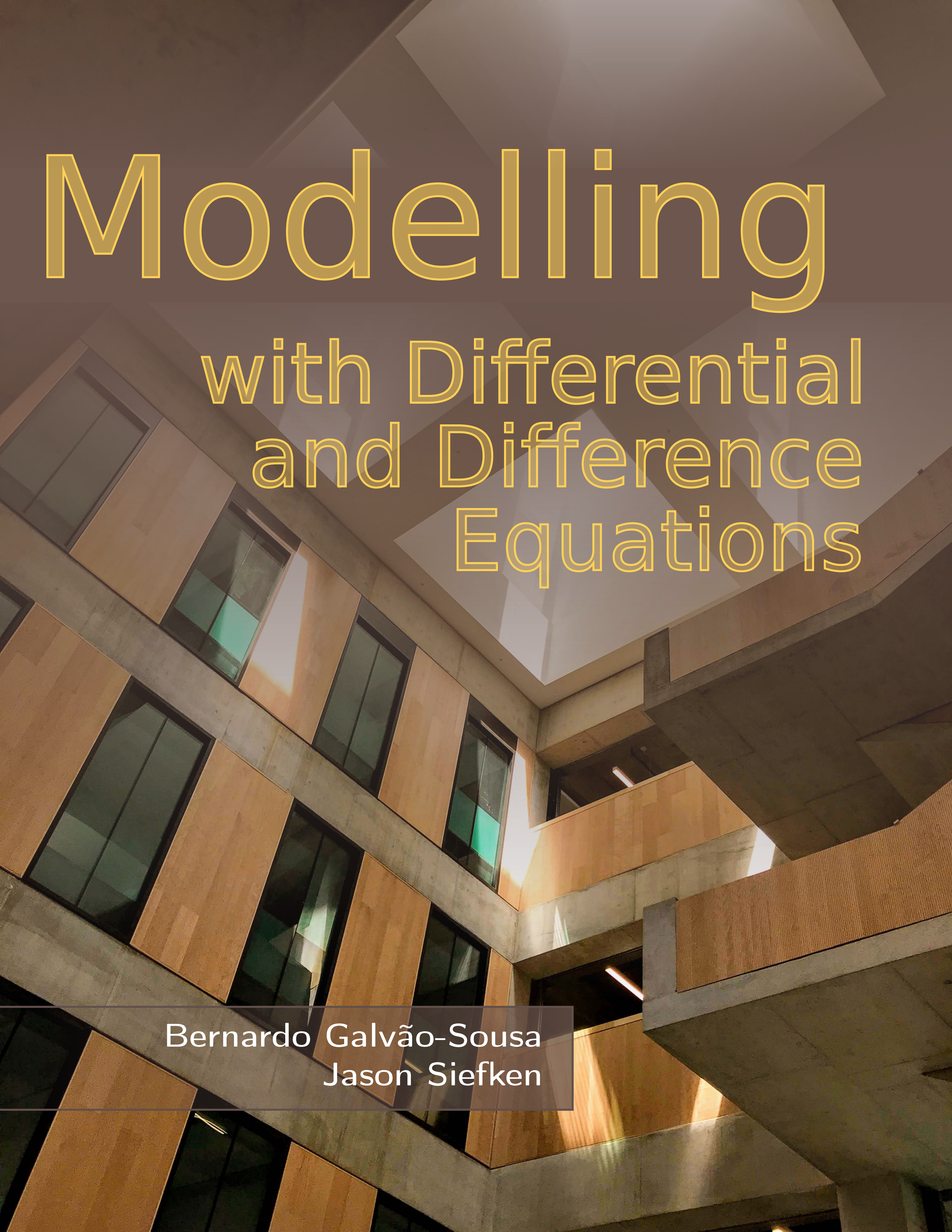

Modelling with Differential and Difference Equations

Bernardo Galvão-Sousa
Jason Siefken

Inquiry Based Modelling with Differential and Difference Equations

For the student

This book is your introductory guide to mathematical modelling and to differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling and differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle though you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a linear algebra concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.
5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.
2. **Work through the core exercises.**
3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.
4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

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Included in this text, in chapter 1, are expositions adapted from the handbook "Math Modeling: Getting Started and Getting Solutions" by K. M. Bliss, K. R. Fowler, and B. J. Gallizzo, published by SIAM in 2014 `https://m3challenge.siam.org/resources/modeling-handbook`.

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/bigfatbernie/IBLModellingDEs/`

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Stephanie Orfano ◦ Yvan Saint-Aubin ◦ Sarah Shujah ◦ Graeme Slaght ◦

Contents

In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analysis of the model
6. Writing a report

In this chapter, we follow the approach of Bliss, Fowler, and Galluzzo from

```
Math Modeling: Getting Started and Getting Solutions, K. M. Bliss,
K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014

  https://m3challenge.siam.org/resources/modeling-handbook
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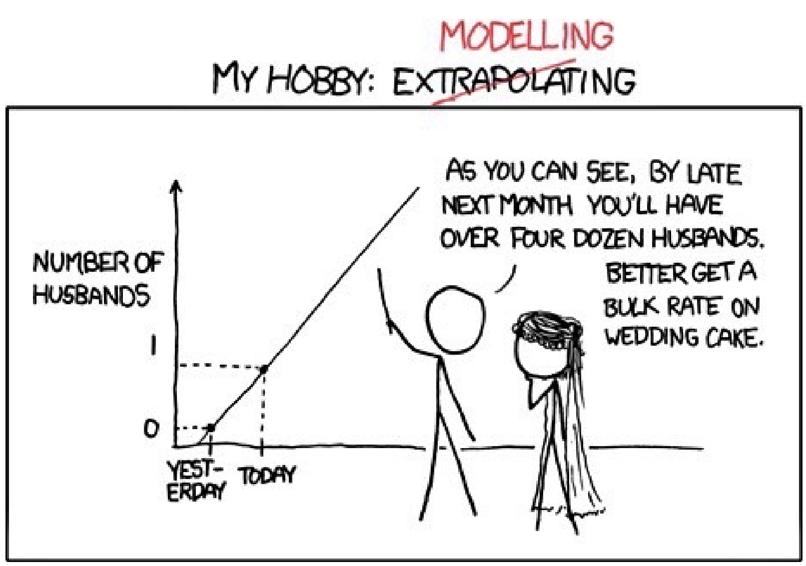


(image from xkcd - comic #605)

Defining the Problem

In this module you will learn

- how to define a problem mathematically.

The first step is to define the problem we want to solve.

To do this, we should start from the end!

We need to decide on what kind of mathematical object we will use in the end to show that we solved the problem we were tasked with.

Once this is done, we can define the problem mathematically.

Example. Your team was tasked with optimizing the layout of an airport.

The team decided to define:

- $T =$ the total time (in minutes) necessary by the average person to walk from their airport transportation (taxi, train, bus) to their gate, disregarding the time spent in security or immigration.

At the end of the project, to show that the team did find a good layout for the airport, the team will show that the new layout reduces the value of T.

Once this decision is made, the problem to solve (or improve) becomes clear:

- Minimize T

There will probably be some constraints, which will be studied in Module 4.

Practice Problems

1 For each part, what "mathematical object" would you use to communicate that you have solved or improved the problem? Then define the problem mathematically.

(a) Help the city of Toronto choose the best recycling system.

(b) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

(c) Create a mathematical model to rank roller coasters according to thrill factor.

(d) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

(e) Is it better to buy or rent?

i. Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

ii. Does the criteria you used to evaluate the previous question change if the question is whether to buy a bicycle or use Bike Share Toronto?

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their “elevator problem”.

This is the email you received:

——— Forwarded Message ———

Date: Mon, 16 September 2019 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they’re still late !?&!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that’s about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

Teamwork.
With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.

2 The mayor of Toronto wants to extend the subway line with a new **orange line** as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

Building a mind map

In this module you will learn

- How to create a mindmap.

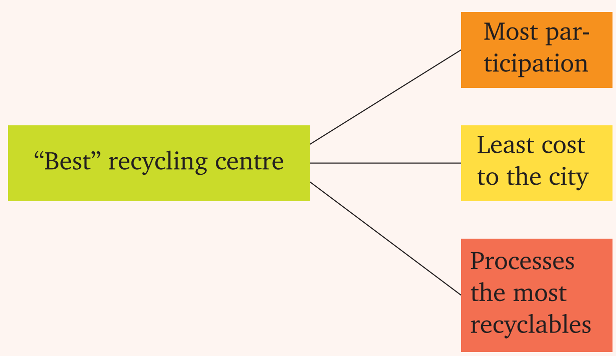
A mind map is a tool to visually outline and organize ideas. Typically a key idea is the centre of a mind map and associated ideas are added to create a diagram that shows the flow of ideas.

Example.

Let us focus on the question: "What is the best recycling system for Toronto?"
Then we can think of many different definitions for what the word "best" means:

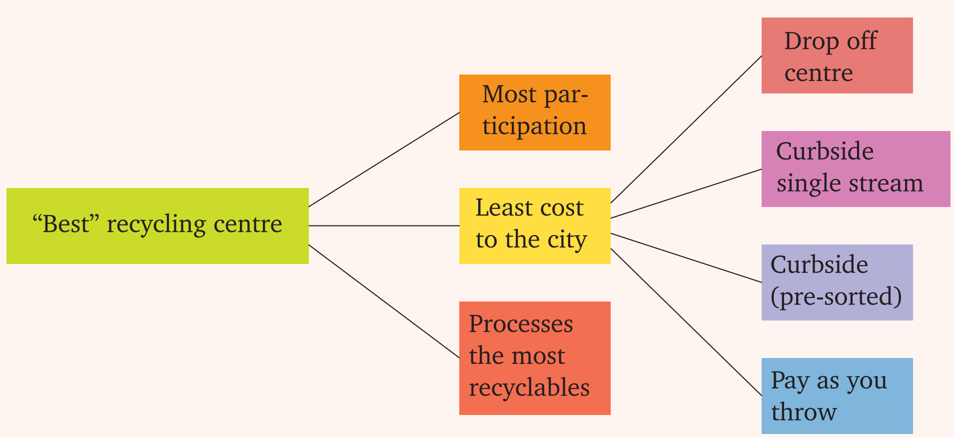
- The system that gets the most participation from the population, which can be measured by the fraction of the Toronto households participating in recycling;
- The system that costs the least amount of money for the city. How can this be measured?
- The system that processes the most amount of recyclables.

In the figure below, we focus on the definition of "best", with these three possible definitions branching off to be further explored.



From here, we can focus our attention on one of the branches at a time.

Let's think about the least-cost option first.
We probably can't determine how much any recycling program costs without knowing more about the recycling program, so a good place to start is to ask the question "What kinds of recycling programs exist?" If we aren't familiar with different types of recycling, we might need to do some research to see what kinds of programs exist.
A possible next step on your mind map for the least-cost approach could be the one shown below.



Important. There is free online software to help creating a mind map. One such is FreeMind (http://freemind.sourceforge.net).

For more details on creating a mind map, check the book:

`Math Modeling: Getting Started and Getting Solutions, K. M. Bliss, K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014`

`https://m3challenge.siam.org/resources/modeling-handbook`

Practice Problems

1 Expand the mind map from the example above by focusing on the other two approaches:

(a) Most participation

(b) Processes the most recyclables

2 For each part, create a mind map. Focus on the same approach you had for question 1 from Module .

(a) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

(b) Create a mathematical model to rank roller coasters according to thrill factor.

(c) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

(d) The mayor of Toronto wants to extend the subway line with a new blue line as in core exercise 2. Is it optimal?

(e) Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $R =$ the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can R be minimized?

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to optimally control the traffic lights.

4.1 What "mathematical" meaning can we give to the word optimal?

4.2 Create a mind map for this problem.

Making assumptions

In this module you will learn

- that we need to make assumptions to be able to create a model
- how to strike a balance between accuracy and solvability

Real problems are complex, so when modelling a real problem mathematically, we must make some assumptions.

The assumptions that we make will affect the problem we are solving and its difficulty, so we need to strike a balance between:

- accuracy – the fewer assumption the better, and
- solvability – the more assumptions the better.

Many assumptions follow naturally when building a mind map.

When figuring which assumptions to make, keep in mind the key-factors of the problem and find data when available (usually online). If not available, measure data when possible, and if it's not possible, make a reasonable assumption on what the data might look like.

Another thing to keep in mind are **time constraints**. Whether in a class, test, or working in a project, there will be deadlines. Your assumptions should take time constraints into consideration.

Example.

Let us revisit the example of the previous module about the "best" recycling centre.
For this example, imagine that the team decided on focusing their attention on the least cost to the city through building drop off centres.

For this, we need to find out how many people would make use of the drop-off centres (termed "likelihood of participation").

The two extremes would be to assume that the 100% of the people near a recycling centre would use it or that none would use it. Neither of these seems like a reasonable assumption, so what would be a better assumption?

Maybe the best idea is do some investigation and see if there has been any successful research on participation rates in drop-off centres.

The team found a study that had been done in Ohio that estimated that about 15% of households participated in drop-off centre recycling, and made an assumption that this rate would hold in every city across the U.S..

One might ask if it is safe to assume that across the U.S. 15% of households will participate in drop-off centre recycling if it is available. Is it true that residents of Arizona will behave the same way residents of Ohio do? Certainly some cities would garner a participation rate much higher than 15%, while other cities would have a significantly lower participation rate. In fact, what are the chances that any city would actually have a participation rate of exactly 15%?
In some sense, one might say that assigning one participation rate to every city across the U.S. is a ridiculous assumption.

In response to that line of thinking, remember two things:

- First, remember that ***one must make assumptions in order to make a model***. It is not practical or feasible to poll every citizen of every city to determine who will bring recyclables to a drop off centre. If we had to rely on data with that level of certainty at every juncture of the modelling process, we would never get any work done.

  It's practical and important to make reasonable assumptions when we cannot find data.

- Second, you are developing a model that is intended to help one understand some complex behaviour or assist in making a complex decision. It is not likely to predict the exact outcome of a situation, only to help provide insight and predict likely outcomes. When you ***provide a list of your assumptions***, you've done your part to inform anyone who might use your model. They can decide whether they think your assumption is or is not appropriate to model the behaviour they are interested in predicting.

Practice Problems

1 For each part, you are required to make an estimate for some quantity. Make assumptions and justify them in order to solve the problem.

(a) What is the number of piano players in Toronto? **(Fermi problem)**

(b) How many linear km of roads are there in Toronto?

(c) How much salt the city of Toronto needs for its roads during the Winter?

(d) The skating season in Canada is shortening: What are the key-factors determining its length?

5 Consider the elevator problem from question 1.

We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie (in Montréal).
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Focus on a **few** parameters and variables. State hypotheses.

6

6.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

6.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

6.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Construct a model

In this module you will learn

- how to build a model based on the previous steps

This is the part of the modelling where we connect all that we have done so far: the problem we defined, the mind map, the assumptions, and all the variables and parameters in a mathematical model to answer the "mathematical" problem defined in Step A.

This usually means writing down mathematical equations, constructing a graph, analyzing a geometric figure, or do some statistical analysis.

Example. Your team is tasked with finding the best recycling centre (we looked at this example in Step B) and your team has chosen to minimize the cost to the city by using drop off centres.

As part of modelling process, your team has made the following assumptions/measurements:

- People would be willing to pay \$2.29 to recycle per month or \$0.53 per week
- People would make bi-weekly trips to the centre
- Gasoline costs around \$1.26 per litre
- On average a passenger car needs 10 litres per hundred kilometres

This means that the (one-way) distance people are willing to travel every week to the drop-off centre is

$$d = \frac{1}{4.3 \text{ trips/month}} \cdot \frac{\$2.29/\text{month}}{(\$1.26/\text{L}) \cdot (0.1 \text{ L / km})} = 4.2 \text{ km/trip}.$$

This should help us figure out the best way to place the drop-off centres:
The Mathematical model might look like this

- Maximize (number of people within a 4.2 km radius of a drop-off centre)
- subject to a certain number of drop-off centres (given by the city budget)

Sometimes, the mathematical tools necessary to tackle the problem are clear, but often they are not. In those cases it may be helpful to analyze some simple cases.

Module 4: Construct a model

Practice Problems

1 For each part, create a model to answer the question. Remember all the previous steps.

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

7 With the same details as before in 5, write down a mathematical model for this problem.

Model Assessment

In this module you will learn

- how to analyze a model to check whether it makes sense

At this point, you have defined a problem statement, and a mind map to help you decide how to approach the problem. You have made assumptions and made note of them and justified them. You finally created a model to solve the problem.

The next step is to analyze the model.

There are two types of analysis:

Superficial assessment. Are the units correct? Are the variables and parameters of a reasonable magnitude? Does it behave as expected? Does it make sense?

In-depth assessment. Once the superficial assessment is verified, we need to understand the model at a deeper level.

What are the model's strengths? What are its weaknesses?

When you change the inputs of the model, how do the outputs change? This is called sensitivity analysis.

Next is a simple example adapted from [**?**].

Example. Modelling the flu

History of the project:

- Split population into two classes: ***infected*** and ***not infected***
- Assume that each infected person infects R number of non infected people every b days
- Define $I(n) =$ number of infected people after n days
- The two previous points imply $I(n \cdot b) = R \cdot I(n)$
- We can then conclude that $I(nb) = (1+R)^n I(0)$ (why?)

