$ per year. The population density, $P(t)$ is found to satisfy the differential equation

$$\frac{dP(t)}{dt} = bP(t) - mP(t)$$

(a) If the population is initially $P(0) = 1000$, find how big the population is in 5 years.

(b) When does the population double?

3.8. **Rodent population.** The per capita birthrate of one species of rodent is 0.05 newborns per day. This means that, on average, each member of the population results in 5 newborn rodents every 100 days. Suppose that over the period of 1000 days there are no deaths, and that the initial population of rodents is 250.

(a) Write a differential equation for the population size $N(t)$ at time $t$ (in days).

(b) Write down the initial condition that $N$ satisfies.

(c) Find the solution, i.e. express $N$ as some function of time $t$ that satisfies your differential equation and initial condition.

(d) How many rodents are there after 1 year ?

3.9. **Growth and extinction of microorganisms.**

(a) The population $y(t)$ of a certain microorganism grows continuously and follows an exponential behaviour over time. Its doubling time is found to be 0.27 hours. What differential equation would you use to describe its growth ?

*Note:* you must find the value of the rate constant, $k$, using the doubling time.

(b) With exposure to ultra-violet radiation, the population ceases to grow, and the microorganisms continuously die off. It is found that the half-life is then 0.1 hours. What differential equation would now describe the population?

3.10. **A bacterial population.** A bacterial population grows at a rate proportional to the population size at time $t$. Let $y(t)$ be the population size at time $t$. By experiment it is determined that the population at $t = 10$ min is $15,000$ and at $t = 30$ min it is $20,000$.

(a) What was the initial population?

(b) What is the population at time $t = 60$min?

3.11. **Antibiotic treatment.** A colony of bacteria is treated with a mild antibiotic agent so that the bacteria start to die. It is observed that the density of bacteria as a function of time follows the approximate relationship $b(t) = 85e^{-0.5t}$ where $t$ is time in hours.

Determine the time it takes for half of the bacteria to disappear; this is called the *half-life*.

Find how long it takes for 99% of the bacteria to die.

3.12. **Two populations.** Two populations are studied. Population **1** is found to obey the differential equation

$$dy_1/dt = 0.2y_1$$

and population **2** obeys

$$dy_2/dt = -0.3y_2$$

where $t$ is time in years.

(a) Which population is growing and which is declining?

(b) Find the doubling time (respectively half-life) associated with the given population.

(c) If the initial levels of the two populations were $y_1(0) = 100$ and $y_2(0) = 10,000$, how big would each population be at time $t$?

(d) At what time would the two populations be exactly equal?

3.13. **The human population.** The human population on Earth doubles roughly every 50 years. In October 2000 there were 6.1 billion humans on earth.

(a) Determine what the human population would be 500 years later under the uncontrolled growth scenario.

(b) How many people would have to inhabit each square kilometer of the planet for this population to fit on earth? (Take the circumference of the earth to be $40,000$ km for the purpose of computing its surface area and assume that the oceans have dried up.)

3.14. **Fish in two lakes.** Two lakes have populations of fish, but the conditions are quite different in these lakes. In the first lake, the fish population is growing and satisfies the differential equation

$$\frac{dy}{dt} = 0.2y$$

where $t$ is time in years. At time $t = 0$ there were 500 fish in this lake. In the second lake, the population is dying due to pollution. Its population satisfies the differential equation

$$\frac{dy}{dt} = -0.1y,$$

and initially there were 4000 fish in this lake.

At what time are the fish populations in the two lakes identical?

3.15. **First order chemical kinetics.** When chemists say that a chemical reaction follows "first order kinetics", they mean that the concentration of the reactant at time $t$, i.e. $c(t)$, satisfies an equation of the form $\frac{dc}{dt} = -rc$ where $r$ is a rate constant, here assumed to be positive. Suppose the reaction mixture initially has concentration 1M ("1 molar") and that after 1 hour there is half this amount.

(a) Find the "half life" of the reactant.

(b) Find the value of the rate constant $r$.

(c) Determine how much is left after 2 hours.

(d) When is only 10% of the initial amount be left?

3.16. **Chemical breakdown.** In a chemical reaction, a substance $S$ is broken down. The concentration of the substance is observed to change at a rate proportional to the current concentration. It was observed that 1 Mole/liter of $S$ decreased to 0.5 Moles/liter in 10 minutes.

(a) How long does it take until only 0.25 Moles per liter remain?

(b) How long does it take until only 1% of the original concentration remains?

3.17. **Half-life.** If 10% of a radioactive substance remains after one year, find its half-life.

3.18. **Carbon 14.** Carbon 14, or $^{14}C$, has a half-life of 5730 years. This means that after 5730 years, a sample of Carbon 14, which is a radioactive isotope of carbon, has lost one half of its original radioactivity.

(a) Estimate how long it takes for the sample to fall to roughly 0.001 of its original level of radioactivity.

(b) Each gram of $^{14}C$ has an activity given here in units of 12 decays per minute. After some time, the amount of radioactivity decreases. For example, a sample 5730 years old has only one half the original activity level, i.e. 6 decays per minute. If a 1 gm sample of material is found to have 45 decays per hour, approximately how old is it?

*Note:* $^{14}C$ is used in radiocarbon dating, a process by which the age of materials containing carbon can be estimated. W. Libby received the Nobel prize in chemistry in 1960 for developing this technique.

3.19. **Strontium-90.** Strontium-90 is a radioactive isotope with a half-life of 29 years. If you begin with a sample of 800 units, how long does it take for the amount of radioactivity of the strontium sample to be reduced to

(a) 400 units

(b) 200 units

(c) 1 unit

3.20. **More radioactivity.** The half-life of a radioactive material is 1620 years.

(a) What percentage of the radioactivity remains after 500 years?

(b) Cobalt 60 is a radioactive substance with half life 5.3 years. It is used in medical application (radiology). How long does it take for 80% of a sample of this substance to decay?

3.21. **Salt in a barrel.** A barrel initially contains 2 $kg$ of salt dissolved in 20 $L$ of water. If water flows in the rate of 0.4 $L$ per minute and the well-mixed salt water solution flows out at the same rate, how much salt is present after 8 minutes?

3.22. **Atmospheric pressure.** Assume the atmospheric pressure $y$ at a height $x$ meters above the sea level satisfies the relation

$$\frac{dy}{dx} = kx.$$

If one day at a certain location the atmospheric pressures are 760 and 675 torr (unit for pressure) at sea level and at 1000 meters above sea level, respectively, find the value of the atmospheric pressure at 600 meters above sea level.

# 4
# Solving differential equations

In Chapter 3, we introduced differential equations to keep track of continuous changes in the growth of a population or the decay of radioactivity. We encountered a differential equation that tracks changes in cell mass due to nutrient absorption and consumption. Finally, we learned that the solutions to a differential equation is a function. In applications studied, that function can be interpreted as predictions of the behaviour of the system or process over time.

In this chapter, we further develop some of these ideas. We explore several techniques for finding and verifying that a given function is a solution to a differential equation. We then examine a simple class of differential equations that have many applications to processes of production and decay, and find their solutions. Finally, we show how an approximation method provides for numerical solutions of such problems.

## 4.1 *Verifying that a function is a solution*

**Section 4.1 Learning goals**

1. Given a function, check whether that function does or does not satisfy a given differential equation.
2. Verify whether a given function does or does not satisfy an initial condition.

In this section we concentrate on analytic solutions to a differential equation. By **analytic solution**, we mean a "formula" such as $y = f(x)$ that satisfies the given differential equation. We saw in Chapter 3 that we can check whether a function satisfies a differential equation (e.g., Example 3.8) by simple differentiation. In this section, we further demonstrate this process.

**Example 4.1** *Show that the function $y(t) = (2t+1)^{1/2}$ is a solution to the* *differential equation and initial condition*

$$\frac{dy}{dt} = \frac{1}{y}, \quad y(0) = 1.$$

**Solution.** First, we check the derivative, obtaining

$$\begin{aligned} \frac{dy(t)}{dt} = \frac{d(2t+1)^{1/2}}{dt} &= \frac{1}{2}(2t+1)^{-1/2} \cdot 2 \\ &= (2t+1)^{-1/2} = \frac{1}{(2t+1)^{1/2}} = \frac{1}{y}. \end{aligned}$$

Hence, the function satisfies the differential equation. We must also verify the initial condition. We find that $y(0) = (2 \cdot 0 + 1)^{1/2} = 1^{1/2} = 1$. Thus the initial condition is also satisfied, and $y(t)$ is indeed a solution. ♢

**Example 4.2** *Consider the differential equation and initial condition*

$$\frac{dy}{dt} = 1 - y, \quad y(0) = y_0. \tag{4.1}$$

**a)** *Show that the function* $y(t) = y_0 e^{-t}$ *is* **not** *a solution to this differential equation.*

**b)** *Show that the function* $y(t) = 1 - (1 - y_0)e^{-t}$ ***is*** *a solution.*

**Solution.**

**a)** To check whether $y(t) = y_0 e^{-t}$ is a solution to the differential equation (4.1), we substitute the function into each side (“left hand side", LHS; “right hand side”. RHS) of the equation. We show the results in the columns of Table 4.1. After some steps in the simplification, we see that the two sides do not match, and conclude that the function is not a solution, as it fails to satisfy the equation

| LHS | RHS |
|---|---|
| $\frac{dy}{dt}$ | $1 - y$ |
| $\frac{d[y_0 e^{-t}]}{dt}$ | $1 - y_0 e^{-t}$ |
| $-y_0 e^{-t}$ | ☹ |

Table 4.1: The function $y(t) = y_0 e^{-t}$ **is not** a solution to the differential equation (4.1). Plugging the function into each side of the DE and simplifying (down the rows) leads to expressions that do not match.

**b)** Similarly, we check the second function. The calculations

are shown in columns of Table 4.2. We find that RHS=LHS, so the differential equation is satisfied. Finally, let us show that the initial condition $y(0) = y_0$ is also satisfied. Plugging in $t = 0$ we have

$$y(0) = 1 - (1 - y_0)e^0 = 1 - (1 - y_0) \cdot 1 = 1 - (1 - y_0) = y_0.$$

Thus, both differential equation and initial condition are satisfied.

| LHS | RHS |
|---|---|
| $\frac{dy}{dt}$ | $1 - y$ |
| $\frac{d}{dt}[1 - (1 - y_0)e^{-t}]$ | $1 - [1 - (1 - y_0)e^{-t}]$ |
| $-(1 - y_0)\frac{de^{-t}}{dt}$ | $(1 - y_0)e^{-t}$ |
| $(1 - y_0)e^{-t}$ | ✓ |

Table 4.2: (b) The function $y(t) = 1 - (1 - y_0)e^{-t}$ is a solution to the differential equation (4.1). The expressions we get by evaluating each side of the differential equation do match.

♢

**Example 4.3 (Height of water draining out of a cylindrical container)** *A cylindrical container with cross-sectional area A has a small hole of area a at its base, through which water leaks out. It can be shown that height of water $h(t)$ in the container satisfies the differential equation*

$$\frac{dh}{dt} = -k\sqrt{h}, \tag{4.2}$$

*(where k is a constant that depends on the size and shape of the cylinder and its hole:* $k = \frac{a}{A}\sqrt{2g} > 0$ *and g is acceleration due to gravity.) Show that the function*

$$h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2 \tag{4.3}$$

*is a solution to the differential equation* (4.2) *and initial condition* $h(0) = h_0$.

Mastered Material Check

1. Draw a diagram of the system described in Example 4.3.
2. What set of units would be reasonable for each of the parameters in Example 4.3.
3. Create a table to organize the calculations for this example, similar to Tables 4.1 and 4.2.

**Solution.** We first easily verify that the initial condition is satisfied. Substitute $t = 0$ into the function (4.3). Then we find $h(0) = h_0$, verifying the initial conditions.

To show that the differential equation (4.2) is satisfied, we differentiate the function in Eqn. (4.3):

$$\begin{aligned} \frac{dh(t)}{dt} = \frac{d}{dt}\left(\sqrt{h_0} - k\frac{t}{2}\right)^2 &= 2\left(\sqrt{h_0} - k\frac{t}{2}\right) \cdot \left(\frac{-k}{2}\right) \\ &= -k\left(\sqrt{h_0} - k\frac{t}{2}\right) = -k\sqrt{h(t)}. \end{aligned}$$

Here we have used the power law and the chain rule, remembering that $h_0, k$ are constants. Now we notice that, using Eqn. (4.3), the expression for $\sqrt{h(t)}$ exactly matches what we have computed for $dh/dt$. Thus, we have shown that the function in Eqn. (4.3) satisfies both the initial condition and the differential equation. ◊

As shown in Examples 4.1- 4.3, if we are told that a function is a solution to a differential equation, we can check the assertion and verify that it is correct or incorrect. A much more difficult task is to find the solution of a new differential equation from first principles.

In some cases, **integration**, learned in second semester calculus, can be used. In others, some transformation that changes the problem to a more familiar one is helpful - an example of this type is presented in Section 4.2. In many cases, particularly those of so-called non-linear differential equations, great expertise and familiarity with advanced mathematical methods are required to find the solution to such problems in an analytic form, i.e. as an explicit formula. In such cases, approximation and numerical methods are helpful.

## 4.2 *Equations of the form $y'(t) = a - by$*

**Section 4.2 Learning goals**

1. Define steady states of a differential equation, and be able to find such special solutions.
2. Starting with the differential equation (4.4) $y' = a - by$, find a new differential equation for the deviation away from a steady state, $z(t)$ and show that it is a simple decay equation.
3. Use the transformed (decay) equation to find the solution for $z(t)$ and, for $y(t)$ in the original equation, (4.4).
4. Explain Newton's Law of Cooling (NLC), and the differential equation of the same type, $y' = a - by$. Find its solution and explain what this solution means.
5. Use the solution to NLC to predict the temperature of a cooling or heating object over time.
6. Describe a variety of related examples, and use the same methods to solve and interpret these (examples include chemical production and decay, the velocity of a skydiver, the concentration of drug in the blood, and others).

In this section we introduce an important class of differential equations that have many applications in physics, chemistry, biology, and other applications. All share a similar structure, namely all are of the form

$$\frac{dy}{dt} = a - by, \quad y(0) = y_0. \tag{4.4}$$

First, we show how a solution to such equation can be found. Then, we examine a number of applications.

### *Special solutions: steady states*

We first ask about "special solutions" to the differential equation (4.4) in which there is no change over time. That is, we ask whether there are values of $y$ for which $dy/dt = 0$.

From (4.4), we find that such solutions would satisfy

$$\frac{dy}{dt} = 0 \quad \Rightarrow \quad a - by = 0 \quad \Rightarrow \quad y = \frac{a}{b}.$$

In other words, if we were to start with the initial value $y(0) = a/b$, then that value would not change, since it satisfies $dy/dt = 0$, so that the solution at all future times would be $y(t) = a/b$. (Of course, this is a perfectly good function; it is simply a function that is alway constant.)

We refer to such constant solutions as **Steady States**.

An explanation of the way we find solutions to equations of the form $\frac{dy}{dt} = a - by$, with $y(0) = y_0$.

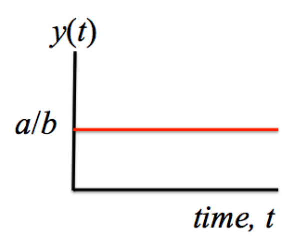



Figure 4.1: $y = a/b$ is a constant solution to the differential equation in (4.4). We call this type of solution a **steady state**.

*Other solutions: away from steady state*

What happens if we start with a value of $y$ that is not exactly at the "special" steady state? Let us rewrite the DE in a more suggestive form,

$$\frac{dy}{dt} = a - by \quad \Rightarrow \quad \frac{dy}{dt} = -b\left(y - \frac{a}{b}\right),$$

(having factored out $-b$). The advantage is that we recognize the expression $(y - \frac{a}{b})$ as the difference, or **deviation** of $y$ away from its steady state value. (That deviation could be either positive or negative, depending on whether $y$ is larger or smaller than $a/b$.) We ask whether this deviation gets larger or smaller as time goes by, i.e., whether $y$ gets further away or closer to its steady state value $a/b$.

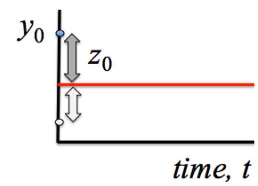



Figure 4.2: We define $z(t)$ as the deviation of $y$ from its steady state value. Here we show two typical initial values of $z$, where $z_0 = y_0 - \frac{a}{b}$.

Define $z(t)$ as that deviation, that is

$$z(t) = y(t) - \frac{a}{b},$$

Then, since $a, b$ are constants, we recognize that

$$\frac{dz}{dt} = \frac{dy}{dt}.$$

Second, the initial value of $z$ follows simply from the initial value of $y$:

$$z(0) = y(0) - \frac{a}{b} = y_0 - \frac{a}{b}.$$

Now we can **transform** the equation (4.4) into a new differential equation for the variable $z$ by using these two facts. We can replace the $y$ derivative by the $z$ derivative, and also, using Eqn. (4.4), find that

$$\frac{dz}{dt} = \frac{dy}{dt} = -b\left(y - \frac{a}{b}\right) = -bz.$$

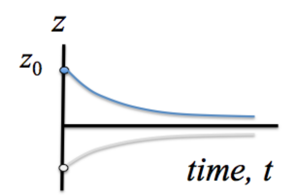



Figure 4.3: The deviation away from steady state (blue, grey curves) is $z(t) = y(t) - a/b$. We can solve the differential equation for $z(t)$ because it is a simple exponential decay equation. Here we show two typical solutions for $z$.

Hence, we have transformed the original DE and IC into the new problem

$$\frac{dz}{dt} = -bz, \quad z(0) = z_0, \qquad \left[\text{where } z_0 = y_0 - \frac{a}{b}\right].$$

But this is the familiar decay initial value problem that we have already solved before. So

$$z(t) = z_0 e^{-bt}.$$

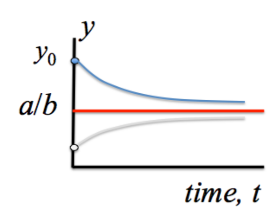



Figure 4.4: Finally, we can determine the solution $y(t)$.

We have arrived at the conclusion that the deviation from steady state **decays exponentially** with time, provided that $b > 0$. Hence, we already know that $y$ should get closer to the constant value $a/b$ as time goes by!

We can do even better than this, by transforming the solution we found for $z(t)$ into an expression for $y(t)$. To do so, use the definition once more, setting

$$z(t) = z_0 e^{-bt} \quad \Rightarrow \quad y(t) - \frac{a}{b} = \left(y_0 - \frac{a}{b}\right) e^{-bt}.$$

Solving for $y(t)$ then leads to

$$y(t) = \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) e^{-bt}. \tag{4.5}$$

Adjust the sliders to see how the parameters $a$ and $b$ and the initial value $y_0$ affect the shape of the function $y(t)$ in the formula (4.5).

**Example 4.4** ($a = b = 1$) *Suppose we are given the differential equation and initial condition*

$$\frac{dy}{dt} = 1 - y, \quad y(0) = y_0. \tag{4.6}$$

*Determine the solution to this differential equation.*

**Solution.**

By substituting $a = 1, b = 1$ in the solution found above, we observe that

$$y(t) = 1 - (1 - y_0)e^{-t}.$$

Representative curves in this **family of solutions** are shown in Figure 4.5 for various initial values $y_0$. ◊

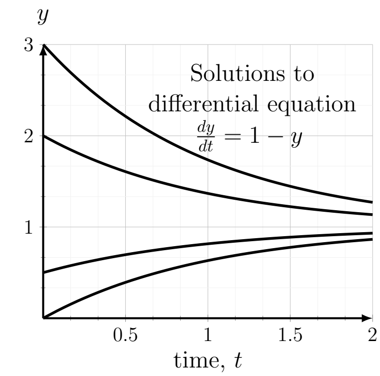



Figure 4.5: Solutions to Eqn. (4.6) are functions that approach $y = 1$.

Mastered Material Check

4. Find the steady state of Eqn. (4.6).
5. From Figure 4.5, determine what were the four different initial conditions used.
6. Rewrite these four initial conditions as the initial deviations away from steady state, that is, give the initial values, $z_0$ of the deviation.

We now apply the methods to a number of examples.

**Featured Problem 4.1 (Predicting the size of a growing cell)** *Find a solution to the differential equation (3.17) for the radius of a growing cell $r(t)$ (in units of $\mu m = 10^{-6} m$) as a function of time $t$ (in hours), that is find $r(t)$ assuming that at time $t = 0$ the cell is $2\mu m$ in radius.*

By solving the above problem, we get a detailed prediction of cell growth based on assumed rates of nutrient intake and consumption.

## *Newton's law of cooling*

Consider an object at temperature $T(t)$ in an environment whose ambient temperature is $E$. Depending on whether the object is cooler or warmer than the environment, it heats up or cools down. From common experience we know that, after a long time, the temperature of the object equilibrates with its environment.

**Isaac Newton** formulated a hypothesis to describe the rate of change of temperature of an object. He assumed that

The rate of change of temperature $T$ of an object is proportional to the difference between its temperature and the ambient temperature, $E$.

To rephrase this statement mathematically, we write

$$\frac{dT}{dt} \text{ is proportional to } (T(t) - E).$$

This implies that the derivative $dT/dt$ is some constant multiple of the term $(T(t) - E)$. However, the sign of that constant requires some discussion. Denote the constant of proportionality by $\alpha$ temporarily, and suppose $\alpha \geq 0$. Let us check whether the differential equation

$$\frac{dT}{dt} = \alpha(T(t) - E),$$

makes physical sense.

Mastered Material Check

7. What can we say about the units of $T$ and $E$?

Suppose the object is warmer than its environment ($T(t) > E$). Then $T(t) - E) > 0$ and $\alpha \geq 0$ implies that $dT/dt > 0$ which says that the temperature of the object should get *warmer*! But this does not agree with our everyday experience: a hot cup of coffee cools off in a chilly room. Hence $\alpha \geq 0$ cannot be correct. Based on this, we conclude that Newton's Law of Cooling, written in the form of a differential equation, should read:

$$\frac{dT}{dt} = k(E - T(t)), \quad \text{where} \quad k > 0. \tag{4.7}$$

*Note:* the sign of the term in braces has been switched.

Typically, given the temperature at some initial time $T(0) = T_0$, we want to predict $T(t)$ for later time.

**Example 4.5** *Consider the temperature $T(t)$ as a function of time. Solve the differential equation for Newton's law of cooling*

$$\frac{dT}{dt} = k(E - T),$$

*together with the initial condition $T(0) = T_0$.*

Mastered Material Check

8. Fill in the details for Example 4.5.
9. In Figure 4.6, what are the five different initial temperatures, $T_0$ corresponding to each solution curve?
10. In Figure 4.6, how many curves represent a heating object and how many a cooling object?

**Solution.** As before, we transform the variable to reduce the differential equation to one that we know how to solve. This time, we select the new variable to be $z(t) = E - T(t)$. Then, by steps similar to previous examples, we find that

$$\frac{dz(t)}{dt} = -kz.$$

We also rewrite the initial condition in terms of $z$, leading to $z(0) = E - T(0) = E - T_0$. After carrying out **Steps 1-3** as before, we find the solution for $T(t)$,

$$T(t) = E + (T_0 - E)e^{-kt}. \tag{4.8}$$

In Figure 4.6 we show a family of curves of the form of Eqn. (4.8) for five different initial temperature values (we have set $E = 10$ and $k = 0.2$ for all these curves). ♢

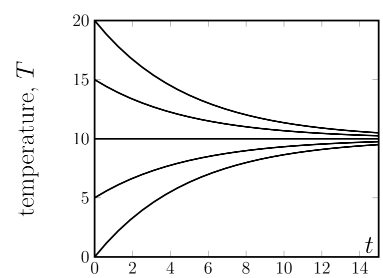



Figure 4.6: Temperature versus time, $T(t)$, for a cooling object.

Next, we interpret the behaviour of these solutions.

**Example 4.6** *Explain (in words) what the form of the solution in Eqn. (4.8) of Newton's law of cooling implies about the temperature of an object as it warms or cools.*

**Solution.** We make the following remarks

- It is straightforward to verify that the initial temperature is $T(0) = T_0$ (substitute $t = 0$ into the solution of Eqn. (4.8)). Now examine the time dependence. Only one term, $e^{-kt}$ depends on time. Since $k > 0$, this is an exponentially decaying function, whose magnitude shrinks with time. The whole term that it multiplies, $(T_0 - E)e^{-kt}$, continually shrinks. Hence,

$$T(t) = E + \quad (T_0 - E)e^{-kt} \quad \Rightarrow \quad \text{as } t \to \infty, \quad e^{-kt} \to 0,$$
$$\text{so } T(t) \to E.$$

Thus the temperature of the object always approaches the ambient temperature. This is evident in the solution curves shown in Figure 4.6.

- We also observe that the direction of approach (decreasing or increasing) depends on the sign of the constant $(T_0 - E)$. If $T_0 > E$, the temperature approaches $E$ from above, whereas if $T_0 < E$, the temperature approaches $E$ from below.

- In the specific case that $T_0 = E$, there is no change at all. $T = E$ satisfies $dT/dt = 0$, and corresponds to a **steady state** of the differential equation, as previously defined.

  Steady states are studied in more detail in Chapter 5.

Mastered Material Check

11. Consider three cups of coffee left in a 20°C room. If one is iced, another is piping hot, and the third is room temperature, which cup will not change temperature? Which, thus, represents a steady state?
12. Convert the temperatures in Example 4.7 to Fahrenheit and repeat.

*Using Newton's law of cooling to solve a mystery*

Now that we have a detailed solution to the differential equation representing Newton's law of cooling, we can apply it to making exact determinations of temperature over time, or of time at which a certain temperature was attained. The following example illustrates an application of this idea.

**Example 4.7 (Murder mystery)** *It is a dark clear night. The air temperature is* 10° *C. A body is discovered at midnight. Its temperature is* 27° *C. One hour later, the body has cooled to* 24° *C. Use Newton's law of cooling to determine the time of death.*

Details of the calculations for Example 4.7.

**Solution.** We assume that body-temperature just before death was 37° C (normal human body temperature). Let $t = 0$ be the time of death. Then the initial temperature is $T(0) = T_0 = 37°$ C. We want to find the time elapsed until the body was found, i.e. time $t$ at which the temperature of the body had cooled down to 27° C. We assume that the ambient temperature, $E = 10$, was constant. From Newton's law of cooling, the body temperature satisfies

$$\frac{dT}{dt} = k(10 - T).$$

From previous work and Eqn. (4.8), the solution to this DE is

$$T(t) = 10 + (37 - 10)e^{-kt}.$$

We do not know the value of the constant $k$, but we have enough information to find it. First, at discovery, the body's temperature was 27°. Hence at time $t$

$$27 = 10 + 27e^{-kt} \quad \Rightarrow \quad 17 = 27e^{-kt}.$$

Also at $t + 1$ (one hour after discovery), the temperature was 24° C, so

$$T(t+1) = 10 + (37 - 10)e^{-k(t+1)} = 24, \quad \Rightarrow \quad 24 = 10 + 27e^{-k(t+1)}.$$

Thus,

$$14 = 27e^{-k(t+1)}.$$

We have two equations for the two unknowns $t$ and $k$. To solve for $k$, take a ratio of the sides of the equations. Then

$$\frac{14}{17} = \frac{27e^{-k(t+1)}}{27e^{-kt}} = e^{-k} \quad \Rightarrow \quad -k = \ln\left(\frac{14}{17}\right) = -0.194\,.$$

This is the constant that describes the rate of cooling of the body.

To find the time of death, $t$, use

$$17 = 27e^{-kt} \quad \Rightarrow \quad -kt = \ln\left(\frac{17}{27}\right) = -0.4626$$

finally, solving for $t$, we get

$$t = \frac{0.4626}{k} = \frac{0.4626}{0.194} = 2.384 \text{ hours.}$$

◊

Mastered Material Check

13. Give the concluding sentence for Example 4.7. Be sure to include an actual time of death, given that the body was discovered at midnight.
14. Use a plotting program to graph $T(t)$ for Example 4.7.
15. Use your plot to estimate how long it took for the body to cool off to 33°C.

### *Related applications and further examples*

Having gained familiarity with specific examples, we now return to the general case and summarize the results.

The differential equation and initial condition

$$\frac{dy}{dt} = a - by, \quad y(0) = y_0 \tag{4.9}$$

has the solution

$$y(t) = \frac{a}{b} - \left(\frac{a}{b} - y_0\right) e^{-bt}. \tag{4.10}$$

Suppose that $a, b > 0$ in Eqn. (4.9). Then we can summarize the behaviour of the solutions (4.10) as follows:

- The time dependence of Eqn. (4.10) is contained in the term $e^{-bt}$, which (for $b > 0$) is exponentially decreasing. As time increases, $t \to \infty$, the exponential term becomes negligibly small, so $y \to a/b$.
- If initially $y(0) = y_0 > a/b$, then $y(t)$ approaches $a/b$ from above, whereas if $y_0 < a/b$, it approaches $a/b$ from below.
- If initially $y_0 = a/b$, there is no change at all ($dy/dt = 0$). Thus $y = a/b$ is a **steady state** of the DE in Eqn. (4.9).

Recognizing such general structure means that we can avoid repeating similar calculations from scratch in related examples. Newton's law of cooling is one representative of the class of differential equations of the form Eqn. (4.9). If we set $a = kE, b = k$ and $T = y$ in Eqn. (4.9), we get back to

Eqn. (4.7). As expected from the general case, $T$ approaches $a/b = E$, the ambient temperature, which corresponds to a steady state of NLC.

Next, we describe other examples that share this structure, and hence similar dynamic behaviour.

**Friction and terminal velocity** A falling object accelerates under the force of gravity, but friction slows down this acceleration. The differential equation satisfied by the velocity $v(t)$ of the falling object with friction is

$$\frac{dv}{dt} = g - kv \tag{4.11}$$

where $g > 0$ is acceleration due to gravity and $k > 0$ is a constant representing the effect of air resistance. (In contrast to the "upwards pointing" coordinate system used in Example **??**, here we focus on how the magnitude of the velocity changes with time.) Usually, a frictional force is assumed to be proportional to the velocity of the object, and to act in a direction that slows it down. (This accounts for the negative sign in Eqn. (4.11).) Parachutes operate on the principle of enhancing that frictional force to damp out the acceleration of a skydiver. Hence, Eqn. (4.11) is often called the **skydiver equation**.

**Note.** Eqn. (4.11) comes from a simple force balance:

$$ma = F_{gravity} - F_{drag},$$

and from the assumption that $F_{drag} = \mu v$, where $\mu > 0$ is the "drag coefficient". Dividing both sides by $m$ and replacing $a$ by $dv/dt$ leads to this equation, with $k = \mu/m$.

**Example 4.8** *Use the general results for Eqn.* (4.9) *to write down the solution to the differential equation* (4.11) *for the velocity of a skydiver given the initial condition* $v(0) = v_0$. *Interpret your results in a simple description of what happens over time.*

**Solution.** Eqn. (4.11) is of the same form as Eqn. (4.9), and has the same type of solutions. We merely have to adjust the notation, by identifying

$$v(t) \to y(t), \quad g \to a, \quad k \to b, \quad v_0 \to y_0.$$

Hence, without further calculation, we can conclude that the solution of (4.11) together with its initial condition is:

$$v(t) = \frac{g}{k} - \left(\frac{g}{k} - v_0\right) e^{-kt}. \tag{4.12}$$

The velocity is initially $v_0$, and eventually approaches $g/k$ which is the **steady state** or **terminal velocity** for the object. Depending on the initial speed, the object either slows down (if $v_0 > g/k$) or speed up (if $v_0 < g/k$) as it approaches the terminal velocity. ◊

Mastered Material Check

16. Assign appropriate units to each of the parameters in Example 4.8.
17. When a sky-diver steps into the void, her initial vertical velocity is zero. Write down her velocity $v(t)$ based on results of Example 4.8 .

**Chemical production and decay.** A chemical reaction inside a fixed reaction volume produces a substance at a constant rate $K_{in}$. A second reaction results in decay of that substance at a rate proportional to its concentration. Let $c(t)$ denote the time-dependent concentration of the substance, and assume that time is measured in units of hours. Then, writing down a balance equation leads to a differential equation of the form

$$\frac{dc}{dt} = K_{\text{in}} - \gamma c. \tag{4.13}$$

Here, the first term is the rate of production and the second term is the rate of decay. The net rate of change of the chemical concentration is then the difference of the two. The constants $K_{\text{in}} > 0, \gamma > 0$ represent the rate of production and decay - recall that the *units of each term in any equation have to match.*For example, if the concentration $c$ is measured in units of milli-Molar (mM), then $dc/dt$ has units of mM/hr, and hence $K_{\text{in}}$ must have units of mM/h and $\gamma$ must have units of 1/hr.

**Example 4.9** *Write down the solution to the DE (4.13) given the initial condition $c(0) = c_0$. Determine the steady state chemical concentration.*

**Solution.** Translating notation from the general case to this example,

$$c(t) \to y(t), \qquad K_{\text{in}} \to a, \qquad \gamma \to b.$$

Then we can immediately write down the solution:

$$c(t) = \frac{K_{\text{in}}}{\gamma} - \left(\frac{K_{\text{in}}}{\gamma} - c_0\right) e^{-\gamma t}. \tag{4.14}$$

Regardless of its initial condition, the chemical concentration will approach a steady state concentration is $c = K_{\text{in}}/\gamma$. ♢

In this section we have seen that the behaviour found in the general case of the differential equation (4.4), can be reinterpreted in each specific situation of interest. This points to one powerful aspect of mathematics, namely the ability to use results in abstract general cases to solve a variety of seemingly unrelated scientific problems that share the same mathematical structure.

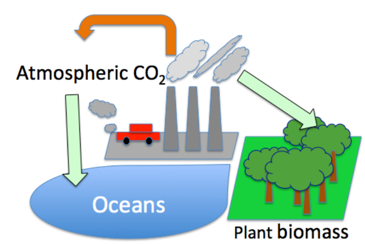



Figure 4.7: $CO_2$ is produced by emissions from burning fossil fuel and other human activities (orange arrow). The oceans and plant biomass are both sinks that absorbs $CO_2$ (light green arrows).

**Featured Problem 4.2 (Greenhouse Gasses and atmospheric $CO_2$)**
*Climate change has been attributed partly to the accumulation of greenhouse gasses (such as carbon dioxide and methane) in the atmosphere.*

*Here we consider a simplified illustrative model for the carbon cycle that tracks the sources and sinks of $CO_2$ in the atmosphere. Consider $C(t)$ as the level of atmospheric carbon dioxide. Define the production rate of $C0_2$ due to utilization of fossil fuel and other human activity to be $E_{FF}$, and let the rate of absorption of $CO_2$ by the oceans be $S_{OCEAN}$. We will also assume that living plants absorb $CO_2$ at a rate proportional to their biomass and to the $CO_2$ level.*

1. *Explain the following differential equation for atmospheric $CO_2$:*

$$\frac{dC}{dt} = E_{FF} - S_{OCEAN} - \gamma PC. \tag{4.15}$$

2. *Assuming that $E_{FF}, S_{OCEAN}, \gamma, P$ are constants, find the steady state level of $CO_2$ in terms of these parameters.*

3. *Find $C(t)$, that is, predict the amount of $CO_2$ over time, assuming that $C(0) = C_0$.*

Hint: $CO_2$ is usually given in units of "parts per million", ppm (=$10^{-6}$) ,1 ppm = 2.1 GtC. (1GtC= 1 gigaton carbon = $10^9$ tons.) Time is typically given in years, so rates are "per year" (yr$^{-1}$).
Approximate parameter values:
$E_{FF} \approx 10$ GtC yr$^{-1}$,
$S_{OCEAN} \approx 3$ GtC yr$^{-1}$,
$P \approx 560$ Gt plant biomass,
$\gamma \approx 1.35 \cdot 10^{-5}$ yr$^{-1}$ Gt$^{-1}$.

4. *Graph the function $C(t)$ for parameter values given in the problem, assuming that $C_0 = 400$ppm= 840 GtC.*

5. *How big an effect would be produced on the $CO_2$ level in 50 years if 15% of the plant biomass is removed to deforestation just prior to $t = 0$?*

Note: Information for Problem 4.2 is adapted from [**?**], and may reflect many simplifications and approximations. In actual fact, most "constants" in the problem are time-dependent, making the real problem of predicting $CO_2$ levels much more challenging.

## 4.3 *Euler's Method and numerical solutions*

**Section 4.3 Learning goals**

1. Explain the idea of a numerical solution to a differential equation and how this compares with an exact or analytic solution.

2. Describe how Euler's method is based on approximating the derivative by the slope of a secant line.

3. Use Euler's method to calculate a numerical solution (using a spreadsheet) to a given initial value problem.

So far, we have explored ways of understanding the behaviour predicted by a differential equation in the form of an **analytic solution**, namely an explicit formula for the solution as a function of time. However, in reality this is typically difficult without extensive training, and occasionally, impossible even for experts. Even if we can find such a solution, it may be inconvenient to determine its numerical values at arbitrary times, or to interpret its behaviour.

For this reason, we sometimes need a method for computing an approximation for the desired solution. We refer to that approximation as a **numerical solution**. The idea is to harness a computational device - computer, laptop, or calculator - to find numerical values of points along the solution curve, rather than attempting to determine the formula for the solution as a function of time. We illustrate this process using a technique called **Euler's method**, which is based on an approximation of a derivative by the slope of a secant line.

Below, we describe how Euler's method is used to approximate the solution to a general initial value problem (differential equation together with initial condition) of the form

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0.$$

**Set up.** We first must pick a "step size," $\Delta t$, and subdivide the $t$ axis into discrete steps of that size. We thus have a set of time points $t_1, t_2, \ldots,$ spaced

$\Delta t$ apart as shown in Figure 4.8. Our procedure starts with the known initial value $y(0) = y_0$, and uses it to generate an approximate value at the next time point ($y_1$), then the next ($y_2$), and so on. We denote by $y_k$ the value of the independent variable generated at the $k$'th time step by Euler's method as an approximation to the (unknown) true solution $y(t_k)$.

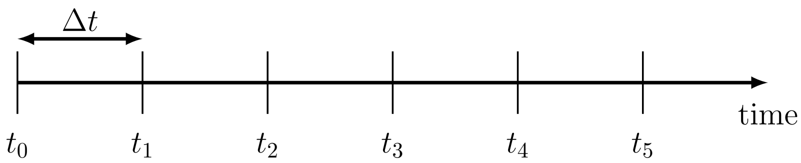



Figure 4.8: The time axis is subdivided into steps of size $\Delta t$.

Mastered Material Check

18. If $\Delta t = 0.1$ and $t_0 = 0$, what are $t_1, t_2$ and $t_3$?
19. Explain the difference between the value $y_1$ and the true solution $y(t_1)$.
20. If $\Delta t$ is not sufficiently small, why might Euler's method give a bad approximation to the solution?

**Method.** We approximate the differential equation by a **finite difference equation**

$$\frac{dy}{dt} = f(y) \quad \text{approximated by} \quad \frac{y_{k+1} - y_k}{\Delta t} = f(y_k).$$

This approximation is reasonable only when $\Delta t$, the time step size, is small. Rearranging this equation leads to a process (also called **recurrence relation**) for linking values of the solution at successive time points,

$$\frac{y_{k+1} - y_k}{\Delta t} = f(y_k), \quad \Rightarrow \quad y_{k+1} = y_k + \Delta t \cdot f(y_k). \tag{4.16}$$

**Application.** We start with the known initial value, $y_0$. Then (setting the index to $k = 0$ in Eqn. (4.16)) we obtain

$$y_1 = y_0 + f(y_0)\Delta t.$$

The quantities on the right are known, so we can compute the value of $y_1$, which is the approximation to the solution $y(t_1)$ at the time point $t_1$. We can then continue to generate the value at the next time point in the same way, by approximating the derivative again as a secant slope. This leads to

$$y_2 = y_1 + f(y_1)\Delta t.$$

The approximation so generated, leading to values $y_1, y_2, \dots$ is called **Euler's method**.

Applying this approximation repeatedly, leads to an **iteration method**, that is, the repeated computation

$$\begin{aligned} y_1 &= y_0 + f(y_0)\Delta t, \\ y_2 &= y_1 + f(y_1)\Delta t, \\ &\vdots \\ y_{k+1} &= y_k + f(y_k)\Delta t. \end{aligned}$$

Mastered Material Check

21. In Euler's method, can you determine $t_2$ directly? That is, without first computing $t_1$?
22. In Euler's method, can you determine $y_2$ directly? That is, without first computing $y_1$?

From this iteration, we obtain the approximate values of the function $y_k \approx y(t_k)$ for as many time steps as desired starting from $t = 0$ in increments of $\Delta t$ up to some final time $T$ of interest.

It is customary to use the following notations:

- $t_0$ : the initial time point, usually at $t = 0$.
- $h = \Delta t$ : common notations for the step size, i.e. the distance between the points along the $t$ axis.
- $t_k$ : the k'th time point. Note that since the points are at multiples of the step size that we have picked, $t_k = k\Delta t = kh$.
- $y(t)$ : the actual value of the solution to the differential equation at time $t$. This is usually not known, but in the examples discussed in this chapter, we can solve the differential equation exactly, so we have a formula for the function $y(t)$. In most hard scientific problems, no such formula is known in advance.
- $y(t_k)$ : the actual value of the solution to the differential equation at one of the discrete time points, $t_k$ (again, not usually known).
- $y_k$ : the approximate value of the solution obtained by Euler's method. We hope that this approximate value is fairly close to the true value, i.e. that $y_k \approx y(t_k)$, but there is always some error in the approximation. More advanced methods that are specifically designed to reduce such errors are discussed in courses on numerical analysis.

### *Euler's method applied to population growth*

We illustrate how Euler's method is used in a familiar example, that of unlimited population growth.

**Example 4.10** *Apply Euler's method to approximating solutions for the simple exponential growth model that was studied in Chapter 3,*

$$\frac{dy}{dt} = ay, \quad y(0) = y_0$$

*where a is a constant (see Eqn 3.2).*

**Solution.** Subdivide the $t$ axis into steps of size $\Delta t$, starting with $t_0 = 0$, and $t_1 = \Delta t, t_2 = 2\Delta t, \ldots$ The first value of $y$ is known from the initial condition,

$$y_0 = y(0) = y_0.$$

We replace the differential equation by the approximation

$$\frac{y_{k+1} - y_k}{\Delta t} = ay_k \quad \Rightarrow \quad y_{k+1} = y_k + a\Delta t y_k, \quad k = 1, 2, \ldots$$

Mastered Material Check

23. Carry our Example 4.10 with $\Delta t = 0.1$, $a = 1$, and $y_0 = 1$.
24. Plot the first 5 points you determine. Compare with the true solution.
25. Solve the initial value problem in Example 4.11 analytically. Compare the points $(0, 100)$, $(0.1, 95)$, $(0.2, 90.25)$ and $(0.3, 85.7375)$ with the true solution at the corresponding $t$ values.

In particular,

$$y_1 = y_0 + a\Delta t y_0 = y_0(1 + a\Delta t),$$
$$y_2 = y_1(1 + a\Delta t),$$
$$y_3 = y_2(1 + a\Delta t),$$

and so on. At every stage, the quantity on the right hand side depends only on value of $y_k$ that as already known from the step before. ♢

The next example demonstrates Euler's method applied to a specific differential equation.

**Example 4.11** *Use Euler's method to find the solution to*

$$\frac{dy}{dt} = -0.5y, \quad y(0) = 100.$$

*Use step size $\Delta t = 0.1$ to approximate the solution for the first two time steps.*

Link to Google Sheets. This spreadsheet implements Euler's method for Example 4.11. You can view the formulae by clicking on a cell in the sheet but you cannot edit the sheet here.

**Solution.** Euler's method applied to this example would lead to

$$y_0 = 100.$$
$$y_1 = y_0(1 + a\Delta t) = 100(1 + (-0.5)(0.1)) = 95, \quad \text{etc.}$$

We show the first five values in Table 4.3. Clearly, these kinds of repeated calculations are best handled on a spreadsheet or similar computer software.

| $k$ | $t_k$ | $y_k$ |
|---|---|---|
| 0 | 0 | 100.00 |
| 1 | 0.1 | 95.00 |
| 2 | 0.2 | 90.25 |
| 3 | 0.3 | 85.74 |
| 4 | 0.4 | 81.45 |
| 5 | 0.5 | 77.38 |

Table 4.3: Euler's method applied to Example 4.11.

### *Euler's method applied to Newton's law of cooling*

We apply Euler's method to Newton's law of cooling. Upon completion, we can directly compare the approximate numerical solution generated by Euler's method to the true (analytic) solution, (4.8), that we determined earlier in this chapter.

**Example 4.12 (Newton's law of cooling)** *Consider the temperature of an object $T(t)$ in an ambient temperature of $E = 10°$. Assume that $k = 0.2$/min. Use the initial value problem*

$$\frac{dT}{dt} = k(E - T), \quad T(0) = T_0$$

*to write the exact solution to Eqn. (4.8) in terms of the initial value $T_0$.*

**Solution.** In this case, the differential equation has the form

$$\frac{dT}{dt} = 0.2(10 - T),$$

and its analytic solution, from Eqn. (4.8), is

$$T(t) = 10 + (T_0 - 10)e^{-0.2t}. \tag{4.17}$$

♢

Below, we use Euler's method to compute a solution from each of several initial conditions, $T(0) = 0, 5, 15, 20$ degrees.

**Example 4.13 (Euler's method applied to Newton's law of cooling)** *Write the Euler's method procedure for the approximate solution to the problem in Example 4.12.*

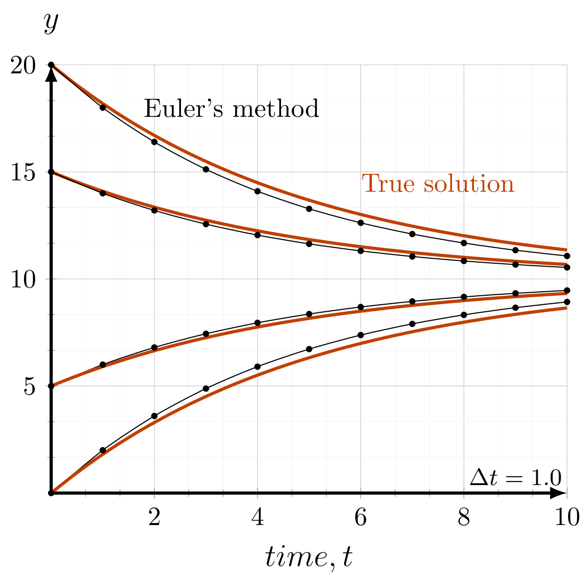
**Solution.** Euler's method approximates the differential equation by

$$\frac{T_{k+1} - T_k}{\Delta t} = 0.2(10 - T_k).$$

or, in simplified form,

$$T_{k+1} = T_k + 0.2(10 - T_k)\Delta t.$$

◊




| time | approx solution | exact soln |
|---|---|---|
| $t_k$ | $T_k$ | $T(t)$ |
| 0.0000 | 0.0000 | 0.0000 |
| 1.0000 | 2.0000 | 1.8127 |
| 2.0000 | 3.6000 | 3.2968 |
| 3.0000 | 4.8800 | 4.5119 |
| 4.0000 | 5.9040 | 5.5067 |
| 5.0000 | 6.7232 | 6.3212 |
| 6.0000 | 7.3786 | 6.9881 |
| 7.0000 | 7.9028 | 7.5340 |
| 8.0000 | 8.3223 | 7.9810 |

Figure 4.9: Euler's method applied to Newton's law of cooling. The graph shows the true solution (red) and the approximate solution (black).

**Example 4.14** *Use Euler's method from Example 4.13 and time steps of size $\Delta t = 1.0$ to find a numerical solution to the the cooling problem. Use a spreadsheet for the calculations. Note that $\Delta t = 1.0$ is not a "small step;" we use it here for illustration purposes.*

**Solution.** The procedure to implement is

$$T_{k+1} = T_k + 0.2(10 - T_k)\Delta t.$$

In Figure 4.9 we show a typical example of the method with initial value $T(0) = T_0$ and with the time step size $\Delta t = 1.0$. Black dots represent the discrete values generated by the Euler method, starting from initial conditions, $T_0 = 0, 5, 15, 20$. Notice that the black curve is simply made up of line segments linking points obtained by the numerical solution. On the same graph, we also show the analytic solution (red curves) given by Eqn. (4.17) with the

Mastered Material Check

26. What change would you make in the process set up in Example 4.14 to improve the approximation made by Euler's method?

same four initial temperatures. We see that the black and red curves start out at the same points (since they both satisfy the same initial conditions). However, the approximate solution obtained with Euler's method is not identical to the true solution. The difference between the two (gap between the red and black curves) is the **numerical error** in the approximation.

## *4.4 Summary*

1. Given a function, we can check whether it is a solution to a differential equation by performing the appropriate differentiation and algebraic simplification.
2. Solutions to differential equations in which there is no change at all ("constant solutions") are referred to as steady states.
3. The differential equations

$$\frac{dy}{dt} = a - by, \qquad y(0) = y_0$$

has a steady state solution $y = a/b$.
4. If we define the deviation from steady state, $z(t) = y(t) - \frac{a}{b}$, we get a decay equation for $z(t)$ that has exponentially decreasing solutions provided $b > 0$. This says that the eviation from steady state always decrease over time.
5. The resulting solution for $y(t)$ is

$$y(t) = \frac{a}{b} - \left(\frac{a}{b} - y_0\right) e^{-bt}.$$

6. For some differential equations, it is not always possible to determine an analytic solution (explicit formula). Numerical solutions can be found using Euler's method, and serve as an approximate solution.
7. Euler's method takes a known initial value $y_0$ and uses the iteration scheme:

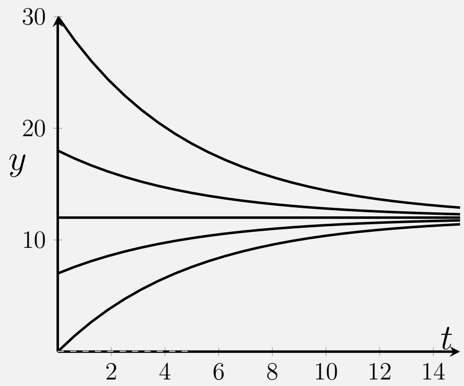
$$y_{k+1} = y_k + f(y_k)\Delta t.$$

to generate successive values of $y_k$ that approximate the solution at time points $t_k = k\Delta t$
8. Applications considered in this chapter included:

   (a) height of water draining out of a cylindrical container (verifying a solution to a differential equation);
   (b) Newton's law of cooling (described by a linear differential equation);
   (c) growth of the radius of a cell;
   (d) the accumulation of greenhouse gasses in the atmosphere;
   (e) friction and terminal velocity; and
   (f) chemical production and decay.

**Quick Concept Checks**

1. Explain why an object at room temperature is at a steady state for Newton's law of cooling.
2. The following graph depicts solution curves to a particular differential equation of the form $dy/dt = a - by$.




   (a) Estimate the value that these solution curves are approaching.
   (b) Which solutions are approaching from above? From below?
3. Consider the following initial value problem:
$$\frac{dy}{dt} = 2 - 4y, \quad y(0) = 4,$$
   (a) What value does its solution curve approach?
   (b) Does its solution approach from above or below?
4. Why is a large value of $\Delta t$ not a good idea when using Euler's method?

---

*Exercises*

4.1. **Water draining form a container.** In Example 4.3, we verified that the function $h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2$ is a solution to the differential equation (4.2). Based on the meaning of the problem, for how long does this solution remain valid?

4.2. **Verifying a solution.** Verify that the function $y(t) = 1 - (1 - y_0)e^{-t}$ satisfies the initial value problem (differential equation and initial condition) (4.6).

4.3. **Linear differential equation.** Consider the differential equation

$$\frac{dy}{dt} = a - by$$

where $a$, $b$ are constants.

(a) Show that the function

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

satisfies the above differential equation for any constant $C$.

(b) Show that by setting

$$C = \frac{a}{b} - y_0$$

we also satisfy the initial condition

$$y(0) = y_0.$$

*Remark:* you have shown that the function

$$y(t) = \left(y_0 - \frac{a}{b}\right)e^{-bt} + \frac{a}{b}$$

is a solution to the *initial value problem* (i.e differential equation plus initial condition)

$$\frac{dy}{dt} = a - by, \quad y(0) = y_0.$$

4.4. **Steps in an example.** Complete the algebraic steps in Example **??** to show that the solution to Eqn. (4.4) can be obtained by the substitution $z(t) = a - by(t)$.

4.5. **Verifying a solution.** Show that the function

$$y(t) = \frac{1}{1-t}$$

is a solution to the differential equation and initial condition

$$\frac{dy}{dt} = y^2, \quad y(0) = 1.$$

Comment on what happens to this solution as $t$ approaches 1.

4.6. **Verifying solutions.** For each of the following, show the given function $y$ is a solution to the given differential equation.

(a) $t \cdot \frac{dy}{dt} = 3y$, $y = 2t^3$.

(b) $\frac{d^2y}{dt^2} + y = 0$, $y = -2\sin t + 3\cos t$.

(c) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 6e^t$, $y = 3t^2e^t$.

4.7. **Verifying a solution.** Show the function determined by the equation $2x^2 + xy - y^2 = C$, where $C$ is a constant and $2y \neq x$, is a solution to the differential equation $(x-2y)\frac{dy}{dx} = -4x - y$.

4.8. **Determining the constant.** Find the constant $C$ that satisfies the given initial conditions.

(a) $2x^2 - 3y^2 = C$, $y|_{x=0} = 2$.

(b) $y = C_1e^{5t} + C_2te^{5t}$, $y|_{t=0} = 1$ and $\frac{dy}{dt}|_{t=0} = 0$.

(c) $y = C_1\cos(t - C_2)$, $y|_{t=\frac{\pi}{2}} = 0$ and $\frac{dy}{dt}|_{t=\frac{\pi}{2}} = 1$.

4.9. **Friction and terminal velocity.** The velocity of a falling object changes due to the acceleration of gravity, but friction has an effect of slowing down this acceleration. The differential equation satisfied by the velocity $v(t)$ of the falling object is

$$\frac{dv}{dt} = g - kv$$

where $g$ is acceleration due to gravity and $k$ is a constant that represents the effect of friction. An object is dropped from rest from a plane.

(a) Find the function $v(t)$ that represents its velocity over time.

(b) What happens to the velocity after the object has been falling for a long time (but before it has hit the ground)?

4.10. **Alcohol level.** Alcohol enters the blood stream at a constant rate $k$ gm per unit time during a drinking session. The liver gradually converts the alcohol to other, non-toxic byproducts. The rate of conversion per unit time is proportional to the current blood alcohol level, so that the differential equation satisfied by the blood alcohol level is

$$\frac{dc}{dt} = k - sc$$

where $k$, $s$ are positive constants. Suppose initially there is no alcohol in the blood.

Find the blood alcohol level $c(t)$ as a function of time from $t = 0$, when the drinking started.

4.11. **Checking a solution.** Check that the differential equation (4.7) has the right sign, so that a hot object cools off in a colder environment.

4.12. **Details of Newtons Law of Cooling.** Fill in the missing steps in the solution to Newton's Law of Cooling in Example 4.5.

4.13. **Newton's Law of Cooling.** Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object, $T$, and the ambient (environmental) temperature, $E$. This leads to the *differential equation*

$$\frac{dT}{dt} = k(E - T)$$

where $k > 0$ is a constant that represents the material properties and, $E$ is the ambient temperature. (We assume that $E$ is also constant.)

(a) Show that the function

$$T(t) = E + (T_0 - E)e^{-kt}$$

which represents the temperature at time $t$ satisfies this equation.

(b) The time of death of a murder victim can be estimated from the temperature of the body if it is discovered early enough after the crime has occurred.

Suppose that in a room whose ambient temperature is $E = 20^\circ$ C, the temperature of the body upon discovery is $T = 30^\circ$ C, and that a second measurement, one hour later is $T = 25^\circ$ C.

Determine the approximate time of death.

*Remark:* use the fact that just prior to death, the temperature of the victim was 37°C.

4.14. **A cup of coffee.** The temperature of a cup of coffee is initially 100 degrees C. Five minutes later, ($t = 5$) it is 50 degrees C. If the ambient temperature is $A = 20$ degrees C, determine how long it takes for the temperature of the coffee to reach 30 degrees C.

4.15. **Newton's Law of Cooling applied to data.** The data presented in Table 4.4 was gathered in producing Figure **??** for cooling milk during yoghurt production. According to Newton's Law of Cooling, this data can be described by the formula

$$T = E + (T(0) - E)\, e^{-kt}.$$

where $T(t)$ is the temperature of the milk (in degrees Fahrenheit) at time $t$ (in min), $E$ is the ambient temperature, and $k$ is some constant that we determine in this exercise.

(a) Rewrite this relationship in terms of the quantity $Y(t) = \ln(T(t) - E)$, and show that $Y(t)$ is related linearly to the time $t$.

(b) Explain how the constant $k$ could be found from this converted form of the relationship.

| **time** (min) | **Temp** |
|---|---|
| 0.0 | 190.0 |
| 0.5 | 185.5 |
| 1.0 | 182.0 |
| 1.5 | 179.2 |
| 2.0 | 176.0 |
| 2.5 | 172.9 |
| 3.0 | 169.5 |
| 3.5 | 167.0 |
| 4.0 | 164.6 |
| 4.5 | 162.2 |
| 5.0 | 159.8 |

Table 4.4: Cooling milk data for Exercise 15.

(c) Use the data in the table and your favourite spreadsheet (or similar software) to show that the data so transformed appears to be close to linear. Assume that the ambient temperature was $E = 20°$F.

(d) Use the same software to determine the constant $k$ by fitting a line to the transformed data.

4.16. **Infant weight gain.** During the first year of its life, the weight of a baby is given by

$$y(t) = \sqrt{3t+64}$$

where $t$ is measured in some convenient unit.

(a) Show that $y$ satisfies the differential equation

$$\frac{dy}{dt} = \frac{k}{y}$$

where $k$ is some positive constant.

(b) What is the value for $k$?

(c) Suppose we adopt this differential equation as a model for human growth. State concisely (that is, in one sentence) one feature about this differential equation which makes it a reasonable model. State one feature which makes it unreasonable.

4.17. **Lake Fishing.** Fish Unlimited is a company that manages the fish population in a private lake. They restock the lake at constant rate (to restock means to add fish to the lake): $N$ fishers are allowed to fish in the lake per day. The population of fish in the lake, $F(t)$ is found to satisfy the differential equation

$$\frac{dF}{dt} = I - \alpha NF \tag{4.18}$$

(a) At what rate are fish added per day according to Eqn. (4.18)? Give both value and units.

(b) What is the average number of fish caught by one fisher? Give both the value and units.

(c) What is being assumed about the fish birth and mortality rates in Eqn. (4.18)?

(d) If the fish input and number of fishers are constant, what is the steady state level of the fish population in the lake?

(e) At time $t = 0$ the company stops restocking the lake with fish. Give the revised form of the differential equation (4.18) that takes this into account, assuming the same level of fishing as before. How long would it take for the fish to fall to 25% of their initial level?

(f) When the fish population drops to the level $F_{low}$, fishing is stopped and the lake is restocked with fish at the same constant rate

(Eqn (4.18), with $\alpha = 0$.) Write down the revised version of Eqn. (4.18) that takes this into account. How long would it take for the fish population to double?

4.18. **Tissue culture.** Cells in a tissue culture produce a cytokine (a chemical that controls the growth of other cells) at a constant rate of 10 nano-Moles per hour (nM/h). The chemical has a half-life of 20 hours.

Give a differential equation (DE) that describes this chemical production and decay. Solve this DE assuming that at $t = 0$ there is no cytokine. [1nM=$10^{-9}$M].

4.19. **Glucose solution in a tank.** A tank that holds 1 liter is initially full of plain water. A concentrated solution of glucose, containing 0.25 gm/cm$^3$ is pumped into the tank continuously, at the rate 10 cm$^3$/min and the mixture (which is continuously stirred to keep it uniform) is pumped out at the same rate.

How much glucose is in the tank after 30 minutes? After a long time? (*hint:* write a differential equation for $c$, the concentration of glucose in the tank by considering the rate at which glucose enters and the rate at which glucose leaves the tank.)

4.20. **Pollutant in a lake.** A lake of constant volume $V$ gallons contains $Q(t)$ pounds of pollutant at time $t$ evenly distributed throughout the lake. Water containing a concentration of $k$ pounds per gallon of pollutant enters the lake at a rate of $r$ gallons per minute, and the well-mixed solution leaves at the same rate.

(a) Set up a differential equation that describes the way that the amount of pollutant in the lake changes.

(b) Determine what happens to the pollutant level after a long time if this process continues.

(c) If $k = 0$ find the time $T$ for the amount of pollutant to be reduced to one half of its initial value.

4.21. **A sugar solution.** Sugar dissolves in water at a rate proportional to the amount of sugar not yet in solution. Let $Q(t)$ be the amount of sugar undissolved at time $t$. The initial amount is 100 kg and after 4 hours the amount undissolved is 70 kg.

(a) Find a differential equation for $Q(t)$ and solve it.

(b) How long does it take for 50 kg to dissolve?

4.22. **Leaking water tank.** A cylindrical tank with cross-sectional area $A$ has a small hole through which water drains. The height of the water in the tank $y(t)$ at time $t$ is given by:

$$y(t) = (\sqrt{y_0} - \frac{kt}{2A})^2$$

where $k, y_0$ are constants.

(a) Show that the height of the water, $y(t)$, satisfies the differential equation

$$\frac{dy}{dt} = -\frac{k}{A}\sqrt{y}.$$

(b) What is the initial height of the water in the tank at time $t = 0$ ?

(c) At what time is the tank be empty ?

(d) At what rate is the **volume** of the water in the tank changing when $t = 0$?

4.23. **Determining constants.** Find those constants $a, b$ so that $y = e^x$ and $y = e^{-x}$ are both solutions of the differential equation

$$y'' + ay' + by = 0.$$

4.24. **Euler's method.** Solve the decay equation in Example (4.11) analytically, that is, find the formula for the solution in terms of a decaying exponential, and then compare your values to the approximate solution values $y_1$ and, $y_2$ computed with Euler's method.

4.25. **Comparing approximate and true solutions:**

(a) Use Euler's method to find an approximate solution to the differential equation

$$\frac{dy}{dx} = y$$

with $y(0) = 1$. Use a step size $h = 0.1$ and find the values of $y$ up to $x = 0.5$. Compare the value you have calculated for $y(0.5)$ using Euler's method with the true solution of this differential equation. What is the **error** i.e. the difference between the true solution and the approximation?

(b) Now use Euler's method on the differential equation

$$\frac{dy}{dx} = -y$$

with $y(0) = 1$. Use a step size $h = 0.1$ again and find the values of $y$ up to $x = 0.5$. Compare the value you have calculated for $y(0.5)$ using Euler's method with the true solution of this differential equation. What is the error this time?

4.26. **Beginning Euler's method.** Give the first 3 steps of Euler's method for the problem in Example 4.13.

4.27. **Euler's method and a spreadsheet.** Use the spreadsheet and Euler's method to solve the differential equation shown below:

$$dy/dt = 0.5y(2-y)$$

Use a step size of $h = 0.1$ and show (on the same graph) solutions for the following four initial values:

$$y(0) = 0.5,\ y(0) = 1,\ y(0) = 1.5,\ y(0) = 2.25$$

For full credit, include a short explanation your process (e.g. 1-2 sentences and whatever equations you implemented on the spreadsheet.)

# 5

# Qualitative methods for differential equations

Not all differential equations are easily solved analytically. Furthermore, even when we find the analytic solution, it is not necessarily easy to interpret, graph, or understand. This situation motivates qualitative methods that promote an overall understanding of behaviour - directly from information in the differential equation - without the challenge of finding a full functional form of the solution.

In this chapter we expand our familiarity with differential equations and assemble new, qualitative techniques for understanding them. We consider differential equations in which the expression on one side, $f(y)$, is **nonlinear**, i.e. equations of the form

$$\frac{dy}{dt} = f(y)$$

in which $f$ is more complicated than the form $a - by$. Geometric techniques, rather than algebraic calculations form the core of the concepts we discuss.

Mastered Material Check

1. What is meant by an analytic solution to a differential equation?
2. What other kind of solutions are possible?
3. Give an example of a nonlinear function $f(y)$.

## 5.1 Linear and nonlinear differential equations

**Section 5.1 Learning goals**

1. Identify the distinction between unlimited and density-dependent population growth. Be able to explain terms in the logistic equation in its original version, Eqn. (5.1), and its rescaled version, Eqn. (5.3).
2. State the definition of a linear differential equation.
3. Explain the law of mass action, and derive simple differential equations for interacting species based on this law.

In the model for population growth in Chapter 3, we encountered the differential equation

$$\frac{dN}{dt} = kN,$$

where $N(t)$ is population size at time $t$ and $k$ is a constant per capita growth rate. We showed that this differential equation has exponential solutions. It means that two behaviours are generically obtained: **explosive growth** if $k > 0$ or **extinction** if $k < 0$.

The case of $k > 0$ is unrealistic, since real populations cannot keep growing indefinitely in an explosive, exponential way. Eventually running out of space or resources, the population growth dwindles, and the population attains some static level rather than expanding forever. This motivates a revision of our previous model to depict **density-dependent growth**.

Mastered Material Check

4. What happens in the case that $k = 0$? Explain under what conditions this might arise and what happens to the population $N(t)$ in this case.

### *The logistic equation for population growth*

Let $N(t)$ represent the size of a population at time $t$, as before. Consider the differential equation

$$\frac{dN}{dt} = rN\frac{(K-N)}{K}. \tag{5.1}$$

We call this differential equation the **logistic equation**.The logistic equation has a long history in modelling population growth of humans, microorganisms, and animals. Here the parameter $r$ is the **intrinsic growth rate** and $K$ is the **carrying capacity**. Both $r, K$ are assumed to be positive constants for a given population in a given environment.

In the form written above, we could interpret the logistic equation as

$$\frac{dN}{dt} = R(N)\cdot N, \quad \text{where } R(N) = \left[r\frac{(K-N)}{K}\right].$$

The term $R(N)$ is a function of $N$ that replaces the constant rate of growth $k$ (found in the unrealistic, unlimited population growth model). $R$ is called the **density dependent growth rate**.

### *Linear versus nonlinear*

The logistic equation introduces the first example of a **nonlinear differential equation**. We explain the distinction between linear and nonlinear differential equations and why it matters.

**Definition 5.1 (Linear differential equation)** *A first order differential equation is said to be linear if it is a linear combination of terms of the form*

$$\frac{dy}{dt}, \quad y, \quad 1$$

*that is, it can be written in the form*

$$\alpha\frac{dy}{dt} + \beta y + \gamma = 0 \tag{5.2}$$

*where $\alpha, \beta, \gamma$ do not depend on $y$. Note that "first order" means that only the first derivative (or no derivative at all) may occur in the equation.*

Mastered Material Check

5. Can the differential equation $\frac{dy}{dt} = a - by$ be written in the form (5.2)? If so, what are the values of $\alpha, \beta, \gamma$?

So far, we have seen several examples of this type with constant coefficients $\alpha$, $\beta$, $\gamma$. For example, $\alpha = 1$, $\beta = -k$, and $\gamma = 0$ in Eqn. 3.2 whereas $\alpha = 1$, $\gamma = -a$, and $\beta = b$ in Eqn. (4.4). A differential equation that is not of this form is said to be nonlinear.

**Example 5.1 (Linear versus nonlinear differential equations)** *Which of the following differential equations are linear and which are nonlinear?*

$$(a)\ \frac{dy}{dt} = y^2, \quad (b)\ \frac{dy}{dt} - y = 5, \quad (c)\ y\frac{dy}{dt} = -1.$$

**Solution.** Any term of the form $y^2$, $\sqrt{y}$, $1/y$, etc. is nonlinear in $y$. A product such as $y\frac{dy}{dt}$ is also nonlinear in the independent variable. Hence equations (a), (c) are nonlinear, while (b) is linear. $\Diamond$

Mastered Material Check

6. For what values of $\alpha$, $\beta$ and $\gamma$ can Example 5.1(b) be put into the form (5.2)?

The significance of the distinction between linear and nonlinear differential equations is that nonlinearities make it much harder to systematically find a solution to the given differential equation by "analytic" methods. Most linear differential equations have solutions that are made of exponential functions or expressions involving such functions. This is not true for nonlinear equations.

