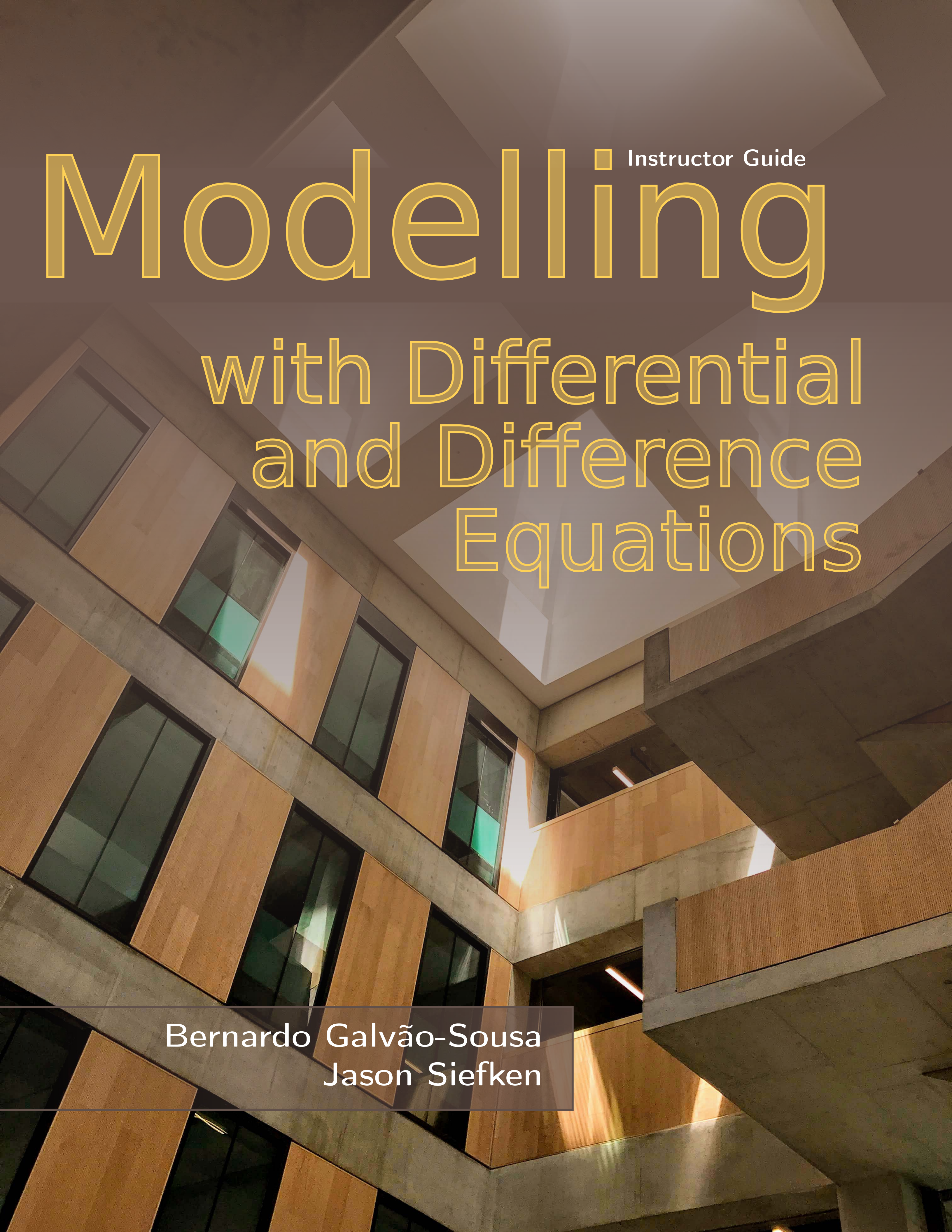

Instructor Guide

Modelling with Differential and Difference Equations

Bernardo Galvão-Sousa
Jason Siefken

Inquiry Based Modelling with Differential and Difference Equations

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For the student

This book is your introductory guide to mathematical modelling and modelling with differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling with differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle through you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time. *practice problems* were designed for students to practice at home by themselves and include some more computational exercises, and *projects* were designed to be solved through teamwork during tutorials with an assistant around to provide guidance.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.
5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

Practical lean. The *projects* have a more open ended or real-world lean. They are meant to give students some practice dealing with "messy" data or having to build their own model and assess it. They should be incorporated into the course alongside the core exercises, so that students get a taste of both the conceptual and practical aspects of modelling. Information is sometimes purposefully lacking and students are encouraged to find it out by themselves: by experimentation, internet consultation, or just by reasoning.

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.
2. **Work through the core exercises.**
3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.
4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

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If you modify this document, you may add your name to the copyright list. Also, if you think your contributions would be helpful to others, consider making a pull request, or opening an *issue* at https://github.com/bigfatbernie/IBLModellingDEs

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Included in this text, in chapter 1, are expositions adapted from the handbook "Math Modeling: Getting Started and Getting Solutions" by K. M. Bliss, K. R. Fowler, and B. J. Gallizzo, published by SIAM in 2014 https://m3challenge.siam.org/resources/modeling-handbook.

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: https://github.com/bigfatbernie/IBLModellingDEs/

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Stephanie Orfano ◦ Yvan Saint-Aubin ◦ Sarah Shujah ◦ Graeme Slaght ◦

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In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analyze the model
6. Write a report

In this chapter, we follow the approach of

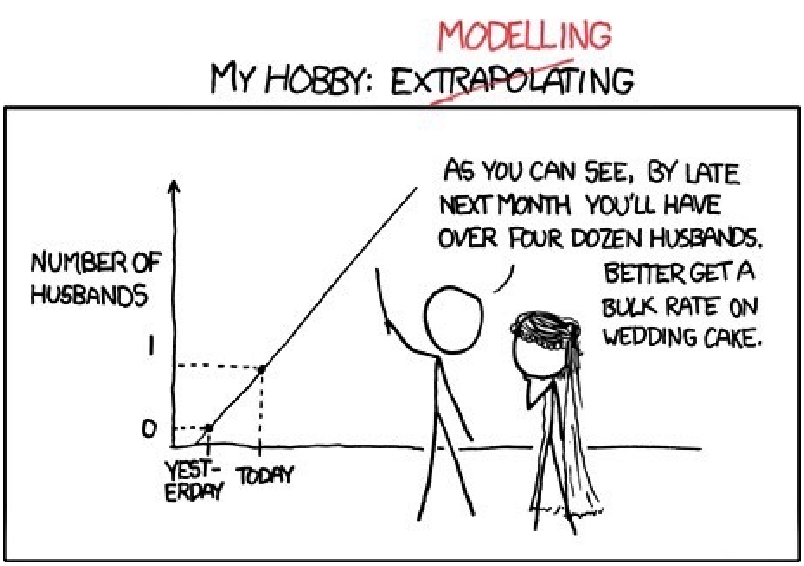
`Math Modeling: Getting Started and Getting Solutions, K. M. Bliss, K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014`

`https://m3challenge.siam.org/resources/modeling-handbook`

and

`GAIMME: Guidelines for Assessment and Instruction in Mathematical Modeling Education, Second Edition, Sol Garfunkel and Michelle Montgomery, editors, COMAP and SIAM, Philadelphia, 2019`

`http://uoft.me/gaimme`



(image adapted from xkcd - comic #605)

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Lesson 1

Defining Problem Statement

Textbook

- Module 1

Objectives

- The first step in Mathematical modelling is to define the problem
- A good way to do this is to figure out what is the “mathematical object” we are looking for at the end of the process

Motivation

Students have heard the words “Model” and “Modelling”, yet they don’t have a good idea what it means.

The goal of this first chapter is to establish a common procedure to approach all (or at least most) modelling tasks.

The very first step in the procedure is to **define the problem** in a clear and Mathematical way.

Notes/Misconceptions

- Students usually want to start thinking of ways to solve the problem.
- They need to learn to tackle a modelling task step-by-step.

Preparation for Class

- Read textbook

Tutorials and Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 10-14
  https://m3challenge.siam.org/resources/modeling-handbook
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors http://uoft.me/gaimme

$M^2(GS)^2$

GAIMME

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1 You are hired by theBigCompany to help with their “elevator problem”.

This is the email you received:

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they’re still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that’s about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

(problem adapted from GAIMME, SIAM `http://uoft.me/gaimme`)

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

Teamwork.
With your team, you must decide on ***one*** answer and be prepared to report on your decision and the reason for your choice.

Make the question precise, bring it into a “mathematical form”.

- Choose a mathematical object best suited for the problem, e.g. a number, a geometric form, a graph, a function, an algorithm, ...

Notes/Misconceptions

- Students will start discussing how to solve the problem
- This question deals with what will happen **after** solving the problem
- The goal of this question is to think about how to best tell a “mathematically-challenged” CEO that you solved the problem
- Student teamwork: “With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.”

2 The mayor of Toronto wants to extend the subway line with a new orange line as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What “mathematical object” would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

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Lesson 2

Building a mind map

Textbook

- Module 2

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions, requirements, other objects that are related, etc.

Motivation

In this step, students are supposed to brainstorm and relate the problem at hand with everything that is affected or can be affected by it.

The idea is to get a simple visual representation of possible solutions, without all the details.

This is a fundamental step in modelling.

Preparation for Class

- Read textbook

Tutorials and Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 10-14
`https://m3challenge.siam.org/resources/modeling-handbook`
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors `http://uoft.me/gaimme`

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3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- T = the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can T be minimized?

Notes/Misconceptions

- Students usually come up with more complicated variations:
  - Money spent on late employees' salaries
  - sum of time in minutes that employees are late counting only employees that are at most 15 minutes late
- Stick with T, a simple first approach

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to *optimally* control the traffic lights.

4.1 What mathematical meaning can we give to the word optimal in this circumstance?

4.2 Create a mind map for this problem.

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Making Assumptions

Textbook

- Module 3

Objectives

- The third step is to decide on a path to the solution and start making assumptions.
- This is a difficult balance between:
  - Accuracy, but difficult to analyze/solve;
  - Simple, easy to analyze/solve but not very accurate.
- Make sure assumptions and conditions of the modelling are clearly mentioned to the future reader/user of the model

Motivation

Students often make assumptions explicitly and implicitly, but they often keep them out of their notes. In the end they forget to include them in the final report.

It is imperative that they include their assumptions in the final report of their model.

Moreover, students should make an effort to find out the implicit assumptions and the conditions that to the model that their assumptions require.

Assumptions. Include the assumptions in the model's final report.

Preparation for Class

- Read textbook

Tutorials and Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 15-19 `https://m3challenge.siam.org/resources/modeling-handbook`
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors `http://uoft.me/gaimme`

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5 Consider the elevator problem from core exercise 1.
We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Hint. Focus on a **few** parameters and variables.

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they're still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

(problem adapted from GAIMME, SIAM `http://uoft.me/gaimme`)

5.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

5.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

5.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Notes/Misconceptions

- Students usually have trouble starting.
- They usually agree that they have to figure out how elevators work, so you can prompt them to be more specific.
- In the end they should come up with questions like these:
  - How fast are the elevators?
  - How much time do elevators take in each floor?
  - How many floors do elevators stop on their way up?
  - How many people fit in the elevator?
  - Should we consider elevator failures?

6 How much would it cost to make a bridge between Toronto and the U.S.?

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Lesson 4

Construct a model

Textbook

- Module 4

Objectives

- The fourth step is to use the mind map created in step 2, the assumptions from step 3, and assemble everything into one model.
- The model is not the solution to the problem: it is the framework to solve the problem.

Motivation

Students usually think that the solution is the model that they need.

Emphasize that students should see the example in the textbook to get an idea of what a model looks like.

Notes/Misconceptions

- Model $\neq$ Solution

Preparation for Class

- Read textbook
- Ask students to prepare the core exercise 7 before class.
- In class, they should combine their mind map(s) and assumptions from the previous lessons and come up with an idea of a model to discuss with their classmates in lecture.

Notes/Misconceptions

There won't be enough time in class to finish the question if students don't prepare in advance.

Tutorials and Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 20-31
`https://m3challenge.siam.org/resources/modeling-handbook`
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors `http://uoft.me/gaimme`

$M^2(GS)^2$

GAIMME

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7 Recall the core exercise 5.

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Write down a mathematical model for this problem.

Teamwork.
Each team should have *one* model and be prepared to present it to the class.

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they're still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

(problem adapted from GAIMME, SIAM `http://uoft.me/gaimme`)

Create a model.

1. Students should join in teams of 4 and come up with *one model* for the team.
2. The model should include:
   - Definition of the problem
   - Mind map
   - Assumptions and conditions
   - Clearly defined path to solve the problem
3. A few teams present their model for everyone else.
4. Full class brainstorm about each model:
   - Does it include all the parts specified?
   - Is it solvable?
   - Could we make some extra assumptions to make it simpler?
   - Is it accurate?
   - Could we change something to make it more accurate without sacrificing simplicity?

Notes/Misconceptions

- Solution should not be included!

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Lesson 5

Model Assessment

Textbook

- Module 5

Objectives

- After creating a model, students should make an analysis of the model
- The analysis is meant to test the model as well as obtain some of its consequences
- If there is a solution to the model, here is where it can be found and analyzed

Motivation

Once a model is created, one must check if the model solves the problem it is meant to solve. There are different types of assessment for a model:

- Check some known cases to see if it works as expected;
- Check some extreme cases to see if it works as expected;
- Check some implications of the model to make sure they are reasonable;
- Check that the assumptions made are reasonable for the problem at hand;
- If possible, use approximation methods to estimate the solution;
- If possible, find the solution and analyze it.

Preparation for Class

- Read textbook

Tutorials and Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 32-39
  `https://m3challenge.siam.org/resources/modeling-handbook`
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors `http://uoft.me/gaimme`

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8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\begin{array}{c}\text{loading time on}\\ \text{ground floor}\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ \text{ground} \to 33\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ 33 \to \text{ground}\end{array}} + \boxed{\begin{array}{c}\text{stop time to}\\ \text{6 of the 11 floors}\end{array}} \quad = \quad 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes
- Accumulated late time $= \boldsymbol{T} = 180 \cdot 20 \cdot 8 + 74 \cdot 20 \cdot 8 = 40\,640$ seconds $= 11\text{h}18\text{m}$

Your task is to assess this "model" (first estimate of the number of minutes employees are late). Be ready to report on your assessment.

Teamwork.
Each team should have *one* assessment and be prepared to present it to the class.

Notes/Misconceptions

Some questions to guide the students:
- What are the strengths of this model?
- What are the weaknesses of this model?
- Is the result around what you expected?

In case students don't realize that something is wrong:
- People start arriving 30 minutes before the starting time, so *almost everybody will be on time?*
- Assume that the CEO of the BigCompany is right: people are arriving late! What's wrong with the model?
- Which assumptions should be relaxed? Or checked?
- If one needs to be replaced, by what?
- Do we need extra assumptions? Which?

First make sure model works. Then try to find a solution.

Notice that the model doesn't attempt to find a solution to the question.

If there weren't any problems with this model, we could then start asking other questions:
- How can we get people in their office faster?
- How will each idea affect the estimate?
- Will they cost money to the company?

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Putting it all together

Textbook

- Module 6

Objectives

- Students practice writing a “technical” report

Motivation

Writing technical reports requires practice and students haven’t had to do this before.

Some time can be devoted to this in lecture or tutorial, but if that’s the case, then students must write a short report before the class and bring it with them.

In class, the instructor can use the Peer-Assisted Reflection (PAR) format of Daniel Reinholz as in the example in the next two pages:

- Students bring their report (1 page);
- Students self assess their own work: students check off the icons for things you think you did well and circle the icons for things they would like feedback on;
- In class students exchange reports and write feedback to each other;
- Students go home and revise their report and submit it online.

Note: Students are also not practiced at giving good feedback, so although the PAR rubric helps mitigate this, we recommend that the instructor spends some lecture/tutorial time showing an example of a question and different answers where students can practice giving good feedback.

PAR

Preparation for Class

- Read textbook;
- In teams of 2–4, write a short report on a modelling problem (e.g. see next two pages).

Projects

- BLABLABLA

Extra Reading

- Math Modelling: Getting started and getting solutions ($M^2(GS)^2$), Bliss-Fowler-Galluzzo – pages 40-44 `https://m3challenge.siam.org/resources/modeling-handbook`
- Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME), Garfunkel-Montgomery, editors `http://uoft.me/gaimme`

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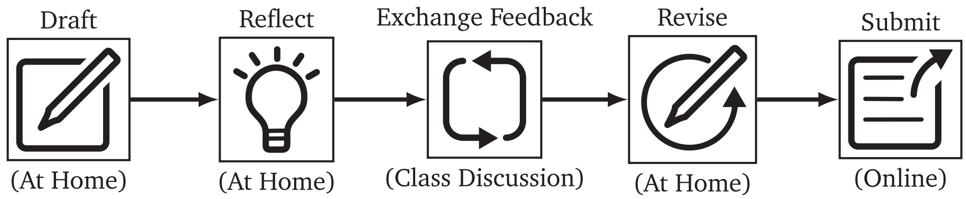
Due: Sep 19 (draft), Sep 21 (final) **Name:**

The PAR Process



Note: Your PAR (both initial and final drafts) must be typed.

