Instructor Guide

Modelling with Differential and Difference Equations

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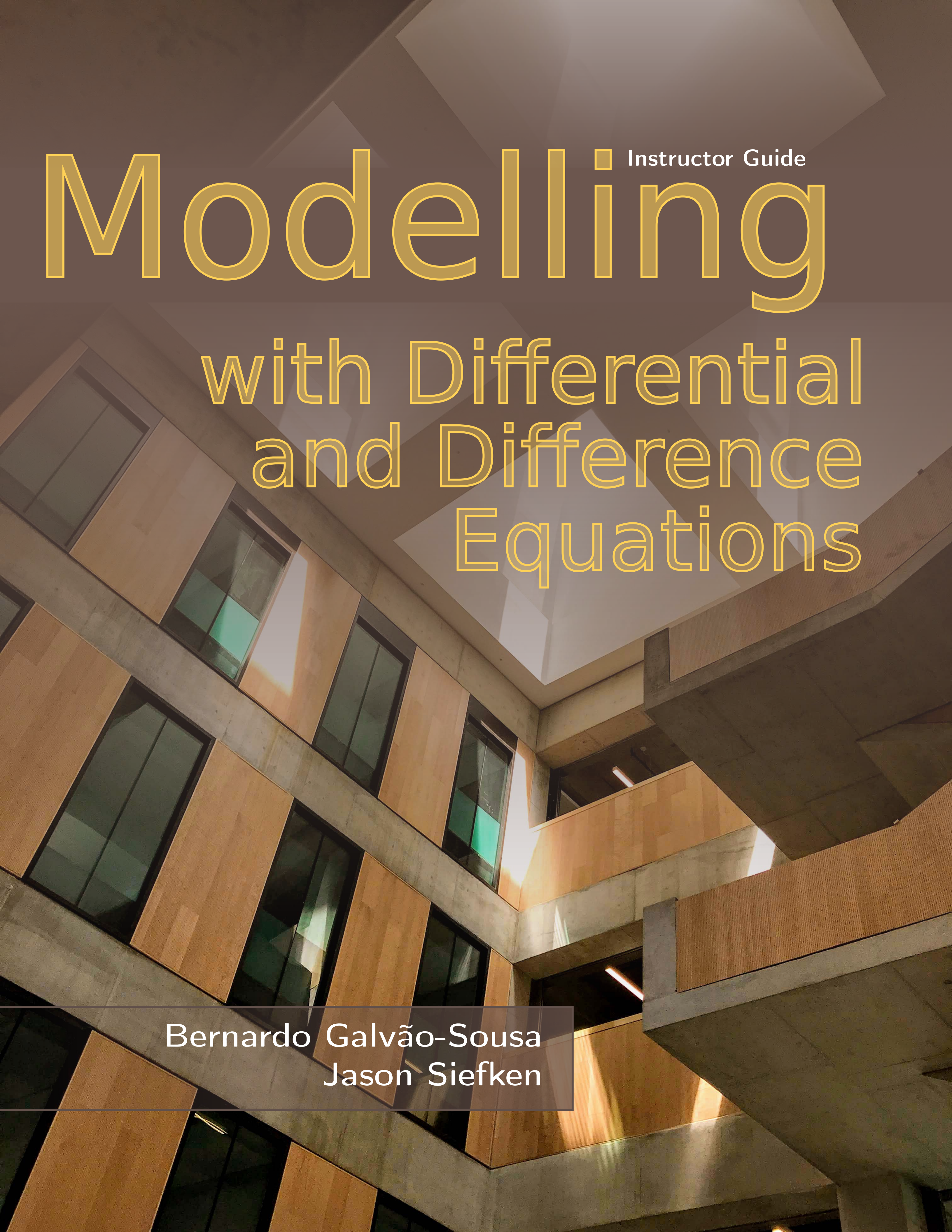

Inquiry Based Modelling with Differential and Difference Equations

For the student

This book is your introductory guide to mathematical modelling and modelling with differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling with differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle through you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.
5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.
2. **Work through the core exercises.**
3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.
4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

License

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/bigfatbernie/IBLModellingDEs/`

Contributors

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In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analysis of the model
6. Writing a report

In this chapter, we follow the approach of Bliss, Fowler, and Galluzzo from

```
Math Modeling: Getting Started and Getting Solutions, K. M. Bliss,
K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014

  https://m3challenge.siam.org/resources/modeling-handbook
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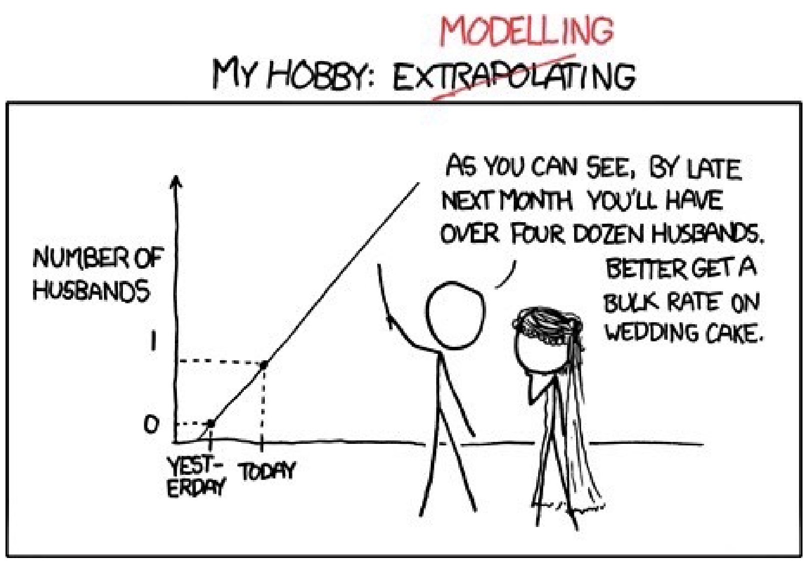

(image from xkcd - comic #605)

Lesson 1

Defining Problem Statement

Objectives

- The first step in Mathematical modelling is to define the problem
- A good way to do this is to figure out what is the “mathematical object” we are looking for at the end of the process
- The second step is to create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they're still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

Make the question precise, bring it into a "mathematical form".

- Choose a mathematical object best suited for the problem, e.g. a number, a geometric form, a graph, a function, an algorithm, ...

Notes/Misconceptions

- Students will start discussing how to solve the problem
- This question deals with what will happen **after** solving the problem
- The goal of this question is to think about how to best tell a "mathematically-challenged" CEO that you solved the problem
- Student teamwork: "With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice."

2 The mayor of Toronto wants to extend the subway line with a new orange line as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

Lesson 2

Defining Problem Statement

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

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3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $T =$ the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can T be minimized?

Notes/Misconceptions

- Students usually come up with more complicated variations:
  - Money spent on late employees' salaries
  - sum of time in minutes that employees are late counting only employees that are at most 15 minutes late
- Stick with T, a simple first approach

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to *optimally* control the traffic lights.

4.1 What mathematical meaning can we give to the word optimal in this circumstance?

4.2 Create a mind map for this problem.

Lesson 3

Making Assumptions

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

5 Consider the elevator problem from core exercise 1.
We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Hint. Focus on a **few** parameters and variables.

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Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

5.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

5.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

5.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Notes/Misconceptions

- Students usually have trouble starting.
- They usually agree that they have to figure out how elevators work, so you can prompt them to be more specific.
- In the end they should come up with questions like these:
  - How fast are the elevators?
  - How much time do elevators take in each floor?
  - How many floors do elevators stop on their way up?
  - How many people fit in the elevator?
  - Should we consider elevator failures?

6 How much would it cost to make a bridge between Toronto and the U.S.?

Lesson 4

Construct a model

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

7 Recall the core exercise 5.

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Write down a mathematical model for this problem.

——— Forwarded Message ———

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From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
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Thanks, the CEO

NEED LOTS OF INSTRUCTIONS FOR INSTRUCTORS HERE!

Lesson 5

Model Assessment

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

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8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\begin{array}{c}\text{loading time on}\\ \text{ground floor}\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ \text{ground} \to 33\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ 33 \to \text{ground}\end{array}} + \boxed{\begin{array}{c}\text{stop time to}\\ \text{6 of the 11 floors}\end{array}} \quad = \quad 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes
- Accumulated late time $= \boldsymbol{T} = 180 \cdot 20 \cdot 8 + 74 \cdot 20 \cdot 8 = 40\,640$ seconds $=$ 11h18m

Your task is to assess this model. Be ready to report on your assessment.

Notes/Misconceptions

Some questions to guide the students:
- What are the strengths?
- What are the weaknesses?
- Is the result around what you expected?

In case students don't realize that something is wrong:
- People start arriving 30 minutes before the starting time, so *almost everybody will be on time?*
- Assume that the CEO of the BigCompany is right: people are arriving late! What's wrong with the model?
- Which assumptions should be relaxed? Or checked?
- If one needs to be replaced, by what?

Lesson 6

Putting it all together

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

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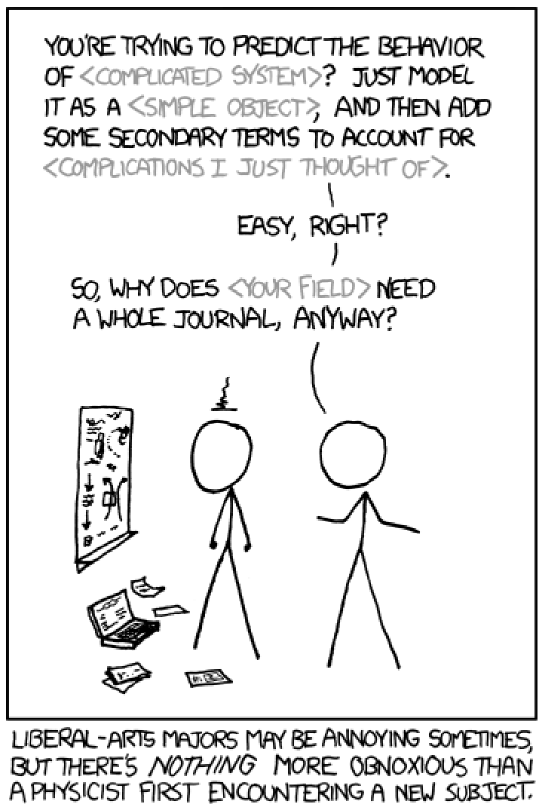
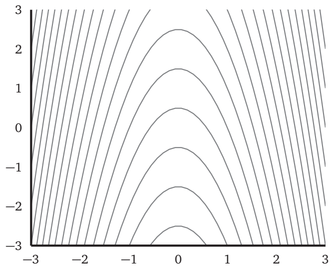


(image from xkcd - comic #793)

Lesson 7

Solutions of Differential Equations

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



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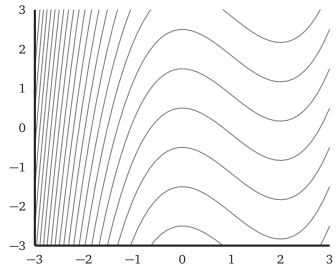

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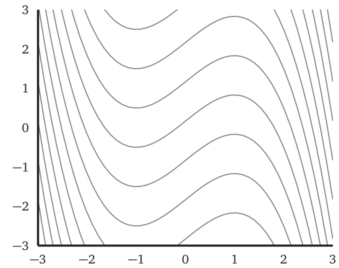

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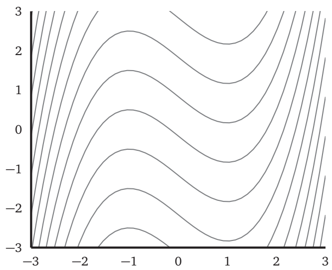

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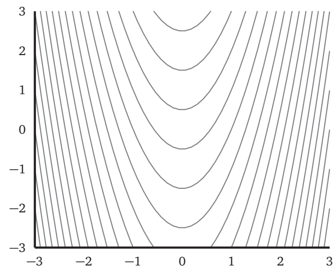

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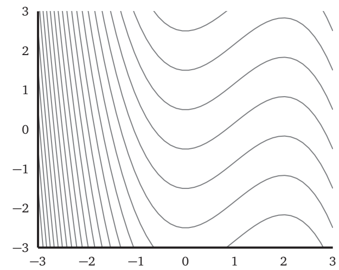


F

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph a function $f(x)$ that would produce such solutions.