However, as we see shortly, geometric methods are very helpful in understanding the behaviour of such nonlinear differential equations.

### *Law of Mass Action*

Nonlinear terms in differential equations arise naturally in various ways. One common source comes from describing interactions between individuals, as the following example illustrates.

In a chemical reaction, molecules of types $A$ and $B$ bind and react to form product $P$. Let $a(t)$, $b(t)$ denote the concentrations of $A$ and $B$. These concentrations depend on time because the chemical reaction uses up both types in producing the product.

The reaction only occurs when $A$ and $B$ molecules "collide" and stick to one another. Collisions occur randomly, but if concentrations are larger, more collisions take place, and the reaction is faster. If either the concentration $a$ or $b$ is doubled, then the reaction rate doubles. But if both $a$ and $b$ are doubled, then the reaction rate should be four times faster, based on the higher chances of collisions between $A$ and $B$. The simplest assumption that captures this dependence is

$$\text{rate of reaction is proportional to } a \cdot b \quad \Rightarrow \quad \text{rate of reaction } = k \cdot a \cdot b$$

where $k$ is some constant that represents the reactivity of the molecules.

We can formally state this result, known as the **Law of Mass Action** as follows:

Mastered Material Check

7. If the concentration of $A$ is tripled, and that of $B$ is doubled, how much faster would we expect the reaction rate to be?
8. Why does the product $a \cdot b$, rather than the sum $a + b$ appear in the Law of Mass Action ?

**The Law of Mass Action:** The rate of a chemical reaction involving an interaction of two or more chemical species is proportional to the *product of the concentrations* of the given species.

**Example 5.2 (Differential equation for interacting chemicals)** *Substance A is added at a constant rate of I moles per hour to a 1-litre vessel. Pairs of molecules of A interact chemically to form a product P. Write down a differential equation that keeps track of the concentration of A, denoted $y(t)$.*

**Solution.** First consider the case that there is no reaction. Then, the addition of $A$ to the reactor at a constant rate leads to changing $y(t)$, described by the differential equation

$$\frac{dy}{dt} = I.$$

When the chemical reaction takes place, the depletion of $A$ depends on interactions of pairs of molecules. By the law of mass action, the rate of reaction is of the form $k \cdot y \cdot y = ky^2$, and as it reduces the concentration, it appear with a minus sign in the DE. Hence

$$\frac{dy}{dt} = I - ky^2.$$

This is a nonlinear differential equation - it contains a term of the form $y^2$. ◊

Mastered Material Check

9. In each of Examples 5.2 and 5.3, clearly identify the constant quantities.

**Example 5.3 (Logistic equation reinterpreted)** *Rewrite the logistic equation in the form*

$$\frac{dN}{dt} = rN - bN^2$$

*(where $b = r/K$ is a positive quantity).*

**a)** *Interpret the meaning of this restated form of the equation by explaining what each of the terms on the right hand side could represent.*

**b)** *Which of the two terms dominates for small versus large population levels?*

**Solution.**

**a)** This form of the equation has growth term $rN$ proportional to population size, as encountered previously in unlimited population growth. However, there is also a quadratic (nonlinear) rate of loss (note the minus sign) $-bN^2$. This term could describe interactions between individuals that lead to mortality, e.g. through fighting or competition.

**b)** From familiarity with power functions (in this case, the functions of $N$ that form the two terms, $rN$ and $bN^2$) we can deduce that the second, quadratic term dominates for larger values of $N$, and this means that when the population is crowded, the loss of individuals is greater than the rate of reproduction. ◊

### *Scaling the logistic equation*

Consider units involved in the logistic equation (5.1):

$$\frac{dN}{dt} = rN\frac{(K-N)}{K}.$$

This equation has two parameters, $r$ and $K$. Since units on each side of an equation must balance, and must be the same for terms that are added or subtracted, we can infer that $K$ has the same units as $N$, and thus it is a population density. When $N = K$, the population growth rate is zero ($dN/dt = 0$).

It turns out that we can understand the behaviour of the logistic equation by converting it to a "generic" form that does not depend on the constant $K$. We do so by transforming variables, which amounts to choosing a convenient way to measure the population size.

**Example 5.4 (Rescaling)** *Define a new variable*

$$y(t) = \frac{N(t)}{K},$$

*with $N(t)$ and $K$ as in the logistic equation. Then $N(t) = Ky(t)$.*

**a)** *Interpret what the transformed variable $y$ represents.*

**b)** *Rewrite the logistic equation in terms of this variable.*

Mastered Material Check

10. Suppose an environment can sustain 2000 aphids per plant, and the current population size on a given plant is 1700. What is $K$, $N$ and $y$ based on this information?
11. This population is at what percent of its carrying capacity?

**Solution.**

**a)** The variable, $y(t)$ represents a scaled version of the population density. Instead of measuring the population in some arbitrary units - such as number of individuals per acre, or number of bacteria per ml - $y(t)$ measures the population in "multiples of the carrying capacity."

For example, if the environment can sustain 1000 aphids per plant (so $K = 1000$ individuals per plant), and the current population size on a given plant is $N = 950$ then the value of the scaled variable is $y = 950/1000 = 0.95$. We would say that "the aphid population is at 95% of its carrying capacity on the plant."

**b)** Since $K$ is assumed constant, it follows that

$$N(t) = Ky(t) \quad \Rightarrow \quad \frac{dN}{dt} = K\frac{dy}{dt}.$$

Using this, we can simplify the logistic equation as follows:

$$\begin{aligned} \frac{dN}{dt} = rN\frac{(K-N)}{K}, \quad &\Rightarrow \quad K\frac{dy}{dt} = r(Ky)\frac{(K-Ky)}{K}, \\ &\Rightarrow \quad \frac{dy}{dt} = ry(1-y). \end{aligned} \tag{5.3}$$

$\Diamond$

Eqn. (5.3) "looks simpler" than Eqn. (5.1) since it depends on only one parameter, $r$. Moreover, by understanding this equation, and transforming back to the original logistic in terms of $N(t) = Ky(t)$, we can interpret results for the original model. While we do not go further with transforming variables at present, it turns out that one can also further reduce the scaled logistic to an equation in which $r = 1$ by "rescaling time units".

Mastered Material Check

12. What are the units of the parameter $r$?
13. How might we use the parameter $r$ to define a time-scale?

## 5.2 *The geometry of change*

**Section 5.2 Learning goals**

1. Explain what is a **slope field** of a differential equation. Given a differential equation (linear or nonlinear), construct such a diagram and use it to sketch solution curves.
2. Describe what a **state-space diagram** is; construct such a diagram and use it to interpret the behaviour of solution curves to a given differential equation.
3. Identify the relationships between a slope field, a state-space diagram, and a family of solution curves to a given differential equation.
4. Identify steady states of a differential equation and determine whether they are stable or unstable.
5. Given a differential equation and initial condition, predict the behaviour of the solution for $t > 0$.

In this section, we introduce a new method for understanding differential equations using graphical and geometric arguments. Such methods circumvent the solutions that we expressed in terms of analytic formulae. We resort to concepts learned much earlier - for example, the derivative as a slope of a tangent line - in order to use the differential equation itself to assemble a sketch of the behaviour that it predicts. That is, rather than writing down $y = F(t)$ as a solution to the differential equation (and then graphing that function) we sketch the qualitative behaviour of such solution curves directly from information contained in the differential equation.

### *Slope fields*

Here we discuss a geometric way of understanding what a differential equation is saying using a **slope field**, also called a **direction field**. We have already seen that solutions to a differential equation of the form

$$\frac{dy}{dt} = f(y)$$

are curves in the $(y,t)$-plane that describe how $y(t)$ changes over time (thus, these curves are graphs of functions of time). Each initial condition $y(0) = y_0$ is associated with one of these curves, so that together, these curves form a *family* of solutions.

What do these curves have in common, geometrically?

- the slope of the tangent line $(dy/dt)$ at any point on any of the curves is related to the value of the $y$-coordinate of that point - as stated in the differential equation.
- at any point $(t, y(t))$ on a solution curve, the tangent line must have slope $f(y)$, which depends only on the $y$ value, and not on the time $t$.

*Note:* in more general cases, the expression $f(y)$ that appears in the differential equation might depend on $t$ as well as $y$. For our purposes, we do not consider such examples in detail.

By sketching slopes at various values of $y$, *we obtain the slope field* through which we can get a reasonable idea of the behaviour of the solutions to the differential equation.

**Example 5.5** *Consider the differential equation*

$$\frac{dy}{dt} = 2y. \tag{5.4}$$

*Compute some of the slopes for various values of y and use this to sketch a slope field for this differential equation.*

Mastered Material Check

14. Solve Differential Eqn. (5.4) analytically.

**Solution.** Equation (5.4) states that if a solution curve passes through a point $(t, y)$, then its tangent line at that point has a slope $2y$, regardless of the value of $t$. This example is simple enough that we can state the following: for positive values of $y$, the slope is positive; for negative values of $y$, the slope is negative; and for $y = 0$, the slope is zero.

We provide some tabulated values of $y$ indicating the values of the slope $f(y)$, its sign, and what this implies about the local behaviour of the solution and its direction. Then, in Figure 5.1 we combine this information to generate

| $y$ | $f(y)$ | slope of tangent line | behaviour of $y$ | direction of arrow |
|---|---|---|---|---|
| -2 | -4 | -ve | decreasing | $\searrow$ |
| -1 | -2 | -ve | decreasing | $\searrow$ |
| 0 | 0 | 0 | no change | $\rightarrow$ |
| 1 | 2 | +ve | increasing | $\nearrow$ |
| 2 | 4 | +ve | increasing | $\nearrow$ |

Table 5.1: Table for the slope field diagram of differential equation (5.4), $\frac{dy}{dt} = 2y$, described in Example 5.5.

the direction field and the corresponding solution curves. Note that the direction of the arrows (rather than their absolute magnitude) provides the most important qualitative tendency for the slope field sketch. $\diamond$

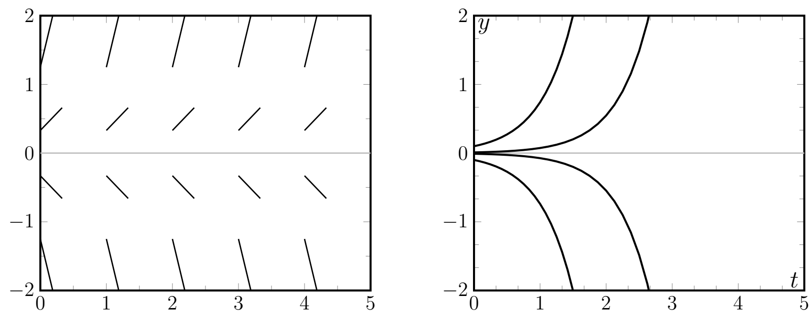



Figure 5.1: Direction field and solution curves for differential equation, $\frac{dy}{dt} = 2y$ described in Example 5.5.

In constructing the slope field and solution curves, the following basic rules should be followed:

1. By convention, time flows from left to right along the $t$ axis in our graphs, so the direction of all arrows (not usually indicated explicitly on the slope field) is always from left to right.

2. According to the differential equation, for any given value of the variable $y$, the slope is given by the expression $f(y)$ in the differential equation. The sign of that quantity is particularly important in determining whether the solution is locally increasing, decreasing, or neither. In the tables, we indicate this in the last column with the notation $\nearrow$, $\searrow$, or $\rightarrow$.

3. There is a *single* arrow at any point in the $ty$-plane, and consequently solution curves cannot intersect anywhere (although they can get arbitrarily close to one another).

We see some implications of these rules in our examples.

**Example 5.6** *Consider the differential equation*

$$\frac{dy}{dt} = f(y) = y - y^3. \tag{5.5}$$

*Create a slope field diagram for this differential equation.*

A summary of steps in creating the slope field for Example 5.6.

**Solution.** Based on the last example, we focus on the sign, rather than the value of the derivative $f(y)$, since that sign determines whether the solutions increase, decrease, or stay constant. Recall that factoring helps to find zeros, and to identify where an expression changes sign. For example,

$$\frac{dy}{dt} = f(y) = y - y^3 = y(1-y^2) = y(1+y)(1-y).$$

The sign of $f$ depends on the signs of the factors $y, (1+y), (1-y)$. For $y < -1$, two factors, $y, (1+y)$, are negative, whereas $(1-y)$ is positive, so that the product is positive overall. The sign of $f(y)$ changes at each of the three points $y = 0, \pm 1$ where one or another of the three factors changes sign,

Mastered Material Check

15. Graph the function $f(y) = y(1+y)(1-y)$ and indiate where it changes sign.
16. Repeat the process for the function $f(y) = y^2(1+y)^2(1-y)$.

as shown in Table 5.2. Eventually, to the right of all three (when $y > 1$), the sign is negative. We summarize these observations in Table 5.2 and show the slopes field and solution curves in Figure 5.2. $\diamond$

| $y$ | sign of $f(y)$ | behaviour of $y$ | direction of arrow |
|---|---|---|---|
| $y < -1$ | +ve | increasing | $\nearrow$ |
| -1 | 0 | no change | $\rightarrow$ |
| -0.5 | -ve | decreasing | $\searrow$ |
| 0 | 0 | no change | $\rightarrow$ |
| 0.5 | +ve | increasing | $\nearrow$ |
| 1 | 0 | no change | $\rightarrow$ |
| $y > 1$ | -ve | decreasing | $\searrow$ |

Table 5.2: Table for the slope field diagram of the DE (5.5) described in Example 5.6.

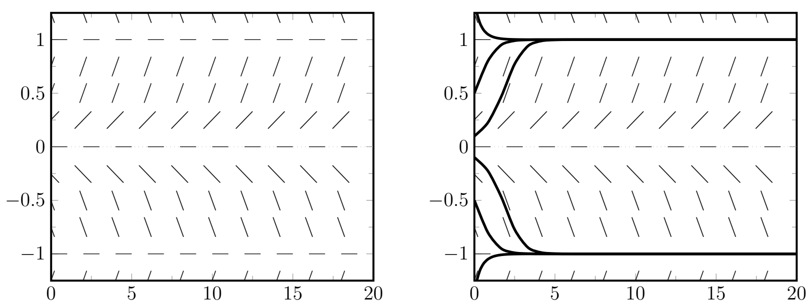



Figure 5.2: Direction field and solution curves for differential equation (5.5) described in Example 5.6.

**Example 5.7** *Sketch a slope field and solution curves for the problem of a cooling object, and specifically for*

$$\frac{dT}{dt} = f(T) = 0.2(10 - T). \tag{5.6}$$

**Solution.** The family of curves shown in Figure 5.3 (also Figure 4.6) are solutions to (5.6). The function $f(T) = 0.2(10 - T)$ corresponds to the slopes of tangent lines to these curves. We indicate the sign of $f(T)$ and thereby the behaviour of $T(t)$ in Table 5.3. Note that there is only one change of sign,

| $T$ | sign of $f(T)$ | behaviour of $T$ | direction of arrow |
|---|---|---|---|
| $T < 10$ | +ve | increasing | $\nearrow$ |
| $T = 10$ | 0 | no change | $\rightarrow$ |
| $T > 10$ | -ve | decreasing | $\searrow$ |

Table 5.3: Table for the slope field diagram of $\frac{dT}{dt} = 0.2(10 - T)$ described in Example 5.7.

at $T = 10$. For smaller $T$, the solution is always increasing and for larger $T$, the solution is always decreasing. The slope field and solution curves are shown in Figure 5.3. In the slope field, one particular value of $t$ is coloured to emphasize the associated changes in $T$, as in Table 5.3. $\diamond$

Mastered Material Check

17. Indicate the regions Figure 5.3 where $T$ is increasing.
18. Where is $T$ not changing in Figure 5.3?

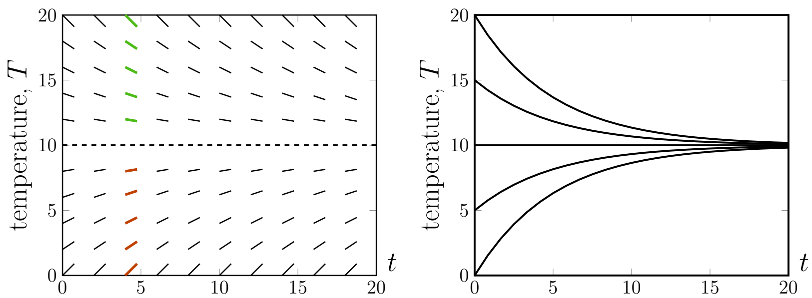



Figure 5.3: Slope field and solution curves for a cooling object that satisfies the differential equation (5.6) in Example 5.7.

We observe an agreement between the detailed solutions found analytically (Example 4.5), found using Euler's method (Example 4.13), and those sketched using the new qualitative arguments (Example 5.7).

### *State-space diagrams*

In Examples 5.5-5.7, we saw that we can understand qualitative features of solutions to the differential equation

$$\frac{dy}{dt} = f(y), \tag{5.7}$$

by examining the expression $f(y)$. We used the sign of $f(y)$ to assemble a slope field diagram and sketch solution curves. The slope field informed us about which initial values of $y$ would increase, decrease or stay constant. We next show another way of determining the same information.

First, let us define a **state space**, also called **phase line**, which is essentially the $y$-axis with superimposed arrows representing the direction of flow.

**Definition 5.2 (State space (or phase line))** *A line representing the dependent variable ($y$) together with arrows to describe the flow along that line (increasing, decreasing, or stationary $y$) satisfying Eqn.* (5.7) *is called the* **state space** *diagram or the* **phase line** *diagram for the differential equation.*

Rather than tabulating signs for $f(y)$, we can arrive at similar conclusions by sketching $f(y)$ and observing where this function is positive (implying that $y$ increases) or negative ($y$ decreases). Places where $f(y) = 0$ ("zeros of $f$") are important since these represent **steady states** ("static solutions", where there is no change in $y$). Along the $y$ axis (which is now on the horizontal axis of the sketch) increasing $y$ means motion to the right, decreasing $y$ means motion to the left.

As we shall see, the information contained in this type of diagram provides a qualitative description of solutions to the differential equation, but with the explicit time behaviour suppressed. This is illustrated by Figure 5.4, where we show the connection between the *slope field diagram* and the *state space diagram* for a typical differential equation.

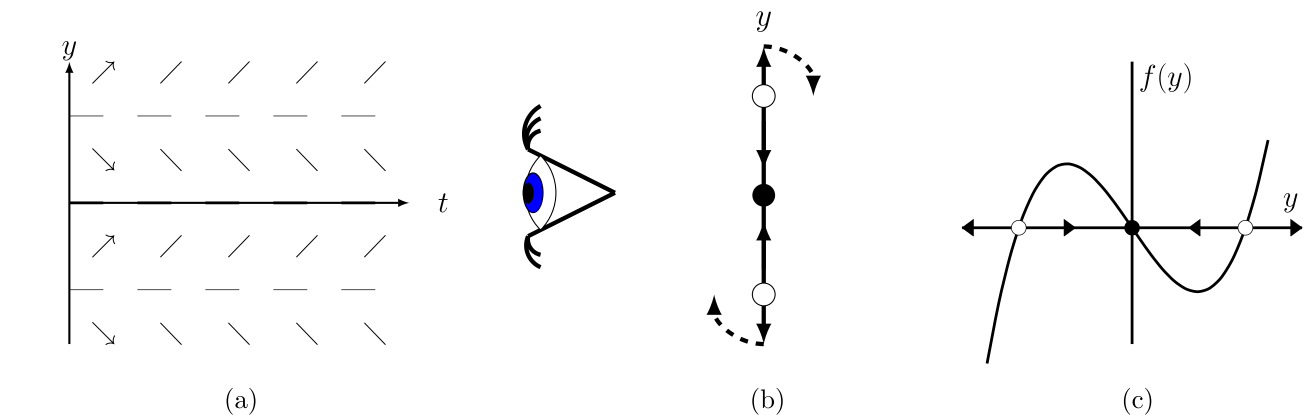



Figure 5.4: The relationship of the slope field and state space diagrams. (a) A typical slope field. A few arrows have been added to indicate the direction of time flow along the tangent vectors. Now consider "looking down the time axis" as shown by the "eye" in this diagram. Then the $t$ axis points towards us, and we see only the $y$-axis as in (b). Arrows on the $y$-axis indicate the directions of flow for various values of $y$ as determined in (a). Now "rotate" the $y$ axis so it is horizontal, as shown in (c). The direction of the arrows exactly correspond to places where $f(y)$, in (c), is *positive* (which implies increasing $y$, $\rightarrow$), or *negative* (which implies decreasing $y$, $\leftarrow$). The state space diagram is the $y$-axis in (b) or (c).

Video explanation of the steps in the solution to Example 5.8.

**Example 5.8** *Consider the differential equation*

$$\frac{dy}{dt} = f(y) = y - y^3. \tag{5.8}$$

*Sketch $f(y)$ versus $y$ and use your sketch to determine where $y$ is static, and where $y$ increases or decreases. Then describe what this predicts starting from each of the three initial conditions:*

**(i)** $y(0) = -0.5$,

**(ii)** $y(0) = 0.3$, *or*

**(iii)** $y(0) = 2$.

**Solution.** From Example 5.6, we know that $f(y) = 0$ at $y = -1, 0, 1$. This means that $y$ does not change at these steady state values, so, if we start a system off with $y(0) = 0$, or $y(0) = \pm 1$, the value of $y$ is static. The three places at which this happens are marked by heavy dots in Figure 5.5(a).

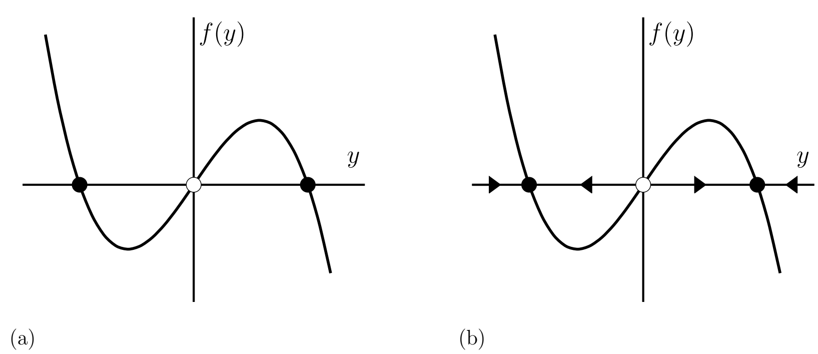



Figure 5.5: Steady states (dots) and intervals for which $y$ increases or decreases for the differential equation (5.8). See Example 5.8.

We also see that $f(y) < 0$ for $-1 < y < 0$ and for $y > 1$. In these intervals, $y(t)$ must be a decreasing function of time ($dy/dt < 0$). On the other hand, for $0 < y < 1$ or for $y < -1$, we have $f(y) > 0$, so $y(t)$ is increasing. See arrows on Figure 5.5(b). We see from this figure that there is a tendency for $y$ to move away from the steady state value $y = 0$ and to approach either of the steady states at 1 or $-1$. Starting from the initial values given above, we have

**(i)** $y(0) = -0.5$ results in $y \to -1$,

**(ii)** $y(0) = 0.3$ leads to $y \to 1$, and

**(iii)** $y(0) = 2$ implies $y \to \infty$. ◊

**Example 5.9 (A cooling object)** *Sketch the same type of diagram for the problem of a cooling object and interpret its meaning.*

**Solution.** Here, the differential equation is

$$\frac{dT}{dt} = f(T) = 0.2(10 - T). \tag{5.9}$$

A sketch of the rate of change, $f(T)$ versus the temperature $T$ is shown in Figure 5.6. We deduce the direction of the flow directly form this sketch. ◊

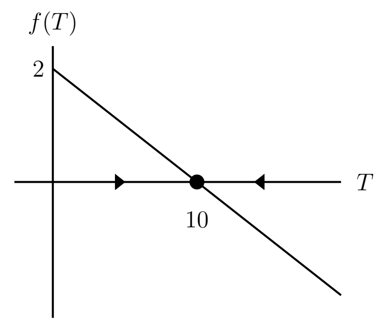



Figure 5.6: Figure for Example 5.9, the differential equation (5.9).

**Example 5.10** *Create a similar qualitative sketch for the more general form of linear differential equation*

$$\frac{dy}{dt} = f(y) = a - by. \tag{5.10}$$

*For what values of y would there be no change?*

**Solution.** The rate of change of $y$ is given by the function $f(y) = a - by$. This is shown in Figure 5.7. The steady state at which $f(y) = 0$ is at $y = a/b$. Starting from an initial condition $y(0) = a/b$, there would be no change. We also see from this figure that $y$ approaches this value over time. After a long time, the value of $y$ will be approximately $a/b$. ◊

Mastered Material Check

19. In Figures 5.6 and 5.7, where is the function positive?
20. Consider Eqn. (5.10) analytically: what value does $y$ approach?

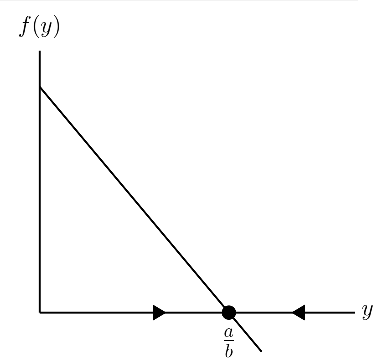



Figure 5.7: Qualitative sketch for Eqn. (5.10) in Example 5.10.

### *Steady states and stability*

From the last few figures, we observe that wherever the function $f$ on the right hand side of the differential equations crosses the horizontal axis (satisfies $f = 0$) there is a steady state. For example, in Figure 5.6 this takes place at $T = 10$. At that temperature the differential equation specifies that $dT/dt = 0$ and so, $T = 10$ is a steady state, a concept we first encountered in Chapter 4.

**Definition 5.3 (Steady state)** *A steady state is a state in which a system is not changing.*

**Example 5.11** *Identify steady states of Eqn. (5.8),*

$$\frac{dy}{dt} = y^3 - y.$$

**Solution.** Steady states are points that satisfy $f(y) = 0$. We already found those to be $y = 0$ and $y = \pm 1$ in Example 5.8. ♢

From Figure 5.5, we see that solutions starting *close to* $y = 1$ tend to get closer and closer to this value. We refer to this behaviour as **stability** of the steady state.

**Definition 5.4 (Stability)** *We say that a steady state is* **stable** *if states that are initially close enough to that steady state will get closer to it with time. We say that a steady state is* **unstable***, if states that are initially very close to it eventually move away from that steady state.*

**Example 5.12** *Determine the stability of steady states of Eqn.* (5.8)*:*

$$\frac{dy}{dt} = y - y^3.$$

**Solution.** From any starting value of $y > 0$ in this example, we see that *after a long time*, the solution curves tend to approach the value $y = 1$. States close to $y = 1$ get closer to it, so this is a stable steady state. For the steady state $y = 0$, we see that initial conditions near $y = 0$ move away over time. Thus, this steady state is unstable. Similarly, the steady state at $y = -1$ is stable. In Figure 5.5 we show the stable steady states with black dots and the unstable steady state with an open dot. ♢

Mastered Material Check

21. In the state space diagram in Figure 5.4, identify the stable steady states.

## 5.3 Applying qualitative analysis to biological models

**Section 5.3 Learning goals**

1. Practice the techniques of slope field, state-space diagram, and steady state analysis on the logistic equation.
2. Explain the derivation of a model for interacting (healthy, infected) individuals based on a set of assumptions.
3. Identify that the resulting set of two ODEs can be reduced to a single ODE. Use qualitative methods to analyse the model behaviour and to interpret the results.

The qualitative ideas developed so far will now be applied to to problems from biology. In the following sections we first use these methods to obtain a thorough understanding of **logistic population growth**. We then derive a model for the spread of a disease, and use qualitative arguments to analyze the predictions of that differential equation model.

### *Qualitative analysis of the logistic equation*

We apply the new methods to the logistic equation.

**Example 5.13** *Find the steady states of the logistic equation, Eqn. (5.1):*

$$\frac{dN}{dt} = rN\frac{(K-N)}{K}.$$

**Solution.** To determine the steady states of Eqn. (5.1), i.e. the level of population that would not change over time, we look for values of $N$ such that

$$\frac{dN}{dt} = 0.$$

This leads to

$$rN\frac{(K-N)}{K} = 0,$$

which has solutions $N = 0$ (no population at all) or $N = K$ (the population is at its carrying capacity). ♢

We could similarly find steady states of the scaled form of the logistic equation, Eqn. (5.3). Setting $dy/dt = 0$ leads to

$$0 = \frac{dy}{dt} = ry(1-y) \quad \Rightarrow \quad y = 0, \text{ or } y = 1.$$

This comes as no surprise since these values of $y$ correspond to the values $N = 0$ and $N = K$.

The scaled logistic equation, its slope field, and steady state values are discussed here.

A second way to analyze the scaled logistic equation, using the phase line approach, and its connection to the slope field method as described in Example 5.14.

**Example 5.14** *Draw a plot of the rate of change $dy/dt$ versus the value of $y$ for the scaled logistic equation,Eqn. (5.3):*

$$\frac{dy}{dt} = ry(1-y).$$

**Solution.** In the plot of Figure 5.8 only $y \geq 0$ is relevant. In the interval $0 < y < 1$, the rate of change is positive, so that $y$ increases, whereas for $y > 1$, the rate of change is negative, so $y$ decreases. Since $y$ refers to population size, we need not concern ourselves with behaviour for $y < 0$.

From Figure 5.8 we deduce that solutions that start with a positive $y$ value approach $y = 1$ with time. Solutions starting at either steady state $y = 0$ or $y = 1$ would not change. Restated in terms of the variable $N(t)$, any initial population should approach its carrying capacity $K$ with time. ♢

We now look at the same equation from the perspective of the slope field.

Mastered Material Check

22. Circle the steady states in Figure 5.8 and identify which one is stable.
23. Why is $y < 0$ not relevant in Example 5.14?

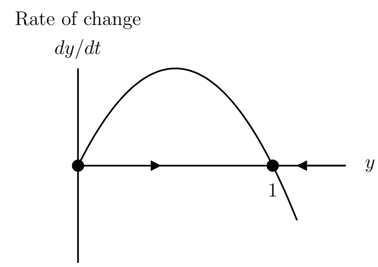



Figure 5.8: Plot of $dy/dt$ versus $y$ for the the scaled logistic equation (5.3).

**Example 5.15** *Draw a slope field for the scaled logistic equation with $r = 0.5$, that is for*

$$\frac{dy}{dt} = f(y) = 0.5 \cdot y(1-y). \tag{5.11}$$

**Solution.** We generate slopes for various values of $y$ in Table 5.4 and plot the slope field in Figure 5.9(a). ♢

Finally, we practice Euler's method to graph the numerical solution to Eqn. (5.11) from several initial conditions.

**Example 5.16 (Numerical solutions to the logistic equation)** *Use Euler's method to approximate the solutions to the logistic equation (5.11).*

| $y$ | sign of $f(y)$ | behaviour of $y$ | direction of arrow |
|---|---|---|---|
| 0 | 0 | no change | $\rightarrow$ |
| $0 < y < 1$ | +ve | increasing | $\nearrow$ |
| 1 | 0 | no change | $\rightarrow$ |
| $y > 1$ | -ve | decreasing | $\searrow$ |

Table 5.4: Table for slope field for the logistic equation (5.11). See Fig 5.9(a) for the resulting diagram.

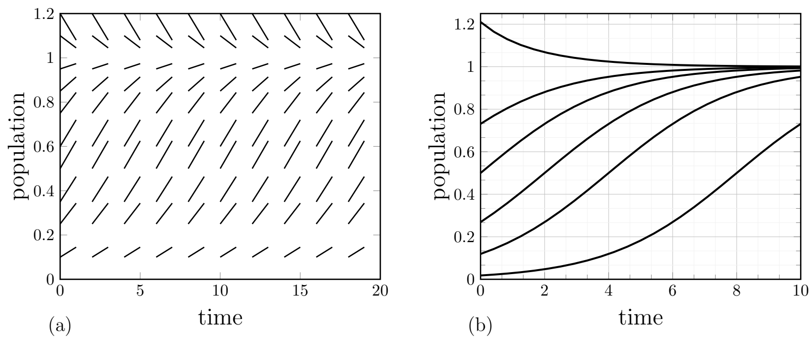



Figure 5.9: (a) Slope field and (b) solution curves for the logistic equation (5.11), $\frac{dy}{dt} = 0.5 \cdot y(1-y)$

Mastered Material Check

24. What initial values $y_0$ were used in drawing the different solution curves depicted in Figure 5.9(b)?

**Solution.** In Figure 5.9(b) we show a set of solution curves, obtained by solving the equation numerically using Euler's method. To obtain these solutions, a value of $h = \Delta t = 0.1$ was used. The solution is plotted for various initial conditions $y(0) = y_0$. The successive values of $y$ were calculated according to

$$y_{k+1} = y_k + 0.5y_k(1-y_k)h, \qquad k = 0, \ldots 100.$$

From Figure 5.9(b), we see that solution curves approach the steady state $y = 1$, meaning that the population $N(t)$ approaches the carrying capacity $K$ for all positive starting values. A link to the spreadsheet that implements Euler's method is included. $\Diamond$

Link to Google Sheets. This spreadsheet implements Euler's method for Example 5.16. A chart showing solutions from four initial conditions is included.

**Example 5.17 (Inflection points)** *Some of the curves shown in Figure 5.9(b) have an inflection point, but others do not. Use the differential equation to determine which of the solution curves have an inflection point.*

**Solution.** We have already established that all initial values in the range $0 < y_0 < 1$ are associated with increasing solutions $y(t)$. Now we consider the concavity of those solutions. The logistic equation has the form

$$\frac{dy}{dt} = ry(1-y) = ry - ry^2$$

Differentiate both sides using the chain rule and factor, to get

$$\frac{d^2y}{dt^2} = r\frac{dy}{dt} - 2ry\frac{dy}{dt} = r\frac{dy}{dt}(1-2y).$$

Mastered Material Check

25. How do we know that initial conditions in the range $0 < y_0 < 1$ lead to increasing solutions?

An inflection point would occur at places where the second derivative changes sign. This is possible for $dy/dt = 0$ or for $(1-2y) = 0$. We have already dismissed the first possibility because we argued that the rate of change is nonzero in the interval of interest. Thus we conclude that an inflection point would occur whenever $y = 1/2$. Any initial condition satisfying $0 < y_0 < 1/2$ would eventually pass through $y = 1/2$ on its way to the steady state level at $y = 1$, and in so doing, would have an inflection point. $\diamond$

### *A changing aphid population*

In Chapters 1 and **??**, we investigated a situation when predation and growth rates of an aphid population exactly balanced. But what happens if these two rates do not balance? We are now ready to tackle this question.

**Featured Problem 5.1 (aphids)** *Consider the aphid-ladybug problem (Example 1.3) with aphid density x, growth rate $G(x) = rx$, and predation rate by a ladybug $P(x)$ as in (1.10). (a) Write down a differential equation for the aphid population. (b) Use your equation, and a sketch of the two functions to answer the following question: What happens to the aphid population starting from various initial population sizes?*

Hint: Growth rate (number of aphids born per unit time) contributes positively, whereas predation rate (number of aphids eaten per unit time) contributes negatively to the rate of change of aphids with respect to time ($dx/dt$).

### *5.3.1 The radius of a growing cell*

In Section 3.4 we examined a cell in which nutrient absorption and consumption each contribute to changing the mass balance of the cell. We first wrote down a differential equation of the form

$$\frac{dm}{dt} = A - C.$$

Assuming the cell was spherical, we showed that this equation results in the differential equation for the cell radius $r(t)$:

$$\frac{dr}{dt} = \frac{1}{\rho}\left(k_1 - \frac{k_2}{3}r\right), \quad k_1, k_2, \rho > 0 \tag{5.12}$$

Using tools in this chapter, we can now understand what this implies about cell size growth.

**Featured Problem 5.2 (How cell radius changes)** *Apply qualitative methods to Eqn. (5.12) so as to determine what happens to cells starting from various initial sizes. Is there a steady state cell size? How do your results compare to our findings in Section 1.2?*

### *A model for the spread of a disease*

In the era of human immunodeficiency virus (HIV), Severe Acute Respiratory Syndrome (SARS), Avian influenza ("bird flu") and similar emerging infectious diseases, it is prudent to consider how infection spreads, and how it could be controlled or suppressed. This motivates the following example.

For a given disease, let us subdivide the population into two classes: healthy individuals who are susceptible to catching the infection, and those that are currently infected and able to transmit the infection to others. We consider an infection that is mild enough that individuals recover at some constant rate, and that they become susceptible once recovered.

*Note:* usually, recovery from an illness leads to partial temporary immunity. While this, too, can be modelled, we restrict attention to the simpler case which is tractable using mathematics we have just introduced.

The simplest case to understand is that of a fixed population (with no birth, death or migration during the timescale of interest). A goal is to predict whether the infection spreads and persists (becomes endemic) in the population or whether it runs its course and disappear. We use the following notation:

A video summary of the model for the spread of a disease, together with its analysis.

$$\begin{aligned} S(t) &= \text{size of population of susceptible (healthy) individuals,} \\ I(t) &= \text{size of population of infected individuals,} \\ N(t) &= S(t) + I(t) = \text{total population size.} \end{aligned}$$

We add a few simplifying assumptions.

1. The population mixes very well, so each individual is equally likely to contact and interact with any other individual. The contact is random.
2. Other than the state ($S$ or $I$), individuals are "identical," with the same rates of recovery and infectivity.
3. On the timescale of interest, there is no birth, death or migration, only exchange between $S$ and $I$.

**Example 5.18** *Suppose that the process can be represented by the scheme*

$$\begin{aligned} S + I &\to I + I, \\ I &\to S \end{aligned}$$

*The first part, transmission of disease from I to S involves interaction. The second part is recovery. Use the assumptions above to track the two populations and to formulate a set of differential equations for $I(t)$ and $S(t)$.*

**Solution.** The following balance equations keeps track of individuals

$$\begin{bmatrix} \text{Rate of} \\ \text{change of} \\ I(t) \end{bmatrix} = \begin{bmatrix} \text{Rate of gain} \\ \text{due to disease} \\ \text{transmission} \end{bmatrix} - \begin{bmatrix} \text{Rate of loss} \\ \text{due to} \\ \text{recovery} \end{bmatrix}$$

According to our assumption, recovery takes place at a constant rate per unit time, denoted by $\mu > 0$ . By the law of mass action, the disease transmission rate should be proportional to the product of the populations, $(S \cdot I)$. Assigning $\beta > 0$ to be the constant of proportionality leads to the following differential equations for the infected population:

$$\frac{dI}{dt} = \beta SI - \mu I.$$

Similarly, we can write a balance equation that tracks the population of susceptible individuals:

$$\begin{bmatrix} \text{Rate of} \\ \text{change of} \\ S(t) \end{bmatrix} = - \begin{bmatrix} \text{Rate of Loss} \\ \text{due to disease} \\ \text{transmission} \end{bmatrix} + \begin{bmatrix} \text{Rate of gain} \\ \text{due to} \\ \text{recovery} \end{bmatrix}$$

Observe that loss from one group leads to (exactly balanced) gain in the other group. By similar logic, the differential equation for $S(t)$ is then

$$\frac{dS}{dt} = -\beta SI + \mu I.$$

We have arrived at a **system of equations** that describe the changes in each of the groups,

$$\frac{dI}{dt} = \beta SI - \mu I, \tag{5.13a}$$
$$\frac{dS}{dt} = -\beta SI + \mu I. \tag{5.13b}$$

◊

Mastered Material Check

26. Identify any constants in Eqns. (5.13)(a) and (b).
27. What are the units of those constants?
28. Why does the hint given in Example 5.19 help?

From Eqns. (5.13) it is clear that changes in one population depend on both, which means that the differential equations are **coupled** (linked to one another). Hence, we cannot "solve one" independently of the other. We must treat them as a pair. However, as we observe in the next examples, we can simplify this system of equations using the fact that the total population does not change.

**Example 5.19** *Use Eqns.(5.13) to show that the total population does not change (*hint: *show that the derivative of $S(t) + I(t)$ is zero).*

Video showing that the population $N(t) = I(t) + S(t)$ is constant.

**Solution.** Add the equations to one another. Then we obtain

$$\frac{d}{dt}[I(t) + S(t)] = \frac{dI}{dt} + \frac{dS}{dt} = \beta SI - \mu I - \beta SI + \mu I = 0.$$

Hence

$$\frac{d}{dt}[I(t) + S(t)] = \frac{dN}{dt} = 0,$$

which mean that $N(t) = [I(t) + S(t)] = N$=constant, so the total population does not change. (In Eqn. (5.1), here $N$ is a constant and $I(t), S(t)$ are the variables.) ◊

**Example 5.20** *Use the fact that $N$ is constant to express $S(t)$ in terms of $I(t)$ and $N$, and eliminate $S(t)$ from the differential equation for $I(t)$. Your equation should only contain the constants $N$, $\beta$, $\mu$.*

**Solution.** Since $N = S(t) + I(t)$ is constant, we can write $S(t) = N - I(t)$. Then, plugging this into the differential equation for $I(t)$ we obtain

$$\frac{dI}{dt} = \beta SI - \mu I, \quad \Rightarrow \quad \frac{dI}{dt} = \beta(N - I)I - \mu I.$$

◊

Mastered Material Check

29. Redo Example 5.20 but eliminate $I(t)$ instead of $S(t)$ .
30. Analyze the equation you get for $dS(t)/dt$ as done for $dI/dt$ in Example 5.21.

**Example 5.21** **a)** *Show that the above equation can be written in the form*

$$\frac{dI}{dt} = \beta I(K - I),$$

*where $K$ is a constant.*

**b)** *Determine how this constant $K$ depends on $N, \beta$, and $\mu$.*

**c)** *Is the constant $K$ positive or negative?*

**Solution.**

**a)** We rewrite the differential equation for $I(t)$ as follows:

$$\frac{dI}{dt} = \beta(N - I)I - \mu I = \beta I\left((N - I) - \frac{\mu}{\beta}\right) = \beta I\left(N - \frac{\mu}{\beta} - I\right).$$

**b)** We identify the constant,

$$K = \left(N - \frac{\mu}{\beta}\right).$$

**c)** Evidently, $K$ could be *either positive or negative*, that is

$$\begin{cases} N \geq \frac{\mu}{\beta} & \Rightarrow K \geq 0, \\ N < \frac{\mu}{\beta} & \Rightarrow K < 0. \end{cases}$$

◊

Using the above process, we have reduced the system of two differential equations for the two variables $I(t), S(t)$ to a *single* differential equation for $I(t)$, together with the statement $S(t) = N - I(t)$. We now examine implications of this result using the qualitative methods of this chapter.

**Example 5.22** *Consider the differential equation for $I(t)$ given by*

$$\frac{dI}{dt} \equiv f(I) = \beta I(K - I), \quad \textit{where} \quad K = \left(N - \frac{\mu}{\beta}\right). \tag{5.14}$$

*Find the steady states of the differential equation* (5.14) *and draw a state space diagram in each of the following cases:*

**(a)** $K \geq 0$,

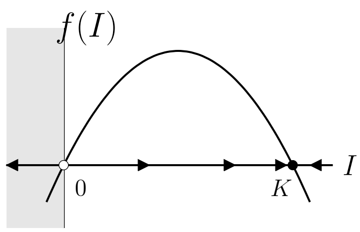
**(b)** $K < 0$.

*Use your diagram to determine which steady state(s) are stable or unstable.*

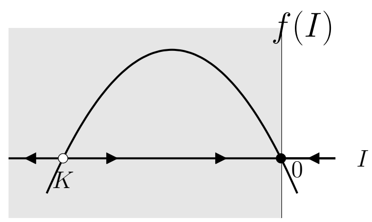
**Solution.** Steady states of Eqn. (5.14) satisfy $dI/dt = \beta I(K-I) = 0$. Hence, these steady states are $I = 0$ (no infected individuals) and $I = K$. The latter only makes sense if $K \geq 0$. We plot the function $f(I) = \beta I(K-I)$ in Eqn. (5.14) against the state variable $I$ in Figure 5.10 (a) for $K \geq 0$ and (b) for $K < 0$. Since $f(I)$ is quadratic in $I$, its graph is a parabola and it opens downwards. We add arrows pointing right ($\rightarrow$) in the regions where $dI/dt > 0$ and arrows pointing left ($\leftarrow$) where $dI/dt < 0$.

Mastered Material Check

31. What is the significance of the grey shaded regions in Fig. 5.10.
32. Draw Fig. 5.10 for $K = 0$.
33. Why is $I = K$ not an admissible steady state if $K < 0$?




(a)




(b)

Figure 5.10: State-space diagrams for differential equation (5.14). Plots of $f(I)$ as a function of $I$ in the cases (a) $K \geq 0$, and (b) $K < 0$. The grey regions are not biologically meaningful since $I$ cannot be negative.

In case (a), when $K \geq 0$, we find that arrows point toward $I = K$, so this steady state is stable. Arrows point away from $I = 0$, so this represents an unstable steady state. In case (b), while we still have a parabolic graph with two steady states, the state $I = K$ is not admissible since $K$ is negative. Hence only one steady state, at $I = 0$ is relevant biologically, and all initial conditions move towards this state. ◊

**Example 5.23** *Interpret the results of the model in terms of the disease, assuming that initially most of the population is in the susceptible S group, and a small number of infected individuals are present at* $t = 0$.

**Solution.** In case (a), as long as the initial size of the infected group is positive ($I > 0$), with time it approaches $K$, that is, $I(t) \rightarrow K = N - \mu/\beta$. The rest of the population is in the susceptible group, that is $S(t) \rightarrow \mu/\beta$ (so that $S(t) + I(t) = N$ is always constant.) This first scenario holds provided $K > 0$ which is equivalent to $N > \mu/\beta$. There are then some infected and some healthy individuals in the population indefinitely, according to the model. In this case, we say that the disease becomes **endemic**.

In case (b), which corresponds to $N < \mu/\beta$, we see that $I(t) \rightarrow 0$ regardless of the initial size of the infected group. In that case, $S(t) \rightarrow N$ so with time, the infected group shrinks and the healthy group grows so that the whole population becomes healthy. From these two results, we conclude that the disease is wiped out in a small population, whereas in a sufficiently large population, it can spread until a steady state is attained where some fraction of the population is always infected. In fact we have identified a *threshold* that separates these two behaviours:

Mastered Material Check

34. In the case that $\beta = 0.001$per person per day and $\mu = 0.1$ per day, how large would the population have to be for the disease to become endemic?
35. Frequent hand-washing can be a protective measure that decreases the spread of disease. Which parameter of the model would this affect and in what way?

$$\frac{N\beta}{\mu} > 1 \quad \Rightarrow \quad \text{disease becomes endemic,}$$
$$\frac{N\beta}{\mu} < 1 \quad \Rightarrow \quad \text{disease is wiped out.}$$

The ratio of constants in these inequalities, $R_0 = N\beta/\mu$ is called the **basic reproduction number** for the disease. Many current and much more detailed models for disease transmission also have such threshold behaviour, and the ratio that determines whether the disease spreads or disappears, $R_0$ is of great interest in vaccination strategies. This ratio represents the number of infections that arise when 1 infected individual interacts with a population of $N$ susceptible individuals.

A video summarizing the interpretation of the model and the meaning of the constant $R_0 = N\beta/\mu$.

## 5.4 Summary

1. A differential equation of the form $\alpha\frac{dy}{dt} + \beta y + \gamma = 0$ is linear (and "first order"). We encountered several examples of nonlinear DEs in this chapter.
2. A (possibly nonlinear) differential equation $\frac{dy}{dt} = f(y)$ can be analyzed qualitatively by observing where $f(y)$ is positive, negative or zero.
3. A slope field (or "direction field") is a collection of tangent vectors for solutions to a differential equation. Slope fields can be sketched from $f(y)$ without the need to solve the differential equation.
4. A solution curve drawn in a slope field corresponds to a single solution to a differential equation, with some initial $y_0$ value given.
5. A state space (or "phase line" diagram) for the differential equation is a $y$ axis, together with arrows describing the flow (increasing/decreasing/stationary) along that axis. It can be obtained from a sketch of $f(y)$.
6. A steady state is stable if nearby states get closer. A steady state is unstable if nearby states get further away with time.
7. Creating/interpreting slope field and state space diagrams is helpful in understanding the behavior of solutions to differential equations.
8. Applications considered in this chapter included:

   (a) the logistic equations for population growth (a nonlinear differential equation, scaling, steady state and slope field demonstration);
   (b) the Law of Mass Action (a nonlinear differential equation);
   (c) a cooling object (state space and phase line diagram demonstration); and
   (d) disease spread model (an extensive exposition on qualitative differential equation methods).

**Quick Concept Checks**

1. Why is it helpful to rescale an equation?

2. Identify which of the following differential equations are linear:

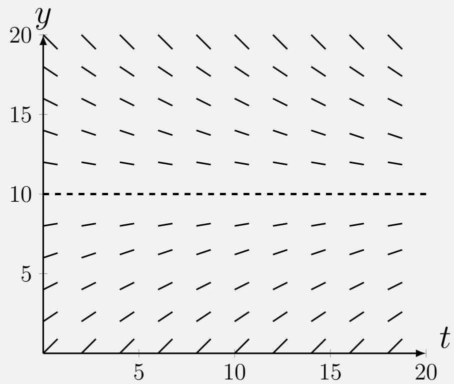
   (a) $5\frac{dy}{dt} - y = -0.5$

   (b) $\left(\frac{dy}{dt}\right)^2 + y + 1 = 0$

   (c) $\frac{dy}{dx} + \pi y + \rho = 3$

   (d) $\frac{dx}{dt} + x + 2 = -3x$

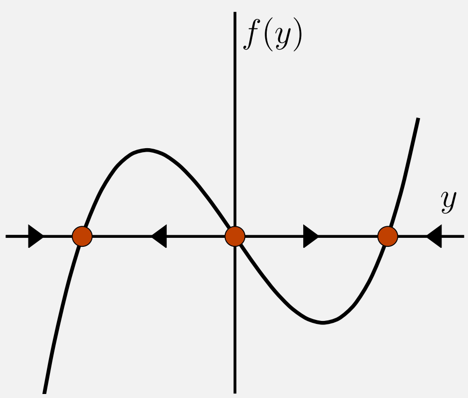
3. Consider the following slope field:




   (a) Where is $y$ decreasing?

   (b) What is $y$ approaching?

4. Circle the **stable** steady states in the following state space diagram



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*Exercises*

5.1. **Explaining connections.** Explain the connection between Eqn. (5.2) and the equations Eqn. 3.2 and Eqn. (4.4).

5.2. **Slope fields.** Consider the differential equations given below. In each case, draw a slope field, determine the values of $y$ for which no change takes place - such values are called steady states - and use your slope field to predict what would happen starting from an initial value $y(0) = 1$.

(a) $\dfrac{dy}{dt} = -0.5y$

(b) $\dfrac{dy}{dt} = 0.5y(2-y)$

(c) $\dfrac{dy}{dt} = y(2-y)(3-y)$

5.3. **Drawing slope fields.** Draw a slope field for each of the given differential equations:

(a) $\frac{dy}{dt} = 2+3y$

(b) $\frac{dy}{dt} = -y(2-y)$

(c) $\frac{dy}{dt} = 2-3y+y^2$

(d) $\frac{dy}{dt} = -2(3-y)^2$

(e) $\frac{dy}{dt} = y^2-y+1$

(f) $\frac{dy}{dt} = y^3-y$

(g) $\frac{dy}{dt} = \sqrt{y}(y-2)(y-3)^2, y \geq 0.$

5.4. **Linear or Nonlinear.** Identify which of the differential equations in Exercise 2 and 3 is linear and which nonlinear.

5.5. **Using slope fields.** For each of the differential equations (a) to (g) in Exercise 3, plot $\frac{dy}{dt}$ as a function of $y$, draw the motion along the $y$-axis, identify the steady state(s) and indicate if the motions are toward or away from the steady state(s).

5.6. **Direction field.** The direction field shown in the figure below corresponds to which differential equation?

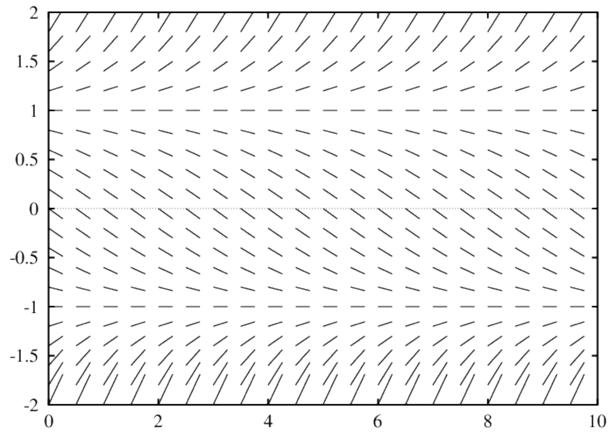
(A) $\frac{dy}{dt} = ry(y+1)$

(B) $\frac{dy}{dt} = r(y-1)(y+1)$

(C) $\frac{dy}{dt} = -r(y-1)(y+1)$

(D) $\frac{dy}{dt} = ry(y-1)$

(E) $\frac{dy}{dt} = -ry(y+1)$




5.7. **Differential equation.** Given the differential equation and initial condition

$$\frac{dy}{dt} = y^2(y-a), y(0) = 2a$$

where $a > 0$ is a constant, the value of the function $y(t)$ would

(A) approach $y = 0$;

(B) grow larger with time;

(C) approach $y = a$;

(D) stay the same;

(E) none of the above.

5.8. **There's a hole in the bucket.** Water flows into a bucket at constant rate $I$. There is a hole in the container. Explain the model

$$\frac{dh}{dt} = I - k\sqrt{h}.$$

Analyze the behaviour predicted. What would the height be after a long time? Is this result always valid, or is an additional assumption needed? (*hint:* recall Example 4.3.)

5.9. **Cubical crystal.** A crystal grows inside a medium in a cubical shape with side length $x$ and volume $V$. The rate of change of the volume is given by

$$\frac{dV}{dt} = kx^2(V_0 - V)$$

where $k$ and $V_0$ are positive constants.

(a) Rewrite this as a differential equation for $\frac{dx}{dt}$.

(b) Suppose that the crystal grows from a very small "seed." Show that its growth rate continually decreases.

(c) What happens to the size of the crystal after a very long time?

(d) What is its size (that is, what is either $x$ or $V$) when it is growing at half its initial rate?

5.10. **The Law of Mass Action.** The Law of Mass Action in Section 5.1 led to the assumption that the rate of a reaction involving two types of molecules (A and B) is proportional to the product of their concentrations, $k \cdot a \cdot b$.

Explain why the sum of the concentrations, $k \cdot (a+b)$ would not make for a sensible assumption about the rate of the reaction.

5.11. **Biochemical reaction.** A biochemical reaction in which a substance $S$ is both produced and consumed is investigated. The concentration $c(t)$ of $S$ changes during the reaction, and is seen to follow the differential equation

$$\frac{dc}{dt} = K_{\max} \frac{c}{k+c} - rc$$

where $K_{\max}, k, r$ are positive constants with certain convenient units. The first term is a concentration-dependent production term and the second term represents consumption of the substance.

(a) What is the maximal rate at which the substance is produced? At what concentration is the production rate 50% of this maximal value?

(b) If the production is turned off, the substance decays. How long would it take for the concentration to drop by 50%?

(c) At what concentration does the production rate just balance the consumption rate?

5.12. **Logistic growth with proportional harvesting.** Consider a fish population of density $N(t)$ growing at rate $g(N)$, with harvesting, so that the population satisfies the differential equation

$$\frac{dN}{dt} = g(N) - h(N).$$

Now assume that the growth rate is logistic, so $g(N) = rN\frac{(K-N)}{K}$ where $r, K > 0$ are constant. Assume that the rate of harvesting is proportional to the population size, so that

$$h(N) = qEN$$

where $E$, the effort of the fishermen, and $q$, the catchability of this type of fish, are positive constants.

Use qualitative methods discussed in this chapter to analyze the behaviour of this equation. Under what conditions does this lead to a sustainable fishery?

5.13. **Logistic growth with constant number harvesting.** Consider the same fish population as in Exercise 12, but this time assume that the rate of harvesting is fixed, regardless of the population size, so that

$$h(N) = H$$

where $H$ is a constant number of fish being caught and removed per unit time. Analyze this revised model and compare it to the previous results.

5.14. **Scaling time in the logistic equation.** Consider the scaled logistic equation (5.3). Recall that $r$ has units of 1/time, so $1/r$ is a quantity with units of time. Now consider scaling the time variable in (5.3) by defining $t = s/r$. Then $s$ carries no units ($s$ is "dimensionless").

Substitute this expression for $t$ in (5.3) and find the differential equation so obtained (for $dy/ds$).

5.15. **Euler's method applied to logistic growth.** Consider the logistic differential equation

$$\frac{dy}{dt} = ry(1-y).$$

Let $r = 1$. Use Euler's method to find a solution to this differential equation starting with $y(0) = 0.5$, and step size $h = 0.2$. Find the values of $y$ up to time $t = 1.0$.

5.16. **Spread of infection.** In the model for the spread of a disease, we used the fact that the total population is constant ($S(t) + I(t) = N =$constant) to eliminate $S(t)$ and analyze a differential equation for $I(t)$ on its own.

Carry out a similar analysis, but eliminate $I(t)$. Then analyze the differential equation you get for $S(t)$ to find its steady states and behaviour, practicing the qualitative analysis discussed in this chapter.

5.17. **Vaccination strategy.** When an individual is vaccinated, he or she is "removed" from the susceptible population, effectively reducing the size of the population that can participate in the disease transmission. For example, if a fraction $\phi$ of the population is vaccinated, then only the remaining $(1-\phi)N$ individuals can be either susceptible or infected, so $S(t) + I(t) = (1-\phi)N$. When smallpox was an endemic disease, it had a basic reproductive number of $R_0 = 7$.

What fraction of the population would have had to be vaccinated to eradicate this disease?

5.18. **Social media.** Sally Sweetstone has invented a new social media App called HeadSpace, which instantly matches compatible mates according to their changing tastes and styles. Users hear about the App from one another by word of mouth and sign up for an account. The account expires randomly, with a half-life of 1 month. Suppose $y_1(t)$ are the number of individuals who are not subscribers and $y_2(t)$ are the number of are subscribers at time $t$. The following model has been suggested for the evolving subscriber population

$$\frac{dy_1}{dt} = by_2 - ay_1y_2,$$
$$\frac{dy_2}{dt} = ay_1y_2 - by_2.$$

(a) Explain the terms in the equation. What is the value of the constant $b$?

(b) Show that the total population $P = y_1(t) + y_2(t)$ is constant.

*Note:* this is a **conservation statement**.

(c) Use the conservation statement to eliminate $y_1$. Then analyze the differential equation you obtain for $y_2$.

(d) Use your model to determine whether this newly launched social media will be successful or whether it will go extinct.

5.19. **A bimolecular reaction.** Two molecules of $A$ can react to form a new chemical, $B$. The reaction is **reversible** so that $B$ also continually decays back into 2 molecules of $A$. The differential equation model proposed for this system is

$$\frac{da}{dt} = -\mu a^2 + 2\beta b$$
$$\frac{db}{dt} = \frac{\mu}{2}a^2 - \beta b,$$

where $a(t), b(t) > 0$ are the concentrations of the two chemicals.

(a) Explain the factor 2 that appears in the differential equations and the conservation statement. Show that the total mass $M = a(t) + 2b(t)$ is constant.

(b) Use the techniques in this chapter to investigate what happens in this chemical reaction, to find any steady states, and to explain the behaviour of the system

# 6

# Areas, volumes and simple sums

## 6.1 Introduction to integral calculus

This chapter has several aims. First, we concentrate here a number of basic formulae for areas and volumes that are used later in developing the notions of integral calculus. Among these are areas of simple geometric shapes and formulae for sums of certain common sequences. An important idea is introduced, namely that we can use the sum of areas of elementary shapes to approximate the areas of more complicated objects, and that the approximation can be made more accurate by a process of refinement.

We show using examples how such ideas can be used in calculating the volumes or areas of more complex objects. In particular, we conclude with a detailed exploration of the structure of branched airways in the lung as an application of ideas in this chapter.

## 6.2 Areas of simple shapes

One of the main goals in this course will be calculating areas enclosed by curves in the plane and volumes of three dimensional shapes. We will find that the tools of calculus will provide important and powerful techniques for meeting this goal. Some shapes are simple enough that no elaborate techniques are needed to compute their areas (or volumes). We briefly survey some of these simple geometric shapes and list what we know or can easily determine about their area or volume.

The areas of simple geometrical objects, such as rectangles, parallelograms, triangles, and circles are given by elementary formulae. Indeed, our ability to compute areas and volumes of more elaborate geometrical objects will rest on some of these simple formulae, summarized below.

**Rectangular areas**

Most integration techniques discussed in this course are based on the idea of carving up irregular shapes into rectangular strips. Thus, areas of rectangles will play an important part in those methods.

- The area of a rectangle with base $b$ and height $h$ is

$$A = b \cdot h$$

- Any parallelogram with height $h$ and base $b$ also has area, $A = b \cdot h$. See Figure 6.1(a) and (b).

**Areas of triangular shapes**

A few illustrative examples in this chapter will be based on dissecting shapes (such as regular polygons) into triangles. The areas of triangles are easy to compute, and we summarize this review material below. However, triangles will play a less important role in subsequent integration methods.

- The area of a triangle can be obtained by slicing a rectangle or parallelogram in half, as shown in Figure 6.1(c) and (d). Thus, any triangle with base $b$ and height $h$ has area

$$A = \frac{1}{2}bh.$$

- In some cases, the height of a triangle is not given, but can be determined from other information provided. For example, if the triangle has sides of length $b$ and $r$ with enclosed angle $\theta$, as shown on Figure 6.1(e) then its height is simply $h = r\sin(\theta)$, and its area is

$$A = (1/2)br\sin(\theta)$$

- If the triangle is isosceles, with two sides of equal length, $r$, and base of length $b$, as in Figure 6.1(f) then its height can be obtained from Pythagoras's theorem, i.e. $h^2 = r^2 - (b/2)^2$ so that the area of the triangle is

$$A = (1/2)b\sqrt{r^2 - (b/2)^2}.$$

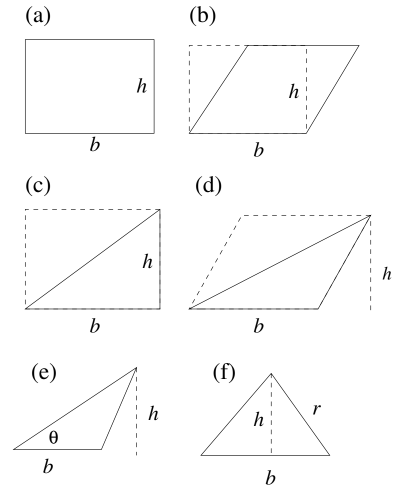


Figure 6.1: Planar regions whose areas are given by elementary formulae.

### *Finding the area of a polygon using triangles: a "dissection" method*

Using the simple ideas reviewed so far, we can determine the areas of more complex geometric shapes. For example, let us compute the area of a regular polygon with $n$ equal sides, where the length of each side is $b = 1$. This example illustrates how a complex shape (the polygon) can be dissected into simpler shapes, namely triangles.

The polygon has $n$ sides, each of length $b = 1$. We dissect the polygon into $n$ isosceles triangles, as shown in Figure 6.2. We do not know the heights of these triangles, but the angle $\theta$ can be found. It is

$$\theta = 2\pi/n$$

since together, $n$ of these identical angles make up a total of 360° or $2\pi$ radians.

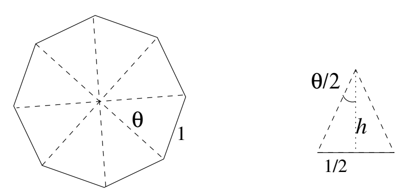


Figure 6.2: An equilateral $n$-sided polygon with sides of unit length can be dissected into $n$ triangles. One of these triangles is shown at right. Since it can be further divided into two Pythagorean triangles, trigonometric relations can be used to find the height $h$ in terms of the length of the base $1/2$ and the angle $\theta/2$.

Let $h$ stand for the height of one of the triangles in the dissected polygon. Then trigonometric relations relate the height to the base length as follows:

$$\frac{\text{opp}}{\text{adj}} = \frac{b/2}{h} = \tan(\theta/2)$$

Using the fact that $\theta = 2\pi/n$, and rearranging the above expression, we get

$$h = \frac{b}{2\tan(\pi/n)}$$

Thus, the area of each of the $n$ triangles is

$$A = \frac{1}{2}bh = \frac{1}{2}b\left(\frac{b}{2\tan(\pi/n)}\right).$$

The statement of the problem specifies that $b = 1$, so

$$A = \frac{1}{2}\left(\frac{1}{2\tan(\pi/n)}\right).$$

The area of the entire polygon is then $n$ times this, namely

$$A_{\text{n-gon}} = \frac{n}{4\tan(\pi/n)}.$$

For example, the area of a square (a polygon with 4 equal sides, $n = 4$) is

$$A_{\text{square}} = \frac{4}{4\tan(\pi/4)} = \frac{1}{\tan(\pi/4)} = 1,$$

where we have used the fact that $\tan(\pi/4) = 1$.

As a second example, the area of a hexagon (6 sided polygon, i.e. $n = 6$) is

$$A_{\text{hexagon}} = \frac{6}{4\tan(\pi/6)} = \frac{3}{2(1/\sqrt{3})} = \frac{3\sqrt{3}}{2}.$$

Here we used the fact that $\tan(\pi/6) = 1/\sqrt{3}$.

### *How Archimedes discovered the area of a circle: dissect and "take a limit"*

As we learn early in school the formula for the area of a circle of radius $r$, $A = \pi r^2$. But how did this convenient formula come about? and how could we relate it to what we know about simpler shapes whose areas we have discussed so far.

Here we discuss how the formula for the area of a circle was determined long ago by Archimedes using a clever "dissection" and approximation trick. We have already seen part of this idea in dissecting a polygon into triangles. Here we see a terrifically important second step that formed the "leap of faith" on which most of calculus is based, namely taking a limit as the number of subdivisions increases.

First, we recall the definition of the constant $\pi$:

**Definition of $\pi$**: In any circle, $\pi$ is the ratio of the circumference to the diameter of the circle. (Comment: expressed in terms of the radius, this assertion states the obvious fact that the ratio of $2\pi r$ to $2r$ is $\pi$.)

Shown in Figure 6.3 is a sequence of regular polygons inscribed in the circle. As the number of sides of the polygon increases, its area gradually becomes a better and better approximation of the area inside the circle. Similar observations are central to integral calculus, and we will encounter this idea often. We can compute the area of any one of these polygons by dissecting into triangles. All triangles will be isosceles, since two sides are radii of the circle, whose length we'll call $r$.

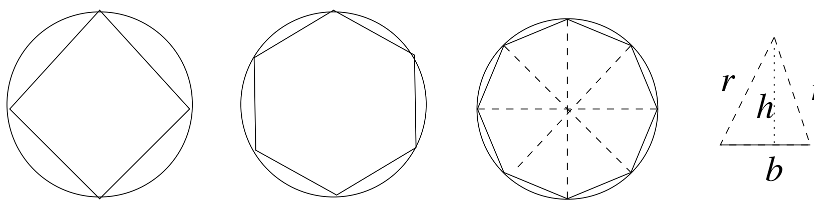



Figure 6.3: Archimedes approximated the area of a circle by dissecting it into triangles.

Let $r$ denote the radius of the circle. Suppose that at one stage we have an $n$ sided polygon. (If we knew the side length of that polygon, then we already have a formula for its area. However, this side length is not known to us. Rather, we know that the polygon should fit exactly inside a circle of radius $r$.) This polygon is made up of $n$ triangles, each one an isosceles triangle with two equal sides of length $r$ and base of undetermined length that we will denote by $b$. (See Figure 6.3.) The area of this triangle is

$$A_{\text{triangle}} = \frac{1}{2}bh.$$

The area of the whole polygon, $A_n$ is then

$$A = n \cdot (\text{area of triangle}) = n\frac{1}{2}bh = \frac{1}{2}(nb)h.$$

We have grouped terms so that $(nb)$ can be recognized as the perimeter of the polygon (i.e. the sum of the n equal sides of length $b$ each). Now consider what happens when we increase the number of sides of the polygon, taking larger and larger $n$. Then the height of each triangle will get closer to the radius of the circle, and the perimeter of the polygon will get closer and closer to the perimeter of the circle, which is (by definition) $2\pi r$. i.e. as $n \to \infty$,

$$h \to r, \quad (nb) \to 2\pi r$$

so

$$A = \frac{1}{2}(nb)h \to \frac{1}{2}(2\pi r)r = \pi r^2$$

We have used the notation "$\to$" to mean that in the limit, as $n$ gets large, the quantity of interest "approaches" the value shown. This argument proves that the area of a circle must be

$$A = \pi r^2.$$

One of the most important ideas contained in this little argument is that by approximating a shape by a larger and larger number of simple pieces (in this case, a large number of triangles), we get a better and better approximation of its area. This idea will appear again soon, but in most of our standard calculus computations, we will use a collection of rectangles, rather than triangles, to approximate areas of interesting regions in the plane.