Problem Statement

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

> ——— Forwarded Message ———
>
> **Date:** Monday, 7 September 2020 21:41:35 + 0000
> **From:** CEO <theCEO@theBigCompany.ca>
> **To:** Human Resources <hr@theBigCompany.ca>
> **Subject:** they're still late!
>
> Hey Shophika!
>
> I still get complaints about staff being late, some by 15 minutes.
> With the staff we have, that's about one salary lost.
> Again the bottleneck of the elevators seems to be the problem.
> Can you suggest solutions?
>
> Thanks, the CEO

(problem adapted from GAIMME, SIAM `http://uoft.me/gaimme`)

Model the current situation at theBigCompany and write a **one-page** report to the CEO about it.

Reflection

Turn the page and check off the icons for things you think you did well; circle the icons for things you would like feedback on.

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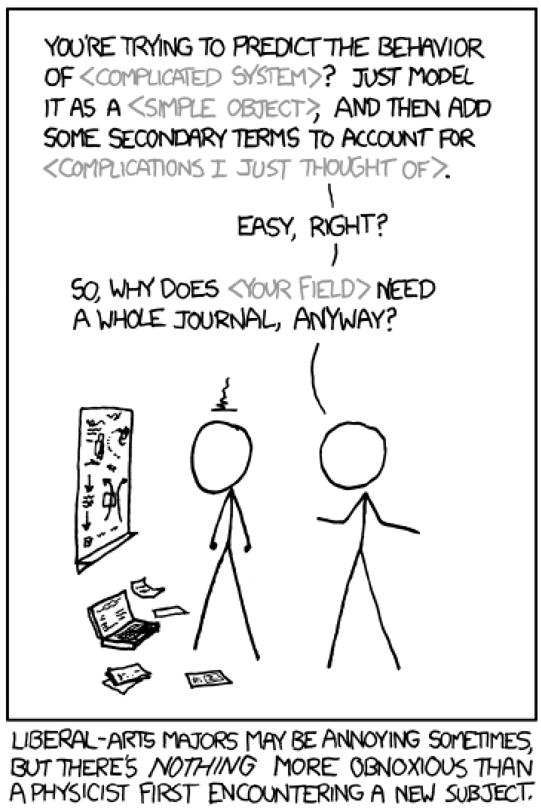
Feedback Provided By: ______________________________

Suggestions	Communication	Strengths
	Show All Steps	
	Explain Why, Not Just What	
	Avoid Pronouns	
	Use Correct Definitions	
	Define Variables, Units, etc.	
	Create Diagrams	

Suggestions	Accuracy	Strengths
	Correct Setup	
	Accurate Calculations	
	Solve Multiple Ways	
	Answer Reasonable	
	Other (Write Below)	



(image from xkcd - comic #793)

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Lesson 7

Solutions of Differential Equations

Textbook

- Modules 7, 8

Objectives

- Identifying the order of a differential equation
- Identifying a linear vs nonlinear differential equation
- Knowing how to check if a function is a solution of a differential equation

Motivation

This is an introduction to differential equations.

Students have different levels of experience with differential equations. We want to establish a common notation.

Preparation for Class

- Students should bring an ODE of their choice to class with some information about it
- Read textbook modules.

Using pre-class ODEs.

- At the start of class, put about 8 student ODEs on the board
- Get students to identify them: order + linearity
- Get some info about the ODEs to the whole class

Tutorials and Projects

- Project 2: Managing a fishery

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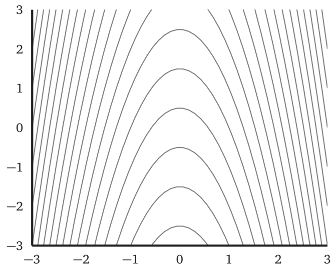
9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



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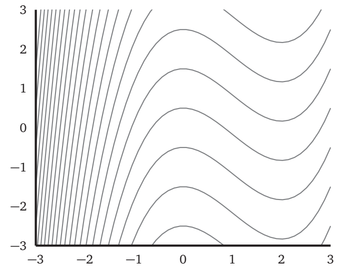

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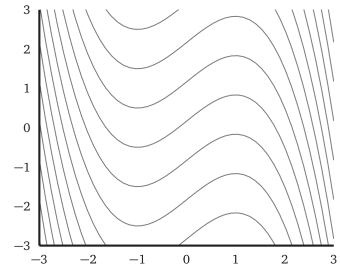

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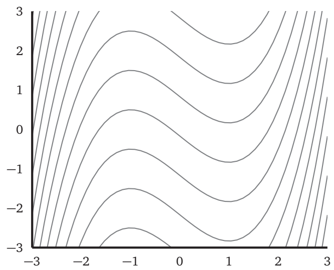

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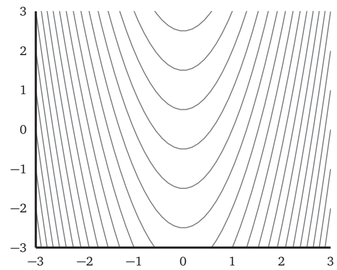

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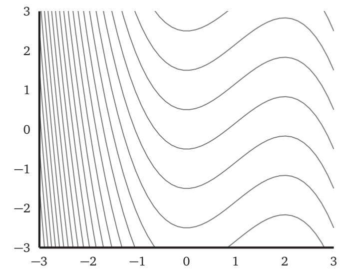


F

Without solving.

- Students should try to answer this question without solving the differential equation.
- Check properties of the ODE:
  - What does $x^2 - 1 = 0$ mean for the solution?
  - When is $y' > 0$? What does that mean for the solution?
  - When is $y' < 0$? What does that mean for the solution?

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph a function $f(x)$ that would produce such solutions.

Get students to figure out this core exercise in two different ways:
- by solving the ODE
- without solving the ODE

Quick exercise

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be decreasing

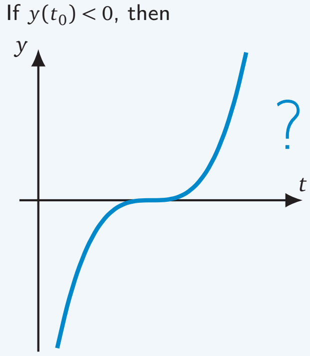
11.2 $y(t)$ must always be increasing

11.3 $y(t)$ must always be positive

11.4 $y(t)$ must always be negative

11.5 $y(t)$ must never change sign.

Stronger students.
The 5th part is for stronger students to think about while they wait for the others to finish.
Will be addressed later in module 13 (Properties of solutions).
They can see that:
- If $y(t_0) > 0$, then $y(t) > 0$ for $t > t_0$
- If $y(t_0) < 0$, then



12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

Similar to some practice problems. Skip if the other exercises take too long.

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Lesson 8

Slope Fields

Textbook

- Module 9

Objectives

- Sketch a slope field
- Use technology to create a slope field: WolframAlpha, Desmos, Geogebra, etc.
- Interpret a slope field
- Deduce properties of slope field from the ODE
- Deduce properties of solution from slope field

Motivation

ODEs are often difficult to solve, so we need tools to be able to interpret their solutions and deduce properties without having to solve them.

Slope fields are such a tool.

Slope fields are not meant to be sketched by hand, so they shouldn't be asked to do that, except at the beginning to learn how they are sketched so they can understand them better.

Slope fields should be used to help interpret solutions of ODEs.

Preparation for Class

- Read textbook
- Watch first video (second is optional)
- Solve the core exercise 13

Tutorials and Projects

There is no project that targets slope fields directly.

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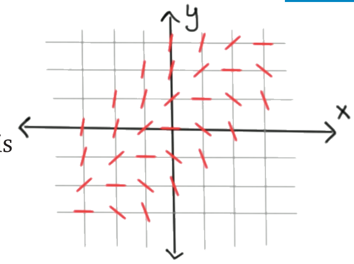
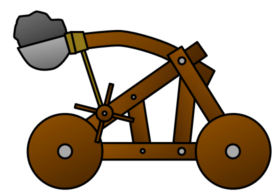
13 Consider the slope field from the first video of the module.



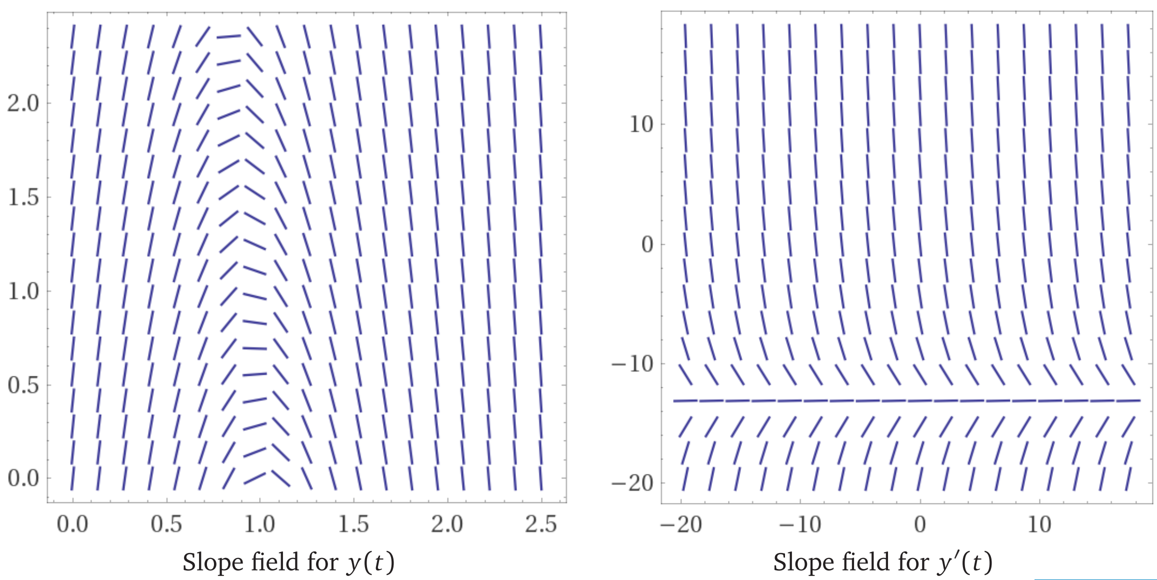
13.1 If $y(0) = 5$, then estimate $y(-7)$.

13.2 If $y(0) = a$, then $y(x) > 0$ for all $x > 0$. For which values of a is this statement true?

14 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



(These slope fields were created using WolframAlpha)

14.1 On the slope field, sketch a *possible* solution.

14.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

- Students should think about the initial conditions. What is a possible value for $y(0)$? What is a possible value for $y'(0)$?
- What does the second slope field tell us? The equilibrium in the slope field for $y'(t)$ is called *terminal velocity*.
- Sketch a possible solution again, but for $t \in [0, 30]$.

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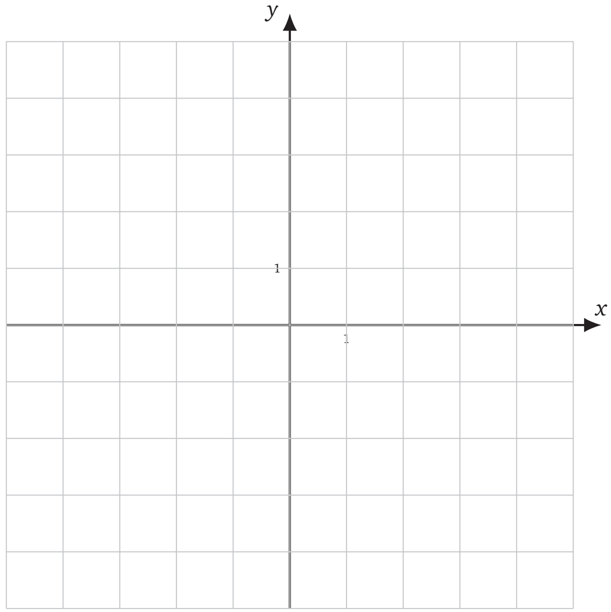
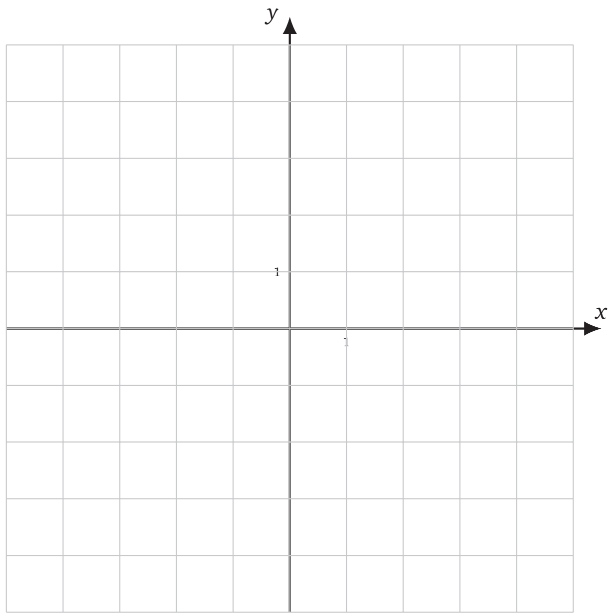
15 Sketch the slope field for the following differential equations.

15.1 $y' = x$



15.2 $y' = y^2$



Symmetry.

- The goal is not to be very accurate, but to capture the symmetry of each of these slope fields.
- Which property of the slope field allowed you to sketch it more quickly?

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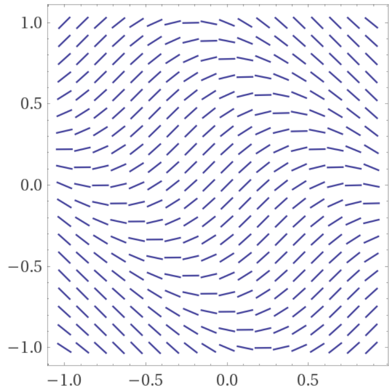
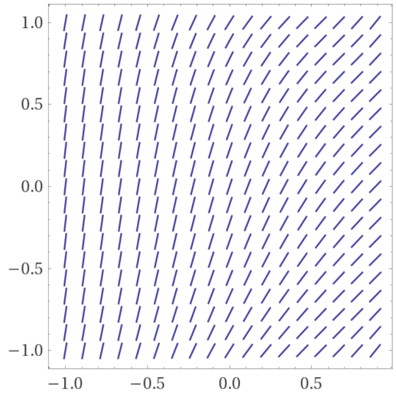
Students should be able to justify their choices .

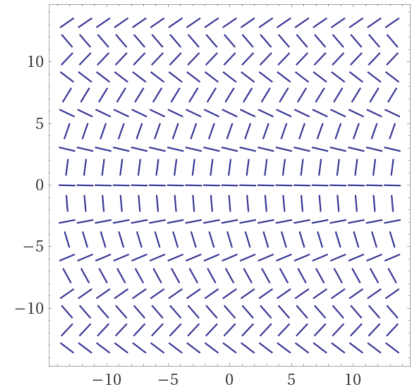
16 Consider the following slope fields:



(A)



(B)



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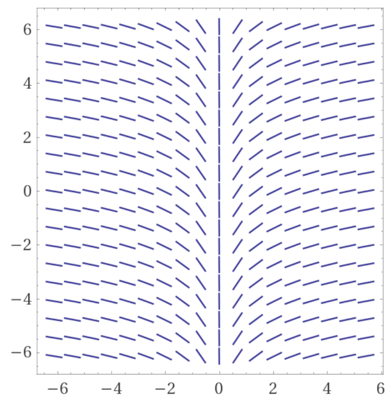
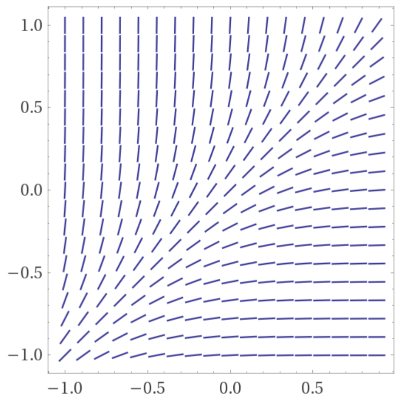
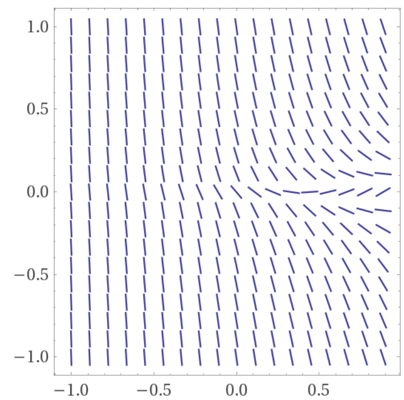


(D)



(E)



(F)

(These slope fields were created using WolframAlpha)

16.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

16.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

16.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

16.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

16.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

16.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \big(m(x,y)\big)^2$?

