2018-09-25 Linear vs Nonlinear ODEs (2.4)

Existence and Uniqueness

Consider the problem

$y\prime = t + \sqrt{y - \pi}$

$y(1) = 4$

Q1. Is there a unique solution? YES.

f(t,y)=t+$\sqrt{y - \pi}$ continuous for all t, for $y \geqslant \pi$ (from the theorem)

$\frac{\partial f}{\partial y}(t, y) = \frac{1}{2\sqrt{y-\pi}}$ continuous for all t, for $y > \pi$ (from the theorem)

Q2. Without solving, what is its domain?

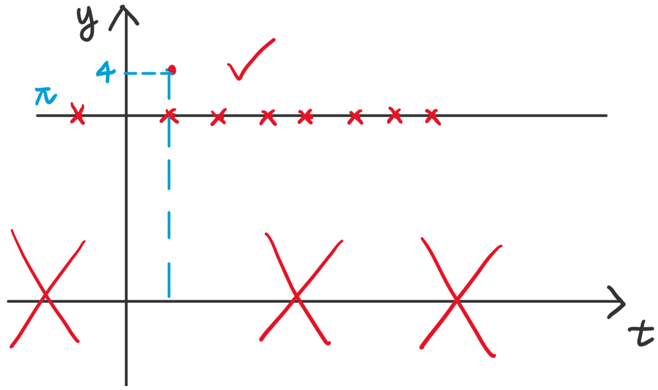
continuous for all t, for $y \geqslant \pi$ for f(t, y)

continuous for all t, for $y > \pi$ for $\frac{\partial f}{\partial y}$(t, y)

* Initial condition: a solution that passes through the point;
The uniqueness is only guaranteed around the initial value (a, b), need to solve the DE for more information like for how long or till where.

What if the solution was continuous for $y \geq \pi$ (solution stops at $y = \pi$) and our given point was (1, π) ? Can we still confirm existence and uniqueness?

No! The given point AND the points around it must be differentiable



Consider the problem (not discussed in class)

$$y' = \sqrt{4-(t^2+y^2)}$$
$$y(1) = 1$$

Q3. Is there a unique solution?

If f(t, y) is differentiable at (1, 1), then f(t, y) must also be continuous at (1, 1).

Since y' exists at (1, 1), f(t, y) is continuous at (1, 1)

$\frac{\partial f}{\partial y}(t,y) = -\frac{2y}{\sqrt{4-t^2-y^2}}$

$\frac{\partial f}{\partial y}(t,y)$ is also continuous near (1, 1)

This satisfies the conditions of the Existence and Uniqueness for Nonlinear Differential Equations Theorem. Therefore, there is a unique solution.

Q4. Without solving, what is its domain?

$f(t,y)$ is continuous for $y \le \sqrt{4-t^2}$ and $-2 \le t \le 2$

$\frac{\partial f}{\partial y}(t,y)$ is continuous for $y < \sqrt{4-t^2}$ and $-2 < t < 2$

The Initial-Value Problem

The equations:

$$y' = -\frac{x}{y}$$
$$y(\tfrac{1}{2}) = \frac{\sqrt{3}}{2}$$

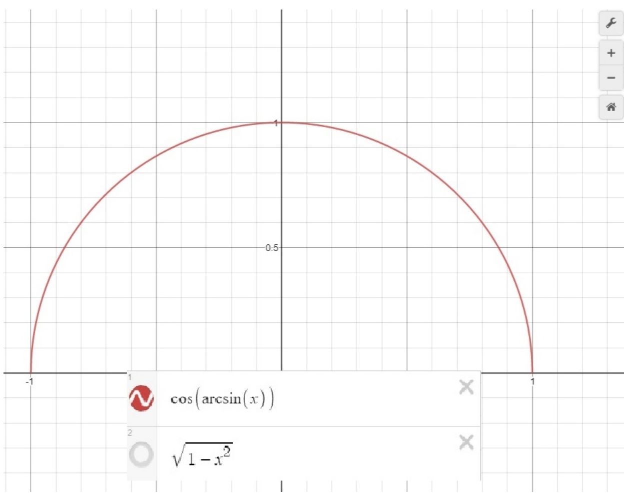
Have the solutions $y_1 = \cos(\arcsin(x))$ and $y_2 = \sqrt{1 - x^2}$