> Get students to figure out this core exercise in two different ways:
> - by solving the ODE
> - without solving the ODE

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be decreasing

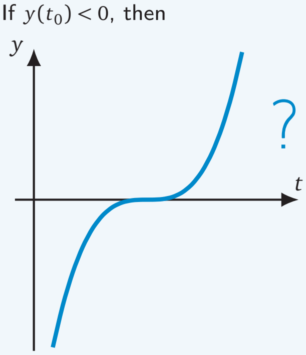
11.2 $y(t)$ must always be increasing

11.3 $y(t)$ must always be positive

11.4 $y(t)$ must always be negative

11.5 $y(t)$ must never change sign.

> The 5th part is for stronger students to think about while they wait for the others to finish.
> Will be addressed later in module ?? (Properties of solutions).
> They can see that:
> - If $y(t_0) > 0$, then $y(t) > 0$ for $t > t_0$
> - If $y(t_0) < 0$, then



12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

> Similar to some practice problems. Skip if the other exercises take too long.

Lesson 8

Slope Fields

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

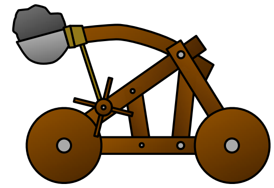
Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

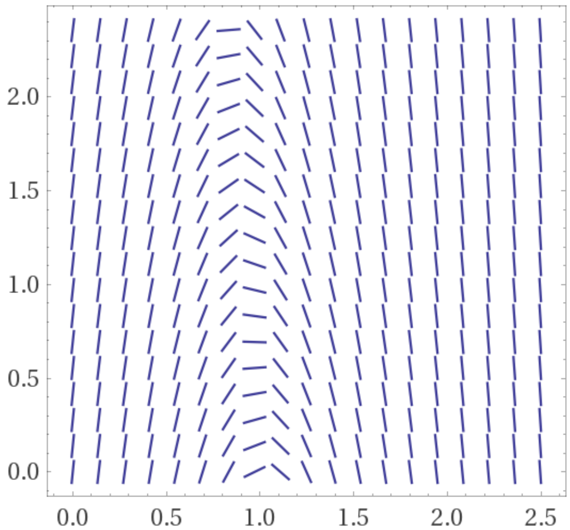
Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

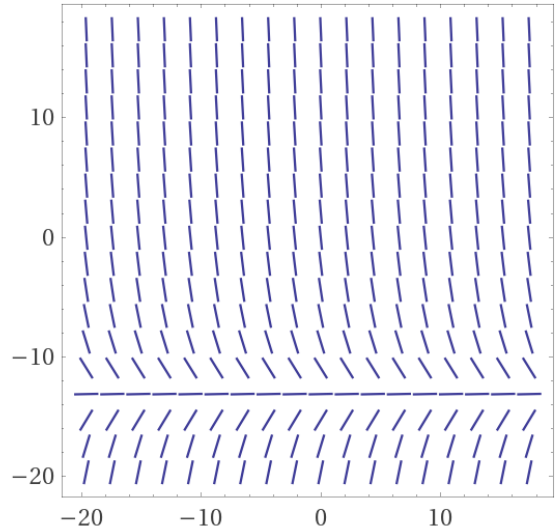
13 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

(These slope fields were created using WolframAlpha)

13.1 On the slope field, sketch a *possible* solution.

13.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

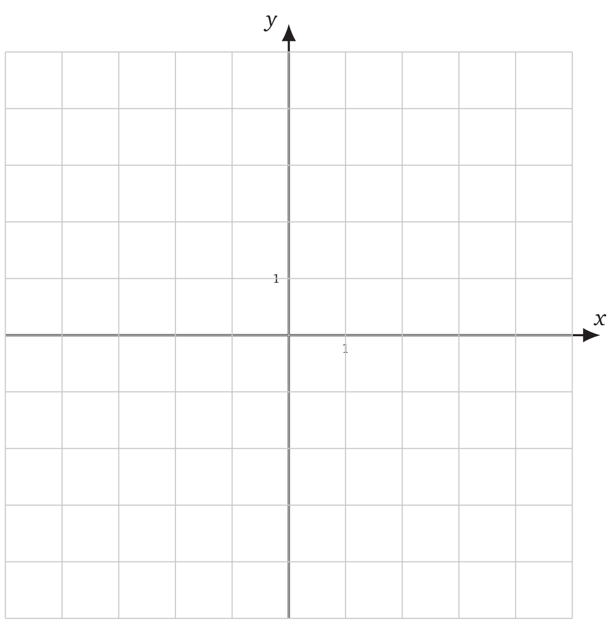
Students should think about the initial conditions. What is a possible value for $y(0)$? What is a possible value for $y'(0)$? Then sketch a possible solution that starts at those values.

The equilibrium in the slope field for $y'(t)$ is called *terminal velocity*. Some students might be able to identify it.

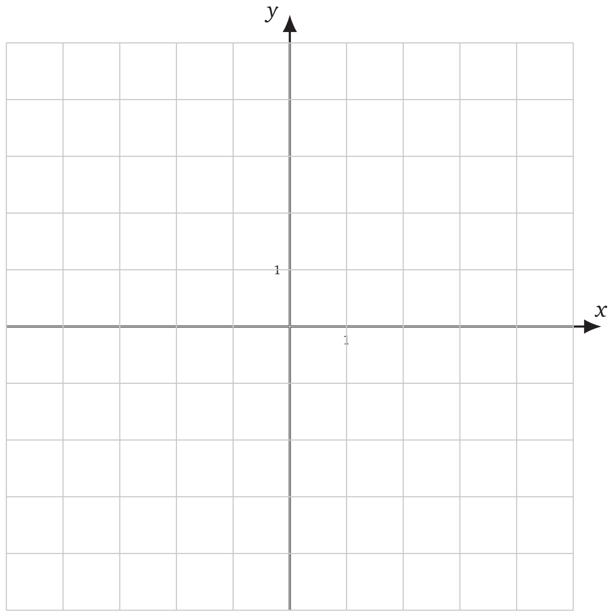
14 Sketch the slope field for the following differential equations.

The goal is not to be very accurate, but to capture the symmetry of each of these slope fields.

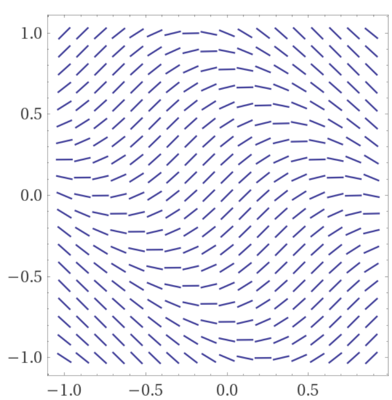
14.1 $y' = x$



14.2 $y' = y^2$



15 Consider the following slope fields:



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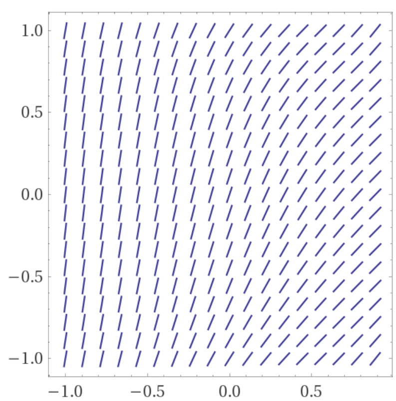
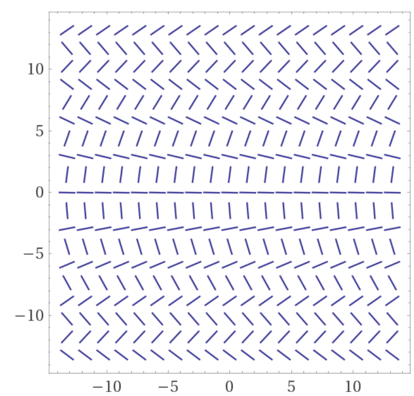
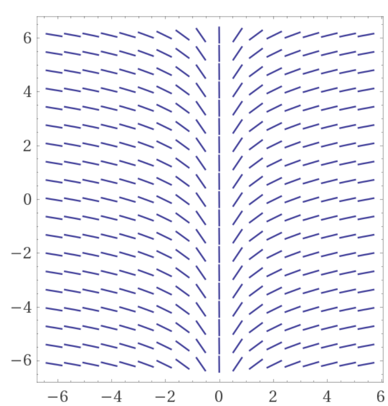


(B)



(C)



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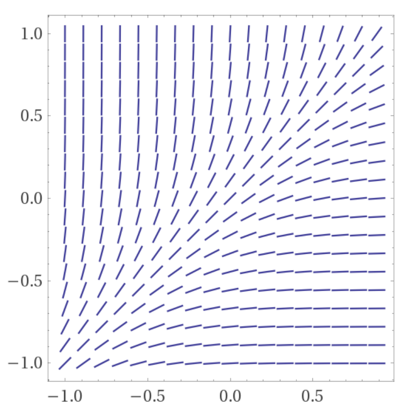

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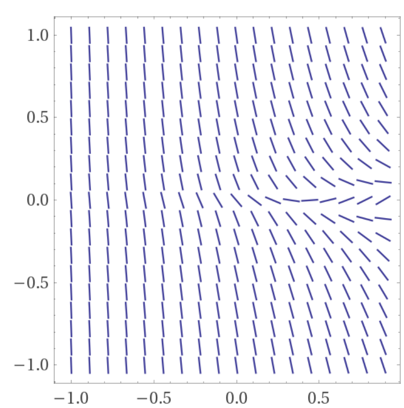


(F)

(These slope fields were created using WolframAlpha)

15.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

15.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

15.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

15.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

15.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

15.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \left(m(x, y)\right)^2$?

Students should be able to justify their choices .

Lesson 9

Approximating Solutions

16 Consider the differential equation

$$y' = y - 2.$$

16.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,3)$.

16.2 Find the solution of the differential equation with the same initial condition.

16.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,1)$.

16.4 Find the solution of the differential equation with the same initial condition.

16.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

16.6 Explain in words why the Method satisfies that property.

- The goal is to have student's recognize that the Euler approximation "curves slower" than the actual solution.
- Students can explain in words why that is the case using the way the approximations are generated.
- For .2 and .4:
  (a) If students learned how to solve ODEs before, then fine!
  (b) If students didn't learn, then ask students:
    - From the approximation, what kind of function does the approximation look like? (exponential)
    - Exponentials pass through $(0,1)$, so this looks like an exponential moved up.
    - Try $y = ae^{bt} + c$ and find a, b, c.

17 Which differential equations will be approximated perfectly using Euler's Method?

- The question is purposefully ambiguous. What do we mean by approximated perfectly?
- Ex: The IVP $y' = \text{sign}(t)$ (assuming $\text{sign}(0) = 1$) with $y(-5) = 5$ has solution $y = |t|$ and it is captured with Euler's method if $\Delta t = \frac{5}{k}$ for any $k \in \mathbb{N}$.
- Once students discuss, they'll find ODE's of the form $y' = c$ for any constant.
- Prompt them to find other types. Show them the example above only after they tried for a bit. Then, let them revise their Conjecture.
- Ex 2: $y' = f(y+t)$ with $y(0) = 0$ and $f(z) = \lfloor z \rfloor$ is approximated perfectly if $\Delta t = 1$, but not if Δt takes any other value.

Lesson 10

Modelling with Differential Equations

18 A pendulum is swinging side to side. We want to model its movement.

18.1 Define the problem. Which function(s) do we want to find in the end?

18.2 Build a mind map.

18.3 Make assumptions. Remember to use your mind map to help structure the problem.

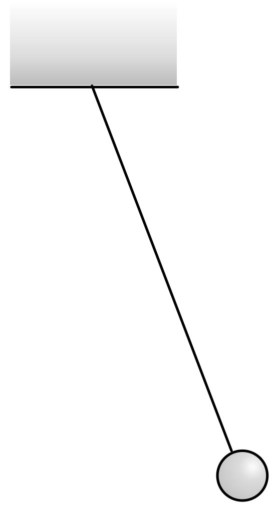
18.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2nd Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in — Rate out).