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Lesson 9

Approximating Solutions

Textbook

- Module 10

Objectives

- Know the idea of Euler's method
- Use Euler's method
- Be aware of the limitations of Euler's method
- Deduce some properties of Euler's method

Motivation

After sketching slope fields and observing that they can be used to get an idea of the solution, the idea to use slope fields to rigorously define an approximation of the solution should come naturally.

Euler's method is just that approximation method.

This is probably one of the most important tools for an Engineer or a Physicist studying Differential equations, since the ODEs that often arise naturally from real problems are too complicated to solve rigorously and approximating the solution might be the only way.

Preparation for Class

- Read textbook
- Watch first video (second is optional – algebraic approach of approximating the derivative)
- Solve the core exercise 17

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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17 Consider the initial-value problem

$$\begin{cases} y' = -\sin(x) + \frac{y}{20} \\ y(-10) = 2 \end{cases}$$

The solution satisfies $y(10) = \frac{20\sin(10)+400\cos(10)-2e(-401-10\sin(10)+200\cos(10))}{401} \approx 6.7738406\ldots$.

17.1 Using some software, approximate the solution at $x = 10$ for different values of Δx.

17.2 Calculate the error between the solution and the approximation at $x = 10$ for the different values of Δx.

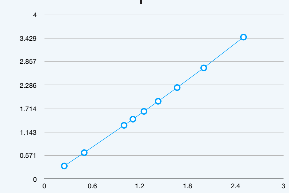
17.3 Plot the error. Is it decreasing as Δx decreases? Does it decrease linearly / quadratically / cubicly as Δx decreases?

Pre-class exercise

In class, quickly solve it.

`https://www.desmos.com/calculator/ebfn5vpudr`

Δx	Error
2.5	3.4552
2	2.7050
1.67	2.2269
1.43	1.8937
1.25	1.6475
1.11	1.4581
1	1.3078
0.5	0.6439
0.25	0.3193



- Linear with Δx: $E \approx C\Delta x$.

18 Consider the differential equation

$$y' = y - 2.$$

18.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,3)$.

18.2 Find the solution of the differential equation with the same initial condition.

18.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,1)$.

18.4 Find the solution of the differential equation with the same initial condition.

18.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

18.6 Explain in words why the Method satisfies that property.

- The goal is to have student's recognize that the Euler approximation "curves slower" than the actual solution.
- Students can explain in words why that is the case using the way the approximations are generated.
- For .2 and .4:
  - If students learned how to solve ODEs before, then fine!
  - If students didn't learn, then tell them to use WolframAlpha: `solve y'=y-2, y(0)=3`

19 Which differential equations will be approximated perfectly using Euler's Method?

Only if there is time.

- The question is purposefully ambiguous. What do we mean by approximated perfectly?
- Ex: The IVP $y' = \operatorname{sign}(t)$ (assuming $\operatorname{sign}(0) = 1$) with $y(-5) = 5$ has solution $y = |t|$ and it is captured with Euler's method if $\Delta t = \frac{5}{k}$ for any $k \in \mathbb{N}$.
- Once students discuss, they'll find ODE's of the form $y' = c$ for any constant.
- Prompt them to find other types. Show them the example above only after they tried for a bit. Then, let them revise their Conjecture.
- Ex 2: $y' = f(y+t)$ with $y(0) = 0$ and $f(z) = \lfloor z \rfloor$ is approximated perfectly if $\Delta t = 1$, but not if Δt takes any other value.

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Lesson 10

Modelling with Differential Equations I

Textbook

- Module 11

Objectives

- Start modelling physical quantities
- Follow the procedure from chapter 1 when creating a model

Motivation

This is one of the main goals of this course.

Students should be given the opportunity to create models in class. They should be encouraged to follow the procedure from chapter 1, as it will improve their models.

Preparation for Class

- Read textbook
- Read the core exercise 20 and solve steps 1 (define problem) and 2 (create a mind map)

Tutorials and Projects

- Project 3: X-ray attenuation
- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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20 A pendulum is swinging side to side. We want to model its movement.

20.1 Define the problem. Which function(s) do we want to find in the end?

20.2 Build a mind map.

20.3 Make assumptions. Remember to use your mind map to help structure the problem.

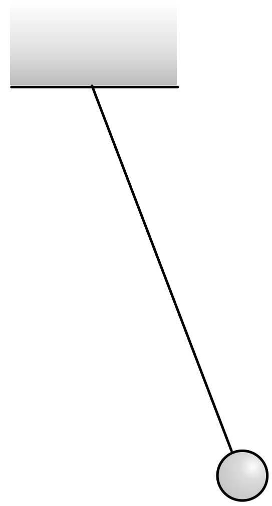
20.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2nd Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in − Rate out).

20.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.



Students will mostly likely identify the goal as finding the position of the ball $\vec{r}(t) = \big(x(t), y(t)\big)$. That's fine!
Later, in Step 3, try to guide the students to recognize the following:

- Rope is massless (negligible)
- Rope doesn't bend (negligible)
- So can assume that the rope is rigid. How does that affect the position of the ball?
- No friction (negligible)
- **Important:** Students always focus on string tension. One can consider it, but it all cancels out. It's one of the exercises of the module (above). For the lecture, don't consider tension.

Then on Step 4, guide students to recognize that they actually only need to find a model for the angle, because the position of the ball really only depends on the angle $\theta(t) : \vec{r}\big(\theta(t)\big)$

Students should finish it at home.

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Lesson 11

Modelling with Differential Equations II

Textbook

- Module 11

Objectives

- Model an open ended problem
- Follow the procedure from chapter 1 when creating a model

Motivation

Students may be used to some models from physics classes or from Calculus classes.

They should practice developing more open ended models.

Preparation for Class

- Read textbook
- Read the core exercise 21 and solve steps 1 (define problem) and 2 (create a mind map)

Tutorials and Projects

- Project 3: X-ray attenuation
- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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21 Model the spreading of a rumour through the students of a school.

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Lesson 12

Solvable Types of Differential Equations

Textbook

- Module 12

Objectives

- Identify a Separable ODE
- Know how to solve a Separable ODE
- Identify a first-order linear ODE
- Know how to solve a first-order linear ODE: Method of the Integrating Factor

Motivation

This is the traditional class on differential equations.

Even though the focus of the course is not on recipes for solving differential equations, when creating a model that involves a differential equation and analyzing it, it is necessary to be aware of some of the basic methods for solving them.

Students should practice these methods more than the limited time devoted to them in lecture. Even though there aren't many practice problems, it is easy to find exercises for this module in any differential equations textbook.

Preparation for Class

- Read textbook
- Watch first video for each method

Tutorials and Projects

- Project 2: Managing a fishery
- Project 3: X-ray attenuation
- Project 4: Predator-prey chase

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22 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

22.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

22.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

22.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

22.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

23

23.1 Calculate $\big(\sin(x)f(x)\big)'$.

23.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

23.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

- First, ask students to identify all the ODEs
- Then, students can start solving them one by one

Notes/Misconceptions

- Make sure there is time for next core exercise. Skip some ODEs if necessary.

Idea of Integrating Factor

By recognizing that the left-hand side is the result of a product rule, it is much easier to find the solution.

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Lesson 13

Properties of Differential Equations

Textbook

- Module 13

Objectives

- Identify which Theorem to use for a given ODE
- Know the Existence and Uniqueness Theorems
- Deduce properties of the solutions from the Theorems

Motivation

This is a little glimpse into the theory of Differential Equations.

Knowing whether a differential equation has solutions, and whether there is one unique solution is important on its own right.

But even from a more applied point-of-view, these Theorems offer consequences to the behaviour of solutions of differential equations that make them very important.

One of these properties is the fact that in the region in the (t, y)-plane where the Theorems' conditions are satisfied, solutions cannot intersect.

Preparation for Class

- Read textbook
- Watch first video: about the Theorem for Nonlinear ODEs
- Watch the second video: short example video
- Solve the core exercise 24

Tutorials and Projects

- Project 2: Managing a fishery
- Project 3: X-ray attenuation
- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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Pre-Class exercise

In class, quickly solve it.

24 Consider the example in the video:

$$\begin{cases} x\dfrac{dy}{dx} = y \\ y(0) = b \end{cases}$$

Without solving, but only according to the Existence and Uniqueness Theorem, what can we conclude?

(a) We can conclude that there is a unique solution.

(b) We can conclude that if $b = 0$ there are many solutions, but if $b \neq 0$, then there are no solutions.

(c) We can conclude that there are many solutions.

(d) We can conclude that there are no solutions.

(e) We can't conclude anything.

25 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

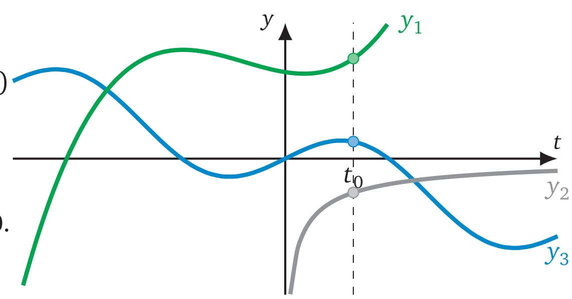
25.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

25.2 $y' = t + \sqrt{y-\pi}$ with $y(1) = 1$

25.3 $y' = \sqrt{4-(t^2+y^2)}$ with $y(1) = 1$

26 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all $t, y \in \mathbb{R}$;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all $t \in \mathbb{R}, y > 0$.



26.1 Can **green** $y1$ and **blue** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.2 Can **green** $y1$ and **gray** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.3 Can **gray** $y2$ and **blue** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

Intersecting solutions?

The goal is for students to understand why solutions cannot touch *IF* the Theorem is valid at the intersection point.

- $y1$, y_3 not ok!
- $y1$, y_2 ok!
- $y2$, y_3 ok!

27 The initial-value problem

$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

$$y_1(x) = \cos\big(\arcsin(x)\big) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

27.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

27.2 What can you conclude?

Hint. "When you have eliminated the impossible, whatever remains, however improbable, must be the truth." – Sherlock Holmes

Existence and Uniqueness Thms

The goal is for the students to realize that if the Theorem's conditions hold, then its result must also hold, even though it might not look like it.
So if the Theorem says that the solution is unique and we see two solutions, then they must be the same function (at least for x near $\frac{1}{2}$).

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Lesson 14

Autonomous Differential Equations

Textbook

- Module 14

Objectives

- Identify an Autonomous ODE
- Find and plot equilibrium solutions
- Sketch solutions of an autonomous ODE without solving it
- Classify equilibrium solutions

Motivation

Even though nonlinear ODEs are in general very hard to study, Autonomous ODEs, which are often nonlinear, are an exception.

These are ODEs of the form $y' = f(y)$, which can usually be thoroughly studied without having to find their solution.

One of the key ways that we use to study Autonomous equations is to find equilibrium points and classify them as stable / unstable. This will be extended to systems of ODEs.

Preparation for Class

- Read textbook
- Watch the video
- Solve the core exercise 28

Tutorials and Projects

- Project 2: Managing a fishery
- Project 3: X-ray attenuation
- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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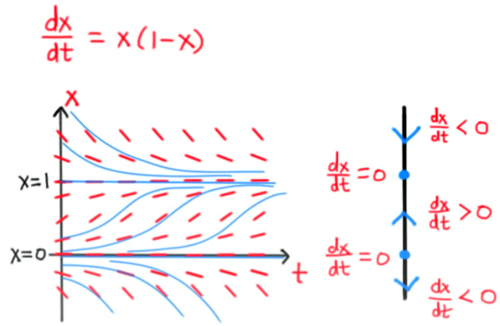
28 Consider the ODE from the video in the module.



Select all the initial conditions that yield a decreasing solution.

(a) $x(-2) = \sqrt{2}$

(b) $x(20\,000) = 0.000000001$

(c) $x(5) = \pi$

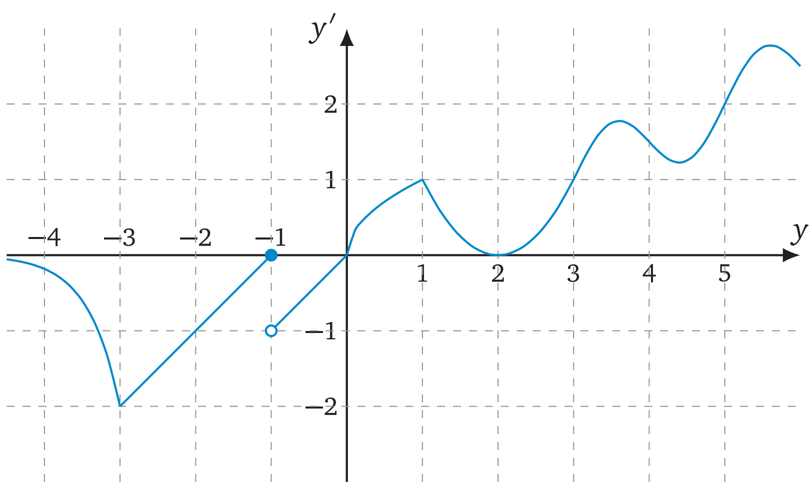
(d) $x(0) = \frac{1}{2}$

(e) $x(3) = -\frac{1}{2}$

(f) $x(1000) = \frac{1}{e}$

Pre-Class exercise
In class, quickly solve it.

29 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



- *1–3*: Students come up with a definition of stable, unstable, semi-stable equilibrium points.
- *4*: Sketch solutions (do (a),(b) in class)
- *5–6*: Find properties of solutions
- *7*: Leave as HW

29.1 What are the equilibrium points?

29.2 Which equilibrium solutions are stable, unstable, or semi-stable?

29.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

29.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

(b) $y(0) = -\frac{1}{4}$.

(c) $y(1) = \frac{1}{4}$.

29.5 If $y(0) = 2$, then $y(t) =$

29.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

29.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$

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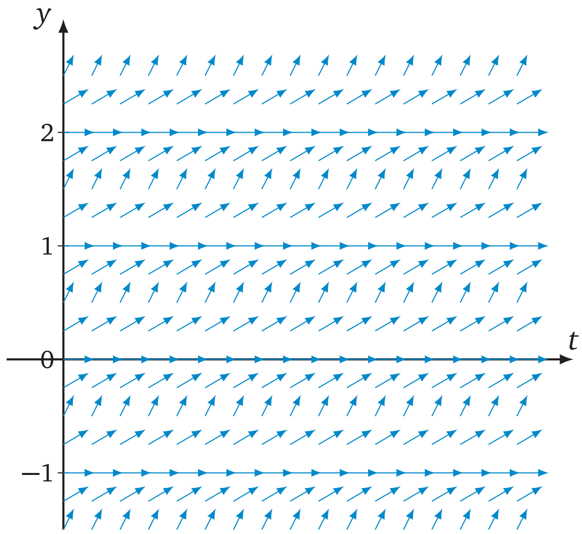
30 Consider a differential equation $y' = f(t, y)$ with the following slope field.



30.1 What are the equilibrium solutions of the ODE?

30.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \dfrac{1}{4} \end{cases}$$

30.3 From the direction field above, circle the correct type(s) of this ODE? Justify your answer.

(a) separable. (c) autonomous.

(b) of first-order and linear. (d) none of the other options.