After plotting the resulting function $I(n)$ (click or follow the QR code on the right), we can assess our model:

Strengths:

- After two days ($b = 2$), there are 6 infected people, so it is following our assumption
- The number of infected people increases faster and faster as expected
- The disease spreads at a constant rate. Also on Desmos, check the infection rate $\dfrac{I(n+b)}{I(n)}$
- We could find an explicit formula for the number of infected individuals $I(n)$

Weaknesses:

- The model is too simple, so it doesn't model the spread of the flu accurately
- The model an exponential rate of infection, which is not possible for very long
- The model predicts that eventually the disease will spread to everyone
- The model assumes that there are only two types of people: infected and susceptible. Do people recover from the disease?

After assessing the model, if time allows, it is important to re-think the model and the assumptions made.

Practice Problems

1 Assess the models created in question **??**:

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\begin{array}{c}\text{loading time on}\\ \text{ground floor}\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ \text{ground} \to 33\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ 33 \to \text{ground}\end{array}} + \boxed{\begin{array}{c}\text{stop time to}\\ \text{6 of the 11 floors}\end{array}} \quad = \quad 106\text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes

Your task is to assess this model. Be ready to report on your assessment.

Putting it all together

In this module you will learn

- how to put all that you have done together into a well structured report

This is the final stage of the modelling project.

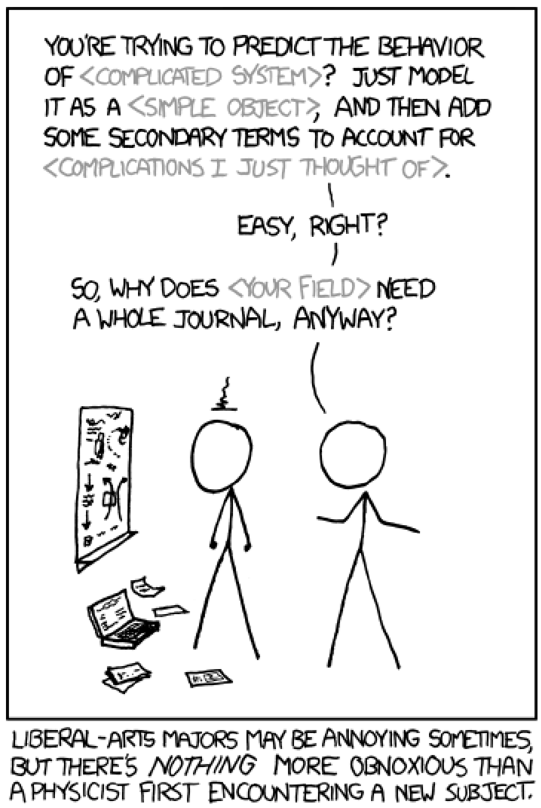
By now, you have started with a mathematically defined problem, with some assumptions, and you have created a mind map to help you navigate the problem. You have also constructed a model and assessed it to make sure it is sound.

All that we have left is to put all this work together into the form of a report.

The report should consist of two parts:

1. **Summary.** Should be at most one page long, and contain a statement of the problem, a brief description of the methods chose to solve it, and some final results and a conclusion. In this part of the report, you should keep mathematical symbols to a minimum, so the reader gets an idea of what to expect in the remainder of the report without getting bogged down in unfamiliar mathematics.
2. **In-depth report.** This is where the details go in. It should start with an introduction to the problem assuming that the reader is not aware of it. It should then be structured according to the steps we did before:
   - Optionally, you can include a mind map with a description of how it guided the whole process
   - Assumptions and variables in the model
   - The model described in detail
   - The solution process
   - The assessment of the model
   - A conclusion, with a description of the results

Example. You can find the report from the winning team of the 2019 M_3C challenge in appendix 6.2.



(image from xkcd - comic #793)

Introduction to Differential Equations

In this module you will learn

- what is a differential equation
- the different types of differential equations

Differential Equation. A ***differential equation*** is an equation involving an unknown function and one or more of its derivatives.

Among differential equations, there are lots of types, that require different approaches, so we need to classify them.

Types of Differential Equations. There are two main types of differential equations:

- ***Ordinary differential equations***, usually denoted as ODEs – when the unknown function is a function of one variable;
- ***Partial differential equations***, usually denoted as PDEs – when the unknown function is a a function of several variables.

In this book, we are going to focus only on ordinary differential equations.
Among ordinary differential equations, we distinguish them according to:

- ***order***: the order of a differential equations is the order of the highest derivative present in the differential equation;
- ***linear*** vs ***nonlinear***: A differential equation $F(t, y, y', \dots, y^{(n)}) = 0$ is called ***linear*** if F is a linear function of $y, y', \dots, y^{(n)}$. Linear ODEs have the form

$$a_0(t)y(t) + a_1(t)y'(t) + \cdots + a_n(t)y^{(n)}(t) = g(t).$$

All other differential equations are called ***nonlinear***.

Roughly, to check whether an ODE is **linear**, we need to check that:

- The unknown y and its derivatives appear with exponent 1;
- The unknown y and its derivatives do not multiply by each other;
- The unknown y and its derivatives are not the objects of other functions – there are no occurrences of things like $\sin(y)$ or $e^{y'}$, $\ln(y'')$, $\sqrt{y^{(3)}}$, etc.

In general, when tackling a differential equation, linear ODEs are easier to solve and study than nonlinear.

In the following chapters, observe how the methods and theory for linear ODEs is much more developed. Nonlinear ODEs are usually tackled on a case-by-case basis, and there is no theory that applies to a class of nonlinear ODEs.

Fortunately, many important problems are modelled by linear equations.

A common approach to nonlinear problems is to “transform” them into a linear problem. This means that the new linear problem is easier to study, but will be an approximation of the original problem, and often that approximation is only reasonable within some restricted conditions.

Example. Consider the nonlinear ODE

$$y' = -\sin(y).$$

This is a nonlinear ODE. However, by Taylor’s Theorem, we can approximate the function $\sin(y)$ by y, as long as $|y|$ is very small.
So we can say that the solution of the original solution is very close to the solution of

$$y' = -y,$$

as long as $|y|$ is very small.

Module 7: Introduction to Differential Equations

Solutions of Differential Equations

In this module you will learn

- what is a solution of a differential equation
- the difference between a solution and an integral curve

Assume that we have found a differential equation that models a situation. Often the goal is to figure out what happens, so we usually attempt to either solve the differential equation and obtain a solution or to find an approximation for the solution.

In this module, we will discuss solutions in more detail.

Solution. Given a differential equation, a ***solution*** is a differentiable function that satisfies the differential equation.

Example. Consider the differential equation

$$t\frac{du}{dt} = u + t^2\cos(t).$$

Then the function

$$u(t) = t\sin(t)$$

is a solution, because

$$t\frac{du}{dt} = t\big(\sin(t) + t\cos(t)\big) = t\sin(t) + t^2\cos(t) = u + t^2\cos(t).$$

Integral curve. We can represent all the solutions geometrically as an infinite family of curves. These curves are called ***integral curves***.

Example. Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

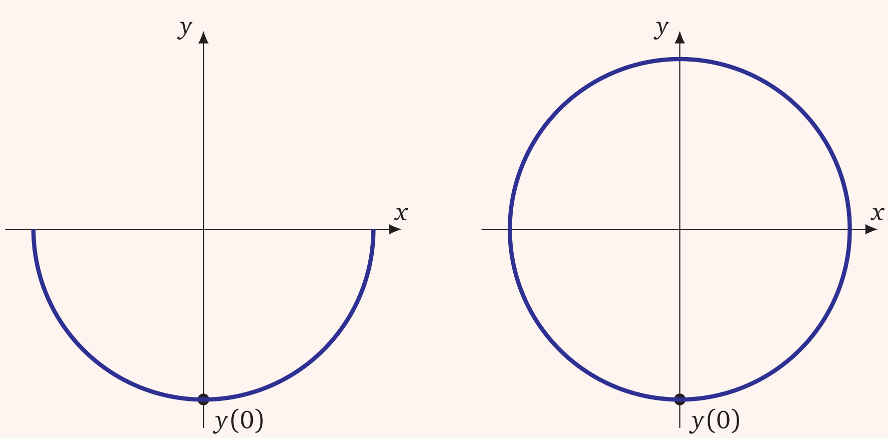
Then, we can check that curves of the form $x^2 + y^2 = C$ satisfy this differential equation.

This gives us the solution

$$y(x) = -\sqrt{9 - x^2}.$$

However, the integral curve for this initial-value problem is the curve

$$x^2 + y^2 = 9$$



Solution of the initial-value problem   Integral curve for the initial-value problem

Practice Problems

1 Check that curves of the form $x^2 + y^2 = C$ satisfy the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

2 Is the piecewise-defined function

$$y(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$?

3 Consider the differential equation

$$y^{(4)} - 8y^{3)} + 26y'' - 40y' + 25y = 0.$$

(a) Is $y = 4e^{2x}\sin(x)$ a solution?

(b) Is $y = -8xe^{2x}\cos(x)$ a solution?

(c) For the two function above, if they are solutions, what are initial conditions of the form

$y(0) =$
$y'(0) =$
$y''(0) =$
$y'''(0) =$

that the solution satisfies?

4 Consider the functions

$$\begin{array}{ll} f(x) = 3x + x^2 & g(x) = e^{-7x} \\ h(x) = \sin(x) & j(x) = \sqrt{x} \\ k(x) = 8e^{3x} & \ell(x) = -2\cos(x) \end{array}$$

Match each differential to one or more functions which are solutions.

(a) $y' = 3y$

(b) $y'' + 9y' + 14y = 0$

(c) $y'' + y = 0$

(d) $2x^2y'' + 3xy' = y$

5 Consider the differential equation $u' = -2(u - 10)$.

(a) Check that the curves of the form $u = 10 + Ce^{-2t}$ satisfy the differential equation.

(b) Sketch one solution of the differential equation.

(c) Sketch all the integral curves for the differential equation.

(d) What is the difference between a solution passing through the point $(1, 20)$ and an integral curve passing through the same point?

6 Consider the differential equation $y'\big(3y^2 - 1\big) = 1$.

(a) Check that the curves of the form $y^3 - y = x + C$ satisfy the differential equation.

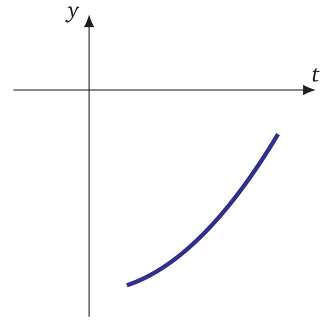
(b) Sketch the solution of the differential equation that passes through $(1, 1)$.

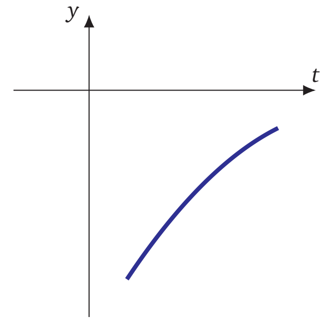
(c) Sketch the integral curve for the differential equation that passes through $(1, 1)$.

(d) What is the difference between a solution passing through the point $(1, 1)$ and an integral curve passing through the same point?

(e) Repeat (b)–(d) with the points $(1, 0)$ and $(1, -1)$ instead of $(1, 1)$.

7 Consider the ODE $y'(t) = \big(y(t)\big)^2$. One of these two graphs **cannot** describe the solution. Which one?



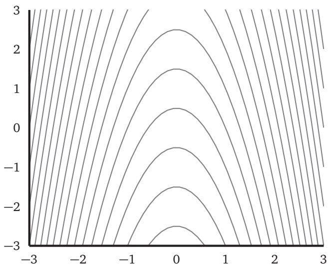


8 We seek a first-order ordinary differential equation $y' = f(\mathbf{y})$ whose solutions satisfy

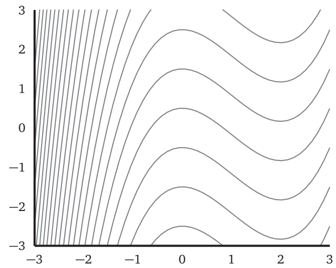
$$\begin{cases} y(x) \text{ is concave up if } y < 1 \\ y(x) \text{ is concave down if } y > 1 \end{cases}$$

Write down or graph a function $f(y)$ that would produce such solutions.

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



A



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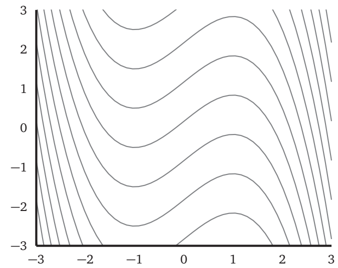

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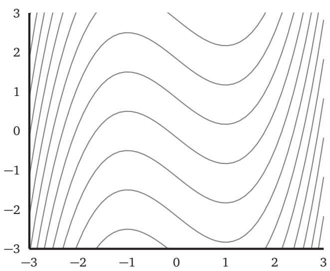

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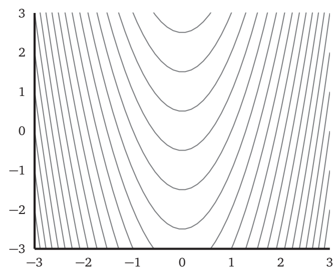

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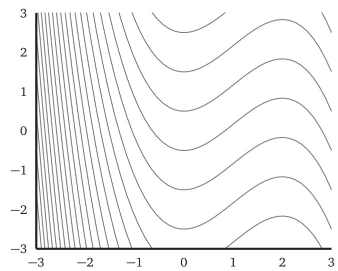


F

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph an $\boldsymbol{f(x)}$ that would produce such solutions.

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be positive

11.2 $y(t)$ must always be negative

11.3 $y(t)$ must always be decreasing

11.4 $y(t)$ must always be increasing

12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

Slope Fields

In this module you will learn

- what is a slope field
- how to sketch a slope field
- to interpret a slope field

As we saw in the previous module, once we have found a differential equation that models a situation, we often want to figure out what happens to the solution.

In this module, we will focus on getting an idea of the solutions and integral curves using what is called a **slope field**.

Slope field. Consider the equation $y' = f(x,y)$. If we evaluate $f(x,y)$ over a rectangular grid of points, and we draw an arrow at each point (x,y) of the grid with slope $f(x,y)$, then the collection of all the arrows is called a ***slope field***.

We can sketch Slope Fields with Wolfram Alpha.

For a differential equation $\dfrac{dy}{dx} = f(x,y)$, we need to input

- Vector Field: $(1, f(x,y))$.

http://www.wolframalpha.com/input/?i=slope+field

Example. Let us take an example from the previous module.

Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

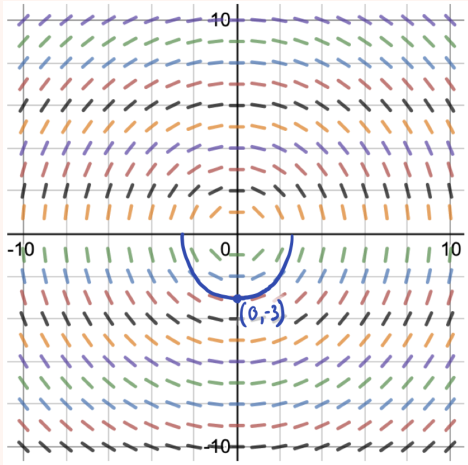
We can use this definition to sketch the slope field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

We now sketch this slope field with Desmos:

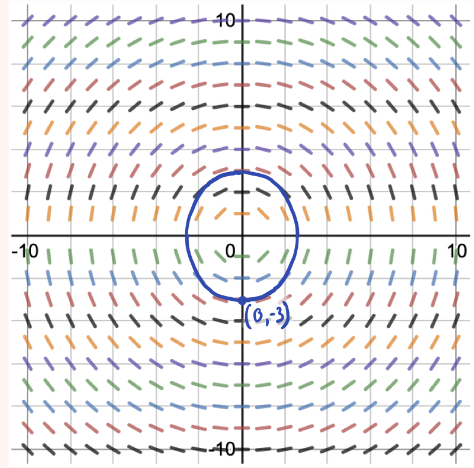
https://www.desmos.com/calculator/scmz6ps0or

Now notice that the arrows have the slope of a solution. This means that solutions will be tangent to the arrows, so we can *roughly* trace the solution by following the arrows.
Below, we did just that starting with the point $(0, -3)$.

Module 9: Slope Fields



approximated solution



approximated integral curve

Important. Remember that this gives us only an approximation of the solution and integral curve. From the approximation, we can tell that the solution seems circular, but we still need to show that it is so.

Video.

- https://youtu.be/MI2xCwBekX4
- https://youtu.be/8Amgakx5aII

Practice Problems

1 Use Wolfram Alpha, Desmos, or another software to sketch the slope field for the following differential equations. Then roughly trace different solutions.