18.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.



Pendulum and Rumours

This should take 2 lectures.
Lecture 1. Focus on Pendulum problem. Students should finish it at home.
Lecture 2. Rumour problem is more open-ended.
Students should follow the structure from chapter 1 to think about this problem.

Students will mostly likely identify the goal as finding the position of the ball $\vec{r}(t) = \big(x(t), y(t)\big)$. That's fine!
Later, in Step 3, try to guide the students to recognize the following:

- Rope is massless (negligible)
- Rope doesn't bend (negligible)
- So can assume that the rope is rigid. How does that affect the position of the ball?
- No friction (negligible)
- **Important:** Students always focus on string tension. One can consider it, but it all cancels out. It's one of the exercises of the module (above). For the lecture, don't consider tension.

Then on Step 4, guide students to recognize that they actually only need to find a model for the angle, because the position of the ball really only depends on the angle $\theta(t) : \vec{r}\big(\theta(t)\big)$

19 Model the spreading of a rumour through the students of a school.

Lesson 11

Solvable Types of Differential Equations

20 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

20.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

20.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

20.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

20.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

21 21.1 Calculate $\big(\sin(x)f(x)\big)'$.

21.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

21.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

Students will mostly likely identify the goal as finding the position of the ball $\vec{r}(t) = \big(x(t), y(t)\big)$. That's fine!
Later, in Step 3, try to guide the students to recognize the following:

- Rope is massless (negligible)
- Rope doesn't bend (negligible)
- So can assume that the rope is rigid. How does that affect the position of the ball?
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Then on Step 4, guide students to recognize that they actually only need to find a model for the angle, because the position of the ball really only depends on the angle $\theta(t) : \vec{r}\big(\theta(t)\big)$

Lesson 12

Properties of Differential Equations

22 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

22.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

22.2 $y' = t + \sqrt{y-\pi}$ with $y(1) = 1$

22.3 $y' = \sqrt{4-(t^2+y^2)}$ with $y(1) = 1$

23 The initial-value problem

$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

$$y_1(x) = \cos\big(\arcsin(x)\big) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

23.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

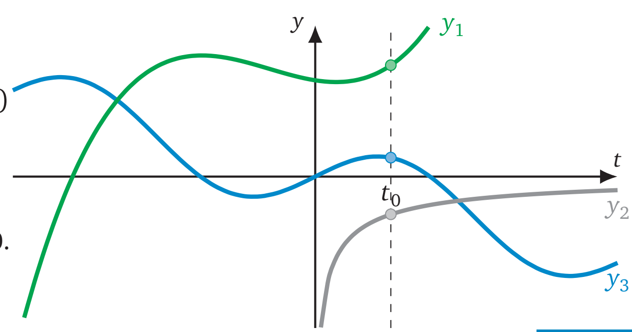
23.2 What can you conclude?

Existence and Uniqueness Thms

The goal is for the students to realize that if the Theorem's conditions hold, then its result must also hold, even though it might not look like it.
So if the Theorem says that the solution is unique and we see two solutions, then they must be the same function (at least for x near $\frac{1}{2}$).

24 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all $t, y \in \mathbb{R}$;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all $t \in \mathbb{R}, y > 0$.



24.1 Can **green** $y1$ and **blue** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

24.2 Can **green** $y1$ and **gray** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

24.3 Can **blue** $y3$ and **gray** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

24.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

Intersecting solutions?

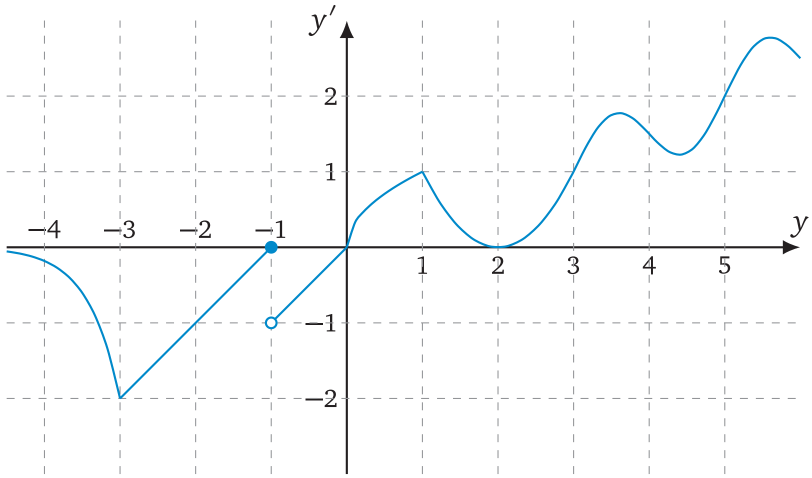
The goal is for students to understand why solutions cannot touch *IF* the Theorem is valid at the intersection point.

- $y1$, y_2 not ok!
- $y1$, y_3 ok!
- $y2$, y_3 ok!

Lesson 13

Autonomous Differential Equations

25 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



25.1 What are the equilibrium points?

25.2 Which equilibrium solutions are stable, unstable, or semi-stable?

25.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

25.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

(b) $y(0) = -\frac{1}{4}$.

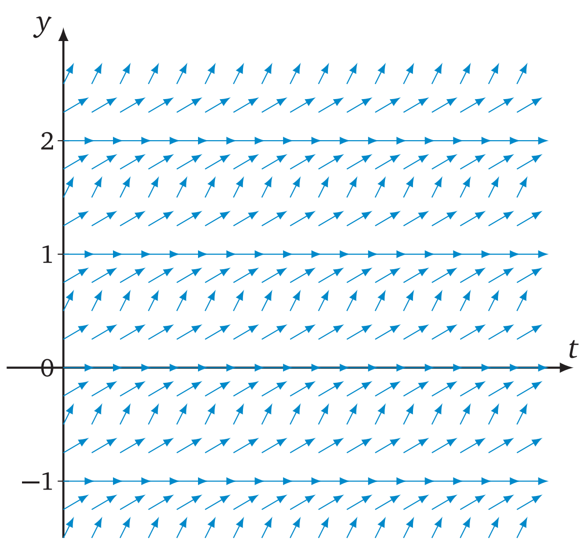
(c) $y(1) = \frac{1}{4}$.

25.5 If $y(0) = 2$, then $y(t) =$

25.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

25.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$

26 Consider a differential equation $y' = f(t, y)$ with the following slope field.



26.1 What are the equilibrium solutions of the ODE?

26.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \dfrac{1}{4} \end{cases}$$

26.3 From the direction field above, circle the correct type(s) of this ODE? Justify your answer.

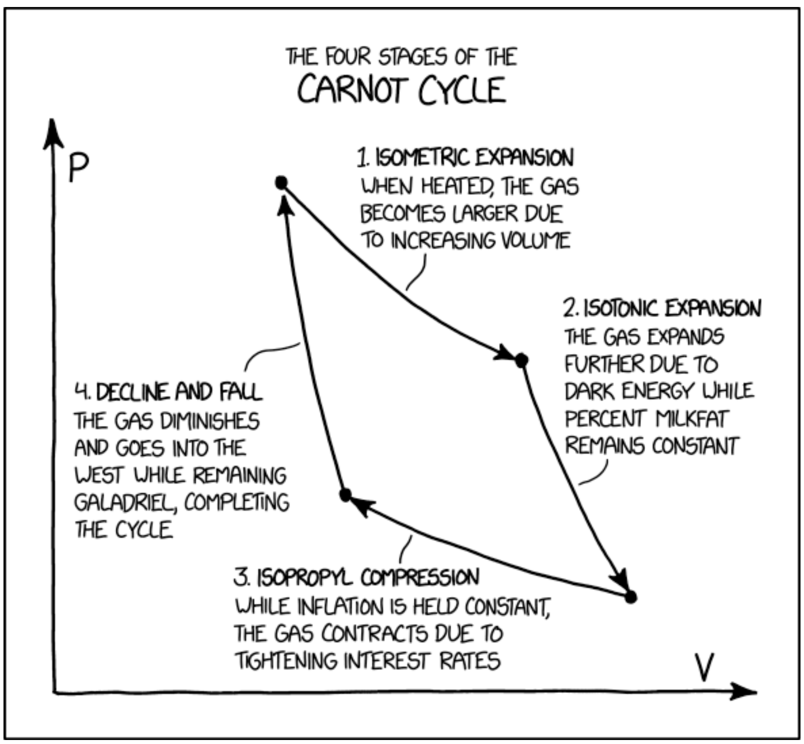
(a) separable. **(c)** autonomous.

(b) of first-order and linear. **(d)** none of the other options.

26.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then

(select all the possible options)

(a) $g(3) < h(3)$ **(b)** $g(3) = h(3)$ **(c)** $g(3) > h(3)$



(image from xkcd - comic #2063)

Lesson 14

Modelling Two Quantities

Objectives

- Bla

Motivation

27 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

27.1 Create a model for these two populations.

27.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

27.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

27.4 Extend the model to include a population of antelopes.

Stress that students should follow the step-by-step approach from chapter 1. *.4* only if there is time. Tell the students to "go nuts" and include everything that relates.

28 A cheetah is chasing an antelope. We want a model of their positions as they run.

This exercise is not required to do in lecture.

Be careful with assumptions! A very general model will be very hard to study.

Allow some brainstorming and try to create a structure for this problem:

- Positions seen from above ($xy-$plane).
- Only need $x_a(t), y_a(t)$ and $x_c(t), y_c(t)$
- Focus on the cheetah: where is she heading to?
- For the antelope, students need to come up with an escape strategy
- Model will be nonlinear!

Lesson 15

Systems of two linear ODEs with constant coefficients

Objectives

- Bla

Motivation

29 Consider a cheetah-lion inspired problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{r}.$$

29.1 Find the two solutions $\vec{r}_1, \vec{r}_2$.

29.2 Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

29.3 Is $\vec{r}_1(t) - \vec{r}_2(t)$ a solution?

29.4 Is $2\vec{r}_1(t) + 3\vec{r}_2(t)$ a solution?

29.5 What is the general solution?

29.6 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

30 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r}.$$

30.1 Find the general solution.

30.2 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

31 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r} - \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

31.1 Find the equilibrium solution.

31.2 Find the general solution.

31.3 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$?

Non-Homogeneous Problem

Students don't know how to solve it yet:

- Equilibrium solution ($\begin{bmatrix} 2 \\ -1 \end{bmatrix}$)
- Show phase portrait using WolframAlpha `https://uoft.me/modelling-sys-nonhom`
- Ask students about properties of the phase portrait (*Goal*: solutions revolve around the equilibrium point)
- Redefine centre: $\vec{r} = \vec{eq} + \vec{p}$. What system foes $\vec{p}$ solve?
- ...

There are practice problems about this.

Lesson 16

Phase Portraits

Objectives

- Bla

Motivation

32 Consider the following model for cheetah's and lions, where

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

The general solution is:

$$\vec{p}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

32.1 Without computing them, what are the eigenvalues and eigenvectors of the matrix?

32.2 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = 0$.

32.3 Sketch the graph of the solution with $c_1 = 0$ and $c_2 = \pm 1$.

32.4 When one constant is set to 0, what is the shape of the graph? Is it always like that? Can you prove it?

32.5 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = \pm 1$.

32.6 Provide an interpretation of the different types of solutions.

Unstable Saddle Point

At the end, let the students know that the equilibrium is called *saddle point* and it is *unstable*, because solutions go away from it.
For the interpretation question, when one population hits zero, it is extinct, so the graph doesn't make sense.
We can interpret that if a population becomes extinct, then the other will behave as it would without competitors: grow exponentially fast!

33 Let us expand the model from the previous exercise to:

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \vec{p} + \begin{bmatrix} -10 \\ 50 \end{bmatrix}.$$

The extra term corresponds to the effect of harvesting 10 lions and bringing in 50 cheetahs every year to the reserve.

The general solution is:

$$\vec{p}(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

33.1 Sketch the phase portrait.

33.2 Provide an interpretation of the different types of solutions.

Get students to compare their results with the previous core exercise.