**Areas of other shapes**
We concentrate here the area of a circle and of other shapes.

- The area of a circle of radius $r$ is
$$A = \pi r^2.$$
- The surface area of a sphere of radius $r$ is
$$S_{\text{ball}} = 4\pi r^2.$$
- The surface area of a right circular cylinder of height $h$ and base radius $r$ is
$$S_{\text{cyl}} = 2\pi rh.$$

The units of area can be meters$^2$ (m$^2$), centimeters$^2$ (cm$^2$), square inches, etc.

## 6.3 *Simple volumes*

Later in this course, we will also be computing the volumes of 3D shapes. As in the case of areas, we collect below some basic formulae for volumes of elementary shapes. These will be useful in our later discussions.

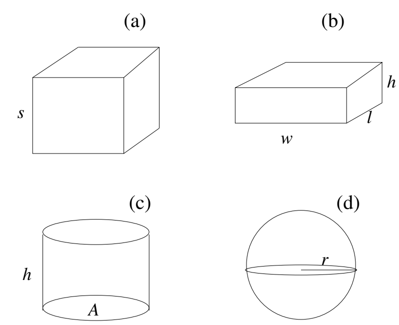


Figure 6.4: 3-dimensional shapes whose volumes are given by elementary formulae

1. The volume of a cube of side length $s$ (Figure 6.4a), is
$$V = s^3.$$
2. The volume of a rectangular box of dimensions $h, w, l$ (Figure 6.4b) is
$$V = hwl.$$
3. The volume of a cylinder of base area $A$ and height $h$, as in Figure 6.4(c), is
$$V = Ah.$$
This applies for a cylinder with flat base of any shape, circular or not.
4. In particular, the volume of a cylinder with a circular base of radius $r$, (e.g. a disk) is
$$V = h(\pi r^2).$$

5. The volume of a sphere of radius $r$ (Figure 6.4d), is

$$V = \frac{4}{3}\pi r^3.$$

6. The volume of a spherical shell (hollow sphere with a shell of some small thickness, $\tau$) is approximately

$$V \approx \tau \cdot (\text{surface area of sphere}) = 4\pi\tau r^2.$$

7. Similarly, a cylindrical shell of radius $r$, height $h$ and small thickness, $\tau$ has volume given approximately by

$$V \approx \tau \cdot (\text{surface area of cylinder}) = 2\pi\tau r h.$$

The units of volume are meters$^3$ (m$^3$), centimeters$^3$ (cm$^3$), inches$^3$, etc.

### *The Tower of Hanoi: a tower of disks*

In this example, we consider how elementary shapes discussed above can be used to determine volumes of more complex objects. The Tower of Hanoi is a shape consisting of a number of stacked disks. It is a simple calculation to add up the volumes of these disks, but if the tower is large, and comprised of many disks, we would want some shortcut to avoid long sums.

(a) Compute the volume of a tower made up of four disks stacked up one on top of the other, as shown in Figure 6.5. Assume that the radii of the disks are 1, 2, 3, 4 units and that each disk has height 1.

(b) Compute the volume of a tower made up of 100 such stacked disks, with radii $r = 1, 2, \ldots, 99, 100$.

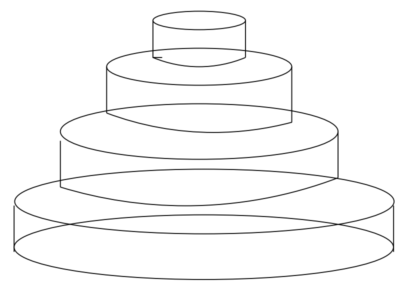


Figure 6.5: Computing the volume of a set of disks. (This structure is sometimes called the tower of Hanoi after a mathematical puzzle by the same name.)

For part a), note that the volume of the four-disk tower is calculated as follows:

$$V = V_1 + V_2 + V_3 + V_4,$$

where $V_i$ is the volume of the $i$'th disk whose radius is $r = i,\ i = 1, 2 \ldots 4$. The height of each disk is $h = 1$, so

$$V = (\pi 1^2) + (\pi 2^2) + (\pi 3^2) + (\pi 4^2) = \pi(1 + 4 + 9 + 16) = 30\pi.$$

For part b), the idea will be the same, but we have to calculate

$$V = \pi(1^2 + 2^2 + 3^2 + \cdots + 99^2 + 100^2).$$

It would be tedious to do this by adding up individual terms, and it is also cumbersome to write down the long list of terms that we will need to add up. This motivates inventing some helpful notation, and finding some clever way of performing such calculations.

## 6.4 Sigma Notation

Let's consider the sequence of squared integers

$$(1^2, 2^2, 3^2, 4^2, 5^2, \ldots) = (1, 4, 9, 16, 25, \ldots)$$

and let's add them up. Okay so the sum of the first square is $1^2 = 1$, the sum of the first two squares is $1^2 + 2^2 = 5$, the sum of the first three squares is $1^2 + 2^2 + 3^2 = 14$, etc. Now suppose we want to sum the first fifteen square integers–how should we write out this sum in our notes? Of course, the sum just looks like

$$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2+13^2+14^2+15^2$$

which equals

$$1+4+9+16+25+36+49+64+81+100+121+144+169+196+225.$$

But let's agree that the above expression is an eyesore. Frankly it occupies more space than it deserves, is confusing to look at, and is logically redundant. All we wanted to do was 'sum the first fifteen squared integers', not 'sum all the phone numbers that occur in the first fifteen pages of the phone book'. A sum with such a simple internal structure should have a simple notation. Mathematicians and physicists have such a notation which is convenient, logical, and simplifies many summations–this is the 'sigma notation'.

The sum of the elements $a_k + a_{k+1} + \cdots + a_n$ will be written $\sum_{j=k}^{n} a_j$, i.e.

$$\sum_{j=k}^{n} a_j := a_k + a_{k+1} + \cdots + a_n.$$

The symbol $\Sigma$ is the Greek letter for 'S' – we think of 'S' as standing for *s*ummation. The expression $\sum_{j=k}^{n} a_j$ represents the sum of the elements $a_k, a_{k+1}, \ldots, a_n$. The letter '$j$' is the *index of summation* and is a dummy-variable, i.e. you are free to replace it by any letter or symbol you want (like $k, \ell, m, n, \heartsuit, \star, \ldots$). Both

$$\sum_{\clubsuit=1}^{2013} a_\clubsuit \quad \text{and} \quad \sum_{\triangle=1}^{2013} a_\triangle$$

stand for the same sum: $a_1 + a_2 + \cdots + a_{2013}$. The notation $j = k$ that appears underneath $\Sigma$ indicates where the sum begins (i.e. which term starts off the series), and the superscript $n$ tells us where it ends. We will be interested in getting used to this notation, as well as in actually computing the value of the desired sum using a variety of shortcuts.

**Simple examples**

1. The convenience of the sigma notation is that the above sum of the first fifteen squared integers has now the more compact form

$$\sum_{j=1}^{15} j^2.$$

We shall find below a closed-formula for this sum which will be easier to derive using the sigma notation.

2. Suppose we want to form the sum of ten numbers, each equal to 1. We would write this as

$$S = 1 + 1 + 1 + \ldots 1 \equiv \sum_{k=1}^{10} 1.$$

The notation $\ldots$ signifies that we have left out some of the terms (out of laziness, or in cases where there are too many to conveniently write down.) We could have just as well written the sum with another symbol (e.g. $n$) as the index, i.e. the same operation is implied by

$$\sum_{n=1}^{10} 1.$$

To compute the value of the sum we use the elementary fact that the sum of ten ones is just 10, so

$$S = \sum_{k=1}^{10} 1 = 10.$$

3. Sum of squares: Expand and sum the following:

$$S = \sum_{k=1}^{4} k^2.$$

*Solution:*

$$S = \sum_{k=1}^{4} k^2 = 1 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30.$$

(We have already seen this sum in part (a) of The Tower of Hanoi.)

4. Common factors: Add up the following list of 100 numbers (only a few of them are shown):

$$S = 3 + 3 + 3 + 3 + \cdots + 3.$$

*Solution:* There are 100 terms, all equal, so we can take out a common factor

$$S = 3 + 3 + 3 + 3 + \cdots + 3 = \sum_{k=1}^{100} 3 = 3 \sum_{k=1}^{100} 1 = 3(100) = 300.$$

5. Find the pattern: Write the following terms in summation notation:

$$S = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}.$$

*Solution:* We recognize that there is a pattern in the sequence of terms, namely, each one is $1/3$ raised to an increasing integer power, i.e.

$$S = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^4.$$

We can represent this with the "Sigma" notation as follows:

$$S=\sum_{n=1}^{4}\left(\frac{1}{3}\right)^{n}.$$

The "index" $n$ starts at 1, and counts up through 2, 3, and 4, while each term has the form of $(1/3)^n$.

This series is a **geometric series**, to be explored shortly. In most cases, a standard geometric series starts off with the value 1. We can easily modify our notation to include additional terms, for example:

$$S=\sum_{n=0}^{5}\left(\frac{1}{3}\right)^{n}=1+\frac{1}{3}+\left(\frac{1}{3}\right)^{2}+\left(\frac{1}{3}\right)^{3}+\left(\frac{1}{3}\right)^{4}+\left(\frac{1}{3}\right)^{5}.$$

Learning how to compute the sum of such terms will be important to us, and will be described later on in this chapter.

6. Often there are different but equivalent ways to represent the same sum:

$$4+5+6+7+8=\sum_{j=4}^{8} j=\sum_{j=1}^{5}(3+j).$$

7. It is useful to have some dexterity in arranging and rearranging the Sigma notations. For instance, the sum

$$1-2+3-4+5-6+\cdots$$

has no upper bound and we may write

$$1-2+3-4+5-6+\cdots=\sum_{j=1}^{\infty}(-1)^{j+1}j=\sum_{j=0}^{\infty}(-1)^{j}(j+1)$$

to highlight the fact that this sum has infinitely many terms.

8.
$$7+9+11+13+15=\sum_{j=0}^{4}(7+2j).$$

9.
$$\sum_{j=n}^{n} a_j=a_n.$$

**Manipulations of sums**

Since addition is commutative and distributive, sums of lists of numbers satisfy many convenient properties. We give a few examples below:

- Simplify the following expression:

$$\sum_{k=1}^{10}2^k-\sum_{k=3}^{10}2^k.$$

*Solution:* We have

$$\sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = (2+2^2+2^3+\cdots+2^{10}) - (2^3+\cdots+2^{10}) = 2+2^2.$$

Alternatively we could have arrived at this conclusion directly from

$$\sum_{k=1}^{10} 2^k - \sum_{k=3}^{10} 2^k = \sum_{k=1}^{2} 2^k = 2+2^2 = 2+4 = 6.$$

The idea is that all but the first two terms in the first sum will cancel. The only remaining terms are those corresponding to $k=1$ and $k=2$.

- Expand the following expression: $\sum_{n=0}^{5}(1+3^n)$.

  *Solution:* We have

$$\sum_{n=0}^{5}(1+3^n) = \sum_{n=0}^{5} 1 + \sum_{n=0}^{5} 3^n.$$

### *Formulae for sums of integers, squares, and cubes*

The general formulae are:

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \tag{6.1}$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6} \tag{6.2}$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2. \tag{6.3}$$

We now provide a justification as to why these formulae are valid. The sum of the first $N$ integers can be seen by the following amusing argument.

**Example 6.1 (The sum of consecutive integers (Gauss' formula))** *We first show that the sum $S_N$ of the first $N$ integers is*

$$S_N = 1+2+3+\cdots+N = \sum_{k=1}^{N} k = \frac{N(N+1)}{2}. \tag{6.4}$$

**Solution.** By aligning two copies of the above sum, one written backwards, we can easily add them up one by one vertically. We see that:

$$\begin{array}{lccccccccc} S_N = & 1 & + & 2 & + & \ldots & + & (N-1) & + & N \\ + & & & & & & & & & \\ S_N = & N & + & (N-1) & + & \ldots & + & 2 & + & 1 \end{array}$$

$$2S_N = \quad (1+N) \quad + \quad (1+N) \quad + \quad \ldots \quad + \quad (1+N) \quad + \quad (1+N)$$

Thus, there are $N$ times the value $(N+1)$ above, so that

$$2S_N = N(1+N), \quad \text{so} \quad S_N = \frac{N(1+N)}{2}.$$

Thus, the formula is confirmed. ◊

**Example 6.2 (Adding up the first 1000 integers)** *Suppose we want to add up the first 1000 integers. This formula is very useful in what would otherwise be a huge calculation. We find that*

$$S = 1+2+3+\cdots+1000 = \sum_{k=1}^{1000} k = \frac{1000(1+1000)}{2} = 500(1001) = 500500.$$

**Example 6.3 (Sums of squares and cubes)** *We now present an argument verifying the above formulae for sums of squares and cubes.*

**Solution.** First, note that

$$(k+1)^3 - (k-1)^3 = 6k^2 + 2,$$

so

$$\sum_{k=1}^{n} \left((k+1)^3 - (k-1)^3\right) = \sum_{k=1}^{n} (6k^2+2).$$

But looking more carefully at the left hand side (LHS), we see that

$$\sum_{k=1}^{n} ((k+1)^3 - (k-1)^3) = 2^3 - 0^3 + 3^3 - 1^3 + 4^3 - 2^3 + 5^3 - 3^3 ... + (n+1)^3 - (n-1)^3$$

so most of the terms cancel, leaving only $-1+n^3+(n+1)^3$. This means that

$$-1+n^3+(n+1)^3 = 6\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2,$$

so

$$\sum_{k=1}^{n} k^2 = \frac{-1+n^3+(n+1)^3-2n}{6} = \frac{2n^3+3n^2+n}{6}.$$

◊ Similarly, the formulae for $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^3$, can be obtained by starting with the identities

$$(k+1)^2 - (k-1)^2 = 4k, \text{ and } (k+1)^4 - (k-1)^4 = 8k^3 + 8k,$$

respectively. We encourage the interested reader to carry out these details.

An alternative approach using a technique called *mathematical induction* to verify the formulae for the sum of squares and cubes of integers is presented at the end of this chapter.

**Example 6.4 (Volume of a Tower of Hanoi, revisited)** *Armed with the formula for the sum of squares, we can now return to the problem of computing the volume of a tower of 100 stacked disks of heights 1 and radii* $r = 1, 2, \ldots, 99, 100$. *We have*

$$V = \pi(1^2+2^2+3^2+\cdots+99^2+100^2) = \pi\sum_{k=1}^{100} k^2$$
$$= \pi\frac{100(101)(201)}{6} = 338{,}350\pi \textit{ cubic units}.$$

**Example 6.5** *Compute the following sum:*

$$S_a = \sum_{k=1}^{20} (2 - 3k + 2k^2).$$

**Solution.** We can separate this into three individual sums, each of which can be handled by algebraic simplification and/or use of the summation formulae developed so far.

$$S_a = \sum_{k=1}^{20} (2 - 3k + 2k^2) = 2\sum_{k=1}^{20} 1 - 3\sum_{k=1}^{20} k + 2\sum_{k=1}^{20} k^2.$$

Thus, we get

$$S_a = 2(20) - 3\left(\frac{20(21)}{2}\right) + 2\left(\frac{(20)(21)(41)}{6}\right) = 5150.$$

$\Diamond$

**Example 6.6** *Compute the following sum:*

$$S_b = \sum_{k=10}^{50} k.$$

**Solution.** We can express the second sum as a difference of two sums:

$$S_b = \sum_{k=10}^{50} k = \left(\sum_{k=1}^{50} k\right) - \left(\sum_{k=1}^{9} k\right).$$

Thus

$$S_b = \left(\frac{50(51)}{2} - \frac{9(10)}{2}\right) = 1275 - 45 = 1230.$$

$\Diamond$

## 6.5 *Summing the geometric series*

Consider a sum of terms that all have the form $r^k$, where $r$ is some real number and $k$ is an integer power. We refer to a series of this type as a **geometric series**. We have already seen one example of this type in a previous section. Below we will show that the sum of such a series is given by:

$$S_N = 1 + r + r^2 + r^3 + \ldots + r^N = \sum_{k=0}^{N} r^k = \frac{1 - r^{N+1}}{1 - r}, \qquad r \neq 1. \qquad (6.5)$$

We call this sum a (finite) geometric series. We would like to find an expression for terms of this form in the general case of any real number $r$, and finite number of terms $N$. First we note that there are $N + 1$ terms in this sum, so that if $r = 1$ then

$$S_N = 1 + 1 + 1 + \ldots 1 = N + 1$$

(a total of $N+1$ ones added.) If $r \neq 1$, we have the following trick:

$$\begin{array}{rcccccccccc} S_N = & 1 & + & r & + & r^2 & + & \dots & + & r^N & \\ - & & & & & & & & & & \\ rS_N = & & & r & + & r^2 & + & \dots & + & r^{N+1} & \end{array}$$

Subtracting leads to

$$S_N - rS_N = (1 + r + r^2 + \cdots + r^N) - (r + r^2 + \cdots + r^N + r^{N+1})$$

Most of the terms on the right hand side cancel, leaving

$$S_N(1-r) = 1 - r^{N+1}.$$

Now dividing both sides by $1-r$ leads to

$$S_N = \frac{1-r^{N+1}}{1-r},$$

which was the formula to be established.

**Example 6.7 (Geometric series)** *Compute the following sum:*

$$S_c = \sum_{k=0}^{10} 2^k.$$

**Solution.** This is a geometric series

$$S_c = \sum_{k=0}^{10} 2^k = \frac{1-2^{10+1}}{1-2} = \frac{1-2048}{-1} = 2047.$$

$\Diamond$

## 6.6 *Prelude to infinite series*

So far, we have looked at several examples of finite series, i.e. series in which there are only a finite number of terms, $N$ (where $N$ is some integer). We would like to investigate how the sum of a series behaves when more and more terms of the series are included. It is evident that in many cases, such as Gauss's series (6.4), or sums of squared or cubed integers (e.g., Eqs. (6.2) and (6.3)), the series simply gets larger and larger as more terms are included. We say that such series *diverge* as $N \to \infty$. Here we will look specifically for series that *converge*, i.e. have a finite sum, even as more and more terms are included.

Let us focus again on the geometric series and determine its behaviour when the number of terms is increased. Our goal is to find a way of attaching a meaning to the expression

$$S = \sum_{k=0}^{\infty} r^k,$$

when the series becomes an *infinite series*. We will use the following definition:

### *The infinite geometric series*

**Definition.** An infinite series that has a unique, finite sum is said to be *convergent*. Otherwise it is *divergent*.

**Definition.** Suppose that $S$ is an (infinite) series whose terms are $a_k$. Then the *partial sums*, $S_n$, of this series are

$$S_n = \sum_{k=0}^{n} a_k.$$

We say that the sum of the infinite series is $S$, and write

$$S = \sum_{k=0}^{\infty} a_k,$$

provided that

$$S = \lim_{n\to\infty} \sum_{k=0}^{n} a_k.$$

That is, we consider the infinite series as the limit of the partial sums as the number of terms $n$ is increased. In this case we also say that the infinite series converges to $S$.

We will see that only under certain circumstances will infinite series have a finite sum, and we will be interested in exploring two questions:

1. Under what circumstances does an infinite series have a finite sum.

2. What value does the partial sum approach as more and more terms are included.

In the case of a geometric series, the sum of the series, (6.5) depends on the number of terms in the series, $n$ via $r^{n+1}$. Whenever $r > 1$, or $r < -1$, this term will get bigger in magnitude as $n$ increases, whereas, for $0 < r < 1$, this term decreases in magnitude with $n$. We can say that

$$\lim_{n\to\infty} r^{n+1} = 0 \;\text{ provided }\; |r| < 1.$$

These observations are illustrated by two specific examples below. This leads to the following conclusion:

The sum of an infinite geometric series,

$$S = 1 + r + r^2 + \cdots + r^k + \cdots = \sum_{k=0}^{\infty} r^k,$$

exists provided $|r| < 1$ and is

$$S = \frac{1}{1-r}. \tag{6.6}$$

**Example 6.8 (A geometric series that converges)**

Consider the geometric series with $r = \frac{1}{2}$, i.e.

$$S_n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots + \left(\frac{1}{2}\right)^n = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k.$$

Then

$$S_n = \frac{1-(1/2)^{n+1}}{1-(1/2)}.$$

We observe that as $n$ increases, i.e. as we retain more and more terms, we obtain

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-(1/2)^{n+1}}{1-(1/2)} = \frac{1}{1-(1/2)} = 2.$$

In this case, we write

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots = 2$$

and we say that "the (infinite) series *converges* to 2".

**Example 6.9 (A geometric series that diverges)**

In contrast, we now investigate the case that $r = 2$: then the series consists of terms

$$S_n = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n = \sum_{k=0}^{n} 2^k = \frac{1-2^{n+1}}{1-2} = 2^{n+1} - 1$$

We observe that as $n$ grows larger, the sum continues to grow indefinitely. In this case, we say that the sum *does not converge*, or, equivalently, that the sum *diverges*.

It is important to remember that an infinite series, i.e. a sum with infinitely many terms added up, can exhibit either one of these two very different behaviours. It may converge in some cases, as the first example shows, or *diverge* (fail to converge) in other cases.

## 6.7 *Application of geometric series to the branching structure of the lungs*

In this section, we will compute the volume and surface area of the branched airways of lungs. We use the summation formulae to arrive at the results, and we also illustrate how the same calculation could be handled using a simple spreadsheet.

Our lungs pack an amazingly large surface area into a confined volume. Most of the oxygen exchange takes place in tiny sacs called *alveoli* at the terminal branches of the airways passages. The bronchial tubes conduct air, and distribute it to the many smaller and smaller tubes that eventually lead to those alveoli. The principle of this efficient organ for oxygen exchange is that these very many small structures present a very large surface area. Oxygen from the air can diffuse across this area into the bloodstream very efficiently.

The lungs, and many other biological "distribution systems" are composed of a branched structure. The initial segment is quite large. It bifurcates into smaller segments, which then bifurcate further, and so on, resulting in a geometric expansion in the number of branches, their collective volume, length, etc. In this section, we apply geometric series to explore this branched structure of the lung. We will construct a simple mathematical model and explore its consequences. The model will consist in some well-formulated assumptions about the way that "daughter branches" are related to their "parent branch". Based on these assumptions, and on tools developed in this chapter, we will then predict properties of the structure as a whole. We will be particularly interested in the volume $V$ and the surface area $S$ of the airway passages in the lungs. The surface area of the bronchial tubes does not actually absorb much oxygen, in humans. However, as an example of summation, we will compute this area and compare how it grows to the growth of the volume from one branching layer to the next.

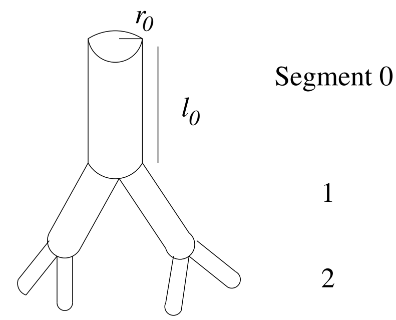



Figure 6.6: Air passages in the lungs consist of a branched structure. The index $n$ refers to the branch generation, starting from the initial segment, labeled 0. All segments are assumed to be cylindrical, with radius $r_n$ and length $\ell_n$ in the $n$'th generation.

*Assumptions*

| | | |
|---|---|---|
| radius of first segment | $r_0$ | 0.5 cm |
| length of first segment | $\ell_0$ | 5.6 cm |
| ratio of daughter to parent length | $\alpha$ | 0.9 |
| ratio of daughter to parent radius | $\beta$ | 0.86 |
| number of branch generations | $M$ | 30 |
| average number daughters per parent | $b$ | 1.7 |

Table 6.1: Typical structure of branched airway passages in lungs.

- The airway passages consist of many "generations" of branched segments. We label the largest segment with index "0", and its daughter segments with index "1", their successive daughters "2", and so on down the structure from large to small branch segments. We assume that there are $M$ "generations", i.e. the initial segment has undergone $M$ subdivisions. Figure 6.6 shows only generations 0, 1, and 2. (Typically, for human lungs there can be up to 25-30 generations of branching.)

- At each generation, every segment is approximated as a cylinder of radius $r_n$ and length $\ell_n$.

- The number of branches grows along the "tree". On average, each parent branch produces $b$ daughter branches. In Figure 6.6, we have illustrated this idea for $b = 2$. A branched structure in which each branch produces two daughter branches is described as a **bifurcating** tree structure (whereas **trifurcating** implies $b = 3$). In real lungs, the branching is slightly irregular. Not every level of the structure bifurcates, but in general, averaging over the many branches in the structure $b$ is smaller than 2. In fact, the rule that links the number of branches in generation $n$, here denoted $x_n$ with the number (of smaller branches) in the next generation, $x_{n+1}$ is

$$x_{n+1} = bx_n. \tag{6.7}$$

We will assume, for simplicity, that $b$ is a constant. Since the number of branches is growing down the length of the structure, it must be true that $b > 1$. For human lungs, on average, $1 < b < 2$. Here we will take $b$ to be constant, i.e. $b = 1.7$. In actual fact, this simplification cannot be precise, because we have just one segment initially ($x_0 = 1$), and at level 1, the number of branches $x_1$ should be some small *integer*, not a number like "1.7". However, as in many mathematical models, some accuracy is sacrificed to get intuition. Later on, details that were missed and are considered important can be corrected and refined.

- The ratios of radii and lengths of daughters to parents are approximated by "proportional scaling". This means that the relationship of the radii and lengths satisfy simple rules: The lengths are related by

  $$\ell_{n+1} = \alpha \ell_n, \tag{6.8}$$

  and the radii are related by

  $$r_{n+1} = \beta r_n, \tag{6.9}$$

  with $\alpha$ and $\beta$ positive constants. For example, it could be the case that the radius of daughter branches is 1/2 or 2/3 that of the parent branch. Since the branches decrease in size (while their number grows), we expect that $0 < \alpha < 1$ and $0 < \beta < 1$.

Rules such as those given by equations (6.8) and (6.9) are often called *self-similar growth* laws. Such concepts are closely linked to the idea of fractals, i.e. theoretical structures produced by iterating such growth laws indefinitely. In a real biological structure, the number of generations is finite. (However, in some cases, a finite geometric series is well-approximated by an infinite sum.)

Actual lungs are not fully symmetric branching structures, but the above approximations are used here for simplicity. According to physiological measurements, the scale factors for sizes of daughter to parent size are in the range $0.65 \leq \alpha, \beta \leq 0.9$. (K. G. Horsfield, G. Dart, D. E. Olson, and G. Cumming, (1971) J. Appl. Phys. 31, 207-217.) For the purposes of this example, we will use the values of constants given in Table 6.1.

### *A simple geometric rule*

The three equations that govern the rules for successive branching, i.e. equations (6.7), (6.8), and (6.9), are examples of a very generic "geometric progression" recipe. Before returning to the problem at hand, let us examine the implications of this recursive rule, when it is applied to generating the whole structure. Essentially, we will see that the rule linking two generations implies an exponential growth. To see this, let us write out a few first terms in the progression of the sequence $\{x_n\}$:

$$\begin{array}{rll} \text{initial value:} & x_0 & \\ \text{first iteration:} & x_1 & = bx_0 \\ \text{second iteration:} & x_2 & = bx_1 = b(bx_0) = b^2x_0 \\ \text{third iteration:} & x_3 & = bx_2 = b(b^2x_0) = b^3x_0 \\ & & \vdots \end{array}$$

By the same pattern, at the $n$'th generation, the number of segments will be

$$x_n = bx_{n-1} = b(bx_{n-2}) = b(b(bx_{n-3})) = \cdots = \underbrace{(b \cdot b \cdots b)}_{n \text{ factors}} x_0 = b^n x_0.$$

We have arrived at a simple, but important result, namely:

The rule linking two generations,

$$x_n = bx_{n-1} \tag{6.10}$$

implies that the $n$'th generation will have grown by a factor $b^n$, i.e.,

$$x_n = b^n x_0. \tag{6.11}$$

This connection between the rule linking two generations and the resulting number of members at each generation is useful in other circumstances. Equation (6.10) is sometimes called a **recursion relation**, and its solution is given by equation (6.11). We will use the same idea to find the connection between the volumes, and surface areas of successive segments in the branching structure.

### *Total number of segments*

We used the result of Section 6.7 and the fact that there is one segment in the 0'th generation, i.e. $x_0 = 1$, to conclude that at the $n$'th generation, the number of segments is

$$x_n = x_0 b^n = 1 \cdot b^n = b^n.$$

For example, if $b = 2$, the number of segments grows by powers of 2, so that the tree bifurcates with the pattern 1, 2, 4, 8, etc.

To determine how many branch segments there are in total, we add up over all generations, $0, 1, \dots M$. This is a geometric series, whose sum we can compute. Using equation (6.5), we find

$$N = \sum_{n=0}^{M} b^n = \left(\frac{1 - b^{M+1}}{1 - b}\right).$$

Given $b$ and $M$, we can then predict the exact number of segments in the structure. The calculation is summarized further on for values of the branching parameter, $b$, and the number of branch generations, $M$, given in Table 6.1.

*Total volume of airways in the lung*

Since each lung segment is assumed to be cylindrical, its volume is

$$v_n = \pi r_n^2 \ell_n.$$

Here we mean just a single segment in the $n$'th generation of branches. (There are $b^n$ such identical segments in the $n$'th generation, and we will refer to the volume of all of them together as $V_n$ below.)

The length and radius of segments also follow a geometric progression. In fact, the same idea developed above can be used to relate the length and radius of a segment in the $n$'th, generation segment to the length and radius of the original 0'th generation segment, namely,

$$\ell_n = \alpha \ell_{n-1} \Rightarrow \quad \ell_n = \alpha^n \ell_0,$$

and

$$r_n = \beta r_{n-1} \Rightarrow \quad r_n = \beta^n r_0.$$

Thus the volume of one segment in generation n is

$$v_n = \pi r_n^2 \ell_n = \pi (\beta^n r_0)^2 (\alpha^n \ell_0) = (\alpha \beta^2)^n \underbrace{(\pi r_0^2 \ell_0)}_{v_0}.$$

This is just a product of the initial segment volume $v_0 = \pi r_0^2 \ell_0$, with the $n$'th power of a certain factor$(\alpha, \beta)$. (That factor takes into account that both the radius and the length are being scaled down at every successive generation of branching.) Thus

$$v_n = (\alpha \beta^2)^n v_0.$$

The total volume of all ($b^n$) segments in the $n$'th layer is

$$V_n = b^n v_n = b^n (\alpha \beta^2)^n v_0 = (\underbrace{b \alpha \beta^2}_{a})^n v_0.$$

Here we have grouped terms together to reveal the simple structure of the relationship: one part of the expression is just the initial segment volume, while the other is now a "scale factor" that includes not only changes in length and radius, but also in the number of branches.

Letting the constant $a$ stand for that scale factor, $a = (b\alpha\beta^2)$ leads to the result that the volume of all segments in the $n$'th layer is

$$V_n = a^n v_0.$$

The total volume of the structure is obtained by summing the volumes obtained at each layer. Since this is a geometric series, we can use the summation formula. i.e., Equation (6.5). Accordingly, total airways volume is

$$V = \sum_{n=0}^{30} V_n = v_0 \sum_{n=0}^{30} a^n = v_0 \left( \frac{1 - a^{M+1}}{1 - a} \right).$$

The similarity of treatment with the previous calculation of number of branches is apparent. We compute the value of the constant $a$ in Table 6.2, and find the total volume in Section 6.7.

### *Total surface area of the lung branches*

The surface area of a single segment at generation $n$, based on its cylindrical shape, is

$$s_n = 2\pi r_n \ell_n = 2\pi(\beta^n r_0)(\alpha^n \ell_0) = (\alpha\beta)^n \underbrace{(2\pi r_0 \ell_0)}_{s_0},$$

where $s_0$ is the surface area of the initial segment. Since there are $b^n$ branches at generation n, the total surface area of all the $n$'th generation branches is thus

$$S_n = b^n (\alpha\beta)^n s_0 = (\underbrace{b\alpha\beta}_{c})^n s_0,$$

where we have let $c$ stand for the scale factor $c = (b\alpha\beta)$. Thus,

$$S_n = c^n s_0.$$

This reveals the similar nature of the problem. To find the total surface area of the airways, we sum up,

$$S = s_0 \sum_{n=0}^{M} c^n = s_0 \left( \frac{1 - c^{M+1}}{1 - c} \right).$$

We compute the values of $s_0$ and $c$ in Table 6.2, and summarize final calculations of the total airways surface area in section 6.7.

*Summary of predictions for specific parameter values*

By setting up the model in the above way, we have revealed that each quantity in the structure obeys a simple geometric series, but with distinct "bases" $b, a$ and $c$ and coefficients $1, v_0$, and $s_0$. This approach has shown that the formula for geometric series applies in each case. Now it remains to merely "plug in" the appropriate quantities. In this section, we collect our results, use the sample values for a model "human lung" given in Table 6.1, or the resulting derived scale factors and quantities in Table 6.2 to finish the task at hand.

| | | |
|---|---|---|
| volume of first segment | $v_0 = \pi r_0^2 \ell_0$ | 4.4 $cm^3$ |
| surface area of first segment | $s_0 = 2\pi r_0 \ell_0$ | 17.6 $cm^2$ |
| ratio of daughter to parent segment volume | $(\alpha\beta^2)$ | 0.66564 |
| ratio of daughter to parent segment surface area | $(\alpha\beta)$ | 0.774 |
| ratio of net volumes in successive generations | $a = b\alpha\beta^2$ | 1.131588 |
| ratio of net surface areas in successive generations | $c = b\alpha\beta$ | 1.3158 |

Table 6.2: Volume, surface area, scale factors, and other derived quantities. Because $a$ and $c$ are bases that will be raised to large powers, it is important to that their values are fairly accurate, so we keep more significant figures.

**Total number of segments**

$$N = \sum_{n=0}^{M} b^n = \left(\frac{1-b^{M+1}}{1-b}\right) = \left(\frac{1-(1.7)^{31}}{1-1.7}\right) = 1.9898 \cdot 10^7 \approx 2 \cdot 10^7.$$

According to this calculation, there are a total of about 20 million branch segments overall (including all layers, from top to bottom) in the entire structure!

**Total volume of airways**

Using the values for $a$ and $v_0$ computed in Table 6.2, we find that the total volume of all segments is

$$V = v_0 \sum_{n=0}^{30} a^n = v_0 \left(\frac{1-a^{M+1}}{1-a}\right) = 4.4\frac{(1-1.131588^{31})}{(1-1.131588)} = 1510.3 \text{ cm}^3.$$

Recall that 1 litre = 1000 $cm^3$. Then we have found that the lung airways contain about 1.5 litres.

**Total surface area of airways**

Using the values of $s_0$ and $c$ in Table 6.2, the total surface area of the tubes that make up the airways is

$$S = s_0 \sum_{n=0}^{M} c^n = s_0 \left(\frac{1-c^{M+1}}{1-c}\right) = 17.6\frac{(1-1.3158^{31})}{(1-1.3158)} = 2.76 \cdot 10^5 \text{ cm}^2.$$

There are 100 cm per meter, and $(100)^2 = 10^4$ $cm^2$ per $m^2$. Thus, the area we have computed is equivalent to about 28 square meters!

### *Exploring the problem numerically*

Up to now, all calculations were done using the formulae developed for geometric series. However, sometimes it is more convenient to devise a computer algorithm to implement "rules" and perform repetitive calculations in a problem such as discussed here. The advantage of that approach is that it eliminates tedious calculations by hand, and, in cases where summation formulae are not know to us, reduces the need for analytical computations. It can also provide a shortcut to visual summary of the results. The disadvantage is that it can be less obvious how each of the values of parameters assigned to the problem affects the final answers.

A spreadsheet is an ideal tool for exploring iterated rules such as those given in the lung branching problem. In Figure 6.7 we show the volumes and surface areas associated with the lung airways for parameter values discussed above. Both layer by layer values and cumulative sums leading to total volume and surface area are shown in each of (a) and (c). In (b) and (d), we compare these results to similar graphs in the case that one parameter, the branching number, $b$ is adjusted from 1.7 (original value) to 2. The contrast between the graphs shows how such a small change in this parameter can significantly affect the results.

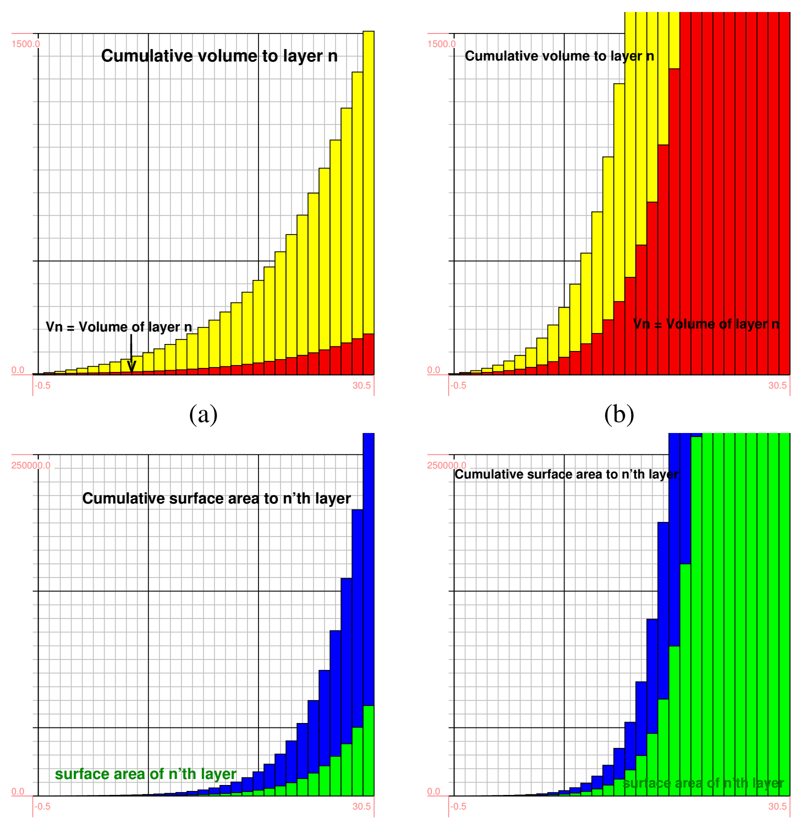



Figure 6.7: (a) $V_n$, the volume of layer n (red bars), and the cumulative volume down to layer n (yellow bars) are shown for parameters given in Table 6.1. (b) Same as (a) but assuming that parent segments always produce two daughter branches (i.e. $b = 2$). The graphs in (a) and (b) are shown on the same scale to accentuate the much more dramatic growth in (b). (c) and (d): same idea showing the surface area of $n$'th layer (green) and the cumulative surface area to layer n (blue) for original parameters (in c), as well as for the value $b = 2$ (in d).

*For further study*

The following problems can be used for further exploration of these ideas.

1. In our model, we have assumed that, on average, a parent branch has only "1.7" daughter branches, i.e. that $b = 1.7$. Suppose we had assumed that $b = 2$. What would the total volume $V$ be in that case, keeping all other parameters the same? Explain why this is biologically impossible in the case $M = 30$ generations. For what value of $M$ would $b = 2$ lead to a reasonable result?

2. Suppose that the first 5 generations of branching produce 2 daughters each, but then from generation 6 on, the branching number is $b = 1.7$. How would you set up this variant of the model? How would this affect the calculated volume?

3. In the problem we explored, the net volume and surface area keep growing by larger and larger increments at each "generation" of branching. We would describe this as "unbounded growth". Explain why this is the case, paying particular attention to the scale factors $a$ and $c$.

4. Suppose we want a set of tubes with a large surface area but small total volume. Which *single* factor or parameter should we change (and how should we change it) to correct this feature of the model, i.e. to predict that the total volume of the branching tubes remains roughly constant while the surface area increases as branching layers are added.

5. Determine how the branching properties of real human lungs differs from our assumed model, and use similar ideas to refine and correct our estimates. You may want to investigate what is known about the actual branching parameter $b$, the number of generations of branches, $M$, and the ratios of lengths and radii that we have assumed. Alternately, you may wish to find parameters for other species and do a comparative study of lungs in a variety of animal sizes.

6. Branching structures are ubiquitous in biology. Many species of plants are based on a regular geometric sequence of branching. Consider a tree that trifurcates (i.e. produces 3 new daughter branches per parent branch, $b = 3$). Explain (a) What biological problem is to be solved in creating such a structure (b) What sorts of constraints must be satisfied by the branching parameters to lead to a viable structure. This is an open-ended problem.

## 6.8 Summary

In this chapter, we collected useful formulae for areas and volumes of simple 2D and 3D shapes. A summary of the most important ones is given below. Table 6.3 lists the areas of simple shapes, Table 6.4 the volumes and Table 6.5 the surface areas of 3D shapes.

We used areas of triangles to compute areas of more complicated shapes, including regular polygons. We used a polygon with $N$ sides to approximate the area of a circle, and then, by letting $N$ go to infinity, we were able to prove that the area of a circle of radius $r$ is $A = \pi r^2$. This idea, and others related to it, will form a deep underlying theme in the next two chapters and later on in this course.

We introduced sum notation for series and collected useful formulae for summation of such series. These are summarized in the table below. Finally, we investigated geometric series and studied a biological application, namely the branching structure of lungs.

| Object | dimensions | area, $A$ |
|---|---|---|
| triangle | base $b$, height $h$ | $\frac{1}{2}bh$ |
| rectangle | base $b$, height $h$ | $bh$ |
| circle | radius $r$ | $\pi r^2$ |

Table 6.3: **Areas of planar regions**

| Object | dimensions | volume, $V$ |
|---|---|---|
| box | base $b$, height $h$, width $w$ | $hwb$ |
| circular cylinder | radius $r$, height $h$ | $\pi r^2 h$ |
| sphere | radius $r$ | $\frac{4}{3}\pi r^3$ |

Table 6.4: **Volumes of 3D shapes**.

| Object | dimensions | surface area, $S$ |
|---|---|---|
| box | base $b$, height $h$, width $w$ | $2(bh+bw+hw)$ |
| circular cylinder | radius $r$, height $h$ | $2\pi rh$ |
| sphere | radius $r$ | $4\pi r^2$ |

Table 6.5: **Surface areas of 3D shapes**

| Sum | Notation | Formula | Comment |
|---|---|---|---|
| $1+2+3+\cdots+N$ | $\sum_{k=1}^{N} k$ | $\frac{N(1+N)}{2}$ | Gauss' formula |
| $1^2+2^2+3^2+\cdots+N^2$ | $\sum_{k=1}^{N} k^2$ | $\frac{N(N+1)(2N+1)}{6}$ | Sum of squares |
| $1^3+2^3+3^3+\cdots+N^3$ | $\sum_{k=1}^{N} k^3$ | $\left(\frac{N(N+1)}{2}\right)^2$ | Sum of cubes |
| $1+r+r^2+r^3\ldots r^N$ | $\sum_{k=0}^{N} r^k$ | $\frac{1-r^{N+1}}{1-r}$ | Geometric sum |

Table 6.6: **Useful summation formulae.**

## *Exercises*

**Exercise 6.1** Answer the following questions:

(a) What is the value of the fifth term of the sum $S = \sum_{k=1}^{20} (5+3k)/k$?

(b) How many terms are there in total in the sum $S = \sum_{k=7}^{17} e^k$?

(c) Write out the terms in $\sum_{n=1}^{5} 2^{n-1}$.

(d) Write out the terms in $\sum_{n=0}^{4} 2^n$.

(e) Write the series $1+3+3^2+3^3$ in summation notation in two equivalent forms.

**Exercise 6.2 Summation notation**

(a) Write $2+4+6+8+10+12+...$ in summation notation.

(b) Write $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$ in summation notation.

(c) Write out the first few terms of $\sum_{i=0}^{100} 3^i$

(d) Write out the first few terms of $\sum_{n=1}^{\infty} \frac{1}{n^n}$

(e) Simplify $\sum_{k=5}^{\infty} \left(\frac{1}{2}\right)^k + \sum_{k=2}^{4} \left(\frac{1}{2}\right)^k$

(f) Simplify $\sum_{x=0}^{50} 3^x - \sum_{x=10}^{50} 3^x$

(g) Simplify $\sum_{n=0}^{100} n + \sum_{n=0}^{100} n^2$

(h) Simplify $2\sum_{y=0}^{100} y + \sum_{y=0}^{100} y^2 + \sum_{y=0}^{100} 1$

**Exercise 6.3** Show that the following pairs of sums are equivalent:

(a) $\sum_{m=0}^{10} (m+1)^2$ and $\sum_{n=1}^{11} n^2$

(b) $\sum_{n=1}^{4} (n^2-2n+1)$ and $\sum_{n=1}^{4} (n-1)^2$

**Exercise 6.4** Compute the following sums: (Some of the calculations will be facilitated by a calculator)

(a) $\sum_{i=1}^{290} 1$ (b) $\sum_{i=1}^{150} 2$ (c) $\sum_{i=1}^{80} 3$

(d) $\sum_{n=1}^{50} n$ (e) $\sum_{n=1}^{60} n$ (f) $\sum_{n=10}^{60} n$

(g) $\sum_{n=20}^{100} n$ (h) $\sum_{n=1}^{25} 3n^2$ (i) $\sum_{n=1}^{20} 2n^2$

(j) $\sum_{i=1}^{55} (i+2)$ (k) $\sum_{i=1}^{75} (i+1)$ (l) $\sum_{k=100}^{500} k$

(m) $\sum_{k=50}^{100} k$ (n) $\sum_{k=2}^{50} (k^2-2k+1)$ (o) $\sum_{k=5}^{50} (k^2-2k+1)$

(p) $\sum_{m=10}^{20} m^3$ (q) $\sum_{m=0}^{15} (m+1)^3$

For the solutions to these, we will use several summation formulae, and the notation shown below for convenience:

$$S_0(n) = \sum_{i=1}^{n} 1 = n \qquad S_1(n) = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$S_2(n) = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad S_3(n) = \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

**Exercise 6.5** Use the sigma summation notation to set up the following problems, and then apply known formulae to compute the sums.

(a) Find the sum of the first 50 even numbers, $2+4+6+...$

(b) Find the sum of the first 50 odd numbers, $1+3+...$

(c) Find the sum of the first 50 integers of the form $n(n+1)$ where $n = 1,2,..,50$.

(d) Consider all the integers that are of the form $n(n-1)$ where $n = 1, 2, 3 \ldots$. Find the sum of the first 50 such numbers.

**Exercise 6.6** Compute the following sum.

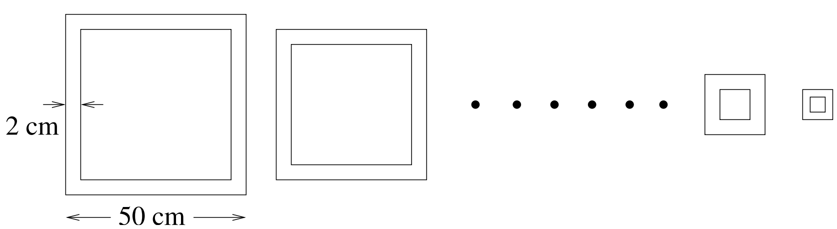
$$S = \sum_{i=1}^{12} i(1-i) + 2^i$$

**Exercise 6.7** A clock at London's Heathrow airport chimes every half hour. At the beginning of the $n$'th hour, the clock chimes $n$ times. (For example, at 8:00 AM the clock chimes 8 times, at 2:00 PM the clock chimes fourteen times, and at midnight the clock chimes 24 times.) The clock also chimes once at half-past every hour. Determine how many times in total the clock chimes in one full day. Use sigma notation to write the form of the series, and then find its sum.

**Exercise 6.8** A set of Japanese lacquer boxes have been made to fit one inside the other. All the boxes are cubical, and they have sides of lengths $1, 2, 3 \ldots 15$ inches. Find the total volume enclosed by all the boxes combined. Ignore the thickness of the walls of the boxes. (Calculator may be helpful.)

**Exercise 6.9** A framing shop uses a square piece of matt cardboard to create a set of square frames, one cut out from the other, with as little wasted as possible. The original piece of cardboard is 50 cm by 50 cm. Each of the "nested" square frames (see Problem 6.8 for the definition of nested) has a border 2 cm thick. How many frames in all can be made from this original piece of cardboard? What is the total area that can be enclosed by all these frames together?

Figure 6.8: For problem 6.9




**Exercise 6.10** The Great Pyramid of Giza, Egypt, built around 2,720-2,560 BC by Khufu (also known as Cheops) has a square base. We will assume that the base has side length 200 meters. The pyramid is made out of blocks of stone whose size is roughly $1 \times 1 \times 0.73$ m$^3$. There are 200 layers of blocks, so that the height of the pyramid is $200 \cdot 0.73 = 146$ m. Assume that the size of the pyramid steps (i.e. the horizontal distance between the end of one step and the beginning of another) is 0.5 m. We will also assume that the pyramid is solid, i.e. we will neglect the (relatively small) spaces that make up passages and burial chambers inside the structure.

(a) How many blocks are there in the layer that makes up the base of the pyramid? How many blocks in the second layer?

(b) How many blocks are there at the very top of the pyramid?

(c) Write down a summation formula for the total number of blocks in the pyramid and compute the total. (Hint: you may find it easiest to start the sum from the top layer and work your way down.)

**Exercise 6.11** Your local produce store has a special on oranges. Their display of fruit is a triangular pyramid with 100 layers, topped with a single orange (i.e. top layer: 1). The layer second from the top has three ($3 = 1+2$) oranges, and the one directly under it has six ($6 = 3+2+1$). The same pattern continues for all 100 layers. (This results in efficient "hexagonal" packing, with each orange sitting in a little depression created by three neighbours right under it.)

(a) How many oranges are there in the fourth and fifth layers from the top? How many in the $N^{\text{th}}$ layer from the top?

(b) If the "pyramid of oranges" only has 3 layers, how many oranges are used in total? What if the pyramid has 4, or 5 layers?

(c) Write down a formula for the sum of the total number of oranges that would be needed to make a pyramid with $N$ layers. Simplify your result so that you can use the summation formulae for $\sum n$ and for $\sum n^2$ to determine the total number of oranges in such a pyramid.

(d) Determine how many oranges are needed for the pyramid with 100 layers.

**Exercise 6.12** A right circular cone (shaped like "an inverted ice cream cone"), has height $h$ and a circular base (radius $r$) which is perpendicular to the cone's axis. In this exercise you will calculate the volume of this cone.

(1) Make $N$ uniform slices of the cone, each one parallel to the bottom, and of height $h/N$. Inside each slice put a cylindrical disk of the same height. The radius of the slices vary from 0 at the top to nearly $r$ at the bottom. (See Figure 6.9, $N = 10$.) Use similar triangles to answer these questions:

  (a) What is the smallest disk radius other than 0?

  (b) What is the radius of the $k$th disk?

(2) Express the total volume of the $N$ disks as a sum and evaluate it.

(3) As $N$ gets larger, what is the limit of this sum? (This is the volume of the cone.)

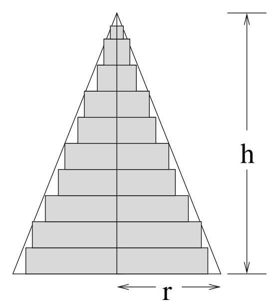



Figure 6.9: For problem 6.12. The cone with approximating $N$ disks.

**Exercise 6.13** A (finite) geometric series with $k$ terms is a sum of the form

$$1+r+r^2+r^3+\dots r^{k-1} = \sum_{n=0}^{k-1} r^n$$

and is given by the formula

$$S = \frac{1-r^k}{1-r} \quad \text{provided} \quad r \neq 1.$$

(a) This formula does not work if $r = 1$. Find the value of the series for $r = 1$.

(b) Express in summation notation and find the sum of the series

$$1+2^1+2^2+2^3+..+2^{10}.$$

(c) Express in summation notation and find the sum of the series

$$1+(0.5)^1+(0.5)^2+(0.5)^3+..(0.5)^{10}.$$

**Exercise 6.14** Use the sum of a geometric series to answer this question (use a calculator).

(a) Find the sum of the first 11 numbers of the form $1.1^k$ for $k=0,1,2,\ldots$. Now find the sum of the first 21 such numbers, the first 31 such numbers, the first 41 such numbers, and the first 51 such numbers.

(b) Repeat the process but now find sums of the numbers $0.9^k$, $k=0,1,2...$

(c) What do you notice about the pattern of results in (a) and (b) ? Can you explain what is happening in each of these cases and why they are different?

(d) Now consider the general problem of finding a value for the sum $\sum_{k=0}^{N} r^k$ when the number $N$ gets larger and larger. Suggest under what circumstances this sum will stay finite, and what value that finite sum will approach. To do this, you should think about the formula for the finite geometric sum and determine how it behaves for various values of $r$ as $N$ gets very large. This idea will be very important when we discuss infinite series.

**Exercise 6.15** According to legend, the inventor of the game of chess (in Persia) was offered a prize for his clever invention. He requested payment in kind, i.e. in kernels of grain. He asked to be paid 1 kernel for the first square of the board, two for the second, four for the third, etc. Use a summation formula to determine the total number of kernels of grain he would have earned in total.

(Hint: a chess board has $8\times 8=64$ squares and the first square contains $2^0=1$ kernel.)

**Exercise 6.16** A branching colony of fungus starts as a single spore with a single segment of filament growing out of it. This will be called generation 0. The tip of the filament branches, producing two new segments. Each tip then branches again and the process repeats. Suppose there have been 10 such branching events. How many tips will there be? If each segment is the same length (1 unit), what will be the total length of all the segments combined after 10 branching events? (Include the length of the initial single segment in your answer.)

**Exercise 6.17** **A branching plant and geometric series** A plant grows by branching, starting with one segment of length $\ell_0$ (in the 0'th generation). Every parent branch has exactly two daughter branches. The length of each daughter branch is (2/5) times the length of the parent branch. (Your answers will be in terms of $\ell_0$.)

(a) Find the total length of just the $12^{\text{th}}$ generation branch segments.

(b) Find the total length of the whole structure including the original segment and all 12 successive generations.

(c) Find the approximate total length of all segments in the whole structure if the plant keeps on branching forever.

**Exercise 6.18** **Branching airways, continued** Consider the branching airways in the lungs. Suppose that the initial bronchial segment has length $\ell_0$ and radius $r_0$. Let $\alpha$ and $\beta$ be the scale factors for the length and radius, respectively, of daughter branches (i.e. in a branching event, assume that $\ell_{n+1} = \alpha \ell_n$ and $r_{n+1} = \beta r_n$ are the relations that link daughters to parent branches, and that $0 < \alpha < 1, 0 < \beta < 1, l_i > 0, r_i > 0$ for all $i$). Let $b$ be the average number of daughters per parent branch. Let $F_n = S_n / V_n$ be the ratio of total surface area to total volume in the $n^{\text{th}}$ layer of the structure (i.e. for the $n^{\text{th}}$ generation branches).

(a) Find $F_n$ in terms of $\ell_0, r_0, b, \beta, \alpha$.

(b) In the lungs, it would be reasonable to expect that the surface area to volume ratio should *increase* from the initial segment down through the layers. What should be true of the parameters for this to be the case?

**Exercise 6.19** **Branching lungs**

(a) Consider branched airways that have the following geometric properties (Table 6.7). Find the total number of branch segments, the volume and

| | | |
|---|---|---|
| radius of first segment | $r_0$ | 0.5 cm |
| length of first segment | $\ell_0$ | 5.0 cm |
| ratio of daughter to parent length | $\alpha$ | 0.8 |
| ratio of daughter to parent radius | $\beta$ | 0.8 |
| number of branch generations | $M$ | 20 |
| average number daughters per parent | $b$ | 2 |

Table 6.7: Branching lungs properties.

the surface area of this branched structure. Note that a calculator will be helpful.

(b) What happens as $M$ gets larger? Will the volume and the surface area approach some finite limit, or will they grow indefinitely? How should the parameter $\beta$ be changed so that the surface area will keep increasing while the volume stays finite as $M$ increases?

### Exercise 6.20 Using simple geometry to compute an area

(a) Find the area of a regular octagon (a polygon that has eight equal sides). Assume that the length of each side is 1 cm.

(b) What is the area of the smallest circle that can be drawn around this octagon?

# 7
# Areas

## 7.1 Areas in the plane

A long-standing problem of integral calculus is how to compute the area of a region in the plane. This type of geometric problem formed part of the original motivation for the development of calculus techniques, and we will discuss it in many contexts in this course. We have already seen examples of the computation of areas of especially simple geometric shapes in Chapter 6. For triangles, rectangles, polygons, and circles, no advanced methods (beyond simple geometry) are needed. However, beyond these elementary shapes, such methods fail, and a new idea is needed. We will discuss such ideas in this chapter, and in Chapter 8.

We now consider the problem of determining the area of a region in the plane that has the following special properties: The region is formed by straight lines on three sides, and by a smooth curve on one of its edges, as shown in Figure 7.1. You might imagine that the shaded portion of this figure is a plot of land bounded by fences on three sides, and by a river on the fourth side. A farmer wishing to purchase this land would want to know exactly how large an area is being acquired. Here we set up the calculation of that area.

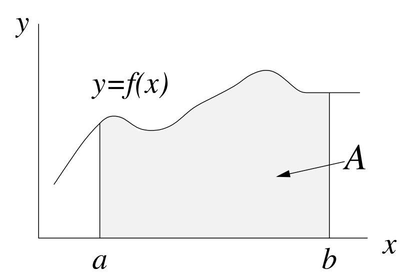



Figure 7.1: We consider the problem of determining areas of regions such bounded by the x axis, the lines $x = a$ and $x = b$ and the graph of some function, $y = f(x)$.

More specifically, we use a cartesian coordinate system to describe the region: we require that it falls between the $x$-axis, the lines $x = a$ and $x = b$, and the graph of a function $y = f(x)$. This is required for the process described below to work (not all planar areas have this property. Later examples indicate how to deal with some that do not.). We will first restrict attention to the case that $f(x) > 0$ for all points in the interval $a \le x \le b$ as we concentrate on "real areas". Later, we generalize our results and lift this restriction.

We will approximate the area of the region shown in Figure 7.1 by dissecting it into smaller regions (rectangular strips) whose areas are easy to determine. We will refer to this type of procedure as a **Riemann sum**. In Figure 7.2, we illustrate the basic idea using a region bounded by the function $y = f(x) = x^2$ on $0 \le x \le 1$. It can be seen that the approximation is fairly coarse when the number of rectangles is small (that is, the area of the rectangles is very different from the area of the region of interest). However, if the number of rectangles is increased, (as shown in subsequent panels of this same figure), we obtain a better and better approximation of the true area. In the limit as $N$, the number of rectangles, approaches infinity, the area of the desired region is obtained. This idea will form the core of this chapter. The reader will note a similarity with the idea we already encountered in obtaining the area of a circle, though in that context, we had used a dissection of the circle into approximating triangles.

With this idea in mind, in Section 7.2, we compute the area of the region shown in Figure 7.2 in two ways. First, we use a simple spreadsheet to do the computations for us. This is meant to illustrate the "numerical approach".

Then, as the alternate analytic approach , we set up the Riemann sum corresponding to the function shown in Figure 7.2. We will find that carefully setting up the calculation of areas of the approximating rectangles will be important. Making a cameo appearance in this calculation will be the formula for the sums of square integers developed in the previous chapter. A new feature will be the limit $N \to \infty$ that introduces the final step of arriving at the smooth region shown in the final panel of Figure 7.2.

## 7.2 *Computing the area under a curve by rectangular strips*

### *First approach: Numerical integration using a spreadsheet*

The same tool that produces Figure 7.2 can be used to calculate the areas of the steps for each of the panels in the figure. To do this, we fix $N$ for a given panel, (e.g. $N = 10, 20$, or 40), find the corresponding value of $\Delta x$, and set up a calculation which adds up the areas of steps, i.e. $\sum x^2 \Delta x$ in a given panel. The ideas are analogous to those described in Section 7.2, but a spreadsheet does the number crunching for us.

Using a spreadsheet, for example, we find the following results at each stage: For $N = 10$ strips, the area is 0.3850 units$^2$, for $N = 20$ strips it is 0.3588, for $N = 40$ strips, the area is 0.3459. If we increase $N$ greatly, e.g. set $N = 1000$ strips, which begins to approximate the limit of $N \to \infty$, then the area obtained is 0.3338 units$^2$. Note that all these values are approximations, correct to 4 decimal places. Compare with the exact calculations in later section.

This example illustrates that areas can be computed "numerically" - indeed many of the laboratory exercises that accompany this course will be based on precisely this idea. The advantage of this approach is that it requires only elementary "programming" - i.e. the assembly of a simple **algorithm**, i.e. a set of instructions. Once assembled, we can use essentially the same algorithm to explore various functions, intervals, number of rectangles, etc. Lab 2 in this course will motivate the student to explore this numerical integration approach, and later labs will expand and generalize the idea to a variety of settings.

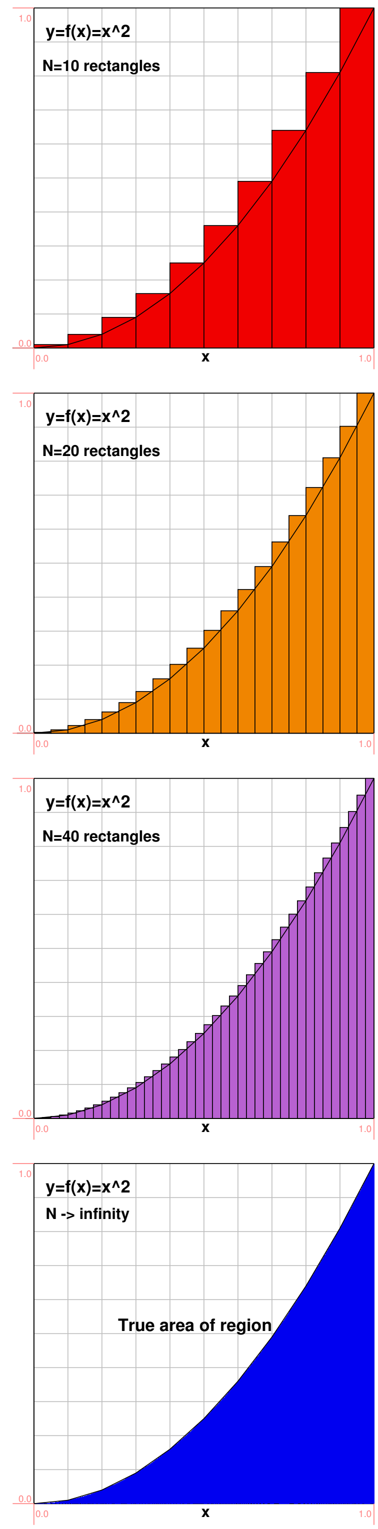



Figure 7.2: The function $y = x^2$ for $0 \le x \le 1$ is shown, with rectangles that approximate the area under its curve. As we increase the number of rectangular strips, the total area of the strips becomes a better and better approximation of the desired "true" area. Shown are the intermediate steps $N = 10$, $N = 20$, $N = 40$ and the true area for $N \to \infty$

In our second approach, we set up the problem analytically. We will find that results are similar. However, we will get deeper insight by understanding what happens in the limit as the number of strips $N$ gets very large.

*Second approach: Analytic computation using Riemann sums*

In this section we consider the detailed steps involved in analytically computing the area of the region bounded by the function

$$y = f(x) = x^2, \quad 0 \le x \le 1.$$

By this we mean that we use "pen-and-paper" calculations, rather than computational aids to determine that area.

We set up the rectangles (as shown in Figure 7.2, with detailed labeling in Figures 7.3), determine the heights and areas of these rectangle, sum their total area, and then determine how this value behaves as the rectangles get more numerous (and thinner).

The interval of interest in this problem is $0 \le x \le 1$. Let us subdivide this interval into $N$ equal subintervals. Then each has width $1/N$. (We will refer to this width as $\Delta x$, as shown in Figure 7.3, as it forms a difference of successive $x$ coordinates.) The coordinates of the endpoints of these subintervals will be labeled $x_0, x_1, \ldots, x_k, \ldots, x_N$, where the value $x_0 = 0$ and $x_N = 1$ are the endpoints of the original interval. Since the points are equally spaced, starting at $x_0 = 0$, the coordinate $x_k$ is just $k$ steps of size $1/N$ along the $x$ axis, i.e. $x_k = k(1/N) = k/N$. In the right panel of Figure 7.3, some of these coordinates have been labeled. For clarity, we show only the first few points, together with a representative pair $x_{k-1}$ and $x_k$ inside the region.

Let us look more carefully at one of the rectangles. Suppose we look at the rectangle labeled $k$. Such a representative k-th rectangle is shown shaded in Figures 7.3. The height of this rectangle is determined by the value of the function, since one corner of the rectangle is "glued" to the curve. The choice shown in Figure 7.3 is to affix the right corner of each rectangle on the curve. This implies that the height of the $k$-th rectangle is obtained from substituting $x_k$ into the function, i.e. height = $f(x_k)$. The base of every rectangle is the same, i.e. base = $\Delta x = 1/N$. This means that the area of the $k$-th rectangle, shown shaded, is

$$a_k = \text{height} \times \text{base} = f(x_k)\Delta x$$

We now use three facts:

$$f(x_k) = x_k^2, \quad \Delta x = \frac{1}{N}, \quad x_k = \frac{k}{N}.$$

Then the area of the $k$'th rectangle is

$$a_k = \text{height} \times \text{base} = f(x_k)\Delta x = \underbrace{\left(\frac{k}{N}\right)^2}_{f(x_k)} \underbrace{\left(\frac{1}{N}\right)}_{\Delta x}.$$

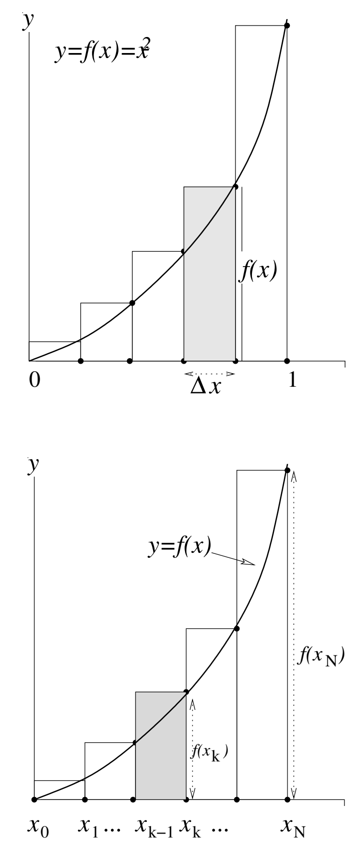



Figure 7.3: The region under the graph of $y = f(x)$ for $0 \le x \le 1$ will be approximated by a set of $N$ rectangles. A rectangle (shaded) has base width $\Delta x$ and height $f(x)$. Since $0 \le x \le 1$, and the all rectangles have the same base width, it follows that $\Delta x = 1/N$. In the panel on the right, the coordinates of base corners and two typical heights of the rectangles have been labeled. Here $x_0 = 0, x_N = 1$ and $x_k = k\Delta x$.

A list of rectangles, and their properties are shown in Table 7.1. This may help the reader to see the pattern that emerges in the summation. (In general this table is not needed in our work, and it is presented for this example only, to help visualize how heights of rectangles behave.) The total area of all

| rectangle ($k$) | right $x$ coord ($x_k$) | height $f(x_k)$ | area $a_k$ |
|---|---|---|---|
| 1 | $(1/N)$ | $(1/N)^2$ | $(1/N)^2\Delta x$ |
| 2 | $(2/N)$ | $(2/N)^2$ | $(2/N)^2\Delta x$ |
| 3 | $(3/N)$ | $(3/N)^2$ | $(3/N)^2\Delta x$ |
| $\vdots$ | | | |
| $k$ | $(k/N)$ | $(k/N)^2$ | $(k/N)^2\Delta x$ |
| $\vdots$ | | | |
| $N$ | $(N/N)=1$ | $(N/N)^2=1$ | $(1)\Delta x$ |

Table 7.1: The label, position, height, and area $a_k$ of each rectangular strip is shown above. Each rectangle has the same base width, $\Delta x = 1/N$. We approximate the area under the curve $y = f(x) = x^2$ by the sum of the values in the last column, i.e. the total area of the rectangles.

rectangular strips (a sum of the values in the right column of Table 7.1) is

$$A_{N \text{ strips}} = \sum_{k=1}^{N} a_k = \sum_{k=1}^{N} f(x_k)\Delta x = \sum_{k=1}^{N} \left(\frac{k}{N}\right)^2 \left(\frac{1}{N}\right). \tag{7.1}$$

The expressions shown in Eqn. (7.1) is a Riemann sum. A recurring theme underlying integral calculus is the relationship between Riemann sums and definite integrals, a concept introduced later on in this chapter.