(a) $y' = 2y - x$

(b) $y' = xy$

(c) $y' = \cos(y)$

(d) $y' = \frac{1}{2} + \cos(y)$

(e) $y' = 1 + \cos(y)$

(f) $y' = 2 + \cos(y)$

(g) $y' = \sin(xy)$

(h) $y' = \tan(x + y)$

2 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where

$$f(x,y) = \begin{cases} -x & \text{if } x < 1 \\ y & \text{if } x \geq 1 \end{cases}$$

3 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where the function $f(x, y)$ satisfies all of the following properties:

(a) $f(x, y)$ is continuous

(b) $f(x, y) > 0$ when $x > 1$ and $y > 1$

(c) $f(x, y) < 0$ when $x < -1$ and $y < -1$

(d) $f(x, y)$ depends only on x when $x < -1$ and $y > 1$

(e) $f(x, y)$ depends only on y when $x > 1$ and $y < -1$

4 (a) On the slope field from the previous problem, show that there must exist a smooth continuous curve with horizontal lines.

(b) Show that the curve divides the (x, y) plane in two parts.

5 Consider a differential equation

$$y' = f(x, y)$$

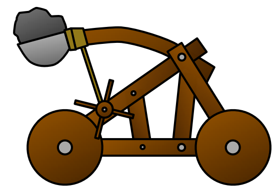
where the solutions satisfy

$$\lim_{x\to\infty} y(x) = 1.$$

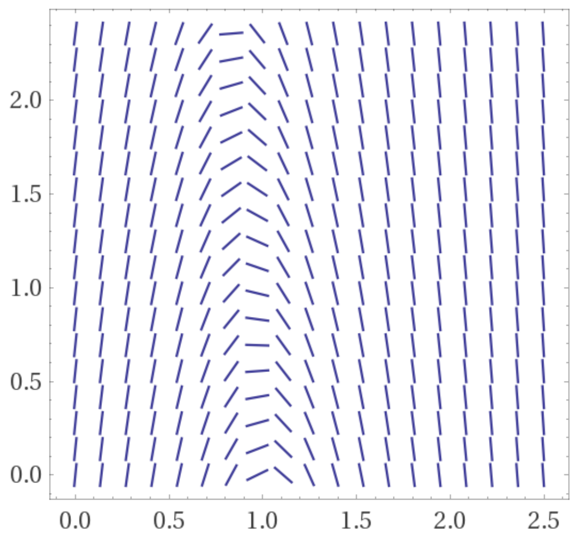
(a) What property must the slope field satisfy?

(b) Sketch a possible slope field for this differential equation.

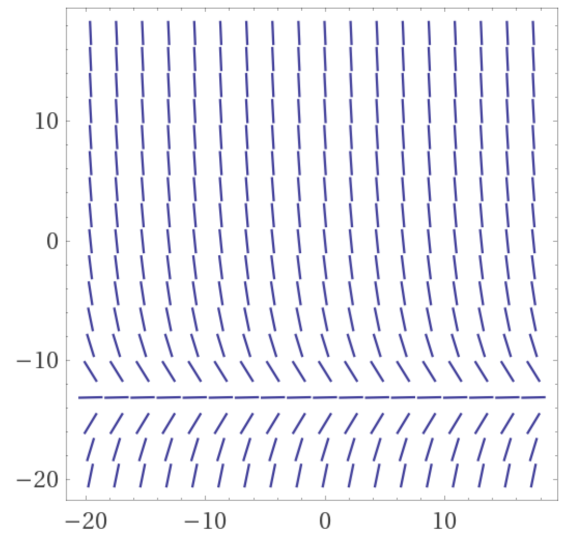
13 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

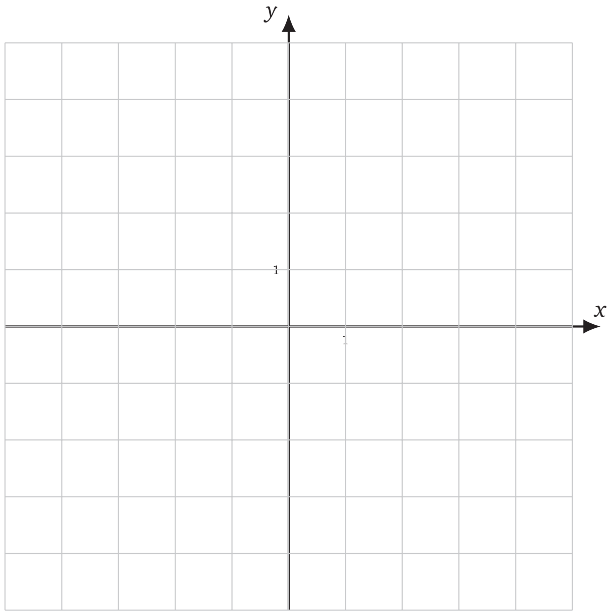
(These slope fields were created using WolframAlpha)

13.1 On the slope field, sketch a *possible* solution.

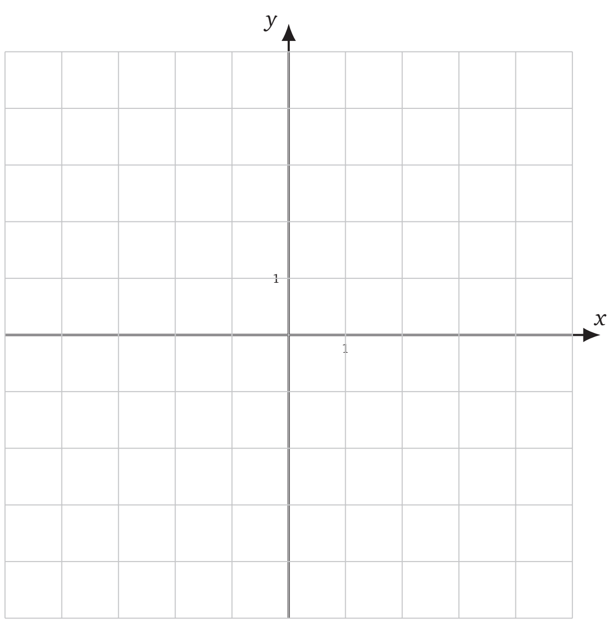
13.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

14 Sketch the slope field for the following differential equations.

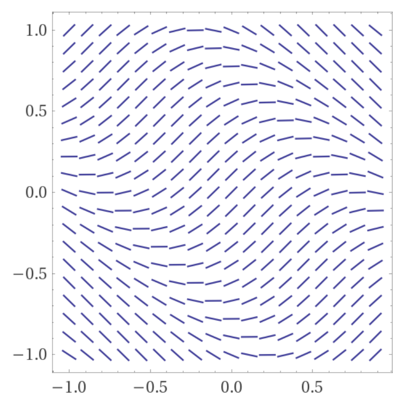
14.1 $y' = x$



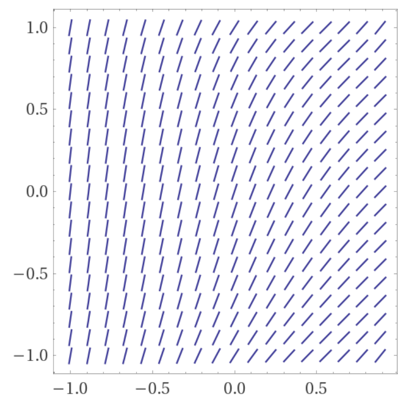
14.2 $y' = y^2$



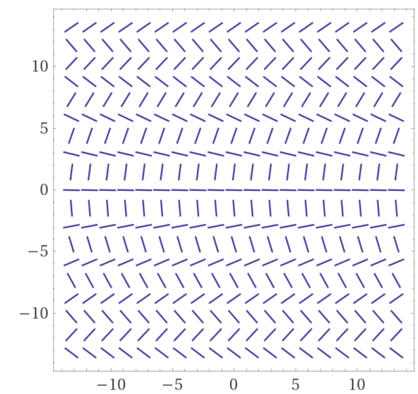
15 Consider the following slope fields:



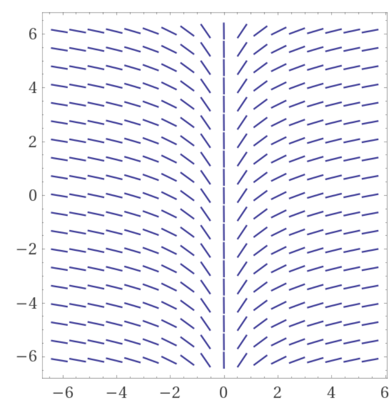
(A)



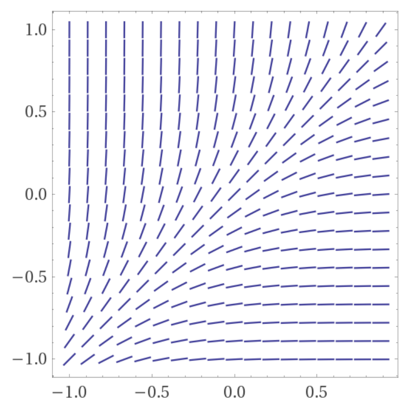
(B)



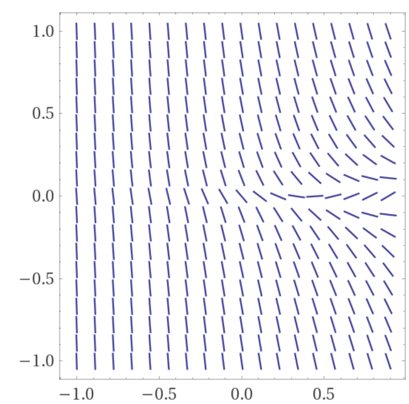
(C)



(D)



(E)



(F)

(These slope fields were created using WolframAlpha)

15.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

15.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

15.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

15.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

15.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

15.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \big(m(x,y)\big)^2$?

Approximating Solutions

In this module you will learn

- to approximate the solutions of differential equations

We just learned to sketch a slope field and how to use it to sketch a rough approximation of a solution of a differential equation.

The method of "following the arrows" of a slope field, when formalized mathematically is called **Euler's Method**.

So let us start with an initial-value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big) \\ y(0) = y_0 \end{cases}$$

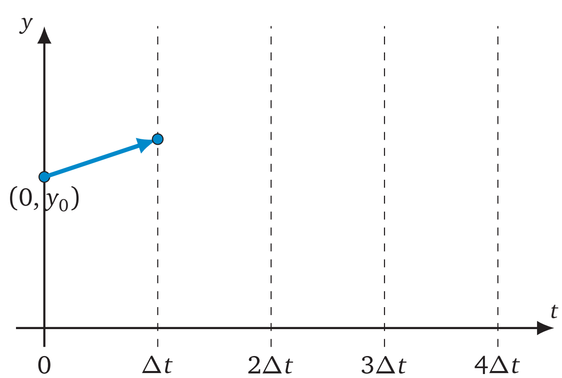
The idea is to follow the directions given by the differential equation, so we know that

- $y(0) = y_0$
- $y'(0) = f(0, y_0)$

This means that we have a starting point $(0, y_0)$. We still need to decide the distance that we want to follow the arrow:

- smaller distance: more accurate approximation, but will take more calculations
- longer distance: less accurate approximation, but will take fewer calculations

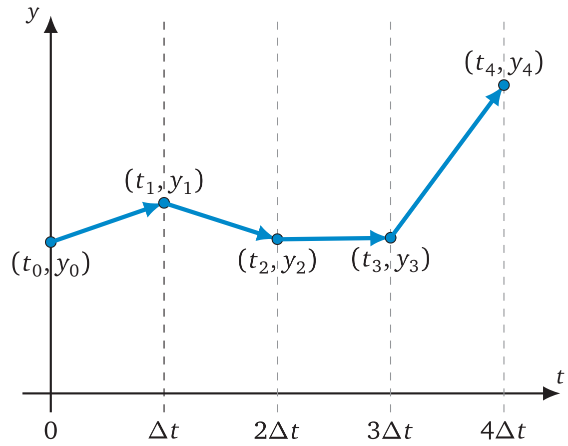
The typical way to decide is to set a parameter Δt, that measures the distance we will travel in the t-axis.



This way we find our second point $(\Delta t, y_1)$ where:

$$\frac{y_1 - y_0}{\Delta t} = \text{slope of the arrow } = f(0, y_0) \quad \Rightarrow \quad y_1 = y_0 + f(0, y_0)\Delta t$$

We continue in this way to find more points (t_i, y_i):



Euler's Method. Let $y'(t) = f(t,y)$ be a first-order differential equation. The ***Euler approximation*** to the initial value problem $y'(t) = (f(t,y)$ and $y(t_0) = y_0$ with step size Δt is the sequence of points (t_i, y_i) given by (t_0, y_0) if $i = 0$ and

- $t_i = t_{i-1} + \Delta t$
- $y_i = y_{i-1} + f\big(t_{i-1}, y_{i-1}\big)\Delta t.$

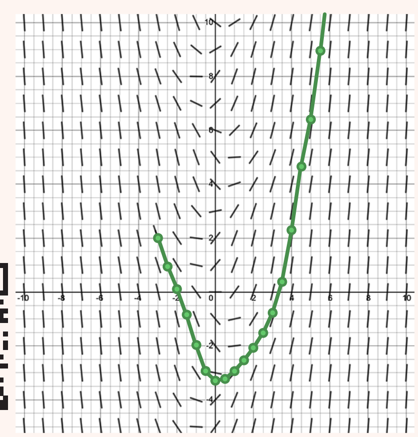
The method used to generate (t_i, y_i) is called ***Euler's Method***.

Example. Consider the initial-value problem

$$\begin{cases} y'(t) = \sin(y) + t \\ y(-3) = 2 \end{cases}$$

Then, we can follow Euler's Method with $h = 0.5$ to obtain:

- $y_0 = 2$
- $y_1 = 2 + \frac{1}{2}\big(\sin(2) - 3\big) \approx 0.95$
- $y_2 = 0.95 + \frac{1}{2}\big(\sin(0.95) - 2.5\big) \approx 0.1$
- $y_3 = 0.1 + \frac{1}{2}\big(\sin(0.1) - 2\big) \approx -0.85$



Here is the link to the desmos graph:

- https://www.desmos.com/calculator/kkgj5jhggd

Video.

- https://youtu.be/q87L9R9v274
- https://youtu.be/g3Xw1r7QGOE
- Euler's Method helping to take a person to the Moon

Practice Problems

1 For the following initial-value problems, approximate their solution with different values of Δt and compare with their exact solutions.

(a) $y' = -y + 5 + t,\ y(0) = 4, 5, 6$

(b) $y' = y + 5 - t,\ y(0) = -4$

(c) $y' = (t - y)\sin(y),\ y(0) = -1$

(d) $y' = \dfrac{y + 3t}{1 + t^2},\ y(0) = -1, 1$

Hint. Write a computer program that does the approximation for you.

2 Consider the differential equation

$$y' = -\frac{x}{y}.$$

(a) Sketch a slope field for this differential equation.

(b) Use Euler's Method to approximate the solution for some values of Δx and for some initial conditions.

(c) Does Euler Method do a good job approximating the solution?

3 In this module, we derived Euler's Method. One of the main steps was obtaining the equation

$$\frac{y_1 - y_0}{\Delta t} = \text{slope of the arrow.}$$

In Euler's Method, we used the slope at the beginning of the arrow. We can derive a new Method where we use the slope at the end of the arrow.

(a) Find a formula and the algorithm for this new method.

(b) Use this method with to approximate the solution of $y' = -y + 5 + t,\ y(0) = 4, 5, 6$ and compare the results with your results from question 1.

(c) Which of these two methods gives a better approximation?

(d) In your opinion, which of these two methods is better? Why?

4 Consider an initial-value problem with solution $y(t)$. If we want to find an approximation for $t \in [0, T]$, we define the error of the approximation $\{y_i^{\Delta t}\}$ by

$$E(\Delta t) = \left| y(T) - y_N^{\Delta t} \right|, \qquad \text{(E)}$$

where $T = N\Delta t$.

(a) For the initial-value problems from the previous question, study what happens when the value of Δt decreases.

(b) What do you expect to happen as Δt converges to 0?

(c) Estimate how fast Euler's method converges. Find a value of p such that

$$E(\Delta t) \le C(\Delta t)^p,$$

where the constant C changes for each ODE, but doesn't change if you keep the same ODE but change only the value of Δt.

5 Using Euler's Method with a step size of $\Delta t = 0.05$, and keeping only three digits throughout your computations, determine the approximations at $T = 0.2, 0.3, 0.4$ for each of the following initial-value problems.

(a) $y' = -y + 5 + t,\ y(0) = 4$

(b) $y' = y + 5 - t,\ y(0) = -4$

Compare the results with what you obtained for problem 1. Where do the differences come from?

6 Round-off errors become important when the value of N is very large, which happens if we want a very accurate approximation. This means that the actual error (E) of the approximation has two components:

$$E(\Delta t) = f(\Delta t) + g(\Delta t),$$

where

- $\lim_{\Delta t \to 0^+} f(\Delta t) = 0$ (approximation error)
- $\lim_{\Delta t \to \infty} f(\Delta t) = \infty$ (approximation error)
- $\lim_{\Delta t \to 0^+} g(\Delta t) = \infty$ (round-off error)
- $\lim_{\Delta t \to \infty} f(\Delta t) = 0$ (round-off error)

Answer the following questions and justify your answers based on these ideas.

(a) Justify why the four limits above make sense.

(b) Does the approximation converge to the solution as $\Delta t \to 0$?

(c) Is there an optimal Δt that gives the best possible approximation?

16 Consider the differential equation

$$y' = y - 2.$$

16.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 3)$.