34 For each of the following general solutions, sketch the phase portrait.

34.1 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}.$

34.2 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}.$

At the end, let the students know what these equilibria are called:

- *source* and it is *unstable*, because solutions go away from it.
- *sink* and it is *asymptotically stable*, because solutions converge to it.

If there is time, students can think about:

- Given a matrix A, which part of A indicates whether the equilibrium is stable / unstable? Which part indicates whether it's a sink/source vs spiral sink/source?

Lesson 17

Analysis of Models with Systems

35 Consider the following model for the sales from a designer clothing brand:

- $x_1(t)$ = purchases by "common mortals" (CM) at time t in years since the beginning of 2015.
- $x_2(t)$ = purchases by "famous people" (FP) at time t.

Our model is based on the following two principles:

(P_1) CM will buy more items from the brand when CM or FP buy more.

(P_2) FP will buy less when CM buy them, but will buy more when FP buy it.

The model we considered is:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \vec{x}(t)$$

35.1 Suppose that at the beginning only CM buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

35.2 Suppose that at the beginning only FP buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

35.3 What conditions on the constants a, b, c, d will guarantee that the solutions will spiral? In that case, is it a spiral source or spiral sink? Is it clockwise or counterclockwise?

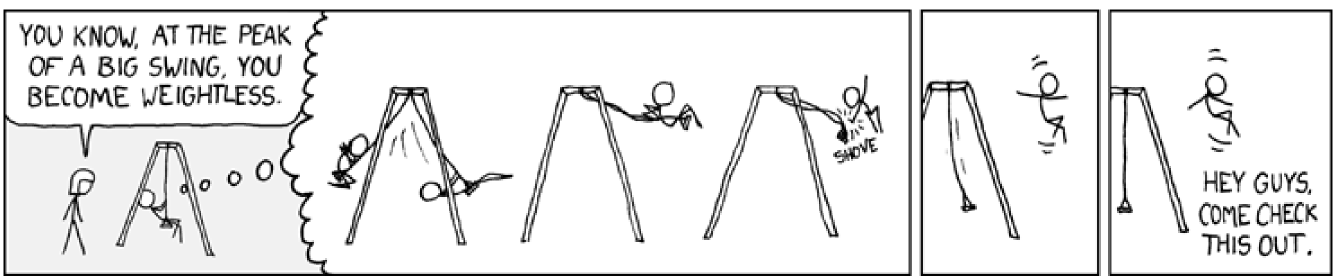
35.4 Are there constants $a, b, c, d > 0$, such that the solution $\vec{x}$ is periodic?

35.5 Consider the constants $a = b = c = d = 1$. Assume that initially CM were buying $c_0 > 0$ items and FP were buying $f_0 > 0$ items. What will happen to $x_1(t)$ and $x_2(t)$ as $t \to \infty$? Explain the results in terms of the evolution of purchases from CM and FP.

35.6 Consider the constants $a = b = c = d = 1$. If $c_0 = 10$, $f_0 = 100$, then at what time will FP stop buying items? And at what time will FP be buying the maximum number of items?

Can leave .6 as a practice problem if there isn't enough time.

Chapter 4 – Higher-Order Models



(image from xkcd - comic #226)

Lesson 18

Modelling with Second-Order ODEs

Objectives

- Bla

Motivation

36 Here are some facts about laptop keys:

(da) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.

(di) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.

(fo) On average, a person exerts the force of 42 N with one finger on a key.

(gr) Gravity is much weaker than the spring that keeps the key in place.

(hl) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").

(lo) Keys last 50 million presses on average.

(ve) Keys can only move vertically.

36.1 Model a laptop keypress.

36.2 What happens if the damping system of the key is broken? What happens if the damping system is too strong? How strong should the damping system be?

36.3 What happens to the key when the spring breaks?

.1 should be very quick, since a very (very) similar example was solved in the module.

37 Model a ball rolling down a ramp.

Ball rolling

Different approach depending on students.

Students need some challenge and have time:

- Ramp $y = f(x)$ makes it simpler
- Need projection (Linear Algebra) to find gravity force along the ramp at $(x_0, y_0) = \big(x_0, f(x_0)\big)$:

$$\ell = (0, -mg)\cdot(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{\sqrt{1+k^2}}$$

where $k = f'(x_0)$. So gravity force along the ramp is:

$$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

- Yields second-order ODE

Weaker students with less time:

- Give the formula for gravity force along the ramp:

$$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

Follow-up question:

- Ramp is $y = (x-1)^2$
- Get second-order ODE for ball position
- Will the ball always move to the right? Justify with the ODE.
- Approximate near the bottom of the ramp: $y' \approx 0 \Leftrightarrow \sqrt{1+(y')^2} \approx 1$ and solve the simpler ODE.
- When is this approximation valid? (when ball oscillates back and forth near the bottom)

Lesson 19

Second-Order Linear ODEs with Constant Coefficients

Objectives

- Bla

Motivation

38 Consider the ODE $y''(t) - 9y(t) = f(t)$.

38.1 Find a complementary solution.

38.2 Find a particular solution for $f(t) = 14e^{-4t}$.

38.3 Find a particular solution for $f(t) = 9e^{-3t}$.

38.4 Find a particular solution for $f(t) = 10\cos(t)$.

---

39 Consider the ODE $y''(t) - 2y'(t) + 5y(t) = f(t)$.

39.1 Find a complementary solution.

39.2 Find a particular solution for $f(t) = \sin(2t)e^t$.

39.3 Find a particular solution for $f(t) = (4t+2)\sin(2t)e^t$.

---

40 Consider the ODE $y'' + 3y' = 3t$.

40.1 Find the complementary solution.

40.2 Find a particular solution.

40.3 Find the solution that also satisfies

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Lesson 20

Analysis of Models with Higher Order ODEs

Objectives

- Bla

Motivation

41 Consider the second-order ODE:

$$y''(t) - 3y(t) = t\big(2 + \sin(t)\big).$$

41.1 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) > 0$ for $t \geq 0$.

41.2 Assume that $y(0) = a < 0$ and $y'(0) = b$. Give an example of a, b such that $y(t)$ is increasing for $t \geq 0$.

41.3 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) < 0$ for all $t > 0$.

Without solution

The goal is to solve this without finding an expression for the solution.
For .2, the idea is to make sure that $y'' < 0$.

42 Consider the second-order ODE:

$$\begin{cases} y''(t) + 4y(t) = f(t) \\ y(0) = y_0 \\ y'(0) = 0 \end{cases}$$

42.1 Let $f(t) = 0$ and $y_0 = 1$. Sketch the solution.

42.2 Let $f(t) = 396\cos(20t)$ and $y_0 = 0$. Sketch the solution.

42.3 Let $f(t) = -4\sin(2t)$ and $y_0 = 1$. Sketch the solution.

42.4 Let $f(t) = 0.39\cos(1.9t)$ and $y_0 = 2$. Sketch the solution.

Hint. $\cos(at) + \cos(bt) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}t\right)$

Goals:

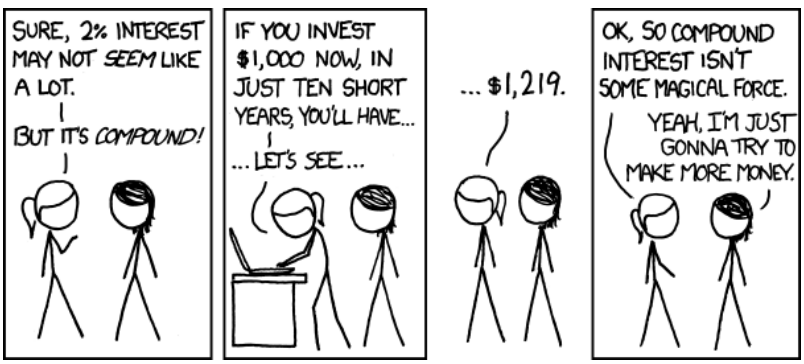
- Learn some different types of behaviour of for second-order ODEs
- Learn how to sketch trig functions combined with linear or other trig functions

.1. Complementary solution!

.2. *Adding* two trig functions: one oscillating slowly and one oscillating quickly

.3. Resonance: t times trig function

.4. Beats: *product* of two trig functions – one oscillating slowly and one oscillating quickly



(image from xkcd - comic #947)

Lesson 21

Introduction to Difference Equations

Objectives

- Bla

Motivation

Lesson 22

Solving Difference Equations

Objectives

- Bla

Motivation

43 Consider the difference equation

$$u_{k+1} = 6u_k - 9u_{k-1}$$

43.1 Find the solution that satisfies $u_0 = 1, u_1 = 3$.

43.2 Find the solution that satisfies $u_0 = 1, u_1 = 4$.

44 Consider a difference equation that has solutions $u_k = r^k$ for $r = 2$ and $r = 3$.

We also have the conditions $u_0 = 7$ and $u_1 = 6$.

What is u_{22}?

Lesson 23

Modelling with Difference Equations

Objectives

- Bla

Motivation

45 Let us expand on the economic example above.

We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

Even though we call $p\%$ the annual interest rate, because it is compounded during the year, at the end of the year the effective annual interest rate $p_{\text{eff}}\%$ is actually higher.

Calculate the effective interest rate $p_{\text{eff}}\%$.

The effective annual interest rate is the interest rate with a compounding period of 1 year that gives the same result s the rate of $p\%$ compounded every α years.

46 The goal of this quesiton is to try to understand the meaning of average lifespan.

46.1 Consider a small tribe, where the people in there died at the ages:

42, 56, 46, 52, 5, 103, 47, 67, 67, 85, 57, 42, 47, 67, 46, 42, 5, 46, 57, 42.

What is the average lifespan of this tribe's population?

46.2 Consider another small tribe, where people recorded their lifespans differently. Below is a table with the percentage of the population that died at each age:

Percentage of population	2%	5%	9%	9%	16%	22%	37%
Age at death	98	82	71	66	61	53	48

What is the average lifespan of this tribe's population?

Pre-class question

The goal of this question is to prepare for calculating the average lifespan in the next page.

Not to do in lecture. Assign to students to solve at home before.

47 Given a population with

- $\mu =$ probability that an individual will die between two seasons.

47.1 Define the following quantity

- $P(k)$ =probability that an individual born at season 0 is alive at the beginning of season k.

Find a model for $P(k)$.

47.2 What is the probability of the individual dying during the k$^{\text{th}}$ season?

47.3 What is the average lifespan of an individual in this population?

Some hints:

- individual dying during season $k \Leftrightarrow$ lifespan $= k$ seasons
- From previous two exercises, deduce that: average lifespan = expected value of lifespan $\ell = E$:

$$E = \sum_{k=0}^{\infty} k\ell(k)$$

- $\displaystyle\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$ for $|r| < 1$.
- End result should be $\frac{1}{\mu}$.

48 Consider a population of special rabbits. Once a pair of rabbits is born, they grow and one year later they are still immature. But two years after they are born they give birth to another pair of rabbits.

Model this population of rabbits.

If there is time, students should show that the Fibonacci sequence does indeed match the number of rabbits.

49 Consider another population of rabbits. This is the lifecycle of a pair of rabbits:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pairs of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Model this population of rabbits.

Students might try to find a pattern. It is possible, but very difficult.
Hint: Use a system of difference equations.

In core exercise 52, the students are asked to prove the formula.

