2018-10-09 Linear Algebra Review (3.1)

3.1 Linear Algebra Review

Recap from Last Day

<u>Consider the differential equation y'=f(y)</u>

1.What are the equilibrium solutions?

-1, 0, 2

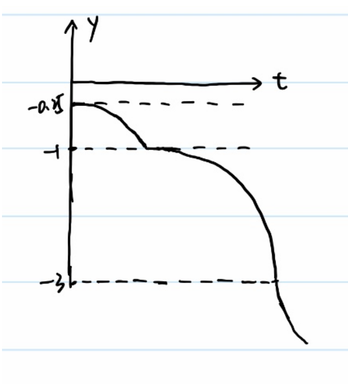
2.Which equilibrium solutions are stable/semi-stable/unstable?

Semi-stable: -1,2

Unstable: 0

3.y(0)=2.5

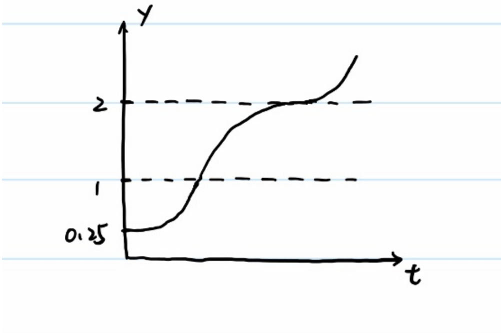
4.y(0)=-0.25



** the graph above touches -1 (the equilibrium) since on the y' by y graph it is discontinuous there so we don't know what happens to the solution. Therefore, the graph can touch.

**an asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity.

5.y(0)=0.25



3.1 Linear Algebra Review

Eigenvector - Eigenvalue

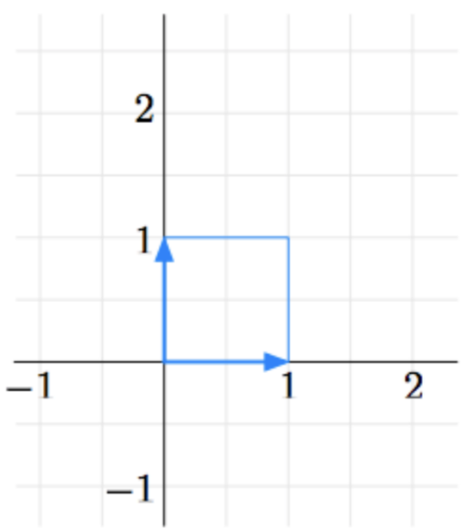
For a linear transformation T, an eigenvector for T is a non-zero vector that doesn't change direction when T is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector of T if

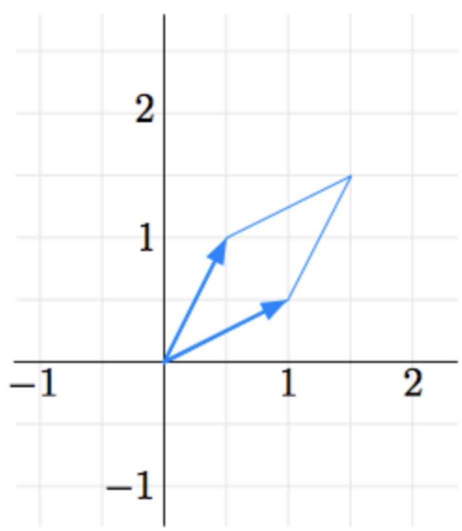
$$T\vec{v} = \lambda\vec{v}$$

for some scalar λ. We call λ the eigenvalue of T corresponding to the eigenvector $\vec{v}$.

Eigenvector-Eigenvalue: $T\vec{v} = \lambda\vec{v}$

The picture shows what the linear transformation T does to the unit square.





1. Give an eigenvector for T. What is the eigenvalue?

Suppose we choose $\vec{v}$ = (1, 2) = $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$T\vec{v}$ = $T\begin{bmatrix}1\\2\end{bmatrix}$ = $T(\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}0\\1\end{bmatrix})$ =$T\begin{bmatrix}1\\0\end{bmatrix}+2T\begin{bmatrix}0\\1\end{bmatrix}$= $\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$ + 2 $\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}$ = $\begin{bmatrix}2\\\frac{5}{2}\end{bmatrix}$

Since $\begin{bmatrix}2\\\frac{5}{2}\end{bmatrix}$ cannot be a scalar multiple of $T\vec{v}$, $\begin{bmatrix}1\\2\end{bmatrix}$cannot be an eigenvector for T.