16.2 Find the solution of the differential equation with the same initial condition.

16.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 1)$.

16.4 Find the solution of the differential equation with the same initial condition.

16.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

16.6 Explain in words why the Method satisfies that property.

17 Which differential equations will be approximated perfectly using Euler's Method?

Modelling with Differential Equations

In this module you will learn

- how to start modelling a physical phenomenon into a differential equation

We started by studying some mathematical modelling in chapter 1. Then, we just used mathematical tools that we learned before.

We now want to focus on mathematical models that arise from physical applications. These will often take the form of one or more differential equations.

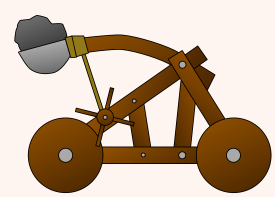
The modelling of the situation will develop in a similar way.

Step 1. Defining the problem

As before, we should start by thinking about what our ultimate goal is. Once we settle on a goal, we define it as the function we want to study.

Example.

In this module, we are going to think about the catapult problem from Module 9 - Slope Fields.



A catapult throws a projectile into the air.
Our goal is to track the height (in metres) of the projectile from the ground.
This means that we have a goal: to find the height of the projectile at every moment in time after it is launched.

This means that we define

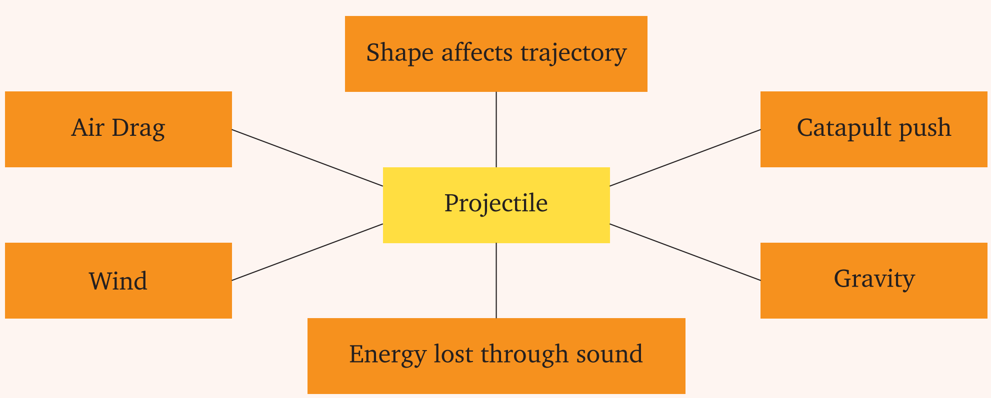
- $y(t) =$ height of the projectile, in metres, t seconds after it was launched from the catapult.

Step 2. Building a mind map

A mind map will help us identify the notions that we want to include in our model.

Example.

In the catapult example, since we decided to study the projectile's height, we need to find everything that affects its height.



We can include more layers to these topics if we want.

Step 3. Make assumptions

This is a fundamental step in any modelling endeavour. The real world is too complicated, so we make assumptions that simplify our model.

This has two main consequences:

1. It makes our model simpler and easier to study;
2. It creates constraints on our model: it is only valid under certain conditions.

Example.

Let us discuss the topics included in the mind map above:

- Catapult Push – the catapult pushes on the projectile for a small period of time when $t < 0$. If we are considering only $t \geq 0$, then this will likely provide us with some starting conditions for the projectile
- Gravity – The height of the projectile is affected by gravity. We have a choice to make:
  - assume that the Earth is flat and gravity is constantly accelerating the projectile downwards;
  - assume that the Earth is spherical and gravity is constantly accelerating the projectile towards the centre of the Earth;
  - assume that the Earth is spherical and gravity is a force accelerating the projectile towards the centre of the Earth with a magnitude that decreases with the square of the distance to the centre of the Earth
  - or other more complicated and more accurate models.
- Air Drag – air is making it hard for the projectile to move forward. We have another choice to make:
  - assume that the air drag is a force that accelerates the projectile in the direction opposite to its movement and with magnitude proportional to its speed;
  - assume that the air drag is a force that accelerates the projectile in the direction opposite to its movement and with magnitude proportional to the square of its speed;
  - or other more complicated and more accurate models.

I will leave it to you to think about the remaining three topics in the mind map.

We now need to make a decision about what to assume.

To keep this model simple, let us assume the following:

1. The projectile's height will stay within a small range: $y(t) \in [0, 100]$. Is this reasonable for a catapult?

   This means that we can consider the first of the gravitational models above: define gravitational acceleration as a constant $-g$.

2. The projectile will not move very fast, so we can approximate the air drag to be directly proportional to the speed: define air drag acceleration as $\pm\gamma v$, where $\gamma > 0$ is a constant that depends on the projectile and v is the velocity of the projectile. Which sign should we have?

3. Again, the projectile will not move very fast, so we can approximate the air drag to use only the vertical speed of the projectile: define air drag acceleration as $\pm\gamma v_y$, where $\gamma > 0$ is a constant that depends on the projectile and v_y is the vertical velocity of the projectile. Which sign should we have?

4. The shape of the projectile will affect air drag in the form of the constant $\gamma > 0$.

5. Assume that for a medieval catapult (as in the drawing above), the other components are negligible.

We come out of this step with some conditions for the validity of our model and some new constants and terms to use in our model.

Step 4. Construct a model

This is the part where we put together the last three steps into one (or a system of) differential equations. This should not be a difficult part if the last three steps were completed carefully.

Example.

Summary of Steps 1–3:

- **Goal:** study $y(t) =$ height of a projectile in metres, t seconds after being released from a catapult
- **Forces:**
  - Gravity: constant acceleration $-g$
  - Air Drag: acceleration $\pm\gamma y'(t)$
- **Conditions:**
  - $y(t) \in [0, 100]$
  - Air drag should really by quadratic, but in this example we will consider this as an academic case.

So the model we end up is:

$$F_y = \overset{\text{vertical component}}{\underset{\text{of force}}{}} = -g \pm \gamma y'.$$

Now we bring a little bit of a Physics class into here: Newton's 2nd Law states that $F = ma$, so we obtain the model

$$my''(t) = -g \pm \gamma y'(t).$$

Step 5. Model Assessment

We just found a differential equation (model) for our situation. It is now time to test it to make sure that it behaves correctly.

For this step, we need to obtain a solution of the differential equation, either by solving it mathematically and finding a formula for the solution, or by approximating the solution numerically (see Module **??**).

Then we need to check if the differential in one of several ways:

- We can test it empirically: make an experiment and compare the results of the experiment with the results of the model
- We can test it mathematically: change the parameters and the initial conditions to make sure that we know how the model should behave and test some qualitative aspects of the model

Important. Even if the model passes all the tests, it might still not be correct.
Also, if it fails one test, it might mean that the model is incorrect, or that it has some limitations that are more subtle and we hadn't thought about them.

Example. We have found the following model:

- $y(t) =$ height of a projectile in metres, t seconds after being released from a catapult
- It satisfies:

  $$my''(t) = -g + \gamma y'(t).$$

  (note that I chose the $+$ sign for the air drag component)
- Constraints:
  - $y(t) \in [0, 100]$;
  - $\gamma > 0$ is the drag constant: more air drag for larger values of γ;

This differential equation tells us what the second derivative, $y''(t)$, of $y(t)$ is given the first derivative $y'(t)$. This means that to start solving the problem, we need to know what the initial values for $y'(t)$ and $y(t)$ are.
Need to know the starting conditions:

- $y(t_0) = y_0$;

- $y'(t_0) = v_0$.

For this example, consider a situation where:

- $g, \gamma > 0$ can take any value.
- $y(0) = 0$;
- $y'(0) = g/\gamma > 0$;

The projectile is being catapulted from the ground with a positive velocity, so we expect it to go up for a while and then come back down to the ground.
What happens is

$$y''(0) = -g + \gamma y'(0) = 0,$$

so the initial acceleration is 0, which means that the velocity is not changing.
The result is a function with constant velocity equal to its initial velocity:

- $y(t) = \frac{g}{\gamma} t$.

This means that the height of the projectile keeps increasing, so the ***projectile never falls back to the ground***!

This means that there is a problem with our differential equation:

- Is the model incorrect?
- Is there a limitation on the initial velocity that we were not aware of?

We must check our process again and correct it.

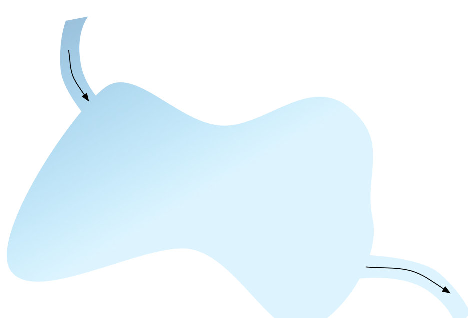
Step 6. Putting it all together in a report

We're not going to elaborate much on this step. For more on the subject, please check Module 8.

Video.

- https://youtu.be/njg8xwMviGQ
- https://youtu.be/nKDsJB8iwb0

Practice Problems



1 Model the pollution in a lake where water flows in and out at the same rate and incoming water is polluted with $2 + \sin(2t)$ kg/L of pollutant, where t is measured in years.

2 Construct a model for a population with a rate of growth proportional to its current size.

3 Find a model for a population that grows proportionally to its current size but with a variable proportion constant. This variable proportion constant should guarantee the following properties for the population:

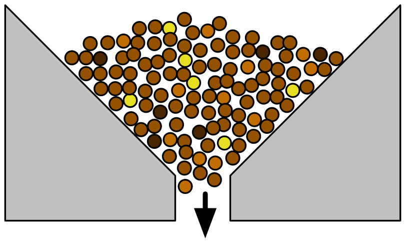
- If the population is too large, then it should decrease;
- If the population is small, then it should increase.

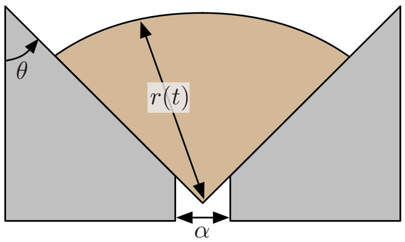
4 Improve the previous model by considering also a **survivability threshold**: if the population is below this value, it should decrease and eventually become extinct.

5 Consider two competing populations, like cheetahs ($c(t)$) and lions ($\ell(t)$): two populations that do not hunt each other, but compete for the same food (prey). Create a model for these two populations that captures how the competition for food affects them.

Hint. It might be helpful to think about how one population would grow in the absence of the other;

and how one population is affected by the competition of the other.





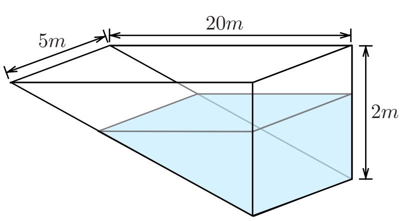
6 People are in a stadium watching cricket match. When the match is over, people leave the stadium.

(a) Model the way people leave the stadium.

To help you with this task, use the fact that in this situation, people behave like a fluid according to Torricelli's Law:

> The area of the region occupied by the fans decreases proportionally to the square root of the radius and also proportionally to the size of the exit.

(b) How do the parameters θ and α affect the total time it will take for the stadium to empty?



7 Consider the pool in the figure. The goal is to track the amount of chlorine in the water for one Summer month. At the beginning of the month, the pool is full and contains 150g of chlorine uniformly mixed in the water. Consider evaporation and rain. To make the model simpler, assume that water evaporates with the chlorine.

8 After solving the core exercise 18 below, we find a property of this model.

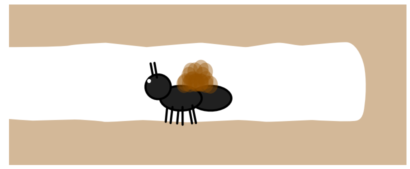
(a) The constants g and L (length of the string) appear only has $\frac{g}{L}$. What does this imply?

(b) We are sending a mission to the Moon and we need to know how a 1m long pendulum behaves on the Moon. To test it, we need to build on Earth a pendulum that behaves in the same way. How long should the length of the string be on Earth?

9 After solving the core exercise 18 below, construct a model for the same problem considering string tension.

(a) Show that you obtain the same model that you get while disregarding tension.

(b) Explain why this makes sense.



10 An ant queen, known affectionately as Aunty Ant, is commissioning a construction assessment for a new tunnel. Aunty Ant's worker ants only know one way to construct a tunnel: they grab some dirt in their pincers, walk the dirt out of the tunnel, deposit it, and then return to grab more dirt.

Prepare a report which uses differential equations to address the following construction scenarios. Include a description of how you modelled the scenario and a graph of tunnel-length vs. worktime. Also make sure to define any variables and constants you are using.

(a) One tireless worker is assigned dig the tunnel. The worker walks the same speed whether she is carrying dirt or not.

(b) One tireless worker is assigned to dig the tunnel, but she can walk twice as fast when she is not carrying dirt as when she is carrying dirt.

(c) Aunty Ant really wants the tunnel to progress linearly after the first day of construction (that is, the graph of tunnel-depth vs. time after the first day should be a straight line). She will give you full control over how many workers are devoted to the tunnel at any given time.

(d) (Optional) A single ant is assigned to dig the tunnel, but she gets fatigued the farther she walks. Her speed after walking a total distance of k units is $1/k$.

11 The alien world of Robotron is inhabited by billions of tiny nanobots. These nanobots all share a common source of power, and their speed is directly proportional to the total amount of energy shared among all the nanobots.

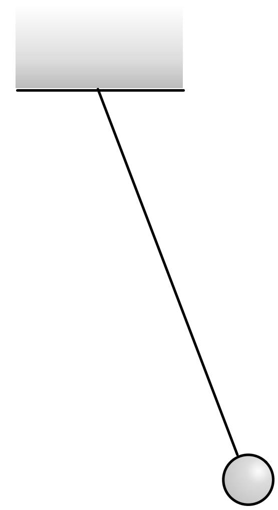
One day the nanobots decide to beam their energy into space. They all form lines, march to the edge of their colony, and send a tiny portion of their shared energy into space. Since the nanobots are very polite, after an individual nanobot has sent its energy into space, it moves aside and lets the next nanobot take a turn.

(a) Suppose the nanobots live in a tube with an opening at only one end. Come up with a differential equation to model the amount of energy left in the nanobot colony over time.

(b) How does your model change if the nanobots live in a disk where energy can be launched from anywhere on the perimeter? What about a sphere?

(c) Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between the object's temperature and the ambient (outside) temperature. Does this law relate to your model for the nanobots? If so, how?

18 A pendulum is swinging side to side. We want to model its movement.



18.1 Define the problem. Which function(s) do we want to find in the end?

18.2 Build a mind map.

18.3 Make assumptions. Remember to use your mind map to help structure the problem.

18.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2nd Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in — Rate out).

18.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.

19 Model the spreading of a rumour through the students of a school.

Solvable Types of ODEs

In this module you will learn

- to identify specific types of differential equations that can be solved rigorously
- how to solve these types of differential equations

We just learned how to model a situation and end up with a differential equation. We will now focus on solving differential equations.

There are a few different techniques that depend on the differential equation.

12.1. Separable Differential Equations

Separable ODE. A differential equation is called ***separable*** if it has the form

$$g(y)y'(t) = h(t),$$

that is if we can separate all the y's into the left-hand side and the all the t's into the right-hand side of the equation. Observe that the y's on the left hand side must all be multiplied by $y'(t)$.

Method of solution. The idea to solve this type of DEs is simple:

1. Integrate both sides with respect to t:
$$\int g(y)y'(t)\,dt = \int h(t)\,dt$$
2. Change variables on the left-hand side to $u = y(t)$, so $du = y'(t)dt$ and we get
$$\int g(u)\,du = \int h(t)\,dt.$$
3. Solve both integrals and we obtain a solution, usually in implicit form:
$$G(u) = H(t) + C.$$
4. To finish, recall that $u = y(t)$, so we obtain
$$G\big(y(t)\big) = H(t) + C.$$

Important. Observe that the solution is given in implicit form. In general, when using this technique, the solution $y(t)$ will be given in implicit form, so there is still some work ahead to find an explicit formula for $y(t)$.

Example. The shape $y(x)$ of a free falling chain under its own weight, called a catenary, satisfies the differential equation:

$$y''(x) = \frac{1}{a}\sqrt{1 + \big(y'(x)\big)^2}.$$

It doesn't seem to be a **separable equation**, but if we can define $z(x) = y'(x)$, which satisfies

$$z'(x) = \frac{1}{a}\sqrt{1 + \big(z(x)\big)^2} \qquad \Leftrightarrow \qquad \frac{1}{\sqrt{1 + \big(z(x)\big)^2}}z'(x) = \frac{1}{a}.$$

This is now clearly in the form of a ***separable ODE***.