Lesson 24

Analysis of Difference Equations

Objectives

- Bla

Motivation

50 Consider the following difference equation:

$$u_{k+1} = a(u_k - b)$$

50.1 What is the equilibrium solution?

50.2 Are there 2-periodic solutions? I.e. satisfying

- $v_0 = v_2 = v_4 = v_6 = \cdots$
- $v_1 = v_3 = v_5 = v_7 = \cdots$
- $v_0 \neq v_1$

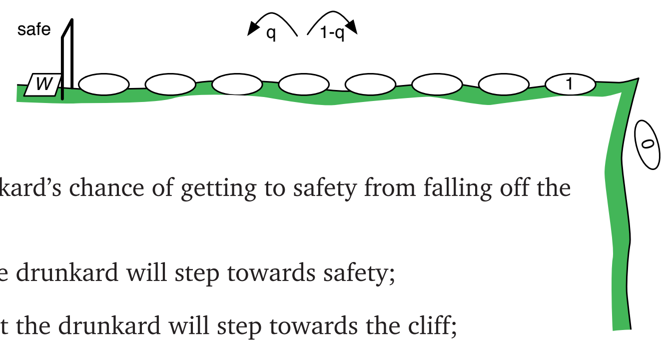
50.3 What happens to the solutions for different values of a?

50.4 What happens to the solutions for different values of b?

- In the calculations for .2, there is a step that involves a division by $(1-a^2)$, so it can only be done for $a \neq \pm 1$.
- The final result for .2 is:

$$\begin{array}{lll} a \neq \pm 1. & \text{periodic} \Rightarrow u_0 = \dfrac{ab}{a-1} & \Rightarrow 1 \\ a = 1. & \text{periodic} \Rightarrow b = 0 & \Rightarrow 1 \\ a = -1. & \text{periodic} \Rightarrow b = 0 & \Rightarrow 2 \end{array}$$

51 Consider a drunkard that is walking randomly near a cliff.



Consider this model for the drunkard's chance of getting to safety from falling off the cliff:

- q is the probability that the drunkard will step towards safety;
- $1-q$ is the probability that the drunkard will step towards the cliff;
- p_n = probability that the drunkard will get to safety if he is in step number n;
- The drunkard will stop moving if he gets to safety (step W) or if he falls out of the cliff (step 0);
- $p_n = qp_{n+1} + (1-q)p_{n-1}$.

51.1 Is p_n increasing or decreasing?

51.2 What is p_0? What is p_W?

51.3 Let $q = \frac{1}{2}$. What is $p_{W/2}$? Is p_n symmetric around $n = \frac{W}{2}$?

51.4 Let $q > \frac{1}{2}$. Is $p_{W/2} > \frac{1}{2}$? Is $p_{W/2} < \frac{1}{2}$?

51.5 How do solutions for $q = \alpha$ and $q = 1-\alpha$ compare?

Question .3 is purposefully ambiguous about symmetry. What kind of symmetry is there? Is there any?

52 Consider a population of rabbits with the following lifecycle:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pair of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

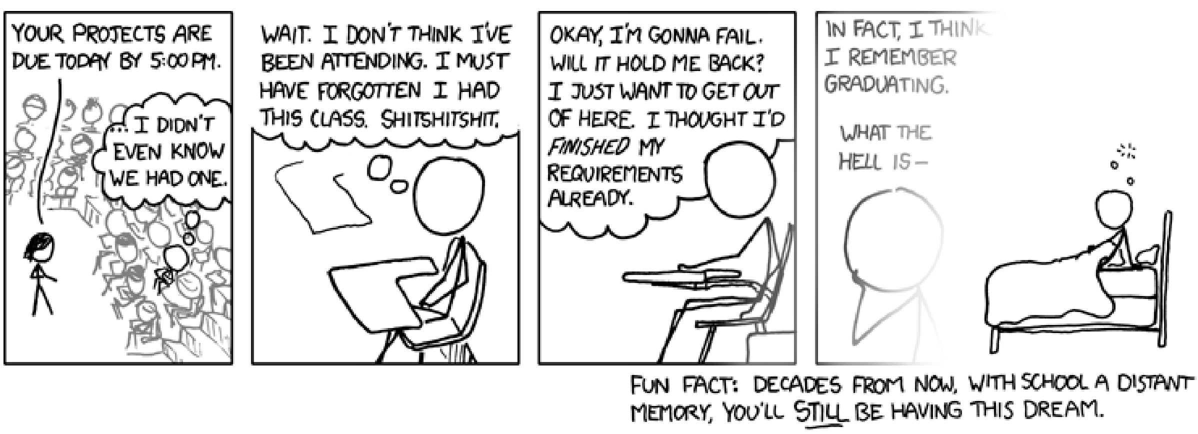
Consider the definitions:

- We start with 1 pair of newborn rabbits in year 0;
- r_n = number of pairs of rabbits alive during year n;
- i_k = number of immature pairs;
- y_k = number of young adult pairs;
- a_k = number of adult pairs;
- o_k = number of old pairs.

52.1 Show that $b_k = b_{k-2} + b_{k-3}$.

52.2 Show that $y_{k+1} = o_k + o_{k+1}$.

52.3 Show that $r_n = r_{n-2} + r_{n-3}$.



(image from xkcd - comic #557)

Managing a fishery

The population $P(t)$ of a species of fish in a finite environment, like a lake is often described by the logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where r is the natural growth rate of the population and K is the maximum number of individuals that the environment can sustain.

If that population is harvested at the rate of $H(t,P)$ fish per year, then it can be modelled by the ODE

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H(P,t).$$

This ODE assumes that the population is harvested continuously throughout the year, which is not exactly true. Later in the course we will see how to make a better model.

Your goal is to figure out the maximum profit you can generate from this population of fish. For that, it should be clear that it is undesirable to harvest too much, since it may lead to the extinction of the fish.

1. Selling fishing licenses. As the manager of the lake, you decide to profit from it by selling fishing licenses to people and allow them to fish there.

(a) Assuming that an average person, after a whole **day** fishing, has an efficiency[1] of $E\%$, what is $H(P,t)$ and what is the ODE that models the fish in the lake?

(b) There is a percentage $E^\star$, such that if $E \geq E^\star$, then the population will become extinct. What is $E^\star$? Explain why the population will become extinct.

Hint. You don't need to solve the ODE.

(c) A sustainable yield Y is the rate at which the fish can be harvested indefinitely: it is the value of $H(P,t)$ which doesn't change with time and for the asymptotically stable population.

Determine the maximum[2] value of E to maximize Y and then find the maximum $Y_{\max}$.

2. Selling fish. As the manager of the lake, you decide to profit from it by harvesting the fish yourself and selling it.

(a) Now the fish are harvested at a constant rate h. What is $H(P,t)$ and what is the ODE that models the fish in the lake?

(b) There is a rate $h^\star$, such that if $h \geq h^\star$, then the population will become extinct. What is $h^\star$? Explain why the population will become extinct.

Hint. You don't need to solve the ODE.

(c) If $h \leq h^\star$, what is the maximum sustainable yield Y_m?

Further Investigation.

1. Can you think of other forms for $H(P,t)$?
2. If, instead of a logistic model, you include an extinction threshold as well, what can you say about the model for constant effort fishing? for constant rate fishing? Is it a useful addition to the model? Have fun with it!

[1] Efficiency of $E\%$ means that after a whole day fishing, that person will have caught $E\%$ of the existing fish in the lake.
[2] This value can be controlled, e.g. by defining the time available for fishing in a day.

X-ray attenuation

An X-ray tube fires X-rays, which travel in a straight line. An X-ray detector will give you the intensity of any X-ray that hits the detector. If there is a vacuum between the X-ray tube and the X-ray detector, then the X-ray will have the same intensity when it hits the detector as when it left the tube. However, when X-rays pass through matter they interact with the atoms in the material and are sometimes deflected off course, or absorbed. We call this phenomenon **attenuation of the X-ray** and it results in a decrease in the intensity of the X-ray beam. When this happens, the X-ray detector will show a lower intensity than the original intensity of the X-ray.

The intensity of an X-ray is measured in keV (kiloelectronvolt).

Task.

1. Experiments indicate that the rate of decrease in the intensity of the X-ray beam as it travels through some matter is proportional to the **linear absorption coefficient** A of the material. Find an ordinary differential equation (ODE) to model the intensity, I, of an X-ray beam fired into some uniform matter with linear absorption coefficient A. Be sure to include an initial condition.
   - What are the units of A?
   - Classify the equation.
   - Solve the equation in terms of the initial condition and A.

2. How far into a material can an X-ray beam travel before its intensity has decreased to $\frac{1}{e}$ times its original intensity.

Now you will explore one of the main ideas behind medical X-ray imaging. In order to do this, you need to know the linear attenuation coefficient of healthy human tissue.

3. You have a 15keV X-ray tube and an X-ray detector. When you fire the X-ray through 10cm of healthy tissue, you measure $\frac{15}{e}$keV on your X-ray detector. When you fire the X-ray through 20cm of healthy tissue, you measure $\frac{15}{e^2}$keV. Using your model of X-ray attenuation, estimate the linear attenuation coefficient of healthy tissue.

This is the basis for using X-ray Computed Tomography (CT) used in medical imaging! We can recognize healthy versus unhealthy tissue by using what we know about their attenuation coefficients. To do this in a human body requires more advanced mathematics such as the Radon transform introduced in 1917 by Johann Radon. However, consider a simple case below.

4. Suppose that you have a 10cm × 2cm rectangle with the same linear attenuation coefficient as healthy tissue, and somewhere inside this square is a circle of unknown size having a linear attenuation coefficient different from that of healthy tissue.
   - Using an X-ray tube and an X-ray detector, can you locate the circle and determine its radius? How?
   - Can you determine the linear attenuation coefficient of the circle? How?

5. In actuality, a more complex model is needed for accurate imaging. The linear attenuation coefficient is actually dependent on the intensity of the X-ray! How does this impact your model? Discuss how this would impact your solutions to the above problems.

"X-Ray Attenuation" is a collaboration between Bernardo Galvão-Sousa and Craig Sinnamon.

Predator-prey chase

Qestion. What is the path of a lion chasing an antelope?

Rules.

- The antelope flees at a constant speed v in a straight line
- The lion chases at a constant speed u

Step-by-step breakdown of the problem.

1. Assume the antelope starts at $(0,0)$ and moves up the y-axis. What is its position?

$$(0, vt)$$

2. Assume the lion's position is $\big(x(t), y(t)\big)$. What happens if $x(0) = 0$?
3. Assume the lion's position is $\big(x(t), y(t)\big)$ with $x(0) \neq 0$. For simplicity, assume that $x(0) < 0$. What is the sign of $x'(t)$?
4. The goal is to find the path, so we are looking for an equation to describe $y(x)$. Using the result from **3.**, explain why the solution will be a function $y(x)$.
5. The lion's speed is u. Express that condition using $x(t)$ and $y(t)$.
6. Find an expression for $\dfrac{dx}{dt}$ without the variable t.
7. Which condition on $\dfrac{dy}{dx}$ do we get from the fact that the lion is chasing the antelope? Draw a picture.
8. Use **7.**, and obtain an expression for $\dfrac{dx}{dt}$ in terms of $\dfrac{d^2y}{dx^2}$.
9. Obtain a Differential Equation that describes the lion's path $y(x)$.
10. For simplicity, assume that the lion starts at $(-1, 0)$. Solve this Initial-Value problem.
11. When does the lion actually catch the antelope?

Further investigation.

1. Program the pursuit (it will be approximated) and check your answer.
2. Can you figure out the path of the lion for other antelope trajectories? Program that pursuit and check your answer.
3. (for fun) Program a "game" where you control the antelope and the computer controls the lion.

Epidemic modelling

Goal. We want to model the spread of the CoViD-19 pandemic in Canada.

SIR Model. This is the typical model for an infectious disease. We start by dividing the population into three groups:

- Susceptible Individuals $S(t) =$ number of people who haven't contracted the disease;
- Infected individuals $I(t) =$ number of people infected;
- Removed individuals $R(t) =$ number of people that either died or recovered from the disease and are now immune to it.

Assumptions.

(a) Population size N is large and constant (no birth, death, or migration);

(b) No latent/incubation period (there is an improved model that includes this - SEIR model);

(c) Homogeneous population;

(d) Recovery rate is constant γ (includes rate at which people die or recover from the disease);

(e) Out of all possible interactions between susceptible and infected individuals $S(t) \cdot I(t)$, there is a proportion $\frac{\beta}{N}$ that will result in the susceptible individual becoming infected;

(f) The probability that an infected person will either die or recover is γ.

