2018-10-22 Systems of ODEs with Complex Eigenvalues (3.4)

3.4 Complex eigenvalues

From video:

Linear Systems Complex Roots

Solve using matrix methods:

x' = -3x - 2y

y' = 5x - y

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 5 & -1 \end{pmatrix} \binom{x}{y}$$

Solve for eigenvalues:

$$\begin{vmatrix} -3-\lambda & -2 \\ 5 & -1-\lambda \end{vmatrix} = 0,\ (3+\lambda)(1+\lambda)+10=0\,,\ \lambda^2+4\lambda+13=0$$

$$\lambda_{\pm} = \frac{-4\pm\sqrt{16-4\cdot13}}{2} = \frac{-4\pm6i}{2} = -2\pm3i$$

$$\lambda_{+} = -2+3i,\ \begin{pmatrix} -1-3i & -2 \\ 5 & 1-3i \end{pmatrix} \binom{a_1}{a_2} = \binom{0}{0},\ (-1-3i)\,a_1 = 2a_2\,,\ v_{+}^{->} = \binom{2}{-1-3i}$$

$$x^{->} = e^{(-2+3i)t}\binom{2}{-1-3i} = e^{-2t}\left[(\cos(3t)+i\cdot sin\,(3t))\binom{2}{-1-3i}\right] =$$
$$e^{-2t}\left[\begin{pmatrix} 2\cos(3t) \\ -\cos(3t)+3\sin(3t) \end{pmatrix} + i\begin{pmatrix} 2\sin(3t) \\ -\sin(3t)-3\cos(3t) \end{pmatrix}\right]$$

$$x' = \begin{bmatrix} 0 & 1 \\ -k & -r \end{bmatrix} x$$

Consider no damping: k=5 and r = 0

1. Find one solution $\overrightarrow{x_1}(t)$ -- note: $\overrightarrow{x}(t)$ is different from the scalar component which is x_1

2. Write the solution in the form: $\overrightarrow{x_1}(t)$ = $\overrightarrow{u}(t)$+ i $\overrightarrow{v}(t)$ where $\overrightarrow{u}(t)$ and $\overrightarrow{v}(t)$ are real-valued.

[Euler`s Formula $e^{i\theta} = \cos(\theta) + i\ \sin(\theta)$]

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix} = \begin{bmatrix} 0-\lambda & 1 \\ -5 & 0-\lambda \end{bmatrix}$$

$$\det = \lambda^2 + 5 = 0$$

$$\lambda = \pm\sqrt{5}i$$

$$\lambda_+ = \sqrt{5}i, \begin{pmatrix} 0-\sqrt{5}i & 1 \\ -5 & 0-\sqrt{5}i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{5}i & 1 \\ -5 & -\sqrt{5}i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, -\sqrt{5}i \cdot v_1 + v_2 = 0$$

$$\lambda_+ = \sqrt{5}i \text{ , } v_+ = \begin{bmatrix} 1 \\ \sqrt{5}i \end{bmatrix}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$x_1(t)$: Position of the Mass

$x_2(t)$: String constant and damping value

$$x_1(t) = C_1 \begin{bmatrix} 1 \\ \sqrt{5}i \end{bmatrix} e^{\lambda_+ t} = C_1 \begin{bmatrix} 1 \\ \sqrt{5}i \end{bmatrix} e^{\sqrt{5}it}$$

$$= C_1 \begin{bmatrix} 1 \\ \sqrt{5}i \end{bmatrix} \left(\cos(\sqrt{5}t) + i\ (\sin(\sqrt{5}t))\right)$$

$$= C_1 \begin{pmatrix} \cos(\sqrt{5}t) + i\ \sin(\sqrt{5}t) \\ -\sqrt{5}\sin(\sqrt{5}t) + i\sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}$$

$$= C_1 \left(\underbrace{\begin{pmatrix} \cos(\sqrt{5}t) \\ -\sqrt{5}\sin(\sqrt{5}t) \end{pmatrix}}_{u(t)} + i \underbrace{\begin{pmatrix} \sin(\sqrt{5}t) \\ \sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}}_{v(t)} \right)$$

$$\vec{x}(t) = \vec{u}(t) - i\vec{v}(t)$$

General solution:

$\vec{x}(t) = C_1\vec{u}(t) + C_2\vec{v}(t)$, where C1 and C2 are real numbers

$$x^{->}(t) = C_1\binom{\cos(\sqrt{5}t)}{-\sqrt{5}\sin(\sqrt{5}t)} + C_2\binom{\sin(\sqrt{5}t)}{\sqrt{5}\cos(\sqrt{5}t)}$$

=> $x(t) = a_1 \begin{bmatrix} 1 \\ -\sqrt{5}i \end{bmatrix} e^{-\sqrt{5}it} + a_2 \begin{bmatrix} 1 \\ \sqrt{5}i \end{bmatrix} e^{\sqrt{5}it}$

Just plug in the eigenvectors and eigenvalues correspondingly into the formula for this type of solution.

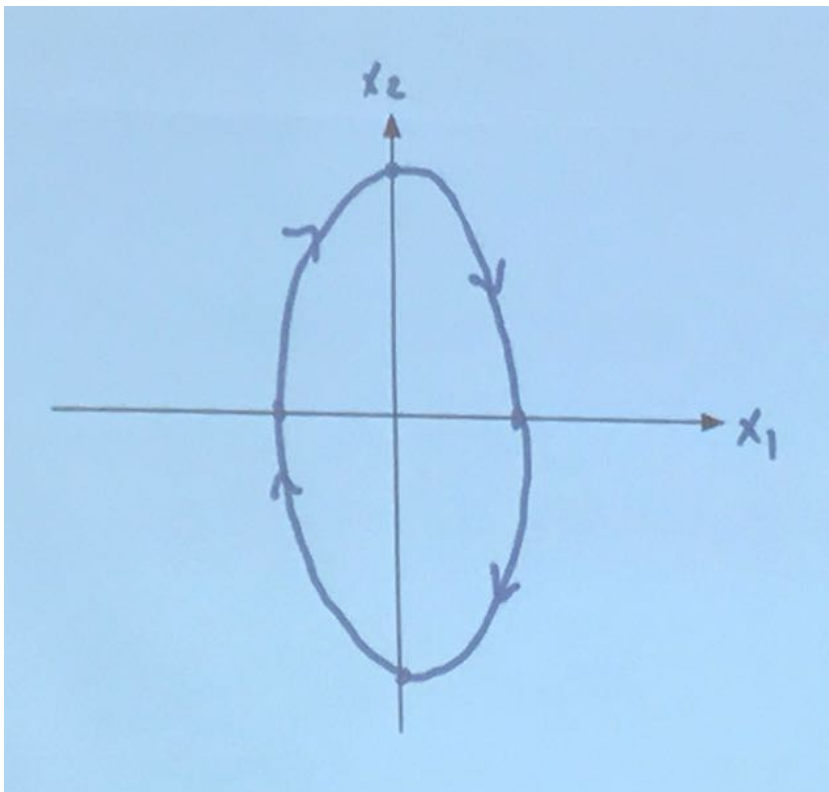
(This is what we know from the last lecture)

Above two general form are the same.

The general form can be defined as $f(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2$

3. Sketch some solutions in the phase plane.

Hint: If the constants are A,B, consider A = 1, B = 0.



- If A = 1/2, B = 0, the solution curve would be an ellipse that lies inside the graph above.
- If A = 3/2, B = 0, the solution curve would be an ellipse that lies outside the graph above.
- All the solution curves must be the same shape (ellipses), because by the Existence and Uniqueness Theorem, solution curves can never touch.
- The eigenvectors with complex eigenvalues would be represented as straight lines that pass through the origin, going either in or out of the page, and they do not change direction.
- The origin is called a center and it is stable, but not asymptotically stable because the trajectories of the solutions will not converge to the equilibrium point (the origin).