We can solve it using the method described above: we need to solve

$$\int \frac{1}{\sqrt{1+z^2}}\,dz \quad = \quad \int \frac{1}{a}\,dx \quad = \quad \frac{x}{a} + C_1$$

The integral on the left can be solved using a hyperbolic substitution $z = \sinh u$:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \int 1\,du = u = \text{arcsinh z.}$$

This means that the solution satisfies

$$\text{arcsinh } z = \frac{x}{a} + C_1 \qquad \Leftrightarrow \qquad z = \sinh\left(\frac{x}{a} + C_1\right).$$

Now recall that $z(x) = y'(x)$, so we need to integrate $z(x)$ to obtain the catenary curve $y(x)$:

$$y(x) = \int z(x)\,dx = a\cosh\left(\frac{x}{a} + C_1\right) + C_2.$$

To find C_1 and C_2, we use the fact that $y'(0) = 0$:

$$y(x) = a\cosh\frac{x}{a} + C_2.$$

(the constant C_2 moves the curve up or down, so it doesn't change the shape).

Video.

- https://youtu.be/txtFH89HwOA
- https://youtu.be/8xG_Xg6X2MQ
- https://youtu.be/ZE1Agfkhr28

12.2. First-Order Linear Differential Equations

First-Order Linear ODE. A differential equation is called ***first-order linear*** if it has the form

$$y'(t) + p(t)y(t) = f(t),$$

that is if we can separate all the y's into the left-hand side and the all the t's into the right-hand side of the equation. Observe that the y's on the left hand side must all be multiplied by $y'(t)$.

The idea to solve this type of DEs is to transform it into the result of a product rule.

Example. Consider the following **first-order linear ODE**

$$t^2\frac{dy}{dt} + 2ty = \sin t.$$

Observe that the left-hand side of the DE is the result of the product rule:

$$\frac{d}{dt}\Big[t^2 y\Big] = \sin t.$$

So we can integrate both sides with respect to t to obtain

$$t^2 y = -\cos t + C \quad \Leftrightarrow \quad y = -\frac{\cos t}{t^2} + \frac{C}{t^2}.$$

Now let us look at another example, where the left-hand side of the ODE is not in the form of the result of a product rule, but can be transformed into one.

Example. Consider the **first-order linear ODE**

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{3}e^{\frac{t}{3}}. \qquad (\star)$$

Again, the "trick" is to look at this equation and realize that the left-hand side can look like the result of the product rule. It's not obvious that this can be done (yet!), but if we multiply the whole ODE by the function

$$e^{\frac{t}{2}},$$

then we obtain

$$e^{\frac{t}{2}}\frac{dy}{dt} + \frac{1}{2}e^{\frac{t}{2}}y = \frac{1}{3}e^{\frac{t}{2}}e^{\frac{t}{3}}$$

and now the left-hand side is the result of a product rule

$$\frac{d}{dt}\Big[e^{\frac{t}{2}}y\Big] = \frac{1}{3}e^{\frac{5}{6}t}.$$

We integrate both sides to obtain

$$e^{\frac{t}{2}}y = \frac{1}{3}\frac{6}{5}e^{\frac{5}{6}t} + c$$

thus

$$y = \frac{2}{5}e^{\frac{t}{3}} + ce^{-\frac{t}{2}}.$$

This last example required us to come up with a function to multiply the ODE so that it becomes of the right form: with a left-hand side that is the result of the product rule.

This function is called the **integrating factor**.

Let us now see how we can find this function in more detail.

Example. Consider the same ODE ($\star$):

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{3}e^{\frac{t}{3}}. \qquad (\star)$$

So we multiply both sides of the equation with an unknown function $\mu(t)$, called the **integrating factor**:

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y = \frac{1}{3}\mu(t)e^{\frac{t}{3}}. \qquad (\#)$$

And we find which $\mu(t)$ makes the left-hand side equal to the product rule:

$$\frac{d}{dt}\big[\mu(t)y\big] = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y,$$

and this needs to equal the left-hand side:

$$\mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y = \mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y \qquad \Leftrightarrow \qquad \mu'(t) = \frac{1}{2}\mu(t).$$

We now need to solve this equation for $\mu(t)$. Fortunately, this is a **separable ODE**:

$$\begin{aligned} \mu'(t) = \frac{1}{2}\mu(t) \quad &\Leftrightarrow \quad \frac{\mu'(t)}{\mu(t)} = \frac{1}{2} \\ &\Leftrightarrow \quad \ln|\mu(t)| = \frac{t}{2} + A \\ &\Leftrightarrow \quad \mu(t) = ae^{\frac{t}{2}}, \end{aligned}$$

where $a = e^A$.
We say that the function $\mu(t) = e^{\frac{t}{2}}$ is an **integrating factor** for the equation ($\star$). Observe that we chose $a = 1$ ($A = 0$), because we only need one function $\mu(t)$ that satisfies our condition $\mu' = \frac{1}{2}\mu$, we don't need to find all possible solutions.

After finding the integrating factor $\mu(t)$, the rest of the solution is the same as in the previous example.

Now that we have a good idea of the method needed to solve these ODEs, let us tackle the general equation.

Method of solution. This method is also known as the **Method of the Integrating Factor**.

1. Multiply both sides by $\mu(t)$, the integrating factor:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t).$$

Note that we don't know what this function is yet. So it is just a placeholder for a function we will find next.

2. Find function $\mu(t)$ which satisfies

$$\mu'(t) = p(t)\mu(t).$$

This is a **separable ODE**, so we can solve it:

$$\mu(t) = Ae^{\int p(t)\,dt}.$$

We only need one function $\mu(t)$, not the general one, so we take $A = 1$ to get

$$\mu(t) = e^{\int p(t)\,dt}.$$

3. Observe that $\mu(t)$ satisfies

$$\mu(t)p(t) = \mu'(t),$$

so we use this in the equation:

$$\frac{d}{dt}\big[\mu(t)y\big] = \mu(t)g(t).$$

4. Integrate the equation:

$$\mu(t)y = \int \mu(t)g(t)\,dt + c,$$

which means that the solution is

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right],$$

where

$$\mu(t) = e^{\int p(t)\,dt}.$$

Important. Observe that the solution is given in explicit form. This is always the case with this type of ODEs.
Also, be careful to add the integration constant as soon as you integrate, so that in the end you will have a term $\frac{c}{\mu(t)}$.

Video.

- https://youtu.be/ezhi3E_bdvk
- https://youtu.be/VdD26Iy4Bkk
- https://youtu.be/GIpOcHNK7eQ

Practice Problems

1 Solve the differential equation $\ln(t)y' + \frac{1}{t}y = 3$.

2 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one type, solve it.

(a) $(t^2+4)y'(t) = \dfrac{2t}{y^2}$

(b) $\dfrac{1}{t^2}y'(t) = 2$

(c) $\dfrac{dy}{dx} = \sqrt{y}(x+1)^2$

(d) $y'(t) = t + y$

(e) $y'(t) = t + y^2$

(f) $y'(t) = \dfrac{t}{y}$

(g) $y'(t) = -\dfrac{1}{t}y$

(h) $y'(t) = 1 - 4t - \dfrac{5}{t}y$

(i) $y'(t) = 5t - 2ty$

(j) $y'(t) = 2 + \cos^2(y)$

(k) $e^{-t}y'(t) - e^{-t}y = 3e^{2t}$

3 Decide whether the following statements are true or false. Give an explanation or a counterexample.

(a) There are differential equations that are both separable and first-order linear.

(b) There are differential equations that are separable, but are not first-order linear.

(c) There are differential equations that are first-order linear, but not separable.

(d) There are first-order differential equations that are neither separable nor linear.

(e) All first-order linear differential equations have solutions defined in the whole real line.

4 Consider the differential equation

$$y' - \frac{y}{2(x+4)} = \frac{1}{2(x+4)}$$

(a) Find the general solution.

(b) Find the solution with initial condition $y(0) = -5$.

(c) What is the domain of the previous solution?

(d) Find the solution with initial condition $y(-5) = -5$.

5 Even though the following differential equation is not linear, find its general solution:

$$2\ln(x)e^{2y}y'(x) + \frac{e^{2y}}{x} = 4x^3.$$

20 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

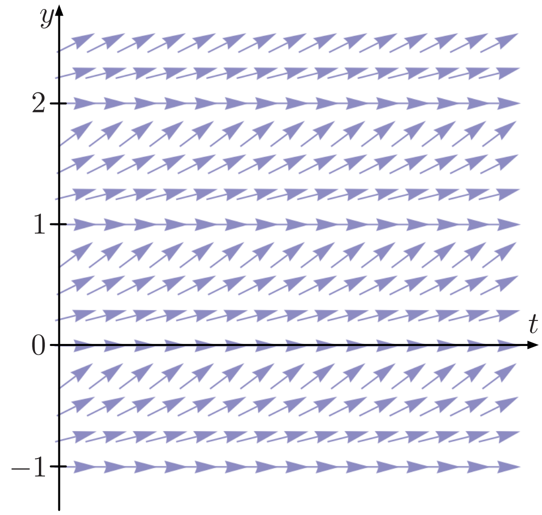
20.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

20.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

20.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

20.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

21 Consider a differential equation $y' = f(t, y)$ with the following slope field.



21.1 What are the equilibrium solutions of the ODE?

21.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \dfrac{1}{4} \end{cases}$$

21.3 From the direction field above, what is the type(s) of this ODE? Justify your answer.

(a) separable.

(b) of first-order and linear.

(c) autonomous.

(d) none of the other options.

21.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then (select all the possible options)

(a) $g(3) < h(3)$ **(b)** $g(3) = h(3)$ **(c)** $g(3) > h(3)$

22 22.1 Calculate $\big(\sin(x)f(x)\big)'$.

22.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

22.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

Properties of Differential Equations

In this module you will learn

- to find some properties of solutions without the need to find a solution or approximating it
- an existence and uniqueness of solution theorem

Until now we studied problems where there was one unique solution. Is this true for every problem?

- There are DEs with no solutions, e.g. $(y')^2 = -1$ or $\sin(y') = 2$.

So if a problem has a solution, is it always unique?

- This is also not true. For example: $ty' = 2y$ with $y(0) = 0$.
  Check that

$$\begin{aligned} y &= 0 && \text{is a solution} \\ y &= t^2 && \text{is also a solution} \end{aligned}$$

It is important (not just to mathematicians) to know whether a problem has solutions or not before trying to solve it. It is also important to know whether there is one unique solution or multiple solutions.

So for **linear differential equations** we have the following theorem.

Theorem. Let p and g be continuous functions in an open interval $I = (a, b)$ containing the point t_0. Then there exists a unique function $y = \phi(t)$ that satisfies

$$\begin{aligned} &y' + p(t)y = g(t) && \text{for each } t \in I, \\ &y(t_0) = y_0, \end{aligned}$$

for any $y_0 \in \mathbb{R}$.

Example. Consider the initial-value problem

$$\begin{cases} y' + \frac{1}{\sin(t)}y = e^t \\ y(1) = 2 \end{cases}$$

We can see that

- $p(t) = \frac{1}{\sin(t)}$, which is continuous for $t \in (0, \pi)$ and $t_0 = 1$ is included in this interval;
- $g(t) = e^t$ is continuous for all values of t.

So we can conclude, from the Theorem, that there is a unique solution $y(t)$ defined for $t \in (0, \pi)$.

Example. We can see why on the previous example $ty' = 2y$, this Theorem doesn't apply. To use the Theorem, we need to write this equation as

$$y' - \frac{2}{t}y = 0,$$

and the function $p(t) = -\frac{2}{t}$ is not continuous at 0.

Example. The equation $y' = \frac{2}{3\sqrt[3]{x}}$ with the condition $y(0) = 0$ has a unique solution:

$$y = x^{\frac{2}{3}}.$$

So even though $g(t) = \frac{2}{3\sqrt[3]{x}}$ is not continuous at 0, the DE still has a unique solution.

The previous Theorem is very restrictive – it only applies to some very particular differential equations. Below, we state another Theorem that applies to a much broader range of differential equations.

Theorem. Let the functions $f(t,y)$ and $\frac{\partial f}{y\partial y}$ be continuous in some rectangle $|t-t_0| \leq a$ and $|y-y_0| \leq b$ for $a, b > 0$.
Then, in some interval (t_0-h, t_0+h), there is a unique solution $y = \phi(t)$ of the IVP

$$\begin{aligned} y' &= f(t,y) \\ y(t_0) &= y_0. \end{aligned}$$

Furthermore, $h \geq \min\{a, b/M\}$ where $M = \max\left|f(t,y)\right|$.

Partial derivative. Consider a function $f(t,y)$. Then its ***partial derivative with respect to*** y at the point (t_0, y_0), denoted by $\frac{\partial f}{\partial y}(t_0,y_0)$ is $g'(y_0)$, the derivative of the function $g(y) = f(t_0,y)$ at the point y_0. Roughly, assume that the variable $t = t_0$ is a fixed number and take the derivative on the variable y.

You should spend some time comparing these two Theorems.

Observe that the last Theorem gives a much weaker result when the differential equation is linear.

Example. Consider the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 3. \end{cases}$$

This problem is nonlinear, so we need to use the second Theorem. To apply, compute

$$\begin{aligned} f(x,y) &= y^2 \\ \frac{\partial f}{\partial y}(x,y) &= 2y, \end{aligned}$$

which are continuous for all $x, y \in \mathbb{R}$.
The previous Theorem guarantees that a solution exists and is unique in some interval around $x = 0$.
Even though the rectangle spans the whole space of x and y, it doesn't mean that the solution exists for all x. The extra part of the Theorem, guarantees that the solution exists for $t < h$ where $h = \frac{1}{b}$ (because $M = b^2$). Since $b \geq y_0$, we know that a solution exists for $t < \frac{1}{y_0} = \frac{1}{3}$.
In fact, this is a separable ODE, so we can find its solution:

$$y(x) = \frac{1}{\frac{1}{3} - x},$$

which is defined only for $x < \frac{1}{3}$.

This kind of Theorems are called **Existence and Uniqueness Theorems**.

Video.

- https://youtu.be/53BPf9JrFcU
- https://youtu.be/GV1gFLZ7V18

Practice Problems

1 For the following initial-value problems, answer the following questions:

(i) Is there a unique solution?

(ii) Without solving, what is its domain?

(a) $y' + y = t$ with $y(0) = 0$.

(b) $y' + \dfrac{1}{e^t} y = t$ with $y(0) = 0$.

(c) $y' + \dfrac{1}{e^t - 2} y = t$ with $y(0) = 0$.

(d) $y' + \ln(t) y = t$ with $y(e) = 1$.

(e) $y' + \dfrac{1}{1+t^2} y = \tan(t)$ with $y(0) = 0$.

(f) $y' + \dfrac{1}{1+t^2} y = \tan(2t)$ with $y(\pi) = 0$.

(g) $y' = \dfrac{1}{1+\sin(t)} y - \tan(t)$ with $y(0) = 0$.

(h) $y' = \dfrac{1}{1+\sin(t)} y - \tan(t)$ with $y(t_0) = 0$.

(i) $y' = \dfrac{1}{1+\sin(t)} y^2 - \tan(t)$ with $y(t_0) = 0$.

(j) $y' + \ln(y) = t$ with $y(e) = 1$.

(k) $y' = \dfrac{ty}{1+y}$ with $y(0) = 0$.

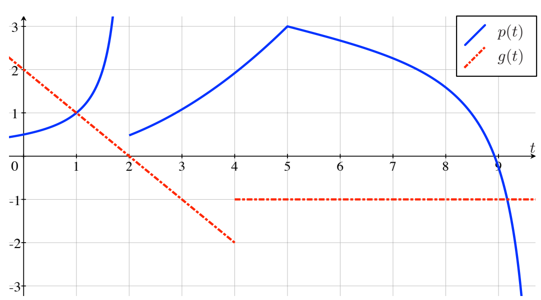
(l) $(t + y^2) y' = ty$ with $y(-1) = 1$.

(m) $y' = \dfrac{t \sin(y)}{y}$ with $y(1) = 0$.

2 Consider the problem

$$y' + p(t) y = g(t) \qquad \text{with} \qquad y(t_0) = y_0,$$

where $p(t)$ and $g(t)$ are graphed below



(a) Is there a unique solution satisfying $y(3) = 2$? If so, what is its domain?

(b) Is there a unique solution satisfying $y(t_0) = -1$ for which values of t_0? If so, what is the domain of these solutions?

3 Consider the problem

$$y' = f(t, y)$$

where $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous for all t, y.

- Assume that $y = \frac{1}{t}$ is a solution for $t > 0$
- Assume that $y = -e^{-t}$ is a solution for all t

Let $y = \phi(t)$ be the solution of this ODE with the initial condition $y(1) = \frac{1}{2}$.
Calculate $\lim\limits_{t \to +\infty} y(t)$.

4 Consider the initial-value problem:

$$\begin{cases} y' = \ln(t+2) y + \frac{1}{t-3} \\ y(0) = 0 \end{cases}$$

(a) Is this ODE linear or nonlinear?

(b) Show that this problem has a unique solution.

(c) Use the Existence and Uniqueness Theorem for **Linear** ODEs. What is the domain of the solution?

(d) Use the Existence and Uniqueness Theorem for **Nonlinear** ODEs. What is the domain of the solution?

(e) Compare both Theorems.

5 Consider the initial-value problem:

$$\begin{cases} y' = \ln(t+2) y + \frac{1}{t-3} \\ y(0) = 0 \end{cases}$$

(a) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Linear** ODEs.

(b) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Nonlinear** ODEs. Simplify the conditions.