ODE. From here we can obtain the SIR model:

$$\begin{aligned} \frac{dS}{dt} &= -\frac{\beta}{N}SI \\ \frac{dI}{dt} &= +\frac{\beta}{N}SI - \gamma I \\ \frac{dR}{dt} &= +\gamma I \end{aligned}$$

Video.

- `https://youtu.be/f1a8JYAixXU`

Important.

- An important constant in this model is $R_0 = \frac{\beta}{\gamma}$, called the basic reproductive number, which informs us about how fast the disease propagates.
- The expected time from infection to recovery (or death) can be proved to be $T = \gamma^{-1}$.

Data. Data from the Public Health Agency of Canada:

- `http://uoft.me/covid19-canada`

Task.

1. Explain how the system of ODEs relates to the assumptions.
2. Estimate the constants N, R_0, β, γ for Canada.
3. Using the idea from Euler's Method (used to approximate the solution of one first-order ODE), create a method to approximate the solution $S(t)$, $I(t)$, $R(t)$ of the SIR model.
4. Compare your approximation from **3** with the actual data.
5. Observe that the data is the result of the lockdown measures imposed in Canada. Find a value for R_0 that best matches your approximation to the data.
6. Study what happens to Canada if the lockdown measures are lifted when the number of infected people is very small vs when the number of infected people is actually zero.

Further Investigation.

1. Study what happens to the model when $R_0 < 1$, $R_0 = 1$ or $R_0 > 1$.
2. Adapt your method to the SEIR model and answer questions **1-6** above for the new model.
3. Improve the SEIR model to better model different lockdown scenarios.

Hunting inspiration

Snow collects on the brim of your fur coat and musket as you stalk your prey through the white woods. A flash of orange, a rustling of branches, and then it's gone. You mutter a curse under your breath. Foxes are scarce this year, and you'll have to explain to the Dutch East India Trading Company why you've come up short. Worse yet, your rival, a trapper who only hunts rabbits, is having a terrific year. You shouldn't have teased him so much when rabbits were down and foxes were up just a few seasons ago.

If only you could somehow predict which game would be plentiful, you could always bid on the easier contract! But how?

Back at camp, amid the crackling of your lonely fire, the answer comes to you. Just two months ago you attended a talk by Dr. Lotka on autocatalytic chemical reactions. It was quite a spectacle when, after Dr. Lotka had finished talking, a Dr. Volterra stood up and proclaimed that he had applied the same model to predator-prey ecology. At the time you were rushed and didn't think much about the proclamation, but now the basic assumptions were making more and more sense:

(a) In the absence of foxes, the rabbit population grows at a rate proportional to the number of rabbits.

(b) In the absence of rabbits, the fox population declines at a rate proportional to the number of foxes.

(c) The population of rabbits declines at a rate proportional to the product of the rabbit and fox populations.

(d) The population of foxes grows at a rate proportional to the product of the rabbit and fox populations.

Pop! A hot coal explodes, snapping you out of your pondering state and into one of action. Grabbing a piece of paper from your limited supplies, you begin to grapple with the consequences of the Lotka–Volterra model.

Task. Let R and F stand for the rabbit and fox populations, respectively, and let α, β, γ, and δ be the constants of proportionality for parts **(a)**–**(d)**.

Work through the following before you begin your report.

1. Write down the Lotka-Volterra system of differential equations. For each of (a)–(d), explain whether or not the assumption is reasonable.
2. When is the fox population increasing or decreasing? Given R and F, could you predict which one is on the rise on which one is on the decline?
3. Is there a steady state for the fox population? Could the fox population remain steady while the rabbit population is changing?
4. Sketch an RF-phase portrait for the Lotka-Volterra system of differential equations with the following constants:

$$\begin{aligned}\alpha &= 0.2 \text{ rabbits per month per rabbit}\\ \beta &= 0.1 \text{ foxes per month per fox}\\ \gamma &= 0.002 \text{ rabbits per month per rabbit-fox}\\ \delta &= 0.001 \text{ foxes per month per rabbit-fox}\end{aligned}$$

   Hint: you will need to consider rabbit and fox populations of well over 100 to see interesting behaviour in your phase portrait.
5. Does your phase portrait have any singular points? What do they mean?
6. Use technology to graph $R(t)$ and $F(t)$ for some initial conditions. Do the initial conditions affect the period of the population increase or decrease? Does this seem reasonable when looking at your phase portrait?

Your writeup should include the following:

- An explanation of the Lotka-Volterra model along with a discussion of whether or not each assumption is reasonable.
- A description of what behaviour you expect from which initial conditions. You may use the parameters specified in question 4.. Include a phase portrait in your description as well as how to interpret the phase portrait, and make sure to point out any critical points.
- Suppose you wanted to legislate limits on the hunting of rabbits and foxes to ensure the population of either never dipped below a certain level. Based on the Lotka-Volterra model, propose legislation. Be specific and comment on whether a flat-out hunting ban would achieve the desired effect.

Be careful with your simulations. Euler's method loses accuracy quickly on Lotka-Volterra–based systems.

"Hunting Inspiration" is a collaboration between Max Brugger and Jason Siefken

Arms race

In this project, you will develop and analyze models for an arms race between two countries.

Define the following:

- $t \geq 0$ represent time in years;
- $x(t)$ and $y(t)$ represent the yearly military budget (in dollars) of countries Blue and Red respectively.

1. **Mutual Fear!** For a first model, assume that each country increases its military budget at a rate directly proportional to the existing military budget of the other nation.
   - **(a)** What are the equations that define this model? **Hint: There should be two constants in your model.**
   - **(b)** Solve the system and sketch a phase portrait.
   - **(c)** What does the model predict about the long term military budgets of the two countries?

2. **The Richardson Model.** Now include some limiting factors in to the model you set up above. Assume that in addition to the budget increases in the mutual fear model, each country's military budget decreases at a rate proportional to it's current military budget and increases at some fixed (independent of military budget) rate due to a long standing grievance.
   - **(a)** What are the equations that define this model?
     Hint. There should be six constants in your model.
   - **(b)** Under what conditions (on the six constants) can the arms race stabilize? By stabilize we mean that the military budgets remain at some fixed amount, or that the budgets approach some constant amounts.
     - There is a line L_B, called the optimal line for Blue, in the phase plane such that if $(x(t), y(t))$ lies on L_B then $x'(t) = 0$. What is the equation of that line?
     - Show that Blue continuously changes its military budget to bring the solution $(x(t), y(t))$ closer to L_B.
     - Repeat the previous two parts for a line L_R, the optimal line for Red.
     - What does the intersection point of L_B and L_R represent?
     - Under what conditions on the constants will the point of intersection lie in the first quadrant $(x > 0, y > 0)$? What will be the long term behaviour of the system for various initial conditions? Explain.
     - Under what conditions on the constants will the point of intersection lie in the third quadrant $(x < 0, y < 0)$? What will be the long term behaviour of the system for various initial conditions? Explain.
   - **(c)** What happens in the long run for various initial conditions if one or both of the "grievance" terms is/are negative? (More of a "good will" term than a "grievance" term!)
   - **(d)** Can L_B and L_R be parallel? What happens in this case?
   - **(e)** Can $L_B = L_R$? What happens in this case?
   - **(f)** Produce examples that demonstrate these various cases and long term behaviours. Plot or sketch their phase portraits.

3. **Real World.** One can argue that in the real world, a runaway arms race is impossible since there is a limit to how much a country can spend. We can add carrying capacities in to the model. Let x_M and y_M be the maximum budgets of the two countries. Then consider the model

$$x'(t) = \left(1 - \frac{x}{x_M}\right)(-ax + by + c)$$
$$y'(t) = \left(1 - \frac{y}{y_M}\right)(dx - ey + f)$$

   Analyze this model.

Further Investigation.

1. **Another Nonlinear Model.** Suppose that the equations underlying the model have the form

$$\begin{cases} x'(t) = -ax + by^2 + c \\ y'(t) = dx^2 - ey + f \end{cases}$$

   where $a, b, d, e, f > 0$. How many stable points are there? There are now optimal curves instead of optimal lines. Discuss the outcomes of such an arms race for various intersections of the optimal curves.

2. **The Richardson Model with Good Will.** If instead of having terms representing increases due to a grievance, what happens if you include terms representing fixed rate decreases in the military budgets of both countries due to good will?
   - What are the equations that define this model?
     Hint. There should be six constants in your model.
   - What possibilities exist for the long term behaviour of the military budgets of the two countries?
   - How do the possibilities for the long term behaviour depend on initial conditions?
   - Produce examples that demonstrate the various long term behaviours.

3. Richardson with carrying capacities.

4. Extend Richardson to three countries.

5. Increase not by absolute level but by amount over stable level.

"Arms Race" is a collaboration between Bernardo Galvão-Sousa and Craig Sinnamon.

Spring data

Statement. A motion sensor was set up to measure the motion in a spring-mass system, but something went wrong and the motion sensor measured the total distance traveled instead of simply measuring the distance from the sensor. The experiment was conducted three times (same spring and same mass) with different initial conditions. The total distance traveled is given as the data sets in the Google Sheet spreadsheet:

- `https://goo.gl/AFMTn8`

The conditions for the three experiments were:

Data Set #1. Initial position: $y(0) = 1$. Initial velocity: $y'(0) = 0$.
Data Set #2. Initial position: $y(0) = 0.5$. Initial velocity: $y'(0) = 1$.
Data Set #3. Initial position: $y(0) = -0.75$. Initial velocity: $y'(0) = -2.5$.

Experimental Setup.

- The sensor gathered data at a rate of 20 samples per second.
- The experiment was run for 5 seconds.
- $y(t)$ is the distance from the equilibrium position of the spring-mass system. Positive values of $y(t)$ indicate that the mass was above the equilibrium position. Negative values of $y(t)$ indicate that the mass was below the equilibrium position.
- There is some noise in the data.

Task.

1. Use the data in the spreadsheet to determine the governing ODE, which should include estimates for the parameters.
2. Use the data in the spreadsheet to determine the height $y(t)$ for the different experiments.

Hint. Recall that if $y(t)$ is displacement and $v(t) = y'(t)$ is velocity then the total distance traveled is given by the function

$$d(t) = \int_0^t \left|v(\tau)\right| d\tau.$$

Further Investigation.

1. How many data sets and how many data points are needed to be able to solve the problem?
2. Add more noise to the data. Can you still solve it? How much noise can you add and still obtain good results?
3. Create your own (fake) data mimicking a spring with different properties (remember to include some noise in the data)? And solve it to show that it can be done.
4. Could you use this to detect an external force acting on the spring-mass system? Try it with two new data sets:

    - `https://goo.gl/TxzQWw`

Wing flutter

Question. Is the airplane wing going to break?

Introduction (adapted from Stuart Lee – Click here for the original).

In early 1959, with great fanfare, Lockheed's new, 4-engine prop-jet, the Electra II, went into service. The Electra looked like a "regular airline", except that the thick prop blades and the four enormous large engine covers (the nacelles and cowlings) that housed the General Electric/Allison jet-turbine driver power plants made the wings seem ever smaller and stubbier. In addition, the fuselage was relatively wide- making it one of the roomiest airliners of its time. But the Electra's appearance seemed slightly off.

The pilots soon got over the appearance and came to respect the airplane, The Electra had incredible power. One pilot remarked that "It climbs like a damned fighter plane!".

In the evening of September 29, 1959, Braniff's spanking new Electra disintegrated in midair (description).

What had caused this brand-new jet prop to disintegrate over Buffalo, Texas?

The investigators combing the wreckage of the Braniff Electra noticed something alarming. The shards of what appeared to be the left wing were found a considerable distance from the rest of the wreckage.

And the story got worse. On March 17, 1960, Northwest Airlines flight 710 left Minneapolis-St. Paul (description). Witnesses on the ground heard tearing sounds in the sky. They looked up and saw the thick fuselage of the Electra emerging from the clouds. The entire right wing was missing, and only a stub of the left wing remained attached to the Electra.

The airliner seemed to float for a while, but then it dipped, diving straight down toward the ground, trailing white smoke and pieces of aircraft. The 63 people entombed in the fuselage struck the muddy ground, vertically, at 618 miles per hour. Rescuers found nothing at the site of impact larger than a spoon.

But 3 km away, they found the wreckage of the left wing.