30.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then

(select all the possible options)

(a) $g(3) < h(3)$ (b) $g(3) = h(3)$ (c) $g(3) > h(3)$

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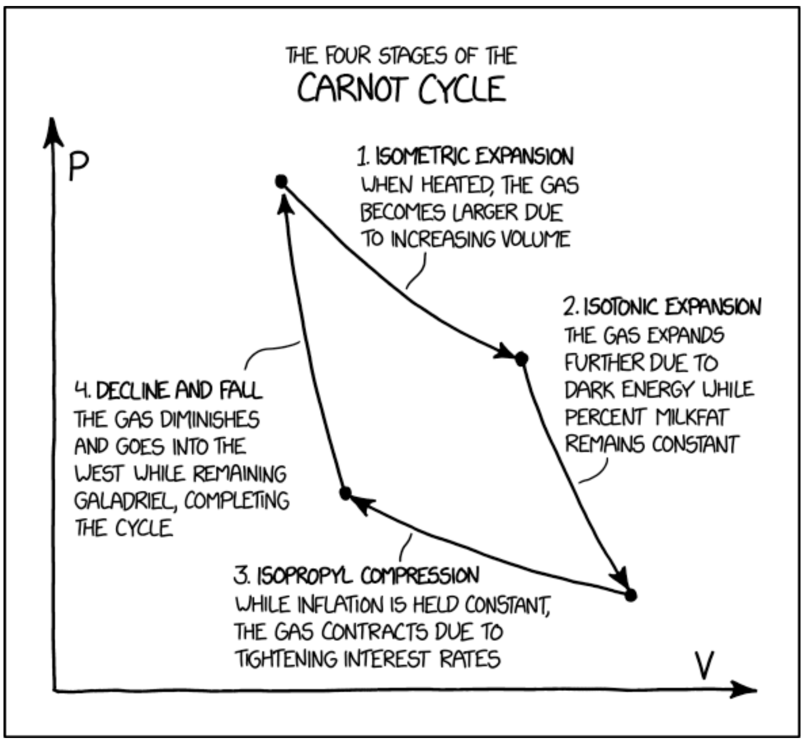

(image from xkcd - comic #2063)

Lesson 15

Modelling Two Quantities I

Textbook

- Module 15

Objectives

- Model with two interconnected quantities
- Follow the step-by-step procedure to create a model

Motivation

Again, this is the main goal of this course: learn how to create models.

In this class, the goal is to study how to model quantities that affect each other (and are affected by external factors).

After the students create a model, they should get used to making sure the model is "good", i.e., using some software look at some approximation of solutions to see the consequences of the model.

Preparation for Class

- Read textbook
- Read the core exercise 31 and solve steps 1 (define problem) and 2 (create a mind map)

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration

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31 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

31.1 Create a model for these two populations.

31.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

31.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

31.4 Extend the model to include a population of antelopes.

Pre-class exercise.
Students bring their mind map and the definition of the functions they need to find.

Stress that students should follow the step-by-step approach from chapter 1. *.4* only if there is time. Tell the students to "go nuts" and include everything that relates.

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Lesson 16

Modelling Two Quantities II

Textbook

- Module 15

Objectives

- Model with two interconnected quantities
- Follow the step-by-step procedure to create a model

Motivation

This class students work on a model that involves a little more calculus (for the cheetah's movement) and also is more **open ended** on how the cheetah will follow the antelope, but mainly on how the antelope will try to escape the cheetah.

The instructor should encourage different solutions from students.

Preparation for Class

- Read textbook
- Read the core exercise 32 and solve steps 1 (define problem) and 2 (create a mind map)

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration
- Project 7: Arms race

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32 A cheetah is chasing an antelope. We want a model of their positions as they run.

Pre-class exercise.

Students bring their mind map and the definition of the functions they need to find.

This exercise is not required to do in lecture.

Be careful with assumptions! A very general model will be very hard to study.

Allow some brainstorming and try to create a structure for this problem:

- Positions seen from above ($xy-$plane).
- Only need $x_a(t), y_a(t)$ and $x_c(t), y_c(t)$
- Focus on the cheetah: where is she heading to?
- For the antelope, students need to come up with an escape strategy
- Model will be nonlinear!

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Lesson 17

Systems of two linear ODEs with constant coefficients I

Textbook

- Module 16

Objectives

- Know how to find eigenvalues and eigenvectors of a matrix
- Know how to find a solution from an eigenvalue and an eigenvector
- Know how to write the general solution to a system of ODEs
- Know how to find the constants given an initial condition

Motivation

Linear Algebra strikes again!!! It will be very important to know how to solve a linear system of equations (although we'll focus on 2×2 systems only) and how to find eigenvalues and eigenvectors.

This is the more computational part of the course.

Even though it's not the focus of the course, students need to know how solutions are found. This will give some insight for the analysis of a system of ODEs later.

We start with problems involving two real distinct eigenvalues.

Preparation for Class

- Review Linear Algebra (Appendix 7.1)
- Read textbook: Two real distinct eigenvalues
- Watch video
- Solve the core exercise 33.1.

Tutorials and Projects

- Project 7: Arms race

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33 Consider a cheetah-lion inspired problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{r}.$$

33.1 Find the two solutions $\vec{r}_1, \vec{r}_2$.

33.2 Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

33.3 Is $\vec{r}_1(t) - \vec{r}_2(t)$ a solution?

33.4 Is $2\vec{r}_1(t) + 3\vec{r}_2(t)$ a solution?

33.5 What is the general solution?

33.6 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

Notes/Misconceptions

Superposition Principle.

After .5, can introduce the principle of superposition:

- If $\vec{r}_1$ and $\vec{r}_2$ are solutions of a linear homogeneous (system of) ODE(s), then $\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2$ is also a solution for any constants c_1, c_2.

Notes/Misconceptions

If there is more time.

Start working on the core exercise for the next lesson.

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Lesson 18

Systems of two linear ODEs with constant coefficients II

Textbook

- Module 16

Objectives

- Know how to find complex eigenvalues and eigenvectors of a matrix
- Know how to find a complex solution from an eigenvalue and an eigenvector
- Know how to re-write the solution using Euler's Formula and only real numbers

Motivation

We now add some complex numbers into the mix! Although students only need a very (Very) superficial knowledge of complex numbers.

We study problems involving two complex eigenvalues or one repeated real eigenvalue.

The third case: One repeated real eigenvalue will not be worked on in lecture. Students should learn it by themselves and practice it.

Preparation for Class

- Read textbook: Two complex eigenvalues
- Watch corresponding video
- Solve the core exercise 34.

Tutorials and Projects

- Project 7: Arms race

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34 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r}.$$

34.1 Find the general solution.

34.2 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

35 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r} - \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

35.1 Find the equilibrium solution.

35.2 Find the general solution.

35.3 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$?

Skip .2 and start working on next core exercise

Non-Homogeneous Problem

Students don't know how to solve it yet:

- Equilibrium solution ($\begin{bmatrix} 2 \\ -1 \end{bmatrix}$)
- Show phase portrait using WolframAlpha `https://uoft.me/modelling-sys-nonhom`
- Ask students about properties of the phase portrait (*Goal*: solutions revolve around the equilibrium point)
- Redefine centre: $\vec{r} = \vec{eq} + \vec{p}$. What system foes $\vec{p}$ solve?
- ...

There are practice problems about this.

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Lesson 19

Phase Portraits I

Textbook

- Module 17

Objectives

- Interpret a phase portrait
- Sketch a phase portrait

Motivation

After learning how to create a model involving a system of ODEs and how to find the solution, we turn our attention to how to represent the solutions.

The phase portrait is a compact way to represent all possible solutions of a system of two ODEs.

Through sketching phase portraits, we are also learning the different possible behaviours of solutions of systems of two ODEs.

We start with systems of two ODEs with two real distinct eigenvalues. It usually takes a while to sketch the first phase portrait, so the first lesson is dedicated to sketching only one.

Preparation for Class

- Read textbook
- Watch first video
- Solve the core exercise 36.

Tutorials and Projects

- Project 6: Hunting inspiration
- Project 7: Arms race

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36 Consider the following model for cheetah's and lions, where

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

The general solution is:

$$\vec{p}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

36.1 Without computing them, what are the eigenvalues and eigenvectors of the matrix?

36.2 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = 0$.

36.3 Sketch the graph of the solution with $c_1 = 0$ and $c_2 = \pm 1$.

36.4 When one constant is set to 0, what is the shape of the graph? Is it always like that? Can you prove it?

36.5 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = \pm 1$.

36.6 Provide an interpretation of the different types of solutions.

Pre-class exercise

Unstable Saddle Point

At the end, let the students know that the equilibrium is called *saddle point* and it is *unstable*, because solutions go away from it.
For the interpretation question, when one population hits zero, it is extinct, so the graph doesn't make sense.
We can interpret that if a population becomes extinct, then the other will behave as it would without competitors: grow exponentially fast!

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Lesson 20

Phase Portraits II

Textbook

- Module 17

Objectives

- Interpret a phase portrait
- Sketch a phase portrait

Motivation

There are a few more different behaviours, so we study some more cases.

We start with a non-homogeneous case and then study some other cases of phase portraits with two real distinct eigenvalues.

Preparation for Class

- Solve the core exercise 37.

Tutorials and Projects

- Project 6: Hunting inspiration
- Project 7: Arms race

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37 Let us expand the model from the previous exercise to:

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \vec{p} + \begin{bmatrix} -10 \\ 50 \end{bmatrix}.$$

The extra term corresponds to the effect of harvesting 10 lions and bringing in 50 cheetahs every year to the reserve.

The general solution is:

$$\vec{p}(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

37.1 Sketch the phase portrait.

37.2 Provide an interpretation of the different types of solutions.

Pre-class exercise

- Solve quickly in class.
- Get students to compare their results with the previous core exercise.
- How does the interpretation of the population dynamics change?

38 For each of the following general solutions, sketch the phase portrait.

38.1 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}.$

38.2 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}.$

At the end, let the students know what these equilibria are called:

- *source* and it is *unstable*, because solutions go away from it.
- *sink* and it is *asymptotically stable*, because solutions converge to it.

If there is time, students can think about:

- Given a matrix A, which part of A indicates whether the equilibrium is stable / unstable? Which part indicates whether it's a sink/source vs spiral sink/source?

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Lesson 21

Analysis of Models with Systems

Textbook

- Module 18

Objectives

- Deduce properties of solutions of systems of ODEs using different approaches

Motivation

Analysis of a system of ODEs is important.

It's not always possible to find the solution of a model, but it's usually possible to sketch a rough phase portrait, to approximate the solution, or to just deduce some properties of the solutions.

Preparation for Class

- Read textbook
- Solve the core exercise 39.1

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 5: Epidemic modelling
- Project 6: Hunting inspiration
- Project 7: Arms race

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39 Consider the following model for the sales from a designer clothing brand:

- $x_1(t) =$ purchases by "common mortals" (CM) at time t in years since the beginning of 2015.
- $x_2(t) =$ purchases by "famous people" (FP) at time t.

Our model is based on the following two principles:

(P_1) CM will buy more items from the brand when CM or FP buy more.

(P_2) FP will buy less when CM buy them, but will buy more when FP buy it.

The model we considered is:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \vec{x}(t)$$

39.1 Suppose that at the beginning only CM buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

39.2 Suppose that at the beginning only FP buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

39.3 What conditions on the constants a, b, c, d will guarantee that the solutions will spiral? In that case, is it a spiral source or spiral sink? Is it clockwise or counterclockwise?

39.4 Are there constants $a, b, c, d > 0$, such that the solution $\vec{x}$ is periodic?

39.5 Consider the constants $a = b = c = d = 1$. Assume that initially CM were buying $c_0 > 0$ items and FP were buying $f_0 > 0$ items. What will happen to $x_1(t)$ and $x_2(t)$ as $t \to \infty$? Explain the results in terms of the evolution of purchases from CM and FP.

39.6 Consider the constants $a = b = c = d = 1$. If $c_0 = 10$, $f_0 = 100$, then at what time will FP stop buying items? And at what time will FP be buying the maximum number of items?

If there isn't enough time to finish:

- Take an extra lecture if there is enough to do
- Can leave .6 as a practice problem

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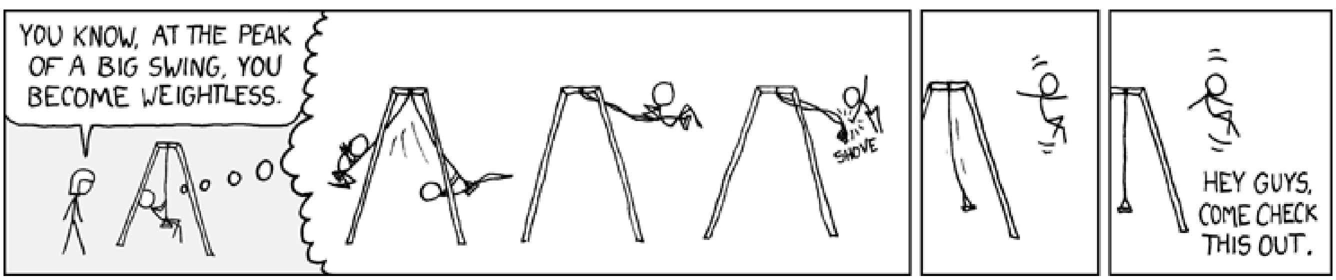

(image from xkcd - comic #226)

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Lesson 22

Modelling with Second-Order ODEs I

Textbook

- Module 19

Objectives

- Model physical phenomena to obtain a second-order ODE
- Understand how to use Newton's Second Law of Motion
- Follow the step-by-step procedure to create a model

Motivation

By this point, the students should have a good idea on how the modelling task goes.

Models about moving objects often require Newton's Second Law of Motion which end up having the form of a second-order ODE.

Engineering and Physics students will find models like this often in their studies.

Preparation for Class

- Read textbook
- Read the core exercise 40 and solve steps 1 (define problem), 2 (create a mind map), and 3 (make assumptions)

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 8: Spring data

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40 Here are some facts about laptop keys:

(da) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.

(di) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.

(fo) On average, a person exerts the force of 42 N with one finger on a key.

(gr) Gravity is much weaker than the spring that keeps the key in place.

(hl) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").

(lo) Keys last 50 million presses on average.

(ve) Keys can only move vertically.

40.1 Model a laptop keypress.

40.2 What happens if the damping system of the key is broken? What happens if the damping system is too strong? How strong should the damping system be?

40.3 What happens to the key when the spring breaks?

.1 should be very quick, since a very (very) similar example was solved in the module.

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Lesson 23

Modelling with Second-Order ODEs II

Textbook

- Module 19

Objectives

- Model physical phenomena to obtain a second-order ODE
- Understand how to use Newton's Second Law of Motion
- Follow the step-by-step procedure to create a model

Motivation

This class, students work on a model that involves a bit more of Linear Algebra (projections) and some "uglier" expressions.

Preparation for Class

- Read textbook
- Read the core exercise 41 and solve steps 1 (define problem), 2 (create a mind map), and 3 (make assumptions)

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 8: Spring data

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41 Model a ball rolling down a ramp.

Ball rolling

Different approach depending on students.

Students need some challenge and have time:

- Ramp $y = f(x)$ makes it simpler
- Need projection (Linear Algebra) to find gravity force along the ramp at $(x_0, y_0) = \big(x_0, f(x_0)\big)$:

  $$\ell = (0, -mg)\cdot(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{\sqrt{1+k^2}}$$

  where $k = f'(x_0)$. So gravity force along the ramp is:

  $$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

- Yields second-order ODE

Weaker students with less time:

- Give the formula for gravity force along the ramp:

  $$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

Follow-up question:

- Ramp is $y = (x-1)^2$
- Get second-order ODE for ball position
- Will the ball always move to the right? Justify with the ODE.
- Approximate near the bottom of the ramp: $y' \approx 0 \Leftrightarrow \sqrt{1+(y')^2} \approx 1$ and solve the simpler ODE.
- When is this approximation valid? (when ball oscillates back and forth near the bottom)

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Lesson 24

Second-Order Linear ODEs with Constant Coefficients I

Textbook

- Module 20

Objectives

- Understand the main idea: assuming the solution is $y = e^{rt}$ and find r
- Know how to find the characteristic equation and its solutions
- Find the general solution of a homogeneous ODE

Motivation

In this section we learn how to solve second-order linear ODEs with constant coefficients.

We do that by assuming that the solution is an exponential of the form $y = e^{rt}$. The intuition behind this comes from the systems of ODEs that we just studied.

Just like with systems, there are three possible cases, but its much simpler to solve them (especially the complex case).

In this lesson, solve the core exercises 42–44 for the complementary solutions.

Preparation for Class

- Read textbook: Homogeneous ODEs
- Watch corresponding video
- Solve the core exercise 42.1

Tutorials and Projects

- Project 8: Spring data

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42 Consider the ODE $y''(t) - 9y(t) = f(t)$.

42.1 Find a complementary solution.

42.2 Find a particular solution for $f(t) = 14e^{-4t}$.

42.3 Find a particular solution for $f(t) = 9e^{-3t}$.

42.4 Find a particular solution for $f(t) = 10\cos(t)$.

---

43 Consider the ODE $y''(t) - 2y'(t) + 5y(t) = f(t)$.

43.1 Find a complementary solution.

43.2 Find a particular solution for $f(t) = \sin(2t)e^{t}$.

43.3 Find a particular solution for $f(t) = (4t+2)\sin(2t)e^{t}$.

---

44 Consider the ODE $y'' + 3y' = 3t$.

44.1 Find the complementary solution.

44.2 Find a particular solution.

44.3 Find the solution that also satisfies

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

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Lesson 25

Second-Order Linear ODEs with Constant Coefficients II

Textbook

- Module 20

Objectives

- Write the solution of a non-homogeneous ODE as $y = y_c + y_p$
- Use the Method of Undetermined Coefficients to find a particular solution

Motivation

This will include some more linear algebra!