We now rewrite this sum in a more convenient form so that summation formulae developed in Chapter 6 can be used. In this sum, only the quantity $k$ changes from term to term. All other quantities are common factors, so that

$$A_{N \text{ strips}} = \left(\frac{1}{N^3}\right) \sum_{k=1}^{N} k^2.$$

The formula (6.2) for the sum of square integers can be applied to the summation, resulting in

$$A_{N \text{ strips}} = \left(\frac{1}{N^3}\right) \frac{N(N+1)(2N+1)}{6} = \frac{(N+1)(2N+1)}{6N^2}. \tag{7.2}$$

In the box below, we use Eqn. (7.2) to compute that approximate area for values of $N$ shown in the first three panels of Fig 7.2. Note that these are comparable to the values we obtained “numerically” in Section 7.2. We plug in the value of $N$ into (7.2) and use a calculator to obtain the results below.

If $N = 10$ **strips** (Figure 7.2a), the width of each strip is 0.1 unit. According to equation 7.2, the area of the 10 strips (shown in red) is

$$A_{10 \text{ strips}} = \frac{(10+1)(2\cdot 10+1)}{6\cdot 10^2} = 0.385.$$

If $N = 20$ **strips** (Figure 7.2b), $\Delta x = 1/20 = 0.05$, and

$$A_{20 \text{ strips}} = \frac{(20+1)(2\cdot 20+1)}{6\cdot 20^2} = 0.35875.$$

If $N = 40$ **strips** (Figure 7.2c), $\Delta x = 1/40 = 0.025$ and

$$A_{40 \text{ strips}} = \frac{(40+1)(2 \cdot 40+1)}{6 \cdot 40^2} = 0.3459375.$$

We will define **the true area** under the graph of the function $y = f(x)$ over the given interval to be:

$$A = \lim_{N \to \infty} A_{N \text{ strips}}.$$

This means that the true area is obtained by letting the number of rectangular strips, $N$, get very large, (while the width of each one, $\Delta x = 1/N$ gets very small.)

In the example discussed in this section, the true area is found by taking the limit as $N$ gets large in equation (7.2), i.e.,

$$A = \lim_{N \to \infty} \left( \frac{1}{N^2} \right) \frac{(N+1)(2N+1)}{6} = \frac{1}{6} \lim_{N \to \infty} \frac{(N+1)(2N+1)}{N^2}.$$

To evaluate this limit, note that when $N$ gets very large, we can use the approximations, $(N+1) \approx N$ and $(2N+1) \approx 2N$ so that (simplifying and cancelling common factors)

$$\lim_{N \to \infty} \frac{(N+1)(2N+1)}{N^2} = \lim_{N \to \infty} \frac{(N)}{N} \frac{(2N)}{N} = 2.$$

The result is:

$$A = \frac{1}{6}(2) = \frac{1}{3} \approx 0.333. \tag{7.3}$$

Thus, the true area of the region (Figure 7.2d) is is 1/3 units$^2$.

## *Riemann sums using left (rather than right) endpoints*

So far, we used the right endpoint of each rectangular strip to assign its height using the given function (see Figs. 7.2, 7.3, 7.6). Restated, we "glued" the top right corner of the rectangle to the graph of the function. This is the so called **right endpoint approximation**. We can just as well use the left corners of the rectangles to assign their heights (**left endpoint approximation**). A comparison of these for the function $y = f(x) = x^2$ is shown in Figs. 7.4 and 7.5.

In the case of the left endpoint approximation, we evaluate the heights of the rectangles starting at $x_0$ (instead of $x_1$, and ending at $x_{N-1}$ (instead of $x_N$). There are still $N$ rectangles. To compare, sum of areas of the rectangles in the left versus the right endpoint approximation is

$$\text{Right endpoints:} \quad A_{N \text{ strips}} = \sum_{k=1}^{N} f(x_k)\Delta x.$$

$$\text{Left endpoints:} \quad A_{N \text{ strips}} = \sum_{k=0}^{N-1} f(x_k)\Delta x.$$

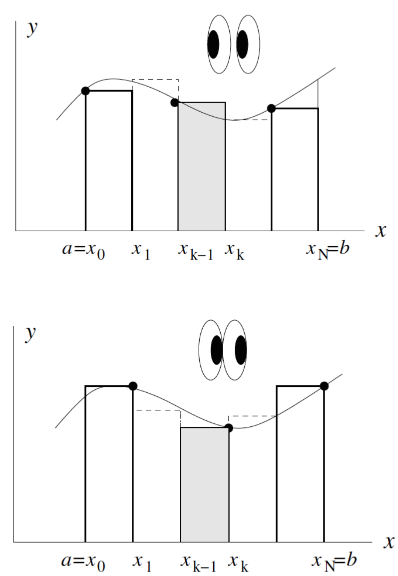



Figure 7.4: The area under the curve $y = f(x)$ over an interval $a \le x \le b$ could be computed by using either a left or right endpoint approximation. That is, the heights of the rectangles are adjusted to match the function of interest either on the right or on their left corner. Here we compare the two approaches. Usually both lead to the same result once a limit is computer to arrive at the "true " area.

We here look again at a simple example, using the quadratic function,

$$f(x) = x^2, \quad 0 \le x \le 1,$$

We now compare the right and left endpoint approximation. These are shown in panels of Figure 7.5. Note that

$$\Delta x = \frac{1}{N}, \quad x_k = \frac{k}{N},$$

The area of the $k$'th rectangle is

$$a_k = f(x_k) \times \Delta x = (k/N)^2 \, (1/N),$$

but now the sum starts at $k = 0$ so

$$A_{N \text{ strips}} = \sum_{k=0}^{N-1} f(x_k)\Delta x = \sum_{k=0}^{N-1} \left(\frac{k}{N}\right)^2 \left(\frac{1}{N}\right) = \left(\frac{1}{N^3}\right) \sum_{k=0}^{N-1} k^2.$$

The first rectangle corresponds to $k = 0$ in the left endpoint approximation (rather than $k = 1$ in the right endpoint approximation). But the $k = 0$ rectangle makes no contribution (as its area is zero in this example) and we have one less rectangle at the right endpoint of the interval, since the N'th rectangle is $k = N - 1$. Then the sum is

$$A_{N \text{ strips}} = \left(\frac{1}{N^3}\right) \frac{(2(N-1)+1)(N-1)(N)}{6} = \frac{(2N-1)(N-1)}{6N^2}.$$

The area, obtained by taking a limit for $N \to \infty$ is

$$A = \lim_{N\to\infty} A_{N \text{ strips}} = \lim_{N\to\infty} \frac{(2N-1)(N-1)}{6N^2} = \frac{2}{6} = \frac{1}{3}.$$

We see that, after computing the limit, the result for the "true area" under the curve is exactly the same as we found earlier in this chapter using the right endpoint approximation.

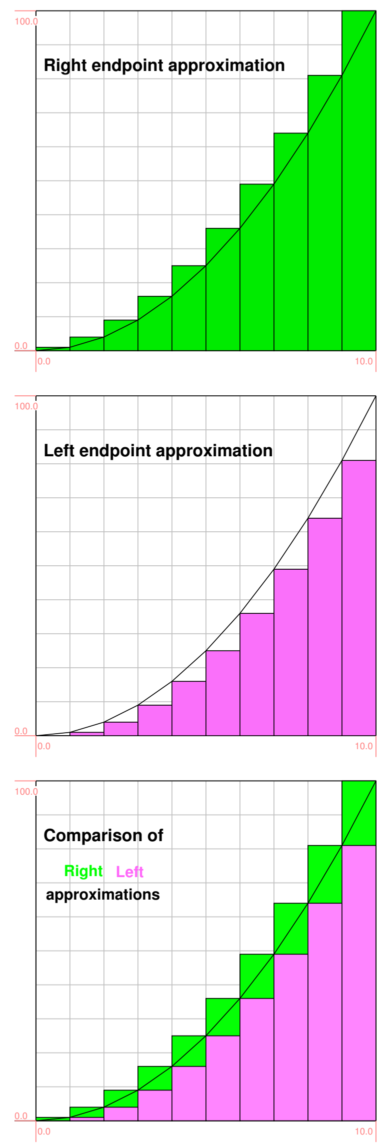



Figure 7.5: Rectangles with left or right corners on the graph of $y = x^2$ are compared in this picture. The approximation shown in pink is "missing" the largest rectangle shown in green. However, in the limit as the number of rectangles, $N \to \infty$, the true area obtained is the same.

**More general interval**

To calculate the area under the curve $y = f(x) = x^2$ over the interval $2 \le x \le 5$ using $N$ rectangles, the width of each one would be $\Delta x = (5 - 2)/N = 3/N$, (i.e., length of interval divided by N). Since the interval starts at $x_0 = 2$, and increments in units of $(3/N)$, the $k$'th coordinate is $x_k = 2 + k(3/N) = 2 + (3k/N)$. The area of the $k$'th rectangle is then $A_K = f(x_k) \times \Delta x = [(2 + (3k/N))^2](3/N)$, and this is to be summed over $k$. A similar algebraic simplification, summation formulae, and limit is needed to calculate the true area.

*Comments*

Many student who have had calculus before in highschool, ask "why do we bother with such tedious calculations, when we could just use integration?".

Indeed, our development of Riemann sums foreshadows and anticipates the idea of a definite integral, and in short order, some powerful techniques will help to shortcut such technical calculations. There are two reasons why we linger on Riemann sums. First, in order to understand integration adequately, we must understand the underlying "technology" and concepts; this proves vital in understanding how to use the methods, and when things can go wrong. It also helps to understand what integrals represent in applications that occur later on. Second, even though we will shortly have better tools for analytical calculations, the ideas of setting up area approximations using rectangular strips is very similar to the way that the spreadsheet computations are designed. (However, the summation is handled automatically using the spreadsheet, and no "formulae" are needed.) In Section 7.2, we gave only few details of the steps involved. The student will find that understanding the ideas of Section 7.2 will go hand-in-hand with understanding the numerical approach of Section 7.2.

The ideas outlined above can be applied to more complicated situations. In the next section we consider a practical problem in which a similar calculation is carried out.

## 7.3 *The area of a leaf*

Leaves act as solar energy collectors for plants. Hence, their surface area is an important property. In this section we use our techniques to determine the area of a rhododendron leaf, shown in Figure 7.6. For simplicity of treatment, we will first consider a function designed to mimic the shape of the leaf in a simple system of units: we will scale distances by the length of the leaf, so that its profile is contained in the interval $0 \le x \le 1$. We later ask how to modify this treatment to describe similarly curved leaves of arbitrary length and width, and leaves that are less symmetric.

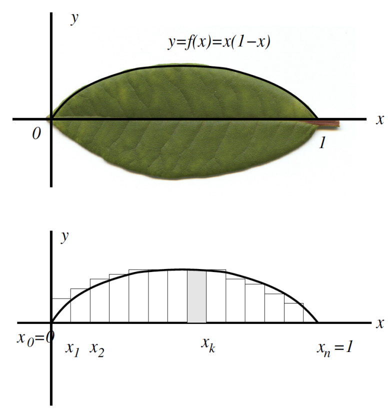



Figure 7.6: In this figure we show how the area of a leaf can be approximated by rectangular strips.

As shown in Figure 7.6, a simple parabola, of the form $y = f(x) = x(1-x)$, provides a convenient approximation to the top edge of the leaf. To check that this is the case, we observe that at $x = 0$ and $x = 1$, the curve intersects the x axis. At $0 < x < 1$, the curve is above the axis. Thus, the area between this curve and the x axis, is one half of the leaf area.

We set up the computation of approximating rectangular strips as before, by subdividing the interval of interest into $N$ rectangular strips. We can set up the calculation systematically, as follows:

$$\text{length of interval} = 1 - 0 = 1$$
$$\text{number of segments}, N$$
$$\text{width of rectangular strips}, \Delta x = \frac{1}{N}$$
$$\text{the } k\text{'th } x \text{ value}, x_k = k\frac{1}{N} = \frac{k}{N}$$
$$\text{height of } k\text{'th rectangular strip}, f(x_k) = x_k(1 - x_k)$$

The representative $k$'th rectangle is shown shaded in Figure 7.6: Its area is

$$a_k = \text{base} \times \text{height} = \Delta x \cdot f(x_k) = \underbrace{\left(\frac{1}{N}\right)}_{\Delta x} \cdot \underbrace{\left(\frac{k}{N}(1-\frac{k}{N})\right)}_{f(x_k)}.$$

The total area of these rectangular strips is:

$$A_{N \text{ strips}} = \sum_{k=1}^{N} a_k = \sum_{k=1}^{N} \Delta x \cdot f(x_k) = \sum_{k=1}^{N} \left(\frac{1}{N}\right) \cdot \left(\frac{k}{N}(1-\frac{k}{N})\right).$$

Simplifying the result (so we can use summation formulae) leads to:

$$A_{N \text{ strips}} = \left(\frac{1}{N}\right) \sum_{k=1}^{N} \left(\frac{k}{N}(1-\frac{k}{N})\right) = \left(\frac{1}{N^2}\right) \sum_{k=1}^{N} k - \left(\frac{1}{N^3}\right) \sum_{k=1}^{N} k^2.$$

Using the summation formulae (6.4) and(6.2) from Chapter 6 results in:

$$A_{N \text{ strips}} = \left(\frac{1}{N^2}\right) \left(\frac{N(N+1)}{2}\right) - \left(\frac{1}{N^3}\right) \left(\frac{(2N+1)N(N+1)}{6}\right).$$

Simplifying, and regrouping terms, we get

$$A_{N \text{ strips}} = \frac{1}{2} \left(\frac{(N+1)}{N}\right) - \frac{1}{6} \left(\frac{(2N+1)(N+1)}{N^2}\right).$$

This is the area for a finite number, $N$, of rectangular strips. As before, the **true area** is obtained as the limit as $N$ goes to infinity, i.e. $A = \lim_{N \to \infty} A_{N \text{ strips}}$. We obtain:

$$A = \lim_{N \to \infty} \frac{1}{2} \left(\frac{(N+1)}{N}\right) - \lim_{N \to \infty} \frac{1}{6} \left(\frac{(2N+1)(N+1)}{N^2}\right) = \frac{1}{2} - \frac{1}{6} \cdot 2 = \frac{1}{6}.$$

Taking the limit leads to

$$A = \frac{1}{2} - \frac{1}{6} \cdot 2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

Thus the area of the entire leaf (twice this area) is 1/3.

**Remark:** The function in this example can be written as $y = x - x^2$. For part of this expression, we have seen a similar calculation in Section 7.2. This example illustrates an important property of sums, namely the fact that we can rearrange the terms into simpler expressions that can be summed individually.

In the homework problems accompanying this chapter, we investigate how to describe leaves with arbitrary lengths and widths, as well as leaves with shapes that are tapered, broad, or less symmetric than the current example.

## 7.4 Area under an exponential curve

In the precious examples, we considered areas under curves described by a simple quadratic functions. Each of these led to calculations in which sums of integers or square integers appeared. Here we demonstrate an example in which a geometric sum will be used. Recall that we derived Eqn. (6.5) in Chapter 6, for a finite geometric sum.

We will find the area under the graph of the function $y = f(x) = e^{2x}$ over the interval between $x = 0$ and $x = 2$. In evaluating a limit in this example, we will also use the fact that the exponential function has a linear approximation as follows:

$$e^z \approx 1 + z$$

(See Linear Approximations in an earlier calculus course.)

As before, we subdivide the interval into N pieces, each of width $2/N$. Proceeding systematically as before, we write

$$\begin{aligned}
&\text{length of interval} = 2 - 0 = 2 \\
&\text{number of segments} = N \\
&\text{width of rectangular strips}, \Delta x = \frac{2}{N} \\
&\text{the } k\text{'th } x \text{ value}, x_k = k\frac{2}{N} = \frac{2k}{N} \\
&\text{height of } k\text{'th rectangular strip}, f(x_k) = e^{2x_k} = e^{2(2k/N)} = e^{4k/N}
\end{aligned}$$

We observe that the length of the interval (here 2) has affected the details of the calculation. As before, the area of the k'th rectangle is

$$a_k = \text{base} \times \text{height} = \Delta x \times f(x_k) = \left(\frac{2}{N}\right) e^{4k/N},$$

and the total area of all the rectangles is

$$A_N \text{ strips} = \left(\frac{2}{N}\right) \sum_{k=1}^{N} e^{4k/N} = \left(\frac{2}{N}\right) \sum_{k=1}^{N} r^k = \left(\frac{2}{N}\right) \left(\sum_{k=0}^{N} r^k - r^0\right),$$

where $r = e^{4/N}$. This is a finite geometric series. Because the series starts with $k = 1$ and not with $k = 0$, the sum is

$$A_N \text{ strips} = \left(\frac{2}{N}\right) \left[\frac{(1 - r^{N+1})}{(1 - r)} - 1\right].$$

After some simplification and using $r = e^{4/N}$, we find that

$$A_N \text{ strips} = \left(\frac{2}{N}\right) e^{4/N} \frac{1 - e^4}{1 - e^{4/N}} = 2\frac{1 - e^4}{N(e^{-4/N} - 1)}.$$

We need to determine what happens when $N$ gets very large. We can use the linear approximation

$$e^{-4/N} \approx 1 - 4/N$$

to evaluate the limit of the term in the denominator, and we find that

$$A = \lim_{N\to\infty} 2\frac{1-e^4}{N(e^{-4/N}-1)} = \lim_{N\to\infty} 2\frac{1-e^4}{-N(1+4/N-1)} = 2\frac{e^4-1}{4} \approx 26.799.$$

## 7.5 *The definite integral*

We now introduce a central concept that will form an important theme in this course, that of the definite integral. We begin by defining a new piece of notation relevant to the topic in this chapter, namely the area associated with the graph of a function.

For a function $y = f(x) > 0$ that is bounded and continuous[1] on an interval $[a,b]$ (also written $a \le x \le b$), we define the *definite integral*,

$$I = \int_a^b f(x)\,dx \tag{7.4}$$

to be the area $A$ of the region under the graph of the function between the endpoints $a$ and $b$. See Figure 7.7.

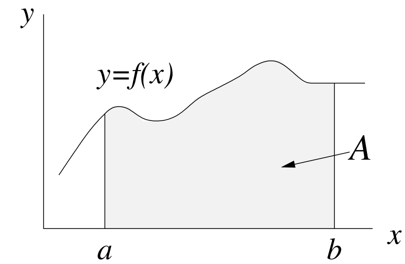



Figure 7.7: The shaded area $A$ corresponds to the definite integral $I$ of the function $f(x)$ over the interval $a \le x \le b$.

[1] A function is said to be bounded if its graph stays between some pair of horizontal lines. It is continuous if there are no "breaks" in its graph.

### *Remarks*

1. The definite integral is a number.
2. The value of the definite integral depends on the function, and on the two end points of the interval.
3. From previous remarks, we have a procedure to calculate the value of the definite integral by dissecting the region into rectangular strips, summing up the total area of the strips, and taking a limit as $N$, the number of strips gets large. (The calculation may be non-trivial, and might involve sums that we have not discussed in our simple examples so far, but in principle the procedure is well-defined.)

### *Examples*

We have calculated the areas of regions bounded by particularly simple functions. To practice notation, we write down the corresponding definite integral in each case. Note that in many of the examples below, we need no elaborate calculations, but merely use previously known or recently derived results, to familiarize the reader with the new notation just defined.

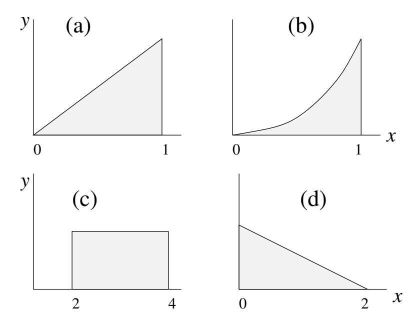



Figure 7.8: Examples (1-4) relate areas shown above to *definite integrals*.

1. The area under the function $y = f(x) = x$ over the interval $0 \le x \le 1$ is triangular, with base and height 1. The area of this triangle is thus $A = (1/2)$base$\times$ height= 0.5 (Figure 7.8a). Hence,

$$\int_0^1 x\,dx = 0.5.$$

2. In Section 7.2, we also computed the area under the function $y = f(x) = x^2$ on the interval $0 \le x \le 1$ and found its area to be 1/3 (See Eqn. (7.3) and Fig. 7.8(b)). Thus
$$\int_0^1 x^2\, dx = 1/3 \simeq 0.333.$$
3. A constant function of the form $y = 1$ over an interval 2 *lex* $\le 4$ would produce a rectangular region in the plane, with base (4-2)=2 and height 1 (Figure 7.8(c)). Thus
$$\int_2^4 1\, dx = 2.$$
4. The function $y = f(x) = 1 - x/2$ (Figure 7.8(d)) forms a triangular region with base 2 and height 1, thus
$$\int_0^2 (1 - x/2)\, dx = 1.$$

## 7.6 *The area as a function*

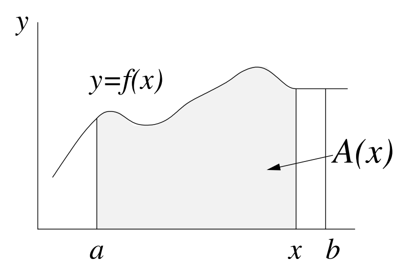


Figure 7.9: We define a new function $A(x)$ to be the area associated with the graph of some function $y = f(x)$ from the fixed endpoint $a$ up to the endpoint $x$, where $a \le x \le b$.

In Chapter 8, we will elaborate on the idea of the definite integral and arrive at some very important connection between differential and integral calculus. Before doing so, we have to extend the idea of the definite integral somewhat, and thereby define a new function, $A(x)$.

We will investigate how the area under the graph of a function changes as one of the endpoints of the interval moves. We can think of this as a function that gradually changes (i.e. the area accumulates) as we sweep across the interval $(a, b)$ from left to right in Figure 7.1. The function $A(x)$ represents the area of the region shown in Figure 7.9.

Extending our definition of the definite integral, we might be tempted to use the notation
$$A(x) = \int_a^x f(x)\, dx.$$
However, there is a slight problem with this notation: the symbol $x$ is used in slightly confusing ways, both as the argument of the function and as the variable endpoint of the interval. To avoid possible confusion, we will prefer the notation
$$A(x) = \int_a^x f(s)\, ds.$$
(or some symbol other than $s$ used as a placeholder instead of $x$.)

An analogue already seen is the sum
$$\sum_{k=1}^{N} k^2$$
where $N$ denotes the "end" of the sum, and $k$ keeps track of where we are in the process of summation. The symbol $s$, sometimes called a "dummy variable" is analogous to the summation symbol $k$.

In the upcoming Chapter 8, we will investigate properties of this new “area function” $A(x)$ defined above. This will lead us to the *Fundamental Theorem of Calculus*, and will provide new and powerful tools to replace the dreary summations that we had to perform in much of Chapter 7. Indeed, we are about to discover the amazing connection between a function, the area $A(x)$ under its curve, and the derivative of $A(x)$.

## *7.7 Summary*

In this chapter, we showed how to calculate the area of a region in the plane that is bounded by the x axis, two lines of the form $x = a$ and $x = b$, and the graph of a positive function $y = f(x)$. We also introduced the terminology “definite integral” (Section 7.5) and the notation (7.4) to represent that area.

One of our main efforts here focused on how to actually compute that area by the following set of steps:

- Subdivide the interval $[a, b]$ into smaller intervals (width $\Delta x$).
- Construct rectangles whose heights approximate the height of the function above the given interval.
- Add up the areas of these approximating rectangles. (Here we often used summation formulae from Chapter 6.) The resulting expression, such as Eqn. (7.1), for example, was denoted a Riemann sum.
- Find out what happens to this total area in the limit when the width $\Delta x$ goes to zero (or, in other words, when the number of rectangles $N$ goes to infinity).

We showed both the analytic approach, using Riemann sums and summation formulae to find areas, as well as numerical approximations using a spreadsheet tool to arrive at similar results. We then used a variety of examples to illustrate the concepts and arrive at computed areas.

As a final important point, we noted that the area “under the graph of a function” can itself be considered a function. This idea will emerge as particularly important and will lead us to the key concept linking the geometric concept of areas with the analytic properties of antiderivatives. We shall see this link in the Fundamental Theorem of Calculus, in Chapter 8.

## *Exercises*

**Exercise 7.1 Areas in the plane 1**

(a) Compute the area of the staircase shown in Figure 7.10.

(b) What would be the area of that region if, instead of the ten steps shown, it consisted of 100 steps, each of width 0.1 and with heights $0.1, 0.2, \ldots 10$?

(c) If there are a very large number of steps of very small width, and very small height increments, what would be the approximate area of the region shaded?

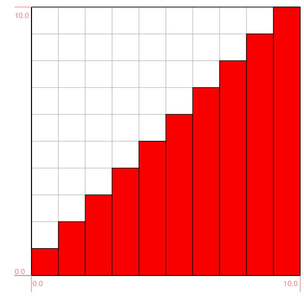


Figure 7.10: Figure for Exercise 7.1.

**Exercise 7.2** Find the area bounded by the $x$-axis, the $y$-axis, and the graph of the function $y = f(x) = 1 - x$. (See Figure 7.11.)

(a) By using your knowledge about the area of a triangular region.

(b) By setting up the problem as a Riemann sum, i.e., as a sum of the areas of $N$ rectangular strips, using the appropriate summation formula, and letting the number of strips ($N$) get larger and larger to arrive at the result. Show that your answer in (b) is then identical to the answer in (a). The point of this exercise is to practice setting up and computing Riemann sum.

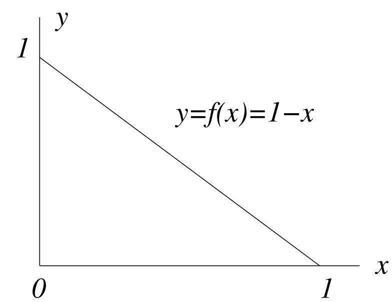



Figure 7.11: Figure for Exercise 7.2

**Exercise 7.3 Areas in the plane 2** Compute the areas of the two shaded regions for the interval $0 < x < 20$ in Figure 7.12. The curve shown on the diagram is $y = x^2$ and there are twenty rectangles forming the staircase. How do the areas of the shaded regions relate to the area $A$ under this curve?

**Exercise 7.4** Estimate the area under the graph of $f(x) = x^2 + 2$ from $x = -1$ to $x = 2$ in each of the following ways, and sketch the graph and the rectangles in each case. (This exercise is less tedious if done on a spread sheet.)

(a) By using three rectangles and left endpoints.

(b) Improve your estimate in (a) by using 6 rectangles.

(c) Repeat part (a) using midpoints.

(d) Repeat part (b) using midpoints.

(e) From your sketches in parts (a), (b), (c) and (d), which appears to be the best estimate?

**Exercise 7.5** Find the area $A$ between the graphs of the functions: $y = f(x) = 2x$ and $y = g(x) = 1 + x^2$ between $x = 0$ and their intersection point.

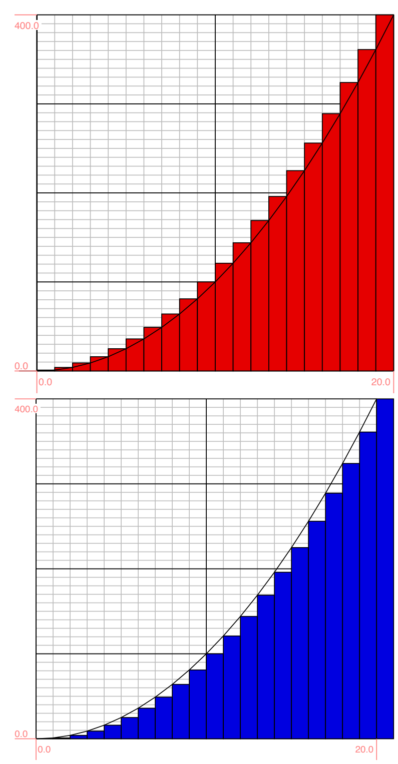


Figure 7.12: Figure for Exercise 7.3

**Exercise 7.6** Consider the function $y = f(x) = e^x$ on the interval $[0, 1]$. Subdivide the interval $[0, 1]$ into 4 equal subintervals of width 0.25 and find an approximation to the area of this region using four rectangular strips.

(a) Use the left endpoints of each interval.

(b) Use the right endpoints of each interval.

(c) Explain why your answer in (a) is different from your answer in (b).

**Exercise 7.7** Find the area between the $x$ axis and the graph of the function $y = f(x) = 2 - x$ between $x = 0$ and $x = 2$.

(a) By using your knowledge about the area of a triangular region.

(b) By setting up the problem as a sum of the areas of $n$ rectangular strips, using the appropriate summation formula, and letting the number of strips ($n$) get larger and larger to arrive at the result. Show that your answer in (b) is then identical to the answer in (a).

**Exercise 7.8** Find the area $A$ between the graphs of the following two functions: $y = f(x) = x^2$ and $y = g(x) = 2 - x^2$. (Hint: what do we mean by "between"? Where does this region begin and where does it end?) This problem should be set up in the form of a sum of areas of rectangular strips. You should *not* use previous familiarity with integration techniques to solve it.

**Exercise 7.9** Determine a function $f(x)$ and an interval on the $x$-axis such that the expression shown below is equal to the area under the graph of $f(x)$ over the given interval:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1 + \frac{3i}{n}}$$

What do the terms that appear in this expression represent?

**Exercise 7.10** **The interval $a \le x \le b$** Use Riemann sums to find an expression for the area under the graph of

$$y = f(x) = x^2 + 2x + 1 \qquad a \le x \le b$$

**Exercise 7.11**

(a) Show that the area of a trapezoid with base $b$ and heights $h_1, h_2$ (region shown on the left in Figure 7.13) is $A_{\text{trapezoid}} = \frac{1}{2}b(h_1 + h_2)$. Hint: consider dividing up the trapezoid into simpler geometric shapes.

(b) Use the result for trapezoids to calculate the region under the graph of the function shown in Figure 7.13.

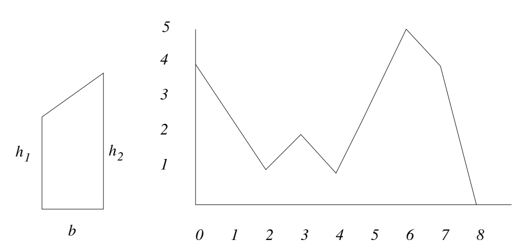



Figure 7.13: For problem 7.11

**Exercise 7.12** Consider the function $y = x^3$. Using the spreadsheet, create one plot which contains all of the following superimposed:

(a) a graph of this function for $0 < x < 1$.

(b) a bar-graph showing 20 rectangular strips, whose top *right* corner lies on the graph of the function.

(c) The function $A(x)$ which adds up the areas of the first strip, the first two strips, the first three strips, etc. (See Figure 2.4 of the lecture notes for an example).

(d) The function $g(x) = x^4/4$ which is the anti-derivative of the original function.

**Exercise 7.13** Determine the values of each of the following definite integrals. Each of these can be done using simple geometry, and need no "integration" techniques. It will be helpful to sketch the regions and functions involved. Recall that we have formulae for areas of rectangles, triangles, and circles.

(a) $\int_0^1 2x\,dx$

(b) $\int_{-1}^1 (1-x)\,dx$

(c) $\int_{-2}^2 \sqrt{4-x^2}\,dx$

**Exercise 7.14** Consider the function $y = f(x) = x^3$. We would like to find the area under this curve for $0 < x < 3$ using the approximation by rectangular strips. Subdivide the interval into $N$ regular subintervals such that $x_0 = 0, x_1 = \Delta x, \ldots x_k = k\Delta x \ldots x_N = 3$.

(a) What is the width of each interval (in terms of $N$)? Express $x_k$ in terms of $k$ and $N$.

(b) Consider $N$ rectangles arranged so that the height of their top right corner is determined by the function $f(x)$. The first rectangle would have height $f(x_1)$, etc. Express the area of the $k^{\text{th}}$ rectangle in terms of $k$ and $N$.

(c) Set up a sum of the areas of all these rectangles, and use the summation formula for the sum of cube integers to "add up" those areas and arrive at a total area $A_N$ associated with those $N$ rectangular strips. Your answer should be expressed in terms of $N$.

(d) Now consider what happens to $A_N$ when the number of rectangles, $N$ gets large. Find the value of the area $A$ under the curve by taking a limit as $N \to \infty$.

**Exercise 7.15** In problem 7.14, we found the area under the function $f(x) = x^3$ for the interval $0 < x < 3$. Use your results from that problem to now determine the area $A$ under the same function but over the interval $2 < x < 3$. (Hint: rather than redoing the entire calculation, think of how you could find this area by subtraction of two areas that start at $x = 0$.)

**Exercise 7.16** **Leaf shape** The function $y = f(x) = Hx(L - x)$, shown in Figure 7.14 could approximately describe the shape of the (top edge) of a symmetric leaf of length $L$ and width $w$ for a particular choice of the constant $H$ (in terms of $w$ and $L$).

(a) Find the appropriate value of $H$. Assume that the width is the distance between the leaf edges at the midpoint of the leaf.

(b) Find the area between the x axis and the function $y = f(x) = Hx(L - x)$.

(c) Use your result from (a) to express the area of the leaf in terms of the width and length of this leaf.

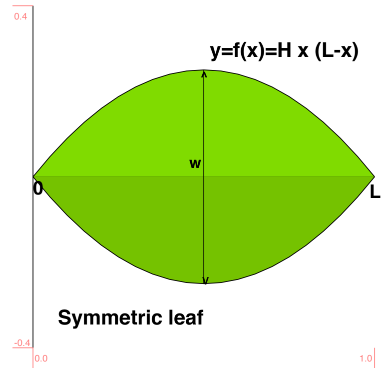



Figure 7.14: The shape of a symmetric leaf of length $L$ and width $w$ is approximated by the quadratic $y = Hx(L - x)$ in Exercise 7.16.

**Exercise 7.17** **Tapered Leaf** Consider the shape of a leaf shown in Figure 7.15, and given by the function $y = f(x) = x^2(1 - x)$. This leaf is not fully symmetric, since it is tapered at one end. By choice of the function that describes its top edge, the length of the leaf is 1 unit.

(a) Find the width of the leaf (distance between edges at the widest place). (Hint: use differential calculus to determine where the widest point occurs.)

(b) Find the area of this shape by dissection into rectangles.

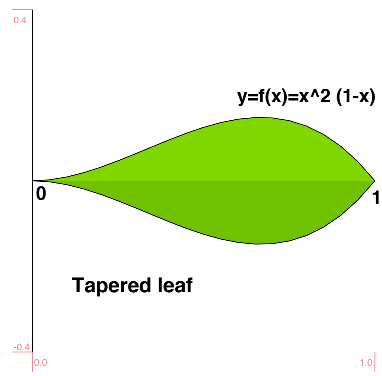



Figure 7.15: The shape of a tapered leaf is approximated by the function $y = x^2(1 - x)$.

# 8

# The Fundamental Theorem of Calculus

In this chapter we will formulate one of the most important results of calculus, the Fundamental Theorem. This result will link together the notions of an integral and a derivative. Using this result will allow us to replace the technical calculations of Chapter 7 by much simpler procedures involving antiderivatives of a function.

## 8.1 The definite integral

In Chapter 7, we defined the definite integral, $I$, of a function $f(x) > 0$ on an interval $[a, b]$ as the area under the graph of the function over the given interval $a \leq x \leq b$. We used the notation

$$I = \int_a^b f(x)dx$$

to represent that quantity. We also set up a technique for computing areas: the procedure for calculating the value of $I$ is to write down a sum of areas of rectangular strips and to compute a limit as the number of strips increases:

$$I = \int_a^b f(x)dx = \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k)\Delta x, \qquad (8.1)$$

where $N$ is the number of strips used to approximate the region, $k$ is an index associated with the $k$'th strip, and $\Delta x = x_{k+1} - x_k$ is the width of the rectangle. As the number of strips increases ($N \to \infty$), and their width decreases ($\Delta x \to 0$), the sum becomes a better and better approximation of the true area, and hence, of the definite integral, $I$. Example of such calculations (tedious as they were) formed the main theme of Chapter 7 .

We can generalize the definite integral to include functions that are not strictly positive, as shown in Figure 8.1. To do so, note what happens as we incorporate strips corresponding to regions of the graph below the $x$ axis: These are associated with negative values of the function, so that the quantity $f(x_k)\Delta x$ in the above sum would be negative for each rectangle in the "negative" portions of the function. This means that regions of the graph below the $x$ axis will contribute negatively to the net value of $I$.

If we refer to $A_1$ as the area corresponding to regions of the graph of $f(x)$ above the $x$ axis, and $A_2$ as the total area of regions of the graph under the $x$ axis, then we will find that the value of the definite integral $I$ shown above will be

$$I = A_1 - A_2.$$

Thus the notion of "area under the graph of a function" must be interpreted a little carefully when the function dips below the axis.

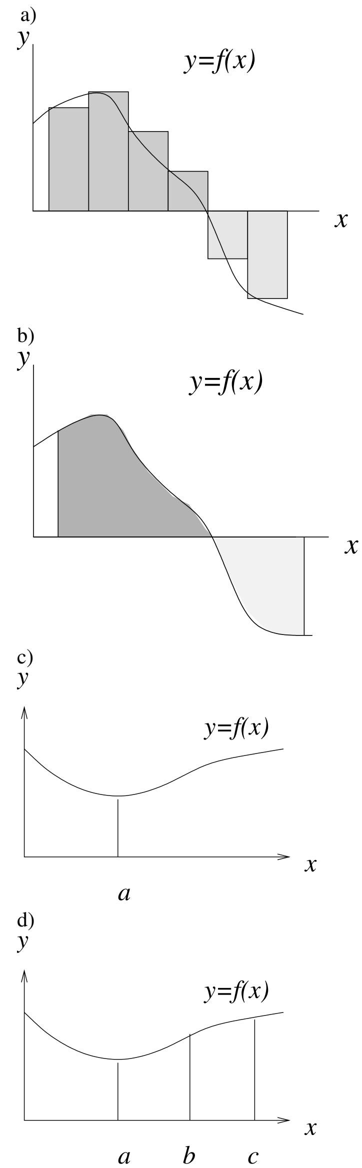



Figure 8.1: (a) If $f(x)$ is negative in some regions, there are terms in the sum (8.1) that carry negative signs: this happens for all rectangles in parts of the graph that dip below the $x$ axis. (b) This means that the definite integral $I = \int_a^b f(x)dx$ will correspond to the difference of two areas, $A_1 - A_2$ where $A_1$ is the total area (dark) of positive regions minus the total area (light) of negative portions of the graph. Properties of the definite integral: (c) illustrates Property 1. (d) illustrates Property 2.

## 8.2 Properties of the definite integral

The following properties of a definite integral stem from its definition, and the procedure for calculating it discussed so far. For example, the fact that summation satisfies the distributive property means that an integral will satisfy the same the same property. We illustrate some of these in Fig 8.1.

1. $\displaystyle\int_a^a f(x)dx = 0,$
2. $\displaystyle\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx,$
3. $\displaystyle\int_a^b Cf(x)dx = C\int_a^b f(x)dx,$
4. $\displaystyle\int_a^b (f(x)+g(x))dx = \int_a^b f(x) + \int_a^b g(x)dx,$
5. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Property 1 states that the "area" of a region with no width is zero. Property 2 shows how a region can be broken up into two pieces whose total area is just the sum of the individual areas. Properties 3 and 4 reflect the fact that the integral is actually just a sum, and so satisfies properties of simple addition. Property 5 is obtained by noting that if we perform the summation "in the opposite direction", then we must replace the previous "rectangle width" given by $\Delta x = x_{k+1} - x_k$ by the new "width" which is of opposite sign: $x_k - x_{k+1}$. This accounts for the sign change shown in Property 5.

## 8.3 The area as a function

In Chapter 7, we investigated how the area under the graph of a function changes as one of the endpoints of the interval moves. We defined a function that represents the area under the graph of a function $f$, from some fixed starting point, $a$ to an endpoint $x$.

$$A(x) = \int_a^x f(t)\,dt.$$

This endpoint is considered as a variable, i.e. we will be interested in the way that this area changes as the endpoint varies (Figure 8.2(a)). Recall that the "dummy variable" $t$ inside the integral is just a "place holder", and is used to avoid confusion with the endpoint of the integral ($x$ in this case). Also note that the value of $A(x)$ does not depend in any way on $t$, so any letter or symbol in its place would do just as well. We will now investigate the interesting connection between $A(x)$ and the original function, $f(x)$.

We would like to study how $A(x)$ changes as $x$ is increased ever so slightly. Let $\Delta x = h$ represent some (very small) increment in $x$. (*Caution: do not confuse h with height here. It is actually a step size along the x axis.*) Then, according to our definition,

$$A(x+h) = \int_a^{x+h} f(t)\, dt.$$

In Figure 8.2(a)(b), we illustrate the areas represented by $A(x)$ and by $A(x+h)$, respectively. The difference between the two areas is a thin sliver (shown in Figure 8.2(c)) that looks much like a rectangular strip (Figure 8.2(d)). (Indeed, if $h$ is small, then the approximation of this sliver by a rectangle will be good.) The height of this sliver is specified by the function $f$ evaluated at the point $x$, i.e. by $f(x)$, so that the area of the sliver is approximately $f(x) \cdot h$. Thus,

$$A(x+h) - A(x) \approx f(x)h$$

or

$$\frac{A(x+h) - A(x)}{h} \approx f(x).$$

As $h$ gets small, i.e. $h \to 0$, we get a better and better approximation, so that, in the limit,

$$\lim_{h\to 0} \frac{A(x+h) - A(x)}{h} = f(x).$$

The ratio above should be recognizable. It is simply the derivative of the area function, i.e.

$$f(x) = \frac{dA}{dx} = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}. \tag{8.2}$$

We have just given a simple argument in support of an important result, called the *Fundamental Theorem of Calculus*, which is restated below..

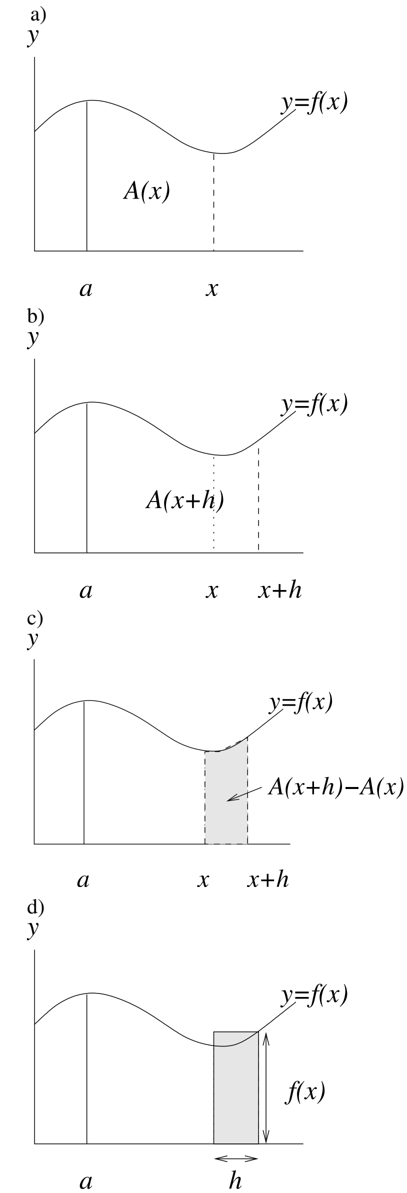



Figure 8.2: When the right endpoint of the interval moves by a distance $h$, the area of the region increases from $A(x)$ to $A(x+h)$. This leads to the important Fundamental Theorem of Calculus, given in Eqn. (8.2).

## 8.4 The Fundamental Theorem of Calculus

### *Fundamental theorem of calculus: Part I*

Let $f(x)$ be a bounded and continuous function on an interval $[a, b]$. Let

$$A(x) = \int_a^x f(t)\, dt.$$

Then for $a < x < b$,

$$\frac{dA}{dx} = f(x).$$

In other words, this result says that $A(x)$ is an "antiderivative" (or "anti-derivative") of the original function, $f(x)$.

For proof, see above argument and Figure 8.2.

### *Example: an antiderivative*

Recall the connection between functions and their derivatives. Consider the following two functions:

$$g_1(x) = \frac{x^2}{2}, \quad g_2 = \frac{x^2}{2} + 1.$$

Clearly, both functions have the same derivative:

$$g_1'(x) = g_2'(x) = x.$$

We would say that $x^2/2$ is an "antiderivative" of $x$ and that $(x^2/2) + 1$ is also an "antiderivative" of $x$. In fact, *any* function of the form

$$g(x) = \frac{x^2}{2} + C \quad \text{where } C \text{ is any constant}$$

is also an "antiderivative" of $x$.

This example illustrates that adding a constant to a given function will not affect the value of its derivative, or, stated another way, antiderivatives of a given function are defined only up to some constant. We will use this fact shortly: if $A(x)$ and $F(x)$ are both antiderivatives of some function $f(x)$, then $A(x) = F(x) + C$.

### *Fundamental theorem of calculus: Part II*

Let $f(x)$ be a continuous function on $[a,b]$. Suppose $F(x)$ is *any* antiderivative of $f(x)$. Then for $a \le x \le b$,

$$A(x) = \int_a^x f(t)\,dt = F(x) - F(a).$$

*Proof.* From comments above, we know that a function $f(x)$ could have many different antiderivatives that differ from one another by some additive constant. We are told that $F(x)$ is an antiderivative of $f(x)$. But from Part I of the Fundamental Theorem, we know that $A(x)$ is also an antiderivative of $f(x)$. It follows that

$$A(x) = \int_a^x f(t)\,dt = F(x) + C, \quad \text{where } C \text{ is some constant.} \tag{8.3}$$

However, by property 1 of definite integrals,

$$A(a) = \int_a^a f(t) = F(a) + C = 0.$$

Thus,

$$C = -F(a).$$

Replacing $C$ by $-F(a)$ in equation 8.3 leads to the desired result. Thus

$$A(x) = \int_a^x f(t)\,dt = F(x) - F(a).$$

**Remark 1: Implications.**

This theorem has tremendous implications, because it allows us to use a powerful new tool in determining areas under curves. Instead of the drudgery of summations in order to compute areas, we will be able to use a shortcut: find an antiderivative, evaluate it at the two endpoints $a$, $b$ of the interval of interest, and subtract the results to get the area. In the case of elementary functions, this will be very easy and convenient.

**Remark 2: Notation.**

We will often use the notation

$$F(t)|_a^x = F(x) - F(a)$$

to denote the difference in the values of a function at two endpoints.

## 8.5 *Review of derivatives (and antiderivatives)*

By remarks above, we see that integration is related to "anti-differentiation". This motivates a review of derivatives of common functions. Table 8.1 lists functions $f(x)$ and their derivatives $f'(x)$ (in the first two columns) and functions $f(x)$ and their antiderivatives $F(x)$ in the subsequent two columns. These will prove very helpful in our calculations of basic integrals.

| function | derivative | | function | antiderivative |
|---|---|---|---|---|
| $f(x)$ | $f'(x)$ | | $f(x)$ | $F(x)$ |
| $Cx$ | $C$ | | $C$ | $Cx$ |
| $x^n$ | $nx^{n-1}$ | | $x^m$ | $\frac{x^{m+1}}{m+1}$ |
| $\sin(ax)$ | $a\cos(ax)$ | | $\cos(bx)$ | $(1/b)\sin(bx)$ |
| $\cos(ax)$ | $-a\sin(ax)$ | | $\sin(bx)$ | $-(1/b)\cos(bx)$ |
| $\tan(ax)$ | $a\sec^2(ax)$ | | $\sec^2(bx)$ | $(1/b)\tan(bx)$ |
| $e^{kx}$ | $ke^{kx}$ | | $e^{kx}$ | $e^{kx}/k$ |
| $\ln(x)$ | $\frac{1}{x}$ | | $\frac{1}{x}$ | $\ln(x)$ |
| $\arctan(x)$ | $\frac{1}{1+x^2}$ | | $\frac{1}{1+x^2}$ | $\arctan(x)$ |
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ | | $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin(x)$ |

Table 8.1: Common functions and their derivatives (on the left two columns) also result in corresponding relationships between functions and their antiderivatives (right two columns). In this table, we assume that $m \neq -1, b \neq 0, k \neq 0$. Also, when using $\ln(x)$ as antiderivative for $1/x$, we assume that $x > 0$.

As an example, consider the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

This polynomial could have many other terms (or even an infinite number of such terms, as we discuss much later, in Chapter **??**). Its antiderivative can be found easily using the "power rule" together with the properties of addition of terms. Indeed, the antiderivative is

$$F(x) = C + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \dots$$

This can be checked easily by differentiation. In fact, it is very good practice to perform such checks.

## 8.6 Examples: Computing areas with the Fundamental Theorem of Calculus

**Example 8.1** *Compute the area under the polynomial* $p(x) = 1 + x + x^2 + x^3$ *between 0 and 1.*

**Solution.** Here we have taken the first few terms from the example of the last section with coefficients all set to 1. Computing

$$I = \int_0^1 p(x)\, dx$$

leads to

$$I = \int_0^1 (1+x+x^2+x^3)\, dx = (x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4)\bigg|_0^1 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \approx 2.083.$$

$\Diamond$

**Example 8.2** *Determine the values of the following definite integrals by finding antiderivatives and using the Fundamental Theorem of Calculus:*

1. $I = \int_0^1 x^2\, dx,$
2. $I = \int_{-1}^1 (1-x^2)\, dx,$
3. $I = \int_{-1}^1 e^{-2x}\, dx,$
4. $I = \int_{-2\pi}^{2\pi} \sin\left(\frac{x}{2}\right)\, dx,$

**Solution.**

1. An antiderivative of $f(x) = x^2$ is $F(x) = (x^3/3)$, thus

$$I = \int_0^1 x^2 dx = F(x)\bigg|_0^1 = (1/3)(x^3)\bigg|_0^1 = \frac{1}{3}(1^3 - 0) = \frac{1}{3}.$$

2. An antiderivative of $f(x) = (1-x^2)$ is $F(x) = x - (x^3/3)$, thus

$$I = \int_{-1}^{1} (1-x^2)\, dx = F(x)\Big|_{-1}^{1} = \left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$
$$= \left(1 - \frac{1^3}{3}\right) - \left((-1) - \frac{(-1)^3}{3}\right) = 4/3$$

See comment 1 below for a simpler way to compute this integral.

3. An antiderivative of $e^{-2x}$ is $F(x) = (-1/2)e^{-2x}$. Thus,

$$I = \int_{-1}^{1} e^{-2x}\, dx = F(x)\Big|_{-1}^{1} = (-1/2)(e^{-2x})\Big|_{-1}^{1} = (-1/2)(e^{-2} - e^{2}).$$

4. An antiderivative of $\sin(x/a)$ is $F(x) = -\cos(x/a)/(1/a) = -a\cos(x/a)$. Thus

$$I = \int_{-2\pi}^{2\pi} \sin\left(\frac{x}{2}\right)\, dx = -2\cos\left(\frac{x}{2}\right)\Big|_{-2\pi}^{2\pi} = -2\left(\cos(\pi) - \cos(-\pi)\right)$$
$$= -2(-1-(-1)) = 0.$$

See comment 2 below for a simpler way to compute this integral. $\Diamond$

**Comment.** The evaluation of Integral 2 in the examples above is tricky only in that signs can easily get garbled when we plug in the endpoint at -1. However, we can simplify our work by noting the symmetry of the function $f(x) = 1 - x^2$ on the given interval. As shown in Fig 8.3, the areas to the right and to the left of $x = 0$ are the same for the interval $-1 \le x \le 1$. This stems directly from the fact that the function considered is **even**. (Recall that a function $f(x)$ is **even** if $f(x) = f(-x)$ for all $x$. A function is **odd** if $f(x) = -f(-x)$.) Thus, we can immediately write

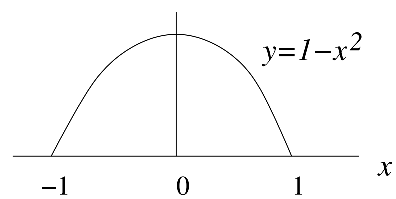


Figure 8.3: We can exploit the symmetry of the function $f(x) = 1 - x^2$ in the second integral of Examples 8.2. We can integrate over $0 \le x \le 1$ and double the result.

$$I = \int_{-1}^{1} (1-x^2)\, dx = 2\int_{0}^{1} (1-x^2)\, dx = 2\left(x - \frac{x^3}{3}\right)\Big|_{0}^{1} = 2\left(1 - \frac{1^3}{3}\right) = 4/3.$$

Note that this calculation is simpler since the endpoint at $x = 0$ is trivial to plug in.

We state the general result we have obtained, which holds true for any function with even symmetry integrated on a symmetric interval about $x = 0$:

If $f(x)$ is an **even** function, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx. \tag{8.4}$$

*Caution:* The function $f(x)$ must be integrable, i.e. $f(x)$ must exist and be defined over the *entire* interval $x \in [-a, a]$. For example, the integrand in

$\int_{-2}^{2} x^{-2} dx$ is even but requires more careful considerations because it does not exist at $x = 0$).

Similarly, if $f(x)$ is an odd function then the symmetry yields an even simpler result on a symmetric interval about $x = 0$:

If $f(x)$ is an **odd** function, then

$$\int_{-a}^{a} f(x)\, dx = 0 \qquad \text{for any } a. \tag{8.5}$$

*Caution:* Again, $f(x)$ must be integrable over the *entire* interval $x \in [-a,a]$. For example, even though the integrand in $\int_{-2}^{2} \frac{1}{x}\, dx$ is odd, the integral requires more careful considerations because the integrand does not exist at $x = 0$).

**Comment.** We can exploit symmetries for a simpler evaluation of Integral 4 above: by realizing that $\sin(\frac{x}{2})$ is an odd function and also noticing that the integration bounds are symmetric about $x = 0$, we immediately conclude that the integral evaluates to zero without requiring any actual calculations.

The definite integral is an area of a somewhat special type of region, i.e., an axis, two vertical lines ($x = a$ and $x = b$) and the graph of a function. However, using additive (or subtractive) properties of areas, we can generalize to computing areas of other regions, including those bounded by the graphs of two functions.

**Example 8.3 (Area between two curves)***(a) Find the area enclosed between the graphs of the functions $y = x^3$ and $y = x^{1/3}$ in the first quadrant.*

*(b) Find the area enclosed between the graphs of the functions $y = x^3$ and $y = x$ in the first quadrant.*

*(c) What is the relationship of these two areas? What is the relationship of the functions $y = x^3$ and $y = x^{1/3}$ that leads to this relationship between the two areas?*

**Solution.**

(a) The two curves, $y = x^3$ and $y = x^{1/3}$, intersect at $x = 0$ and at $x = 1$ in the first quadrant. Thus the interval that we will be concerned with is $0 < x < 1$. On this interval, $x^{1/3} > x^3$, so that the area we want to find can be expressed as:

$$A_1 = \int_0^1 \left(x^{1/3} - x^3\right) dx.$$

Thus,

$$A_1 = \left.\frac{x^{4/3}}{4/3}\right|_0^1 - \left.\frac{x^4}{4}\right|_0^1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

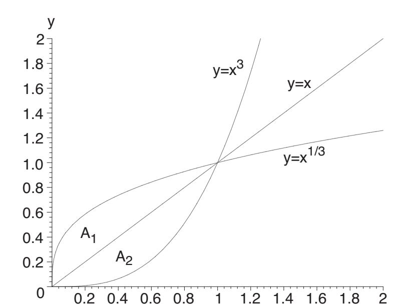



Figure 8.4: In Example 3, we compute the areas $A_1$ and $A_2$ shown above.

(b) The two curves $y = x^3$ and $y = x$ also intersect at $x = 0$ and at $x = 1$ in the first quadrant, and on the interval $0 < x < 1$ we have $x > x^3$. The area can be represented as

$$A_2 = \int_0^1 \left(x - x^3\right) dx.$$

$$A_2 = \left.\frac{x^2}{2}\right|_0^1 - \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

(c) The area calculated in (a) is twice the area calculated in (b). The reason for this is that $x^{1/3}$ is the inverse of the function $x^3$, which means geometrically that the graph of $x^{1/3}$ is the mirror image of the graph of $x^3$ reflected about the line $y = x$. Therefore, the area $A_1$ between $y = x^{1/3}$ and $y = x^3$ is twice as large as the area $A_2$ between $y = x$ and $y = x^3$ calculated in part (b): $A_1 = 2A_2$ (see Figure 8.4).

◊

**Example 8.4 (Area of land)** *Find the exact area of the piece of land which is bounded by the y axis on the west, the x axis in the south, the lake described by the function* $y = f(x) = 100 + (x/100)^2$ *in the north and the line* $x = 1000$ *in the east.*

**Solution.** The area is

$$A = \int_0^{1000} \left(100 + \left(\frac{x}{100}\right)^2\right) dx. = \int_0^{1000} \left(100 + \left(\frac{1}{10000}\right) x^2\right) dx.$$

Note that the multiplicative constant (1/10000) is not affected by integration. The result is

$$A = 100x\Big|_0^{1000} + \left.\frac{x^3}{3}\right|_0^{1000} \cdot \left(\frac{1}{10000}\right) = \frac{4}{3}10^5.$$

◊

## 8.7 *Qualitative ideas*

In some cases, we are given a sketch of the graph of a function, $f(x)$, from which we would like to construct a sketch of the associated function $A(x)$. This sketching skill is illustrated in the figures shown in this section.

Suppose we are given a function as shown in the top left hand panel of Figure 8.5. We would like to assemble a sketch of

$$A(x) = \int_a^x f(t)dt$$

which corresponds to the area associated with the graph of the function $f$. As $x$ moves from left to right, we show how the "area" accumulated along the graph gradually changes. (See $A(x)$ in bottom panels of Figure 8.5): We start with no area, at the point $x = a$ (since, by definition $A(a) = 0$) and gradually build up to some net positive amount, but then we encounter a portion of the graph of $f$ below the $x$ axis, and this subtracts from the amount accrued. (Hence the graph of $A(x)$ has a little peak that corresponds to the point at which $f = 0$.) Every time the function $f(x)$ crosses the $x$ axis, we see that $A(x)$

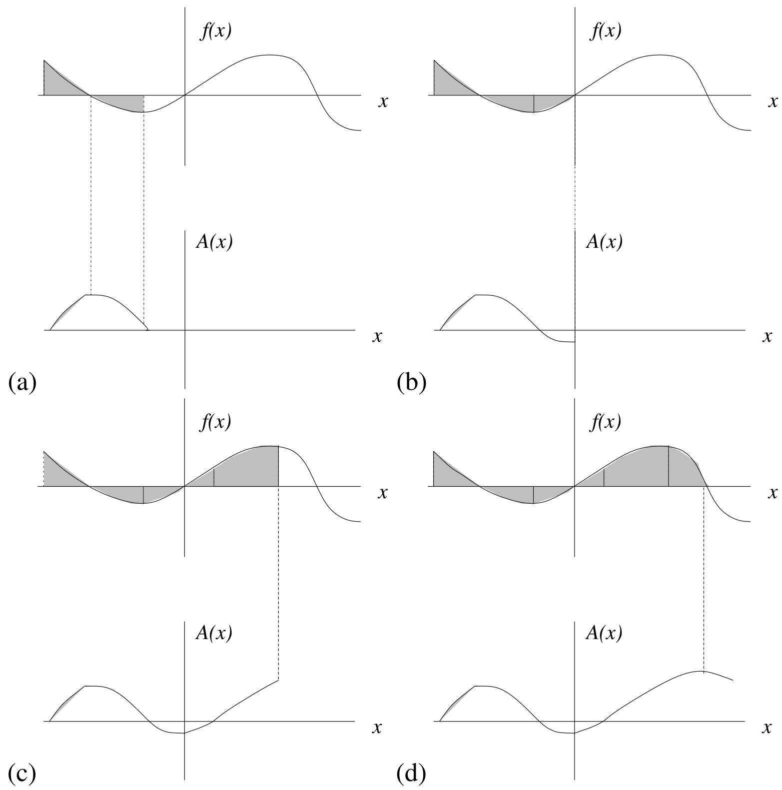



Figure 8.5: Given a function $f(x)$, we here show how to sketch the corresponding "area function" $A(x)$. (The relationship is that $f(x)$ is the derivative of $A(x)$

has either a maximum or minimum value. This fits well with our idea of $A(x)$ as the antiderivative of $f(x)$: Places where $A(x)$ has a critical point coincide with places where $dA/dx = f(x) = 0$.

Sketching the function $A(x)$ is thus analogous to sketching a function $g(x)$ when we are given a sketch of its derivative $g'(x)$. Recall that this was one of the skills we built up in learning the connection between functions and their derivatives in a first semester calculus course.

**Remarks**

The following remarks may be helpful in gaining confidence with sketching the "area" function $A(x) = \int_a^x f(t)\, dt$, from the original function $f(x)$:

1. The endpoint of the interval, $a$ on the x axis indicates the place at which $A(x) = 0$. This follows from Property 1 of the definite integral, i.e. from the fact that $A(a) = \int_a^a f(t)\, dt = 0$.

2. Whenever $f(x)$ is positive, $A(x)$ is an increasing function - this follows

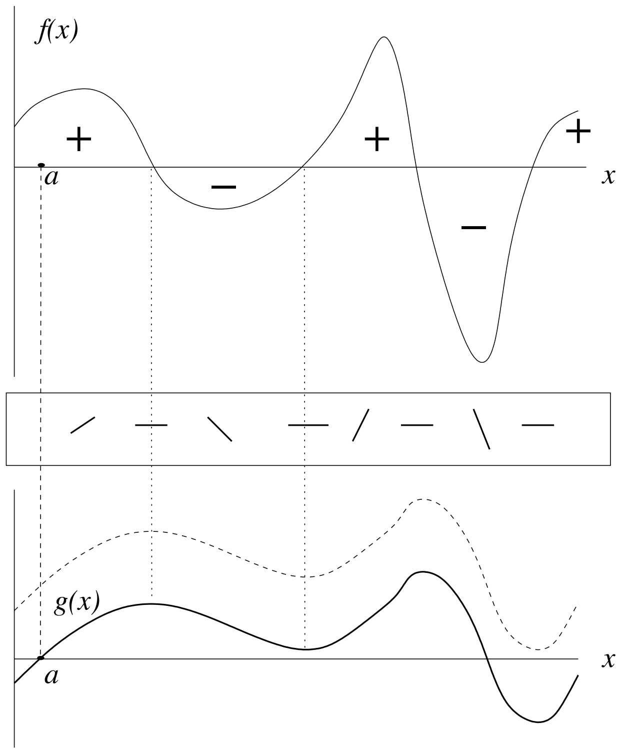



Figure 8.6: Given a function $f(x)$ (top, solid line), we assemble a plot of the corresponding function $g(x) = \int_a^x f(t)dt$ (bottom, solid line). $g(x)$ is an antiderivative of $f(x)$. Whether $f(x)$ is positive (+) or negative (-) in portions of its graph, determines whether $g(x)$ is increasing or decreasing over the given intervals. Places where $f(x)$ changes sign correspond to maxima and minima of the function $g(x)$ (Two such places are indicated by dotted vertical lines). The box in the middle of the sketch shows configurations of tangent lines to $g(x)$ based on the sign of $f(x)$. Where $f(x) = 0$, those tangent lines are horizontal. The function $g(x)$ is drawn as a smooth curve whose direction is parallel to the tangent lines shown in the box. While the function $f(x)$ has many antiderivatives (e.g., dashed curve parallel to $g(x)$), only one of these satisfies $g(a) = 0$ as required by Property 1 of the definite integral. (See dashed vertical line at $x = a$). This determines the height of the desired function $g(x)$.

from the fact that the area continues to accumulate as we "sweep across" positive regions of $f(x)$.

3. Wherever $f(x)$, changes sign, the function $A(x)$ has a local minimum or maximum. This means that either the area stops increasing (if the transition is from positive to negative values of $f$), or else the area starts to increase (if $f$ crosses from negative to positive values).

4. Since $dA/dx = f(x)$ by the Fundamental Theorem of Calculus, it follows that (taking a derivative of both sides) $d^2A/dx^2 = f'(x)$. Thus, when $f(x)$ has a local maximum or minimum, (i.e. $f'(x) = 0$), it follows that $A''(x) = 0$. This means that at such points, the function $A(x)$ would have an inflection point.

Given a function $f(x)$, Figure 8.6 shows in detail how to sketch the corresponding function

$$g(x) = \int_a^x f(t)dt.$$

**Example 8.5 (Sketching the antideriavtive)** *Consider the $f(x)$ whose graph is shown in the top part of Figure 8.7. Sketch the corresponding function $g(x) = \int_a^x f(x)dx$.*

**Solution.** See Figure 8.7. ◊

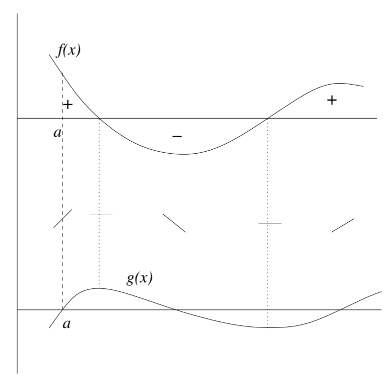



Figure 8.7: The original functions, $f(x)$ is shown above. The corresponding functions $g(x)$ is drawn below.

## 8.8 *Prelude to improper integrals*

The Fundamental Theorem has a number of restrictions that must be satisfied before its results can be applied. In this section we look at some examples in which care must be used. The examples of this section should be compared with our chapter on Improper Integrals. In particular one must be careful in applying the Fundamental theorem when the integrand has a discontinuity within the region of integration.

### *Function unbounded I*

Consider the definite integral

$$\int_0^2 \frac{1}{x}\, dx.$$

The function $f(x) = \frac{1}{x}$ is discontinuous at $x = 0$, and unbounded on any interval that contains the point $x = 0$, i.e. $\lim_{x\to 0^\pm} \frac{1}{x} = \pm\infty$. Hence we cannot directly evaluate this integral using the Fundamental theorem. However we can show that $\lim_{\varepsilon\to 0^+} \int_\varepsilon^2 \frac{1}{x}\, dx = \infty$. Indeed the integrand is continuous on the interval $(\varepsilon, 2)$ for all $\varepsilon > 0$. The Fundamental theorem therefore tells us

$$\int_\varepsilon^2 \frac{1}{x}\, dx = \log(2) - \log(\varepsilon).$$

But $\lim_{\varepsilon\to 0^+} \log(\varepsilon) = -\infty$, and therefore $\lim_{\varepsilon\to 0^+} \int_\varepsilon^2 \frac{1}{x}\, dx = \infty$, as claimed. We say that the integral $\int_0^2 \frac{1}{x}\, dx$ diverges.

### *Function unbounded II*

Consider the definite integral

$$\int_{-1}^1 \frac{1}{x^2} dx.$$

As in the previous example the integrand $\frac{1}{x^2}$ is discontinuous and unbounded at $x = 0$. To 'blindly' apply the Fundamental theorem (which in this case is unjustified) yields

$$-\frac{1}{x}\Big|_{x=-1}^{1} = -2.$$

However the integral $\int_{-1}^1 \frac{1}{x^2} dx$ certainly cannot be negative, since we are integrating a positive function over a finite domain (and the graph bounds some positive – possibly infinite – area).

Splitting the integral as a sum

$$\int_{-1}^1 \frac{1}{x^2} dx = \int_{-1}^0 \frac{1}{x^2} dx + \int_0^1 \frac{1}{x^2} dx,$$

we can see by an argument entirely similar to our previous example that seperately each improper integral $\int_{-1}^0 \frac{1}{x^2} dx, \int_0^1 \frac{1}{x^2} dx$ diverges, i.e. both

$$\lim_{\varepsilon\to 0} \int_{-1}^{-|\varepsilon|} \frac{1}{x^2} dx, \qquad \lim_{\varepsilon\to 0} \int_{|\varepsilon|}^1 \frac{1}{x^2} dx$$

diverge to $\infty$. Consequently our integral – the sum of two integrals diverging to $+\infty$ – diverges.