Then, we choose $\vec{v}$ = $\begin{bmatrix}1\\1\end{bmatrix}$:

$T\vec{v}$ = $T\begin{bmatrix}1\\1\end{bmatrix}$ = $\begin{bmatrix}1.5\\1.5\end{bmatrix}$ = $1.5\times\begin{bmatrix}1\\1\end{bmatrix}$ = 1.5 $\vec{v}$

Given $\vec{v}$ = $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for T, the corresponding eigenvalue is λ = 1.5

NOTE:

If $v^{->}$ is an eigenvector of T, then T$v^{->}$ is parallel to $v^{->}$.

From the figure above, we first get the eigenvector then its corresponding eigenvalue.

Whereas for $det(A-\lambda I)=0$ and $(A-\lambda I)\vec{v}=0$ calculations, we usually get the eigenvalue first and then its corresponding eigenvector.

Using this method:

$$det(\begin{bmatrix}1 & 0.5\\0.5 & 1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix})=det(\begin{bmatrix}1-\lambda & 0.5\\0.5 & 1-\lambda\end{bmatrix})=(1-\lambda)(1-\lambda)-0.5^2=\lambda^2-2\lambda+0.75$$

eigenvalues: $\lambda_1=1.5,\ \lambda_2=0.5$

Use this to find the corresponding eigenvectors for the eigenvalues: $[A-I\lambda]\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$\begin{bmatrix}-0.5 & 0.5\\0.5 & -0.5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$ -0.5x+0.5y=0, y=x

$$The\ corresponding\ eigenvector\ for\ \lambda_1\ is\ \overrightarrow{v_1}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$The\ corresponding\ eigenvector\ for\ \lambda_2\ is\ \overrightarrow{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. Can you find another?

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is another eigenvector for T, which is perpendicular to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For some matrix A, $A \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \frac{2}{3} \end{bmatrix}$.

3. Give an eigenvector and corresponding eigenvalue for A.

$\vec{v}$ = $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$, the corresponding eigenvalue is λ = $\frac{2}{3}$.

or $\vec{v}$ = $\begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix}$, the corresponding eigenvalue is λ = $\frac{2}{3}$. (This solution should be wrong since v is a scalar multiple of the vector $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ by multiplying a constant k = 1/3. This is also the reason why that the resulting λ = $\frac{2}{9}$ is a scalar multiple with a constant k = 1/3 of the eigenvalue λ = $\frac{2}{3}$ as well. Essentially, $\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ \frac{1}{3} \end{bmatrix}$ are the same $\vec{v}$ with the same λ, which is consistent with "**no eigenvectors can have two distinct eigenvalues".**)

Review on eigenvectors and eigenvalues:

1. Except for zero vector, **no eigenvectors can have two distinct eigenvalues**.

Suppose that, $A \cdot x = \lambda_1 x \ A \cdot x = \lambda_2 x$

where λ_1 and λ_2 are two arbitrary distinct numbers, x is not a zero vector

The System of Equation then becomes: $A \cdot x = \lambda_1 x$ [1]

$$A \cdot x = \lambda_2 x \quad [2]$$

We subtract equation [2] from equation [1]

We get $$0 = (\lambda_1 - \lambda_2) \cdot x$$

Since x is not a zero vector, then λ_1 - λ_2 should equal to zero, which proves that one non-zero eigenvector cannot have two distinct eigenvalues.

2. A scalar multiple of a vector (eigenvector) is still the vector(eigenvector) itself. The scalar multiple only changes its magnitude but not its direction.

NOTE:
(more information on eigenvectors and eigenvalues from the video)

- There could be no eigenvalues.
  For example, for a matrix [0 -1]
  [1 0]
  we get $\lambda^2 + 1 = 0$
  => $\lambda = i$ (or) $\lambda = -i$
- For a diagonal matrix, the diagonals are the eigenvalues.