The Method of Undetermined Coefficients is very computational once the students have some practice with it. A good way to learn it is by giving only partial information and let students struggle a little before showing how to proceed:

- that's the idea behind core exercise 42, where the first two exercises are straight forwards, but then without more information the students will struggle to find the particular solution.
- After a little struggle in lecture, the instructor can then show the way.

In this class, solve the core exercises 42–44 for the particular solutions.

Preparation for Class

- Read textbook: Non-Homogeneous ODEs
- Watch corresponding video
- Solve the core exercise 43.2-3.

Tutorials and Projects

- Project 8: Spring data
- Project 9: Wing flutter

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Lesson 26

Analysis of Models with Higher Order ODEs I

Textbook

- Module 21

Objectives

- Deduce properties of solutions of second-order of ODEs using different approaches

Motivation

As we have seen in the previous chapters, it is important to know how to analyze ODEs.

In this first lesson about the Analysis of ODEs, we start with some more qualitative properties deduced without finding the solution.

Preparation for Class

- Read textbook
- Solve the core exercise 45.1

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 8: Spring data
- Project 9: Wing flutter

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45 Consider the second-order ODE:

$$y''(t) - 3y(t) = t\big(2 + \sin(t)\big).$$

45.1 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) > 0$ for $t \geq 0$.

45.2 Assume that $y(0) = a < 0$ and $y'(0) = b$. Give an example of a, b such that $y(t)$ is increasing for $t \geq 0$.

45.3 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) < 0$ for all $t > 0$.

Without solution

The goal is to solve this without finding an expression for the solution.
For .2, the idea is to make sure that $y'' < 0$.

If there is time, start the next core exercise.

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Lesson 27

Analysis of Models with Higher Order ODEs II

Textbook

- Module 21

Objectives

- Deduce interesting behaviours:
  - Resonance
  - Beats

Motivation

As we have seen in the previous chapters, it is important to know how to analyze ODEs.

In this case, even after we find the solutions, the way they behave is hidden in the formula. We still need some work to find out the interesting behaviours that arise.

Preparation for Class

- Read textbook
- Solve the core exercise 46.1–2

Tutorials and Projects

- Project 4: Predator-prey chase
- Project 8: Spring data
- Project 9: Wing flutter

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46 Consider the second-order ODE:

$$\begin{cases} y''(t)+4y(t)=f(t) \\ y(0)=y_0 \\ y'(0)=0 \end{cases}$$

46.1 Let $f(t)=0$ and $y_0=1$. Sketch the solution.

46.2 Let $f(t)=396\cos(20t)$ and $y_0=0$. Sketch the solution.

46.3 Let $f(t)=-4\sin(2t)$ and $y_0=1$. Sketch the solution.

46.4 Let $f(t)=0.39\cos(1.9t)$ and $y_0=2$. Sketch the solution.

Hint. $\cos(at)+\cos(bt)=2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}t\right)$

Goals:

- Learn some different types of behaviour of for second-order ODEs
- Learn how to sketch trig functions combined with linear or other trig functions

.1. Complementary solution!

.2. *Adding* two trig functions: one oscillating slowly and one oscillating quickly

.3. Resonance: t times trig function

.4. Beats: *product* of two trig functions – one oscillating slowly and one oscillating quickly

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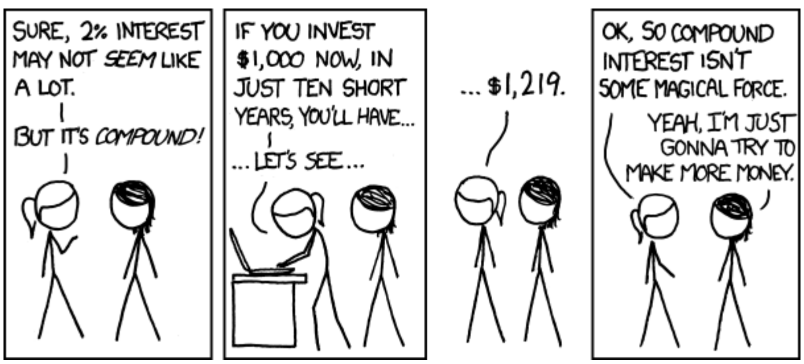

(image from xkcd - comic #947)

Lesson 28

Solving Difference Equations

Textbook

- Module 23

Objectives

- Understand the two main ideas: expansion or educated guessing
- Use mathematical induction to show that the formula found is indeed the solution

Motivation

Difference equations are a bit different from Differential equations.

One of the possible ways to solve difference equations is very simple in its idea, but it is actually hard to do in practice.

The second method to solve this type of equations is similar to the method we used before for systems and second-order ODEs.

However, showing that a solution is indeed a solution is quite a bit more difficult. It involves using the Principle of Mathematical Induction (appendix 7.2).

Preparation for Class

- Read textbook
- Watch videos
- Solve the core exercise 47

Tutorials and Projects

- Project 12: Math of lungs

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47 Consider the difference equation

$$u_{k+1} = 6u_k - 9u_{k-1}$$

47.1 Find the solution that satisfies $u_0 = 1, u_1 = 3$.

47.2 Find the solution that satisfies $u_0 = 1, u_1 = 4$.

48 Consider a difference equation that has solutions $u_k = r^k$ for $r = 2$ and $r = 3$ and satisfies the conditions $u_0 = 7$ and $u_1 = 6$.

What is u_{22}?

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Lesson 29

Modelling with Difference Equations I

Textbook

- Module 24

Objectives

- Model economic quantities to obtain a difference equation

Motivation

Economic quantities often change in bursts, not continuously. That's what happens to a savings account of the example in the textbook. So difference equations are a good tool to model these quantities.

In lecture, the goal is to study the concept of effective annual interest.

Preparation for Class

- Read textbook: Economic Models
- Watch the corresponding video
- Bring definition of `effective annual interest`

Tutorials and Projects

- Project 10 Bullwhip effect
- Project 11: Approximating the temperature of a thin sheet
- Project 12:Math of lungs
- Project 13: Dark day
- Project 14: Maximus vs Commodus

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49 Let us expand on the economic example above.

We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

The effective annual interest rate is the interest rate with a compounding period of 1 year that gives the same result s the rate of $p\%$ compounded every α years.

Even though we call $p\%$ the annual interest rate, because it is compounded during the year, at the end of the year the effective annual interest rate $p_{\text{eff}}\%$ is actually higher.

Calculate the effective interest rate $p_{\text{eff}}\%$.

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Lesson 30

Modelling with Difference Equations II

Textbook

- Module 24

Objectives

- Model a population to obtain a difference equation

Motivation

We have modelled populations using differential equations. Populations can be modelled using both differential or difference equations. Which kind of equations to use depends on the goal of the model and the assumptions that we make.

In lecture, the goal is to study the concept of average lifespan.

Preparation for Class

- Read textbook: Population Models
- Watch the first corresponding video
- Solve core exercise 50

Tutorials and Projects

- Project 10 Bullwhip effect
- Project 11: Approximating the temperature of a thin sheet
- Project 12:Math of lungs
- Project 13: Dark day
- Project 14: Maximus vs Commodus

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50 The goal of this question is to try to understand the meaning of average lifespan.

50.1 Consider a small tribe, where the people in there died at the ages:

42, 56, 46, 52, 5, 103, 47, 67, 67, 85, 57, 42, 47, 67, 46, 42, 5, 46, 57, 42.

What is the average lifespan of this tribe's population?

50.2 Consider another small tribe, where people recorded their lifespans differently. Below is a table with the percentage of the population that died at each age:

Percentage of population	2%	5%	9%	9%	16%	22%	37%
Age at death	98	82	71	66	61	53	48

What is the average lifespan of this tribe's population?

Pre-class question

The goal of this question is to prepare for calculating the average lifespan in the next page.

In class do it quickly.

Some hints:

- individual dying during season $k \iff$ lifespan $= k$ seasons
- From previous two exercises, deduce that: average lifespan = expected value of lifespan $\ell = E$:

$$E = \sum_{k=0}^{\infty} k\ell(k)$$

- $\displaystyle\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$ for $|r| < 1$.
- End result should be $\frac{1}{\mu}$.

51 Given a population with

- $\mu =$ probability that an individual will die between two seasons.

51.1 Define the following quantity

- $P(k) =$probability that an individual born at season 0 is alive at the beginning of season k.

Find a model for $P(k)$.

51.2 What is the probability of the individual dying during the k$^{\text{th}}$ season?

51.3 What is the average lifespan of an individual in this population?

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Lesson 31

Modelling with Difference Equations III

Textbook

- Module 24

Objectives

- Model a population to obtain a system of difference equations

Motivation

We now turn our attention to a different type of population model. It's still a population model, but the reproduction of the population changes depending on the age. This means that a system of difference equations is a good way to model this.

Preparation for Class

- Watch the second video about Population Models
- Solve core exercise 52

Tutorials and Projects

- Project 10 Bullwhip effect
- Project 11: Approximating the temperature of a thin sheet
- Project 12:Math of lungs
- Project 13: Dark day
- Project 14: Maximus vs Commodus

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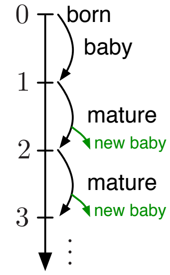
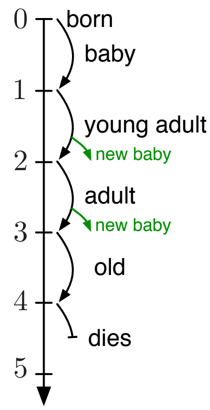


Pre-class exercise.

- In class solve it quickly.
- Let students know that they should use Mathematical Induction to prove that the Fibonacci sequence does indeed match the number of rabbits.

52 Consider a population of special rabbits. Once a pair of rabbits is born, they grow and one year later they are still immature. But two years after they are born they give birth to another pair of rabbits. Model this population of rabbits.

53 Consider another population of rabbits. This is the lifecycle of a pair of rabbits:



(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pairs of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Model this population of rabbits.

- Students might try to find a pattern.
- It is possible, but very difficult.
- Hint: Use a system of difference equations.
- In core exercise 56, the students are asked to prove the formula.

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Lesson 32

Analysis of Difference Equations I

Textbook

- Module 25

Objectives

- Deduce properties of solutions of difference equations using different approaches

Motivation

Analyzing difference equations is similar to the different ways that we analyzed differential equations.

Approximating solutions though is much easier as a difference equation is already in the form of a numerical method.

Preparation for Class

- Read textbook
- Solve the core exercise 54.1–2.

Tutorials and Projects

- Project 10 Bullwhip effect
- Project 11: Approximating the temperature of a thin sheet
- Project 12:Math of lungs
- Project 13: Dark day
- Project 14: Maximus vs Commodus

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54 Consider the following difference equation:

$$u_{k+1} = a(u_k - b)$$

54.1 What is the equilibrium solution?

54.2 Are there 2-periodic solutions? I.e. satisfying

- $v_0 = v_2 = v_4 = v_6 = \cdots$
- $v_1 = v_3 = v_5 = v_7 = \cdots$
- $v_0 \neq v_1$

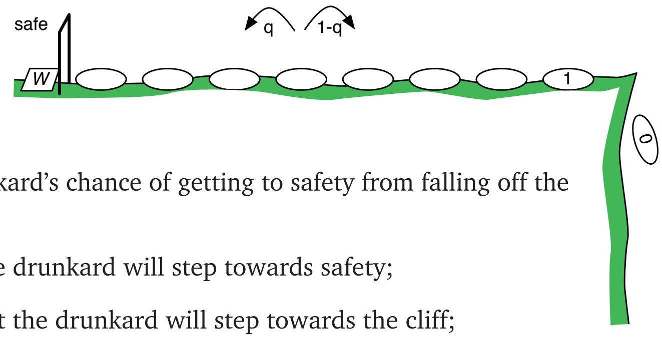
54.3 What happens to the solutions for different values of a?

54.4 What happens to the solutions for different values of b?

- In the calculations for .2, there is a step that involves a division by $(1-a^2)$, so it can only be done for $a \neq \pm 1$.
- The final result for .2 is:

$$\begin{aligned} a \neq \pm 1. \quad & \text{periodic} \Rightarrow u_0 = \frac{ab}{a-1} \\ & \Rightarrow 1-\text{periodic} \\ a = 1. \quad & \text{periodic} \Rightarrow b = 0 \\ & \Rightarrow 1-\text{periodic} \\ a = -1. \quad & \text{periodic} \Rightarrow b = 0 \\ & \Rightarrow 2-\text{periodic if } u_0 \neq 0 \end{aligned}$$

55 Consider a drunkard that is walking randomly near a cliff.



Consider this model for the drunkard's chance of getting to safety from falling off the cliff:

- q is the probability that the drunkard will step towards safety;
- $1-q$ is the probability that the drunkard will step towards the cliff;
- p_n = probability that the drunkard will get to safety if he is in step number n;
- The drunkard will stop moving if he gets to safety (step W) or if he falls out of the cliff (step 0);

- $p_n = qp_{n+1} + (1-q)p_{n-1}$.

55.1 Is p_n increasing or decreasing?

55.2 What is p_0? What is p_W?

55.3 Let $q = \frac{1}{2}$. What is $p_{W/2}$? Is p_n symmetric around $n = \frac{W}{2}$?

55.4 Let $q > \frac{1}{2}$. Is $p_{W/2} > \frac{1}{2}$? Is $p_{W/2} < \frac{1}{2}$?

55.5 How do solutions for $q = \alpha$ and $q = 1-\alpha$ compare?

Question .3 is purposefully ambiguous about symmetry. What kind of symmetry is there? Is there any?

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Lesson 33

Analysis of Difference Equations II

Textbook

- Module 25

Objectives

- Deduce properties of solutions of difference equations using different approaches

Motivation

Analyzing difference equations is similar to the different ways that we analyzed differential equations.

Approximating solutions though is much easier as a difference equation is already in the form of a numerical method.

Preparation for Class

- Read textbook
- Finish the core exercise 55.

Tutorials and Projects

- Project 10 Bullwhip effect
- Project 11: Approximating the temperature of a thin sheet
- Project 12:Math of lungs
- Project 13: Dark day
- Project 14: Maximus vs Commodus

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56 Consider a population of rabbits with the following lifecycle:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pair of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Consider the definitions:

- We start with 1 pair of newborn rabbits in year 0;
- r_n = number of pairs of rabbits alive during year n;
- i_k = number of immature pairs;
- y_k = number of young adult pairs;
- a_k = number of adult pairs;
- o_k = number of old pairs.

56.1 Show that $b_k = b_{k-2} + b_{k-3}$.

56.2 Show that $y_{k+1} = o_k + o_{k+1}$.

56.3 Show that $r_n = r_{n-2} + r_{n-3}$.

$$r_k = r_{k-1} - o_{k-1} + b_k$$

On the other hand,

$$\begin{aligned} r_{k-1} - o_{k-1} &= b_{k-2} + 2y_{k-2} + a_{k-2} \\ &= r_{k-2} + y_{k-2} - o_{k-2} \end{aligned}$$

and

$$\begin{aligned} b_k &= b_{k-2} + b_{k-3} \\ &= y_{k-3} + a_{k-3} + b_{k-3} \\ &= r_{k-3} - o_{k-3} \end{aligned}$$

So we have

$$\begin{aligned} r_k &= r_{k-2} + \cancel{y}_{k-2} - \cancel{o}_{k-2} + r_{k-3} - \cancel{o}_{k-3} \\ &= r_{k-2} + r_{k-3} \end{aligned}$$

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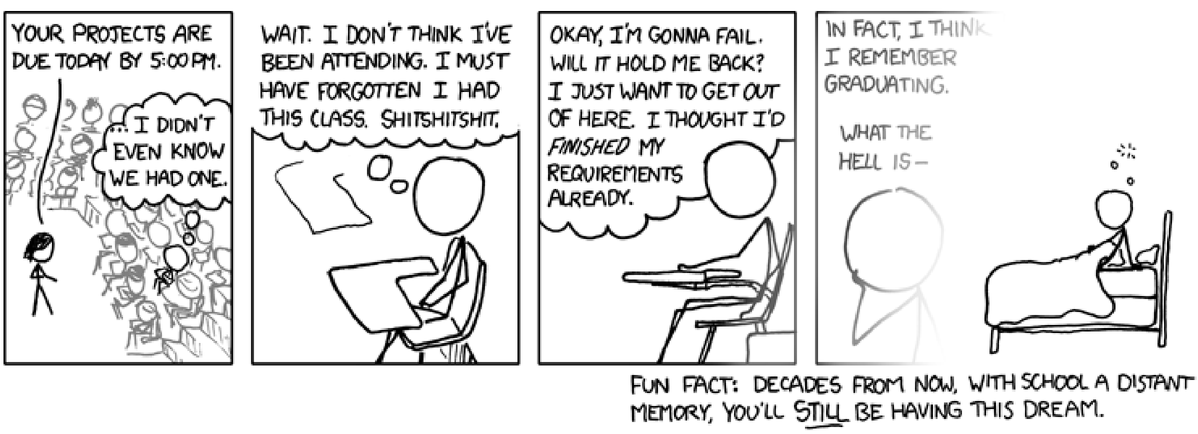

(image from xkcd - comic #557)

Elephants: too much is too much!