### *Function discontinuous or with distinct parts*

Suppose we are given the integral

$$I = \int_{-1}^{2} |x|\, dx.$$

This function is actually made up of two distinct parts, namely

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0. \end{cases}$$

The integral $I$ must therefore be split up into two parts, namely

$$I = \int_{-1}^{2} |x|\, dx = \int_{-1}^{0} (-x)\, dx + \int_{0}^{2} x\, dx.$$

We find that

$$I = -\left.\frac{x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{2} = -\left[0 - \frac{1}{2}\right] + \left[\frac{4}{2} - 0\right] = 2.5$$

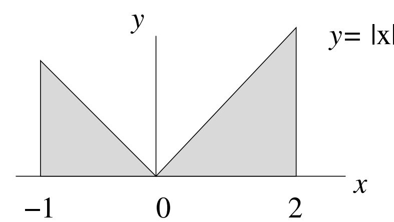


Figure 8.8: In this example, to compute the integral over the interval $-1 \le x \le 2$, we must split up the region into two distinct parts.

### *Function undefined*

Now let us examine the integral

$$\int_{-1}^{1} x^{1/2}\, dx.$$

We see that there is a problem here. Recall that $x^{1/2} = \sqrt{x}$. Hence, the function is not defined for $x < 0$ and the interval of integration is inappropriate. Hence, this integral does not make sense.

### *Integrating over an infinite domain*

Consider the integral

$$I = \int_{0}^{b} e^{-rx}\, dx, \quad \text{where } r > 0, \text{ and } b > 0 \text{ are constants.}$$

Simple integration using the antiderivative in Table 8.1 (for $k = -r$) leads to the result

$$I = \left.\frac{e^{-rx}}{-r}\right|_{0}^{b} = -\frac{1}{r}\left(e^{-rb} - e^{0}\right) = \frac{1}{r}\left(1 - e^{-rb}\right).$$

This is the area under the exponential curve between $x = 0$ and $x = b$. Now consider what happens when $b$, the upper endpoint of the integral increases, so that $b \to \infty$. Then the value of the integral becomes

$$I = \lim_{b\to\infty} \int_{0}^{b} e^{-rx}\, dx = \lim_{b\to\infty} \frac{1}{r}\left(1 - e^{-rb}\right) = \frac{1}{r}(1 - 0) = \frac{1}{r}.$$

(We used the fact that $e^{-rb} \to 0$ as $b \to \infty$.) We have, in essence, found that

$$I = \int_0^\infty e^{-rx}\, dx = \frac{1}{r}. \tag{8.6}$$

An integral of the form (8.6) is called an **improper integral**. Even though the domain of integration of this integral is infinite, $(0, \infty)$, observe that the value we computed is finite, so long as $r \neq 0$. Not all such integrals have a bounded finite value. Learning to distinguish between those that do and those that do not will form an important theme in Chapter **??**.

### *Regions that need special treatment*

So far, we have learned how to compute areas of regions in the plane that are bounded by one or more curves. In all our examples so far, the basis for these calculations rests on imagining rectangles whose heights are specified by one or another function. Up to now, all the rectangular strips we considered had bases (of width $\Delta x$) on the $x$ axis. In Figure 8.9 we observe an example in which it would not be possible to use this technique.

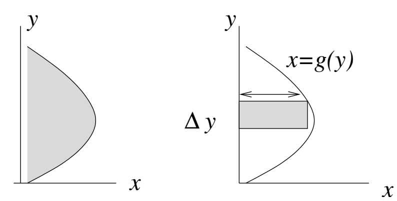



Figure 8.9: The area in the region shown here is best computed by integrating in the $y$ direction. If we do so, we can use the curved boundary as a single function that defines the region. (Note that the curve cannot be expressed in the form of a function in the usual sense, $y = f(x)$, but it can be expressed in the form of a function $x = f(y)$.)

We are asked to find the area between the curve $y^2 - y + x = 0$ and the $y$ axis. However, one and the same curve, $y^2 - y + x = 0$ forms the boundary from both the top and the bottom of the region. We are unable to set up a series of rectangles with bases along the $x$ axis whose heights are described by this curve. This means that our definite integral (which is really just a convenient way of carrying out the process of area computation) has to be handled with care.

Let us consider this problem from a "new angle", i.e. with rectangles based on the $y$ axis, we can achieve the desired result. To do so, let us express our curve in the form

$$x = g(y) = y - y^2.$$

Then, placing our rectangles along the interval $0 < y < 1$ on the y axis (each having base of width $\Delta y$) leads to the integral

$$I = \int_0^1 g(y)\, dy = \int_0^1 (y - y^2) dy = \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

## 8.9 *Summary*

In this chapter we first recapped the definition of the definite integral, recalled its connection to an area in the plane under the graph of some function $f(x)$, and examined its basic properties.

If one of the endpoints, $x$ of the integral is allowed to vary, the area it represents, $A(x)$, becomes a function of $x$. Our construction in Figure 8.2 showed that there is a connection between the derivative $A'(x)$ of the area and the function $f(x)$. Indeed, we showed that $A'(x) = f(x)$ and argued that this makes $A(x)$ an antiderivative of the function $f(x)$.

This important connection between integrals and antiderivatives is the crux of Integral Calculus, forming the Fundamental Theorem of Calculus. Its significance is that finding areas need not be as tedious and labored as the calculation of Riemann sums that formed the bulk of Chapter 7. Rather, we can take a shortcut using antidifferentiation.

Motivated by this very important result, we reviewed some common functions and derivatives, and used this to relate functions and their antiderivatives in Table 8.1. We used these antiderivatives to calculate areas in several examples. Finally, we extended the treatment to include qualitative sketches of functions and their antiderivatives.

As we will see in upcoming chapters, the ideas presented here have a much wider range of applicability than simple area calculations. Indeed, we will shortly show that the same concepts can be used to calculate net changes in continually varying processes, to compute volumes of various shapes, to determine displacement from velocity, mass from densities, as well as a host of other quantities that involve a process of accumulation. These ideas will be investigated in the next chapter.

## *Exercises*

**Exercise 8.1**

(a) Give a concise statement of the Fundamental Theorem of Calculus.

(b) Why is it a useful practical tool?

**Exercise 8.2** Consider the function $y = f(x) = e^x$ on the interval $[0,1]$. Find the area under the graph of this function over this interval using the Fundamental Theorem of Calculus.

**Exercise 8.3** Determine the values of the integrals shown below, using the Fundamental Theorem of Calculus (i.e. find the anti-derivative of each of the functions and evaluate at the two endpoints.)

(a) $\int_0^1 2x dx$ (b) $\int_{-1}^1 (1-x) dx$

**Exercise 8.4** Use the Fundamental Theorem of Calculus to compute each of the following integrals. The last few are a little more challenging, and will require special care. Some of these integrals may not exist. Explain why.

(a) $\int_0^{\pi} \sin(x)\,dx$ (b) $\int_0^{\frac{\pi}{4}} 2\sin(x)\,dx$ (c) $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$

(d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2\cos(x)\,dx$ (e) $\int_2^3 (x-2)\,dx$ (f) $\int_{-1}^1 (x-1)\,dx$

(g) $\int_{-1}^1 (x^2+1)\,dx$ (h) $\int_0^4 (x+1)^2\,dx$ (i) $\int_0^4 x^{1/2}\,dx$

(j) $\int_0^1 3x^{1/2}\,dx$ (k) $\int_1^4 (1+\sqrt{x})\,dx$ (l) $\int_1^2 \frac{3}{x}\,dx$

(m) $\int_1^3 \frac{2}{x}\,dx$ (n) $\int_0^1 2e^x\,dx$ (o) $\int_{-2}^{-3} x^{1/3}\,dx$

(p) $\int_{-1}^0 x^{1/2}\,dx$ (q) $\int_{-1}^1 x^{-2}\,dx$ (r) $\int_{-1}^1 2|x|\,dx$

(s) $\int_0^2 \frac{1}{x}\,dx$ (t) $\int_{-1}^1 \frac{2}{x}$

**Exercise 8.5** Find the following integrals using the Fundamental theorem of Calculus.

(a) $\int_a^x e^{kt}\,dt$ (b) $\int_0^x A\cos(ks)\,ds$ (c) $\int_b^x Ct^m\,dt$

(d) $\int_0^x \frac{1}{aq}\,dq$ (e) $\int_c^T \sec^2(5x)\,dx$ (f) $\int_1^x \frac{2}{1+t^2}\,dt$

(g) $\int_b^x \frac{3}{s^2}\,ds$ (h) $\int_a^T \frac{1}{x^{1/2}}\,dx$ (i) $\int_0^x \sin(3y)\,dy$

(j) $\int_b^x 3\,dt$

**Exercise 8.6**

(a) Use an integral to estimate the sums $\sum_{k=1}^{N} \sqrt{k}$.

(b) For $N = 4$ draw a sketch in which the value computed by summing the four terms is compared to the value found by the integration. (Your sketch should show the graph of the appropriate function and a set of steps that represent the above sum.)

**Exercise 8.7** Find the area under the graphs of these functions:

(a) $f(x) = 1/x$ between $x = 1$ and $x = 3$.

(b) $v(t) = at$ between $t = 0$ and $t = T$, where $a > 0$ and $T$ are fixed constants.

(c) $h(u) = u^3$ between $u = 1/2$ and $u = 2$.

**Exercise 8.8** Find the area $S$ between the two curves $y = 1 - x$ and $y = x^2 - 1$ for $x > 0$. Explain the relationship of your answer to the two integrals $I_1 = \int_0^1 (1 - x)dx$ and $I_2 = \int_0^1 (x^2 - 1)dx$.

**Exercise 8.9** Find the area $S$ between the graphs of $y = f(x) = 2 - 3x^2$ and $y = -x^2$.

**Exercise 8.10**

(a) Find the area enclosed between the graphs of the functions $y = f(x) = x^n$ and the straight line $y = x$ in the first quadrant. (Note that we are considering positive values of $n$ and that for $n = 1$ the area is zero.)

(b) Use your answer in part (a) to find the area between the graphs of the functions $y = f(x) = x^n$ and $y = g(x) = x^{1/n}$ in the first quadrant. (Hint: what is the relationship between these two functions and what sort of symmetry do their graphs satisfy?)

**Exercise 8.11** A piece of tin shaped like a leaf blade is to be cut from a square 1m×1m sheet of tin. The shape of one of the sides is given by the function $y = x^2$ and the shape is to be symmetric about the line $y = x$, see Figure 8.10.

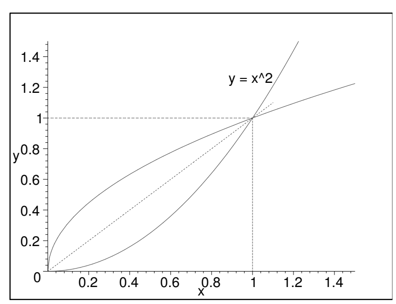


Figure 8.10: For Exercise 8.11

(a) How much tin goes into making the shape?

(b) How much is left over?

Assume that the thickness of the sheet is such that each square cm weighs one gram (1g), and that $y$ and $x$ are in meters.

**Exercise 8.12** Find the area $S$ between the graphs of the functions $y = f(x) = 2x$, $y = g(x) = 1 + x^2$ and the $y$ axis.

**Exercise 8.13** Find the area $A$ of the finite plane region bounded by the parabola $y_1 = 6 - x^2$ and the parabola $y_2 = x^2 - 4x$.

**Exercise 8.14** Find the area $A$ of the shape shown in Figure 8.11.

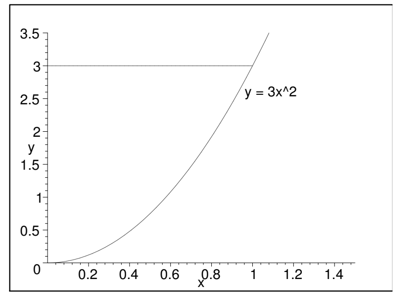


Figure 8.11: For Exercise 8.14

**Exercise 8.15** Find the area $A$ under the function shown in Figure 8.12 between $x = 1$ and $x = 5$. To do so, determine the equations of the line segments making up this graph and use integration methods.

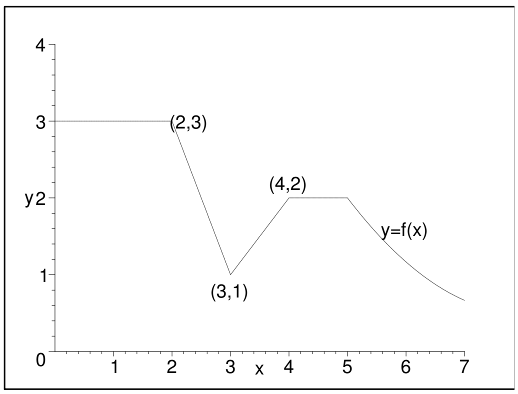


Figure 8.12: For problem 8.15

**Exercise 8.16** Let $g(x) = \int_0^x f(t)dt$, where $f(t)$ is the function shown in Figure 8.13.

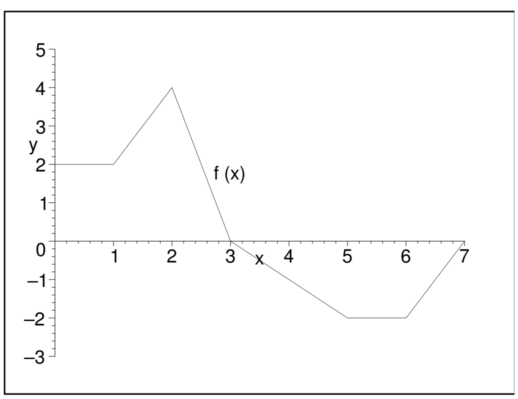


Figure 8.13: For problem 8.16

(a) Evaluate $g(0)$, $g(1)$, $g(2)$, $g(3)$ and $g(6)$.

(b) On what intervals is $g(x)$ increasing?

(c) Where does $g(x)$ have a maximum value?

(d) Sketch a rough graph of $g(x)$.

**Exercise 8.17** Consider the functions shown in Figure 8.14 (a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)\,dt$. (Assume that $F(-1) = -4$ in (a), and $F(x) = -0.5$ at the left end of the interval in (b).)

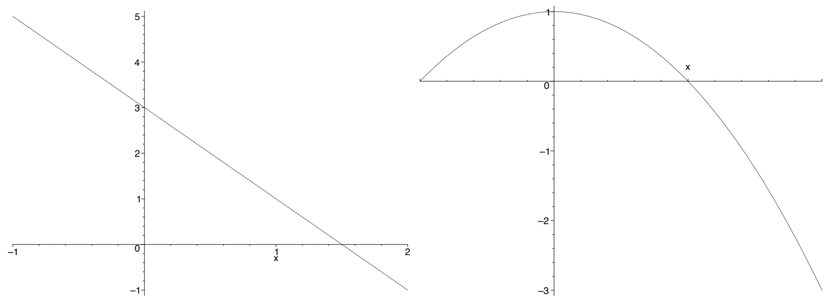



Figure 8.14: For problem 8.17

**Exercise 8.18** Consider the functions shown in Figure 8.15 (a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t) dt$.

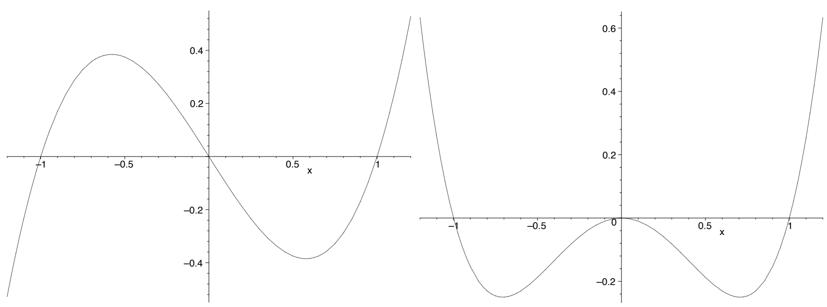



Figure 8.15: For problem 8.18

**Exercise 8.19** Consider the functions shown in Figure 8.16 (a) and (b). In each case, use the sketch of this function $y = f(x)$ to draw a sketch of the graph of the related function $F(x) = \int_0^x f(t)\,dt$. (Assume that $F(0) = 0$ in both (a) and (b).)

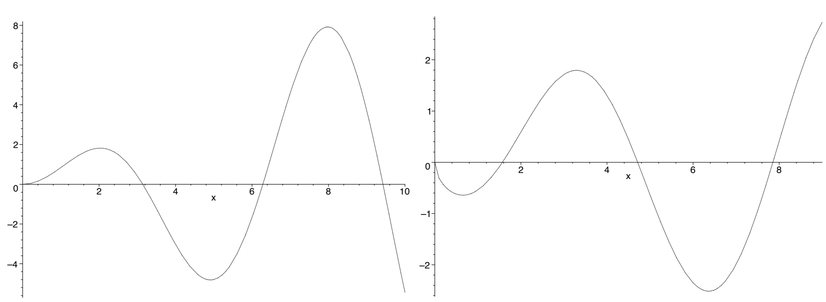



Figure 8.16: For problem 8.19

**Exercise 8.20** **Leaves revisited** Use the Fundamental Theorem of Calculus (i.e. integration techniques) to find the areas of the leaves shown in Figure 8.17. These leaves are generated by the functions

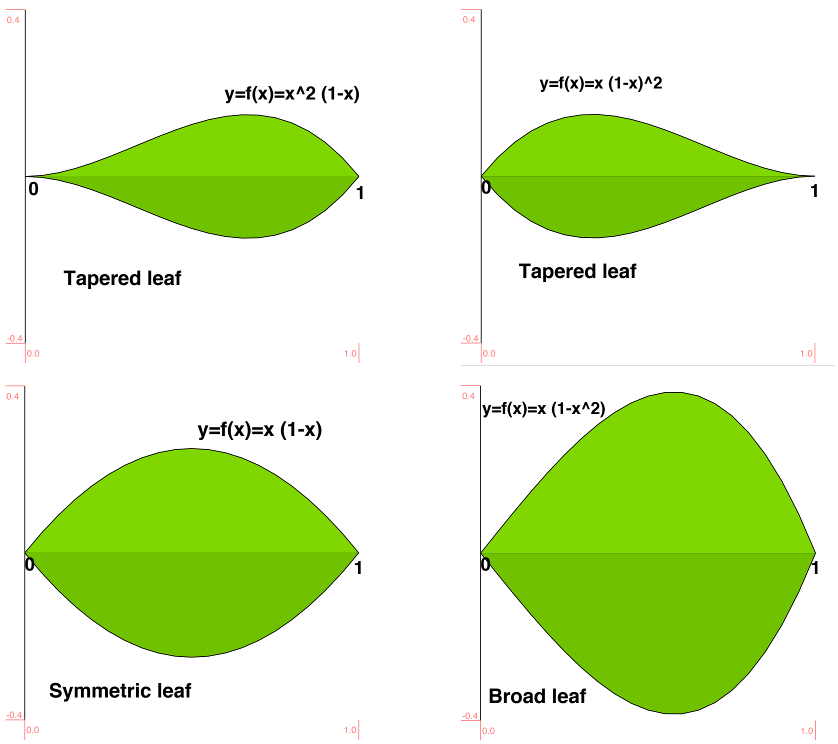



Figure 8.17: The shape of leaves for problem 8.20.

(a) $y = x^2(1-x)$ (b) $y = x(1-x)^2$ (c) $y = x(1-x)$ (d) $y = x(1-x^2)$

# 9

# Applications of the definite integral to velocities and rates

## 9.1 Introduction

In this chapter, we encounter a number of applications of the definite integral to practical problems. We will discuss the connection between acceleration, velocity and displacement of a moving object, a topic we visited in an earlier, Differential Calculus Course. Here we will show that the notion of antiderivatives and integrals allows us to deduce details of the motion of an object from underlying Laws of Motion. We will consider both uniform and accelerated motion, and recall how air resistance can be described, and what effect it induces.

An important connection is made in this chapter between a rate of change (e.g. rate of growth) and the total change (i.e. the net change resulting from all the accumulation and loss over a time span). We show that such examples also involve the concept of integration, which, fundamentally, is a cumulative summation of infinitesimal changes. This allows us to extend the utility of the mathematical tools to a variety of novel situations. We will see examples of this type in Sections 9.3 and 9.4.

Several other important ideas are introduced in this chapter. We encounter for the first time the idea of spatial density, and see that integration can also be used to "add up" the total amount of material distributed over space. In Section **??**, this idea is applied to the density of cars along a highway. We also consider mass distributions and the notion of a center of mass.

Finally, we also show that the definite integral is useful for determining the average value of a function, as discussed in Section 9.5. In all these examples, the important step is to properly set up the definite integral that corresponds to the desired net change. Computations at this stage are relatively straightforward to emphasize the process of setting up the appropriate integrals and understanding what they represent.

## 9.2 *Displacement, velocity and acceleration*

Recall from our study of derivatives that for $x(t)$ the position of some particle at time $t$, $v(t)$ its velocity, and $a(t)$ the acceleration, the following relationships hold:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = a.$$

(**Velocity** is the derivative of position and **acceleration** is the derivative of velocity.) This means that position is an anti-derivative of velocity and velocity is an anti-derivative of acceleration.

Since position, $x(t)$, is an anti-derivative of velocity, $v(t)$, by the Fundamental Theorem of Calculus, it follows that over the time interval $T_1 \le t \le T_2$,

$$\int_{T_1}^{T_2} v(t)\, dt = x(t)\Big|_{T_1}^{T_2} = x(T_2) - x(T_1). \tag{9.1}$$

The quantity on the right hand side of Eqn. (9.1) is a **displacement**,, i.e., the difference between the position at time $T_1$ and the position at time $T_2$. In the case that $T_1 = 0, T_2 = T$, we have

$$\int_0^T v(t)\, dt = x(T) - x(0),$$

as the displacement over the time interval $0 \le t \le T$.

Similarly, since velocity is an anti-derivative of acceleration, the Fundamental Theorem of Calculus says that

$$\int_{T_1}^{T_2} a(t)\, dt = v(t)\Big|_{T_1}^{T_2} = v(T_2) - v(T_1). \tag{9.2}$$

as above, we also have that

$$\int_0^T a(t)\, dt = v(t)\Big|_0^T = v(T) - v(0)$$

is the net change in velocity between time 0 and time $T$, (though this quantity does not have a special name).

### *Geometric interpretations*

Suppose we are given a graph of the velocity $v(t)$, as shown on the left of Figure 9.1. Then by the definition of the definite integral, we can interpret $\int_{T_1}^{T_2} v(t)\, dt$ as the "area" associated with the curve (counting positive and negative contributions) between the endpoints $T_1$ and $T_2$. Then according to the above observations, this area represents the displacement of the particle between the two times $T_1$ and $T_2$.

Similarly, by previous remarks, the area under the curve $a(t)$ is a geometric quantity that represents the net change in the velocity, as shown on the right of Figure 9.1.

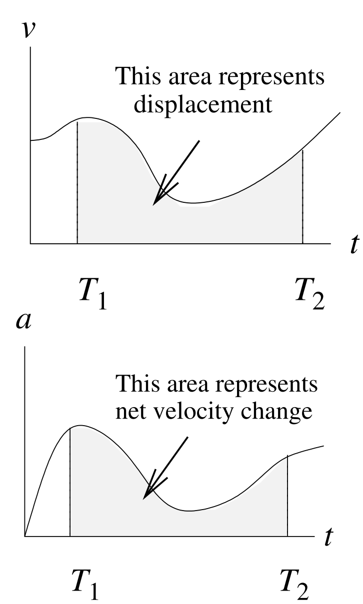



Figure 9.1: The total area under the velocity graph represents net displacement, and the total area under the graph of acceleration represents the net change in velocity over the interval $T_1 \le t \le T_2$.

Next, we consider two examples where either the acceleration or the velocity is constant. We use the results above to compute the displacements in each case.

*Displacement for uniform motion*

We first examine the simplest case that the velocity is constant, i.e. $v(t) = v =$ constant. Then clearly, the acceleration is zero since $a = dv/dt = 0$ when $v$ is constant. Thus, by direct antidifferentiation,

$$\int_0^T v\,dt = vt\Big|_0^T = v(T-0) = vT.$$

However, applying result (9.1) over the time interval $0 \le t \le T$ also leads to

$$\int_0^T v\,dt = x(T) - x(0).$$

Therefore, it must be true that the two expressions obtained above must be equal, i.e.

$$x(T) - x(0) = vT.$$

Thus, for uniform motion, the displacement is proportional to the velocity and to the time elapsed. The final position is

$$x(T) = x(0) + vT.$$

This is true for all time $T$, so we can rewrite the results in terms of the more familiar (lower case) notation for time, $t$, i.e.

$$x(t) = x(0) + vt. \tag{9.3}$$

*Uniformly accelerated motion*

In this case, the acceleration $a$ is a constant. Thus, by direct antidifferentiation,

$$\int_0^T a\,dt = at\Big|_0^T = a(T-0) = aT.$$

However, using Equation (9.2) for $0 \le t \le T$ leads to

$$\int_0^T a\,dt = v(T) - v(0).$$

Since these two results must match, $v(T) - v(0) = aT$ so that

$$v(T) = v(0) + aT.$$

Let us refer to the initial velocity $V(0)$ as $v_0$. The above connection between velocity and acceleration holds for any final time $T$, i.e., it is true for all $t$ that:

$$v(t) = v_0 + at. \tag{9.4}$$

This just means that velocity at time $t$ is the initial velocity incremented by an increase (over the given time interval) due to the acceleration. From this we can find the displacement and position of the particle as follows: Let us call the initial position $x(0) = x_0$. Then

$$\int_0^T v(t)\, dt = x(T) - x_0. \tag{9.5}$$

But

$$I = \int_0^T v(t)\, dt = \int_0^T (v_0 + at)\, dt = \left( v_0 t + a\frac{t^2}{2} \right) \Big|_0^T = \left( v_0 T + a\frac{T^2}{2} \right). \tag{9.6}$$

So, setting Equations (9.5) and (9.6) equal means that

$$x(T) - x_0 = v_0 T + a\frac{T^2}{2}.$$

But this is true for *all* final times, $T$, i.e. this holds for any time $t$ so that

$$x(t) = x_0 + v_0 t + a\frac{t^2}{2}. \tag{9.7}$$

This expression represents the position of a particle at time $t$ given that it experienced a constant acceleration. The initial velocity $v_0$, initial position $x_0$ and acceleration $a$ allowed us to predict the position of the object $x(t)$ at any later time $t$. That is the meaning of Eqn. (9.7)[1].

[1] Of course, Eqn. (9.7) only holds so long as the object is accelerating. Once the a falling object hits the ground, for example, this equation no longer holds.

### *Non-constant acceleration and terminal velocity*

In general, the acceleration of a falling body is not actually uniform, because frictional forces impede that motion. A better approximation to the rate of change of velocity is given by the **differential equation**

$$\frac{dv}{dt} = g - kv. \tag{9.8}$$

We will assume that initially the velocity is zero, i.e. $v(0) = 0$.

This equation is a mathematical statement that relates changes in velocity $v(t)$ to the constant acceleration due to gravity, $g$, and drag forces due to friction with the atmosphere. A good approximation for such drag forces is the term $kv$, proportional to the velocity, with $k$, a positive constant, representing a frictional coefficient. Because $v(t)$ appears both in the derivative and in the expression $kv$, we cannot apply the methods developed in the previous section directly. That is, we do not have an expression that depends on time whose antiderivative we would calculate. The derivative of $v(t)$ (on the left) is connected to the unknown $v(t)$ on the right.

Finding the velocity and then the displacement for this type of motion requires special techniques. In Chapter 12, we will develop a systematic approach, called Separation of Variables to find analytic solutions to equations such as (9.8).

Here, we use a special procedure that allows us to determine the velocity in this case. We first recall the following result from first term calculus material:

The differential equation and initial condition

$$\frac{dy}{dt} = -ky, \quad y(0) = y_0 \tag{9.9}$$

has a solution

$$y(t) = y_0 e^{-kt}. \tag{9.10}$$

Equation (9.8) implies that

$$a(t) = g - kv(t),$$

where $a(t)$ is the acceleration at time $t$. Taking a derivative of both sides of this equation leads to

$$\frac{da}{dt} = -k\frac{dv}{dt} = -ka.$$

We observe that this equation has the same form as equation (9.9) (with $a$ replacing $y$), which implies (according to 9.10) that $a(t)$ is given by

$$a(t) = C\, e^{-kt} = a_0\, e^{-kt}.$$

Initially, at time $t = 0$, the acceleration is $a(0) = g$ (since $a(t) = g - kv(t)$, and $v(0) = 0$). Therefore,

$$a(t) = g\, e^{-kt}.$$

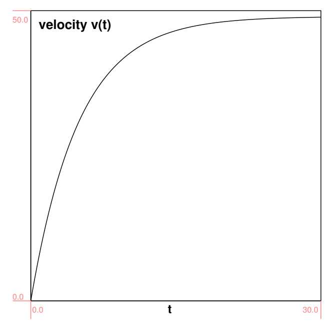


Figure 9.2: Terminal velocity (m/s) for acceleration due to gravity $g$=9.8 m/s$^2$, and $k = 0.2$/s. The velocity reaches a near constant 49 m/s by about 20 s.

Since we now have an explicit formula for acceleration vs time, we can apply direct integration as we did in the examples in Sections 9.2 and 9.2. The result is:

$$\begin{aligned} \int_0^T a(t)\, dt &= \int_0^T g\, e^{-kt}\, dt = g \int_0^T e^{-kt}\, dt \\ &= g \left[\frac{e^{-kt}}{-k}\right]\Bigg|_0^T = g\, \frac{(e^{-kT} - 1)}{-k} = \frac{g}{k}\, \left(1 - e^{-kT}\right). \end{aligned}$$

In the calculation, we have used the fact that the antiderivative of $e^{-kt}$ is $e^{-kt}/k$. (This can be verified by simple differentiation.)

As before, based on equation (9.2) this integral of the acceleration over $0 \le t \le T$ must equal $v(T) - v(0)$. But $v(0) = 0$ by assumption, and the result is true for *any* final time $T$, so, in particular, setting $T = t$, and combining both results leads to an expression for the velocity at any time:

$$v(t) = \frac{g}{k}\, \left(1 - e^{-kt}\right). \tag{9.11}$$

We will study the differential equation (9.8) again in Section 12.3, in the context of a more detailed discussion of differential equations

From our result here, we can also determine how the velocity behaves in the long term: observe that for $t \to \infty$, the exponential term $e^{-kt} \to 0$, so that

$$v(t) \to \frac{g}{k}(1 - \text{ very small quantity}) \approx \frac{g}{k}.$$

Thus, when drag forces are in effect, the falling object does not continue to accelerate indefinitely: it eventually attains a **terminal velocity**. We have seen that this limiting velocity is $v = g/k$. The object continues to fall at this (approximately constant) speed as shown in Figure 9.2. The terminal velocity is also a steady state value of Eqn. (9.8), i.e. a value of the velocity at which no further change occurs.

## 9.3 From rates of change to total change

In this section, we examine several examples in which the rate of change of some process is specified. We use this information to obtain the total change[2] that occurs over some time period.

[2] We will use the terminology "total change" and "net change" interchangeably in this section.

We must carefully distinguish between information about the time dependence of some function, from information about the rate of change of some function. Here is an example of these two different cases, and how we would handle them.

**Example 9.1 (Changing temperature)***(a) The temperature of a cup of juice is observed to be*

$$T(t) = 25(1 - e^{-0.1t})^\circ \textit{Celcius}$$

*where $t$ is time in minutes. Find the change in the temperature of the juice between the times $t = 1$ and $t = 5$.*

*(b) The* **rate of change** *of temperature of a cup of coffee is observed to be*

$$f(t) = 8e^{-0.2t\circ} \textit{Celcius per minute}$$

*where $t$ is time in minutes. What is the* **total change** *in the temperature between $t = 1$ and $t = 5$ minutes ?*

**Solution.**

(a) In this case, we are given the temperature as a function of time. To determine what **net change** occurred between times $t = 1$ and $t = 5$, we find the temperatures at each time point and subtract: That is, the change in temperature between times $t = 1$ and $t = 5$ is simply

$$T(5) - T(1) = 25(1 - e^{-0.5}) - 25(1 - e^{-0.1}) = 25(0.94 - 0.606) = 7.47^\circ\text{C}.$$

(b) Here, we do not know the temperature at any time, but we are given information about **the rate of change**. (Carefully note the subtle difference in the wording.) To get the total change, we would sum up all the small changes, $f(t)\Delta t$ (over $N$ subintervals of duration $\Delta t = (5-1)/N = 4/N$) for $t$ starting at 1 and ending at 5 min. We obtain a sum of the form $\sum f(t_k)\Delta t$ where $t_k$ is the k'th time point. Finally, we take a limit as the number of subintervals increases ($N \to \infty$). By now, we recognize that this amounts to a process of integration. Based on this variation of the same concept we can take the usual shortcut of integrating the rate of change, $f(t)$, from $t = 1$ to $t = 5$. To do so, we apply the Fundamental Theorem as before, reducing the amount of computation to finding antiderivatives. We compute:

$$I = \int_1^5 f(t)\, dt = \int_1^5 8e^{-0.2t} dt = -40e^{-0.2t}\Big|_1^5 = -40e^{-1} + 40e^{-0.2},$$

$$I = 40(e^{-0.2} - e^{-1}) = 40(0.8187 - 0.3678) = 18.$$

Only in the second case did we need to use a definite integral to find a net change, since we were given the way that the *rate of change* depended on time. Recognizing the subtleties of the wording in such examples will be an important skill that the reader should gain. ◊

**Example 9.2 (Tree growth rates)** *The rate of growth in height for two species of trees (in feet per year) is shown in Figure 9.3. If the trees start at the same height, which tree is taller after 1 year? After 4 years?*

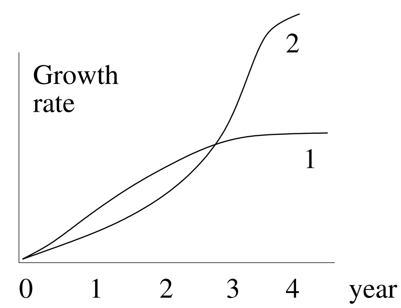


Figure 9.3: Growth rates of two trees over a four year period. Tree 1 initially has a higher growth rate, but tree 2 catches up and grows faster after year 3.

**Solution.** In this problem we are provided with a sketch, rather than a formula for the growth rate of the trees. Our solution will thus be *qualitative* (i.e. descriptive), rather than *quantitative*. (This means we do not have to calculate anything; rather, we have to make some important observations about the behaviour shown in Fig 9.3.)

We recognize that the net change in height of each tree is of the form

$$H_i(T) - H_i(0) = \int_0^T g_i(t)dt, \quad i = 1, 2.$$

where $i = 1$ for tree 1, $i = 2$ for tree 2, $g_i(t)$ is the growth rate as a function of time (shown for each tree in Figure 9.3) and $H_i(t)$ is the height of tree "$i$" at time $t$. But, by the Fundamental Theorem of Calculus, this definite integral corresponds to the area under the curve $g_i(t)$ from $t = 0$ to $t = T$. Thus we must interpret the net change in height for each tree as the area under its growth curve. We see from Figure 9.3 that at $t = 1$ year, the area under the curve for tree 1 is greater, so it has grown more. At $t = 4$ years the area under the second curve is greatest so tree 2 has grown the most by that time. ◊

**Example 9.3 (Radius of a tree trunk)** *The trunk of a tree, assumed to have the shape of a cylinder, grows incrementally, so that its cross-section consists of "rings". In years of plentiful rain and adequate nutrients, the tree grows faster than in years of drought or poor soil conditions. Suppose the rainfall* *pattern has been cyclic, so that, over a period of 14 years, the growth rate of the radius of the tree trunk (in cm/year) is given by the function*

$$f(t) = 1.5 + \sin(\pi t/5),$$

*as shown in Figure 9.4. Let the height of the tree trunk be approximately constant over this ten year period, and assume that the density of the trunk is approximately 1 gm/cm*$^3$.

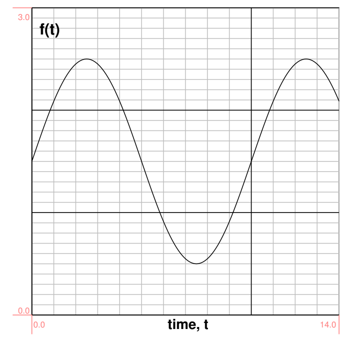



Figure 9.4: Rate of change of radius, $f(t)$ for a growing tree over a period of 14 years.

(a) *If the radius was initially* $r_0$ *at time* $t = 0$, *what will the radius of the trunk be at time t later?*

(b) *What is the ratio of the mass of the tree trunk at* $t = 10$ *years and* $t = 0$ *years? (i.e. find the ratio mass(10)/mass(0).)*

**Solution.**

(a) Let $R(t)$ denote the trunk's radius at time t. The rate of change of the radius of the tree is given by the function $f(t)$, and we are told that at $t = 0$, $R(0) = r_0$. A graph of this growth rate over the first fifteen years is shown in Figure 9.4. The net change in the radius is

$$R(t) - R(0) = \int_0^t f(s)\, ds = \int_0^t (1.5 + \sin(\pi s/5))\, ds.$$

Integrating the above, we get

$$R(t) - R(0) = \left(1.5t - \frac{\cos(\pi s/5)}{\pi/5}\right)\Bigg|_0^t.$$

Here we have used the fact that the antiderivative of $\sin(ax)$ is $-(\cos(ax)/a)$.

Thus, using the initial value, $R(0) = r_0$ (which is a constant), and evaluating the integral, leads to

$$R(t) = r_0 + 1.5t - \frac{5\cos(\pi t/5)}{\pi} + \frac{5}{\pi}.$$

(The constant at the end of the expression stems from the fact that $\cos(0) = 1$.) A graph of the radius of the tree over time (using $r_0 = 1$) is shown in Figure 9.5. This function is equivalent to the area associated with the function shown in Figure 9.4. Notice that Figure 9.5 confirms that the radius keeps growing over the entire period, but that its growth rate (slope of the curve) alternates between higher and lower values.

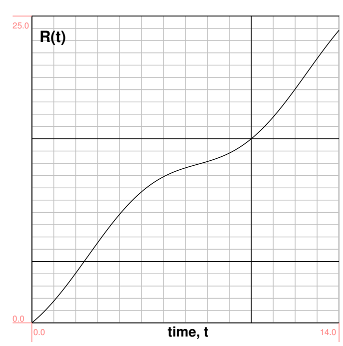



Figure 9.5: The radius of the tree, $R(t)$, as a function of time, obtained by integrating the rate of change of radius shown in Fig. 9.4.

After ten years we have

$$R(10) = r_0 + 15 - \frac{5}{\pi}\cos(2\pi) + \frac{5}{\pi}.$$

But $\cos(2\pi) = 1$, so

$$R(10) = r_0 + 15.$$

(b) The mass of the tree is density times volume, and since the density in this example is constant, 1 gm/cm$^3$, we need only obtain the volume at $t = 10$. Taking the trunk to be cylindrical means that the volume at any given time is

$$V(t) = \pi[R(t)]^2 h.$$

The ratio we want is then

$$\frac{V(10)}{V(0)} = \frac{\pi[R(10)]^2 h}{\pi r_0^2 h} = \frac{[R(10)]^2}{r_0^2} = \left(\frac{r_0 + 15}{r_0}\right)^2.$$

In this problem we used simple anti-differentiation to compute the desired total change. We also related the graph of the radial growth rate in Fig. 9.4 to that of the resulting radius at time $t$, in Fig. 9.5. ♢

**Example 9.4 (Birth rates and total births)** *After World War II, the birth rate in western countries increased dramatically. Suppose that the number of babies born (in millions per year) was given by*

$$b(t) = 5 + 2t, \ \ 0 \le t \le 10,$$

*where t is time in years after the end of the war.*

*(a) How many babies in total were born during this time period (i.e in the first 10 years after the war)?*

*(b) Find the time $T_0$ such that the total number of babies born from the end of the war up to the time $T_0$ was precisely 14 million.*

**Solution.**

(a) To find the number of births, we would integrate the birth rate, $b(t)$ over the given time period. The *net change* in the population due to births (neglecting deaths) is

$$P(10) - P(0) = \int_0^{10} b(t)\, dt = \int_0^{10} (5 + 2t)\, dt = (5t + t^2)|_0^{10} = 50 + 100 = 150[\text{million babies}].$$

(b) Denote by $T$ the time at which the total number of babies born was 14 million. Then, [in units of million]

$$I = \int_0^T b(t)\, dt = 14 = \int_0^T (5 + 2t)\, dt = 5T + T^2$$

equating $I = 14$ leads to the quadratic equation, $T^2 + 5T - 14 = 0$, which can be written in the factored form, $(T - 2)(T + 7) = 0$. This has two solutions, but we reject $T = -7$ since we are looking for time after the War. Thus we find that $T = 2$ years, i.e it took two years for 14 million babies to have been born. ♢

While this problem involves simple integration, we had to solve for a quantity ($T$) based on information about behaviour of that integral. Many problems in real application involve such slight twists on the ideas of integration.

## *9.4 Production and removal*

The process of integration can be used to convert rates of production and removal into net amounts present at a given time. The example in this section is of this type. We investigate a process in which a substance accumulates as it is being produced, but disappears through some removal process. We would like to determine when the quantity of material increases, and when it decreases.

**Circadean rhythm in hormone levels**

Consider a hormone whose level in the blood at time $t$ will be denoted by $H(t)$. We will assume that the level of hormone is regulated by two separate processes: one might be the secretion rate of specialized cells that produce the hormone. (The production rate of hormone might depend on the time of day, in some cyclic pattern that repeats every 24 hours or so.) This type of cyclic pattern is called *circadean* rhythm. A competing process might be the removal of hormone (or its deactivation by some enzymes secreted by other cells.) In this example, we will assume that both the production rate, $p(t)$, and the removal rate, $r(t)$, of the hormone are time-dependent, periodic functions with somewhat different phases.

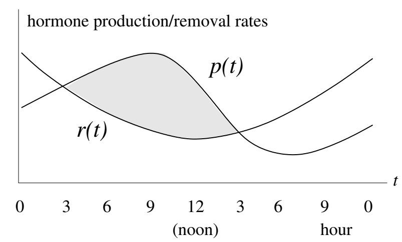



Figure 9.6: The rate of hormone production $p(t)$ and the rate of removal $r(t)$ are shown here. We want to use these graphs to deduce when the level of hormone is highest and lowest.

A typical example of two such functions are shown in Figure 9.6. This figure shows the production and removal rates over a period of 24 hours, starting at midnight. Our first task will be to use properties of the graph in Figure 9.6 to answer the following questions:

1. When is the production rate, $p(t)$, maximal?
2. When is the removal rate $r(t)$ minimal?
3. At what time is the hormone level in the blood highest?
4. At what time is the hormone level in the blood lowest?
5. Find the maximal level of hormone in the blood over the period shown, assuming that its basal (lowest) level is $H = 0$.

**Solutions**

1. We see directly from Fig. 9.6 that production rate is maximal at about 9:00 am.
2. Similarly, removal rate is minimal at noon.
3. To answer this question we note that the total amount of hormone produced over a time period $a \le t \le b$ is

$$P_{\text{total}} = \int_a^b p(t)dt.$$

The total amount removed over time interval $a \le t \le b$ is

$$R_{\text{total}} = \int_a^b r(t)dt.$$

This means that the net change in hormone level over the given time interval (amount produced minus amount removed) is

$$H(b) - H(a) = P_{\text{total}} - R_{\text{total}} = \int_a^b (p(t) - r(t))dt.$$

We interpret this integral as the *area between the curves* $p(t)$ and $r(t)$. But we must use caution here: For any time interval over which $p(t) > r(t)$, this integral will be positive, and the hormone level will have increased. Otherwise, if $r(t) < p(t)$, the integral yields a negative result, so that the hormone level has decreased. This makes simple intuitive sense: If production is greater than removal, the level of the substance is accumulating, whereas in the opposite situation, the substance is decreasing. With these remarks, we find that the hormone level in the blood will be *greatest* at 3:00 pm, when the greatest (positive) area between the two curves has been obtained.

4. Similarly, the least hormone level occurs after a period in which the removal rate has been larger than production for the longest stretch. This occurs at 3:00 am, just as the curves cross each other.

5. Here we will practice integration by actually fitting some cyclic functions to the graphs shown in Figure 9.6. Our first observation is that if the length of the cycle (also called the *period*) is 24 hours, then the *frequency* of the oscillation is $\omega = (2\pi)/24 = \pi/12$.

   We further observe that the functions shown in the Figure 9.7 have the form
$$p(t) = A(1+\sin(\omega t)), \qquad r(t) = A(1+\cos(\omega t)).$$

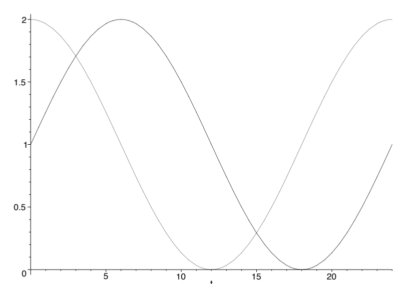



Figure 9.7: The functions shown above are trigonometric approximations to the rates of hormone production and removal from Figure 9.6.

   Intersection points occur when

$$\begin{aligned} p(t) &= r(t) \\ A(1+\sin(\omega t)) &= A(1+\cos(\omega t)) \\ \sin(\omega t) &= \cos(\omega t) \\ \tan(\omega t) &= 1. \end{aligned}$$

   This last equality leads to $\omega t = \pi/4, 5\pi/4$. But then, the fact that $\omega = \pi/12$ implies that $t = 3, 15$. Thus, over the time period $3 \le t \le 15$ hrs, the hormone level is increasing. For simplicity, we will take the amplitude $A = 1$. (In general, this would just be a multiplicative constant in whatever solution we compute.) Then the net increase in hormone over this period is calculated from the integral

$$H_{\text{total}} = \int_3^{15} [p(t) - r(t)]\, dt = \int_3^{15} [(1+\sin(\omega t)) - (1+\cos(\omega t))]\, dt$$

In the problem set, the reader is asked to compute this integral and to show that the amount of hormone that accumulated over the time interval $3 \le t \le 15$, i.e. between 3:00 am and 3:00 pm is $24\sqrt{2}/\pi$.

## 9.5 *Average value of a function*

In this final example, we apply the definite integral to computing the average height of a function over some interval. First, we define what is meant by average value in this context.

Given a function

$$y = f(x)$$

over some interval $a \le x \le b$, we will define average value of the function as follows:

**Definition.** The average value of $f(x)$ over the interval $a \le x \le b$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

**Example 9.5** *Find the average value of the function $y = f(x) = x^2$ over the interval $2 < x < 4$.*

**Solution.**

$$\bar{f} = \frac{1}{4-2}\int_2^4 x^2\,dx = \frac{1}{2}\frac{x^3}{3}\bigg|_2^4 = \frac{1}{6}\left(4^3 - 2^3\right) = \frac{28}{3}$$

$\Diamond$

**Example 9.6 (Day length over the year)** *Find the average length of the day during summer and spring.*

**Solution.** We will assume that day length follows a simple periodic behaviour, with a cycle length of 1 year (365 days). Let us measure time $t$ in days, with $t = 0$ at the spring equinox, i.e. the date in spring when night and day lengths are equal, each being 12 hrs. We will refer to the number of daylight hours on day $t$ by the function $f(t)$. (We will also call $f(t)$ the day-length on day $t$.

A simple function that would describe the cyclic changes of day length over the seasons is

$$f(t) = 12 + 4\sin\left(\frac{2\pi t}{365}\right).$$

This is a periodic function with period 365 days as shown in Figure 9.8.

Its maximal value is $16h$ and its minimal value is $8h$. The average day length over spring and summer, i.e. over the first $(365/2) \approx 182$ days is:

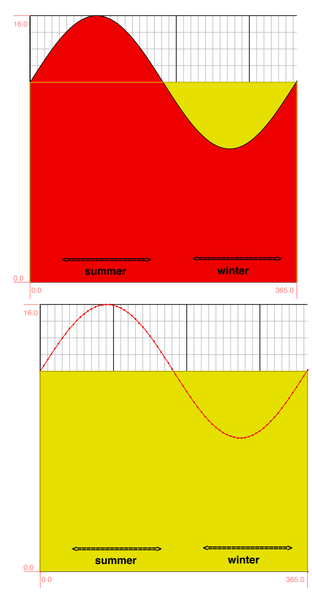


Figure 9.8: We show the variations in day length (cyclic curve) as well as the average day length (height of rectangle) in this figure.

$$
\begin{aligned}
\bar{f} &= \frac{1}{182}\int_0^{182} f(t)dt \\
&= \frac{1}{182}\int_0^{182}\left(12+4\sin(\frac{\pi t}{182})\right)\, dt \\
&= \frac{1}{182}\left(12t-\frac{4\cdot 182}{\pi}\cos(\frac{\pi t}{182})\right)\Big|_0^{182} \\
&= \frac{1}{182}\left(12\cdot 182-\frac{4\cdot 182}{\pi}[\cos(\pi)-\cos(0)]\right) \\
&= 12+\frac{8}{\pi}\approx 14.546 \text{ hours} \qquad (9.12)
\end{aligned}
$$

Thus, on average, the day is 14.546 hrs long during the spring and summer. $\Diamond$

In Figure 9.8, we show the entire day length cycle over one year. It is left as an exercise for the reader to show that the average value of $f$ over the entire year is 12 hrs. (This makes intuitive sense, since overall, the short days in winter will average out with the longer days in summer.)

Figure 9.8 also shows geometrically what the average value of the function represents. Suppose we determine the area associated with the graph of $f(x)$ over the interval of interest. (This area is painted red (dark) in Figure 9.8, where the interval is $0 \le t \le 365$, i.e. the whole year.) Now let us draw in a rectangle over the same interval ($0 \le t \le 365$) having the same total area. (See the big rectangle in Figure 9.8, and note that its area matches with the darker, red region.) The height of the rectangle represents the average value of $f(t)$ over the interval of interest.

## 9.6 *Application: Flu Vaccination*

Suppose Health Canada has been monitoring flu outbreaks continuously over the last 100 years. They have found that the number of infections follows an annual (seasonal) cycle and a twenty-year-cycle. In all, the number of infections $I(t)$ are well-approximated by the function:

$$
I(t) = \underbrace{\cos\left(\frac{\pi}{6}t\right)}_{\text{annual cycle}} + \underbrace{\cos\left(\frac{\pi}{120}t\right)}_{\text{20 year cycle}} + 2 \qquad (9.13)
$$

where $t$ is measured in months and $I(t)$ given in units of $100{,}000$ individuals.

Figure 9.9 depicts the number of infections $I(t)$ over time and illustrates the superposition of the annual cycle of seasonal flu outbreaks modulated by slower fluctuations with a longer period of 10 years.

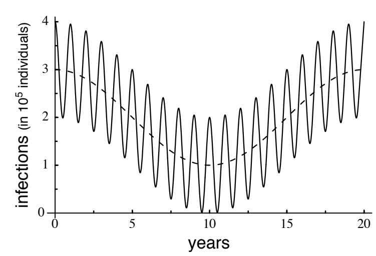



Figure 9.9: Flu infections over time – the number of infections undergoes seasonal fluctuations, which are superimposed on slower fluctuations of a twenty-year-cycle (dashed line).

**Example 9.7** *Health Canada decides to eradicate the flu. This is estimated to take* 5 *years of intensive vaccination and quarantine. In order to use the least amount of resources, they decide to start their erradication program at the start of a* 5*-year term where the flu virus is naturally at a* 5*-year minimum* *average. Supposing that $t = 0$ is January $1^{st}$, 2013, when should they start this program?*

**Solution.**

*Average:* First, find 5-year average, $\bar{I}(t)$, starting at $t$:

$$\begin{aligned}
\bar{I}(t) &= \frac{1}{60}\int_t^{t+60} I(s)\,ds = \frac{1}{60}\int_t^{t+60} \cos\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{120}t\right) + 2\,ds \\
&= \frac{1}{60}\left[\sin\left(\frac{\pi}{6}s\right)\frac{6}{\pi} + \sin\left(\frac{\pi}{120}s\right)\frac{120}{\pi} + 2s\right]\Bigg|_t^{t+60} \\
&= \frac{1}{60}\left[\sin\left(\frac{\pi}{6}(t+60)\right)\frac{6}{\pi} + \sin\left(\frac{\pi}{120}(t+60)\right)\frac{120}{\pi} + 120\right. \\
&\qquad \left. - \sin\left(\frac{\pi}{6}t\right)\frac{6}{\pi} - \sin\left(\frac{\pi}{120}t\right)\frac{120}{\pi}\right]. \qquad (9.14)
\end{aligned}$$

Equation 9.14 could be further simplified using trigonometric identities but for our purposes this will do. $\bar{I}(t)$ is shown in 9.10.

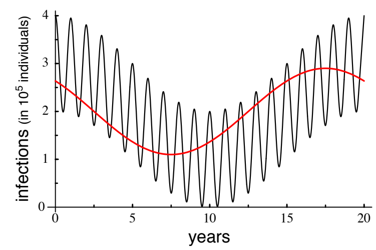



Figure 9.10: 5-year average of flu infections over time (red line), averaging starts at $t$ until $t+60$ months. As a reference, the number of infections are also shown (black line).

*Minimum:* Second, find the minimum of the 5-year average by solving $\dfrac{d\bar{I}(t)}{dt} = 0$:

$$\begin{aligned}
\frac{d\bar{I}(t)}{dt} &= \frac{1}{60}\left[\cos\left(\frac{\pi}{6}(t+60)\right) + \cos\left(\frac{\pi}{120}(t+60)\right) - \cos\left(\frac{\pi}{6}t\right) - \cos\left(\frac{\pi}{120}t\right)\right] \\
&= \frac{1}{60}\left[\cos\left(\frac{\pi}{6}t\right)\cos(10\pi) - \sin\left(\frac{\pi}{6}t\right)\sin(10\pi) + \cos\left(\frac{\pi}{120}t\right)\cos\left(\frac{\pi}{2}\right)\right. \\
&\qquad \left. - \sin\left(\frac{\pi}{120}t\right)\sin\left(\frac{\pi}{2}\right) - \cos\left(\frac{\pi}{6}t\right) - \cos\left(\frac{\pi}{120}t\right)\right] \\
&= \frac{1}{60}\left[-\sin\left(\frac{\pi}{120}t\right) - \cos\left(\frac{\pi}{120}t\right)\right]. \qquad (9.15)
\end{aligned}$$

Note, for the second equality we have used the trigonometric identity $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$. Hence, the start of a 5-year minimum (or maximum) average period is marked by the condition $\cos\left(\frac{\pi}{120}t\right) = -\sin\left(\frac{\pi}{120}t\right)$. The equality $\cos\alpha = -\sin\alpha$ holds for $\alpha = \frac{3\pi}{4}$ and $\alpha = \frac{7\pi}{4}$ (as well as when adding multiples of $2\pi$ to $\alpha$). Since $\cos(\alpha)$ is a decreasing function for $0 < \alpha < \pi$, we expect that $\alpha = \frac{3\pi}{4}$ marks a minimum. Solving

$$\frac{\pi}{120}t = \frac{3\pi}{4}$$

yields $t = 90$ months. Indeed this indicates the start of a 5-year *minimum* average because

$$\left.\frac{d^2\bar{I}(t)}{dt^2}\right|_{t=90} = \frac{\pi}{120} > 0.$$

Hence the earliest intervention could start on June $1^{st}$, 2020, i.e. 90 months after January $1^{st}$, 2013.

*Shortcut:* Even though there is nothing wrong with the previous two steps, it is nevertheless important and educational to realize that we first calculated an antiderivative only to take the derivative of it! While this is a good exercise it is actually unnecessary and we could have used the Fundamental Theorem of Calculus (*FTC*) instead:

$$\begin{aligned}\frac{d\bar{I}(t)}{dt} &= \frac{d}{dt}\left[\frac{1}{60}\int_t^{t+60} I(s)\,ds\right]\\ &= \frac{1}{60}\frac{d}{dt}\left[F(t+60)-F(t)\right], \quad \text{where } F(t) \text{ is any antiderivative of } I(t)\\ &= \frac{1}{60}\left[I(t+60)-I(t)\right]. \end{aligned} \tag{9.16}$$

Note, for the second equality we have used the *FTC* part *II* and for the last equality the *FTC*, part *I*. After inserting $I(t)$ and some algebraic manipulations this immediately leads to 9.15 and the remaining calculations remain the same as before.

*Check:* Looking at 9.9 we note that the the slow fluctuations (dashed line) have a minimum after 10 years. Therefore we would expect that the 5-year minimum average would be centered at 10 years and hence that the averaging window starts 2.5 years earlier, i.e. after 7.5 years or 90 months. In the present case this estimate turns out to be accurate – however, this is only true because in the present case the peak of the slow fluctations coincides with the peak of seasonal flu infections. ◊

## 9.7 Summary

In this chapter, we arrived at a number of practical applications of the definite integral.

1. In Section 9.2, we found that for motion at constant acceleration $a$, the displacement of a moving object can be obtained by integrating twice: the definite integral of acceleration is the velocity $v(t)$, and the definite integral of the velocity is the displacement.

$$v(t) = v_0 + \int_0^t a\,ds. \quad x(t) = x(0) + \int_0^t v(s)\,ds.$$

   (Here we use the "dummy variable" $s$ inside the integral, but the meaning is, of course, the same as in the previous presentation of the formulae.) We showed that at any time $t$, the position of an object initially at $x_0$ with velocity $v_0$ is

$$x(t) = x_0 + v_0 t + a\frac{t^2}{2}.$$

2. We extended our analysis of a moving object to the case of non-constant acceleration (Section 9.2), when air resistance tends to produce a drag

force to slow the motion of a falling object. We found that in that case, the acceleration gradually decreases, $a(t) = ge^{-kt}$. (The decaying exponential means that $a \to 0$ as $t$ increases.) Again, using the definite integral, we could compute a velocity,

$$v(t) = \int_0^t a(s)\, ds = \frac{g}{k}(1 - e^{-kt}).$$

3. We illustrated the connection between rates of change (over time) and total change (between on time point and another). In general, we saw that if $r(t)$ represents a rate of change of some process, then

$$\int_a^b r(s)\, ds = \text{Total change over the time interval } a \le t \le b.$$

   This idea was discussed in Section 9.3.

4. In the concluding Section 9.5, we discussed the average value of a function $f(x)$ over some interval $a \le x \le b$,

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

   In the next few chapters we encounters a vast assortment of further examples and practical applications of the definite integral, to such topics as mass, volumes, length, etc. In some of these we will be called on to "dissect" a geometric shape into pieces that are not simple rectangles. The essential idea of an integral as a sum of many (infinitesimally) small pieces will, nevertheless be the same.

## *Exercises*

**Exercise 9.1** Two cars, labeled 1 and 2 start side by side and accelerate from rest. Figure 9.11 shows a graph of their velocity functions, with $t$ measured in minutes.

(a) At what time(s) do the cars have equal velocities?

(b) At what time(s) do the cars have equal accelerations?

(c) Which car is ahead after one minute?

(d) Which car is ahead after 3 minutes?

(e) When does one car overtake the other? (Give an approximate answer)

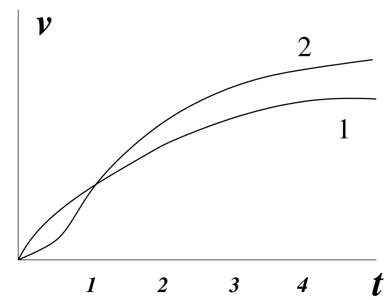


Figure 9.11: For problem 9.1

**Exercise 9.2** The speed of a car (km/h) is given by the expression

$$v(t) = 2t^2 + 5t, \qquad 0 < t < 1$$

where $t$ is time in hours. Use this expression to find

(a) The acceleration over this time period.

(b) The total displacement over the same time period.

**Exercise 9.3** The velocity of a boat moving through water is found to be $v(t) = 10(1 - e^{-t})$.

(a) Find the acceleration of the boat and show that it satisfies a differential equation, i.e an equation of the form $d[a(t)]/dt = -k \cdot a(t)$ for some value of the constant $k$ (i.e. find that value of $k$).

(b) Find the displacement of the boat at time $t$.

**Exercise 9.4** A particle in a force field accelerates so that its acceleration is described by the function

$$a(t) = t - t^2, \qquad 0 < t < 1$$

(a) Find the velocity of the particle $v(t)$ for $0 < t < 1$ assuming that it starts at rest.

(b) Find the total displacement of the particle over the time interval $0 < t < 1$.

**Exercise 9.5** An express mail truck delivers mail to various companies situated along a central avenue and often goes back and forth as new mail arrives. Over some period of time, $0 < t < 10$, its velocity (in km per hour) can be described by the function:

$$v(t) = t^2 - 9t + 14.$$

(a) Find the displacement over this period of time. (Hint: recall that if you leave home in the morning, travel to work, and then go back home, then your net distance traveled, or total displacement, over this full period of time is zero.)

(b) How much gasoline was consumed during this period of time if the vehicle uses $1/2$ liters per km. (Hint: To answer (b), you will need to find the total distance that the vehicle actually covered during its trip.)

**Exercise 9.6** The reaction time of a driver (time it takes to notice and react to danger in the road ahead) is about 0.5 seconds. When the brakes are applied, it then takes the car some time to decelerate and come to a full stop. If the deceleration rate is $a = -8$ m/sec$^2$, how long would it take the driver to stop from an initial speed of 100 km per hour? (Include both reaction time and deceleration time, and use the Fundamental Theorem of Calculus to arrive at your answer.)

**Exercise 9.7** You are driving your car quickly (at speed 100 km/h) to catch your flight to Hawaii for mid-term break. A pedestrian runs across the road, forcing you to brake hard. Suppose it takes you 1 second to react to the danger, and that when you apply your brakes, you slow down at the rate $a = -10$ m/s$^2$.

(a) How long will it take you to stop?

(b) How far will your car move from the instant that the danger is sighted until coming to a complete stop?

Use the Fundamental Theorem of Calculus to arrive at your answer.

**Exercise 9.8** The growth rate of a pair of twins (in mm/day) is shown in Figure 9.12. Suppose both children have the same weight at time $t = 0$.

(a) Which one is taller at age 0.5 years?

(b) Which one is taller at age 1 year?

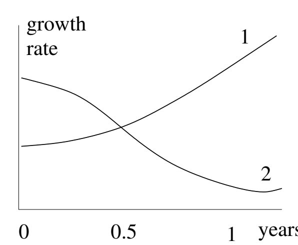


Figure 9.12: For problem 9.8

**Exercise 9.9** The flow rate of blood through the heart can be described approximately as a periodic function of the form $F(t) = A(1 + \sin(0.15t))$, where $t$ is time in seconds and $A$ is a constant in units of cubic cm per second. (Thus, at time $t$, $F(t)$ cm$^3$ of blood flow through the heart per second.) Find the total volume of blood that flows through the heart between $t = 0$ and $t = 1$. (Express your answer in terms of $A$.)

**Exercise 9.10** During a gambling session lasting 6 hours, the rate of winnings at a casino in dollars per hour are seen to follow the formula $w(t) = 2000t(1 - (t/6))$. Find the total winnings during that whole session.

**Exercise 9.11** At time $t$, an intravenous infusion delivers a flow rate of $y = 100(1-t^3)\ cm^3/hr$, where $t$ is time in hours. The infusion contains a drug at concentration 0.1 $mg/cm^3$. Find the total volume of fluid and the total amount of drug delivered to the patient over a 1 hour period from $t = 0$ to $t = 1$.

**Exercise 9.12** Oil leaks out of an oil tanker at the rate $f(t) = 10 - 0.2t^2$ (where $f$ is in 10 thousand barrels per hour and $t$ is in hours). (Note: This function only makes sense as long as $f(t) \geq 0$ since a negative flow of oil is meaningless in this case.)

(a) At what time will the flow be zero?

(b) What is the total amount that has leaked out between $t = 0$ and the time you found in (a)?

**Exercise 9.13** After a heavy rainfall, the rate of flow of water into a lake is found to satisfy the relationship $F(t) = 4 - \left(\frac{t}{10} - 1\right)^2$ where $t$ is time in hours, $0 \leq t \leq 30$ and $F$ is in units of 100,000 gallons/h.

(a) Find the time, $t_1$ at which the rate of flow is greatest.

(b) Find the time, $t_2$ at which the flow is zero.

(c) Find how much water in total has flowed into the lake between these two times.

**Exercise 9.14** The growth rate of a crop is known to depend on temperature during the growing season. Suppose the growth rate of the crop in tons per day is given by $g(t) = 0.1(T(t) - 18)$ where $T(t)$ is temperature in degrees Celsius. Suppose the temperature record during the 90 days of the season was $T(t) = 22 + 0.1t + 4\sin(2\pi t/60)$ where $t$ is time in days. Find the total growth (in tons) that would have occurred over the whole season.

**Exercise 9.15** This question concerns cumulative exposure to radiation experienced by people living near nuclear waste disposal sites.

(a) Recall from last term that radioactive material decays according to a negative exponential: $m(t) = m_0 \cdot e^{-rt}$, where $m(t)$ is the mass of the radioactive material at time $t$, $m_0 = m(0)$ is the initial mass at time $t = 0$, and $r$ is the rate of decay. The half-life is the time it takes for the material to decay to one half its initial mass. Suppose that $t$ is measured in months. Determine the rate of decay $r$ if the half-life is 1 month.

(b) Assume that at any time, the amount of radiation is proportional to the mass of the radioactive material. If initially the radiation level is 0.5 rads per month, how could we describe the radiation level as a function of time?

(c) Assume now that there is radioactive material in your backyard of the type considered in (a) above, i.e. that it has a half-life of one month. Calculate the cumulative exposure in rads that would occur if you lived in your house for 10 years.

**Exercise 9.16** The level of glucose in the blood depends on the rate of intake from ingestion of food and on the rate of clearance due to glucose metabolism. Shown in Figure 9.13 are two functions, $I(t)$ and $C(t)$ for the intake and clearance **rates** over a period of time after fasting. Both rates are functions of time $t$. Suppose that at time 0 there is no glucose in the blood.

(a) Express the level of blood glucose as a definite integral.

(b) At what approximate time was the intake rate maximal?

(c) At what approximate time was the clearance rate maximal?

(d) When was the blood level of glucose maximal?

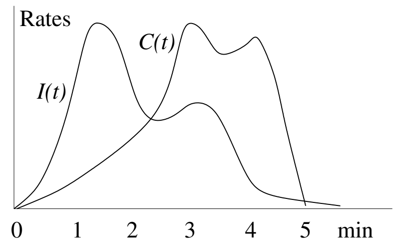



Figure 9.13: For problem 9.16

**Exercise 9.17** The rate at which water flows in and out of Capilano Reservoir is described by two functions. $I(t)$ is the rate at which water flows in to the reservoir (in gallons per day) and $O(t)$ is the rate at which water flows out (in gallons per day). See sketch below in Figure 9.14. Assume that there is water in the reservoir at time $t = 0$.

(a) Express the quantity of water $Q(t)$ in the reservoir as a definite integral. (i.e. $Q(0) > 0$).

(b) When is the quantity of water in the reservoir greatest and when is it smallest?

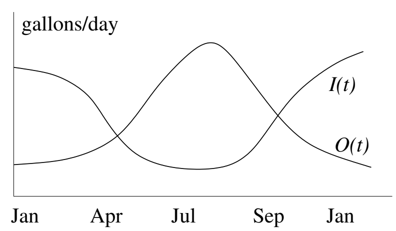



Figure 9.14: For problem 9.17

**Exercise 9.18** The rate at which animals migrate into and out of a wildlife reserve is described by two functions shown in Figure 9.15. $I(t)$ is the rate at which animals enter the reserve and $O(t)$ is the rate at which they leave (both in number per day).

(a) Express the number of animals in the reserve as a definite integral.

(b) When is the number of animals in the reserve greatest and when is it smallest?