(c) Compare the conditions of both theorems.

6 Consider the initial-value problem:

$$\begin{cases} y' + p(t) y = g(t) \\ y(0) = 0 \end{cases}$$

(a) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Linear** ODEs.

(b) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Nonlinear** ODEs. Simplify the conditions.

(c) Compare the conditions of both theorems.

23 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

23.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

23.2 $y' = t + \sqrt{y-\pi}$ with $y(1) = 1$

23.3 $y' = \sqrt{4-(t^2+y^2)}$ with $y(1) = 1$

24 The initial-value problem

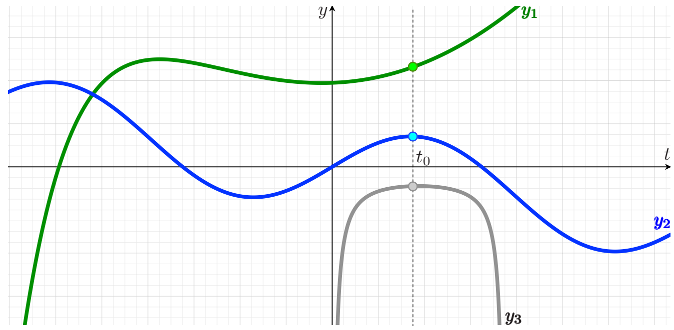
$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

$$y_1(x) = \cos\left(\arcsin(x)\right) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

24.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

24.2 What can you conclude?



25 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all t, y;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all t, y.

25.1 Can **green** $y1$ and **blue** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.2 Can **green** $y1$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.3 Can **blue** $y2$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

Autonomous Differential Equations

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

In this module, we focus on another type of differential equations. The ultimate goal of this module is to learn that with some creativity and observation of the differential equation, it is possible to study solutions without actually solving them.

We start by defining autonomous equations.

Autonomous differential equations. A first-oder DE is called ***autonomous*** if it has the form

$$y' = f(y).$$

These are basically ODEs where the rate of change does not depend on time, meaning that the nature of the ODE always stays the same.

Observe that autonomous ODEs are also Separable ODEs.

So let us look at an autonomous ODE and think what happens when $f(y_0) = 0$?

Then if the solution is unique (what are the conditions that will guarantee that?), then if $y(t_0) = y_0$, that means that

$$y'(t_0) = f(y_0) = 0.$$

So we can find one immediate solution:

$$y(t) = y_0,$$

a constant solution. Since the solution is unique, that must be **the** solution.

This is a property of autonomous ODEs:

Equilibrium points. Consider an autonomous ODE $y'(f(y)$. The zeros of the function f are called ***critical points***. They can also be called ***equilibrium*** or ***stationary*** points.

Important. Consider an autonomous ODE $y' = f(y)$ and let c be a zero of f, i.e. $f(c) = 0$. Then the constant function $y(t) = c$ is a solution of the ODE, called an ***equilibrium solution***.

In an ODE where solutions are unique, these equilibrium solutions are extremely important, as they give bounds for all other solutions.

Example. Consider the autonomous ODE

$$y' = \sin(2y).$$

The ***equilibrium solutions*** for this ODE are

$$y = k\pi,$$

for all values $k \in \mathbb{Z}$.
That means that even without solving, we can infer that the solution passing through $y(0) = 1$, must satisfy

$$y(t) \in (0, \pi),$$

for all t.

Equilibrium solutions are even more important. In fact, we can also study what happens between equilibrium solutions without having to actually solve the ODE.

If the function $f(y)$ is continuous, then its sign cannot change between equilibrium points, so the solutions will be monotonic between equilibrium solutions.

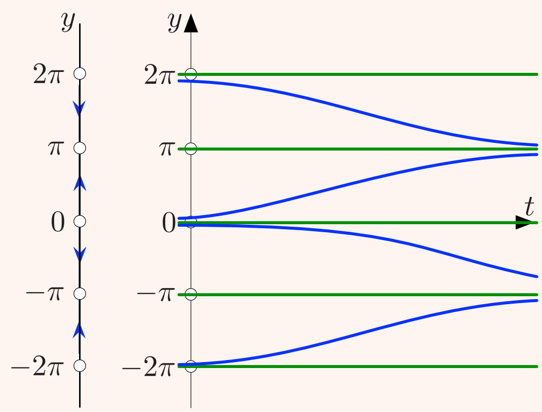
Example. Consider the same ODE:

$$y' = \sin(2y).$$

We can study whether solutions will be increasing or decreasing by studying the function $f(y)$.

y	$\cdots$	-2π		$-\pi$		0		π		2π	$\cdots$
$y' = \sin(2y)$		0	$+$	0	$-$	0	$+$	0	$-$	0	
$y(t)$	$\cdots$	c	$\nearrow$	c	$\searrow$	c	$\nearrow$	0	$\searrow$	c	$\cdots$

In a graph, we have



We can infer that the graphs will approach the constant solutions without touching because the derivative y' will become smaller and smaller the more they approach the equilibria. We also know that solutions cannot touch each other.

There is also a distinction that we make about equilibrium points that helps us understand the behaviour of solutions. We will study this distinction in the core exercises.

Population Models. The fact that these differential equations keep the same rate of change independently of time, makes them an ideal candidate when studying populations.

Video.

- https://youtu.be/swt-let4pCI

Practice Problems

1 Show that all autonomous differential equations are separable.

2 Consider the ODE $y' = \dfrac{1}{y}$. Then which of the statements below are true or false and justify your choice.

(a) The solutions always stay positive.

(b) The solutions always stay negative.

(c) The solutions never change sign.

(d) The solutions always change sign.

(e) Some solutions change sign, and some don't.

3 Sketch a slope field for the autonomous ODE

$$y' = \cos(y).$$

4 Sketch a slope field for the autonomous ODE

$$y' = \tan(y).$$

5 What is a common property of all slope fields of autonomous ODEs?

6 Consider the ODE

$$y' = \operatorname{sign}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases}.$$

(a) What are the equilibrium solutions?

(b) Find two solutions that satisfy $y(0) = 0$.

(c) Are there solutions that satisfy $\lim_{t\to\infty} y(t) = \pi$?

(d) What are the possible limits of solutions as $t \to \infty$?

7 Consider a function $f(y)$ such that

- $f(1) = 0$;
- f' is continuous for all y;
- $f'(1) < 0$.

(a) Show that there is an open interval (a, b) satisfying $1 \in (a, b)$ and $f'(y) < 0$ for all $y \in (a, b)$.

(b) Show that
  - $f(y) > 0$ if $y \in (a, 1)$,
  - $f(y) < 0$ if $y \in (1, b)$.

(c) Consider the initial-value problem $y' = f(y)$ with $y(0) = y_0$. Show that this problem has a unique solution.

(d) Show that if $y_0 \in (a, b)$, then $\lim_{t\to\infty} y(t) = 1$.

(e) Write a Theorem about equilibrium points based on the results of this question.

8 Consider a function $f(y)$ such that

- $f(2) = 0$;
- f' is continuous for all y;
- $f'(2) > 0$.

Complete a study similar to question 7 for this function f.

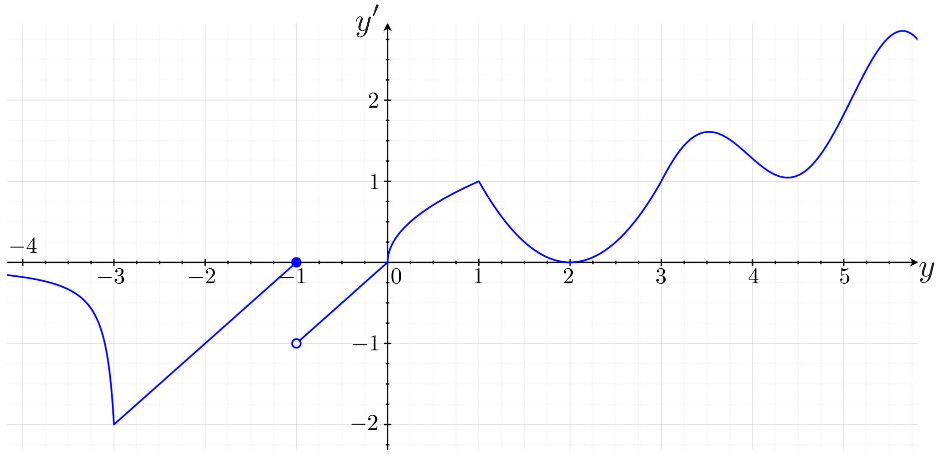
9 Consider an autonomous ODE $y' = f(y)$ with two stable equilibrium solutions $y = 1$ and $y = 2$ and where f is continuous.

(a) Show that there must exist another equilibrium point $c \in (1, 2)$.

(b) Show that if $f(y) \not\equiv 0$ in $(1, 2)$, then there must exist an equilibrium point $c \in (1, 2)$ that is either semi-stable or unstable.

Module 14: Autonomous Differential Equations

26 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



26.1 What are the equilibrium points?

26.2 Which equilibrium solutions are stable, unstable, or semi-stable?

26.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

26.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

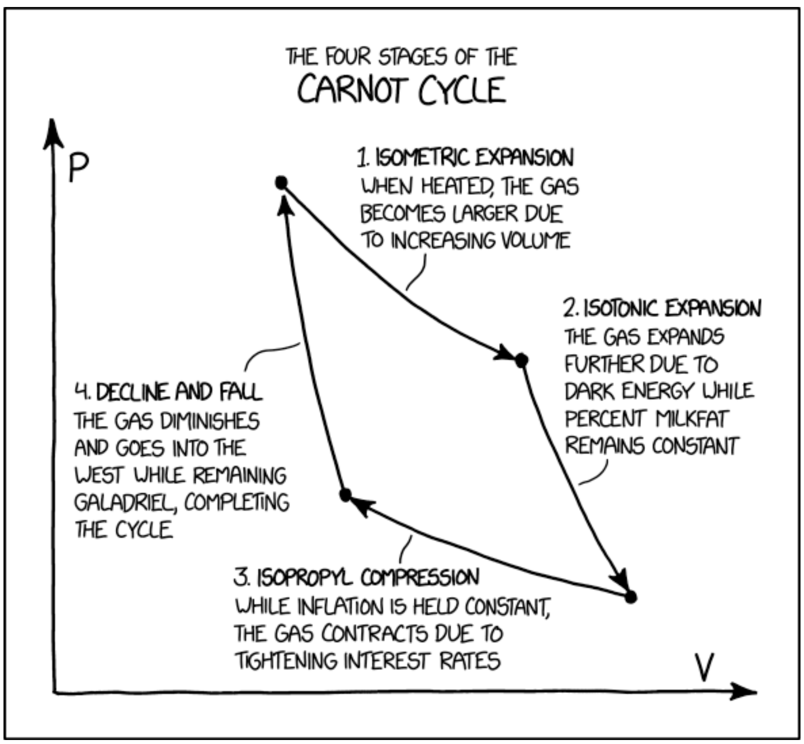
(b) $y(0) = -\frac{1}{4}$.

(c) $y(1) = \frac{1}{4}$.

26.5 If $y(0) = 2$, then $y(t) =$

26.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

26.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$



(image from xkcd - comic #2063)

Modelling Two Quantities

In this module you will learn

- how to model two or more inter-dependent quantities using systems

Often, when modelling something, we are faced with two or more quantities that depend on each other. This means that one equation is not enough, so we need to learn how to deal with a system of equations.

Just like we did in module 25, we will follow the step by step procedure developed in chapter 1.

Step 1. Define the problem

Example. We want to model two interacting populations, like the populations of bears and salmon in a specific natural park.

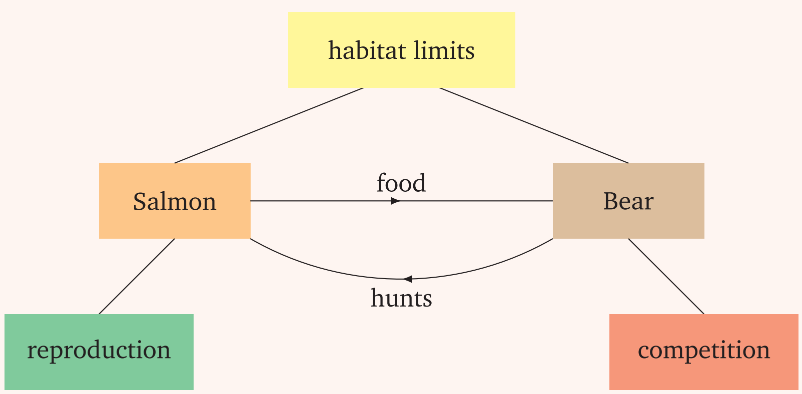
The first step is to decide on what we want to find at the end of the process. In this case, we want to know the number of individuals in each population and how they change as time passes. So we define:

- $b(t) =$ number of bears in the natural park at time t;
- $s(t) =$ number of salmon in the natural park at time t.

Step 2. Build a mind map

Example. We start with both species in the centre:

We can start brainstorming about the things that affect these populations:



Step 3. Make assumptions

Example. In this step, we discuss which of the boxes in the mind map we want to actually consider in our model, and which assumptions we need to make to consider them.

Let us start with how these species interact with each other:

1. Salmon provide food for bears: the bear population profits from each encounter with salmon. How does each bear-salmon encounter affect the bear population?
2. Bears hunt salmon: the salmon population is likely to decrease with each encounter with a bear. How

does each bear-salmon encounter affect the salmon population?

These two components are essential in our model, so we need to include them. It still leaves some freedom on how to do this.

There are other elements that we might want to include in our model:

3. Salmon reproduction: in the absence of predators and under ideal conditions, salmon should grow according to the Malthusian model, i.e. the rate of growth is proportional to the number of salmon;

4. Bear competition: bears are mainly predators, so without salmon, their numbers will decrease, also according to the Malthusian model;

5. Habitat limits: these species live in habitats that have limited resources, so we can consider a carrying capacity for each species.

To make the model simpler, we will ***ignore habitat limits***. This means that this model will not be accurate if the populations become very large.

Step 4. Construct a model

Example. We will start with our populations:

- $b(t)$
- $s(t)$

and we will start adding components to each of these one by one.

For the first two items, we need to estimate the number of encounters salmon-bear. We assume that the number of encounters is proportional to the number of all possible encounters: $b(t)s(t)$.

1. Salmon provide food for bears: for every possible salmon-bear encounter, there is a probability that a bear actually encounters a salmon, and then there is a chance that the bear will catch the salmon. Each catch improves the possibility that the bear population will increase. All these put together means that this factor should increase the bear growth rate by $a\,b(t)s(t)$, where the constant a needs to be found.

2. Bears hunt salmon: similarly to the previous item, for every possible encounter, there is a probability that the bear actually encounters a salmon, and then there is a chance that the bear will catch the salmon. Every catch will decrease the salmon population, so the salmon growth rate will decrease by $c\,b(t)s(t)$, there the constant b needs to be found.

Right now we have the following model:

$$\begin{cases} b'(t) = a\,b(t)s(t) + \cdots \\ s'(t) = -c\,b(t)s(t) + \cdots \end{cases}$$

We continue with the other elements:

3. Salmon reproduction: this was explained before and should contribute to the salmon growth rate with the term $ds(t)$.

4. Bear competition: this was also explained above and should contribute to the bear growth rate with the term $-eb(t)$.

5. Habitat limits: we decided to ignore this.

We have the model:

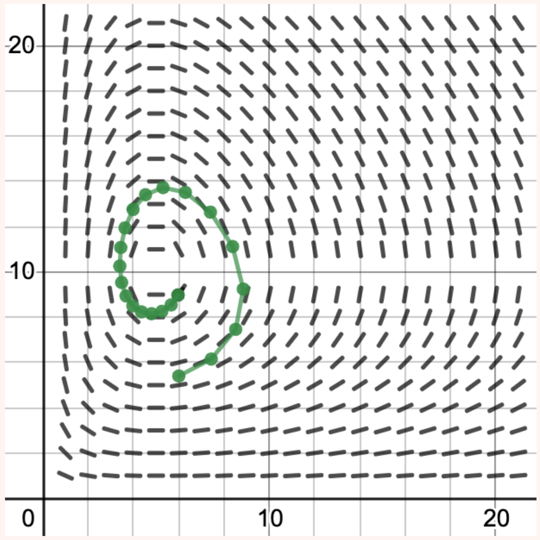
$$\begin{cases} b'(t) = a\,b(t)s(t) - e\,b(t) \\ s'(t) = -c\,b(t)s(t) + d\,s(t) \end{cases}$$

To find the constants a, c, d, e, we would probably need to go back to Step 3 and make further assumptions related to the way that we are measuring them.

Step 5. Model assessment

Example.