A large national park in South Africa is home to around 11 000 elephants. Wildlife management policies establish that a healthy environment for this herd will keep it at 11 000 head. Each year the park officials assess the elephant population. During the past 20 years, part of the herd has had to be removed to keep the population as close to the 11 000 number as possible. "Withdrawing" part of the herd means either slaughtering the animals or moving them. In the latter case, 600 to 800 animals are moved each year.

Recently, environmental and animal welfare groups have protested against killing elephants. In addition, it is no longer possible to move even a small population of elephants each year. A contraceptive dart has however been developed which prevents a mature elephant from conceiving for a period of two years.

Here is some information about the elephants in the park:

- There is very little migration of elephants;
- The gender ratio is close to 1: 1 and control measures should be chosen to maintain this ratio;
- The gender ratio of baby elephants is also 1:1. Twins are born in 1.35% of pregnancies;
- The elephant's maturity period begins between 10 and 12 years and continues until the age of 60. They produce an average of one baby elephant every 3.5 years. Gestation lasts approximately 22 months;
- Elephants who have received a contraceptive dart are in estrus once a month, but cannot conceive. An elephant (stingless) usually has only one seduction and mating period per 3.5 years; the monthly heat period can therefore cause additional stress;
- An elephant can receive a sting each year without negative effects. A mature elephant will not be able to conceive during a period of 2 years from the moment when she received the sting;
- About 70% to 80% of baby elephants reach the age of one year. Afterwards, the survival rate is uniform up to the age of 60 and very high (above 95%). Few elephants are over 70 years of age;
- There is no hunting or poaching in the park;

The park management team has **data** on the approximate age and gender of elephants relocated outside the park during the past two years. Unfortunately, no data is available on the elephants that have been slaughtered or that constitute the herd present in the park.

Task. The overall goal of the project is to investigate the feasibility of a herd control program using contraceptive darts. More specifically:

1. Develop a model and use it to describe the probable survival rate of elephants from 2 to 60 years old. Also speculate on the demographic curve of the herd present in the park.
2. Estimate how many elephants need a dart each year to keep the herd at around 11,000 head. Show how the uncertainties in the data at your disposal can influence your estimate. Comment on possible changes in the demographic curve and how these changes could have an impact on the tourism industry over a 30 to 60 year horizon.
3. Some opponents of the contraceptive dart technique claim that, if there was a sudden disappearance of a large part of the herd (for example following an illness or poaching out of control), the ability of the population to return to its optimal level would be seriously understood, even if the dart program was quickly discontinued. Investigate this statement and respond to this concern.

"Elephants: too much is too much!" is a collaboration with Yvan Saint-Aubin.

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Managing a fishery

The population $P(t)$ of a species of fish in a finite environment, like a lake is often described by the logistic equation

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right),$$

where r is the natural growth rate of the population and K is the maximum number of individuals that the environment can sustain.

If that population is harvested at the rate of $H(t, P)$ fish per year, then it can be modelled by the ODE

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) - H(P, t).$$

This ODE assumes that the population is harvested continuously throughout the year, which is not exactly true. Later in the course we will see how to make a better model.

Task. Your goal is to figure out the maximum profit you can generate from this population of fish. For that, it should be clear that it is undesirable to harvest too much, since it may lead to the extinction of the fish.

1. Selling fishing licenses. As the manager of the lake, you decide to profit from it by selling fishing licenses to people and allow them to fish there.

(a) Assuming that an average person, after a whole **day** fishing, has an efficiency[1] of E%, what is $H(P, t)$ and what is the ODE that models the fish in the lake?

(b) There is a percentage $E^\star$, such that if $E \geq E^\star$, then the population will become extinct. What is $E^\star$? Explain why the population will become extinct.

Hint. You don't need to solve the ODE.

(c) A sustainable yield Y is the rate at which the fish can be harvested indefinitely: it is the value of $H(P, t)$ which doesn't change with time and for the asymptotically stable population.

Determine the maximum[2] value of E to maximize Y and then find the maximum Y_{max}.

2. Selling fish. As the manager of the lake, you decide to profit from it by harvesting the fish yourself and selling it.

(a) Now the fish are harvested at a constant rate h. What is $H(P, t)$ and what is the ODE that models the fish in the lake?

(b) There is a rate $h^\star$, such that if $h \geq h^\star$, then the population will become extinct. What is $h^\star$? Explain why the population will become extinct.

Hint. You don't need to solve the ODE.

(c) If $h \leq h^\star$, what is the maximum sustainable yield Y_m?

Further Investigation.

1. Can you think of other forms for $H(P, t)$?
2. If, instead of a logistic model, you include an extinction threshold as well, what can you say about the model for constant effort fishing? for constant rate fishing? Is it a useful addition to the model? Have fun with it!

[1] Efficiency of E% means that after a whole day fishing, that person will have caught E% of the existing fish in the lake.
[2] This value can be controlled, e.g. by defining the time available for fishing in a day.

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X-ray attenuation

An X-ray tube fires X-rays, which travel in a straight line. An X-ray detector will give you the intensity of any X-ray that hits the detector. If there is a vacuum between the X-ray tube and the X-ray detector, then the X-ray will have the same intensity when it hits the detector as when it left the tube. However, when X-rays pass through matter they interact with the atoms in the material and are sometimes deflected off course, or absorbed. We call this phenomenon **attenuation of the X-ray** and it results in a decrease in the intensity of the X-ray beam. When this happens, the X-ray detector will show a lower intensity than the original intensity of the X-ray.

The intensity of an X-ray is measured in keV (kiloelectronvolt).

Task.

1. Experiments indicate that the rate of decrease in the intensity of the X-ray beam as it travels through some matter is proportional to the **linear absorption coefficient** A of the material. Find an ordinary differential equation (ODE) to model the intensity, I, of an X-ray beam fired into some uniform matter with linear absorption coefficient A. Be sure to include an initial condition.
   - What are the units of A?
   - Classify the equation.
   - Solve the equation in terms of the initial condition and A.

2. How far into a material can an X-ray beam travel before its intensity has decreased to $\frac{1}{e}$ times its original intensity.

Now you will explore one of the main ideas behind medical X-ray imaging. In order to do this, you need to know the linear attenuation coefficient of healthy human tissue.

3. You have a 15keV X-ray tube and an X-ray detector. When you fire the X-ray through 10cm of healthy tissue, you measure $\frac{15}{e}$keV on your X-ray detector. When you fire the X-ray through 20cm of healthy tissue, you measure $\frac{15}{e^2}$keV. Using your model of X-ray attenuation, estimate the linear attenuation coefficient of healthy tissue.

This is the basis for using X-ray Computed Tomography (CT) used in medical imaging! We can recognize healthy versus unhealthy tissue by using what we know about their attenuation coefficients. To do this in a human body requires more advanced mathematics such as the Radon transform introduced in 1917 by Johann Radon. However, consider a simple case below.

4. Suppose that you have a 10cm × 2cm rectangle with the same linear attenuation coefficient as healthy tissue, and somewhere inside this square is a circle of unknown size having a linear attenuation coefficient different from that of healthy tissue.
   - Using an X-ray tube and an X-ray detector, can you locate the circle and determine its radius? How?
   - Can you determine the linear attenuation coefficient of the circle? How?

5. In actuality, a more complex model is needed for accurate imaging. The linear attenuation coefficient is actually dependent on the intensity of the X-ray! How does this impact your model? Discuss how this would impact your solutions to the above problems.

"X-ray attenuation" is a collaboration with Craig Sinnamon.

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Predator-prey chase

Question. What is the path of a lion chasing an antelope?

Rules.

- The antelope flees at a constant speed v in a straight line
- The lion chases at a constant speed u

Task.

1. Assume the antelope starts at $(0,0)$ and moves up the y-axis. What is its position?

$$(0, vt)$$

2. Assume the lion's position is $\big(x(t), y(t)\big)$. What happens if $x(0) = 0$?
3. Assume the lion's position is $\big(x(t), y(t)\big)$ with $x(0) \neq 0$. For simplicity, assume that $x(0) < 0$. What is the sign of $x'(t)$?
4. The goal is to find the path, so we are looking for an equation to describe $y(x)$. Using the result from **3.**, explain why the solution will be a function $y(x)$.
5. The lion's speed is u. Express that condition using $x(t)$ and $y(t)$.
6. Find an expression for $\dfrac{dx}{dt}$ without the variable t.
7. Which condition on $\dfrac{dy}{dx}$ do we get from the fact that the lion is chasing the antelope? Draw a picture.
8. Use **7.**, and obtain an expression for $\dfrac{dx}{dt}$ in terms of $\dfrac{d^2y}{dx^2}$.
9. Obtain a Differential Equation that describes the lion's path $y(x)$.
10. For simplicity, assume that the lion starts at $(-1, 0)$. Solve this Initial-Value problem.
11. When does the lion actually catch the antelope?

Further investigation.

1. Program the pursuit (it will be approximated) and check your answer.
2. Can you figure out the path of the lion for other antelope trajectories? Program that pursuit and check your answer.
3. (for fun) Program a "game" where you control the antelope and the computer controls the lion.

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Epidemic modelling

Goal. We want to model the spread of the CoViD-19 pandemic in Canada.

SIR Model. This is the typical model for an infectious disease. We start by dividing the population into three groups:

- Susceptible Individuals $S(t) =$ number of people who haven't contracted the disease;
- Infected individuals $I(t) =$ number of people infected;
- Removed individuals $R(t) =$ number of people that either died or recovered from the disease and are now immune to it.

Assumptions.

(a) Population size N is large and constant (no birth, death, or migration);

(b) No latent/incubation period (there is an improved model that includes this - SEIR model);

(c) Homogeneous population;

(d) Recovery rate is constant γ (includes rate at which people die or recover from the disease);

(e) Out of all possible interactions between susceptible and infected individuals $S(t)\cdot I(t)$, there is a proportion $\frac{\beta}{N}$ that will result in the susceptible individual becoming infected;

(f) The probability that an infected person will either die or recover is γ.

ODE. From here we can obtain the SIR model:

$$\begin{aligned}
\frac{dS}{dt} &= -\frac{\beta}{N}SI \\
\frac{dI}{dt} &= +\frac{\beta}{N}SI - \gamma I \\
\frac{dR}{dt} &= +\gamma I
\end{aligned}$$

Video.

- `https://youtu.be/f1a8JYAixXU`

Important.

- An important constant in this model is $R_0 = \frac{\beta}{\gamma}$, called the basic reproductive number, which informs us about how fast the disease propagates.
- The expected time from infection to recovery (or death) can be proved to be $T = \gamma^{-1}$.

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Data. Data from the Public Health Agency of Canada:

- http://uoft.me/covid19-canada

Task.

1. Explain how the system of ODEs relates to the assumptions.
2. Estimate the constants N, R_0, β, γ for Canada.
3. Using the idea from Euler's Method (used to approximate the solution of one first-order ODE), create a method to approximate the solution $S(t)$, $I(t)$, $R(t)$ of the SIR model.
4. Compare your approximation from **3** with the actual data.
5. Observe that the data is the result of the lockdown measures imposed in Canada. Find a value for R_0 that best matches your approximation to the data.
6. Study what happens to Canada if the lockdown measures are lifted when the number of infected people is very small vs when the number of infected people is actually zero.

Further Investigation.

1. How does the model change if some people can get reinfected? What happens if there is a vaccine available but it doesn't ork for everyone? What happens when there had been lockdown and it is slowly (or brutally) lifted?

   Hint. All these require new hypotheses, changing the differential equations, exploring the new solutions, etc.
2. Study what happens to the model when $R_0 < 1$, $R_0 = 1$ or $R_0 > 1$.
3. Adapt your method to the SEIR model and answer questions **1-6** above for the new model.
4. Improve the SEIR model to better model different lockdown scenarios.

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Hunting inspiration

Snow collects on the brim of your fur coat and musket as you stalk your prey through the white woods. A flash of orange, a rustling of branches, and then it's gone. You mutter a curse under your breath. Foxes are scarce this year, and you'll have to explain to the Dutch East India Trading Company why you've come up short. Worse yet, your rival, a trapper who only hunts rabbits, is having a terrific year. You shouldn't have teased him so much when rabbits were down and foxes were up just a few seasons ago.
If only you could somehow predict which game would be plentiful, you could always bid on the easier contract! But how?
Back at camp, amid the crackling of your lonely fire, the answer comes to you. Just two months ago you attended a talk by Dr. Lotka on autocatalytic chemical reactions. It was quite a spectacle when, after Dr. Lotka had finished talking, a Dr. Volterra stood up and proclaimed that he had applied the same model to predator-prey ecology. At the time you were rushed and didn't think much about the proclamation, but now the basic assumptions were making more and more sense:

> In the absence of foxes, the rabbit population grows at a rate proportional to the number of rabbits.
>
> In the absence of rabbits, the fox population declines at a rate proportional to the number of foxes.
>
> The population of rabbits declines at a rate proportional to the product of the rabbit and fox populations.
>
> The population of foxes grows at a rate proportional to the product of the rabbit and fox populations.

Pop! A hot coal explodes, snapping you out of your pondering state and into one of action. Grabbing a piece of paper from your limited supplies, you begin to grapple with the consequences of the Lotka–Volterra model.

Task. Let R and F stand for the rabbit and fox populations, respectively, and let α, β, γ, and δ be the constants of proportionality for parts **(a)**–**(d)**.

Work through the following before you begin your report.

1. Write down the Lotka-Volterra system of differential equations. For each of (a)–(d), explain whether or not the assumption is reasonable.
2. When is the fox population increasing or decreasing? Given R and F, could you predict which one is on the rise on which one is on the decline?
3. Is there a steady state for the fox population? Could the fox population remain steady while the rabbit population is changing?
4. Sketch an RF-phase portrait for the Lotka-Volterra system of differential equations with the following constants:

$$\begin{aligned}\alpha &= 0.2 \text{ rabbits per month per rabbit}\\ \beta &= 0.1 \text{ foxes per month per fox}\\ \gamma &= 0.002 \text{ rabbits per month per rabbit-fox}\\ \delta &= 0.001 \text{ foxes per month per rabbit-fox}\end{aligned}$$

   Hint: you will need to consider rabbit and fox populations of well over 100 to see interesting behaviour in your phase portrait.
5. Does your phase portrait have any singular points? What do they mean?
6. Use technology to graph $R(t)$ and $F(t)$ for some initial conditions. Do the initial conditions affect the period of the population increase or decrease? Does this seem reasonable when looking at your phase portrait?

Your writeup should include the following:

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- An explanation of the Lotka-Volterra model along with a discussion of whether or not each assumption is reasonable.
- A description of what behaviour you expect from which initial conditions. You may use the parameters specified in question 4.. Include a phase portrait in your description as well as how to interpret the phase portrait, and make sure to point out any critical points.
- Suppose you wanted to legislate limits on the hunting of rabbits and foxes to ensure the population of either never dipped below a certain level. Based on the Lotka-Volterra model, propose legislation. Be specific and comment on whether a flat-out hunting ban would achieve the desired effect.

Be careful with your simulations. Euler's method loses accuracy quickly on Lotka-Volterra–based systems.

"Hunting Inspiration" is a collaboration with Max Brugger.

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Arms race

In this project, you will develop and analyze models for an arms race between two countries.

Define the following:

- $t \geq 0$ represent time in years;
- $x(t)$ and $y(t)$ represent the yearly military budget (in dollars) of countries Blue and Red respectively.

Task 1. Mutual Fear! For a first model, assume that each country increases its military budget at a rate directly proportional to the existing military budget of the other nation.

(a) What are the equations that define this model?

Hint. There should be two constants in your model.

(b) Solve the system and sketch a phase portrait.

(c) What does the model predict about the long term military budgets of the two countries?

Task 2. The Richardson Model. Now include some limiting factors in to the model you set up above. Assume that in addition to the budget increases in the mutual fear model, each country's military budget decreases at a rate proportional to it's current military budget and increases at some fixed (independent of military budget) rate due to a long standing grievance.

(a) What are the equations that define this model?

Hint. There should be six constants in your model.

(b) Under what conditions (on the six constants) can the arms race stabilize? By stabilize we mean that the military budgets remain at some fixed amount, or that the budgets approach some constant amounts.