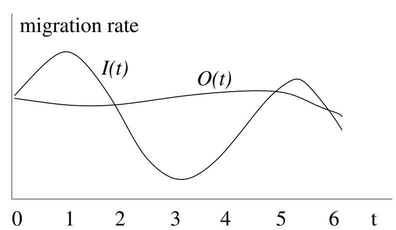



Figure 9.15: For problem 9.18

**Exercise 9.19** During a particularly soggy week in Vancouver, rainfall reached epic proportions. The rainfall pattern was as follows: A constant 20 mm over the first day, a steady increase from 20 up to 50 mm over the next day (assume linear increase), 50 mm rain over the next day, a steady drop from 50 down to 40 mm over the next day, and a flat 30 mm over the next day.

(a) Determine the total amount of rain during this period.

(b) Calculate the average daily rainfall for the same period.

**Exercise 9.20** Find the average value of the function $f(x) = \sin(\frac{\pi x}{2})$ over the interval $[0,2]$.

**Exercise 9.21**

(a) Find the average value of $x^n$ over the interval $[0,1]$.

(b) What happens as $n$ becomes arbitrarily large (that is, $n \to \infty$)?

(c) Explain your answer to part (b) by considering the graphs of these functions.

(d) Repeat parts (a) - (c) using the functions $x^{1/n}$.

**Exercise 9.22** An object starts from rest at $t = 0$ and accelerates so that $a = \frac{dv}{dt} = ce^{-t}$ where $c$ is a positive constant.

(a) Find the average velocity over the first $t$ seconds.

(b) What happens to this average velocity as $t$ becomes very large?

**Exercise 9.23 Symmetry**

(a) Find the average value of the function $\sin x$ over the interval $[-\pi, \pi]$.

(b) Find the average value of the function $x^3$ over the interval $[-1,1]$.

(c) Find the average value of the function $x^3 - x$ over the interval $[-1,1]$.

(d) Explain these results graphically.

(e) Find the average value of an odd function $f(x)$ over the interval $[-a,a]$. (Remember that $f(x)$ is odd if $f(-x) = -f(x)$.)

(f) Suppose now that $f(x)$ is an even function (that is, $f(-x) = f(x)$) and its average value over the interval $[0,1]$ is 2. Find its average value over the interval $[-1,1]$.

**Exercise 9.24** The intensity of light cast by a street lamp at a distance $x$ (in meters) along the street from the base of the lamp is found to be approximately $I(x) = 400 - x^2$ in arbitrary units for $-20 < x < 20$.

(a) Find the average intensity of the light over the interval $-5 < x < 5$.

(b) Find the average intensity over $-7 < x < 7$.

(c) Find the value of $b$ such that the average intensity over $[-b,b]$ is $I_{\text{av}} = 10$.

**Exercise 9.25** **Rates of hormone production and removal** Consider the rate of hormone production $p(t)$ and the rate of removal $r(t)$ given by

$$p(t) = A(1+\sin(\omega t)), \qquad r(t) = A(1+\cos(\omega t)).$$

for $\omega = \pi/12$. Calculate the net increase in hormone over the time period $3 \le t \le 15$ (in hours).

We find that

$$\begin{aligned}
H &= \int_3^{15} [p(t) - r(t)]\,dt = \int_3^{15} [(1+\sin(\omega t)) - (1+\cos(\omega t))]\,dt \\
&= \int_3^{15} (\sin(\omega t) - \cos(\omega t))\,dt = \left(-\frac{\cos(\omega t)}{\omega} - \frac{\sin(\omega t)}{\omega}\right)\Big|_3^{15} \\
&= -\frac{12}{\pi}\left((\cos\frac{15\pi}{12} + \sin\frac{15\pi}{12}) - (\cos\frac{3\pi}{12} + \sin\frac{3\pi}{12})\right) \\
&= -\frac{12}{\pi}\left((\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}) - (\cos\frac{\pi}{4} + \sin\frac{\pi}{4})\right) \\
&= -\frac{12}{\pi}\left((-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}) - (\cos\frac{\pi}{4} + \sin\frac{\pi}{4})\right) \\
&= -\frac{12}{\pi}\left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4} - \cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right) \\
&= -\frac{12}{\pi}\left(-2\sin\frac{\pi}{4} - 2\sin\frac{\pi}{4}\right) = -\frac{-12}{\pi}\left(\frac{-4}{\sqrt{2}}\right) = \frac{24\sqrt{2}}{\pi}
\end{aligned}$$

We used the fact that $5\pi/4$ is an angle in the third quadrant, so that $\cos(5\pi/4) = -\cos(\pi/4)$ and $\sin(5\pi/4) = -\sin(\pi/4)$.

Thus the amount of hormone that accumulated over the time interval $3 \le t \le 15$, i.e. between 3:00 am and 3:00 pm is $24\sqrt{2}/\pi$.

**Exercise 9.26** The length of time from sunrise to sunset (in hours) $t$ days after the Spring Equinox is given by

$$l(t) = 12 + 4\sin\left(\frac{\pi t}{182}\right)$$

(a) Explain the meaning of the word "equinox" and describe what happens on that day according to the above formula.

(b) What is the length of the shortest and the longest day and when do these occur according to this formula?

(c) How long is one complete cycle in this expression?

(d) Sketch $l(t)$ as a function of $t$.

(e) Find the average day-length over the month immediately following the equinox.

(f) Find the average day length over the whole year. Explain your result with a simple geometric or intuitive argument.

**Exercise 9.27** Consider the periodic function,

$$f(t) = \sin(2t) + \cos(2t).$$

(a) What is the frequency, the amplitude, and the length of one cycle in this function? (You are asked to express $f(t)$ in the form $A\sin(\omega t + \varphi)$. Then we use the terminology $A =$ amplitude, $\varphi =$ phase shift, $\omega =$ frequency.

(b) How would you define the average value of this function over *one cycle* ?

(c) Compute this average value and show that it is zero. Now explain why this is true using a geometric argument.

**Exercise 9.28** The current in an AC electric circuit is given by

$$I(t) = A\cos(\omega t)$$

The power in the circuit is defined as $P(t) = I^2(t)$.

(a) What is meant by *one cycle* in this situation?

(b) Sketch graphs of $I(t)$ and $P(t)$. Explain why $P(t)$ is always positive, and indicate how its zeros are related to zeros of $I(t)$. What are the maximal and minimal values of each of these functions?

(c) Find the average power and the average current over half a cycle. (Note: in computing the average power, you will need to use the trick $cos^2(\omega t) = \frac{1}{2}(1+\cos(2\omega t))$).

**Exercise 9.29** **Food surplus** Thousands of years ago, in the Fertile Crescent, the land was fertile and food production rate was plentiful and constant over time, but the population was growing steadily. Suppose that the rate of food production (in units of kg per year) is denoted $P$, and that the population size was $N(t) = N_0 e^{rt}$ where $t$ is time in months and $r$ is a per capita growth rate of the population. Assume that each person consumes food at the rate $f$ kg/year. In year 0, it was realized that there was a food surplus, and a large mud-brick structure was built to store the surplus food. (The surplus food is any food left over after consumption by the population that year).

(a) Write down an expression that represents the rate of accumulation of food surplus (as a function of time). Use that expression to determine the net surplus, $S(t)$ at time $t = T$.

(b) At what year was the food production rate the same as the food consumption rate?

(c) When was the food surplus greatest?

(d) How large was the maximal stored surplus?

# 10
# Applications of the definite integral to volume, mass, and length

## 10.1 Introduction

In this chapter, we consider applications of the definite integral to calculating geometric quantities such as volumes of geometric solids, masses and lengths of curves.

In this chapter, we will consider how to dissect certain three dimensional solids into a set of simpler parts whose volumes are easy to compute. We will use familiar formulae for the volumes of disks and cylindrical shells, and carefully construct a summation to represent the desired volume. The volume of the entire object will then be obtained by summing up volumes of a stack of disks or a set of embedded shells, and considering the limit as the thickness of the dissection cuts gets thinner. There are some important differences between material in this chapter and in previous chapters. Calculating volumes will stretch our imagination, requiring us to visualize 3-dimensional (3D) objects, and how they can be subdivided into shells or slices. Most of our effort will be aimed at understanding how to set up the needed integral. We provide a number of examples of this procedure.

## 10.2 Volumes of solids of revolution

We now turn to the problem of calculating volumes of 3D solids. Here we restrict attention to symmetric objects denoted *solids of revolution*. The outer surface of these objects is generated by revolving some curve around a coordinate axis. In Figure 10.2 we show one such curve, and the surface it forms when it is revolved about the $y$ axis.

### Volumes of cylinders and shells

Before starting the calculation, let us recall the volumes of some of the geometric shapes that are to be used as elementary pieces into which our shapes will be carved. See Figure 10.1.

1. The **volume of a cylinder** of height $h$ having circular base of radius $r$, is

$$V_{\text{cylinder}} = \pi r^2 h.$$

2. The **volume of a circular disk** of thickness $\tau$, and radius $r$ (shown on the left in Figure 10.1 ), is a special case of the above,

$$V_{\text{disk}} = \pi r^2 \tau.$$

3. The **volume of a cylindrical shell** of height $h$, with circular radius $r$ and small thickness $\tau$ (shown on the right in Figure 10.1) is

$$V_{\text{shell}} = 2\pi r h \tau.$$

(This approximation holds for $\tau \ll r$.)

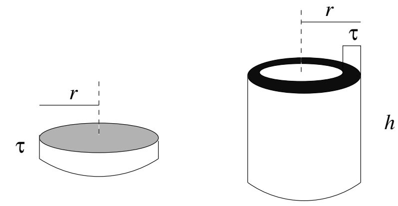



Figure 10.1: The volumes of these simple 3D shapes are given by simple formulae. We use them as basic elements in computing more complicated volumes. Here we will present examples based on disks. At the end of this chapter we give an example based on shells.

*Computing the Volumes*

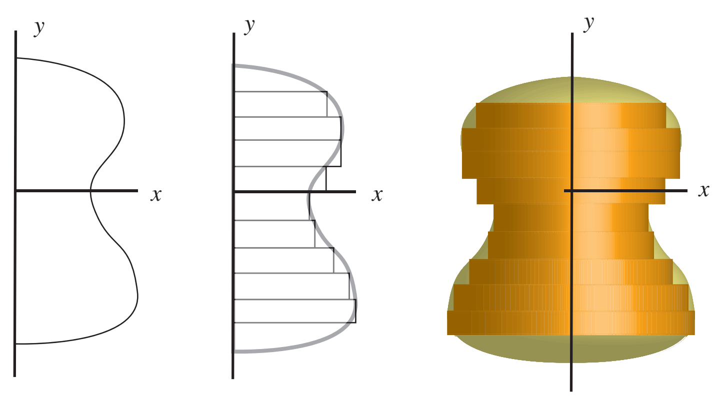



Figure 10.2: A solid of revolution is formed by revolving a region in the xy-plane about the y-axis. We show how the region is approximated by rectangles of some given width, and how these form a set of approximating disks for the 3D solid of revolution.

Consider the curve in Figure 10.2 and the surface it forms when it is revolved about the $y$ axis. In the same figure, we also show how a set of approximating rectangular strips associated with the planar region (grey rectangles) lead to a set of stacked disks (orange shapes) that approximate the volume of the solid (greenish object in Fig. 10.2). The total volume of the disks is not the same as the volume of the object but if we make the thickness of these disks very small, the approximation of the true volume is good. In the limit, as the thickness of the disks becomes infinitesimal, we arrive at the true volume of the solid of revolution. The reader should recognize a familiar theme. We used the same concept in computing areas using Riemann sums based on rectangular strips in Chapter 7.

Fig. 10.3 similarly shows a volume of revolution obtained by revolving the graph of the function $y = f(x)$ about the x axis. We note that if this surface is cut into slices, the radius of the cross-sections depend on the position of the cut. Let us imagine a stack of disks approximating this volume. One such disk has been pulled out and labeled for our inspection. We note that its radius (in the y direction) is given by the height of the graph of the function, so that $r = f(x)$. The thickness of the disk (in the x direction) is $\Delta x$. The volume of this single disk is then $v = \pi[f(x)]^2\Delta x$. Considering this disk to be based at the k'th coordinate point in the stack, i.e. at $x_k$, means that its volume is

$$v_k = \pi[f(x_k)]^2\Delta x.$$

Summing up the volumes of all such disks in the stack leads to the total volume of disks

$$V_{\text{disks}} = \sum_k \pi[f(x_k)]^2\Delta x.$$

When we increase the number of disks, making each one thinner so that $\Delta x \to 0$, we arrive at a definite integral,

$$V = \int_a^b \pi[f(x)]^2 dx.$$

In most of the examples discussed in this chapter, the key step is to make careful observation of the way that the radius of a given disk depends on the function that *generates* the surface. (By this we mean the function that specifies the curve that forms the surface of revolution.) We also pay attention to the dimension that forms the disk thickness, $\Delta x$.

Some of our examples will involve surfaces revolved about the $x$ axis, and others will be revolved about the $y$ axis. In setting up these examples, a diagram is usually quite helpful.

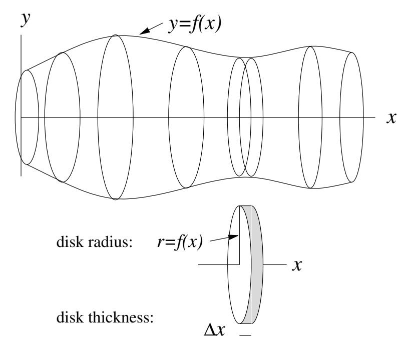



Figure 10.3: Here the solid of revolution is formed by revolving the curve $y = f(x)$ about the y axis. A typical disk used to approximate the volume is shown. The radius of the disk (parallel to the y axis) is $r = y = f(x)$. The thickness of the disk (parallel to the x axis) is $\Delta x$. The volume of this disk is hence $v = \pi[f(x)]^2\Delta x$

*Volume of a sphere*

We can think of a sphere of radius $R$ as a solid whose outer surface is formed by rotating a semi-circle about its long axis. A function that describe a semi-circle (i.e. the top half of the circle, $y^2 + x^2 = R^2$) is

$$y = f(x) = \sqrt{R^2 - x^2}.$$

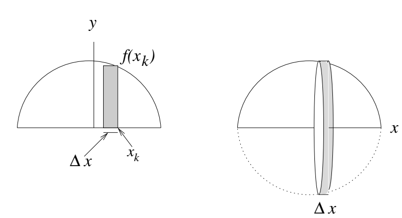



Figure 10.4: When the semicircle (on the left) is rotated about the x axis, it generates a sphere. On the right, we show one disk generated by the revolution of the shaded rectangle.

In Figure 10.4, we show the sphere dissected into a set of disks, each of width $\Delta x$. The disks are lined up along the $x$ axis with coordinates $x_k$, where $-R \le x_k \le R$. These are just integer multiples of the slice thickness $\Delta x$, so for example,

$$x_0 = -R, \quad x_1 = -R + \Delta x, \quad \dots, \quad x_k = -R + k\Delta x\ .$$

The radius of the disk depends on its position[1]. Indeed, the radius of a disk through the $x$ axis at a point $x_k$ is specified by the function $r_k = f(x_k)$. The volume of the $k$'th disk is

$$V_k = \pi r_k^2 \Delta x.$$

[1] Note that the radius is oriented along the y axis, so sometimes we may write this as $r_k = y_k = f(x_k)$.

By the above remarks, using the fact that the function $f(x)$ determines the radius, we have

$$V_k = \pi[f(x_k)]^2 \Delta x,$$

$$V_k = \pi \left[\sqrt{R^2 - x_k^2}\right]^2 \Delta x = \pi(R^2 - x_k^2)\Delta x.$$

The total volume of *all the disks* is

$$V = \sum_k V_k = \sum_k \pi[f(x_k)]^2 \Delta x = \pi \sum_k (R^2 - x_k^2)\Delta x.$$

as $\Delta x \to 0$, this sum becomes a definite integral, and represents the true volume. We start the summation at $x = -R$ and end at $x_N = R$ since the semi-circle extends from $x = -R$ to $x = R$. Thus

$$V_{\text{sphere}} = \int_{\text{-R}}^{R} \pi[f(x_k)]^2 \, dx = \pi \int_{\text{-R}}^{R} (R^2 - x^2) \, dx.$$

We compute this integral using the Fundamental Theorem of calculus, obtaining

$$V_{\text{sphere}} = \pi \left(R^2 x - \frac{x^3}{3}\right)\Bigg|_{-R}^{R}.$$

Observe that this is twice the volume obtained for the interval $0 < x < R$,

$$V_{\text{sphere}} = 2\pi \left(R^2 x - \frac{x^3}{3}\right)\Bigg|_{0}^{R} = 2\pi \left(R^3 - \frac{R^3}{3}\right).$$

We often use such symmetry properties to simplify computations. After simplification, we arrive at the familiar formula

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3.$$

*Volume of a paraboloid*

Consider the curve

$$y = f(x) = 1 - x^2.$$

If we rotate this curve about the $y$ axis, we will get a paraboloid, as shown in Figure 10.5. In this section we show how to compute the volume by dissecting into disks stacked up along the $y$ axis.

The object has the y axis as its axis of symmetry. Hence disks are stacked up along the y axis to approximate this volume. This means that the width of each disk is $\Delta y$. This accounts for the $dy$ in the integral below. The volume of each disk is

$$V_{\text{disk}} = \pi r^2 \Delta y,$$

where the radius, $r$ is now in the direction parallel to the $x$ axis. Thus we must express radius as

$$r = x = f^{-1}(y),$$

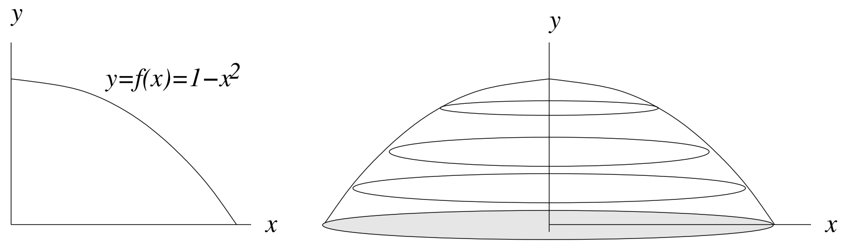



Figure 10.5: The curve that generates the shape of a paraboloid (left) and the shape of the paraboloid (right).

i.e, we invert the relationship to obtain $x$ as a function of $y$. From $y = 1 - x^2$ we have $x^2 = 1 - y$ so $x = \sqrt{1-y}$. The radius of a disk at height $y$ is therefore $r = x = \sqrt{1-y}$. The shape extends from a smallest value of $y = 0$ up to $y = 1$. Thus the volume is

$$V = \pi \int_0^1 [f(y)]^2 \, dy = \pi \int_0^1 [\sqrt{1-y}]^2 \, dy.$$

It is helpful to note that once we have identified the thickness of the disks ($\Delta y$), we are guided to write an integral in terms of the variable $y$, i.e. to reformulate the equation describing the curve. We compute

$$V = \pi \int_0^1 (1-y) \, dy = \pi \left( y - \frac{y^2}{2} \right) \Big|_0^1 = \pi \left( 1 - \frac{1}{2} \right) = \frac{\pi}{2}.$$

The above example was set up using disks. However, there are other options. In Section 10.3.1 we show yet another method, comprised of *cylindrical shells* to compute the volume of a cone. In some cases, one method is preferable to another, but here either method works equally well.

**Example 10.1** *Find the volume of the surface formed by rotating the curve*

$$y = f(x) = \sqrt{x}, \quad 0 \le x \le 1$$

*(a) about the x axis;*

*(b) about the y axis.*

**Solution.**

(a) If we rotate this curve about the $x$ axis, we obtain a bowl shape; dissecting this surface leads to disks stacked along the $x$ axis, with thickness $\Delta x \to dx$, with radii in the $y$ direction, i.e. $r = y = f(x)$, and with $x$ in the range $0 \le x \le 1$. The volume will thus be

$$V = \pi \int_0^1 [f(x)]^2 \, dx = \pi \int_0^1 [\sqrt{x}]^2 \, dx = \pi \int_0^1 x \, dx = \pi \frac{x^2}{2} \Big|_0^1 = \frac{\pi}{2}.$$

(b) When the curve is rotated about the $y$ axis, it forms a surface with a sharp point at the origin. The disks are stacked along the $y$ axis, with thickness

$\Delta y \to dy$, and radii in the $x$ direction. We must rewrite the function in the form

$$x = g(y) = y^2.$$

We now use the interval along the $y$ axis, i.e. $0 < y < 1$ The volume is then

$$V = \pi \int_0^1 [f(y)]^2 \, dy = \pi \int_0^1 [y^2]^2 \, dy = \pi \int_0^1 y^4 \, dy = \pi \frac{y^5}{5}\Big|_0^1 = \frac{\pi}{5}.$$

◊

## 10.3 Length of a curve: Arc length

Analytic geometry provides a simple way to compute the length of a straight line segment, based on the distance formula. Recall that, given points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the length of the line joining those points is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Things are more complicated for "curves" that are not straight lines, but in many cases, we are interested in calculating the length of such curves. In this section we describe how this can be done using the definite integral "technology".

The idea of dissection also applies to the problem of determining the length of a curve. In Figure 10.6, we see the general idea of subdividing a curve into many small "arcs". Before we look in detail at this construction, we consider a simple example, shown in Figure 10.7. In the triangle shown, by the Pythagorean theorem we have the length of the sloped side related as follows to the side lengths $\Delta x, \Delta y$:

$$\Delta \ell^2 = \Delta x^2 + \Delta y^2,$$

$$\Delta \ell = \sqrt{\Delta x^2 + \Delta y^2} = \left(\sqrt{1 + \frac{\Delta y^2}{\Delta x^2}}\right) \Delta x = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \; \Delta x.$$

We now consider a curve given by some function

$$y = f(x) \quad a < x < b,$$

as shown in Figure 10.6(a). We will approximate this curve by a set of line segments, as shown in Figure 10.6(b). To obtain these, we have selected some step size $\Delta x$ along the $x$ axis, and placed points on the curve at each of these $x$ values. We connect the points with straight line segments, and determine the lengths of those segments. (The total length of the segments is only an approximation of the length of the curve, but as the subdivision gets finer and finer, we will arrive at the true total length of the curve.)

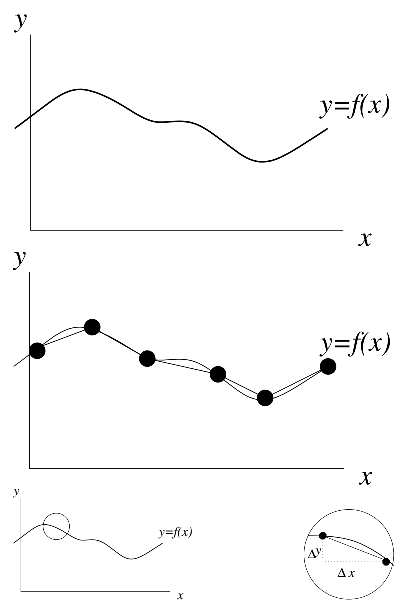



Figure 10.6: Top: Given the graph of a function, $y = f(x)$ (at left), we draw secant lines connecting points on its graph at values of $x$ that are multiples of $\Delta x$ (right). Bottom: a small part of this graph is shown, and then enlarged, to illustrate the relationship between the arc length and the length of the secant line segment.

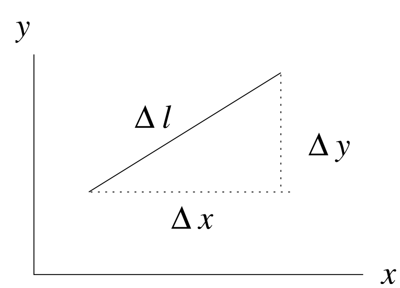



Figure 10.7: The basic idea of arclength is to add up lengths $\Delta l$ of small line segments that approximate the curve.

We show one such segment enlarged in the circular inset in Figure 10.6. Its slope, shown at right is given by $\Delta y/\Delta x$. According to our remarks, above, the length of this segment is given by

$$\Delta\ell = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x.$$

As the step size is made smaller and smaller $\Delta x \to dx$, $\Delta y \to dy$ and

$$\Delta\ell \to \sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx.$$

We recognize the ratio inside the square root as as the derivative, $dy/dx$. If our curve is given by a function $y = f(x)$ then we can rewrite this as

$$d\ell = \sqrt{1+(f'(x))^2}\ dx.$$

Thus, the length of the entire curve is obtained from summing (i.e. adding up) these small pieces, i.e.

$$L = \int_a^b \sqrt{1+(f'(x))^2}\ dx. \tag{10.1}$$

**Example 10.2** *Find the length of a line whose slope is $-2$ given that the line extends from $x = 1$ to $x = 5$.*

**Solution.** We could find the equation of the line and use the distance formula. But for the purpose of this example, we apply the method of Equation (10.1): we are given that the slope $f'(x)$ is -2. The integral in question is

$$L = \int_1^5 \sqrt{1+(f'(x))^2}\,dx = \int_1^5 \sqrt{1+(-2)^2}\,dx = \int_5^1 \sqrt{5}\,dx.$$

We get

$$L = \sqrt{5}\int_1^5 dx = \sqrt{5}x\Big|_1^5 = \sqrt{5}[5-1] = 4\sqrt{5}.$$

◊

**Example 10.3** *Find an integral that represents the length of the curve that forms the graph of the function*

$$y = f(x) = x^3, \quad 1 < x < 2.$$

**Solution.** We find that

$$\frac{dy}{dx} = f'(x) = 3x^2.$$

Thus, the integral is

$$L = \int_1^2 \sqrt{1+(3x^2)^2}\,dx = \int_1^2 \sqrt{1+9x^4}\,dx.$$

At this point, we will not attempt to find the actual length, as we must first develop techniques for finding the anti-derivative for functions such as $\sqrt{1+9x^4}$. ◊

### *10.3.1 The shell method for computing volumes*

Earlier, we used dissection into small disks to compute the volume of solids of revolution. Here we use an alternative dissection into shells.

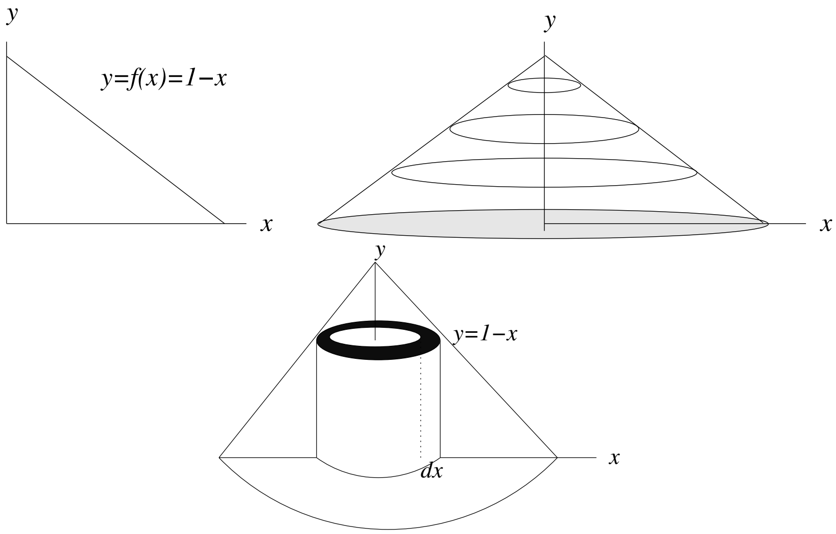



Figure 10.8: Top: The curve that generates the cone (left) and the shape of the cone (right). Bottom: the cone showing one of the series of shells that are used in this example to calculate its volume.

We use the shell method to find the volume of the cone formed by rotating the curve

$$y = 1 - x$$

about the $y$ axis.

We show the cone and its generating curve in Figure 10.8, together with a representative shell used in the calculation of total volume. The volume of a cylindrical shell of radius $r$, height $h$ and thickness $\tau$ is

$$V_{\text{shell}} = 2\pi r h \tau.$$

We will place these shells one inside the other so that their radii are parallel to the $x$ axis (so $r = x$). The heights of the shells are determined by their $y$ value (i.e. $h = y = 1 - x = 1 - r$). For the tallest shell $r = 0$, and for the flattest shell $r = 1$. The thickness of the shell is $\Delta r$. Therefore, the volume of one shell is

$$V_{\text{shell}} = 2\pi r(1 - r)\ \Delta r.$$

The volume of the object is obtained by summing up these shell volumes. In the limit, as $\Delta r \to dr$ gets infinitesimally small, we recognize this as a process of integration. We integrate over $0 \le r \le 1$, to obtain:

$$V = 2\pi \int_0^1 r(1 - r)\ dr = 2\pi \int_0^1 (r - r^2)\ dr.$$

We find that

$$V = 2\pi \left( \frac{r^2}{2} - \frac{r^3}{3} \right) \Bigg|_0^1 = 2\pi \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{3}.$$

## 10.4 *Miscellaneous examples and related problems*

For the continuous distributions, we determine the total mass by integration. Underlying the integral is the idea that the interval could be "dissected" into small parts (of width $\Delta x$), and a sum of pieces transformed into an integral. In the next examples, we consider similar ideas, but instead of dissecting the region into 1-dimensional intervals, we have slightly more interesting geometries.

### 10.4.1 *A glucose density gradient*

A cylindrical test-tube of radius $r$, and height $h$, contains a solution of glucose which has been prepared so that the concentration of glucose is greatest at the bottom and decreases gradually towards the top of the tube. (This is called a *density gradient*). Suppose that the concentration $c$ as a function of the depth $x$ is $c(x) = 0.1 + 0.5x$ grams per centimeter$^3$. ($x = 0$ at the top of the tube, and $x = h$ at the bottom of the tube.) In Figure 10.9 we show a schematic version of what this gradient might look like. (In reality, the transition between high and low concentration would be smoother than shown in this figure.) Determine the total amount of glucose in the tube (in gm). Neglect the "rounded" lower portion of the tube: i.e. assume that it is a simple cylinder.

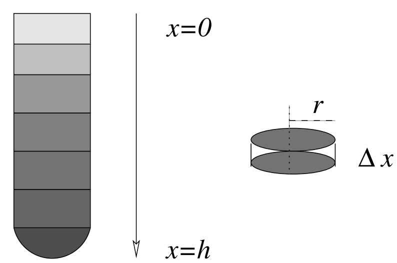


Figure 10.9: A test-tube of radius $r$ containing a gradient of glucose. A disk-shaped slice of the tube with small thickness $\Delta x$ has approximately constant density.

We assume a simple cylindrical tube and consider imaginary "slices" of this tube along its vertical axis, here labeled as the "$x$" axis. Suppose that the thickness of a slice is $\Delta x$. Then the volume of each of these (disk shaped) slices is $\pi r^2 \Delta x$. The amount of glucose in the slice is approximately equal to the concentration $c(x)$ multiplied by the volume of the slice, i.e. the small slice contains an amount $\pi r^2 \Delta x c(x)$ of glucose. In order to sum up the total amount over all slices, we use a definite integral. (As before, we imagine $\Delta x \to dx$ becoming "infinitesimal" as the number of slices increases.) The integral we want is

$$G = \pi r^2 \int_0^h c(x)\, dx.$$

Even though the geometry of the test-tube, at first glance, seems more complicated than the one-dimensional highway described in Chapter 9, we observe here that the integral that computes the total amount is still a sum over a single spatial variable, $x$. (Note the resemblance between the integrals

$$I = \int_0^L C(x)\, dx \quad \text{and} \quad G = \pi r^2 \int_0^h c(x)\, dx,$$

here and in the previous example.) We have neglected the complication of the rounded bottom portion of the test-tube, so that integration over its length (which is actually summation of disks shown in Figure 10.9) is a one-dimensional problem.

In this case the total amount of glucose in the tube is

$$G = \pi r^2 \int_0^h (0.1 + 0.5x)dx = \pi r^2 \left(0.1x + \frac{0.5x^2}{2}\right)\bigg|_0^h = \pi r^2 \left(0.1h + \frac{0.5h^2}{2}\right).$$

Suppose that the height of the test-tube is $h = 10$ cm and its radius is $r = 1$ cm. Then the total mass of glucose is

$$G = \pi \left(0.1(10) + \frac{0.5(100)}{2}\right) = \pi (1 + 25) = 26\pi \text{ gm}.$$

In the next example, we consider a circular geometry, but the concept of dissecting and summing is the same. Our task is to determine how to set up the problem in terms of an integral, and, again, we must imagine which type of subdivision would lead to the summation (integration) needed to compute total amount.

### *10.4.2 A circular colony of bacteria*

A circular colony of bacteria has radius of 1 *cm*. At distance $r$ from the center of the colony, the density of the bacteria, in units of one million cells per square centimeter, is observed to be $b(r) = 1 - r^2$ (Note: $r$ is distance from the center in cm, so that $0 \le r \le 1$). What is the total number of bacteria in the colony?

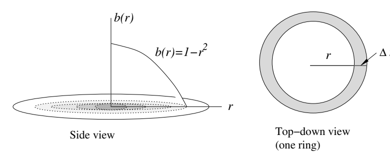



Figure 10.10: A colony of bacteria with circular symmetry. A ring of small thickness $\Delta r$ has roughly constant density. The superimposed curve on the left is the bacterial density $b(r)$ as a function of the radius $r$.

Figure 10.10 shows a rough sketch of a flat surface with a colony of bacteria growing on it. We assume that this distribution is radially symmetric. The density as a function of distance from the center is given by $b(r)$, as shown in Figure 10.10. Note that the function describing density, $b(r)$ is smooth, but to accentuate the strategy of dissecting the region, we have shown a top-down view of a ring of nearly constant density on the right in Figure 10.10. We see that this ring occupies the region between two circles, e.g. between a circle of radius $r$ and a slightly bigger circle of radius $r + \Delta r$. The area of that "ring"[2] would then be the area of the larger circle minus that of the smaller circle, namely

$$A_{\text{ring}} = \pi(r + \Delta r)^2 - \pi r^2 = \pi(2r\Delta r + (\Delta r)^2).$$

However, if we make the thickness of that ring really small ($\Delta r \to 0$), then the quadratic term is very very small so that

$$A_{\text{ring}} \approx 2\pi r \Delta r.$$

Consider all the bacteria that are found inside a "ring" of radius $r$ and thickness $\Delta r$ (see Figure 10.10.) The total number within such a ring is the product of the density, $b(r)$ and the area of the ring, i.e.

$$b(r) \cdot (2\pi r \Delta r) = 2\pi r(1 - r^2)\Delta r.$$

[2] Students commonly make the error of writing $A_{\text{ring}} = \pi(r + \Delta r - r)^2 = \pi(\Delta r)^2$. This trap should be avoided! It is clear that the correct expression has additional terms, since we really are computing a difference of two circular areas.

To get the total number in the colony we sum up over all the rings from $r = 0$ to $r = 1$ and let the thickness, $\Delta r \to dr$ become very small. But, as with other examples, this is equivalent to calculating a definite integral, namely:

$$B_{\text{total}} = \int_0^1 (1-r^2)(2\pi r)\, dr = 2\pi \int_0^1 (1-r^2) r dr = 2\pi \int_0^1 (r - r^3) dr.$$

We calculate the result as follows:

$$B_{\text{total}} = 2\pi \left( \frac{r^2}{2} - \frac{r^4}{4} \right) \Big|_0^1 = (\pi r^2 - \pi \frac{r^4}{2}) \Big|_0^1 = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

Thus the total number of bacteria in the entire colony is $\pi/2$ million which is approximately 1.57 million cells.

### *10.4.3 How the alligator gets its smile*

The American alligator, *Alligator mississippiensis* has a set of teeth best viewed at some distance. The regular arrangement of these teeth, i.e. their spacing along the jaw is important in giving the reptile its famous bite. We will concern ourselves here with how that pattern of teeth is formed as the alligator develops from its embryonic stage to that of an adult. As is the case in humans, the teeth on an alligator do not form or sprout simultaneously. In the development of the baby alligator, there is a sequence of initiation of teeth, one after the other, at well-defined positions along the jaw.

Paul Kulesa, a former student of James D Muray, set out to understand the pattern of development of these teeth, based on data in the literature about what happens at distinct stages of embryonic growth. Of interest in his research were several questions, including what determines the positions and timing of initiation of individual teeth, and what mechanisms lead to this pattern of initiation. One theory proposed by this group was that chemical signals that diffuse along the jaw at an early stage of development give rise to instructions that are interpreted by jaw cells: where the signal is at a high level, a tooth will start to initiate.

While we will not address the details of the mechanism of development here, we will find a simple application of the ideas of arclength in the developmental sequence of teething. Shown in Figure 10.11 is a smiling baby alligator (no doubt thinking of some future tasty meal). A close up of its smile (at an earlier stage of development) reveals the shape of the jaw, together with the sites at which teeth are becoming evident. (One of these sites, called primordia, is shown enlarged in an inset in this figure.)

Paul Kulesa found that the shape of the alligator's jaw can be described remarkably well by a parabola. A proper choice of coordinate system, and some experimentation leads to the equation of the best fit parabola

$$y = f(x) = -ax^2 + b$$

where $a = 0.256$, and $b = 7.28$ (in units not specified).

We show this curve in Figure 10.12(a). Also shown in this curve is a set of points at which teeth are found, labelled by order of appearance. In Figure 10.12(b) we see the same curve, but we have here superimposed the function $L(x)$ given by the arc length along the curve from the front of the jaw (i.e. the top of the parabola), i.e.

$$L(x) = \int_0^x \sqrt{1 + [f'(s)]^2}\, ds.$$

This curve measures distance along the jaw, from front to back. The distances of the teeth from one another, or along the curve of the jaw can be determined using this curve if we know the $x$ coordinates of their positions.

The table below gives the original data, courtesy of Dr. Kulesa, showing the order of the teeth, their $(x, y)$ coordinates, and the value of $L(x)$ obtained from the arclength formula. We see from this table that the teeth do not appear randomly, nor do they fill in the jaw in one sweep. Rather, they appear in several stages.

In Figure 10.12(c), we show the pattern of appearance: Plotting the distance along the jaw of successive teeth reveals that the teeth appear in waves of nearly equally-spaced sites. (By equally spaced, we refer to distance along the parabolic jaw.) The first wave (teeth 1, 2, 3) are followed by a second wave (4, 5, 6, 7), and so on. Each wave forms a linear pattern of distance from the front, and each successive wave fills in the gaps in a similar, equally spaced pattern.

The true situation is a bit more complicated: the jaw grows as the teeth appear as shown in 10.12(c). This has not been taken into account in our simple treatment here, where we illustrate only the essential idea of arc length application.

| Tooth number | position | | distance along jaw |
|---|---|---|---|
| | $x$ | $y$ | $L(x)$ |
| 1 | 1.95 | 6.35 | 2.1486 |
| 2 | 3.45 | 4.40 | 4.7000 |
| 3 | 4.54 | 2.05 | 7.1189 |
| 4 | 1.35 | 6.95 | 1.4000 |
| 5 | 2.60 | 5.50 | 3.2052 |
| 6 | 3.80 | 3.40 | 5.4884 |
| 7 | 5.00 | 1.00 | 8.4241 |
| 8 | 3.15 | 4.80 | 4.1500 |
| 9 | 4.25 | 2.20 | 6.3923 |
| 10 | 4.60 | 1.65 | 7.3705 |
| 11 | 0.60 | 7.15 | 0.6072 |
| 12 | 3.45 | 4.05 | 4.6572 |
| 13 | 5.30 | 0.45 | 9.2644 |

Table 10.1: Data for the appearance of teeth, in the order in which they appear as the alligator develops. We can use arc-length computations to determine the distances between successive teeth.

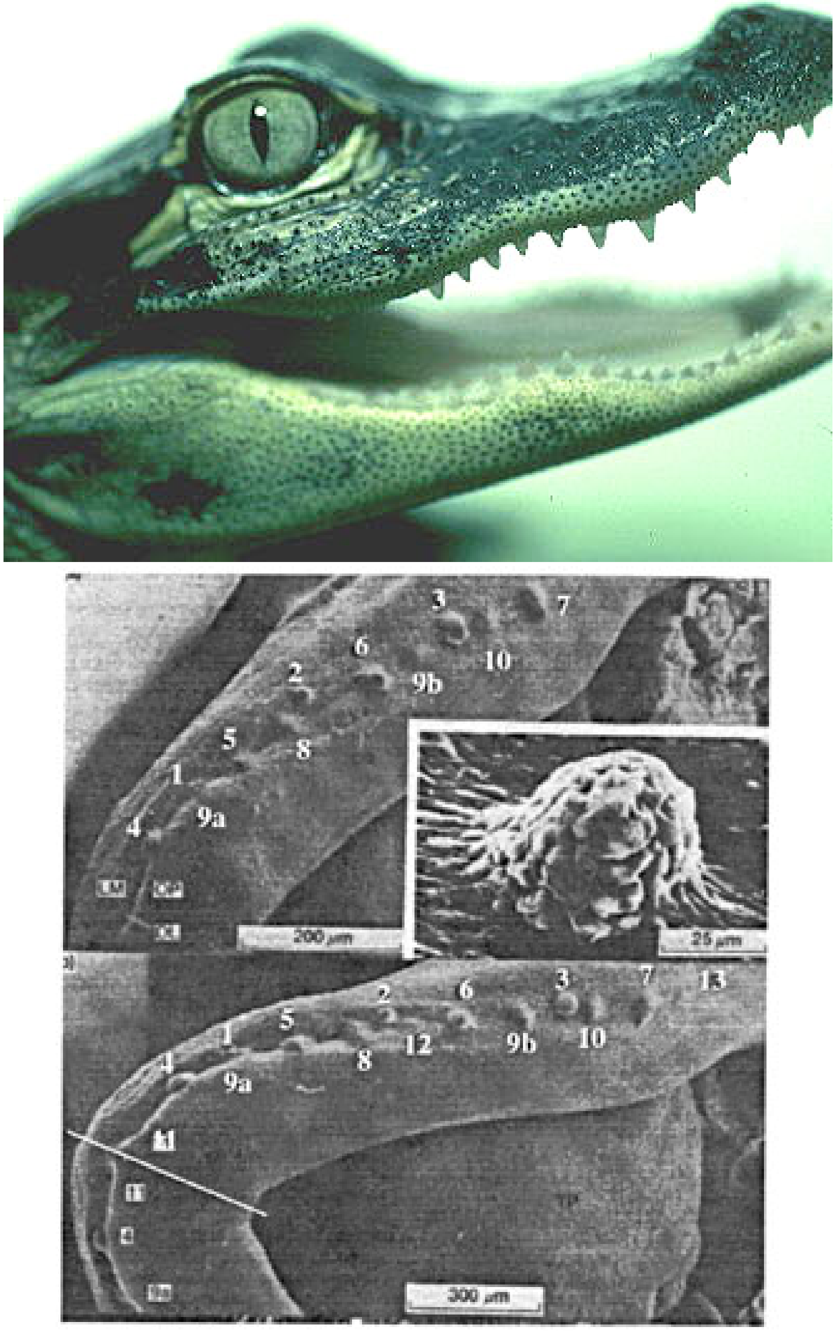



Figure 10.11: *Alligator mississippiensis* and its teeth

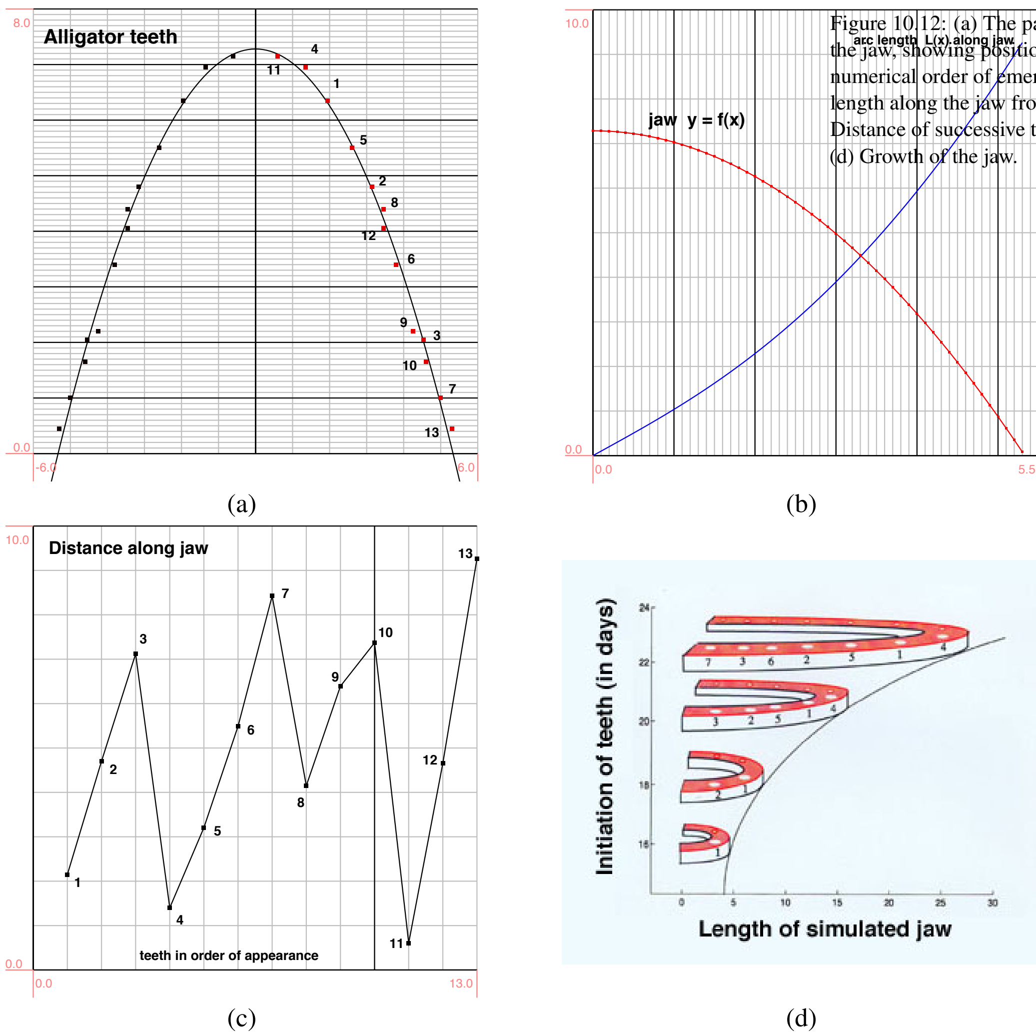



Figure 10.12: (a) The parabolic shape of the jaw, showing positions of teeth and numerical order of emergence. (b) Arc length along the jaw from front to back. (c) Distance of successive teeth along the jaw. (d) Growth of the jaw.

### *10.4.4 References*

1. P.M. Kulesa and J.D. Murray (1995). Modelling the Wave-like Initiation of Teeth Primordia in the Alligator. FORMA. Cover Article. Vol. 10, No. 3, 259-280.

2. J.D. Murray and P.M. Kulesa (1996). On A Dynamic Reaction-Diffusion Mechanism for the Spatial Patterning of Teeth Primordia in the Alligator. Journal of Chemical Physics. J. Chem. Soc., Faraday Trans., 92 (16), 2927-2932.

3. P.M. Kulesa, G.C. Cruywagen, S.R. Lubkin, M.W.J. Ferguson and J.D. Murray (1996). Modelling the Spatial Patterning of Teeth Primordia in the Alligator. Acta Biotheoretica, 44, 153-164.

## *Exercises*

**Exercise 10.1** The density of a band of protein along a one-dimensional strip of gel in an electrophoresis experiment is given by $p(x) = 2(x-1)(2-x)$ for $1 \le x \le 2$, where $x$ is the distance along the strip in cm and $p(x)$ is the protein density (i.e. protein mass per cm) at distance $x$.

(a) Graph the density $p$ as a function of $x$.

(b) Find the total mass of the protein in the band for $1 \le x \le 2$.

(Hint: simplify the function first.)

**Exercise 10.2** The air density $h$ meters above the earth's surface is

$$p(h) = Ae^{-ah}\text{kg/m}^3.$$

Find the mass of a cylindrical column of air of radius $r = 2$ meters and height $H = 25000$ meters. Let $A = 1.28$ (kg/m$^3$), $a = 0.000124$ per meter. (Hint: set the integral up as a Riemann sum).

**Exercise 10.3** A test-tube contains a solution of glucose which has been prepared so that the concentration of glucose is greatest at the bottom and decreases gradually towards the top of the fluid. (This is called a *density gradient*). Suppose that the concentration $c$ as a function of the depth $x$ is $c(x) = x/10$ (in units of g/cm$^3$). The radius of the tube is 2 cm and the height of the glucose-containing solution is 10 cm. Determine the total amount of glucose in the tube (in g).

**Exercise 10.4** To investigate changes in the Earth's weather, scientists examine the distribution of pollen grains in a 1 dimensional drilled "core sample", i.e. a sample of the Earth's crust that contains archaeological deposits of soil from many thousands of years. Assume that the core sample is a cylinder of unit cross-sectional area and length $L$. Suppose that pollen grain density $p(x)$ at a point $x$ in this sample is in a core sample of length $L$ is given by

$$p(x) = A\sin(ax), \qquad 0 < x < L = \frac{\pi}{a}.$$

where $p(x)$ are the number of particles per unit volume at a distance $x$ from one end of the sample.

(a) Where is the pollen grain most concentrated along this one dimensional sample?

(b) Find the average density of pollen grains along the length of the sample.

(c) Find the center of mass of the pollen grain distribution. (Note: you can use the fact that the density is distributed symmetrically to avoid having to integrate.)

**Exercise 10.5** **Population density** The population density of inhabitants living along the banks of the river Nile at a distance $x$ km from its mouth is found to be $n(x) = 1000e^{-kx}$. Twenty km from the river mouth, the population density is half as large as it is at the mouth of the river. Find the total population of people living along the Nile. (Hint: consider the Nile being "very long", i.e. let $x \to \infty$ be the "length" for the purpose of the integration.)

**Exercise 10.6** Find the volume of a cone whose height $h$ is equal to its base radius $r$, by using the disc method. We will place the cone on its side, as shown in the Figure 10.13, and let $x$ represent position along its axis.

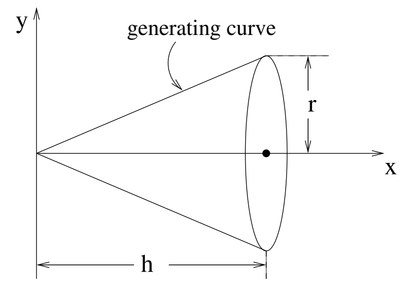


Figure 10.13: For problem 10.6

(a) Using the diagram shown below (Figure 10.13), explain what kind of a curve in the $x-y$-plane we would use to *generate* the surface of the cone as a surface of revolution.

(b) Using the proportions given in the problem, specify the exact function $y = f(x)$ that we need to describe this "curve".

(c) Now find the volume enclosed by this surface of revolution for $0 \le x \le 1$.

(d) Show that, in this particular case, we would have gotten the same geometric object, and also the same enclosed volume, if we had rotated the "curve" about the y axis.

**Exercise 10.7** Find the volume of the cone generated by revolving the curve $y = f(x) = 1 - x$ (for $0 < x < 1$) about the $y$ axis. Use the disk method, with disks stacked up along the $y$ axis.

**Exercise 10.8** Find the volume of the "bowl" obtained by rotating the curve $y = 4x^2$ about the $y$ axis for $0 \le x \le 1$.

**Exercise 10.9** On his wedding day, Kepler wanted to calculate the amount of wine contained inside a wine barrel whose shape is shown below in Figure 10.14. Use the disk method to compute this volume. You may assume that the function that generates the shape of the barrel (as a surface of revolution) is $y = f(x) = R - px^2$, for $-1 < x < 1$ where $R$ is the radius of the widest part of the barrel. ($R$ and $p$ are both positive constants.)

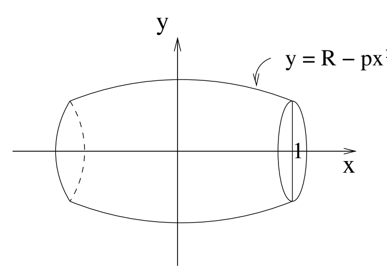


Figure 10.14: For problem 10.9

**Exercise 10.10** Consider the curve

$$y = f(x) = 1 - x^2 \qquad 0 < x < 1$$

rotated about the $y$-axis. Recall that this will form a shape called a paraboloid. Use the cylindrical shell method to calculate the volume of this shape.

**Exercise 10.11** Find the volume of the solid obtained by rotating the region bounded by the given curves $f(x)$ and $g(x)$ about the specified line.

(a) $f(x) = \sqrt{x-1}$, $g(x) = 0$, from $x = 2$ to $x = 5$, about the $x$-axis.

(b) $f(x) = \sqrt{x}$, $g(x) = x/2$, about the $y$-axis.

(c) $f(x) = 1/x$, $g(x) = x^3$, from $x = 1/10$ to $x = 1$, about the $x$-axis.

**Exercise 10.12** Let $R$ denote the region contained between $y = \sin(x)$ and $y = \cos(x)$ for $0 \le x \le \pi/2$. Write down the expression for the volume obtained by rotating $R$ about

(a) the $x$-axis

(b) the line $y = -1$.

Do not integrate.

**Exercise 10.13** Suppose a lake has a depth of 40 meters at its deepest point and is bowl-shaped, with the surface of the bowl generated by rotating the curve $z = x^2/10$ around the $z$-axis. Here $z$ is the height in meters above the lowest point of the bowl. The distribution of sediment in the lake is stratified by height along the water column. In other words, the density of sediment (in mass per unit volume) is a function of the form $s(z) = C(40 - z)$, where $z$ is again vertical height in meters from the point at the bottom of the lake. Find the total mass of sediment in the lake (Your answer will have the constant $C$ in it.). The volume of the lake is the volume above the curve $z = x^2/10$ and below $z = 40$.

**Exercise 10.14** In this problem you are asked to find the volume of a height $h$ pyramid with a square base of width $w$. (This is related to the Cheops pyramid problem, but we will use calculus.) Let the variable $z$ stand for distance **down** the axis of the pyramid with $z = 0$ at the top, and consider "slicing" the pyramid along this axis (into horizontal slices). This will produce square "slices" (having area $A(z)$ and some width $\Delta z$). Calculate the volume of the pyramid as an integral by figuring out how $A(z)$ depends on $z$ and integrating this function.

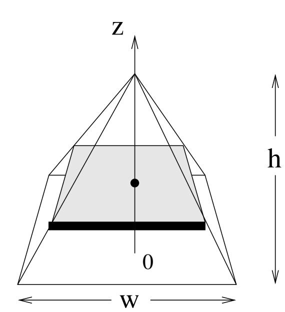


Figure 10.15: For problem 10.14

**Exercise 10.15** Set up the integral that represents the length of the following curves: Do not attempt to calculate the integral in any of these cases

(a) $y = f(x) = \sin(x) \qquad 0 < x < 2\pi$.

(b) $y = f(x) = \sqrt{x} \qquad 0 < x < 1$.

(c) $y = f(x) = x^n \qquad -1 < x < 1$.

**Exercise 10.16** Compute the length of the line $y = 2x + 1$ for $-1 < x < 1$ using the arc-length formula. Check your work by using the simple distance formula (or Pythagorean theorem).

**Exercise 10.17** **Work of spring** A spring has a natural length of 16 cm. When it is stretched $x$ cm beyond that, Hooke's Law states that the spring pulls back with a restoring force $F = kx$ dyne, where the constant $k$ is called the spring constant, and represents the stiffness of the spring. For the given spring, 8 dyne of force are required to hold it stretched by 2 cm. How much work (dyne-cm) is done in stretching this spring from its natural length to a length 24 cm? (Note: use integration to set up this problem.)

**Exercise 10.18** **Work of pump** Calculate the work done in pumping water out of a parabolic container up to the height $h = 10$ units. Assume that the container is a surface of revolution generated by rotating the curve $y = x^2$ about the $y$ axis, that the height of the water in the container is 10 units, that the density of water is 1 g/cm$^3$ and that the force due to gravity is $F = mg$ where $m$ is mass and $g = 9.8$ m/s$^2$.

# 11
# Techniques of Integration

In this chapter, we expand our repertoire for antiderivatives beyond the "elementary" functions discussed so far. A review of the table of elementary antiderivatives (found in Chapter 8) will be useful. Here we will discuss a number of methods for finding antiderivatives. We refer to these collected "tricks" as methods of integration. As will be shown, in some cases, these methods are systematic (i.e. with clear steps), whereas in other cases, guesswork and trial and error is an important part of the process.

A primary method of integration to be described is **substitution**. A close relationship exists between the chain rule of differential calculus and the substitution method. A second very important method is **integration by parts**. Aside from its usefulness in integration per se, this method has numerous applications in physics, mathematics, and other sciences. Many other techniques of integration used to form a core of methods taught in such courses in integral calculus. Many of these are quite technical. Nowadays, with sophisticated mathematical software packages (including Maple and Mathematica), integration can be carried our automatically via computation called "symbolic manipulation", reducing our need to dwell on these technical methods.

## 11.1 Differential notation

We begin by familiarizing the reader with notation that appears frequently in substitution integrals, i.e. differential notation. Consider a straight line

$$y = mx + b.$$

Recall that the slope of the line, $m$, is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$

This relationship can also be written in the form

$$\Delta y = m\Delta x.$$

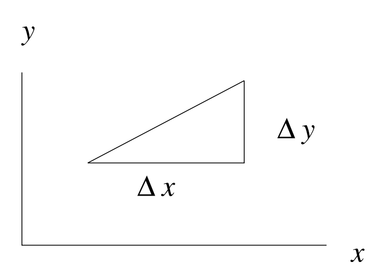



Figure 11.1: The slope of the line shown here is $m = \Delta y/\Delta x$. This means that the small quantities $\Delta y$ and $\Delta x$ are related by $\Delta y = m\Delta x$.

If we take a very small step along this line in the $x$ direction, call it $dx$ (to remind us of an "infinitesimally small" quantity), then the resulting change in the $y$ direction, (call it $dy$) is related by

$$dy = m\,dx.$$

Now suppose that we have a curve defined by some arbitrary function, $y = f(x)$ which need not be a straight line.For a given point $(x, y)$ on this curve, a step $\Delta x$ in the x direction is associated with a step $\Delta y$ in the y direction. The

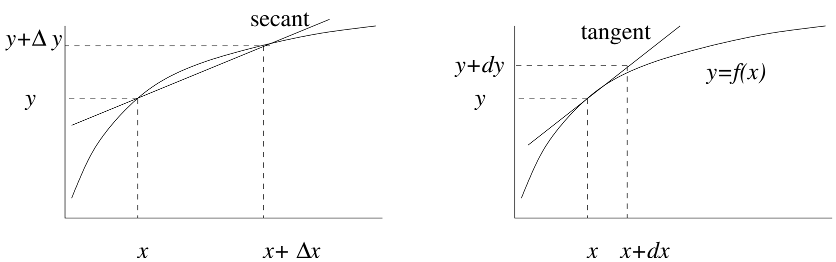



Figure 11.2: On this figure, the graph of some function is used to illustrate the connection between differentials $dy$ and $dx$. Note that these are related via the slope of a tangent line, $m_t$ to the curve, in contrast with the relationship of $\Delta y$ and $\Delta x$ which stems from the slope of the secant line $m_s$ on the same curve.

relationship between the step sizes is:

$$\Delta y = m_s \Delta x,$$

where now $m_s$ is the slope of a secant line (shown connecting two points on the curve in Figure 11.2). If the sizes of the steps are small ($dx$ and $dy$), then this relationship is well approximated by the slope of the tangent line, $m_t$ as shown in Figure 11.2 i.e.

$$dy = m_t\,dx = f'(x)dx.$$

The quantities $dx$ and $dy$ are called **differentials**. In general, they link a small step on the $x$ axis with the resulting small change in height along the tangent line to the curve (shown in Figure 11.2). We might observe that the ratio of the differentials, i.e.

$$\frac{dy}{dx} = f'(x),$$

appears to link our result to the definition of the derivative. We remember, though, that the derivative is actually defined as a limit:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

When the step size $\Delta x$ is quite small, it is approximately true that

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}.$$

This notation will be useful in substitution integrals.

We give some examples of functions, their derivatives, and the differential notation that goes with them.

1. The function $y = f(x) = x^3$ has derivative $f'(x) = 3x^2$. Thus
$$dy = 3x^2\ dx.$$
2. The function $y = f(x) = \tan(x)$ has derivative $f'(x) = \sec^2(x)$. Therefore
$$dy = \sec^2(x)\ dx.$$
3. The function $y = f(x) = \ln(x)$ has derivative $f'(x) = \frac{1}{x}$ so
$$dy = \frac{1}{x}\ dx.$$

With some practice, we can omit the intermediate step of writing down a derivative and go directly from function to differential notation. Given a function $y = f(x)$ we will often write

$$df(x) = \frac{df}{dx}dx$$

and occasionally, we use just the symbol $df$ to mean the same thing. The following examples illustrate this idea with specific functions.

$$d(\sin(x)) = \cos(x)\ dx, \quad d(x^n) = nx^{n-1}\ dx, \quad d(\arctan(x)) = \frac{1}{1+x^2}\ dx.$$

Moreover, some of the basic rules of differentiation translate directly into rules for handling and manipulating differentials. We give a list of some of these elementary rules below.

**Rules for derivatives and differentials**

1. $\frac{d}{dx}C = 0, \qquad dC = 0.$
2. $\frac{d}{dx}(u(x) + v(x)) = \frac{du}{dx} + \frac{dv}{dx} \qquad d(u + v) = du + dv.$
3. $\frac{d}{dx}u(x)v(x) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad d(uv) = u\ dv + v\ du.$
4. $\frac{d}{dx}(Cu(x)) = C\frac{du}{dx}, \qquad d(Cu) = C\ du$

## *11.2 Antidifferentiation and indefinite integrals*

In Chapters 7 and 8, we defined the concept of the **definite integral**, which represents a number. It will be useful here to consider the idea of an **indefinite integral**, which is a function, namely an antiderivative.

If two functions, $F(x)$ and $G(x)$, have the same derivative, say $f(x)$, then they differ at most by a constant, that is $F(x) = G(x) + C$, where $C$ is some constant.

**Proof** Since $F(x)$ and $G(x)$ have the same derivative, we have

$$\frac{d}{dx}F(x) = \frac{d}{dx}G(x),$$

$$\frac{d}{dx}F(x) - \frac{d}{dx}G(x) = 0,$$

$$\frac{d}{dx}(F(x) - G(x)) = 0.$$

This means that the function $F(x) - G(x)$ should be a constant, since its derivative is zero. Thus

$$F(x) - G(x) = C,$$

so

$$F(x) = G(x) + C,$$

as required. $F(x)$ and $G(x)$ are called antiderivatives of $f(x)$, and this confirms, once more, that any two antiderivatives differ at most by a constant.

In another terminology, which means the same thing, we also say that $F(x)$ (or $G(x)$) is the integral of the function $f(x)$, and we refer to $f(x)$ as the *integrand*. We write this as follows:

$$F(x) = \int f(x)\, dx.$$

This notation is sometimes called "an *indefinite integral*" because it does not denote a specific numerical value, nor is an interval specified for the integration range. An indefinite integral is a function with an arbitrary constant. (Contrast this with the definite integral studied in our last chapters: in the case of the definite integral, we specified an interval, and interpreted the result, a number, in terms of areas associated with curves.) We also write

$$\int f(x)\, dx = F(x) + C,$$

if we want to indicate the form of all possible functions that are antiderivatives of $f(x)$. $C$ is referred to as a *constant of integration*.

### *Integrals of derivatives*

Suppose we are given an integral of the form

$$\int \frac{df}{dx} dx,$$

or alternately, the same thing written using differential notation,

$$\int df.$$

How do we handle this? We reason as follows. The $df/dx$ (a quantity that is, itself, a function) is the derivative of the function $f(x)$. That means that $f(x)$ is the antiderivative of $df/dx$. Then, according to the Fundamental Theorem of Calculus,

$$\int \frac{df}{dx}dx = f(x) + C.$$

We can write this same result using the differential of $f$, as follows:

$$\int df = f(x) + C.$$

The following examples illustrate the idea with several elementary functions.

1. $\int d(\cos x) = \cos x + C$.
2. $\int dv = v + C$.
3. $\int d(x^3) = x^3 + C$.

## *11.3 Simple substitution*

In this section, we observe that the forms of some integrals can be simplified by making a judicious substitution, and using our familiarity with derivatives (and the chain rule). The idea rests on the fact that in some cases, we can spot a “helper function”

$$u = f(x),$$

such that the quantity

$$du = f'(x)dx$$

appears in the integrand. In that case, the substitution will lead to eliminating $x$ entirely in favour of the new quantity $u$, and simplification may occur.

Suppose we are given the function

$$f(x) = (x+1)^{10}.$$

Then its antiderivative (indefinite integral) is

$$F(x) = \int f(x)\,dx = \int (x+1)^{10}\,dx.$$

We could find an antiderivative by expanding the integrand $(x+1)^{10}$ into a degree 10 polynomial and using methods already known to us; but this would be laborious. Let us observe, however, that if we define

$$u = (x+1),$$

then

$$du = \frac{d(x+1)}{dx}dx = \left(\frac{dx}{dx} + \frac{d(1)}{dx}\right)dx = (1+0)dx = dx.$$

Now replacing $(x+1)$ by $u$ and $dx$ by the equivalent $du$ we get:

$$F(x) = \int u^{10} du.$$

An antiderivative to this can be easily found, namely,

$$F(x) = \frac{u^{11}}{11} = \frac{(x+1)^{11}}{11} + C.$$

In the last step, we converted the result back to the original variable, and included the arbitrary integration constant. A very important point to remember is that we can always check our results by differentiation:

**Check.** Differentiate $F(x)$ to obtain

$$\frac{dF}{dx} = \frac{1}{11}(11(x+1)^{10}) = (x+1)^{10}.$$

*How to handle endpoints*

We consider how substitution type integrals can be calculated when we have endpoints, i.e. in evaluating definite integrals. Consider the example:

$$I = \int_1^2 \frac{1}{x+1} dx.$$

This integration can be done by making the substitution $u = x+1$ for which $du = dx$. We can handle the endpoints in one of two ways:

**Method 1: Change the endpoints.**

We can change the integral over entirely to a definite integral in the variable $u$ as follows: Since $u = x+1$, the endpoint $x = 1$ corresponds to $u = 2$, and the endpoint $x = 2$ corresponds to $u = 3$, so changing the endpoints to reflect the change of variables leads to

$$I = \int_2^3 \frac{1}{u} du = \ln|u| \Big|_2^3 = \ln 3 - \ln 2 = \ln\frac{3}{2}.$$

In the last steps we have plugged in the new endpoints (appropriate to $u$).

**Method 2: Change back to $x$ before evaluating at endpoints.**

Alternately, we could rewrite the antiderivative in terms of $x$.

$$\int \frac{1}{u} du = \ln|u| + C = \ln|x+1| + C$$

and then evaluate this function at the original endpoints.

$$\int_1^2 \frac{1}{x+1} dx = \ln|x+1| \Big|_1^2 = \ln\frac{3}{2}$$

Here we plugged in the original endpoints (as appropriate to the variable $x$). Notice that when evaluating the antiderivative at the endpoints the integration constant can be ignored as it cancels (c.f. second part of Fundamental Theorem of Calculus).

**Example 11.1 (Substitution type integrals)** *Find a simple substitution and determine the antiderivatives (indefinite or definite integrals) of the following functions:*

1. $I = \int \frac{2}{x+2}\, dx.$
2. $I = \int_0^1 x^2 e^{x^3}\, dx$
3. $I = \int \frac{1}{(x+1)^2+1}\, dx.$
4. $I = \int (x+3)\sqrt{x^2+6x+10}\, dx.$
5. $I = \int_0^\pi \cos^3(x)\sin(x)\, dx.$
6. $I = \int \frac{1}{ax+b}\, dx$
7. $I = \int \frac{1}{b+ax^2}\, dx.$

**Solution.**

1. $I = \int \frac{2}{x+2}\, dx$. Let $u = x+2$. Then $du = dx$ and we get
$$I = \int \frac{2}{u}\, du = 2\int \frac{1}{u}\, du = 2\ln|u| = 2\ln|x+2| + C.$$

2. $I = \int_0^1 x^2 e^{x^3}\, dx$. Let $u = x^3$. Then $du = 3x^2\, dx$. Here we use method 2 for handling endpoints.
$$\int e^u \frac{du}{3} = \frac{1}{3}e^u = \frac{1}{3}e^{x^3} + C.$$
Then
$$I = \int_0^1 x^2 e^{x^3}\, dx = \frac{1}{3}e^{x^3}\Big|_0^1 = \frac{1}{3}(e-1).$$
(We converted the antiderivative to the original variable, $x$, before plugging in the original endpoints.)