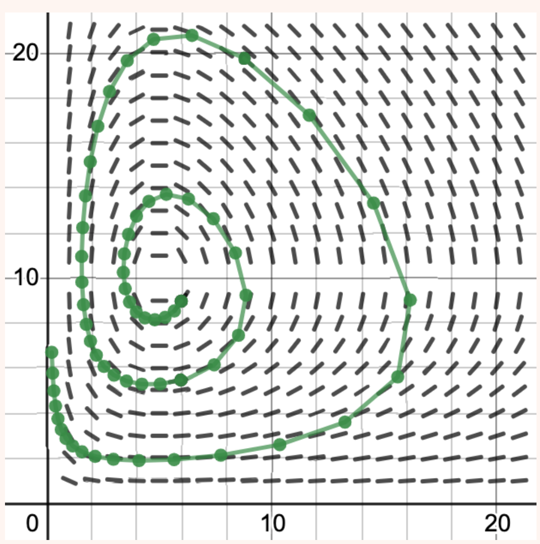
We can do several things here. I'll let you brainstorm and think of ways you can assess this model. One of the things that we can do is approximate its solutions using Euler's Method discussed in Module 15. Let us assume, for this example the constants: $a = 1$, $e = 10$, $c = 1$, $d = 5$ and a time step $\Delta t = 0.5$ and we assumed an initial population of $b(0) = 6$ and $s(0) = 9$. Then we obtain the graph below:



- https://www.desmos.com/calculator/zywspwstwk

The x—axis is the bear population while the y—axis is the salmon population. Each dot gives gives an approximation of the populations $\Delta t = 0.5$ time units after the previous approximation.

From this approximation, we can say infer that this model creates a population cycle, but it seems to spiral outwards:



- Is having a population cycle a feature that our model should have?
- Is the spiralling outwards a feature we want in our model?
- Is the spiralling a feature of the model or the approximation? If it's from the approximation, how does the model behave?

There are lots of other tools to create slope fields and approximate solutions of systems of ODEs.

- GeoGebra approximation of the same model, called the Lotka-Volterra model:

  https://www.geogebra.org/m/KqNV7eHB

- WolframAlpha slope field of the same model:

  https://uoft.me/modelling-sys-wa

- WolframAlpha stream plot of the same model:

  https://uoft.me/modelling-sys-wa2

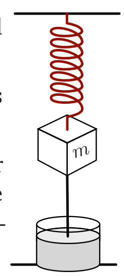
Step 6. Putting it all together in a report

We'll skip this part here.

Practice Problems

1 Create a model for two cooperating populations, like sharks and remoras.

2 We have a spring attached to a mass and with a dashpot.



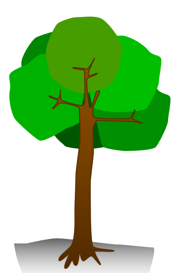
(a) Model the position of the mass as time changes.

(b) Obtain a system of two first-order ODEs. Remember to explain how the new functions relate to the spring-mass-dashpot system.

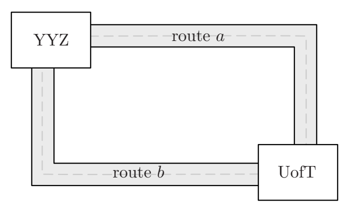
3 Model a vehicle with a special engine that provides an acceleration to the car proportional to the fuel left.

4 Imagine two twin babies and model their crying volume. Assume that they naturally become tired and stop crying if alone, but they cry more if the other twin is crying.

5 Create a simplified model for a tree, considering the height of the tree and its leaf area and how they affect each other.



6 Create a model on how a student's confidence in her own ability affects her learning/knowledge of a subject. Remember the Ebbinghaus' "forgetting curve".



7 Two ways to travel from UofT to Toronto's Pearson airport (YYZ). Both paths take the same time if there is no traffic. You want to direct people on the fastest path. Create a model for choosing the fastest path.

8 Create a model for the sales of a specific brand of sneakers. The goal is to capture the influence of famous people and non-famous people on each other's purchases.

9 Create a model on how the population and the cost of living in Toronto affect each other.

27 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

27.1 Create a model for these two populations.

27.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

27.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

27.4 Extend the model to include a population of antelopes.

28 A cheetah is chasing an antelope. We want a model of their positions as they run.

Systems of two linear ODEs with constant coefficients

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

29 Core Exercise with several parts

29.1 Part 1

29.2 Part 2

30 One more core exercise

Phase Portraits

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

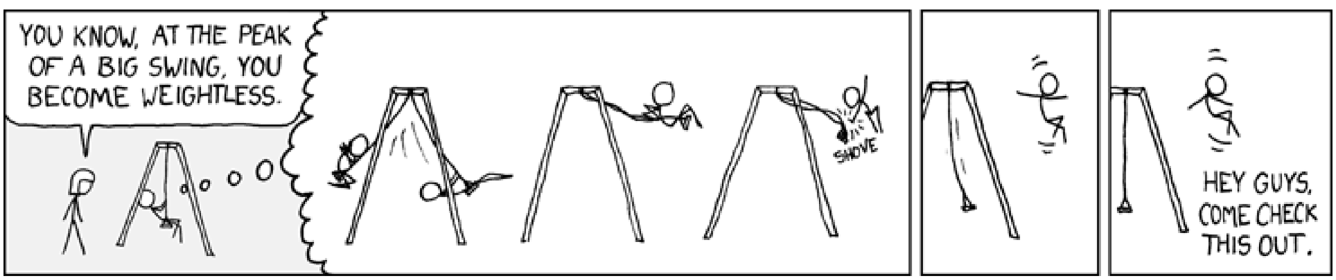
Module 17: Phase Portraits

31 Core Exercise with several parts

31.1 Part 1

31.2 Part 2

32 One more core exercise



(image from xkcd - comic #226)

Modelling with Second-Order ODEs

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

33 Core Exercise with several parts

33.1 Part 1

33.2 Part 2

34 One more core exercise

Second-Order Linear ODEs with Constant Coefficients

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

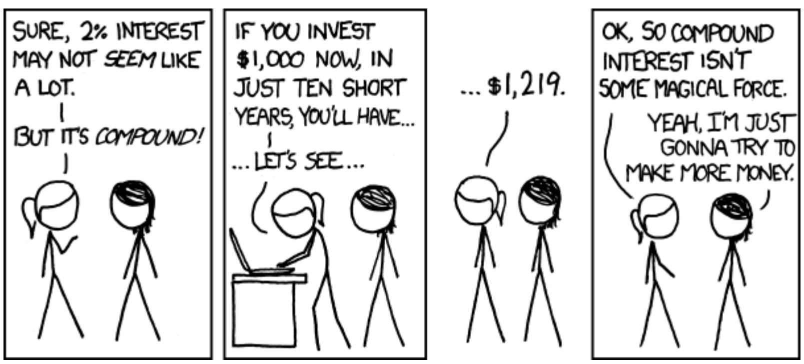
35 Core Exercise with several parts

35.1 Part 1

35.2 Part 2

36 One more core exercise

Difference Equations



(image from xkcd - comic #947)

Introduction to Difference Equations

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

37 Core Exercise with several parts

37.1 Part 1

37.2 Part 2

38 One more core exercise

Modelling with Difference Equations

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1. Show that all autonomous differential equations are separable.

39 Core Exercise with several parts

39.1 Part 1

39.2 Part 2

40 One more core exercise

Solving Difference Equations

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

41 Core Exercise with several parts

41.1 Part 1

41.2 Part 2

42 One more core exercise

Models for Two or More Interconnected Quantities

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

43 Core Exercise with several parts

43.1 Part 1

43.2 Part 2

44 One more core exercise

Nonlinear Models

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

Practice Problems

1 Show that all autonomous differential equations are separable.

45 Core Exercise with several parts

45.1 Part 1

45.2 Part 2

46 One more core exercise

6.1 Linear Algebra Review

Algebra of Solving Systems of 2 Linear Equations

We can write a linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

into matrix form

$$\mathbf{A}\vec{x} = \vec{b},$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We can solve a system like this one in several different ways.

Example. Solve the system

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ 2x_1 + 3x_2 &= 8. \end{aligned}$$

Solution by substitution.. We can write

$$x_2 = \frac{7 - 3x_1}{2},$$

and use this on the second equation

$$2x_1 + 3\frac{7-3x_1}{2} = 8 \quad \Leftrightarrow \quad 4x_1 + 21 - 9x_1 = 16 \quad \Leftrightarrow \quad -5x_1 = -5 \quad \Leftrightarrow \quad x_1 = 1$$

Then re-use the first equation we obtained to get $x_2 = 2$.

Solution by Cramer's rule.
Using the same method of substitution on the general system, we obtain

$$a_{12}x_2 = b_1 - a_{11}x_1,$$

and we use this into the second equation (after multiplying by a_{12})

$$a_{12}a_{21}x_1 = a_{22}b_1 - a_{11}a_{22}x_1 = a_{12}b_2$$

This implies

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Then we use this to obtain

$$x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Important. This implies that there is a unique solution of the system if and only if

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

Solution by inverse matrix. A matrix is **invertible** or **nonsingular** iff $\mathbf{A}^{-1}$ exists iff $\det(\mathbf{A}) \neq 0$.
If the matrix $\mathbf{A}$ is invertible, then we can write

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

We can now use this to solve the system of equations:

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \mathbf{A}^{-1}\mathbf{A}\vec{x} &= \mathbf{A}^{-1}\vec{b} \\ \mathbf{I}\vec{x} &= \mathbf{A}^{-1}\vec{b} \\ \vec{x} &= \mathbf{A}^{-1}\vec{b} \end{aligned}$$

Homogeneous Systems.
A system of equations is called ***homogeneous*** if $\vec{x} = \vec{0}$ is a solution, which means that $\vec{b} = 0$:

$$\mathbf{A}\vec{x} = \vec{0}.$$

Otherwise, it is called ***nonhomogeneous***.

Eigenvalues and Eigenvectors

We can think of the matrix multiplication $\vec{y} = \mathbf{A}\vec{x}$ as a mapping or transformation: given a vector $\vec{x}$ it transforms it into a different vector $\vec{y}$.

In many applications, it is important to know which vectors $\vec{x}$ are transformed into multiples of themselves.

These vectors satisfy the property

$$\mathbf{A}\vec{x} = \lambda\vec{x} \qquad \Leftrightarrow \qquad (\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}.$$

One such vector is $\vec{x} = \vec{0}$. But that's not very interesting. We want to look for nonzero vectors that satisfy this property.

These vectors are called **eigenvectors** and the corresponding λ is called an **eigenvalue**.

Important. The second formulation above implies that the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular, otherwise the unique solution would be $\vec{x} = \vec{0}$. So that implies that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \qquad \textbf{(characteristic equation)}$$

Example. Let us find the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{pmatrix} 1 & 8 \\ 4 & 5 \end{pmatrix}.$$

First we solve the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 8 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 32 = 0 \qquad \Leftrightarrow \qquad \lambda^2 - 6\lambda - 27 = 0$$

which implies that

$$\lambda = 3 \pm \sqrt{9+27} = 3 \pm 6$$

Eigenvalue $\lambda_1 = -3$. To find the eigenvector, we write its equation

$$(\mathbf{A} - \lambda_1\mathbf{I})\vec{x} = \vec{0} \qquad \Leftrightarrow \qquad \begin{pmatrix} 4 & 8 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that

$$4x_1 + 8x_2 = 0 \qquad \Leftrightarrow \qquad x_1 = -2x_2.$$

So one eigenvector for this eigenvalue is

$$\vec{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 9$. To find the eigenvector, we write its equation

$$\begin{pmatrix} -8 & 8 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that

$$-8x_1 + 8x_2 = 0 \qquad \Longleftrightarrow \qquad x_1 = x_2.$$

So one eigenvector for this eigenvalue is

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Theorem. Let $\mathbf{A}$ have real or complex eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$ and let the corresponding eigenvectors be

$$\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \qquad \text{and} \qquad \vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

If $\mathbf{X}$ is the matrix with columns taken from the eigenvectors:

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

then

$$\det(\mathbf{X}) \neq 0.$$

6.2 2019 M_3C competition report from the winning team

In the following pages you can find an abridged version of the full report. The full report can be found at https://uoft.me/modelling-app-report.

MathWorks Math Modeling Challenge 2019

High Technology High School–
Team # 12038 Lincroft, New Jersey
Coach: Raymond Eng
Students: Eric Chai, Gustav Hansen, Emily Jiang, Kyle Lui, Jason Yan

MathWorks Math Modeling Challenge Champions

$20,000 Team Prize

MathWorks Math Modeling Challenge

***Note: This cover sheet has been added by SIAM to identify the winning team after judging was completed. Any identifying information other than team # on a MathWorks Math Modeling Challenge submission is a rules violation.

***Note: This paper underwent a light edit by SIAM staff prior to posting.

Substance Use and Abuse

Executive Summary

In recent years, substance abuse has intensified to an alarming degree in the United States. In particular, the rise of vaping, a new form of nicotine consumption, is dangerously exposing drug abuse to a new generation. With the need to understand how substance use spreads and impacts individuals differently, our team seeks to provide a report with mathematically founded insights on this prevalent issue.

We first strove to predict the spread of nicotine use due to both vaping and cigarettes over the next decade. By comparing the spread of nicotine use to an infectious disease, we modified the SIRS epidemiology model to create our adapted SIRI model in which individuals are divided into four compartments: infected (drug users), recovered (users who quit drugs), susceptible (potential drug users), and nonsusceptible (those who will never use drugs). People progress from susceptible to infected to recovered, but may relapse into their old habits, causing them to re-enter the infected population. Birth and death rates of our designated population were modeled with linear equations. We solved a system of differential equations to determine e-cigarette and cigarette use in 2029: 26.63% of the American population will vape and 6.45% will smoke cigarettes. These results align with the expectation that vaping will increase in popularity while cigarette smoking will decline.

Substance abuse is associated with numerous social factors and personal attributes. We incorporated those determinants to create a second mathematical model that computes the probability that an individual will use nicotine, marijuana, alcohol, and unprescribed opioids. A binary multivariate logistic model was used to assess the effects of age, gender, ethnicity, income, parental status, friendship, opinion about school, overall health, weapon possession, and bullying on substance use. To demonstrate our model, we coded and executed a Monte Carlo simulation that created 300 high school seniors with varying attributes. We found that 46.3% of the students would use nicotine, 17.3% would use marijuana, 66.0% would use alcohol, and 0.0% would use opiates.

Substance use has far-reaching implications in personal and societal spheres. It is crucial to rank substances based on their overall impact in order to assess necessary government action regarding drug abuse. To address this issue, we developed a robust metric to rank the effects of nicotine, marijuana, alcohol, and opioid abuse. Our model and ranking considers physical harm, dependence, social harm, and economic impact of the drugs. The former three factors were measured on a scale of 0 to 3 based on psychiatrist surveys. Then economic impact was defined as GDP loss from the decrease in life expectancy caused by drug abuse. After applying risk factors obtained from the amount of people that use each drug, the four substances were ranked. From highest to lowest individual impact, the ranking was opioids, alcohol, cigarettes, and marijuana. From highest to lowest total societal impact, the ranking was alcohol, cigarette, marijuana, and opioids.

The repercussions of substance abuse are reverberating and remain with an individual for life. However, drugs not only severely affect the user but also cause extensive societal harm. Increased understanding of the projected spread and impact of substance abuse, as well as the underlying factors that lead to poor judgment, are needed to optimize measures to restrict consumption. Ultimately, we believe that our models provide novel insight into the nationwide issue of substance use and abuse.

1 Introduction

This section delineates the components of the modeling problem and their objectives. Global assumptions applying to the entire modeling process are also listed.

1.1 Restatement of the Problem

The problem we are tasked with addressing is as follows:

1. Build a mathematical model that predicts the spread of nicotine use due to vaping over the next 10 years. Analyze how this growth compares to that of cigarettes.

2. Create a model that simulates the likelihood that a given individual will use a given substance, accounting for social influence, characteristic traits, and properties of the drug itself. Demonstrate the model by predicting how many students among a class of 300 high school seniors with varying characteristics will use nicotine, marijuana, alcohol, and unprescribed opioids.

3. Develop a metric for the impact of substance use, considering both financial and nonfinancial factors. Use the metric to rank the substances listed in Part II.

1.2 Global Assumptions

1. *The current drug scene remains constant.* We assume that there will be no radical changes in the recreational drug industry, such as new drugs or drug products. This assumption is imperative because attempting to account for unpredictable and volatile factors would make model development virtually impossible.

2. *All vapes count as e-cigarettes.* Some people distinguish between e-cigarettes and vaping. For the purposes of this model, e-cigarettes and vapes will be considered synonymous.

3. *People respond honestly to surveys.* Our model is dependent on survey results to calculate weight constants. Because we have no way of determining the accuracy of the survey responses, we will assume that they are accurate and without bias for simplicity.

2 Part 1: Darth Vapor

First commercialized in 2003, electronic cigarettes have become an increasingly popular product among youth [1]. Although they are advertised as safer alternatives to traditional cigarettes, e-cigarettes contain high doses of nicotine and have introduced a new generation to tobacco products. This section outlines a mathematical model for predicting the change in nicotine use in the United States due to vaping compared to the change due to cigarettes.

2.1 Assumptions

1. *Nicotine use can be modeled as an infectious disease.* Like an epidemic, nicotine use is prevalent and contagious, reflected in the surge in popularity of smoking due to peer pressure, advertisements, and social media. Additionally, the U.S. Surgeon General declared youth vaping a nationwide epidemic in 2018 [2].

2. *Individuals can smoke from age 11 until death.* Peak years for first trying nicotine products is 6th or 7th grade [3].