This was beyond, alarming. In a period of less than six months, two brand-new Electras lost their wings and disintegrated with much loss of life. What could have caused this? Could it have been severe clear-air turbulence (CAT), or was there something drastically wrong with these airliners.

The airlines who had Electra fleets were nearly panicking. Meetings were quickly set up with the FAA. Investigations were set up. Boeing lent staff, simulators, and a wind tunnel to Lockheed. Douglas contributed engineers and equipment; most notably flutter vanes that, when attached to the ends of the wings, could induce serious oscillation.

The investigation, occurring in the early sixties, was the first serious use of computer stress analysis in this field.

Electras were test flown in every possible form of turbulence. Test pilots tried to destroy the Electra by ramming it into the severe Sierra Madre air waves, over and over again. Electras were put through every possible flight maneuver that would normally cause a wing failure. Super severe wind tunnel winds were shot out at Electras and mock-ups. Over and over, every possible test was done to try and break the Electra.

Finally, on May 5, 1960, an engineer stood up at a Lockheed meeting and announced: “We’re pretty sure we’ve found it!”.

Basically, the problem was a high-speed aircraft in a conventional design. Every aircraft wing is flexible to some degree. And wing vibration, oscillation, or flutter is inherent in the design. Flutter is expected on wings. In engineering terms, there are more than 100 different types of flutter – or “modes” – in which metal can vibrate. The “mode” that destroyed the Electras was “whirl mode”.

Whirl mode was nothing new. It was not a mysterious phenomenon. As a matter of fact, it is a form of vibrating motion inherent in any piece of rotating machinery such as oil drills, table fans, and automobile drive shafts.

The theory was devastatingly simple. A propeller has gyroscopic tendencies. In other words, it will stay in a smooth plane of rotation unless it is displaced by some strong external force, just as a spinning top can be made to wobble if a finger is placed firmly against it. The moment such a force is applied to a propeller, it reacts in the opposite direction.

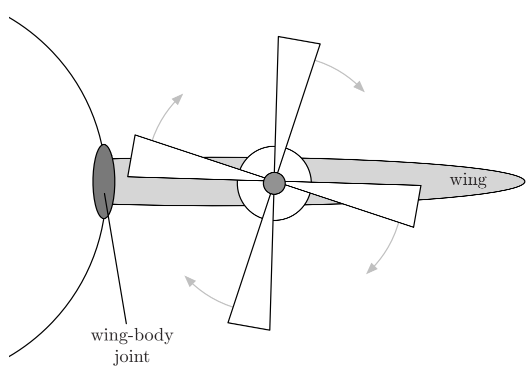
Now suppose the force drives the propeller upward. The stiffness that is part of its structure promptly resists the force and pitches the prop downward. Each succeeding upward force is met by a protesting downward motion. The battle of vibration progresses. The propeller continues to rotate in one direction, but the rapidly developing whirl mode is vibrating in the opposite direction. The result, if the mode is not checked, is a wildly wobbling gyroscope that eventually begins to transmit its violent motion to a natural outlet: the wing.

Whirl mode did occasionally develop in propeller-driver airliners. It always encountered the powerful stiffness of the entire engine package, the nacelles and the engine mounting, the mounting being a bar truss holding the engine to the wing. No problem usually. But on painful microscopic examination of the crash wreckage of the eight Electra engines, it was found that something caused the engines to loosen and wobble, causing severe whirl mode, which tore off the Electra’s wings. Specifically, the investigation centred on the outboard engines.

What the investigators found was that the engine mounts weren’t strong enough to dampen the whirl mode that originated in the outboard engine nacelles. The oscillation transmitted to the wings caused severe up-and-down vibration, which grew until the wings tore right off.

Project. In this project we will study mechanical resonance of an airplane wing due to a vibrating propeller. We use Differential Equations to create a simple model of the wing flutter.

Start with a picture of a propeller mounted on a wing.



We want to keep the model simple, so we consider only the wing’s centre of mass. This implies that the wing behaves as a **spring-mass system**: the spring is the wing-body joint that allows the centre of mass to move up and down.[3] The forcing function is the vibrational force that results from the motion of the propeller.

[3] The centre of mass actually moves on an arch, but we consider only its vertical motion.

For this example assume that the wing has a mass of 900 kg and the wing-body joint acts as a spring with constant 8100 N/m. Also assume that the damping forces are negligible and the wing is at rest when the propeller begins to vibrate.

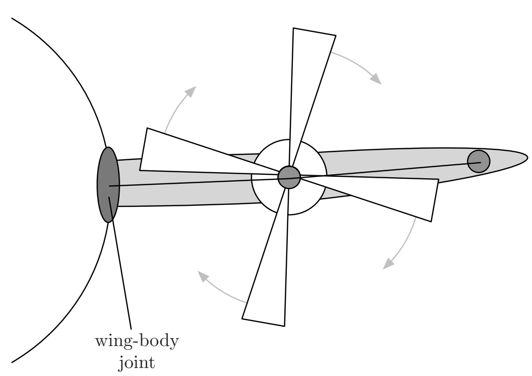
Task.

1. Let $y(t)$ be the position of the centre of mass of the wing and $f(t)$ the vertical vibrational force from the propeller. Write an IVP (Initial-Value Problem) that models the movement of the wing.
2. Assuming that the propeller vibrates with a force $f_1(t) = 1800\sin(6t)$ (in N). Find the position of the wing's centre of mass and plot it.

   Describe the position of the wing's centre of mass as t grows large ($0 \le t \le 25$).
3. Just before wing-failure, the propeller actually slowed down. Let us simulate this by changing the forcing function to $f_2(t) = 1800\sin(3t)$ (in N).
   - **(a)** Find the equation of motion using the new forcing function.
   - **(b)** Plot the solution.
   - **(c)** Describe the position of the wing's centre of mass as t grows large. What consequences does this have for the wing?
4. It is very unlikely that the frequency of the propeller will match exactly this, so assume that $f_3(t) = 1800\sin(3.5t)$ (in N).
   - **(a)** Find the equation of motion using the new forcing function.
   - **(b)** Plot the solution.
   - **(c)** Using a trigonometric identity, re-write your solution as a product of two trig functions. Describe how this new form for the solution explains the plot.
   - **(d)** Describe the position of the wing's centre of mass as t grows large. What consequences does this have for the wing?

Further Investigation.

1. If you were the Lead Engineer in charge of fixing this problem, what would you do? How would that change the Differential Equation? Using the new differential equation, show that it would indeed solve the problem.
2. What happens if there are two propellers (like the actual Lockheed Electra)?
3. Can you model wing flex?



Bullwhip effect

Goal. Understand that Supply Chain Management is hard(!) and attempt to model it.

Watch the short movie:

- `https://youtu.be/2nlmkTYZG5s`

to understand a bit better about the bullwhip effect.

Read also:

- `http://forio.com/about/blog/bullwhips-and-beer/`

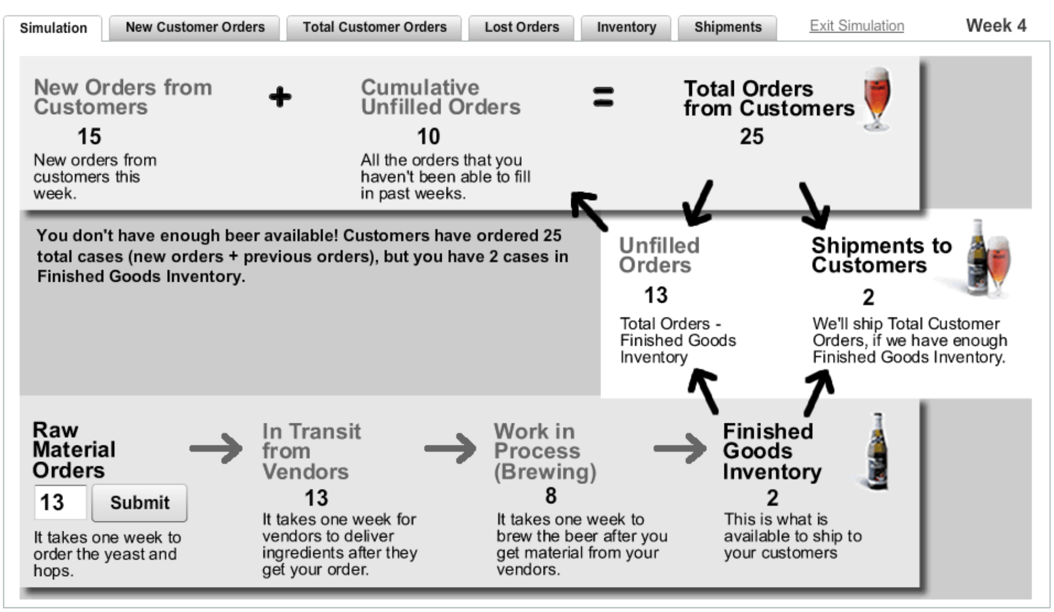
Near Beer Game (Novice). You own a (tiny) Beer Store.

You start with a stable situation where your customers have been asking for 10 cases of beer every week, and your inventory and orders match the situation (so you don't run low on inventory and you don't accumulate either).

From the second week, your customers start ordering 15 cases of beer instead.

It is you job to stabilize the whole supply chain as soon as possible.

Below is a screen from the "game".



- New Orders from Customers: Number of beer cases your new customers want this week
- Cumulative Unfilled Orders: Number of beer cases that your

Task. Go to

- `https://forio.com/simulate/mbean/near-beer-game/run/`

The goal of the "game" is to try and stabilize the number of customer orders, your inventory, arriving orders, and your order, so that you end up with the following situation

- Customer Orders: 15 cases every week (with no unfilled orders)
- Inventory: 15 cases
- Arriving Order: 15 cases
- Order 15 cases

1. Play the "game" on **`Novice`** as a group and see how many weeks it takes to stabilize the situation.

   Consider the following sequences:
   - c_n = number of beer cases ordered by customers
   - u_n = number of cases ordered previously but not fulfilled yet
   - i_n = number of cases in inventory
   - o_n = number of cases ordered
   - r_n = number of cases of beer produced

   where n is the number of weeks elapsed since the beginning of the "game".

   (a) What are the initial conditions $(n = 0)$?
   (b) What is the formula for c_n?
   (c) What is the formula for r_n?
   (d) What is i_n?
   (e) What is u_n?
   (f) Confirm that your modelling is correct, that is, that your variables follow the outcome of the game.
   (g) Decide on a strategy for ordering beer cases. Decide on a formula for o_n that can depend on n, c_n, u_n, i_n, r_n. Explain your choice.
   (h) What is the result of your strategy? Does it go "amuck" – bullwhip effect[4]? Or does it control the supply chain nicely?

[4] It's ok if it goes "amuck"! The goal is to see the Bullwhip Effect in action... Now try to fix it!

2. Play the "game" on **`Expert`** as a group and see how many weeks it takes to stabilize the situation.

   Consider the same sequences as for **1.**

   The difference between `Novice` and `Expert` is that the customer orders go from 10 → 50 and every week 25% of unfilled orders are cancelled.

   (a) What are the initial conditions ($n = 0$) ?
   (b) What is the formula for c_n?
   (c) What is the formula for r_n?
   (d) What is i_n?
   (e) What is u_n?
   (f) Confirm that your modelling is correct, that is, that your variables follow the outcome of the game.
   (g) Decide on a strategy for ordering beer cases. Decide on a formula for o_n that can depend on n, c_n, u_n, i_n, r_n. Explain your choice.
   (h) What is the result of your strategy? Does it go "amuck" – bullwhip effect? Or does it control the ordering nicely?

Further Investigation.

1. There is a more complex version of the game

   - `https://beergame.pipechain.com/`

   which includes 1–4 players from 4 different stages of the supply chain. It takes 2 weeks for orders to arrive to a different stage and it takes 2 weeks to fulfill a request.

   (a) Play the game with 2 players[5] who do not communicate with each other, i.e., two-stage supply chain.
   (b) Define the new sequences
   - c_n = number of beer cases ordered by customers
   - o_n = number of cases ordered by the retailer
   - s_n = number of cases in the retailer's stock
   - p_n = number of cases ordered by the producer
   - q_n = number of cases in the producer's stock

   (c) Make a similar study for this case. Observe that now you have to decide on the strategy for both o_n and p_n.