3. $I = \int \frac{1}{(x+1)^2+1}\, dx$. Let $u = x+1$, then $du = dx$ so we have
$$I = \int \frac{1}{u^2+1}\, du = \arctan(u) = \arctan(x+1) + C.$$

4. $I = \int (x+3)\sqrt{x^2+6x+10}\, dx$. Let $u = x^2+6x+10$. Then $du = (2x+6)\, dx = 2(x+3)\, dx$. With this substitution we get
$$I = \int \sqrt{u}\, \frac{du}{2} = \frac{1}{2}\int u^{1/2}\, du = \frac{1}{2}\frac{u^{3/2}}{3/2} = \frac{1}{3}u^{3/2} = \frac{1}{3}(x^2+6x+10)^{3/2} + C.$$

5. $I = \int_0^\pi \cos^3(x)\sin(x)\,dx$. Let $u = \cos(x)$. Then $du = -\sin(x)\,dx$. Here we use method 1 for handling endpoints. For $x = 0, u = \cos 0 = 1$ and for $x = \pi, u = \cos\pi = -1$, so changing the integral and endpoints to u leads to

$$I = \int_1^{-1} u^3(-du) = -\left.\frac{u^4}{4}\right|_1^{-1} = -\frac{1}{4}((-1)^4 - 1^4) = 0.$$

Here we plugged in the new endpoints that are relevant to the variable $u$.

6. $\int \frac{1}{ax+b}\,dx$. Let $u = ax+b$. Then $du = a\,dx$, so $dx = du/a$. Substitute the above equations into the first equation and simplify to get

$$I = \int \frac{1}{u}\frac{du}{a} = \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}ln|u| + C.$$

Substitute $u = ax+b$ back to arrive at the solution

$$I = \int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| \tag{11.1}$$

7. $I = \int \frac{1}{b+ax^2}\,dx = \frac{1}{b}\int \frac{1}{1+(a/b)x^2}\,dx$. This can be brought to the form of an arctan type integral as follows: Let $u^2 = (a/b)x^2$, so $u = \sqrt{a/b}x$ and $du = \sqrt{a/b}dx$. Now substituting these, we get

$$I = \frac{1}{b}\int \frac{1}{1+u^2}\frac{du}{\sqrt{a/b}} = \sqrt{b/a}\frac{1}{b}\int \frac{1}{1+u^2}\,du$$

$$I = \frac{1}{\sqrt{ba}}\arctan(u)\,du = \frac{1}{\sqrt{ba}}\arctan(\sqrt{a/b}\,x) + C.$$

$\diamond$

*When simple substitution fails*

Not every integral can be handled by simple substitution. Let us see what could go wrong. Consider the case

$$F(x) = \int \sqrt{1+x^2}\,dx = \int (1+x^2)^{1/2}\,dx.$$

A “reasonable” guess for substitution might be

$$u = (1+x^2).$$

Then

$$du = 2x\,dx,$$

and $dx = du/2x$. Attempting to convert the integral to the form containing $u$ would lead to

$$I = \int \sqrt{u}\,\frac{du}{2x}.$$

We have not succeeded in eliminating $x$ entirely, so the expression obtained contains a mixture of two variables. We can proceed no further. This substitution did not simplify the integral and we must try some other technique.

*Checking your answer*

Finding an antiderivative can be tricky. (To a large extent, methods described in this chapter are a "collection of tricks".) However, it is always possible (and wise) to check for correctness, by differentiating the result. This can help uncover errors.

For example, suppose that (in the previous example) we had incorrectly guessed that the antiderivative of

$$\int (1+x^2)^{1/2}\, dx$$

might be

$$F_{\text{guess}}(x) = \frac{1}{3/2}(1+x^2)^{3/2}.$$

The following check demonstrates the incorrectness of this guess: Differentiate $F_{\text{guess}}(x)$ to obtain

$$F'_{\text{guess}}(x) = \frac{1}{3/2}(3/2)(1+x^2)^{(3/2)-1} \cdot 2x = (1+x^2)^{1/2} \cdot 2x$$

The result is clearly not the same as $(1+x^2)^{1/2}$, since an "extra" factor of $2x$ appears from application of the chain rule: this means that the trial function $F_{\text{guess}}(x)$ was not the correct antiderivative. (We can similarly check to confirm correctness of any antiderivative found by following steps of methods here described. This can help to uncover sign errors and other algebraic mistakes.)

## *11.4 More substitutions*

In some cases, rearrangement is needed before the form of an integral becomes apparent. We give some examples in this section. The idea is to reduce each one to the form of an elementary integral, whose antiderivative is known.

**Standard integral forms**

1. $I = \int \frac{1}{u}\, du = \ln|u| + C.$
2. $I = \int u^n\, du = \frac{u^{n+1}}{n+1}.$
3. $I = \int \frac{1}{1+u^2}\, du = \arctan(u) + C.$

However, finding which of these forms is appropriate in a given case will take some ingenuity and algebra skills. Integration tends to be more of an art than differentiation, and recognition of patterns plays an important role here.

**Example 11.2 (Perfect square in denominator)** *Find the antiderivative for*

$$I = \int \frac{1}{x^2 - 6x + 9}\, dx.$$

**Solution.** We observe that the denominator of the integrand is a perfect square, i.e. that $x^2 - 6x + 9 = (x-3)^2$. Replacing this in the integral, we obtain

$$I = \int \frac{1}{x^2 - 6x + 9}\, dx = \int \frac{1}{(x-3)^2}\, dx.$$

Now making the substitution $u = (x-3)$, and $du = dx$ leads to a power type integral

$$I = \int \frac{1}{u^2}\, du = \int u^{-2}\, du = -u^{-1} = -\frac{1}{(x-3)} + C.$$

$\Diamond$

**Example 11.3 (Completing the square)** *A small change in the denominator will change the character of the integral, as shown by this example:*

$$I = \int \frac{1}{x^2 - 6x + 10}\, dx.$$

**Solution.**

Here we use "completing the square" to express the denominator in the form $x^2 - 6x + 10 = (x-3)^2 + 1$. Then the integral takes the form

$$I = \int \frac{1}{1 + (x-3)^2}\, dx.$$

Now a substitution $u = (x-3)$ and $du = dx$ will result in

$$I = \int \frac{1}{1+u^2}\, du = \arctan(u) = \arctan(x-3) + C.$$

$\Diamond$ Remark: in cases where completing the square gives rise to a constant other than 1 in the denominator, we use the technique illustrated in Example 11.1 Eqn. (11.1) to simplify the problem.

**Example 11.4 (Factoring the denominator)** *A change in one sign can also lead to a drastic change in the antiderivative. Consider*

$$I = \int \frac{1}{1 - x^2}\, dx.$$

**Solution.** In this case, we can factor the denominator to obtain

$$I = \int \frac{1}{(1-x)(1+x)}\, dx.$$

We will show shortly that the integrand can be simplified to the sum of two fractions, i.e. that

$$I = \int \frac{1}{(1-x)(1+x)}\, dx = \int \frac{A}{(1-x)} + \frac{B}{(1+x)}\, dx,$$

where $A, B$ are constants. The algebraic technique for finding these constants, and hence of forming the simpler expressions, called *Partial fractions*, will be discussed in an upcoming section. Once these constants are found, each of the resulting integrals can be handled by substitution. $\Diamond$

## 11.5 Trigonometric substitutions

Trigonometric functions provide a rich set of interconnected functions that show up in many problems. It is useful to remember three very important trigonometric identities that help to simplify many integrals. These are:

**Essential trigonometric identities**

1. $\sin^2(x)+\cos^2(x)=1$
2. $\sin(A+B)=\sin(A)\cos(B)+\sin(B)\cos(A)$
3. $\cos(A+B)=\cos(A)\cos(B)-\sin(A)\sin(B)$.

In the special case that $A=B=x$, the last two identities above lead to:

**Double angle trigonometric identities**

1. $\sin(2x)=2\sin(x)\cos(x)$.
2. $\cos(2x)=\cos^2(x)-\sin^2(x)$.

From these, we can generate a variety of other identities as special cases. We list the most useful below. The first two are obtained by combining the double-angle formula for cosines with the identity $\sin^2(x)+\cos^2(x)=1$.

**Useful trigonometric identities**

1. $\cos^2(x)=\dfrac{1+\cos(2x)}{2}$.
2. $\sin^2(x)=\dfrac{1-\cos(2x)}{2}$.
3. $\tan^2(x)+1=\sec^2(x)$.

**Example 11.5 (Simple trigonometric substitution)** *Find the antiderivative of*

$$I=\int \sin(x)\cos^2(x)\,dx.$$

**Solution.** This integral can be computed by a simple substitution, similar to Example 5 of Section 11.3. We let $u=\cos(x)$ and $du=-\sin(x)dx$ to get the integral into the form

$$I=-\int u^2\,du=\frac{-u^3}{3}=\frac{-\cos^3(x)}{3}+C.$$

We need none of the trigonometric identities in this case. Simple substitution is always the easiest method to use. It should be the first method attempted in each case. ◊

**Example 11.6 (Using trigonometric identities (1))** *Find the antiderivative of*

$$I = \int \cos^2(x)\, dx.$$

**Solution.** This is an example in which the "Useful trigonometric identity" 1 leads to a simpler integral. We write

$$I = \int \cos^2(x)\, dx = \int \frac{1+\cos(2x)}{2}\, dx = \frac{1}{2}\int (1+\cos(2x))\, dx.$$

Then clearly,

$$I = \frac{1}{2}\left(x + \frac{\sin(2x)}{2}\right) + C.$$

◊

**Example 11.7 (Using trigonometric identities (2))** *Find the antiderivative of*

$$I = \int \sin^3(x)\, dx.$$

**Solution.** We can rewrite this integral in the form

$$I = \int \sin^2(x)\sin(x)\, dx.$$

Now using the trigonometric identity $\sin^2(x)+\cos^2(x) = 1$, leads to

$$I = \int (1-\cos^2(x))\sin(x)\, dx.$$

This can be split up into

$$I = \int \sin(x)\, dx - \int \sin(x)\cos^2(x)\, dx.$$

The first part is elementary, and the second was shown in a previous example. Therefore we end up with

$$I = -\cos(x) + \frac{\cos^3(x)}{3} + C.$$

Note that it is customary to combine all constants obtained in the calculation into a single constant, $C$ at the end. ◊

Aside from integrals that, themselves, contain trigonometric functions, there are other cases in which use of trigonometric identities, though at first seemingly unrelated, is crucial. Many expressions involving the form $\sqrt{1 \pm x^2}$ or the related form $\sqrt{a \pm bx^2}$ will be simplified eventually by conversion to trigonometric expressions!

**Example 11.8 (Converting to trigonometric functions)** *Find the antiderivative of*

$$I = \int \sqrt{1-x^2}\, dx.$$

**Solution.** The simple substitution $u = 1 - x^2$ will not work, (as shown by a similar example in Section 11.3). However, converting to trigonometric expressions will do the trick. Let

$$x = \sin(u), \quad \text{then } dx = \cos(u)du.$$

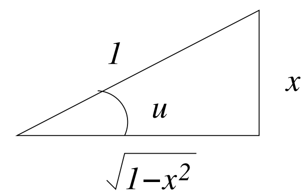



Figure 11.3: This triangle helps to convert the (trigonometric) functions of $u$ to the original variable $x$ in Example 11.8.

In Figure 11.3, we show this relationship on a triangle. This diagram is useful in reversing the substitutions after the integration step.

Then $1 - x^2 = 1 - \sin^2(u) = \cos^2(u)$, so the substitutions lead to

$$I = \int \sqrt{\cos^2(u)} \cos(u)\, du = \int \cos^2(u)\, du.$$

From a previous example, we already know how to handle this integral. We find that

$$I = \frac{1}{2}\left(u + \frac{\sin(2u)}{2}\right) = \frac{1}{2}\left(u + \sin(u)\cos(u)\right) + C.$$

(In the last step, we have used the double angle trigonometric identity. We will shortly see why this simplification is relevant.)

We now desire to convert the result back to a function of the original variable, $x$. We note that $x = \sin(u)$ implies $u = \arcsin(x)$. To convert the term $\cos(u)$ back to an expression depending on $x$ we can use the relationship $1 - \sin^2(u) = \cos^2(u)$, to deduce that

$$\cos(u) = \sqrt{1 - \sin^2(u)} = \sqrt{1-x^2}.$$

$\Diamond$

It is sometimes helpful to use a Pythagorean triangle, as shown in Figure 11.3, to rewrite the antiderivative in terms of the variable $x$. The idea is this: We construct the triangle in such a way that its side lengths are related to the "angle" $u$ according to the substitution rule. In this example, $x = \sin(u)$ so the sides labeled $x$ and 1 were chosen so that their ratio ("opposite over hypotenuse" coincides with the sine of the indicated angle, $u$, thereby satisfying $x = \sin(u)$. We can then determine the length of the third leg of the triangle (using the Pythagorean formula) and thus all other trigonometric functions of $u$. For example, we note that the ratio of "adjacent over hypotenuse" is $\cos(u) = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$. Finally, with these reverse substitutions, we find that,

$$I = \int \sqrt{1-x^2}\, dx = \frac{1}{2}\left(\arcsin(x) + x\sqrt{1-x^2}\right) + C.$$

**Remark.** In computing a definite integral of the same type, we can circumvent the need for the conversion back to an expression involving $x$ by using the appropriate method for handling endpoints. For example, the integral

$$I = \int_0^1 \sqrt{1-x^2}\, dx$$

can be transformed to

$$I = \int_0^{\pi/2} \sqrt{\cos^2(u)} \cos(u)\, du,$$

by observing that $x = \sin(u)$ implies that $u = 0$ when $x = 0$ and $u = \pi/2$ when $x = 1$. Then this means that the integral can be evaluated directly (without changing back to the variable $x$) as follows:

$$I = \int_0^{\pi/2} \sqrt{\cos^2(u)} \cos(u)\, du = \frac{1}{2}\left(u + \frac{\sin(2u)}{2}\right)\Big|_0^{\pi/2} = \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin(\pi)}{2}\right) = \frac{\pi}{4}$$

where we have used the fact that $\sin(\pi) = 0$.

Some subtle points about the domains of definition of inverse trigonometric functions will not be discussed here in detail. (See material on these functions in a first term calculus course.) Suffice it to say that some integrals of this type will be undefined if this endpoint conversion cannot be carried out (e.g. if the interval of integration had been $0 \le x \le 2$, we would encounter an impossible relation $2 = \sin(u)$. Since no value of $u$ satisfies this relation, such a definite integral has no meaning, i.e. "does not exist".)

**Example 11.9 (tan and sec substitution)** *Find the antiderivative of*

$$I = \int \sqrt{1+x^2}\, dx.$$

**Solution.** We aim for simplification by the identity $1 + \tan^2(u) = \sec^2(u)$, so we set

$$x = \tan(u), \quad dx = \sec^2(u) du.$$

Then the substitution leads to

$$I = \int \sqrt{1+\tan^2(u)}\ \sec^2(u)\, du = \int \sqrt{\sec^2(u)}\ \sec^2(u)\, du = \int \sec^3(u)\, du.$$

This integral will require further work, and will be partly calculated by *Integration by Parts* in section **??**. In this example, the triangle shown in Figure 11.4 shows the relationship between $x$ and $u$ and will help to convert other trigonometric functions of $u$ to functions of $x$. ◊

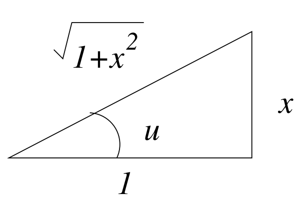



Figure 11.4: As in Figure 11.3 but for example 11.9.

## 11.6 Partial fractions

In this section, we show a simple algebraic trick that helps to simplify an integrand when it is in the form of some *rational function* such as

$$f(x) = \frac{1}{(ax+b)(cx+d)}.$$

The idea is to break this up into simpler rational expressions by finding constants $A$, $B$ such that

$$\frac{1}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}.$$

Each part can then be handled by a simple substitution, as shown in Example 11.1, Eqn. (11.1). We give several examples below.

**Example 11.10 (Partial fractions (1))** *Find the antiderivative of*

$$I = \int \frac{1}{x^2-1}.$$

**Solution.** Factoring the denominator, $x^2 - 1 = (x-1)(x+1)$, suggests breaking up the integrand into the form

$$\frac{1}{x^2-1} = \frac{A}{(x+1)} + \frac{B}{(x-1)}.$$

The two sides are equal provided:

$$\frac{1}{x^2-1} = \frac{A(x-1)+B(x+1)}{x^2-1}.$$

This means that

$$1 = A(x-1)+B(x+1)$$

must be true for all $x$ values. We now ask what values of $A$ and $B$ make this equation hold for any $x$. Choosing two "easy" values, namely $x = 1$ and $x = -1$ leads to isolating one or the other unknown constants, $A, B$, with the results:

$$1 = -2A, \quad 1 = 2B.$$

Thus $B = 1/2, A = -1/2$, so the integral can be written in the simpler form

$$I = \frac{1}{2}\left(\int \frac{-1}{(x+1)}\,dx + \int \frac{1}{(x-1)}\,dx\right).$$

(A common factor of $(1/2)$ has been taken out.) Now a simple substitution will work for each component. (Let $u = x+1$ for the first, and $u = x-1$ for the second integral.) The result is

$$I = \int \frac{1}{x^2-1} = \frac{1}{2}\left(-\ln|x+1| + \ln|x-1|\right) + C.$$

◊

**Example 11.11 (Partial fractions (2))** *Find the antiderivative of*

$$I = \int \frac{1}{x(1-x)}\, dx.$$

**Solution.** This example is similar to the previous one. We set

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{(1-x)}.$$

Then

$$1 = A(1-x) + Bx.$$

This must hold for all $x$ values. In particular, convenient values of $x$ for determining the constants are $x = 0,\ 1$. We find that

$$A = 1,\ \ B = 1.$$

Thus

$$I = \int \frac{1}{x(1-x)}\, dx = \int \frac{1}{x}\, dx + \int \frac{1}{1-x}\, dx.$$

Simple substitution now gives

$$I = \ln|x| - \ln|1-x| + C.$$

$\Diamond$

**Example 11.12 (Partial fractions (3))** *Find the antiderivative of*

$$I = \int \frac{x}{x^2+x-2}.$$

**Solution.** The rational expression above factors into $x^2+x-2 = (x-1)(x+2)$, leading to the expression

$$\frac{x}{x^2+x-2} = \frac{A}{(x-1)} + \frac{B}{(x+2)}.$$

Consequently, it follows that

$$A(x+2) + B(x-1) = x.$$

Substituting the values $x = 1, -2$ into this leads to $A = 1/3$ and $B = 2/3$. The usual procedure then results in

$$I = \int \frac{x}{x^2+x-2} = \frac{1}{3}\ln|x-1| + \frac{2}{3}\ln|x+2| + C.$$

Another example of the technique of partial fractions is provided in section **??**. $\Diamond$

## 11.7 Integration by parts

The method described in this section is important as an additional tool for integration. It also has independent theoretical stature in many applications in mathematics and physics. The essential idea is that in some cases, we can exchange the task of integrating a function with the job of differentiating it.

The idea rests on the product rule for derivatives. Suppose that $u(x)$ and $v(x)$ are two differentiable functions. Then we know that the derivative of their product is

$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx},$$

or, in the differential notation:

$$d(uv) = v\,du + u\,dv,$$

Integrating both sides, we obtain

$$\int d(uv) = \int v\,du + \int u\,dv$$

i.e.

$$uv = \int v\,du + \int u\,dv.$$

We write this result in the more suggestive form

$$\int u\,dv = uv - \int v\,du.$$

The idea here is that if we have difficulty evaluating an integral such as $\int u\,dv$, we may be able to "exchange it" for a simpler integral in the form $\int v\,du$. This is best illustrated by the examples below.

**Example 11.13 (Integration by parts (1))** *Compute*

$$I = \int_1^2 \ln(x)\,dx.$$

**Solution.** Let $u = \ln(x)$ and $dv = dx$. Then $du = (1/x)\,dx$ and $v = x$.

$$\int \ln(x)\,dx = x\ln(x) - \int x(1/x)\,dx = x\ln(x) - \int\,dx = x\ln(x) - x.$$

We now evaluate this result at the endpoints to obtain

$$I = \int_1^2 \ln(x)\,dx = (x\ln(x) - x)\Big|_1^2 = (2\ln(2) - 2) - (1\ln(1) - 1) = 2\ln(2) - 1.$$

(Where we used the fact that $\ln(1) = 0$.) $\Diamond$

**Example 11.14 (Integration by parts (2))** *Compute*

$$I = \int_0^1 xe^x\,dx.$$

**Solution.** At first, it may be hard to decide how to assign roles for $u$ and $dv$. Suppose we try $u = e^x$ and $dv = xdx$. Then $du = e^x\, dx$ and $v = x^2/2$. This means that we would get the integral in the form

$$I = \frac{x^2}{2}e^x - \int \frac{x^2}{2}e^x\, dx.$$

This is certainly *not* a simplification, because the integral we obtain has a higher power of $x$, and is consequently harder, not easier to integrate. This suggests that our first attempt was not a helpful one. (Note that integration often requires trial and error.)

Let $u = x$ and $dv = e^x\, dx$. This is a wiser choice because when we differentiate $u$, we reduce the power of $x$ (from 1 to 0), and get a simpler expression. Indeed, $du = dx$, $v = e^x$ so that

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C.$$

To find a definite integral of this kind on some interval (say $0 \leq x \leq 1$), we compute

$$I = \int_0^1 xe^x\, dx = (xe^x - e^x)\Big|_0^1 = (1e^1 - e^1) - (0e^0 - e^0) = 0 + e^0 = e^0 = 1.$$

Note that all parts of the expression are evaluated at the two endpoints. ◊

**Example 11.15 (Integration by parts (2b))** *Compute*

$$I_n = \int x^n e^x\, dx.$$

**Solution.** We can calculate this integral by repeated application of the idea in the previous example. Letting $u = x^n$ and $dv = e^x\, dx$ leads to $du = nx^{n-1}$ and $v = e^x$. Then

$$I_n = x^n e^x - \int nx^{n-1}e^x\, dx = x^n e^x - n\int x^{n-1}e^x\, dx.$$

Each application of integration by parts, reduces the power of the term $x^n$ inside an integral by one. The calculation is repeated until the very last integral has been simplified, with no remaining powers of $x$. This illustrates that in some problems, integration by parts is needed more than once. ◊

**Example 11.16 (Integration by parts (3))** *Compute*

$$I = \int \arctan(x)\, dx.$$

**Solution.** Let $u = \arctan(x)$ and $dv = dx$. Then $du = (1/(1+x^2))\, dx$ and $v = x$ so that

$$I = x\arctan(x) - \int \frac{1}{1+x^2}x\, dx.$$

The last integral can be done with the simple substitution $w = (1+x^2)$ and $dw = 2x\,dx$, giving

$$I = x\arctan(x) - (1/2)\int (1/w)dw.$$

We obtain, as a result

$$I = x\arctan(x) - \frac{1}{2}\ln(1+x^2).$$

$\Diamond$

**Example 11.17 (Integration by parts (3b))** *Compute*

$$I = \int \tan(x)\,dx.$$

**Solution.** We might try to fit this into a similar pattern, i.e. let $u = \tan(x)$ and $dv = dx$. Then $du = \sec^2(x)\,dx$ and $v = x$, so we obtain

$$I = x\tan(x) - \int x\sec^2(x)\,dx.$$

This is not really a simplification, and we see that integration by parts will not necessarily work, even on a seemingly related example. However, we might instead try to rewrite the integral in the form

$$I = \int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx.$$

Now we find that a simple substitution will do the trick, i.e. that $w = \cos(x)$ and $dw = -\sin(x)\,dx$ will convert the integral into the form

$$I = \int \frac{1}{w}\,(-dw) = -\ln|w| = -\ln|\cos(x)|.$$

This example illustrates that we should always try substitution, first, before attempting other methods. $\Diamond$

**Example 11.18 (Integration by parts (4))** *Compute*

$$I_1 = \int e^x \sin(x)\,dx.$$

*We refer to this integral as $I_1$ because a related second integral, that we'll call $I_2$ will appear in the calculation.*

**Solution.** Let $u = e^x$ and $dv = \sin(x)\,dx$. Then $du = e^x\,dx$ and $v = -\cos(x)\,dx$. Therefore

$$I_1 = -e^x\cos(x) - \int(-\cos(x))e^x\,dx = -e^x\cos(x) + \int \cos(x)e^x\,dx.$$

We now have another integral of a similar form to tackle. This seems hopeless, as we have not simplified the result, but let us not give up! In this case, another application of integration by parts will do the trick. Call $I_2$ the integral

$$I_2 = \int \cos(x)e^x \, dx,$$

so that

$$I_1 = -e^x \cos(x) + I_2.$$

Repeat the same procedure for the new integral $I_2$, i.e. Let $u = e^x$ and $dv = \cos(x)\, dx$. Then $du = e^x\, dx$ and $v = \sin(x)\, dx$. Thus

$$I_2 = e^x \sin(x) - \int \sin(x)e^x \, dx = e^x \sin(x) - I_1.$$

This appears to be a circular argument, but in fact, it has a purpose. We have determined that the following relationships are satisfied by the above two integrals:

$$I_1 = -e^x \cos(x) + I_2$$

$$I_2 = e^x \sin(x) - I_1.$$

We can eliminate $I_2$, obtaining

$$I_1 = -e^x \cos(x) + I_2 = -e^x \cos(x) + e^x \sin(x) - I_1.$$

that is,

$$I_1 = -e^x \cos(x) + e^x \sin(x) - I_1.$$

Rearranging (taking $I_1$ to the left hand side) leads to

$$2I_1 = -e^x \cos(x) + e^x \sin(x),$$

and thus, the desired integral has been found to be

$$I_1 = \int e^x \sin(x)\, dx = \frac{1}{2}\left(-e^x \cos(x) + e^x \sin(x)\right) = \frac{1}{2}e^x(\sin(x) - \cos(x)) + C.$$

(At this last step, we have included the constant of integration.) Moreover, we have also found that $I_2$ is related, i.e. using $I_2 = e^x \sin(x) - I_1$ we now know that

$$I_2 = \int \cos(x)e^x \, dx = \frac{1}{2}e^x\left(\sin(x) + \cos(x)\right) + C.$$

$\Diamond$

## *11.8 Summary*

In this chapter, we explored a number of techniques for computing antiderivatives. We here summarize the most important results:

1. Substitution is the first method to consider. This method works provided the change of variable results in elimination of the original variable and leads to a simpler, more elementary integral.

2. When using substitution on a definite integral, endpoints can be converted to the new variable (Method 1) or the resulting antiderivative can be converted back to its original variable before plugging in the (original) endpoints (Method 2).

3. The integration by parts formula for functions $u(x), v(x)$ is

$$\int u\,dv = uv - \int v\,du.$$

   Integration by parts is useful when $u$ is easy to differentiate (but not easy to integrate). It is also helpful when the integral contains a product of elementary functions such as $x^n$ and a trigonometric or an exponential function. Sometimes more than one application of this method is needed. Other times, this method is combined with substitution or other simplifications.

4. Using integration by parts on a definite integral means that both parts of the formula are to be evaluated at the endpoints.

5. Integrals involving $\sqrt{1 \pm x^2}$ can be simplified by making a trigonometric substitution.

6. Integrals with products or powers of trigonometric functions can sometimes be simplified by application of trigonometric identities or simple substitution.

7. Algebraic tricks, and many associated manipulations are often applied to twist and turn a complicated integral into a set of simpler expressions that can each be handled more easily.

8. Even with all these techniques, the problem of finding an antiderivative can be very complicated. In some cases, we resort to handbooks of integrals, use symbolic manipulation software packages, or, if none of these work, calculate a given definite integral numerically using a spreadsheet.

**Table of elementary antiderivatives**

1. $\int \frac{1}{u}\, du = \ln|u| + C.$
2. $\int u^n\, du = \frac{u^{n+1}}{n+1} + C$
3. $\int \frac{1}{1+u^2} = \arctan(u) + C$
4. $\int \frac{1}{\sqrt{1-x^2}} = \arcsin(u) + C$
5. $\int \sin(u)\, du = -\cos(u) + C$
6. $\int \cos(u)\, du = \sin(u) + C$
7. $\int \sec^2(u)\, du = \tan(u) + C$

**Additional useful antiderivatives**

1. $\int \tan(u)\, du = \ln|\sec(u)| + C.$
2. $\int \cot(u)\, du = \ln|\sin(u)| + C$
3. $\int \sec(u) = \ln|\sec(u) + \tan(u)| + C$

## *Exercises*

**Exercise 11.1 Differential Notation** The differential of the function $y(x)$ is defined as $dy = y'(x)dx$, i.e. as the product between $y'(x)$ and the differential of $x$, $dx$. For example, given

$$y(x) = 3x + \sin(2x),$$

its differential is

$$dy = y'(x)dx = (3 + 2\cos(2x))dx.$$

Calculate the differentials of the following functions:

| | | | | | |
|---|---|---|---|---|---|
| (a) | $f(x) = e^{x^2}$ | (b) | $f(x) = (x+1)^2$ | (c) | $f(x) = \sqrt{x}$ |
| (d) | $f(x) = \arcsin(x)$ | (e) | $f(x) = x^2 + 3x + 1$ | (f) | $f(x) = \cos(2x)$ |
| (g) | $y = x^6 + 2x^4 - 2x$ | (h) | $y = (x-2)^2(x+1)^5$ | (i) | $y = \dfrac{x}{x+3}$ |

**Exercise 11.2 More Differential Notation** Given the differential of a function $y$ in terms of the product between its derivative $y'(x)$ and the differential of $x$, ($dx$), express the same differential in terms of the the differential of $y$ itself, i.e., $dy$. Then, use the result combined with the fundamental theorem of calculus to calculate the corresponding indefinite integrals. For example, if we know that the differential of a function $y(x)$ is given by

$$dy = (3 + 2\sin(2x))dx, \qquad \text{(or, equivalently, } y(x) = \int (3 + 2\sin(2x))dx)$$

we need to figure out that $y(x) = 3x - \cos(2x)$ and express the differential given above in terms of the differential of $y$ itself

$$(3 + 2\sin(2x))dx = d(3x - \cos(2x)).$$

Using this result and the fundamental theorem of calculus, we can solve the following

$$\int (3 + 2\sin(2x))dx = \int d(3x - \cos(2x)) = 3x - \cos(2x) + C.$$

(Those marked by a star are a little bit more difficult since they involve composite functions.)

(a) $x^3\,dx = ?,\quad \int x^3\,dx = ?$ (b) $x^{3/2}\,dx = ?,\quad \int x^{3/2}\,dx = ?$

(c) $\frac{1}{x^3}\,dx = ?,\quad \int \frac{1}{x^3}\,dx = ?$ (d) $\sqrt{x}\,dx = ?,\quad \int \sqrt{x}\,dx = ?$

(e) $\frac{1}{\sqrt{x}}\,dx = ?,\quad \int \frac{1}{\sqrt{x}}\,dx = ?$ (f) $e^{-2x}\,dx = ?,\quad \int e^{-2x}\,dx = ?$

(g) $(x+1)^2\,dx = ?,\quad \int (x+1)^2\,dx = ?$ (h) $\frac{1}{(x+3)^2}\,dx = ?,\quad \int \frac{1}{(x+3)^2}\,dx = ?$

(i) $\cos(2x)\,dx = ?,\quad \int \cos(2x)\,dx = ?$ (j) $\sec^2(x)\,dx = ?,\quad \int \sec^2(x)\,dx = ?$

(k) $\frac{1}{x}dx = ?,\quad \int \frac{1}{x}dx = ?$ (l) $\frac{1}{1+x^2}dx = ?,\quad \int \frac{1}{1+x^2}dx = ?$

(m)* $2xe^{x^2}\,dx = ?,\quad \int 2xe^{x^2}\,dx = ?$ (n)* $3x^2\cos(x^3)\,dx = ?,\quad \int 3x^2\cos(x^3)\,dx = ?$

(o)* $\frac{2x}{1+x^2}\,dx = ?,\quad \int \frac{2x}{1+x^2}\,dx = ?$ (p)* $\frac{2x}{1+x^4}dx = ?,\quad \int \frac{2x}{1+x^4}dx = ?$

**Exercise 11.3 Substitutions**

(a) Evaluate the following indefinite integrals using the substitution method:

(i) $\sin(3x)\,dx$ (ii) $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

(iii) $\int x^3\sqrt{x^4+1}\,dx$ (iv) $\int \frac{4}{1+2x}\,dx$

(b) Calculate the following definite integrals using the substitution rule:

(i) $\int_0^5 (\sqrt{3+2x})\,dx$ (ii) $\int_0^{\pi/4} \sin(4t)\,dt$ (iii) $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

**Exercise 11.4 More substitutions** Compute the following integrals using substitution.

(a) $\int_1^p \frac{1}{1+y^2}\,dy$ (b) $\int x^2(x^3+1)^6\,dx$ (c) $\int e^x\sqrt{1+2e^x}\,dx$

(d) $\int \frac{1}{x\ln(x)}\,dx$ (e) $\int \frac{1}{1-y}\,dy$ (f) $\int_1^S \frac{k_1}{k_2-n}\,dn,\ k_1,k_2>0$

(g) $\int \frac{x^2}{1-x^3}\,dx$ (h) $\int \sqrt{3x+1}\,dx$ (i) $\int \frac{x}{\sqrt{4+x^2}}\,dx$

(j) $\int \cos(x)\sin^5(x)\,dx$ (k) $\int \frac{3}{4+5x}\,dx$ (l) $\int \cot(\theta)\,d\theta$

(m) $\int \frac{\sec^2(x)}{\sqrt{2+\tan(x)}}\,dx$ (n) $\int \frac{2}{4+x^2}\,dx$

**Exercise 11.5 Trigonometric Substitution** The integral $\int \sin(x)\cos(x)dx$ can be computed in several ways:

(a) using the substitution $u = \sin(x)$ (or $u = \cos(x)$)

(b) by first using the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ and then integrating.

Show that the two answers are equivalent. (Hint: you will find that this is a good opportunity to review trigonometric identities.)

**Exercise 11.6 More Trigonometric substitution** The integral $\int_0^1 \sqrt{1-x^2}\,dx$ can be computed using the trigonometric substitution $x = \sin(u)$.

(a) Show that this reduces the integral to $\displaystyle\int \cos^2(u)\,du$.

(b) Use the identity $\cos^2(u) = \dfrac{1+\cos(2u)}{2}$ to re-express the integral in a simpler form. Then integrate.

(c) Explain why the answer is the same as the area of $1/4$ of a circle of radius 1.

**Exercise 11.7 Partial Fractions** Compute the following integrals. Use factoring and/or completing the square and partial fractions, or some other technique if necessary.

(a) $\displaystyle\int \frac{1}{x^2-x-20}\,dx$ (b) $\displaystyle\int \frac{3}{x^2+6x+9}\,dx$

(c) $\displaystyle\int \frac{-1}{x^2+4x+14}\,dx$ (d) $\displaystyle\int \frac{2}{x^2-6x+8}\,dx$

**Exercise 11.8 More Partial Fractions** The two integrals shown below may look very similar, but in fact they lead to quite different results:

(a) $\displaystyle\int \frac{dx}{a^2+x^2}$ (b) $\displaystyle\int \frac{dx}{a^2-x^2}$

(Hint: (a) can be reduced to an inverse tangent type integral by a bit of algebraic rearrangement and a substitution of the form $u = x/a$ and (b) can be integrated by factoring the expression $a^2-x^2$ and using the method of partial fractions.)

**Exercise 11.9 Integration by Parts** Use integration by parts to show that $\displaystyle\int x^n e^x dx = x^n e^x - n\int x^{n-1}e^x dx$

**Exercise 11.10 More Integration by Parts** First make a substitution and then use integration by parts to evaluate the following integrals:

(a) $\displaystyle\int \sin(\sqrt{x})\,dx$ (b) $\displaystyle\int_1^4 e^{\sqrt{x}}\,dx$

(c) $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} x^3\cos\left(x^2\right)\,dx$ (d) $\displaystyle\int x^5 e^{x^2}\,dx$

**Exercise 11.11** Compute the following integrals.

(a) $\displaystyle\int \frac{3}{x^2+4x+6}\,dx$ (b) $\displaystyle\int \frac{2}{(x+1)(x+2)}\,dx$ (c) $\displaystyle\int \frac{1}{x^2+6x+8}\,dx$

(d) $\displaystyle\int_1^p \frac{1}{1-y^2}\,dy$ (e) $\displaystyle\int_0^T te^{-2t}\,dt$ (f) $\displaystyle\int_0^\pi x\sin\left(\frac{x}{2}\right)\,dx$

(g) $\displaystyle\int x^2\ln(x)\,dx$

**Exercise 11.12** **Integration drills** Note: some simplifications in a few of these will make your work much easier.

(1) $\displaystyle\int \frac{1}{x}\ln x\,dx$

(2) $\displaystyle\int \frac{2}{2+3x}\,dx$

(3) $\displaystyle\int x^2\sin\left(3x^3+1\right)\,dx$

(4) $\displaystyle\int_1^2 2t\,e^{3t^2}\,dt$

(5) $\displaystyle\int_0^1 \frac{x}{1+x^2}\,dx$

(6) $\displaystyle\int \frac{1}{\sqrt{2-3x^2}}\,dx$

(7) $\displaystyle\int \frac{2}{3+4x^2}\,dx$

(8) $\displaystyle\int \frac{1}{\sqrt{25-4x^2}}\,dx$

(9) $\displaystyle\int x\frac{1}{\sqrt{9+x^2}}\,dx$

(10) $\displaystyle\int \cos(3x)(1-\sin(3x))\,dx$

(11) $\displaystyle\int \cos(2x)(1-\sin(2x))\,dx$

(12) $\displaystyle\int \sec^2(x)\sqrt{\tan(x)+1}\,dx$

(13) $\displaystyle\int_0^{\pi/2} (1-\sin^2(t))\sin(t)\,dt$

(14) $\displaystyle\int 2x(1-\sin(2x))\,dx$

(15) $\displaystyle\int \sec(x)\tan(x)\,dx$
(Hint: use trig identities)

(16) $\displaystyle\int x\sin(x+1)\,dx$

(17) $\displaystyle\int (x\sin^2(x)+x\cos^2(x))^5\,dx$
(Hint: look carefully!)

(18) $\displaystyle\int \sec^2(t)\,dt$

(19) $\displaystyle\int \frac{1}{2x^2+12x+18}\,dx$
(Hint: some algebra, please)

(20) $\displaystyle\int \frac{1}{(x-5)(x+1)}\,dx$

(21) $\displaystyle\int \frac{1}{(x+3)(2x-1)}\,dx$

(22) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx$

(23) $\displaystyle\int \frac{3x^2+1}{x^2(x^2+1)^2}\,dx$

(24) $\displaystyle\int \frac{1}{x^2-3x+2}\,dx$
(Hint: try factoring)

(25) $\displaystyle\int \frac{3}{x^2+3x+15}\,dx$

(26) $\displaystyle\int x\sec^2(x)\,dx$

(27) $\displaystyle\int xe^{x^2}\,dx$

(28) $\displaystyle\int x^2e^x\,dx$

(29) $\displaystyle\int \arctan(x)\,dx$

(30) $\displaystyle\int x\cos(x)e^x\,dx$

# 12 Differential Equations

## 12.1 Introduction

A differential equation is a relationship between some (unknown) function and one of its derivatives. Examples of differential equations were encountered in an earlier calculus course in the context of population growth, temperature of a cooling object, and speed of a moving object subjected to friction. In Section 9.2, we reviewed an example of a differential equation for velocity, (9.8), and discussed its solution, but here, we present a more systematic approach to solving such equations using a technique called **separation of variables**. In this chapter, we apply the tools of integration to finding solutions to differential equations. The importance and wide applicability of this topic cannot be overstated.

In this course, since we are concerned only with functions that depend on a single variable, we discuss **ordinary differential equations** (ODE's), whereas later, after a multivariate calculus course where partial derivatives are introduced, a wider class, of **partial differential equations** (PDE's) can be studied. Such equations are encountered in many areas of science, and in any quantitative analysis of systems where rates of change are linked to the state of the system. Most laws of physics are of this form; for example, applying the familiar Newton's law, $F = ma$, links the position of a pendulum's mass to its acceleration (second derivative of position).[1] Many biological processes are also described by differential equations. The rate of growth of a population $dN/dt$ depends on the size of that population at the given time $N(t)$.

[1] Newton's law states that force is proportional to acceleration. For a pendulum, the force is due to gravity, and the acceleration is a second derivative of the x or y coordinate of the bob on the pendulum.

Constructing the differential equation that adequately represents a system of interest is an art that takes some thought and experience. In this process, which we call "modeling", many simplifications are made so that the essential properties of a given system are captured, leaving out many complicating details. For example, friction might be neglected in "modeling" a perfect pendulum. The details of age distribution might be neglected in modeling a growing population. Now that we have techniques for integration, we can devise a new approach to computing solutions of differential equations.

Given a differential equation and a starting value, the goal is to make a prediction about the future behaviour of the system. This is equivalent to identifying the function that satisfies the given differential equation and initial value(s). We refer to such a function as the **solution** to the **initial value problem** (IVP). In differential calculus, our exploration of differential equations was limited to those whose solution could be guessed, or whose solution was supplied in advance. We also explored some of the fascinating geometric and qualitative properties of such equations and their predictions.

Now that we have techniques of integration, we can find the analytic solution to a variety of simple first-order differential equations (i.e. those involving the first derivative of the unknown function). We will describe the technique of **separation of variables**. This technique works for examples that are simple enough that we can isolate the dependent variable (e.g. $y$) on one side of the equation, and the independent variable (e.g. time $t$) on the other side.

## *12.2 Unlimited population growth*

We start with a simple example that was treated thoroughly in the differential calculus semester of this course. We consider a population with per capita birth and mortality rates that are constant, irrespective of age, disease, environmental changes, or other effects. We ask how a population in such ideal circumstances would change over time. We build up a simple model (i.e. a differential equation) to describe this ideal case, and then proceed to find its solution. Solving the differential equation is accomplished by a new technique introduced here, namely separation of variables. This reduces the problem to integration and algebraic manipulation, allowing us to compute the population size at any time $t$. By going through this process, we essentially convert information about the rate of change and starting level of the population to a detailed prediction of the population at later times.[2]

[2] Of course, we must keep in mind that such predictions are based on simplifying assumptions, and are to be taken as an approximation of any real population growth.

### *A simple model for population growth*

Let $y(t)$ represent the size of a population at time $t$. We will assume that at time $t = 0$, the population level is specified, i.e. $y(0) = y_0$ is some given constant. We want to find the population at later times, given information about birth and mortality rates, (both of which are here assumed to be constant over time).

The population changes through births and mortality. Suppose that $b > 0$ is the per capita average birth rate, and $m > 0$ the per capita average mortality rate. The assumption that $b$, $m$ are both constants is a simplification that neglects many biological effects, but will be used for simplicity in this first example.

The statement that the population increases through births and decreases due to mortality, can be restated as

$$\text{rate of change of } y = \text{rate of births} - \text{rate of mortality}$$

where the rate of births is given by the product of the per capita average birth rate $b$ and the population size $y$. Similarly, the rate of mortality is given by $my$. Translating the rate of change into the corresponding derivative of $y$ leads to

$$\frac{dy}{dt} = by - my = (b-m)y.$$

Let us define the new constant,

$$k = b - m.$$

Then $k$ is the *net per capita growth rate* of the population. We can distinguish two possible cases: $b > m$ means that there are more births then deaths, so we expect the population to grow. $b < m$ means that there are more deaths than births, so that the population will eventually go extinct. There is also a marginal case that $b = m$, for which $k = 0$, where the population does not change at all. To summarize, this simple model of unlimited growth leads to the differential equation and initial condition:

$$\frac{dy}{dt} = ky, \quad y(0) = y_0. \tag{12.1}$$

Recall that a differential equation together with an initial condition is called an initial value problem. To find a solution to such a problem, we look for the function $y(t)$ that describes the population size at any future time $t$, given its initial size at time $t = 0$.

### *Separation of variables and integration*

We here introduce the technique, **separation of variables**, that will be used in all the examples described in this chapter. Since the differential equation (12.1) is relatively simple, this first example will be relatively straightforward. We would like to determine $y(t)$ given the differential equation

$$\frac{dy}{dt} = ky.$$

Rather than integrating this equation as is[3], we use an alternate approach, considering $dt$ and $dy$ as "differentials" in the sense defined in Section 11.1. We rearrange and rewrite the above equation in the form

$$\frac{1}{y}\, dy = k\, dt, \tag{12.2}$$

This step of putting expressions involving the independent variable $t$ on one side and expressions involving the dependent variable $y$ on the opposite side gives rise to the name "separation of variables".

[3] We may be tempted to integrate both sides of this equation with respect to the independent variable $t$, e.g. writing $\int \frac{dy}{dt}\, dt = \int ky\, dt + C$, (where $C$ is some constant), but this is not very useful, since the integral on the right hand side (RHS) can only be carried out if we know the function $y = y(t)$, which we are trying to determine.

Now, the LHS of Eqn. (12.2) depends only on the variable $y$, and the RHS only on $t$. The constant $k$ will not interfere with any integration step. Moreover, integrating each side of Eqn. (12.2) can be carried out independently.

To determine the appropriate intervals for integration, we observe that when time sweeps over some interval $0 \le t \le T$ (from initial to final time), the value of $y(t)$ will change over a corresponding interval $y_0 \le y \le y(T)$. Here $y_0$ is the given starting value of $y$ (prescribed by the initial condition in (12.1)). We do not yet know $y(T)$, but our goal is to find that value, i.e to predict the future behaviour of $y$. Integrating leads to

$$\int_{y_0}^{y(T)} \frac{1}{y}\, dy = \int_0^T k\, dt = k\int_0^T dt,$$

$$\ln|y| \Big|_{y_0}^{y(T)} = kt\Big|_0^T,$$

$$\ln|y(T)| - \ln|y(0)| = k(T-0),$$

$$\ln\left|\frac{y(T)}{y_0}\right| = kT,$$

$$\frac{y(T)}{y_0} = e^{kT},$$

$$y(T) = y_0 e^{kT}.$$

But this result holds for any arbitrary final time, $T$. In other words, since this is true for any time we chose, we can set $T = t$, arriving at the desired solution

$$y(t) = y_0 e^{kt}. \tag{12.3}$$

The above formula relates the predicted value of $y$ at any time $t$ to its initial value, and to all the parameters of the problem. Observe that plugging in $t = 0$, we get $y(0) = y_0 e^{kt} = y_0 e^0 = y_0$, so that the solution (12.3) satisfies the initial condition. We leave as an exercise for the reader[4] to validate that the function in(12.3) also satisfies the differential equation in (12.1).

[4] This kind of check is good practice and helps to spot errors. Simply differentiate Eqn. (12.3) and show that the result is the same as $k$ times the original function, as required by the equation (12.1).

By solving the initial value problem (12.1), we have determined that, under ideal conditions, when the net per capita growth rate $t$ is constant, a population will grow exponentially with time. Recall that this validates results that we had encountered in our first calculus course.

## *12.3 Terminal velocity and steady states*

Here we revisit the equation for velocity of a falling object that we first encountered in Section 9.2. We wish to derive the appropriate differential equation governing that velocity, and find the solution $v(t)$ as a function of time. We will first reconsider the simplest case of uniformly accelerated motion (i.e. where friction is neglected), as in Section 9.2. We then include friction, as in Section 9.2 and use the new technique of separation of variables to shortcut the method of solution.

*Ignoring friction: the uniformly accelerated case*

Let $v(t)$ and $a(t)$ be the velocity and the acceleration, respectively of an object falling under the force of gravity at time $t$. We take the positive direction to be downwards, for convenience. Suppose that at time $t = 0$, the object starts from rest, i.e. the initial velocity of the object is known to be $v(0) = 0$. When friction is neglected, the object will accelerate,

$$a(t) = g,$$

which is equivalent to the statement that the velocity increases at a constant rate,

$$\frac{dv}{dt} = g. \tag{12.4}$$

Because $g$ is constant, *we do not need to use separation of variables*, i.e. we can integrate each side of this equation directly[5]. Writing

$$\int \frac{dv}{dt}\,dt = \int g\,dt + C = g\int\,dt + C,$$

where $C$ is an integration constant, we arrive at

$$v(t) = gt + C. \tag{12.5}$$

Here we have used (on the LHS) that $v$ is the antiderivative of $dv/dt$. (equivalently, we can simplify the integral $\int \frac{dv}{dt}\,dt = \int\,dv = v$). Plugging in $v(0) = 0$ into Eqn. (12.5) leads to $0 = g \cdot 0 + C = C$, so the constant we need is $C = 0$ and the velocity satisfies

$$v(t) = gt.$$

We have just arrived at a result that parallels Eqn. (9.4) of Section 9.2 (in slightly different notation).

[5] It is important to note the distinction between this simple example and other cases where separation of variables is required. It would not be *wrong* to use separation of variables to find the solution for Eqn. (12.4), but it would just be "overkill", since simple integration of the each side of the equation "as is" does the job.

*Including friction: the case of terminal velocity*

When a falling object experiences the force of friction, it cannot accelerate indefinitely. In fact, a frictional force retards the downwards motion. To a good approximation, that force is proportional to the velocity.

A force balance for the falling object leads to

$$ma(t) = mg - \gamma v(t),$$

where $\gamma$ is the frictional coefficient. For an object of constant mass, we can divide through by $m$, so

$$a(t) = g - \frac{\gamma}{m}v(t).$$

Let $k = \gamma/m$. Then, the velocity at any time satisfies the differential equation and initial condition

$$\frac{dv}{dt} = g - kv, \quad v(0) = 0. \tag{12.6}$$

We can find the solution to this differential equation and predict the velocity at any time $t$ using separation of variables.

Consider a time interval $0 \le t \le T$, and suppose that, during this time interval, the velocity changes from an initial value of $v(0) = 0$ to the final value, $v(T)$ at the final time, $T$. Then using separation of variables and integration, we get

$$\frac{dv}{dt} = g - kv,$$

$$\frac{dv}{g - kv} = dt,$$

$$\int_0^{v(T)} \frac{dv}{g - kv} = \int_0^T dt.$$

Substitute $u = g - kv$ for the integral on the left hand side. Then $du = -kdv$, $dv = (-1/k)du$, so we get an integral of the form

$$-\frac{1}{k}\int \frac{1}{u}\, du = -\frac{1}{k}\ln|u|.$$

After replacing $u$ by $g - kv$, we arrive at

$$-\frac{1}{k}\ln|g - kv|\Big|_0^{v(T)} = t\Big|_0^T.$$

We use the fact that $v(0) = 0$ to write this as

$$-\frac{1}{k}\left(\ln|g - kv(T)| - \ln|g|\right) = T,$$

$$-\frac{1}{k}\left(\ln\left|\frac{g - kv(T)}{g}\right|\right) = T,$$

$$\ln\left|\frac{g - kv(T)}{g}\right| = -kT.$$

We are finished with the integration step, but the function we are trying to find, $v(T)$ is still tangled up inside an expression involving the natural logarithm. Extricating it will involve some subtle reasoning about signs because there is an absolute value to contend with. As a first step, we exponentiate both sides to remove the logarithm.

$$\left|\frac{g - kv(T)}{g}\right| = e^{-kT} \quad \Rightarrow \quad |g - kv(T)| = ge^{-kT}.$$

Because the constant $g$ is positive, we could remove absolute values signs from it. To simplify further, we have to consider the sign of the term inside the absolute value in the numerator. In the case we are considering here, $v(0) = 0$. This will mean that the quantity $g - kv(T)$ is always be non-negative (i.e. $g - kv(T) \ge 0$). We will verify this fact shortly. For the moment, supposing this is true, we can write

$$|g - kv(T)| = g - kv(T) = ge^{-kT},$$

and finally solve for $v(T)$ to obtain our final result,

$$v(T) = \frac{g}{k}(1 - e^{-kT}).$$

Here we note that $v(T)$ can never be larger than $g/k$ since the term $(1 - e^{-kT})$ is always $\leq 1$. Hence, we were correct in assuming that $g - kv(T) \geq 0$.

As before, the above formula relating velocity to time holds for any choice of the final time $T$, so we can write, in general,

$$v(t) = \frac{g}{k}(1 - e^{-kt}). \tag{12.7}$$

This is the solution to the initial value problem (12.6). It predicts the velocity of the falling object through time. Note that we have arrived once more at the result obtained in Eqn. (9.11), but using the technique of separation of variables. (It often happens that a differential equation can be solved using several different methods.).

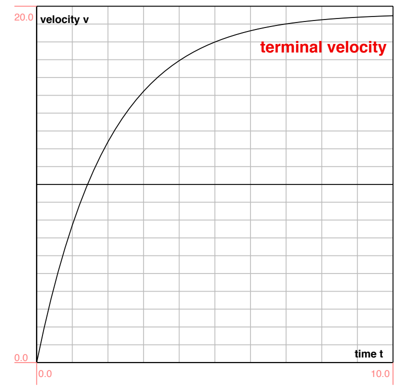



Figure 12.1: The velocity $v(t)$ as a function of time given by Eqn. (12.7) as found in Section 12.3. Note that as time increases, the velocity approaches some constant terminal velocity. The parameters used were $g = 9.8$ m/s$^2$ and $k = 0.5$.

We graph the expression given in (12.7) in Figure 12.1. Note that as $t$ increases, the term $e^{-kt}$ decreases rapidly, so that the velocity approaches a constant whose value is

$$v(t) \to \frac{g}{k}.$$

We call this the *terminal velocity*[6].

[6] A similar plot of the solution of the differential equation (12.6) could be assembled using Euler's method, as studied in differential calculus. That is the numerical method alternative to the analytic technique discussed in this chapter. The student may wish to review results obtained in a previous semester to appreciate the correspondence.

### *Steady state*

We might observe that the terminal velocity can also be found quite simply and directly from the *differential equation itself*: it is the **steady state** of the differential equation, i.e. the value for which no further change takes place. The steady state can be found by setting the derivative in the differential equation, to zero, i.e. by letting

$$\frac{dv}{dt} = 0.$$

When this is done, we arrive at

$$g - kv = 0 \quad \Rightarrow \quad v = \frac{g}{k}.$$

Thus, at steady state, the velocity of the falling object is indeed the same as the terminal velocity that we have just discovered.

## *12.4 Related problems and examples*

The example discussed in Section 12.3 belongs to a class of problems that share many common features. Generally, this class is represented by linear differential equations of the form

$$\frac{dy}{dt} = a - by, \tag{12.8}$$

with given initial condition $y(0) = y_0$. Properties of this equation were studied in the context of differential calculus in a previous semester. Now, with the same method as we applied to the problem of terminal velocity, we can integrate this equation by separation of variables, writing

$$\frac{dy}{a - by} = dt$$

and proceeding as in the previous example. We arrive at its solution,

$$y(t) = \frac{a}{b} + \left(y_0 - \frac{a}{b}\right) e^{-bt}. \tag{12.9}$$

The steps are left as an exercise for the reader.

We observe that the steady state of the above equation is obtained by setting

$$\frac{dy}{dt} = a - by = 0, \quad \text{i.e.} \quad y = \frac{a}{b}.$$

Indeed the solution given in the formula (12.9) has the property that as $t$ increases, the exponential term $e^{-bt} \to 0$ so that the term in large brackets will vanish and $y \to a/b$. This means that from any initial value, $y$ will approach its steady state level.

This equation has a number of important applications that arise in a variety of context. A few of these are mentioned below.

### *Blood alcohol*

Let $y(t)$ be the level of alcohol in the blood of an individual during a party. Suppose that the average rate of drinking is gradual and constant (i.e. small sips are continually taken, so that the rate of input of alcohol is approximately constant). Further, assume that alcohol is detoxified in the liver at a rate proportional to its blood level. Then an equation of the form (12.8) would describe the blood level over the period of drinking. $y(0) = 0$ would signify the absence of alcohol in the body at the beginning of the evening. The constant $a$ would reflect the rate of intake per unit volume of the individual's blood: larger people take longer to "get drunk" for a given amount consumed[7]. The constant $b$ represents the rate of decay of alcohol per unit time due to degradation by the liver, assumed constant[8]; young healthy drinkers have a higher value of $b$ than those who can no longer metabolize alcohol as efficiently.

The solution (12.9) has several features of note: it illustrates the fact that alcohol would increase from the initial level, but only up to a maximum of $a/b$, where the intake and degradation balance. Indeed, the level $y = a/b$ represents a steady state level (as long as drinking continues). Of course, this level could be toxic to the drinker, and the assumptions of the model may break down in that region!

In the phase of "recovery", after drinking stops, the above differential equation no longer describes the level of blood alcohol. Instead, the process

[7] Of course, we are here assuming a constant intake rate, as though the alcohol is being continually sipped all evening at a uniform rate. Most people do not drink this way, instead quaffing a few large drinks over some hour(s). It is possible to describe this, but we will not do so in this chapter.

[8] This is also a simplifying assumption, as the rate of metabolism can depend on other factors, such as food intake.

of recovery is represented by

$$\frac{dy}{dt} = -by, \quad y(0) = y_0. \tag{12.10}$$

The level of blood alcohol then decays exponentially with rate $b$ from its level at the moment that drinking ends. We show this typical pattern in Figure 12.2.

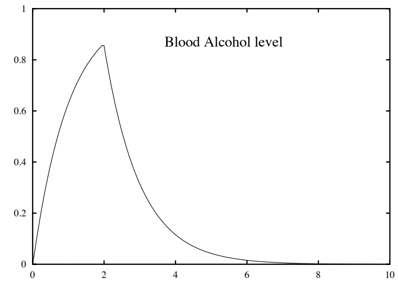



Figure 12.2: The level of alcohol in the blood is described by Eqn. (12.8) for the first two hours of drinking. At $t = 2$h, the drinking stopped (so $a = 0$ from then on). The level of alcohol in the blood then decays back to zero, following Eqn. (12.10).

*Chemical kinetics*

The same ideas apply to any chemical substance that is formed at a constant rate (or supplied at a constant rate) $a$, and then breaks down with rate proportional to its concentration. We then call the constant $b$ the "decay rate constant".

The variable $y(t)$ represents the concentration of chemical at time $t$, and the same differential equation describes this chemical process. As above, given any initial level of the substance, $y(t) = y_0$, the level of $y$ will eventually approach the steady state, $y = a/b$.

## 12.5 Emptying a container

In this section we investigate a new problem in which the differential equation that describes a process will be derived from basic physical principles. We will look at the flow of fluid leaking out of a container, and use mass balance to derive a differential equation model. When this is done, we will also use separation of variables to predict how long it takes for the container to be emptied. This example is particularly instructive. First, it shows precisely how physical laws can be combined to formulate a model, then it shows how the problem can be recast as a single ODE in one dependent variable. Finally, it illustrates a slightly different integral.

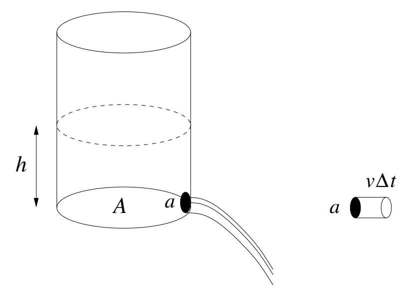



Figure 12.3: We investigate the time it takes to empty a container full of fluid by deriving a differential equation model and solving it using the methods developed in this chapter. $A$ is the cross-sectional area of the cylindrical tank, $a$ is the cross-sectional area of the hole through which fluid drains, $v(t)$ is the velocity of the fluid, and $h(t)$ is the time dependent height of fluid remaining in the tank (indicated by the dashed line). The volume of fluid leaking out in a time span $\Delta t$ is $av\Delta t$ - see small cylindrical volume indicated on the right.

We will assume that the container has a small hole at its base. The rate of emptying of the container will depend on the height of fluid in the container above the hole (as we have assumed that the hole is at $h = 0$, we henceforth consider the height of the fluid surface, $h(t)$ to be the same as "the height of fluid above the hole".). We can derive a simple differential equation that describes the rate that the height of the fluid changes using the following physical argument.

*Conservation of mass*

Suppose that the container is a cylinder, with a constant cross sectional area $A > 0$, as shown in Fig. 12.3. Suppose that the area of the hole is $a$. The rate that fluid leaves through the hole must balance with the rate that fluid decreases in the container. This principle is called **mass balance**. We will here assume that the density of water is constant, so that we can talk about the net changes in volume (rather than mass).

We refer to $V(t)$ as the volume of fluid in the container at time $t$. Note that for the cylindrical container, $V(t) = Ah(t)$ where $A$ is the cross-sectional area and $h(t)$ is the height of the fluid at time $t$. The rate of change of $V$ is

$$\frac{dV}{dt} = -(\text{rate volume lost as fluid flows out}).$$

(The minus sign indicates that the volume is decreasing).

At every second, some amount of fluid leaves through the hole. Suppose we are told that the velocity of the water molecules leaving the hole is precisely $v(t)$ in units of cm/sec. (We will find out how to determine this velocity shortly.) Then in one second, those particles have moved a distance $v$ cm/sec $\cdot$ 1 sec $= v$ cm. In fact, all the particles in a little cylinder of length $v$ behind these molecules have also left the hole. Indeed, if we know the area of the hole, we can determine precisely what volume of water exits through the hole each second, namely

$$\text{rate volume lost as fluid flows out} = va.$$

(The small inset in Fig. 12.3 shows a little "cylindrical unit" of fluid that flows out of the hole per second. The area is $a$ and the length of that little volume is $v$. Thus the volume leaving per second is $va$.)

So far we have a relationship between the volume of fluid in the tank and the velocity of the water exiting the hole:

$$\frac{dV}{dt} = -av.$$

Now we need to determine the velocity $v$ of the flow to complete the formulation of the problem.

### *Conservation of energy*

The fluid "picks up speed" because it has "dropped" by a height $h$ from the top of the fluid surface to the hole. In doing so, a small mass of water has simply exchanged some potential energy (due to its relative height above the hole) for kinetic energy (expressed by how fast it is moving). Potential energy of a small mass of water ($m$) at height $h$ will be $mgh$, whereas when the water flows out of the hole, its kinetic energy is given by $(1/2)mv^2$ where $v$ is velocity. Thus, for these to balance (so that total energy is conserved) we have

$$\frac{1}{2}mv^2 = mgh.$$

(Here $v = v(t)$ is the instantaneous velocity of the fluid leaving the hole and $h = h(t)$ is the height of the water column.) This allows us to relate the velocity of the fluid leaving the hole to the height of the water in the tank, i.e.

$$v^2 = 2gh \quad \Rightarrow \quad v = \sqrt{2gh}. \tag{12.11}$$

In fact, both the height of fluid and its exit velocity are constantly changing as the fluid drains, so we might write $[v(t)]^2 = 2gh(t)$ or $v(t) = \sqrt{2gh(t)}$. We have arrived at this result using an **energy balance** argument.

### *Putting it together*

We now combine the various pieces of information to arrive at the model, a differential equation for a single (unknown) function of time. There are three time-dependent variables that were discussed above, the volume $V(t)$, the height $h(t)$, of the velocity $v(t)$. It proves convenient to express everything in terms of the height of water in the tank, $h(t)$, though this choice is to some extent arbitrary. Keeping units in an equation consistent is essential. Checking for unit consistency can help to uncover errors in equations, including differential equations.

Recall that the volume of the water in the tank, $V(t)$ is related to the height of fluid $h(t)$ by

$$V(t) = Ah(t),$$

where $A > 0$ is a constant, the cross-sectional area of the tank. We can simplify as follows:

$$\frac{dV}{dt} = \frac{d(Ah(t))}{dt} = A\frac{d(h(t))}{dt}.$$

But by previous steps and Eqn. (12.11)

$$\frac{dV}{dt} = -av = -a\sqrt{2gh}.$$

Thus

$$A\frac{d(h(t))}{dt} = -a\sqrt{2gh},$$

or simply put,

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh} = -k\sqrt{h}. \tag{12.12}$$

where $k$ is a constant that depends on the size and shape of the cylinder and its hole:

$$k = \frac{a}{A}\sqrt{2g}.$$

If the area of the hole is very small relative to the cross-sectional area of the tank, then $k$ will be very small, so that the tank will drain very slowly (i.e. the rate of change in $h$ per unit time will not be large). On a planet with a very high gravitational force, the same tank will drain more quickly. A taller column of water drains faster. Once its height has been reduced, its rate of draining also slows down. We comment that Equation (12.12) has a minus sign, signifying that the height of the fluid decreases.

Using simple principles such as conservation of mass and conservation of energy, we have shown that the height $h(t)$ of water in the tank at time $t$ satisfies the differential equation (12.12). Putting this together with the initial condition (height of fluid $h_0$ at time $t = 0$), we arrive at initial value problem to solve:

$$\frac{dh}{dt} = -k\sqrt{h}, \quad h(0) = h_0. \tag{12.13}$$

Clearly, this equation is valid only for $h$ non-negative. We also remark that Eqn. (12.13) is **nonlinear**[9] as it involves the variable $h$ in a nonlinear term, $\sqrt{h}$. Next, we use separation of variables to find the height as a function of time.

[9] In many cases, nonlinear differential equations are more challenging than linear ones. However, examples chosen in this chapter are simple enough that we will not experience the true challenges of such nonlinearities.

### *Solution by separation of variables*

The equation (12.13) shows how height of fluid is related to its rate of change, but we are interested in an explicit formula for fluid height $h$ versus time $t$. To obtain that relationship, we must determine the solution to this differential equation. We do this using separation of variables. (We will also use the initial condition $h(0) = h_0$ that accompanies Eqn. (12.13).) As usual, rewrite the equation in the separated form,

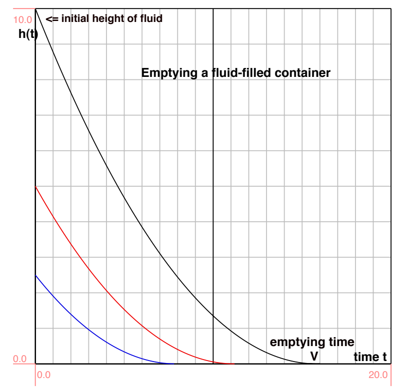



Figure 12.4: Solution curves obtained by plotting Eqn. (12.14) for three different initial heights of fluid in the container, $h_0 = 2.5,\ 5,\ 10$. The parameter $k = 0.4$ in each case. The "V" points to the time it takes the tank to empty starting from a height of $h(t) = 10$.

$$\frac{dh}{\sqrt{h}} = -kdt,$$

We integrate from $t = 0$ to $t = T$, during which the height of fluid that started as $h_0$ becomes some new height $h(T)$ to be determined.

$$\int_{h_0}^{h(T)} \frac{1}{\sqrt{h}}\, dh = -k \int_0^T dt.$$

Now integrate both sides and simplify:

$$\left.\frac{h^{1/2}}{(1/2)}\right|_{h_0}^{h(T)} = -kT$$

$$2\left(\sqrt{h(T)} - \sqrt{h_0}\right) = -kT$$

$$\sqrt{h(T)} = -k\frac{T}{2} + \sqrt{h_0}$$

$$h(T) = \left(\sqrt{h_0} - k\frac{T}{2}\right)^2.$$

Since this is true for any time $t$, we can also write the form of the solution as

$$h(t) = \left(\sqrt{h_0} - k\frac{t}{2}\right)^2. \tag{12.14}$$

Eqn. (12.14) predicts fluid height remaining in the tank versus time $t$. In Fig. 12.4 we show some of the "solution curves"[10], i.e. functions of the form Eqn. (12.14) for a variety of initial fluid height values $h_0$. We can also use our results to predict the emptying time, as shown in the next section.

[10] As before, this figure was produced by plotting the analytic solution (12.14). A numerical method alternative would use Euler's Method and the spreadsheet to obtain the (approximate) solution directly from the initial value problem (12.13).

*How long will it take the tank to empty?*

The tank will be empty when the height of fluid is zero. Setting $h(t) = 0$ in Eqn. 12.14

$$\left(\sqrt{h_0} - k\frac{t}{2}\right)^2 = 0.$$

Solving this equation for the emptying time $t_e$, we get

$$k\frac{t_e}{2} = \sqrt{h_0} \quad \Rightarrow \quad t_e = \frac{2\sqrt{h_0}}{k}.$$

The time it takes to empty the tank depends on the initial height of water in the tank. Three examples are shown in Figure 12.4 for initial heights of $h_0 = 2.5,\ 5,\ 10$. The emptying time depends on the square-root of the initial height. This means, for instance, that doubling the height of fluid initially in the tank only increases the time it takes by a factor of $\sqrt{2} \approx 1.41$. Making the hole smaller has a more direct "proportional" effect, since we have found that $k = (a/A)\sqrt{2g}$.