3. *Rate of entry into pre-adolescence in the U.S. is 0.00103.* [4] Our model defines "birth" as reaching an age at which substance use becomes possible—around 11 years. Thus, we assumed the current birth rate to be constant for the past 11 years, assuming no children die before they turn 11. The current birth rate is 1.03 people/month/person.

4. *Death rate in the U.S. is constant and equal to 0.0007 people per month per person.*[4] Our model assumes that individuals have the capacity to use drugs until their death.

5. *Individuals can only start smoking due to influence from other smokers.* To model substance use as an infectious disease, we must assume that susceptible individuals can become infected only from contact with the already infected. This assumption is valid because peer influence and social media presence are the driving factors behind the popularity of smoking [5].

6. *Individuals are either not susceptible to, susceptible to, infected by, or recovered from substance abuse.* As in the SIR epidemiology model, we assume that people are either unwilling to smoke (not susceptible), open to smoking (susceptible), regular smokers (infected), or past smokers who have quit (recovered).

7. *The infection rate is constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the infection rate will also not drastically change.

8. *The percentage of susceptible people will stay constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the number of people susceptible to it will also not drastically change.

9. *Nobody starts as recovered.* At the start of the model, we do not consider any individuals to be former smokers who have quit.

10. *The recovery and relapse constant for cigarette and e-cigarette users are the same.* The two contain similar amounts of nicotine, which acts as the addictive agent. Thus, the recovery and relapse constants are assumed to be the same.

2.2 Model Development

The surge in popularity of conventional cigarettes in the mid-20th century, as well as the current boom of vaping among American youth, is comparable to the spread of an infectious disease during an epidemic. As stated in assumption 1, we model nicotine use as a disease because it rapidly spreads as a result of interpersonal communications (in-person peer pressure to try a drug as well as social media prevalence); additionally, substance use is a condition from which individuals can recover (by quitting smoking).

Our model is a derivation of the SIRS epidemiological model, a technique used to map the spread of infectious diseases such as influenza. We also consider birth and death rate, since population naturally changes over time. The model separates individuals in a population into four categories: NS for Not Susceptible, S for Susceptible, I for Infected, and R for Recovered. At the start of the model, individuals are either in NS, S, or I, since nobody starts off as recovered. While those in NS remain there permanently, individuals in S can move to I, who can then move to R.

The additional S in SIRS represents the possibility of returning to the Susceptible compartment—in this case, a regular user quitting but relapsing. However, we modified the classic SIRS model by recognizing that a relapsing individual would re-enter the Infected category rather than Susceptible, since they will once again become smokers rather than people merely open to smoking. Thus, we renamed the traditional epidemiology model as SIRI to represent this adjustment. Figure 2.2.1 diagrams the aforementioned movement of individuals between categories, while Table 2.2.1 defines and details values for variables and constants used in the SIRI model for both e-cigarette and cigarette smoking.

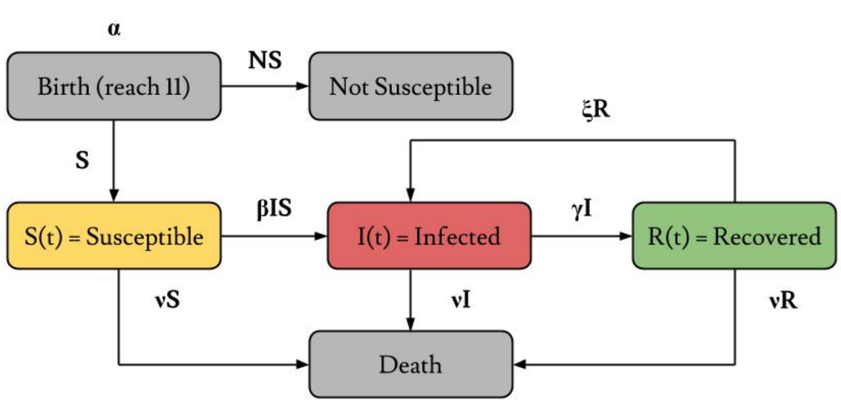


Figure 2.2.1: Diagram of the SIRI Model for Spread of Nicotine Use

2.2.1 Parameters in SIRI Model

Proportion of infected people (I_0). The total number of people that currently vape is approximately 10.8 million [6]. Dividing by the total population of America, 325.7 million

[7], results in an I_0 value of 0.0332 for e-cigarettes. The total number of people that currently smoke cigarettes is approximately 34.3 million [8], resulting in an I_0 value of 0.1053.

Proportion of recovered people (R_0). As per assumption 9, without loss of generality, R_0 was assumed to be 0 at time = 0.

Proportion of susceptible people (S_0). Because I, R, and S are proportions of the total population, their sums must add to 1. Thus, $S_0 = 1 - R - I$, resulting in 0.9667 for e-cigarettes and 0.8947 for cigarettes.

Susceptibility (S). A 2016 Surgeon General report stated that 32% of people are considered susceptible to e-cigarette use [5], while a 2012 report stated that 20% of people are susceptible to cigarettes, which correspond to the S values [9].

Infection constant (β). This was determined based on responses to the survey question "If one of your best friends offered you a cigarette, would you smoke it?" For e-cigarettes, the chance of infection was taken from a 2016 U.S. Surgeon General report that indicated that 18% of young adults responded "yes" to the question [5]. For cigarettes, we obtained β by adding the percentages of the responses "Definitely Yes" and "Probably Yes," from the 2014 National Survey on Drug Use and Health, to get 0.3%, which represented the infection constant [10].

Recovery constant (γ). In a given year, around 40% of smokers attempt to quit [11]. Therefore, in a month, $1.40^{1/12} = 1.0284$ recover, so the recovery rate is 0.0284.

Relapse constant (ξ). In a given year, approximately 6% of attempts to quit smoking succeed and 94% of attempts failed and the person relapsed [12]. Therefore, in a month, $1.94^{1/12} = 1.0568$ fail, so the relapse constant is 0.0568.

Infection rate (y_{inf}). In accordance with assumption 4, we assume that people will only start smoking if they are influenced by a current smoker. In other words, a susceptible person can only become infected if they come into contact with an infected person, which occurs at a rate proportional to $I \cdot S$. The infection constant β represents the likelihood that a susceptible person becomes infected when influenced by a smoker. Thus, infection rate is as follows:

$$y_{inf} = \beta \cdot I \cdot S \tag{1}$$

Recovery rate (y_{rec}). Unlike infection rate, the recovery rate is dependent only on the average probability of an individual quitting. The recovery constant γ multiplied by the proportion of people that currently are infected gives the recovery rate:

$$y_{rec} = \gamma \cdot I \tag{2}$$

Relapse rate (y_{rel}). The relapse rate is dependant only on the average probability of an individual relapsing. The relapse constant is much higher than the infection rate, which is logical because an individual who was previously a regular smoker will be more likely to succumb to the addictive cycle again [12]. Designating ξ as the relapse constant, relapse rate is given by

$$y_{rel} = \xi \cdot R \tag{3}$$

Birth rate (α). The birth rate, as defined by assumption 3, is 1.03 people/month/person.

Death rate (μ). From assumption 4, the death rate is assumed to be constant and equal to 0.0007 people per month per person. Therefore, the number of people dead for each category will be the death rate multiplied by the proportion of the people in each category.

$$\mu_S = v \cdot S \tag{4}$$

$$\mu_I = v \cdot I \tag{5}$$

$$\mu_R = v \cdot R \tag{6}$$

Table 2.2.1 Variables and Constants of SIRI Model for E-Cigarettes and Cigarettes

Variable	Definition	E-Cigarette Values	Cigarette Values
I	Proportion of infected people	$I_0 = 0.0332$	$I_0 = 0.1053$
R	Proportion of recovered people	$R_0 = 0$	$R_0 = 0$
S	Proportion of susceptible people	$S_0 = 0.9667$	$S_0 = 0.8947$
N	Proportion of total individuals in SIR cycle	$N_0 = 0.32$	$N_0 = 0.20$
α	Birth rate	0.00103	0.00103
β	Infection constant	0.18	0.003
γ	Recovery constant	0.0284	0.0284
ξ	Relapse constant	0.0568	0.0568
μ	Death rate	0.0007	0.0007

2.2.2 Differential Equations for SIRI Model

The change in each of the dependent variables S, I, and R is equal to the sum of the input of the respective category minus the sum of its output, as diagrammed by the arrows entering and leaving each box in Figure 2.2.1. Thus, our SIRI model is summarized by the set of ordinary differential equations below:

$$\frac{dS}{dt} = \alpha - \beta \cdot I \cdot S - \mu \cdot S \tag{7}$$

$$\frac{dI}{dt} = B \cdot I \cdot S - \gamma \cdot I + \xi \cdot R - \mu \cdot I \tag{8}$$

$$\frac{dR}{dt} = \gamma \cdot I - \xi \cdot R - \mu \cdot R \tag{9}$$

2.3 Results

With the SIRI model established, we utilized it to predict the change in nicotine use due to e-cigarettes and cigarettes in the next decade. We coded and executed a Python program to solve the system of differential equations, with appropriate constants for each product, and graph the proportion of compartments over time. Figures 2.3.1 and 2.3.2 graph the proportion of the total population falling under each of the SIR categories for both tobacco products, respectively, over a 10-year time period. Table 2.3.1 enumerates the proportion of the population that is susceptible, infected, and recovered for vaping and cigarettes in 2029.

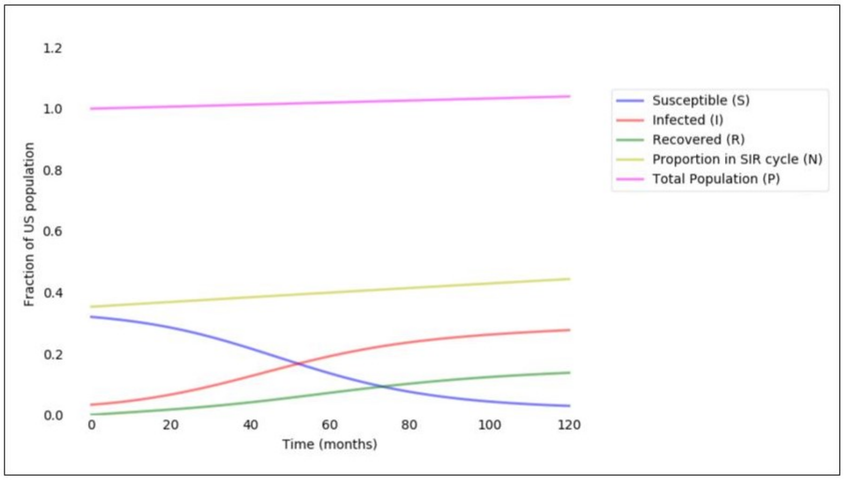


Figure 2.3.1: Graph of SIRI Compartments for E-Cigarettes over Ten Years

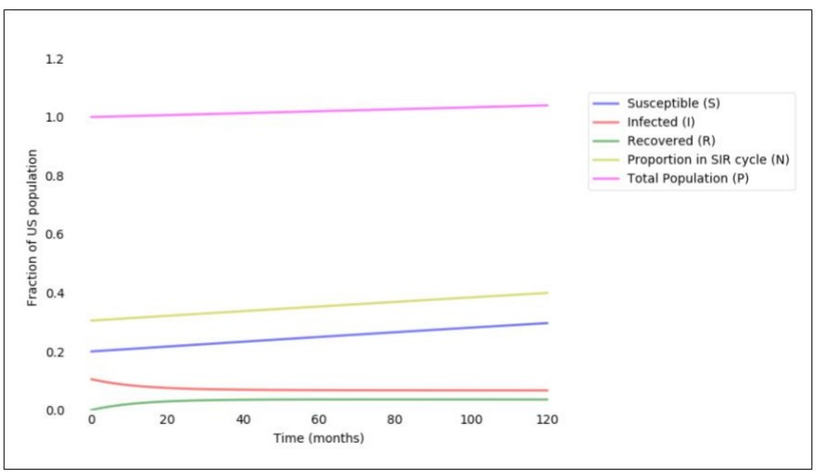


Figure 2.3.2: Graph of SIRI Compartments for Cigarettes over Ten Years

Table 2.3.1 SIR Distribution of 2029 Population for E-Cigarettes and Cigarettes

	Susceptible	Infected	Recovered
E-Cigarettes	2.82%	26.63%	13.21%
Cigarettes	28.53%	6.45%	3.45%

Our model concludes that in 2029, 26.63% of the population will use e-cigarettes, while 6.45% will use cigarettes. This disparity is consistent with previously researched trends, which suggest that as e-cigarettes gain popularity amongst teens, regular cigarettes decrease in popularity [13].

2.4 Sensitivity Analysis

Table 2.4.1 shows the sensitivity analysis for our SIRI model based on an independent increase and decrease of 10% of the infection constant β, recovery constant γ, and relapse constant ξ.

Table 2.4.1 Sensitivity Analysis for Part I

Constant	% Change in Constant	% Change in Vaping (I)	% Change in Cigarette Use (I)
β	10%	1.014%	0.6202%
β	−10%	−1.615%	−0.6202%
γ	10%	−3.492%	−3.566%
γ	−10%	4.018%	3.721%
ξ	10%	3.098%	3.367%
ξ	−10%	−3.496%	−3.905%

Positive changes in the infection or relapse constants resulted in positive changes in the percentage of infected people for both vaping and cigarette use. This is consistent with our predictions because the rate of infection for susceptible and recovered people is increasing. In contrast, a positive change in recovery constant resulted in a decrease in percent infected because the rate at which people are leaving the infected population is increasing.

2.5 Strengths and Weaknesses

Our model is resilient to small changes and outputs sensible results. As demonstrated in the sensitivity analysis, a 10% change in each of the infection, recovery, and relapse constants accounts for less than 5% change in final vaping and cigarette use after a decade. Changes in the model's output due to shifts are consistent with expected trends as well. SIRS is also an established mathematical modeling technique that we adapted to fit our own aims, lending credence to the validity of our model. Additionally, our model is comprehensive, accounting for many contributing factors such as population change, nonsusceptible individuals, and the possibility of relapse for smokers who have attempted to quit.

The model's weaknesses lie in its inability to account for the introduction of new forms of drugs or rapid changes in popularity of existing forms, as stated in global assumption 1. Specifically, a surge in use of a particular drug would likely impact vaping and cigarette use in unforeseen ways that our model will not accurately predict. Furthermore, our model does not consider the association between vaping and cigarette use, and how the growth or decline of one product would influence the other. This is unrealistic because the popularity of e-cigarettes among youth has led many to smoke traditional cigarettes and prompted cigarette smokers to transition to vaping [13]; however, the opposite effects of these two phenomena can reasonably counterbalance each other.

5 Conclusion

5.1 Further Studies

Our first model does not currently account for the introduction of new drugs in the industry, which would greatly impact the change in usage for pre-existing substances. Taking these market changes into account would greatly strengthen our model. The second model used survey data from 2005–2006. The resulting model fits well for this time period, but requires more recent data to reflect recent trends. Applying the same modeling approach for 2019 would create a more accurate model that is applicable to today. Finally, the third model is heavily based on the personal opinions of psychiatrists. Recreating the model to account for each factor with independent methods would greatly complicate the model, but make it more flexible for incorporating newer drugs into our ranking.

5.2 Summary

The first model focuses on comparing the percent of e-cigarette users versus cigarette users in the next ten years. The SIRS epidemic model was used as the basis for ours. People were split into four main categories: infected (those that used drugs), recovered (those that quit using drugs), susceptible (those that may use drugs in the future), and non-susceptible (those that will never use drugs). Birth rate and death rate were both modeled with linear equations. Simultaneous differential equations were solved to determine the number of "infected" people in 2029. According to our model, 26.63% of the American population will vape in 2029 and 6.45% will smoke cigarettes. The results correspond with observed increasing popularity of e-cigarettes and decreasing popularity of regular cigarettes.

The second model determines the probability of a student using nicotine, marijuana, alcohol, and opioids and applies itself to a randomly generated sample of 300 high school seniors. A binary multivariate logistic regression was used to create the model based on an HBSC survey. A machine learning algorithm using an L2 regression was used to calculate the weights and bias in our logistic model. Using a Monte Carlo simulation, 300 random seniors were created based on response frequencies to each of questions necessary for our model. Running this sample of high school seniors through our model, we found 46.33% would use nicotine, 17.33% would use marijuana, 66.00% would use alcohol, and 0.00% would use opiates.

The third and final model focuses on ranking nicotine, marijuana, alcohol, and opioids based on their financial and nonfinancial effects. Factors were analyzed in four main categories: physical harm, dependence, social harm, and economic impact. These factors were further split into 2–3 subcategories each that were each assigned scores on a scale from 0.0 to 3.0 based on expert surveys. To calculate the impact of drugs on GDP, the average annual GDP per person was multiplied by the average decrease in life as a result of using drugs. The impact of drugs on GDP was then rescaled from 0.0 to 3.0 to make them comparable to the other factors. Each of the four main categories was averaged for a total harm score for each of the four drugs. The total harm score was multiplied by a risk factor based on the number of people that used each drug to obtain a final score for each drug that could be used for ranking purposes. This model showed that opioids had the greatest substance harm per person, but since relatively few people use opiods, it had a lower total detriment score. Marijuana had the lowest substance harm per person and the second lowest total impact. Alcohol had the highest total impact, while cigarettes had the second highest because of the great number of people using these substances.

6 References

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