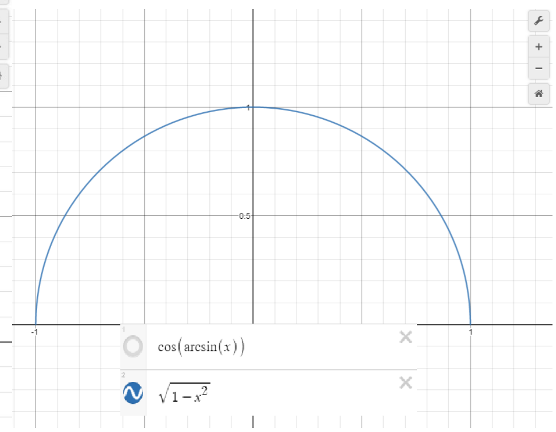
- **Does the problem satisfy the conditions of the Theorem?**
    - YES. $\frac{\partial f}{\partial y}(t, y) = \frac{x}{y^2}$, which is continuous near $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
    - In the initial value, Y is not 0 so this is continuous -- the condition is satisfied.
- **What can you conclude?**

    You can conclude that there is one unique solution because the partial derivative is continuous near the initial value. Thus, we can conclude that these 2 solutions must be identical.

    => $y_1(x) = y_2(x)$

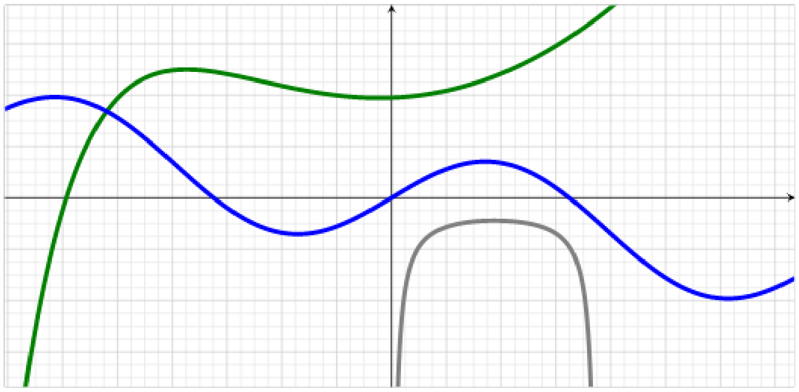
    Woah, they have the same graph! :D





Consider the problem $y' = f(t, y)$ where $f(t, y)$ and $\frac{\partial t}{\partial y}(t, y)$ are continuous for all t, y.

- (graph shown)

- 

- **Could this be the graph of 3 solutions with 3 different initial conditions?**
  - No, this can not be--The thereom says that for any given point, only 1 solution curve can pass through it. Meaning: none of the solutions of a DE can intersect. In this graph, two of the solutions overlap and therefore when the initial value is set to be the intersection, they will not be unique.

- **Theorem (Existence and Uniqueness for Nonlinear DEs):**
  - Given $y' = f(t, y)$ with $y(t_0) = y_0$

    If $f(t, y)$ and $\frac{df}{dy}(t, y)$are continuous near (t_0, y_0)

    then there is one unique solution $y = \phi(t)$defined for t near t_0.

- - **Unique definition:** there is only one function that satisfies the function and the initial value that is given.
  - The theorem only works when the function is continuous around the point (initial value), not just continuous at that point.
  - We can only conclude from the theorem that if a function satisfies the initial value theorem, then there is a unique solution; if not, we don't know anything about the solution.