## *12.6 Density dependent growth*

The simple model discussed in Section 12.2 for population growth has an unrealistic feature of unlimited explosive exponential growth. To correct for this unrealistic feature, a common assumption is that the rate of growth is "density dependent". In this section, we consider a revised differential equation that describes such growth, and use the new tools to analyze its predictions. In place of our previous notation we will now use $N$ to represent the size of a population.

### *The logistic equation*

The logistic equation is the simplest density dependent growth equation, and we study its behaviour below.

Let $N(t)$ be the size of a population at time $t$. Clearly, we expect $N(t) \geq 0$ for all time $t$, since a population cannot be negative. We will assume that the initial population is known, $N(0) = N_0$. The logistic differential equation states that the rate of change of the population is given by

$$\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right). \tag{12.15}$$

Here $r > 0$ is called the **intrinsic growth rate** and $K > 0$ is called the **carrying capacity**. $K$ reflects that size of the population that can be sustained by the given environment. We can understand this equation as a modified growth law in which the "density dependent" term, $r(K-N)/K$, replaces the previous constant net growth rate $k$.

### *Scaling the equation*

The form of the equation can be simplified if we measure the population in units of the carrying capacity, instead of "numbers of individuals". i.e. if we define a new quantity

$$y(t) = \frac{N(t)}{K}.$$

This procedure is called **scaling**. To see this, consider dividing each side of the logistic equation (12.15) by the constant $K$. Then

$$\frac{1}{K}\frac{dN}{dt} = \frac{r}{K}N\left(\frac{K-N}{K}\right).$$

We now group terms conveniently, forming

$$\frac{d(\frac{N}{K})}{dt} = r\left(\frac{N}{K}\right)\left(1-\left(\frac{N}{K}\right)\right).$$

Replacing $(N/K)$ by $y$ in each case, we obtain the scaled equation and initial condition given by

$$\frac{dy}{dt} = ry(1-y), \quad y(0) = y_0. \tag{12.16}$$

Now the variable $y(t)$ measures population size in "units" of the carrying capacity, and $y_0 = N_0/K$ is the scaled initial population level. Here again is an initial value problem, like Eqn. (12.13), but unlike Eqn. (12.1), the logistic differential equation is nonlinear. That is, the variable $y$ appears in a nonlinear expression (in fact a quadratic) in the equation.

### *Separation of variables*

Here we will solve Eqn. (12.16) by separation of variables. The idea is essentially the same as our previous examples, but is somewhat more involved. To show an alternative method of handling the integration, we will treat both sides as indefinite integrals. Separating the variables leads to

$$\frac{1}{y(1-y)}\,dy = r\,dt$$

$$\int \frac{1}{y(1-y)}\,dy = \int r\,dt + K.$$

The integral on the right will lead to $rt+K$ where $K$ is some constant of integration that we need to incorporate since we do not have endpoints on our integrals. But we must work harder to evaluate the integral on the left. We can do so by partial fractions, the technique described in Section 11.6. Details are given in Section 12.6.

*Application of partial fractions*

Let
$$I = \int \frac{1}{y(1-y)}\, dy.$$
Then for some constants $A, B$ we can write
$$I = \int \frac{A}{y} + \frac{B}{1-y}\, dy = A \ln|y| - B \ln|1-y|.$$
(The minus sign in front of $B$ stems from the fact that letting $u = 1-y$ would lead to $du = -dy$.) We can find $A, B$ from the fact that
$$\frac{A}{y} + \frac{B}{1-y} = \frac{1}{y(1-y)},$$
so that
$$A(1-y) + By = 1.$$
This must be true for all $y$, and in particular, substituting in $y = 0$ and $y = 1$ leads to $A = 1, B = 1$ so that
$$I = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|.$$

*The solution of the logistic equation*

We now have to extract the quantity $y$ from the equation
$$\ln\left|\frac{y}{1-y}\right| = rt + C_0.$$
That is, we want $y$ as a function of $t$. After exponentiating both sides we need to remove the absolute value. We will now assume that $y$ is initially smaller than 1, and show that it remains so. In that case, everything inside the absolute value is positive, and we can write
$$\frac{y(t)}{1-y(t)} = e^{rt+C_0} = e^{C_0} e^{rt} = Ce^{rt}.$$
In the above step, we have simply renamed the constant, $e^{C_0}$ by the new name $C$ for simplicity. $C > 0$ is now also an arbitrary constant whose value will be determined from the initial conditions. Indeed, if we substitute $t = 0$ into the most recent equation, we find that
$$\frac{y(0)}{1-y(0)} = Ce^0 = C,$$
so that
$$C = \frac{y_0}{1-y_0}.$$

We will use this fact shortly. What remains now is some algebra to isolate the desired function $y(t)$

$$y(t) = (1 - y(t))Ce^{rt}$$
$$y(t)\,(1 + Ce^{rt}) = Ce^{rt}$$
$$y(t) = \frac{Ce^{rt}}{(1 + Ce^{rt})} = \frac{1}{\frac{1}{C}e^{-rt} + 1}. \tag{12.17}$$

The desired function is now expressed in terms of the time $t$, and the constants $r, C$. We can also express it in terms of the initial value of $y$, i.e. $y_0$, by using what we know to be true about the constant $C$, i.e. $C = y_0/(1 - y_0)$. When we do so, we arrive at

$$y(t) = \frac{1}{\frac{1-y_0}{y_0}e^{-rt} + 1} = \frac{y_0}{y_0 + (1 - y_0)e^{-rt}}. \tag{12.18}$$

Some typical solution curves of the logistic equation are shown in Fig. 12.5.

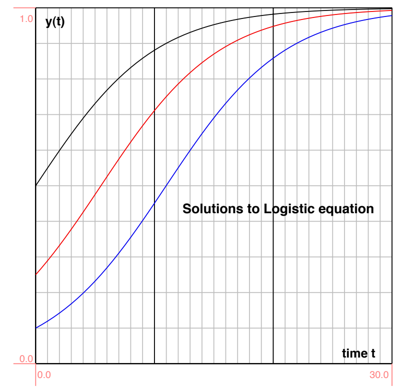



Figure 12.5: Solution curves for $y(t)$ in the scaled form of the logistic equation based on (12.19). We show the predicted behaviour of $y(t)$ as given by Eqn. (12.18) for three different initial conditions, $y_0 = 0.1, 0.25, 0.5$. Note that all solutions approach the value $y = 1$.

### *What this solution tells us*

We have arrived at the function that describes the scaled population as a function of time as predicted by the scaled logistic equation, (12.16). The level of population (in units of the carrying capacity $K$) follows the time-dependent function

$$y(t) = \frac{y_0}{y_0 + (1 - y_0)e^{-rt}}. \tag{12.19}$$

We can convert this result to an equivalent expression for the unscaled total population $N(t)$ by recalling that $y(t) = N(t)/K$. Substituting this for $y(t)$, and noting that $y_0 = N_0/K$ leads to

$$N(t) = \frac{N_0}{N_0 + (K - N_0)e^{-rt}}K. \tag{12.20}$$

It is left as an exercise for the reader to check this claim.

Now recall that $r > 0$. This means that $e^{-rt}$ is a decreasing function of time. Therefore, (12.19) implies that, after a long time, the term $e^{-rt}$ in the denominator will be negligibly small, and so

$$y(t) \to \frac{y_0}{y_0} = 1,$$

so that $y$ will approach the value 1. This means that

$$\frac{N}{K} \to 1 \quad \text{or simply} \quad N(t) \to K.$$

The population will thus settle into a constant level, i.e., a **steady state**, at which no further change will occur.

As an aside, we observe that this too, could have been predicted directly from the differential equation. By setting $dy/dt = 0$, we find that

$$0 = ry(1 - y),$$

which suggests that $y = 1$ is a steady state. (This is also true for the less interesting case of no population, i.e. $y = 0$ is also a steady state.) Similarly, this could have been found by setting the derivative to zero in Eqn. (12.15), the original, unscaled logistic differential equation. Doing so leads to

$$\frac{dN}{dt} = 0 \quad \Rightarrow \quad rN\left(\frac{K-N}{K}\right) = 0.$$

If $r > 0$, the only values of $N$ satisfying this steady state equation are $N = 0$ or $N = K$. This implies that either $N = 0$ or $N = K$ are steady states. The former is not too interesting. It states the obvious fact that if there is no population, then there can be no population growth. The latter reflects that $N = K$, the carrying capacity, is the population size that will be sustained by the environment.

In summary, we have shown that the behaviour of the logistic equation for population growth is more realistic than the simpler exponential growth we studied earlier. We saw in Figure 12.5, that a small population will grow, but only up to some constant level (the carrying capacity). Integration, and in particular the use of partial fractions allowed us to make a full prediction of the behaviour of the population level as a function of time, given by Eqn. (12.20).

## 12.7 *Extensions and other population models: the "Law of Mortality"*

There are many variants of the logistic model that are used to investigate the growth or mortality of a population. Here we extend tools to another example, the gradual decline of a group of individuals born at the same time. Such a group is called a "cohort". In 1825, Gompertz suggested that the rate of mortality, $m$ would depend on the age of the individuals. Because we consider a group of people who were born at the same time, we can trade "age" for "time". Essentially, Gompertz assumed that mortality is not constant: it is low at first, and increase as individuals age. Gompertz argued that mortality increases exponentially. This turns out to be equivalent to the assumption that the logarithm of mortality increases linearly with time. (In actual fact, this is likely true for some range of ages. Infant mortality is generally higher than mortality for young children, whereas mortality levels off or even decreases slightly for those oldest old who have survived past the average lifespan.) It is easy to see that these two statements are equivalent: Suppose we assume that for some constants $A > 0, \mu > 0$,

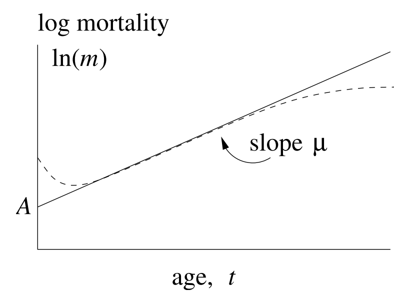



Figure 12.6: In the Gompertz Law of Mortality, it is assumed that the log of mortality increases linearly with time, as depicted by Eqn. 12.21 and by the solid curve in this diagram. Here the slope of $\ln(m)$ versus time (or age) is $\mu$. For real populations, the mortality looks more like the dashed curve.

$$\ln(m(t)) = A + \mu t. \qquad (12.21)$$

Then Eqn. (12.21) means that

$$m(t) = e^{A+\mu t} = e^A e^{\mu t}$$

Since $A$ is constant, so is $e^A$. For simplicity we define Let us define $m_0 = e^A$. ($m_0 = m(0)$ is the so-called "birth mortality" i.e. value of $m$ at age 0.) Thus, the time-dependent mortality is

$$m = m(t) = m_0 e^{\mu t}. \tag{12.22}$$

*Aging and Survival curves for a cohort:*

We now study a population model having Gompertz mortality, together with the following additional assumptions.

1. All individuals are assumed to be identical.
2. There is "natural" mortality, but no other type of removal. This means we ignore the mortality caused by epidemics, by violence and by wars.
3. We consider a single cohort, and assume that no new individuals are introduced (e.g. by immigration). Note that new births would contribute to other cohorts.

We will now study the size of a "cohort", i.e. a group of people who were born in the same year. We will denote by $N(t)$ the number of people in this group who are alive at time $t$, where $t$ is time since birth, i.e. age. Let $N(0) = N_0$ be the initial number of individuals in the cohort.

### *Gompertz Model*

All the people in the cohort were born at time (age) $t = 0$, and there were $N_0$ of them at that time. That number changes with time due to mortality. Indeed,

$$\begin{aligned} \text{The rate of change of cohort size} &= -[\text{number of deaths per unit time}] \\ &= -[\text{mortality rate}] \cdot [\text{cohort size}] \end{aligned}$$

Translating to mathematical notation, we arrive at the differential equation

$$\frac{dN(t)}{dt} = -m(t)N(t),$$

and using information about the size of the cohort at birth leads to the initial condition, $N(0) = N_0$. Together, this leads to the initial value problem

$$\frac{dN(t)}{dt} = -m(t)N(t), \quad N(0) = N_0.$$

Note similarity to Eqn. (12.1), but now mortality is time-dependent.

In the Problem set, we apply separation of variables and integrate over the time interval $[0, T]$: to show that the remaining population at age $t$ is

$$N(t) = N_0 e^{-\frac{m_0}{\mu}(e^{\mu t} - 1)}.$$

## 12.8 Summary

In this chapter, we used integration methods to find the analytical solutions to a variety of differential equations where initial values were prescribed.

We investigated a number of **population growth** models:

1. Exponential growth, given by $\frac{dy}{dt} = ky$, with initial population level $y(0) = y_0$ was investigated (Eqn. (12.1)). This model had an unrealistic feature that growth is unlimited.

2. The Logistic equation $\frac{dN}{dt} = rN\left(\frac{K-N}{K}\right)$ was analyzed (Eqn. (12.15)), showing that density-dependent growth can correct for the above unrealistic feature.

3. The Gompertz equation, $\frac{dN(t)}{dt} = -m(t)N(t)$, was solved to understand how age-dependent mortality affects a cohort of individuals.

In each of these cases, we used separation of variables to "integrate" the differential equation, and predict the population as a function of time.

We also investigated several other **physical models** in this chapter, including the velocity of a falling object subject to drag force. This led us to study a differential equation of the form $\frac{dy}{dt} = a - by$. By slight reinterpretation of terms in this equation, we can use results to understand chemical kinetics and blood alcohol levels, as well as a host of other scientific applications.

Section 12.5, the "centerpiece" of this chapter, illustrated the detailed steps that go into the formulation of a differential equation model for flow of liquid out of a container. Here we saw how conservation statements and simplifying assumptions are interpreted together, to arrive at a differential equation model. Such ideas occur in many scientific problems, in chemistry, physics, and biology.

## *Excercises*

**Exercise 12.1** Use separation of variables to solve the following differential equations with given initial conditions.

(a) $\frac{dy}{dt} = -2ty, \quad y(0) = 10$

(b) $\frac{dy}{dt} = y(1-y), \quad y(0) = 0.5.$ (Hint: $\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}$.)

**Exercise 12.2** Consider the differential equation $\frac{dy}{dt} = 1 - y$ and initial condition $y(0) = 0.5$.

(a) Solve for $y$ as a function of $t$ using separation of variables.

(b) Use the initial condition to find the value of $C$.

**Exercise 12.3** Consider the differential equation $\frac{dy}{dt} = 1 + y^2$ and initial condition $y(0) = 0.5$.

(a) Solve for $y$ as a function of $t$ using separation of variables.

(b) Use the initial condition to find the value of $C$.

**Exercise 12.4** Consider the differential equation $\frac{dy}{dt} = 1 - y^2$ and initial condition $y(0) = 0.5$. Solve using separation of variables (Hint: use partial fractions.)

**Exercise 12.5** Solve the following differential equation $\frac{dy}{dt} = ky^{2/3}$ and initial condition $y(0) = y_0$, assuming that $k > 0$ is a constant.

**Exercise 12.6** A certain cylindrical water tank has a hole in the bottom, out of which water flows. The height of water in the tank, $h(t)$, can be described by the differential equation $\frac{dh}{dt} = -k\sqrt{h}$ where $k$ is a positive constant. If the height of the water is initially $h_0$, determine how much time elapses before the tank is empty.

**Exercise 12.7** The position of a particle is described by

$$\frac{dx}{dt} = \sqrt{1-x^2}, \quad 0 \le t \le \frac{\pi}{2}$$

(a) Find the position as a function of time given that the particle starts at $x = 0$ initially.

(b) Where is the particle when $t = \pi/2$?

(c) At which position(s) is the particle moving the fastest? The slowest?

**Exercise 12.8** In an experiment involving yeast cells, it was determined that mortality of the cells increased at a linear rate, i.e. that at time $t$ after the beginning of the experiment the mortality rate was $m(t) = m_0 + rt$. The experiment was started with $N_0$ cells at time $t = 0$. Let $N(t)$ represent the population size of the cells at time $t$.

(a) If no "birth" occurs, then

$$\frac{dN(t)}{dt} = -m(t)N(t).$$

Solve this differential equation by separation of variables, i.e find $N(t)$ as a function of time.

(b) Suppose cells are "born" at a constant rate $b$, so that the differential equation is

$$\frac{dN(t)}{dt} = -m(t)N(t) + bN(t).$$

Determine how this affects the population size at time $t$.

**Exercise 12.9** Muscle cells are known to be powered by filaments of the protein actin which slide past one another. The filaments are moved by cross-bridges of myosin, that act like little motors, which attach, pull the filaments, and then detach. Let $n$ be the fraction of myosin cross-bridges that are attached at time $t$. A model for cross-bridge attachment is:

$$\frac{dn}{dt} = k_1(1-n) - k_2 n, \quad n(0) = n_0$$

where $k_1 > 0, k_2 > 0$ are constants.

(a) Solve this differential equation and determine the fraction of attached cross-bridges $n(t)$ as a function of time $t$.

(b) What value does $n(t)$ approach after a long time?

(c) Suppose $k_1 = 1.0$ and $k_2 = 0.2$. Starting from $n(0) = 0$, how long does it take for 50% of the cross-bridges to become attached ?

**Exercise 12.10** The velocity of an object falling under the effect of gravity and air resistance is given by: $\frac{dv}{dt} = f(v) = g - kv$ with $v(0) = v_0$, where $g > 0$ denotes the acceleration due to gravity and $k > 0$ is a frictional coefficient (both constant).

(a) Sketch the function $f(v)$ as a function of $v$. Identify a value of $v = v^*$ for which no change occurs, i.e., for which $f(v^*) = 0$. This is called a *steady state value* of the velocity or a *fixed point*. Interpret what this value represents.

(b) Explain what happens if the initial velocity is larger or smaller than this steady state value. Will the velocity increase or decrease? (Recall that the sign of the derivative $dv/dt$ tells us whether $v(t)$ is an increasing or decreasing function of $t$).

(c) Use separation of variables to find the function $v(t)$.

**Exercise 12.11** A model for the velocity of a sky diver (slightly different from the model we have already studied) is

$$\frac{dv}{dt} = 9 - v^2$$

(a) What is this skydiver's "terminal velocity", $v_\infty$? That is, near what velocity will the sky diver eventually stabilize?

(b) Starting from rest, how long will it take to reach half of $v_\infty$?

**Exercise 12.12** **Newton's Law of Cooling** Consider the differential equation for Newton's Law of Cooling

$$\frac{dT}{dt} = -k(T - E)$$

for a constant $k > 0$, temperature of the environment $E$ and the initial condition: $T(0) = T_0$. Solve this differential equation by the method of separation of variables i.e. find $T(t)$. Interpret your result.

**Exercise 12.13** Let $V(t)$ be the volume of a spherical cell that is expanding by absorbing water from its surface area. Suppose that the rate of increase of volume is simply proportional to the surface area of the sphere, and that, initially, the volume is $V_0$. Find a differential equation that describes the way that $V$ changes. Use the connection between surface area and volume in a sphere to rewrite your differential equation in terms of the volume alone. You should get an equation in the form

$$\frac{dV}{dt} = kV^{2/3},$$

where $k$ is some constant. Solve the differential equation to show how the volume changes as a function of the time. (Hint: recall that for a sphere, $V = (4/3)\pi r^3$, $S = 4\pi r^2$.)

**Exercise 12.14** According to Klaassen and Lindstrom (1996) J theor. Biol. 183:29-34, the "fuel load" (nectar) carried by a hummingbird, $F(t)$ depends on the rate of intake (from flowers) and the rate of consumption due to metabolism. They assume that intake takes place at a constant rate $\alpha$. They also assume that consumption increases when the bird is heavier (carrying more fuel). Suppose that fuel is consumed at a rate proportional to the amount of fuel being carried (with proportionality constant $\beta$).

(a) What features are being neglected or simplified in this model?

(b) Write down the differential equation model for $F(t)$.

(c) Find $F(t)$ as a function of time $t$.

(d) Klaassen and Lindstrom determined that for a hummingbird, $\alpha = 0.48$ g fuel per day and $\beta = 0.09$ per day. Determine the steady state level of fuel carried by the bird. (Note: use the fact that at steady state, the rate of change of the fuel level is zero.)

**Exercise 12.15** **Gompertz growth law** The Gompertz growth equation for a cohort with age-dependent mortality is

$$\frac{dN(t)}{dt} = -m(t)N(t),$$

where $m(t) = m_0 e^{\mu t}$. Use separation of variables to find $N(t)$, given that the starting population is $N(0) = N_0$.

**Exercise 12.16** **Bacterial growth** In an experiment involving a bacteria population, let $N(t)$ represent the size of the population (measured in thousands of individuals) as a function of time, starting at $t = 0$. The initial population is $N(0) = N_0$.

(a) Suppose the population growth is governed by the diferential equation

$$\frac{dN}{dt} = \frac{N}{t+1}$$

Find $N(t)$ given that $N(0) = 2$.

(b) Suppose the population growth is governed by the diferential equation

$$\frac{dN}{dt} = \frac{N^2}{(t+1)^2}.$$

Find $N(t)$ if $N(0) = 2$.

(c) What happens to the solution from part (b) as $t \to 0$? Can you find an initial population $N_0$ for which this problem doesn't occur for any time t?

# Index

# *Index of Applications*

*Answers and Solutions*

- **Problem 1.1:**
- **Problem 1.2:**

  (a) Stretched in $y$ direction by factor $A$; (b) Shifted up by $a$; (c) Shifted in positive $x$ direction by $b$.
- **Problem 1.3:**

  Not Provided
- **Problem 1.4:**
- **Problem 1.5:**

  (a) $x = 0, (3/2)^{1/3}$; (b) $x = 0, x = \pm\sqrt{1/4}$.
- **Problem 1.6:**

  (b) $a < 0$: $x = 0$; $a \geq 0$: $x = 0, \pm a^{1/4}$; (c) $a > 0$.
- **Problem 1.7:**

  if $m-n$ even: $x = \pm\left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$; if $m-n$ odd: $x = \left(\frac{A}{B}\right)^{1/(m-n)}, x = 0$
- **Problem 1.8:**

  (a) $(0,0)$ and $(1,1)$; (b) $(0,0)$; (c) $(\frac{\sqrt{7}}{2}, \frac{3}{4})$, $(-\frac{\sqrt{7}}{2}, \frac{3}{4})$, and $(0,-1)$.
- **Problem 1.9:**

  (a) $x = I/\gamma$, (b) $x = \dfrac{\gamma \pm \sqrt{\gamma^2 - 4I\varepsilon}}{2\varepsilon}$.
- **Problem 1.10:**
- **Problem 1.11:**

  $y = x^n$; $y = x^{-n}$; $y = x^{1/n}, n = 2, 4, 6, \ldots$; $y = x^{-n}, n = 1, 2, 3, \ldots$
- **Problem 1.12:**

  $m > -1$
- **Problem 1.13:**
- **Problem 1.14:**

  Not Provided
- **Problem 1.15:**

  $x = \left(\dfrac{B}{A}\right)^{\frac{1}{b-a}}$
- **Problem 1.16:**

  (a) $x = 0, -1, 3$; (b) $x = 1$; (c) $x = -2, 1/3$; (d) $x = 1$.

- **Problem 1.17:**

  (a) Intersections $x = -1, 0, 1$.

- **Problem 1.18:**

  (a) $V$; (b) $\frac{V}{S} = \frac{1}{6}a, a > 0$; (c) $a = V^{\frac{1}{3}}$; $a = \left(\frac{1}{6}S\right)^{\frac{1}{2}}$; $a = 10$ cm; $a = \frac{\sqrt{15}}{3}$ cm.

- **Problem 1.19:**

  (a) $V$; (b) $\frac{r}{3}$; (c) $r = \left(\frac{3}{4\pi}\right)^{1/3} V^{1/3}$; $r = \left(\frac{1}{4\pi}\right)^{1/2} S^{1/2}$; $r \approx 6.2035$ cm; $r \approx 0.8921$ cm.

- **Problem 1.20:**

  $r = 2k_1/k_2 = 12\mu$m.

- **Problem 1.21:**

  (a) $T = \left(\frac{(1-a)S}{\varepsilon\sigma}\right)^{1/4}$.

- **Problem 1.22:**

  (a) $P = C\left(\frac{R}{A}\right)^{d/b}$; (b) $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$.

- **Problem 1.23:**

  (a) $a$: $Ms^{-1}$, $b$: $s^{-1}$; (b) $b = 0.2$, $a = 0.002$; (c) $v = 0.001$.

- **Problem 1.24:**

  (a) $v \approx K$, (b) $v = K/2$.

- **Problem 1.25:**

  $K \approx 0.0048$, $k_n \approx 77$ nM

- **Problem 1.26:**

  (a) $x = -1, 0, 1$ (b) 1 (c) $y_1$ (d) $y_2$.

- **Problem 1.27:**

  Line of slope $a^3/A$ and intercept $1/A$

- **Problem 1.28:**

  $K = 0.5$, $a = 2$

- **Problem 1.29:**

  Not Provided

- **Problem 1.30:**

  $m \approx 67$, $b \approx 1.2$, $K \approx 0.8$, $k_n \approx 56$

- **Problem 1.31:**

  $x = \left(\frac{R}{A}\right)^{\frac{1}{r-a}}$.

- **Problem 2.1:**

  $2^{35} \approx 32 \cdot 10^9$.

- **Problem 2.2:**

  $2^{64} \approx 1.6 \cdot 10^{19}$.

- **Problem 2.3:**

  $11.312, 0.7072, 0.03125$

- **Problem 2.4:**

  Not provided

- **Problem 2.5:**

  $\frac{dy}{dx} = C_3 \cdot 3^x$ and $C_3 = \ln(3) = 1.098$.

- **Problem 2.6-2.7 :**

  Not Provided

- **Problem 2.8:**

  (a) $5^{0.75} > 5^{0.65}$; (b) $0.4^{-0.2} > 0.4^{0.2}$; (c) $1.001^2 < 1.001^3$; (d) $0.999^{1.5} > 0.999^{2.3}$.

- **Problem 2.9:**

  Not Provided

- **Problem 2.10:**

  (a) $x = a^2 b^3$; (b) $x = \frac{b}{c^{\frac{2}{3}}}$.

- **Problem 2.11:**

  Not Provided

- **Problem 2.12:**

  (a) $x = \frac{3 - \ln(5)}{2}$; (b) $x = \frac{e^4 + 1}{3}$; (c) $x = e^{(e^2)} = e^{e \cdot e}$; (d) $x = \frac{\ln(C)}{a - b}$.

- **Problem 2.13:**

  (a) $\frac{dy}{dx} = \frac{6}{2x+3}$; (b) $\frac{dy}{dx} = \frac{6[\ln(2x+3)]^2}{2x+3}$; (c) $\frac{dy}{dx} = -\frac{1}{2}\tan\frac{1}{2}x$; (d) $\frac{dy}{dx} = \frac{3x^2 - 2}{(x^3 - 2x)\ln a}$; (e) $\frac{dy}{dx} = 6xe^{3x^2}$; (f) $\frac{dy}{dx} = -\frac{1}{2}a^{-\frac{1}{2}x}\ln a$; (g) $\frac{dy}{dx} = x^2 2^x (3 + x\ln 2)$; (h) $\frac{dy}{dx} = e^{e^x + x}$; (i) $\frac{dy}{dx} = \frac{4}{(e^t + e^{-t})^2}$.

- **Problem 2.14:**

  (a) min.: $x = \frac{2}{\sqrt{3}}$; max.: $x = -\frac{2}{\sqrt{3}}$; infl.pt.: $x = 0$; (b) min.: $x = \frac{1}{\sqrt[3]{3}}$; (c) max.: $x = 1$; inf.pt.: $x = 2$; (d) min.: $x = 0$; (e) min.: $x = 1$; max.: $x = -1$; (f) min.: $x = \ln(2)$; infl. pt.: $x = \ln(4)$.

- **Problem 2.15:**

  $C = 4$, $k = -0.5$

- **Problem 2.16:**

  (a) decreasing; (b) increasing; $y_1(0) = y_2(0) = 10$; $y_1$ half-life $= 10\ln(2)$; $y_2$ doubling-time $= 10\ln(2)$

- **Problem 2.17:**

  41.45 months.

- **Problem 2.18:**

  (c) $r = 0.0101$ per year, $C = 0.7145$ billions.

- **Problem 2.19:**

  Not provided

- **Problem 2.20:**

  $r_1 \approx -14.5, r_2 \approx -0.9$ per unit time, $C_1 = 55, C_2 = 45$.

- **Problem 2.21- 2.23:**

  Not Provided

- **Problem 2.24:**

  crit.pts.: $x = 0$, $x \approx \pm 1.64$; $f(0) = 1$; $f(\pm 1.64) \approx -0.272$

- **Problem 2.25:**

  (a) $x = 1/\beta$, (b) $x = \ln(\alpha)/\beta$.

- **Problem 2.26:**

  (a) $x = r$; (c) $x = \dfrac{ar}{a-r} \ln\left(\dfrac{R}{A}\right)$; (d) decrease; (e) decrease.

- **Problem 2.27:**

  $x = b\sqrt{\ln((a^2+b^2)/b^2)}$

- **Problem 2.28:**

  Not Provided

- **Problem 3.1:**

  Not Provided

- **Problem 3.2:**

  (a) $C$ any value, $k = -5$; (b) $C$ any value, $k = 3$.

- **Problem 3.3:**

  Not Provided

- **Problem 3.4:**

  (a) $y(t) = Ce^{-t}$; (b) $c(x) = 20e^{-0.1x}$; (c) $z(t) = 5e^{3t}$.

- **Problem 3.5:**

  $e^{2.08t}$.

- **Problem 3.6:**

  Not provided

- **Problem 3.7:**

  (a) $P(5) \approx 1419$; (b) $t \approx 9.9$ years.

- **Problem 3.8:**

  $\dfrac{dN}{dt} = 0.05N$; $N(0) = 250$; $N(t) = 250e^{0.05t}$; $2.1 \times 10^{10}$ rodents

- **Problem 3.9:**

  (a) $dy/dt = 2.57y$; (b) $dy/dt = -6.93y$.

- **Problem 3.10:**

  (a) 12990; (b) 30792 bacteria.

- **Problem 3.11:**

  1.39 hours; 9.2 hours

- **Problem 3.12:**

  (a) $y_1$ growing, $y_2$ decreasing; (b) 3.5, 2.3; (c) $y_1(t) = 100e^{0.2t}$, $y_2(t) = 10000e^{-0.3t}$; (d) $t \approx 9.2$ years.

- **Problem 3.13:**

  12265 people/km$^2$

- **Problem 3.14:**

  6.93 years

- **Problem 3.15:**

  (a) 1 hour; (b) $r = \ln(2)$; (c) 0.25 M; (d) $t = 3.322$ hours.

- **Problem 3.16:**

  20 min; 66.44 min

- **Problem 3.17:**

  $\tau = \dfrac{\ln(10)}{2}$

- **Problem 3.18:**

  (a) 57300 years; (b) 22920 years

- **Problem 3.19:**

  (a) 29 years; (b) 58 years; (c) 279.7 years.

- **Problem 3.20:**

  (a) 80.7%; (b) 12.3 years.

- **Problem 3.21:**

  1.7043 kg

- **Problem 3.22:**

  $y \approx 707.8$ torr

- **Problem 4.1:**

  Until $t = \dfrac{2\sqrt{h_0}}{k}$.

- **Problem 4.2:**

- **Problem 4.3-4.7:**

  Not Provided

- **Problem 4.8:**

  (a) $C = -12$; (b) $C_1 = 1, C_2 = -5$; (c) $C_1 = -1, C_2 = 0$.

- **Problem 4.9:**

  (a) $v(t) = -\dfrac{g}{k}e^{-kt} + \dfrac{g}{k}$; (b) $v = \dfrac{g}{k}$.

- **Problem 4.10:**

  $c(t) = -\dfrac{k}{s}e^{-st} + \dfrac{k}{s}$

- **Problem 4.11-4.12::**

  Not provided

- **Problem 4.13:**

  (b) 46 minutes before discovery.

- **Problem 4.14:**

  10.6 min

- **Problem 4.15:**

  (a) $Y = Y_0 - kt$, (d) $k = 0.0333$ per min.

- **Problem 4.16:**

  (b) $k = 3/2$.

- **Problem 4.17:**

  (a) Input rate $I$, $\alpha F$ fish caught per day. Birth and mortality neglected. (b) Steady state level $F = I/\alpha N$. (c) $2\ln(2)/\alpha N$ days. (d) $t = F_{low}/I$ days.

- **Problem 4.18:**

  Not provided

- **Problem 4.19:**

  64.795 gm, 250 gm

- **Problem 4.20:**

  (a) $Q'(t) = kr - \dfrac{Q}{V}r = -\dfrac{r}{V}[Q - kV]$; (b) $Q = kV$; (c) $T = V\ln 2/r$.

- **Problem 4.21:**

  (a) $\dfrac{dQ}{dt} = kQ$; $Q(t) = 100e^{(-8.9\times10^{-2})t}$; (b) 7.77 hr.

- **Problem 4.22:**

  (b) $y_0$; (c) $t = \dfrac{2A\sqrt{y_0}}{k}$; (d) $-k\sqrt{y_0}$.

- **Problem 4.23:**

  $a = 0$, $b = -1$

- **Problem 4.24:**

  $95.12, 90.48$.

- **Problem 4.25:**

  (a) $y_5 = 1.61051$; $y(0.5) = 1.6487213$; error $= 0.03821$; (b) $y_5 = 0.59049$; $y(0.5) = 0.60653$; error $= 0.01604$.

- **Problem 4.26-4.27:**

  Not Provided

- **Problem 5.1:** Not provided
- **Problem 5.2:**

(a) Steady state: $y = 0$. If $y(0) = 1$, $y \to 0$.
(b) Steady states: $y = 0, 2$. If $y(0) = 1$, $y \to 2$.
(c) Steady states: $y = 0, 2, 3$. If $y(0) = 1$, $y \to 2$.

- **Problem 5.3:**

(a) At $y = -\frac{2}{3}$, TL slope is 0. If $y < -\frac{2}{3}$, TL slope negative. If $y > \frac{2}{3}$, TL slope positive.
(b) Slope = 0 at $y = 0$ and $y = 2$. If $y < 0$, slope positive. If $0 < y < 2$, slopes negative. If $y > 2$, slopes are positive.
(c) Slope = 0 at $y = 1$ and $y = 2$. If $y < 1$, the slopes are positive. If $1 < y < 2$, the slopes are negative. If $y > 2$, the slopes are positive.
(d) Slope = 0 at $y = 3$. If $y < 3$, slopes are negative. If $y > 3$, slopes are also negative.
(e) $y(t)$ increases for all real $t$.
(f) The slopes are zero at $y = -1, 0, 1$. If $y < -1$, the slopes are negative. If $-1 < y < 0$, the slopes are positive. If $0 < y < 1$, the slopes are negative. If $y > 1$, the slopes are positive.
(g) If $0 < y < 2$, the slopes are negative. If $2 < y < 3$, the slopes positive. If $y > 3$, the slopes are positive.

- **Problem 5.4:**

  Linear differential equations: only (a)

- **Problem 5.5:**

  Not Provided

- **Problem 5.6:**

  (B)

- **Problem 5.7:**

  (B)

- **Problem 5.8:**

  $h(t) \to (I/K)^2$.

- **Problem 5.9:**
  (a) $\dfrac{dx}{dt} = \dfrac{k}{3}(V_0 - x^3)$; (d) $V = \dfrac{1}{2}V_0$.
- **Problem 5.10:**

  Not provided

- **Problem 5.11:**

  (a) $K_{\max}$, $c = k$; (b) $\ln(2)/r$; (c) $c = 0$, $c = \dfrac{K_{\max}}{r} - k$.

- **Problem 5.12-5.13:**

  Not provided

- **Problem 5.14:**

  $dy/ds = y(1-y)$

- **Problem 5.15:**

  0.55, 0.5995, 0.6475, 0.6932, 0.7357

- **Problem 5.16:**

  Not provided

- **Problem 5.17:**

  $\phi = 0.857 = 86\%$.

- **Problem 5.18:**

  (c) Steady states at at $y_2 = 0$ and $y_2 = P - a/b$. (d) Social media persists if $Pb/a > 1$.

- **Problem 5.19:**

  (b) Stable steady state at $a = \dfrac{\beta}{2\mu}\left(-1 + \sqrt{1 + 4\mu M/\beta}\right)$

### Solution to 6.1

(a) $a_5 = 4$ (b) 11terms (c) $1+2+2^2+2^3+2^4$

(d) $1+2+2^2+2^3+2^4$ (e) $\sum_{n=0}^{3} 3^n$

### Solution to 6.2

(a) $\sum_{n=1}^{\infty} 2n$ (b) $\sum_{n=1}^{\infty} \frac{1}{n}$

(c) $1+3+9+27+81+243+\dots$ (d) $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\dots$

(e) $\sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^k$ (f) $\sum_{x=0}^{9} 3^x$

(g) $\sum_{n=0}^{100} (n+n^2)$ (h) $\sum_{y=0}^{100} (y+1)^2$

### Solution to 6.3

(a) each sum represents $1+2^2+3^2+4^2+5^2+...+11^2$

(b) each sum represents $0+1+4+9$

### Solution to 6.4

(a) 290 (b) 300 (c) 240 (d) 1275 (e) 1830
(f) 1785 (g) 4860 (h) 16575 (i) 5740 (j) 1650
(k) 2925 (l) 120300 (m) 3825 (n) 40425 (o) 40411
(p) 42075 (q) 18496

### Solution to 6.5

(a) 2550 (b) 2500 (c) 44200 (d) 41650

**Solution to 6.6** $S = 7618$

**Solution to 6.7** The clock chimes 324 times.

**Solution to 6.8** The total enclosed volume is 14400 cubic inches.

**Solution to 6.9** 12 frames can be made enclosing a total of 9200 cm$^2$

### Solution to 6.10

(a) 39601 (b) 1 (c) $2,686,700$

**Solution to 6.11**

(a) $\dfrac{k(k+1)}{2}$ (b) $10, 20, 35$ (c) $\dfrac{N(N+1)(N+2)}{6}$ (d) $171700$

**Solution to 6.12**

(1)(a) The radius of the smallest disk is $r_1 = \dfrac{r}{N}$.

(b) The radius of the $k$-th disk is $r_k = \dfrac{kr}{N}$.

(2) $V = \pi \sum_{k=1}^{N} \left(\dfrac{kr}{N}\right)^2 \dfrac{h}{N} = \pi r^2 h \dfrac{2N^2 + 3N + 1}{6N^2}$.

(3) The volume converges to $V \to \dfrac{1}{3}\pi r^2 h$.

**Solution to 6.13**

(a) $k$ (b) $2047$ (c) $2\left(1 - \dfrac{1}{2^{11}}\right)$

**Solution to 6.14**

(a) Approximately $18.531, 64.002, 181.943, 487.852$, and $1281.299$, respectively.

(b) Approximately $6.862, 8.906, 9.618, 9.867$, and $9.954$, respectively.

(c) In (a), for $|r| > 1$, the sums increase (rapidly) with $N$ whereas in (b), for $|r| < 1$, they increase only very slowly.

(d) For $N \to \infty$, the sum converges for $|r| < 1$ and diverges otherwise.

**Solution to 6.15** If there had been enough grain on Earth, the inventor would have received $2^{64} - 1 \approx 1.845\ 10^{19}$ grains.

**Solution to 6.16** The total length of segments is 2047 units.

**Solution to 6.17**

(a) $\left(\dfrac{4}{5}\right)^{12} \ell_0$ (b) $\dfrac{1 - \left(\frac{4}{5}\right)^{13}}{1 - \frac{4}{5}} \ell_0$ (c) $5\ell_0$

**Solution to 6.18**

(a) $F_n = \dfrac{2}{r_0 \beta^n}$ (b) $\beta < 1$

**Solution to 6.19**

(a) 2097151 branches; volume 105.6 $cm^3$; surface 9950.57 $cm^2$

(b) $0.625 < \beta < 0.7905$

**Solution to 6.20**

(a) $A = 4.829\,\text{cm}^2$ (b) $A = 5.36\,\text{cm}^2$

**Solution to 7.1**

(a) 55 (b) 50.5 (c) 50

**Solution to 7.2**

(a) $\frac{1}{2}$ (b) $\lim_{N\to\infty} 1 - \frac{1}{2} + \frac{1}{2N^2} = \frac{1}{2}$

**Solution to 7.3** $2470 < A < 2870$

**Solution to 7.4**

(a) 8 (b) 8.375 (c) 8.75

(d) 8.9375 (e) $(d)$

**Solution to 7.5** $A = \frac{1}{3}$

**Solution to 7.6**

(a) $\approx 1.51$ (b) $\approx 1.942$ (c) (a) and (b) are under- and overestimates

**Solution to 7.7**

(a) $A = 2$ (b) $A_n = 4 - 2\frac{n+1}{n}, \lim_{n\to\infty} A_n = 2$

**Solution to 7.8** $A = \frac{8}{3}$

**Solution to 7.9** $\int_0^3 \sqrt{1+x}\,dx$ (other solutions possible)

**Solution to 7.10**

$$A(x) = (a+1)^2(b-a) + (a+1)(b-a)^2 + \frac{(b-a)^3}{3}$$

**Solution to 7.11**

(a) $\frac{1}{2}b(h_1+h_2)$ (b) 20.5

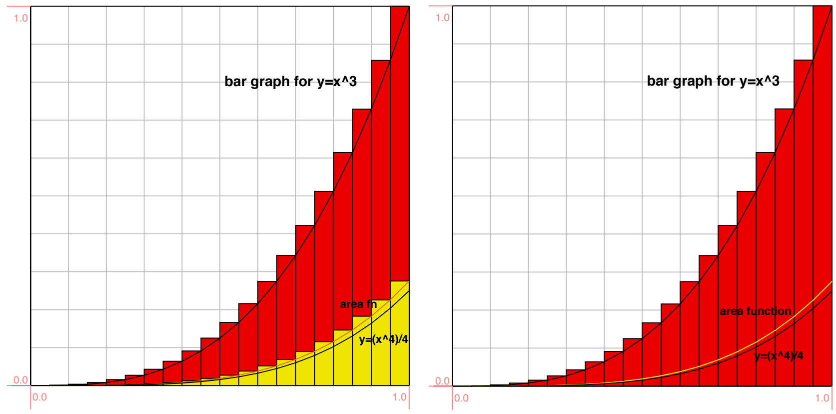



Figure 12.7: Two acceptable solutions to problem 7.12.

### Solution to 7.12

### Solution to 7.13

(a) 1 (b) 2 (c) $2\pi$

### Solution to 7.14

(a) $x_k = \dfrac{3k}{N}$ (b) $\left(\dfrac{3}{N}\right)^4 k^3$ (c) $\dfrac{3^4}{4}\left(\dfrac{N+1}{N}\right)^2$ (d) 182.25

### Solution to 7.15 $A = 16.25$

### Solution to 7.16

(a) $\dfrac{2w}{L^2}$ (b) $\dfrac{HL^3}{6}$ (c) $\dfrac{wL}{3}$

### Solution to 7.17

(a) $\dfrac{2}{3}\left(1-\left(\dfrac{2}{3}\right)^{\frac{1}{2}}\right)$ (b) $\dfrac{1}{6}$

### Solution to 8.1

(a) The Fundamental Theorem of Calculus states that if $F(x)$ is any anti-derivative of $f(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$

(b) the Fundamental Theorem of Calculus provides a shortcut for calculating integrals. Without it, we would have to use tedious summation and limits to compute areas, volumes, etc. of irregular regions.

### Solution to 8.2 $e - 1$

### Solution to 8.3

(a) 1 (b) 2

### Solution to 8.4

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| (a) | 2 | (b) | $2-\sqrt{2}$ | (c) | 1 | (d) | $2\sqrt{2}$ | (e) | $\frac{1}{2}$ |
| (f) | $-2$ | (g) | $\frac{8}{3}$ | (h) | $\frac{124}{3}$ | (i) | $\frac{16}{3}$ | (j) | 2 |
| (k) | $\frac{23}{3}$ | (l) | $3\ln 2$ | (m) | $2\ln(3)$ | (n) | $2(e-1)$ | (o) | $\frac{3}{4}\left(3\cdot 3^{1/3}-2\cdot 2^{1/3}\right)$ |
| (q) | DNE | (q) | DNE | (r) | 2 | (s) | DNE | (t) | DNE |

DNE: does not exist.

### Solution to 8.5

| | | | | | |
|---|---|---|---|---|---|
| (a) | $\frac{1}{k}\left(e^{kx}-e^{ka}\right)$ | (b) | $\frac{A}{k}\sin(kx)$ | (c1) | $\frac{C}{m+1}\left(x^{m+1}-b^{m+1}\right),\ m\neq -1$ |
| (c2) | $C(\ln(x)-\ln(b)),\ m=-1, b\neq 0$ | (d) | DNE | (e) | $\frac{1}{5}(\tan(5T)-\tan(5c))$ |
| (f) | $2\arctan(x)-\frac{\pi}{2}$ | (g) | $3\left(\frac{1}{b}-\frac{1}{x}\right)$ | (h) | $2(\sqrt{T}-\sqrt{a}),\ T>0, a>0$ |
| (i) | $\frac{1}{3}(1-\cos(3x))$ | (j) | $3x-3b$ | | |

DNE: does not exist.

### Solution to 8.6

(a) $\frac{2}{3}N^{3/2}$ (b) see Figure 12.8

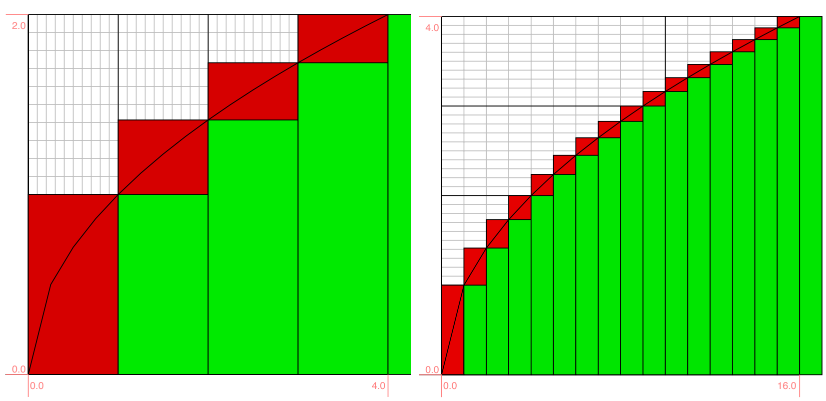



Figure 12.8: Solution to problem 8.6 for $N=4$ and $N=16$.

### Solution to 8.7

(a) $\ln 3$ (b) $\frac{aT^2}{2}$ (c) $\frac{255}{64}$

**Solution to 8.8** $S = \frac{7}{6}$

**Solution to 8.9** $S = \frac{8}{3}$

**Solution to 8.10**

(a) $\frac{n-1}{2(n+1)}$ (b) twice the area of (a).

**Solution to 8.11**

(a) $\frac{10}{3}$kg (b) $\frac{20}{3}$kg

**Solution to 8.12** $S = \frac{1}{3}$

**Solution to 8.13** $S = \frac{64}{3}$

**Solution to 8.14** $A = 2$

**Solution to 8.15** $A = 8.5$

**Solution to 8.16**

(a) $2, 5, 7, 3$ (b) $[0, 3]$ (c) $3$

**Solution to 8.17** For solution, see Figure 12.9.

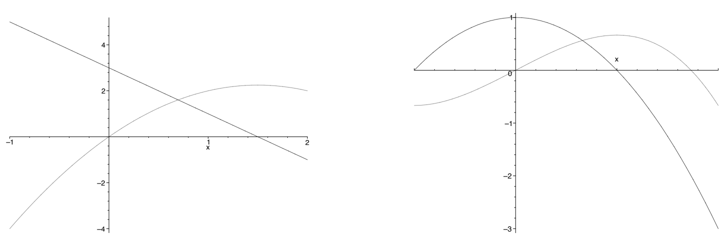


Figure 12.9: Solution to problem 8.17

**Solution to 8.18** For solution, see Figure 12.10.

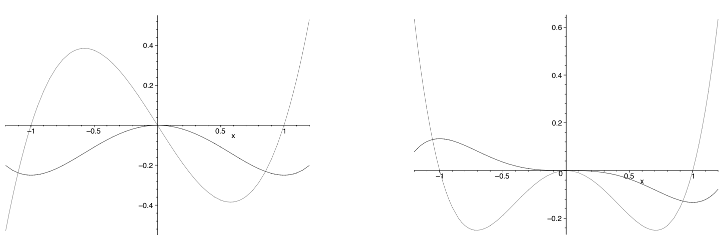


Figure 12.10: Solution to problem 8.18

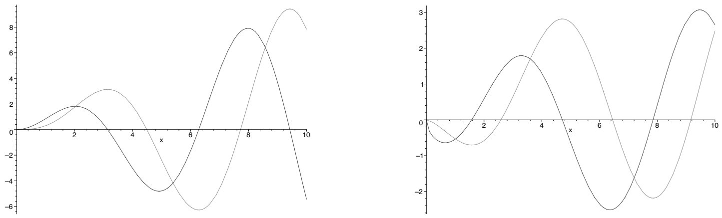


Figure 12.11: Solution to problem 8.19

**Solution to 8.19** For solution, see Figure 12.11.

**Solution to 8.20**

(a) $\frac{1}{6}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

**Solution to 9.1**

(a) At time $t \approx 1.1$.

(b) Whenever slopes are equal, e.g. at time $t \approx 3.3$.

(c) Car 1.

(d) Car 2.

(e) At time $t \approx 2.2$.

**Solution to 9.2**

(a) $a(t) = 4t + 5$ (b) $\frac{19}{6}$

**Solution to 9.3**

(a) $k = 1$ (b) $10(e^{-t} + t - 1)$

**Solution to 9.4**

(a) $v(t) = \frac{t^2}{2} - \frac{t^3}{3}$ (b) $\frac{1}{12}$

**Solution to 9.5**

(a) $23\frac{1}{3}$km (b) 32.51

**Solution to 9.6** $\frac{143}{36}$

**Solution to 9.7**

(a) $\frac{34}{9}$s (b) $\frac{5375}{81}$m

**Solution to 9.8**

(a) child2 (b) child1

**Solution to 9.9** $A\left(\frac{23}{3}-\frac{3}{20}\cos(0.15)\right)$

**Solution to 9.10** 12000\$

**Solution to 9.11** 7.5 mg

**Solution to 9.12**

(a) $t=5\sqrt{2}$h (b) $\frac{100\sqrt{2}}{3}\approx 471.4\cdot 10^3$ barrels

**Solution to 9.13**

(a) $t=10$ h (b) $t_2=30$ h (c) $\frac{160}{3}10^5$ gal

**Solution to 9.14** $0.1\left(765+\frac{240}{\pi}\right)\approx 84.14$ t

**Solution to 9.15**

(a) $r=\ln 2$ (b) $R(t)=0.5e^{-t\ln 2}$ (c) $\approx 0.7213$ rad

**Solution to 9.16**

(a) $\int_0^t I(s)-C(s)\,ds$ (b) $\approx 1.5$ min (c) $\approx 3$ min (d) $\approx 2.5$ min

**Solution to 9.17**

(a) $Q(T)=Q(0)+\int_0^T I(t)-O(t)\,dt$ (b) $A$ max., $B$ min. (see Figure 12.12)

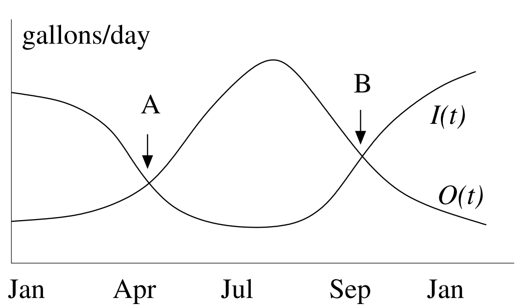



Figure 12.12: Solution for problem 9.17.

**Solution to 9.18**

(a) $P(t) = P_0 + \int_0^t I(s) - O(s)\,ds$ (b) $t \approx 2$ greatest, $t \approx 5$ smallest

**Solution to 9.19**

(a) 18cm (b) $\frac{18}{5}$cm/day

**Solution to 9.20** $\bar{f} = \frac{2}{\pi}$

**Solution to 9.21**

(a) $\bar{f} = \frac{1}{n+1}$, for $n \neq -1$.

(b) $\bar{f} \to 0$.

(c) area under curve decreases.

(d) $\bar{g} = \frac{n}{n+1}$, for $n \neq -1$; $\bar{g} \to 1$; area under curve approaches 1.

**Solution to 9.22**

(a) $\bar{v} = c\left(1 + \frac{e^{-t}}{t} - \frac{1}{t}\right)$ m/s (b) $\bar{v} \to c$m/s

**Solution to 9.23**

(a) 0 (b) 0 (c) 0

(d) odd functions (symmetric about origin) (e) 0 (f) 2

**Solution to 9.24**

(a) $\bar{I} = \frac{1175}{3}$ (b) $\bar{I} = \frac{1151}{3}$ (c) $b = \sqrt{1170}$

**Solution to 9.25** $24\frac{\sqrt{2}}{\pi}$

**Solution to 9.26**

(a) Equal length of day and night, i.e. 12 h, and hence $t = 0$ (spring) or $t = 182$ (fall).

(b) Longest day has 16 h at $t = 91$; shortest day has 8 h at $t = 273$.

(c) 364

(d) See Figure 12.13.

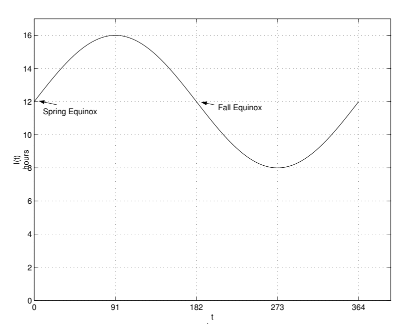


Figure 12.13: Solution for problem 9.26 (d)

(e) $12 + \frac{364}{15\pi}\left(1 - \cos\left(\frac{15\pi}{91}\right)\right) \approx 13$ h.

(f) 12 h for 364 days; 12.0009 h for 365 days.

**Solution to 9.27**

(a) $\varphi = \frac{\pi}{4}, A = \sqrt{2}, \omega = 2$

(b) $\bar{f} = \frac{1}{\pi}\int_0^{\pi} \sin(2t) + \cos(2t)\, dt.$

(c) $\bar{f} = 0$; equal areas above and below $x$-axis.

**Solution to 9.28**

(a) $P(t) = \frac{A^2}{2}(1 + \cos(2\omega t))$

(b) Because $P(t)$ is the square of $I(t)$ it must be $\geq 0$. $I(t)$ and $P(t)$ have the same zeroes. Maximal and minimal values: $-A \leq I(t) \leq A$, $0 \leq P(t) \leq A^2$.

(c) $\bar{P} = \frac{A^2}{2}$ W, $\bar{I} = 0$ A.

**Solution to 9.29**

(a) $\frac{dS}{dt} = P - fN_0e^{rt}$, $S(t) = Pt - \frac{f}{r}N_0(e^{rt} - 1)$.

(b) $t = \frac{1}{r}\ln\left(\frac{P}{fN_0}\right)$.

(c) $t_{\max} = \frac{1}{r}\ln\left(\frac{P}{fN_0}\right)$ (see (b)).

(d) $S_{\max} = \frac{P}{r}\left(\ln\left(\frac{P}{fN_0}\right) - 1\right) + \frac{f}{r}N_0$.

**Solution to 10.1**

(a) see Figure 12.14 (b) $\frac{1}{3}$

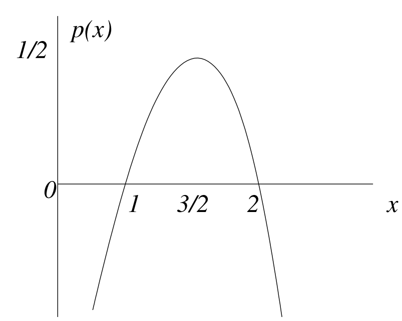


Figure 12.14: Solution for problem 10.1

**Solution to 10.2** 123873.71 kg

**Solution to 10.3** $20\pi$ g

**Solution to 10.4**

(a) $x = \frac{\pi}{2a}$ (b) $\bar{p} = \frac{2A}{\pi}$ (c) $\bar{x} = \frac{\pi}{2a}$

**Solution to 10.5** $N = \frac{20000}{\ln 2} \approx 28854$

**Solution to 10.6**

(a) straight line (b) $y = x$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{3}$

**Solution to 10.7** $V = \frac{\pi}{3}$

**Solution to 10.8** $V = 2\pi$

**Solution to 10.9** $V = 2\pi\left(R^2 - \frac{2Rp}{3} + \frac{p^2}{5}\right)$

**Solution to 10.10** $V = \frac{\pi}{2}$

**Solution to 10.11**

(a) $V = 7.5\pi$ (b) $V = \frac{64}{15}\pi$ (c) $\frac{\pi}{7}(62 + \frac{1}{10^7})$

**Solution to 10.12**

(a) $V = 2\int_0^{\frac{\pi}{4}} \pi\cos(2x)\,dx.$

(b) $V = 2\int_0^{\frac{\pi}{4}} \pi\left((1+\cos(x))^2 - (1+\sin(x))^2\right)\,dx.$

**Solution to 10.13** $16000\pi C\left(20 - \frac{40}{3}\right) \approx 335103C$

**Solution to 10.14** $V = \frac{w^2h}{3}$

**Solution to 10.15**

(a) $L = \int_0^{2\pi} \sqrt{1+\cos^2(x)}\,dx.$

(b) $L = \int_0^{1} \sqrt{1+\frac{1}{4x}}\,dx.$

(c) $L = \int_{-1}^{1} \sqrt{1+n^2x^{2n-2}}\,dx.$

**Solution to 10.16** $L = 2\sqrt{5}$

**Solution to 10.17** $W = 128$ dynes·cm

**Solution to 10.18** $W = 980\pi \cdot 10^3\left(\frac{1}{2} - \frac{1}{3}\right) \approx 5.131 \times 10^5$ g cm$^2$/s$^2$

### Solution to 11.1

(a) $d(e^{x^2}) = 2xe^{x^2}\,dx$  (b) $d((x+1)^2) = 2(x+1)\,dx$

(c) $d(\sqrt{x}) = \frac{1}{2}x^{-1/2}\,dx$  (d) $d(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}\,dx$

(e) $d(x^2+3x+1) = (2x+3)\,dx$  (f) $d(\cos(2x)) = -2\sin(2x)\,dx$

(g) $dy = (6x^5+8x^3-2)\,dx$  (h) $dy = (x-2)(x+1)^4(7x-8)\,dx$

(i) $dy = \frac{3}{(x+3)^2}\,dx$

### Solution to 11.2

(a) $\frac{x^4}{4}+C$  (b) $\frac{2}{5}x^{5/2}+C$  (c) $1\frac{1}{2x^2}$

(d) $\frac{2}{3}x^{3/2}+C$  (e) $2\sqrt{x}+C$  (f) $-\frac{e^{-2x}}{2}+C$

(g) $\frac{(x+1)^3}{3}+C$  (h) $-\frac{1}{x+3}+C$  (i) $\frac{\sin(2x)}{2}+C$

(j) $\tan x+C$  (k) $\ln x+C$  (l) $\arctan x+C$

(m)* $e^{x^2}+C$  (n)* $\sin(x^3)+C$  (o)* $\ln(1+x^2)+C$

(p)* $\arctan(x^2)+C$

$C$: integration constant.

### Solution to 11.3

(a)

(i) $-\frac{\cos(3x)}{3}+C$  (ii) $2\sin(\sqrt{x})+C$

(iii) $\frac{1}{6}(x^4+1)^{3/2}+C$  (iv) $2\ln|1+2x|+C$

(b)

(i) $\approx 13.89$  (ii) $\frac{1}{2}$  (iii) $0$

### Solution to 11.4

(a) $\arctan(p)-\frac{\pi}{4}$  (b) $\frac{1}{21}(x^3+1)^7+C$  (c) $\frac{1}{3}(1+2e^x)^{3/2}+C$

(d) $\ln|\ln|x||+C$  (e) $-\ln|1-y|+C$  (f) $k_1\ln\left|\frac{k_2-1}{k_2-S}\right|$

(g) $-\frac{1}{3}\ln\left|1-x^3\right|+C$  (h) $\frac{2}{9}(3x+1)^{3/2}+C$  (i) $\sqrt{4+x^2}+C$

(j) $\frac{1}{6}\sin^6 x+C$  (k) $\frac{3}{5}\ln|4+5x|+C$  (l) $\ln|\sin(\theta)|+C$

(m) $2(2+\tan x)^{1/2}+C$  (n) $\arctan(\frac{x}{2})+C$

**Solution to 11.5** $\frac{1}{2}\sin^2 x + C$

**Solution to 11.6**

(a) substitute $x = \sin(u)$.

(b) $I = \frac{1}{2}\int (1+\cos(2u))\,du \quad \Rightarrow \quad I = \frac{1}{2}(\arcsin x + x\sqrt{1-x^2})\big|_0^1 = \frac{\pi}{4}$.

(c) $f(x) = \sqrt{1-x^2}$ forms a half-circle of radius 1 for $-1 < x < 1$.

**Solution to 11.7**

(a) $\frac{1}{9}\ln\left|\frac{x-5}{x+4}\right| + C$ (b) $-\frac{3}{x+3} + C$

(c) $-\frac{1}{\sqrt{10}}\arctan\left(\frac{x+2}{\sqrt{10}}\right) + C$ (d) $\ln\left|\frac{x-4}{x-2}\right| + C$

**Solution to 11.8**

(a) $\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$ (b) $\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

**Solution to 11.9** Set $u = x^n$ and $dv = e^x$.

**Solution to 11.10**

(a) $-2\sqrt{x}\cos\left(\sqrt{x}\right) + 2\sin\left(\sqrt{x}\right) + C$ (b) $2e^2$

(c) $-\frac{\pi}{4} - \frac{1}{2}$ (d) $\frac{1}{2}e^{x^2}\left(x^4 - 2x^2 + 2\right) + C$

**Solution to 11.11**

(a) $\frac{3}{\sqrt{2}}\arctan\left(\frac{x+2}{\sqrt{2}}\right) + C$ (b) $2\ln\left|\frac{x+1}{x+2}\right| + C$ (c) $\frac{1}{2}\ln\left|\frac{x+2}{x+4}\right| + C$

(d) DNE (e) $\frac{1}{4}\left(1 - 2Te^{-2T} - e^{-2T}\right)$ (f) 4

(g) $\ln|x|\frac{x^3}{3} - \frac{x^3}{9} + C$

DNE: does not exist.

### Solution to 11.12

(1) $\dfrac{\ln(x)^2}{2}+C$

(2) $\dfrac{2}{3}\ln|2+3x|+C$

(3) $-\dfrac{1}{9}\cos(3x^3+1)+C$

(4) $\dfrac{1}{3}\left(e^{12}-e^3\right)$

(5) $\dfrac{1}{2}\ln 2$

(6) $\dfrac{1}{\sqrt{3}}\arcsin\left(\dfrac{\sqrt{6}}{2}x\right)+C$

(7) $\dfrac{1}{\sqrt{3}}\arctan\left(\dfrac{2x}{\sqrt{3}}\right)+C$

(8) $\dfrac{1}{2}\arcsin\left(\dfrac{2x}{5}\right)+C$

(9) $\sqrt{9+x^2}+C$

(10) $-\dfrac{1}{6}(\sin^2(3x)-2\sin(3x))+C$

(11) $\dfrac{\sin(2x)}{2}+\dfrac{\cos(4x)}{8}+C$

(12) $\dfrac{2}{3}(\tan x+1)^{3/2}+C$

(13) $\dfrac{1}{3}$

(14) $x^2+x\cos(2x)-\dfrac{\sin(2x)}{2}+C$

(15) $\sec x+C$

(16) $-x\cos(x+1)+\sin(x+1)+C$

(17) $\dfrac{x^6}{6}+C$

(18) $\tan t+C$

(19) $-\dfrac{1}{2(x+3)}+C$

(20) $\dfrac{1}{6}\ln\left|\dfrac{x-5}{x+1}\right|+C$

(21) $-\dfrac{1}{7}\ln\left|\dfrac{x+3}{2x-1}\right|+C$

(22) $\ln\left|\dfrac{x+2}{x+3}\right|+C$

(23) $-\dfrac{1}{x^3+x}+C$

(24) $\ln\left|\dfrac{x-2}{x-1}\right|+C$

(25) $\dfrac{2\sqrt{51}}{17}\arctan\left(\dfrac{1}{\sqrt{51}}(2x+3)\right)+C$

(26) $x\tan x+\ln|\cos x|+C$

(27) $\dfrac{1}{2}e^{x^2}+C$

(28) $e^x(x^2-2x+2)+C$

(29) $x\arctan x-\dfrac{1}{2}\ln(1+x^2)+C$

(30) $\dfrac{e^x}{2}(x\cos x+x\sin x-\sin x)+C$

### Solution to 12.1

(a) $y(t)=10e^{-t^2}$ (b) $y(t)=\dfrac{1}{1+e^{-t}}$

### Solution to 12.2

(a) $y(t)=1-Ce^{-t}$ (b) $C=\dfrac{1}{2}$

### Solution to 12.3

(a) $y(t)=\tan(t+C)$ (b) $C=\arctan\left(\dfrac{1}{2}\right)$

**Solution to 12.4** $y = \dfrac{3e^{2t}-1}{3e^{2t}+1}$

**Solution to 12.5** $y(t) = \left(\frac{1}{3}kt + y_0^{1/3}\right)^3$

**Solution to 12.6** $T = \dfrac{2\sqrt{h_0}}{k}$

**Solution to 12.7**

(a) $x(t) = \sin t$ (b) $x = 1$ (c) $v = \sqrt{1-x^2}$, $x_{\max} = 0$, $x_{\min} = 1$

**Solution to 12.8**

(a) $N(t) = N_0 e^{-m_0 t - rt^2/2}$ (b) $N(t) = N_0 e^{-(m_0+b)t - rt^2/2}$

**Solution to 12.9**

(a) $n(t) = \frac{-Ce^{-(k_1+k_2)t}+k_1}{k_1+k_2}$, $C = k_1 - n_0(k_1+k_2)$,

(b) $n(t) \to \frac{k_1}{k_1+k_2}$,

(c) $t = -\frac{\ln(0.4)}{1.2}$.

**Solution to 12.10**

(a) see Figure 12.15, $v^* = \dfrac{g}{k}$ (b) $v$ decreases/increases towards $v^*$

(c) $v(t) = \dfrac{g}{k} - \left(\dfrac{g}{k} - v_0\right)e^{-kt}$

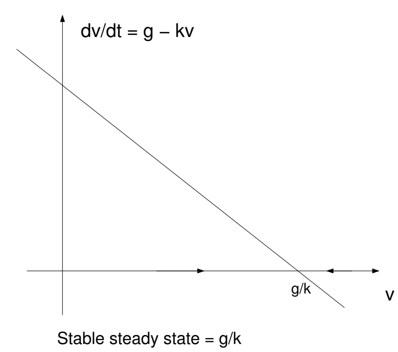


Figure 12.15: Plot for problem 12.10

**Solution to 12.11**

(a) $v_\infty = 3$ m/s (b) $T = \dfrac{1}{6}\ln 3 \approx 0.183$ s

**Solution to 12.12** $T(t) = E + (T_0 - E)e^{-kt}$. For $t \to \infty$, $T \to E$ regardless of initial temperature.

**Solution to 12.13** $\dfrac{dV}{dt} = \alpha S = \alpha 4\pi \left(\dfrac{3V}{4\pi}\right)^{2/3} = kV^{2/3}$.

Solution: $V(t) = (\frac{1}{3}kt + V_0^{1/3})^3$ (c.f. problem 12.5).

**Solution to 12.14**

(a) constant rates; no search/harvest efforts (b) $\dfrac{dF}{dt} = \alpha - \beta F$

(c) $F(t) = (\alpha - Ce^{-\beta t})/\beta$ (d) $F = \dfrac{\alpha}{\beta}$

**Solution to 12.15** $N(t) = N_0 e^{-\frac{m_0}{\mu}(e^{\mu t}-1)}$

**Solution to 12.16**

(a) $N(t) = 2(t+1)$ (b) $N(t) = \dfrac{2(t+1)}{t-1}$

(c) $C(t+1) \neq 1$ must hold – true because $0 \leq C = 1 - \dfrac{1}{N_0} < 1$