- There is a line L_B, called the optimal line for Blue, in the phase plane such that if $(x(t), y(t))$ lies on L_B then $x'(t) = 0$. What is the equation of that line?
- Show that Blue continuously changes its military budget to bring the solution $(x(t), y(t))$ closer to L_B.
- Repeat the previous two parts for a line L_R, the optimal line for Red.
- What does the intersection point of L_B and L_R represent?
- Under what conditions on the constants will the point of intersection lie in the first quadrant $(x > 0, y > 0)$? What will be the long term behaviour of the system for various initial conditions? Explain.
- Under what conditions on the constants will the point of intersection lie in the third quadrant $(x < 0, y < 0)$? What will be the long term behaviour of the system for various initial conditions? Explain.

(c) What happens in the long run for various initial conditions if one or both of the "grievance" terms is/are negative? (More of a "good will" term than a "grievance" term!)

(d) Can L_B and L_R be parallel? What happens in this case?

(e) Can $L_B = L_R$? What happens in this case?

(f) Produce examples that demonstrate these various cases and long term behaviours. Plot or sketch their phase portraits.

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Task 3. Real World. One can argue that in the real world, a runaway arms race is impossible since there is a limit to how much a country can spend. We can add carrying capacities in to the model. Let x_M and y_M be the maximum budgets of the two countries. Then consider the model

$$x'(t) = \left(1 - \frac{x}{x_M}\right)(-ax + by + c)$$
$$y'(t) = \left(1 - \frac{y}{y_M}\right)(dx - ey + f)$$

Analyze this model.

Further Investigation.

1. **Another Nonlinear Model.** Suppose that the equations underlying the model have the form

$$\begin{cases} x'(t) = -ax + by^2 + c \\ y'(t) = dx^2 - ey + f \end{cases}$$

where $a, b, d, e, f > 0$. How many stable points are there? There are now optimal curves instead of optimal lines. Discuss the outcomes of such an arms race for various intersections of the optimal curves.

2. **The Richardson Model with Good Will.** If instead of having terms representing increases due to a grievance, what happens if you include terms representing fixed rate decreases in the military budgets of both countries due to good will?
   - What are the equations that define this model?
     Hint. There should be six constants in your model.
   - What possibilities exist for the long term behaviour of the military budgets of the two countries?
   - How do the possibilities for the long term behaviour depend on initial conditions?
   - Produce examples that demonstrate the various long term behaviours.

3. Richardson with carrying capacities.

4. Extend Richardson to three countries.

5. Increase not by absolute level but by amount over stable level.

"Arms Race" is a collaboration with Craig Sinnamon.

Spring data

Statement. A motion sensor was set up to measure the motion in a spring-mass system, but something went wrong and the motion sensor measured the total distance traveled instead of simply measuring the distance from the sensor. The experiment was conducted three times (same spring and same mass) with different initial conditions. The total distance traveled is given as the data sets in the Google Sheet spreadsheet:

- `https://goo.gl/AFMTn8`

The conditions for the three experiments were:

Data Set #1. Initial position: $y(0) = 1$. Initial velocity: $y'(0) = 0$.
Data Set #2. Initial position: $y(0) = 0.5$. Initial velocity: $y'(0) = 1$.
Data Set #3. Initial position: $y(0) = -0.75$. Initial velocity: $y'(0) = -2.5$.

Experimental Setup.

- The sensor gathered data at a rate of 20 samples per second.
- The experiment was run for 5 seconds.
- $y(t)$ is the distance from the equilibrium position of the spring-mass system. Positive values of $y(t)$ indicate that the mass was above the equilibrium position. Negative values of $y(t)$ indicate that the mass was below the equilibrium position.
- There is some noise in the data.

Task.

1. Use the data in the spreadsheet to determine the governing ODE, which should include estimates for the parameters.
2. Use the data in the spreadsheet to determine the height $y(t)$ for the different experiments.

Hint. Recall that if $y(t)$ is displacement and $v(t) = y'(t)$ is velocity then the total distance traveled is given by the function

$$d(t) = \int_0^t \left|v(\tau)\right| d\tau.$$

Further Investigation.

1. How many data sets and how many data points are needed to be able to solve the problem?
2. Add more noise to the data. Can you still solve it? How much noise can you add and still obtain good results?
3. Create your own (fake) data mimicking a spring with different properties (remember to include some noise in the data)? And solve it to show that it can be done.
4. Could you use this to detect an external force acting on the spring-mass system? Try it with two new data sets:

   - `https://goo.gl/TxzQWw`

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Wing flutter

Question. Is the airplane wing going to break?

Introduction (adapted from Stuart Lee – Click here for the original).

In early 1959, with great fanfare, Lockheed's new, 4-engine prop-jet, the Electra II, went into service. The Electra looked like a "regular airline", except that the thick prop blades and the four enormous large engine covers (the nacelles and cowlings) that housed the General Electric/Allison jet-turbine driver power plants made the wings seem ever smaller and stubbier. In addition, the fuselage was relatively wide- making it one of the roomiest airliners of its time. But the Electra's appearance seemed slightly off.

The pilots soon got over the appearance and came to respect the airplane, The Electra had incredible power. One pilot remarked that "It climbs like a damned fighter plane!".

In the evening of September 29, 1959, Braniff's spanking new Electra disintegrated in midair (description).

What had caused this brand-new jet prop to disintegrate over Buffalo, Texas?

The investigators combing the wreckage of the Braniff Electra noticed something alarming. The shards of what appeared to be the left wing were found a considerable distance from the rest of the wreckage.

And the story got worse. On March 17, 1960, Northwest Airlines flight 710 left Minneapolis-St. Paul (description). Witnesses on the ground heard tearing sounds in the sky. They looked up and saw the thick fuselage of the Electra emerging from the clouds. The entire right wing was missing, and only a stub of the left wing remained attached to the Electra.

The airliner seemed to float for a while, but then it dipped, diving straight down toward the ground, trailing white smoke and pieces of aircraft. The 63 people entombed in the fuselage struck the muddy ground, vertically, at 618 miles per hour. Rescuers found nothing at the site of impact larger than a spoon.

But 3 km away, they found the wreckage of the left wing.

This was beyond, alarming. In a period of less than six months, two brand-new Electras lost their wings and disintegrated with much loss of life. What could have caused this? Could it have been severe clear-air turbulence (CAT), or was there something drastically wrong with these airliners.

The airlines who had Electra fleets were nearly panicking. Meetings were quickly set up with the FAA. Investigations were set up. Boeing lent staff, simulators, and a wind tunnel to Lockheed. Douglas contributed engineers and equipment; most notably flutter vanes that, when attached to the ends of the wings, could induce serious oscillation.

The investigation, occurring in the early sixties, was the first serious use of computer stress analysis in this field.

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Electras were test flown in every possible form of turbulence. Test pilots tried to destroy the Electra by ramming it into the severe Sierra Madre air waves, over and over again. Electras were put through every possible flight maneuver that would normally cause a wing failure. Super severe wind tunnel winds were shot out at Electras and mock-ups. Over and over, every possible test was done to try and break the Electra.

Finally, on May 5, 1960, an engineer stood up at a Lockheed meeting and announced: "We're pretty sure we've found it!".

Basically, the problem was a high-speed aircraft in a conventional design. Every aircraft wing is flexible to some degree. And wing vibration, oscillation, or flutter is inherent in the design. Flutter is expected on wings. In engineering terms, there are more than 100 different types of flutter – or "modes" – in which metal can vibrate. The "mode" that destroyed the Electras was "whirl mode".

Whirl mode was nothing new. It was not a mysterious phenomenon. As a matter of fact, it is a form of vibrating motion inherent in any piece of rotating machinery such as oil drills, table fans, and automobile drive shafts.

The theory was devastatingly simple. A propeller has gyroscopic tendencies. In other words, it will stay in a smooth plane of rotation unless it is displaced by some strong external force, just as a spinning top can be made to wobble if a finger is placed firmly against it. The moment such a force is applied to a propeller, it reacts in the opposite direction.

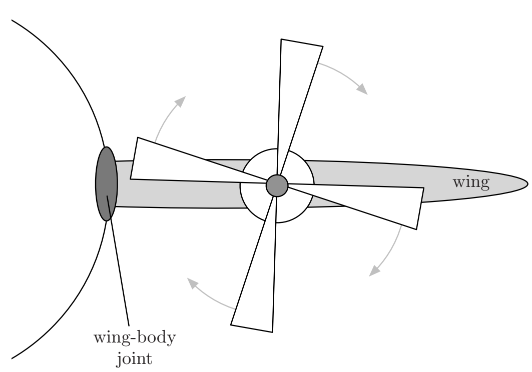
> Now suppose the force drives the propeller upward. The stiffness that is part of its structure promptly resists the force and pitches the prop downward. Each succeeding upward force is met by a protesting downward motion. The battle of vibration progresses. The propeller continues to rotate in one direction, but the rapidly developing whirl mode is vibrating in the opposite direction. The result, if the mode is not checked, is a wildly wobbling gyroscope that eventually begins to transmit its violent motion to a natural outlet: the wing.

Whirl mode did occasionally develop in propeller-driver airliners. It always encountered the powerful stiffness of the entire engine package, the nacelles and the engine mounting, the mounting being a bar truss holding the engine to the wing. No problem usually. But on painful microscopic examination of the crash wreckage of the eight Electra engines, it was found that something caused the engines to loosen and wobble, causing severe whirl mode, which tore off the Electra's wings. Specifically, the investigation centred on the outboard engines.

What the investigators found was that the engine mounts weren't strong enough to dampen the whirl mode that originated in the outboard engine nacelles. The oscillation transmitted to the wings caused severe up-and-down vibration, which grew until the wings tore right off.

Project. In this project we will study mechanical resonance of an airplane wing due to a vibrating propeller. We use Differential Equations to create a simple model of the wing flutter.

Start with a picture of a propeller mounted on a wing.



We want to keep the model simple, so we consider only the wing's centre of mass. This implies that the wing behaves as a **spring-mass system**: the spring is the wing-body joint that allows the centre of mass to move up and down.[3] The forcing function is the vibrational force that results from the motion of the propeller.

[3]The centre of mass actually moves on an arch, but we consider only its vertical motion.

For this example assume that the wing has a mass of 900 kg and the wing-body joint acts as a spring with constant 8100 N/m. Also assume that the damping forces are negligible and the wing is at rest when the propeller begins to vibrate.

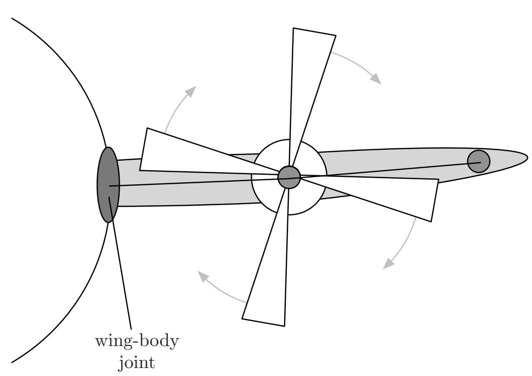
Task.

1. Let $y(t)$ be the position of the centre of mass of the wing and $f(t)$ the vertical vibrational force from the propeller. Write an IVP (Initial-Value Problem) that models the movement of the wing.
2. Assuming that the propeller vibrates with a force $f_1(t) = 1800\sin(6t)$ (in N). Find the position of the wing's centre of mass and plot it.

   Describe the position of the wing's centre of mass as t grows large ($0 \leq t \leq 25$).
3. Just before wing-failure, the propeller actually slowed down. Let us simulate this by changing the forcing function to $f_2(t) = 1800\sin(3t)$ (in N).
   (a) Find the equation of motion using the new forcing function.
   (b) Plot the solution.
   (c) Describe the position of the wing's centre of mass as t grows large. What consequences does this have for the wing?
4. It is very unlikely that the frequency of the propeller will match exactly this, so assume that $f_3(t) = 1800\sin(3.5t)$ (in N).
   (a) Find the equation of motion using the new forcing function.
   (b) Plot the solution.
   (c) Using a trigonometric identity, re-write your solution as a product of two trig functions. Describe how this new form for the solution explains the plot.
   (d) Describe the position of the wing's centre of mass as t grows large. What consequences does this have for the wing?

Further Investigation.

1. If you were the Lead Engineer in charge of fixing this problem, what would you do? How would that change the Differential Equation? Using the new differential equation, show that it would indeed solve the problem.
2. What happens if there are two propellers (like the actual Lockheed Electra)?
3. Can you model wing flex?



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Bullwhip effect

Goal. Understand that Supply Chain Management is hard(!) and attempt to model it.

Video.
Watch the short video:

- `https://youtu.be/2nlmkTYZG5s`

to understand a bit better about the bullwhip effect.

Read.

- `http://forio.com/about/blog/bullwhips-and-beer/`

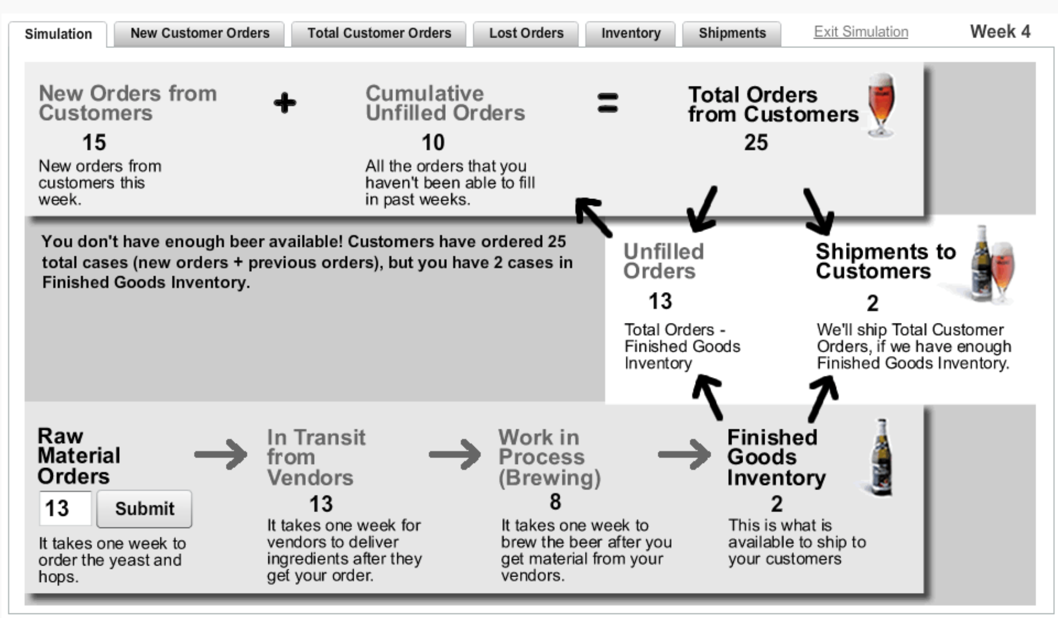
Near Beer Game (Novice). You own a (tiny) Beer Store.

You start with a stable situation where your customers have been asking for 10 cases of beer every week, and your inventory and orders match the situation (so you don't run low on inventory and you don't accumulate either).

From the second week, your customers start ordering 15 cases of beer instead.

It is you job to stabilize the whole supply chain as soon as possible.

Below is a screen from the "game".



- New Orders from Customers: Number of beer cases your new customers want this week
- Cumulative Unfilled Orders: Number of beer cases that your

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Go to

- `https://forio.com/simulate/mbean/near-beer-game/run/`

The goal of the "game" is to try and stabilize the number of customer orders, your inventory, arriving orders, and your order, so that you end up with the following situation

- Customer Orders: 15 cases every week (with no unfilled orders)
- Inventory: 15 cases
- Arriving Order: 15 cases
- Order 15 cases

Task 1. Play the "game" on `Novice` as a group and see how many weeks it takes to stabilize the situation.

Consider the following sequences:

- c_n = number of beer cases ordered by customers
- u_n = number of cases ordered previously but not fulfilled yet
- i_n = number of cases in inventory
- o_n = number of cases ordered
- r_n = number of cases of beer produced

where n is the number of weeks elapsed since the beginning of the "game".

(a) What are the initial conditions $(n = 0)$?

(b) What is the formula for c_n?

(c) What is the formula for r_n?

(d) What is i_n?

(e) What is u_n?

(f) Confirm that your modelling is correct, that is, that your variables follow the outcome of the game.

(g) Decide on a strategy for ordering beer cases. Decide on a formula for o_n that can depend on n, c_n, u_n, i_n, r_n. Explain your choice.

(h) What is the result of your strategy? Does it go "amuck" – bullwhip effect[4]? Or does it control the supply chain nicely?

[4] It's ok if it goes "amuck"! The goal is to see the Bullwhip Effect in action... Now try to fix it!