2. In the article suggested at the beginning

   - `http://forio.com/about/blog/bullwhips-and-beer/`

   the author describes ways a few ways to reduce the Bullwhip effect. Program each of them with your sequence o_n and study how well they reduce the effect.

3. You can avoid the Bullwhip effect completely with perfect information about the supply chain and the future customer demand. In reality, we can predict the customer demand, but it won't match exactly the prediction. Add a little noise to customer demand and try to avoid the Bullwhip effect. You can still use the fact that customer demand will still be close to 15 cases every week.

[5] Create a game and then use another computer to join the same game

Approximating the temperature of a thin sheet

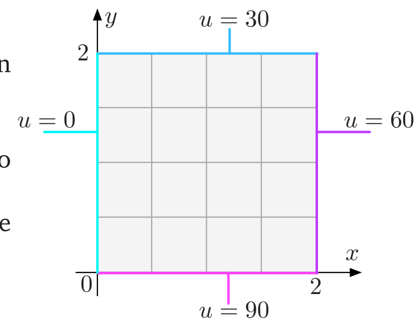
Goal. We want to approximate solutions of a PDE.

The heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where $u(x, y)$ is the equilibrium temperature at the position (x, y) given some boundary conditions.

This is a Partial Differential Equation (PDE), which we don't know how to solve. We can however obtain an approximation of the solution.
In this example, the domain is shown on the right $\Omega = [0,2] \times [0,2]$ and the initial conditions are the following



$$u(x,0) = 90 \quad , \quad u(x,2) = 30 \quad , \quad u(0,y) = 0 \quad , \quad u(2,y) = 60.$$

To approximate the solution, we divide the domain in N small pieces. In the example $N = 4$ and $\Delta = \frac{2-0}{N} = \frac{1}{2}$.
Then we define the points

$$\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots, \vec{p}_M,$$

as the points in the interior of the domain (usually by moving left→right and bottom→top).

1. What are the points $\vec{p}_n$? What is M?

Then we define

$$u_n = u(\vec{p}_n),$$

where $u(x, y)$ is the solution of the initial-value problem above (PDE with boundary conditions).

The next step is to approximate the PDE itself. We do that by approximating the derivatives:

$$\frac{\partial u}{\partial x}(x_0, y_0) \approx \frac{u(x_0 + \Delta, y_0) - u(x_0, y_0)}{\Delta}.$$

2. What is the approximation for $\frac{\partial u}{\partial x}(\vec{p}_5)$? What is the approximation for $\frac{\partial u}{\partial x}(\vec{p}_3)$?

3. What is an approximation for $\frac{\partial u}{\partial y}(x_0, y_0)$? What is the approximation for $\frac{\partial u}{\partial y}(\vec{p}_8)$?

From here, we define the second derivative in a similar fashion:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}(x_0, y_0) &\approx \frac{\frac{u(x_0 + \Delta, y_0) - u(x_0, y_0)}{\Delta} - \frac{u(x_0, y_0) - u(x_0 - \Delta, y_0)}{\Delta}}{\Delta} \\ &= \frac{u(x_0 + \Delta, y_0) - 2u(x_0, y_0) + u(x_0 - \Delta, y_0)}{\Delta^2}.\end{aligned}$$

4. What is the approximation for $\frac{\partial^2 u}{\partial x^2}(\vec{p}_5)$? What is the approximation for $\frac{\partial^2 u}{\partial x^2}(\vec{p}_3)$?

5. What is an approximation for $\frac{\partial^2 u}{\partial y^2}(x_0, y_0)$? What is the approximation for $\frac{\partial^2 u}{\partial y^2}(\vec{p}_8)$?

We are now ready to put it all together.

The PDE applies to all points in the domain. Instead of applying the PDE to all points $(x, y) \in \Omega$, we apply the approximation of the (second) derivatives to all the points $\vec{p}_n$.

6. What is the equation that we obtain for the point $\vec{p}_5$?

7. What is the equation for each point $\vec{p}_n$?

These equations form a linear system of equations. Define a vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$

8. Write the system of equations in matrix form $\mathbf{A}\vec{u} = \vec{b}$.

9. Solve it and plot the solution. (You should use some software to solve this!)

MATLAB. Here is a quick introduction to some tools in MATLAB that are useful for this problem.

- Define a matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by

```
» A=[1,2;3,4]
```

- Define a vector $\vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ by

```
» b=[5;6]
```

- Solve the system $\mathbf{A}\vec{u} = \vec{b}$ by defining $\vec{u} = \mathbf{A}^{-1}\vec{b}$

```
» u=A\b        or        » u=inv(A)*b
```

- To plot a 3D plot like this, define a matrix for the solutions and write

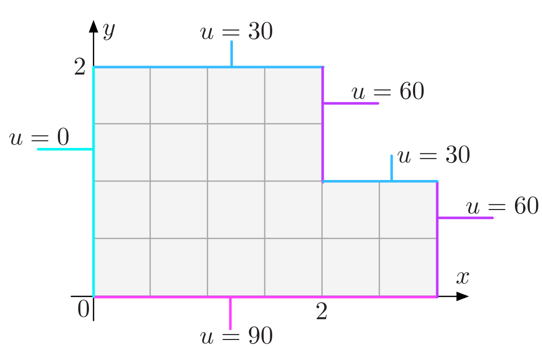
```
» surf(p)
```

  To use the typical colouring for the heat equation, type

```
» colormap(cool)
```

Further Investigation.

1. Approximate the solution for the domain and boundary conditions



2. Formulate the procedure for a general N.

3. Formulate the procedure for a different Δx and Δy.

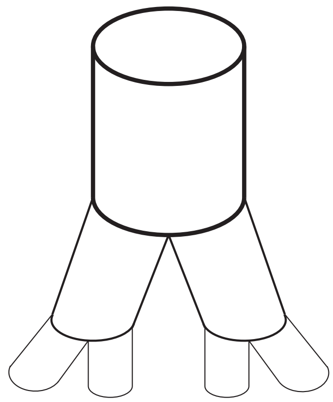
4. This method can be adapted to what kind of domains? And what kind of boundary conditions?

Math of lungs

Lungs are somewhat important to human beings! They are the source of oxygen to our bodies, so it is important to maximize the amount of oxygen that can be absorbed within the available volume.

Task 1. Let us find out the volume and surface area of the lungs.

1. The lungs are composed of a branched structure as in the figure below.

   The first segment is very large, then it bifurcates into smaller segments in a geometrical pattern.

   Here is some data about human lungs:

   - radius of first segment: $r_0 = 0.5\text{cm}$
   - length of first segment: $\ell_0 = 5.6\text{cm}$
   - ratio of daughter to parent length: $\alpha = 0.9$
   - ratio of daughter to parent radius: $\beta = 0.86$
   - number of branch generations: $M = 30$
   - average number of daughters per parent: $b = 1.7$

   In the figure, there are 2 daughters per parent, in real lungs, it isn't perfectly regular, so we have an average that is not a whole number.

2. Calculate the volume inside the segments. What is the limit as the number of segments gets larger and larger?

3. Calculate the surface area inside the segments. What is the limit as the number of segments gets larger and larger?

4. If we had $b = 2$ instead and the same number of generations, would that be possible? If not, how many generations would be possible?

Task 2. Let us model the gas exchange that happens inside the lungs.

1. Suppose that a lung has a volume of 3L when full. With each breath, 0.6L of the air is exhaled and replaced by 0.6L of outside air.

   After exhaling the volume is 2.4L and it returns to 3L after inhaling.

   Suppose further that the lung contains a chemical with a concentration of 2 milimoles per litre before exhaling (a mole is a chemical unit for 6.02×10^{23} molecules). The ambient air has a concentration of 5mmol/L of the same chemical.

   What is the concentration after one breath? What is the concentration after n breaths?

2. Update your model to match the oxygen exchange inside a real human lung.

3. The model above ignored the fact that the body absorbs some of the oxygen. Assume now that the lungs absorb 30% of the oxygen with each breath. Update your model.

4. What is the equilibrium concentration of oxygen in the lungs? Find the graph of the equilibrium concentration as a function of the fraction of oxygen absorbed with each breath.

You may choose to model the gas exchange in the lungs using either a discrete time ***difference equation*** or a continuous time ***differential equation***.

(a) If you choose a difference equation, you might assume that every breath the concentration of the chemical changes. If $c(n)$ or c_n represented the concentration of chemical ager n breaths, your difference equation might look like

- $\Delta c(n) = c(n) - c(n-1) =$ some function at time n, where n is only allowed to take whole numbers.

(b) If you choose to use differential equations, the analogous equation would look like

- $c'(t) =$ some function at t, where t can take any positive real value.

Task 3. Assume that the lungs only absorb a fraction of the air in contact with its surface. Combine the two previous tasks.

Further Investigation:

1. Investigate how the what is known about human lungs. Compare how the branching of real human lungs differs from this model. Refine the estimate.

2. Investigate how the what is known about human lungs. Compare how the gas exchange of real human lungs differs from this model. Refine the model.

"Math of Lungs" is a collaboration between Bernardo Galvão-Sousa, Kseniya Garaschuk, and Miroslav Lovric.

Dark day

At one in the morning, your phone goes off. After three attempts to turn off your alarm clock, you finally realize that it is a call—a call from a number you had promised to always answer. By 1:15, you're dressed and out the door where a black SUV is idling, waiting for you. After a transfer to a government plane, you touch down in Washington, D.C., and as the sun finally begins to rise, you traverse down the Secret Service tunnels to a large conference room lined with leather chairs.

"I'm not going to mince words," a voice says, from the other side of the room. The chair at the end of the table swivels round and you see the President emerge from the shadows. "It's bad. It's worse than bad. It's, uh... it's zombies."

A revelation like that would have thrown a lesser scientist, but you're a professional. You've been preparing for this for years, urging your colleagues to take the threat seriously.

"Where is the origin? Have we identified a patient zero in the US or are there multiple sources? What are the parameters of the disease?" you ask.

"Straight to work, okay," the President says, looking pleased. "Chicago police began reporting violent attacks a few days ago. It started with a single report in the navy shipyard. The shipyard has been quarantined, but we're now getting reports from all over the city. These attacks are carried out by humans who always attempt to bite their victims. Those bitten begin to show symptoms within a matter of hours, but those who are attacked but escape without being bitten appear normal. I've sent in the Marines, and they estimate that an infected person dies after eight days. They also predict about 3,000 individuals have been exposed, five days after the initial report."

"Has any quarantine been successful?" you ask.

"No." The President pauses and the gravity of what he has just said starts to sink in. "There are approximately 9.7 million people living in the Chicago metro area. Obviously time is of the essence. Now, I've been told that you're the best epidemiologist we have. I need to know if the region has a chance of survival, and if it does, what the impact will be. Is there any hope for Chicago?"

You may choose to model the zombie outbreak using either a discrete time ***difference equation*** or a continuous time ***differential equation***.

(a) If you choose a difference equation, you might assume that every day or every hour, the number of zombies, humans, and dead increment. If $Z(n)$ represented the number of zombies at time n, one of your difference equations might look like

- $\Delta Z(n) = Z(n) - Z(n-1) =$ some function of zombies, humans, and dead at time n, where n is only allowed to take whole numbers.

(b) If you choose to use differential equations, the analogous equation would look like

- $Z'(t) =$ some function of zombies, humans, and dead at time t, where t can take any positive real value.

(c) Modelling with a difference equation or a differential equation should give you similar results (why?).

Task. Model the zombie infection. Make sure to address the following in your report:

- What situation are you trying to model?
- What equations are you using, and what does each variable in each equation represent (for example, "In this model, $Z(t)$ is the number of zombies at t hours from initial outbreak).
- Justification for any constants that you use and how you estimated them.
- Is there any hope for Chicago?

Do not attempt to find an equation that solves your differential equations—this is really hard. Instead, rely on estimates and simulations. You can use any computer program you like to assist you in estimating how the zombie outbreak spreads and whether your constants match with the known information. Including plots and figures in your report will make explaining things easier.

"Dark Day" is a collaboration between Max Brugger and Jason Siefken.