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Task 2. Play the "game" on `Expert` as a group and see how many weeks it takes to stabilize the situation.

Consider the same sequences as for **1.**

The difference between `Novice` and `Expert` is that the customer orders go from 10 → 50 and every week 25% of unfilled orders are cancelled.

(a) What are the initial conditions ($n = 0$) ?

(b) What is the formula for c_n?

(c) What is the formula for r_n?

(d) What is i_n?

(e) What is u_n?

(f) Confirm that your modelling is correct, that is, that your variables follow the outcome of the game.

(g) Decide on a strategy for ordering beer cases. Decide on a formula for o_n that can depend on n, c_n, u_n, i_n, r_n. Explain your choice.

(h) What is the result of your strategy? Does it go "amuck" – bullwhip effect? Or does it control the ordering nicely?

Further Investigation.

1. There is a more complex version of the game

   - `https://beergame.pipechain.com/`

   which includes 1–4 players from 4 different stages of the supply chain. It takes 2 weeks for orders to arrive to a different stage and it takes 2 weeks to fulfill a request.

   (a) Play the game with 2 players[5] who do not communicate with each other, i.e., two-stage supply chain.

   (b) Define the new sequences
   - c_n = number of beer cases ordered by customers
   - o_n = number of cases ordered by the retailer
   - s_n = number of cases in the retailer's stock
   - p_n = number of cases ordered by the producer
   - q_n = number of cases in the producer's stock

   (c) Make a similar study for this case. Observe that now you have to decide on the strategy for both o_n and p_n.

2. In the article suggested at the beginning

   - `http://forio.com/about/blog/bullwhips-and-beer/`

   the author describes ways a few ways to reduce the Bullwhip effect. Program each of them with your sequence o_n and study how well they reduce the effect.

3. You can avoid the Bullwhip effect completely with perfect information about the supply chain and the future customer demand. In reality, we can predict the customer demand, but it won't match exactly the prediction. Add a little noise to customer demand and try to avoid the Bullwhip effect. You can still use the fact that customer demand will still be close to 15 cases every week.

[5] Create a game and then use another computer to join the same game

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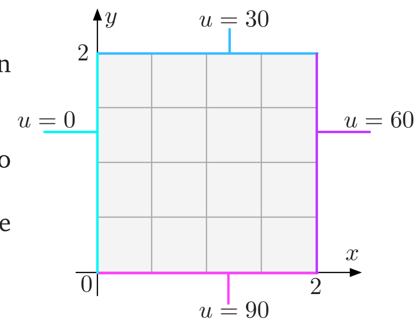
Approximating the temperature of a thin sheet

Task. We want to approximate solutions of a PDE.

The heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where $u(x,y)$ is the equilibrium temperature at the position (x,y) given some boundary conditions.



This is a Partial Differential Equation (PDE), which we don't know how to solve. We can however obtain an approximation of the solution.
In this example, the domain is shown on the right $\Omega = [0,2]\times[0,2]$ and the initial conditions are the following

$$u(x,0) = 90 \quad , \quad u(x,2) = 30 \quad , \quad u(0,y) = 0 \quad , \quad u(2,y) = 60.$$

To approximate the solution, we divide the domain in N small pieces. In the example $N = 4$ and $\Delta = \frac{2-0}{N} = \frac{1}{2}$.
Then we define the points

$$\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots, \vec{p}_M,$$

as the points in the interior of the domain (usually by moving left→right and bottom→top).

1. What are the points $\vec{p}_n$? What is M?

Then we define

$$u_n = u(\vec{p}_n),$$

where $u(x,y)$ is the solution of the initial-value problem above (PDE with boundary conditions).

The next step is to approximate the PDE itself. We do that by approximating the derivatives:

$$\frac{\partial u}{\partial x}(x_0,y_0) \approx \frac{u(x_0+\Delta, y_0) - u(x_0,y_0)}{\Delta}.$$

2. What is the approximation for $\frac{\partial u}{\partial x}(\vec{p}_5)$? What is the approximation for $\frac{\partial u}{\partial x}(\vec{p}_3)$?
3. What is an approximation for $\frac{\partial u}{\partial y}(x_0,y_0)$? What is the approximation for $\frac{\partial u}{\partial y}(\vec{p}_8)$?

From here, we define the second derivative in a similar fashion:

$$\frac{\partial^2 u}{\partial x^2}(x_0,y_0) \approx \frac{\dfrac{u(x_0+\Delta, y_0) - u(x_0,y_0)}{\Delta} - \dfrac{u(x_0,y_0) - u(x_0-\Delta, y_0)}{\Delta}}{\Delta}$$
$$= \frac{u(x_0+\Delta, y_0) - 2u(x_0,y_0) + u(x_0-\Delta, y_0)}{\Delta^2}.$$

4. What is the approximation for $\frac{\partial^2 u}{\partial x^2}(\vec{p}_5)$? What is the approximation for $\frac{\partial^2 u}{\partial x^2}(\vec{p}_3)$?
5. What is an approximation for $\frac{\partial^2 u}{\partial y^2}(x_0,y_0)$? What is the approximation for $\frac{\partial^2 u}{\partial y^2}(\vec{p}_8)$?

We are now ready to put it all together.

The PDE applies to all points in the domain. Instead of applying the PDE to all points $(x,y)\in\Omega$, we apply the approximation of the (second) derivatives to all the points $\vec{p}_n$.

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6. What is the equation that we obtain for the point $\vec{p}_5$?
7. What is the equation for each point $\vec{p}_n$?

These equations form a linear system of equations. Define a vector $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$

8. Write the system of equations in matrix form $\mathbf{A}\vec{u} = \vec{b}$.
9. Solve it and plot the solution. (You should use some software to solve this!)

MATLAB. Here is a quick introduction to some tools in MATLAB that are useful for this problem.

- Define a matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ by

```
» A=[1,2;3,4]
```

- Define a vector $\vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ by

```
» b=[5;6]
```

- Solve the system $\mathbf{A}\vec{u} = \vec{b}$ by defining $\vec{u} = \mathbf{A}^{-1}\vec{b}$

```
» u=A\b        or        » u=inv(A)*b
```

- To plot a 3D plot like this, define a matrix for the solutions and write

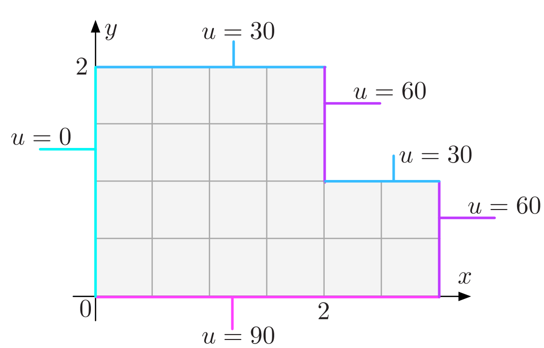
```
» surf(p)
```

To use the typical colouring for the heat equation, type

```
» colormap(cool)
```

Further Investigation.

1. Approximate the solution for the domain and boundary conditions



2. Formulate the procedure for a general N.
3. Formulate the procedure for a different Δx and Δy.
4. This method can be adapted to what kind of domains? And what kind of boundary conditions?

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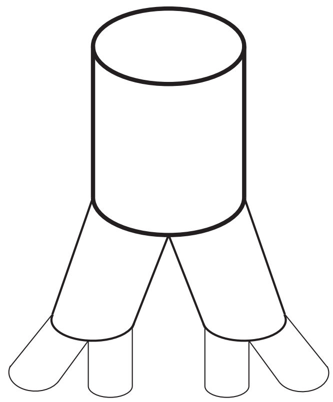
Math of lungs

Lungs are somewhat important to human beings! They are the source of oxygen to our bodies, so it is important to maximize the amount of oxygen that can be absorbed within the available volume.

Task 1. Let us find out the volume and surface area of the lungs.

(a) The lungs are composed of a branched structure as in the figure below.



The first segment is very large, then it bifurcates into smaller segments in a geometrical pattern.

Here is some data about human lungs:

- radius of first segment: $r_0 = 0.5\text{cm}$
- length of first segment: $\ell_0 = 5.6\text{cm}$
- ratio of daughter to parent length: $\alpha = 0.9$
- ratio of daughter to parent radius: $\beta = 0.86$
- number of branch generations: $M = 30$
- average number of daughters per parent: $b = 1.7$

In the figure, there are 2 daughters per parent, in real lungs, it isn't perfectly regular, so we have an average that is not a whole number.

(b) Calculate the volume inside the segments. What is the limit as the number of segments gets larger and larger?

(c) Calculate the surface area inside the segments. What is the limit as the number of segments gets larger and larger?

(d) If we had $b = 2$ instead and the same number of generations, would that be possible? If not, how many generations would be possible?

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Task 2. Let us model the gas exchange that happens inside the lungs.

(a) Suppose that a lung has a volume of 3L when full. With each breath, 0.6L of the air is exhaled and replaced by 0.6L of outside air.

After exhaling the volume is 2.4L and it returns to 3L after inhaling.

Suppose further that the lung contains a chemical with a concentration of 2 milimoles per litre before exhaling (a mole is a chemical unit for 6.02×10^{23} molecules). The ambient air has a concentration of 5mmol/L of the same chemical.

What is the concentration after one breath? What is the concentration after n breaths?

(b) Update your model to match the oxygen exchange inside a real human lung.

(c) The model above ignored the fact that the body absorbs some of the oxygen. Assume now that the lungs absorb 30% of the oxygen with each breath. Update your model.

(d) What is the equilibrium concentration of oxygen in the lungs? Find the graph of the equilibrium concentration as a function of the fraction of oxygen absorbed with each breath.

You may choose to model the gas exchange in the lungs using either a discrete time ***difference equation*** or a continuous time ***differential equation***.

(a) If you choose a difference equation, you might assume that every breath the concentration of the chemical changes. If $c(n)$ or c_n represented the concentration of chemical after n breaths, your difference equation might look like

- $\Delta c(n) = c(n) - c(n-1) =$ some function at time n, where n is only allowed to take whole numbers.

(b) If you choose to use differential equations, the analogous equation would look like

- $c'(t) =$ some function at t, where t can take any positive real value.

Task 3. Assume that the lungs only absorb a fraction of the air in contact with its surface. Combine the two previous tasks.

Further Investigation:

1. Investigate how the what is known about human lungs. Compare how the branching of real human lungs differs from this model. Refine the estimate.

2. Investigate how the what is known about human lungs. Compare how the gas exchange of real human lungs differs from this model. Refine the model.

"Math of Lungs" is a collaboration with Kseniya Garaschuk and Miroslav Lovric.

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Dark day

At one in the morning, your phone goes off. After three attempts to turn off your alarm clock, you finally realize that it is a call – -a call from a number you had promised to always answer. By 1:15, you're dressed and out the door where a black SUV is idling, waiting for you. After a transfer to a government plane, you touch down in Washington, D.C., and as the sun finally begins to rise, you traverse down the Secret Service tunnels to a large conference room lined with leather chairs.

"I'm not going to mince words," a voice says, from the other side of the room. The chair at the end of the table swivels round and you see the President emerge from the shadows. "It's bad. It's worse than bad. It's, uh... it's zombies."

A revelation like that would have thrown a lesser scientist, but you're a professional. You've been preparing for this for years, urging your colleagues to take the threat seriously.

"Where is the origin? Have we identified a patient zero in the US or are there multiple sources? What are the parameters of the disease?" you ask.

"Straight to work, okay," the President says, looking pleased. "Chicago police began reporting violent attacks a few days ago. It started with a single report in the navy shipyard. The shipyard has been quarantined, but we're now getting reports from all over the city. These attacks are carried out by humans who always attempt to bite their victims. Those bitten begin to show symptoms within a matter of hours, but those who are attacked but escape without being bitten appear normal. I've sent in the Marines, and they estimate that an infected person dies after eight days. They also predict about 3,000 individuals have been exposed, five days after the initial report."

"Has any quarantine been successful?" you ask.

"No." The President pauses and the gravity of what he has just said starts to sink in. "There are approximately 9.7 million people living in the Chicago metro area. Obviously time is of the essence. Now, I've been told that you're the best epidemiologist we have. I need to know if the region has a chance of survival, and if it does, what the impact will be. Is there any hope for Chicago?"

You may choose to model the zombie outbreak using either a discrete time ***difference equation*** or a continuous time ***differential equation***.

(a) If you choose a difference equation, you might assume that every day or every hour, the number of zombies, humans, and dead increment. If $Z(n)$ represented the number of zombies at time n, one of your difference equations might look like

- $\Delta Z(n) = Z(n) - Z(n-1) =$ some function of zombies, humans, and dead at time n, where n is only allowed to take whole numbers.

(b) If you choose to use differential equations, the analogous equation would look like

- $Z'(t) =$ some function of zombies, humans, and dead at time t, where t can take any positive real value.

(c) Modelling with a difference equation or a differential equation should give you similar results (why?).

Task. Model the zombie infection. Make sure to address the following in your report:

(a) What situation are you trying to model?

(b) What equations are you using, and what does each variable in each equation represent (for example, "In this model, $Z(t)$ is the number of zombies at t hours from initial outbreak).

(c) Justification for any constants that you use and how you estimated them.

(d) Is there any hope for Chicago?

Do not attempt to find an equation that solves your differential equations—this is really hard. Instead, rely on estimates and simulations. You can use any computer program you like to assist you in estimating how the zombie outbreak spreads and whether your constants match with the known information. Including plots and figures in your report will make explaining things easier.

"Dark Day" is a collaboration with Max Brugger.

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Maximus vs Commodus

Maximus, the Roman general turned slave turned gladiator is seeking **vengeance!** (In this life or the next).

He has managed to make his way to the Roman Forum, and now all that stands between him and nefarious Emperor Commodus is the **Prætorian Guard**.

Maximus must make his way across the Forum, defeating all Prætorians in his way, if he is to achieve his goal in this lifetime.

Conditions.

- The forum can be represented by the horizontal x-axis, in the interval $[1, 1823]$;
- Maximus starts at the left side of the forum;
- Maximus moves at a constant rate across the forum - we call this M_s (Maximus' speed);
- The arrival of Prætorian guards follows a Poisson process with rate P_r (Prætorian arrival rate);
- Prætorians appear at the right side of the forum, and move at a constant rate toward Maximus. This rate is called P_s (Prætorian speed);
- Both Maximus and the Prætorian(s) will move toward each other until blocked (by Maximus in the case of the front Prætorian, or by another Prætorian if there's a lineup waiting to turn Maximus into pulp);
- The maximum number of Prætorians is limited by the size of the forum.

When Maximus meets a Prætorian. Fisticuffs ensue!

- Maximus and the Prætorians both start with a fixed health value, let's call this H (initial health).
- Maximus strikes on the Prætorians are another Poisson process, with rate M_{hr} (Maximus hit-rate) Prætorians strike Maximus also according to a Poisson process, with rate P_{hr} (Prætorian hit-rate)
- When Maximus hits a Prætorian, he causes damage in the amount $d = M_{hv} \cdot \frac{M_h}{H}$.

  That is, the damage is given by a parameter M_{hv} (Maximus hit-value) multiplied by M_h (Maximus' current health) divided by the initial (maximum) health. It's a tough world and Maximus' strikes get weaker the weaker he gets!
- The same is true for the Prætorians (with their own hit-value and corresponding health), but there's an additional factor here - it's Maximus we're talking about! He's trained in the arena and defeated the most skilled gladiators! Therefore, he is able to dodge many of the Prætorians' strikes!

  Prætorians land a blow on Maximus only with probability $1/M_{tf}$ where M_{tf} is Maximus' training factor. The better trained Maximus is, the harder it is for the Prætorians to actually strike him.
- If at any point the health of the Prætorian goes below zero, the soldier dies and Maximus can continue his progress along the Forum (at least until he meets the next Prætorian).

If at any point Maximus' health goes below zero, it's over and he'll have to get his Vengeance on the next life.

Happily for Maximus, as long as he's just walking (not fighting), his health regenerates at a rate given by M_{rr} (Maximus' health recovery rate). How much he'll recover depends on how far he can get before meeting his next foe!

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To help you visualize the task, here is some code you can run on Octave or Matlab.

- https://uoft.me/maximus

Task.

1. Using the code supplied above, find some values for which sometimes Maximus achieves vengeance in this life, and sometimes only in the next.

   Hint. To speed the simulation, you might want to disable the graphics.

2. Model the fight with deterministic processes instead of the Poisson processes.

3. Fix all values and find the **critical** training value for Maximus with the deterministic processes.

4. Model the actual fight.

Will Maximus make his way across the Forum and get his vengeance on Emperor Commodus? And more importantly…

ARE YOU NOT ENTERTAINED?????!!!!!!

"Maximus vs Commodus" is a collaboration with Francisco Estrada.

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