2018-10-16 Systems of ODEs with Real Eigenvalues (3.3)

3.3 (Real Eigenvalues)

Q. Consider a lion-cheetah example without "harvesting":

$\frac{dp}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} p$

Look for solutions that look like: $p^{->}(t) = v^{->} e^{rt}$

1. What problem is satisfied by v and r? --> v = eigenvector; r = eigenvalue

2. Find possible values for v and r.

3. What is the solution p(t)?

(Equilibrium in this case $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, equilibrium solution is the center of the solution.)

$$p(t) = v^{\rightharpoonup} e^{rt}$$

$$\frac{dp}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} p$$

$$\begin{bmatrix} 3-\lambda & -2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$(3-\lambda)(4-\lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 5)(\lambda - 2) = 0$$

$$\lambda = 5, \lambda = 2$$

$$\lambda_1 = 5\text{:} \begin{pmatrix} 3-5 & -2 \\ -1 & 4-5 \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$-y_1 - y_2 = 0$$

$$\lambda_1 = 5, v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2\text{:} \begin{pmatrix} 3-2 & -2 \\ -1 & 4-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}$$

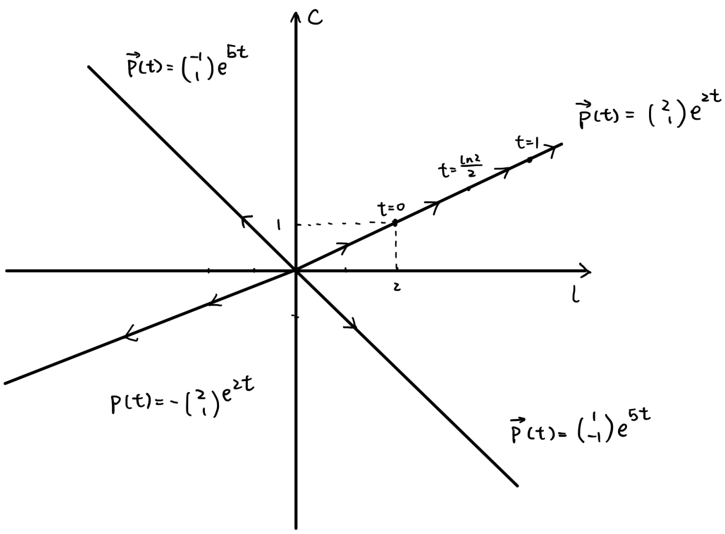
$$y_3 - 2y_4 = 0$$

$$\lambda_2 = 2,\ v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$p(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} c_1 e^{5t} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} c_2 e^{2t}$$

4. Sketch the solution for C1 = $\pm$ 1, C2 = 0; C1 =0, C2 = $\pm$ 1 in the phase plane.

The first component is always twice of the second component, no matter what t is.



NOTE:

Two equations that we should know:

ln(e) = 1

i=$\sqrt{-1}$, i^2 = -1

$$\vec{p} = A \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + B \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}$$

5. Sketch the solution for A = $\pm$1, B = $\pm$1 in the phase plane.

a. Plot eigenvectors

- **Note: Eigenvectors will always be a straight line.**

b. Plot easy points

- Use easy A and B values like 1 or -1. Plot the point t=0 then plot the point t = (ln2) / 2 and see which direction and eigenvector the solution moves toward

c. Plot easy solution curves

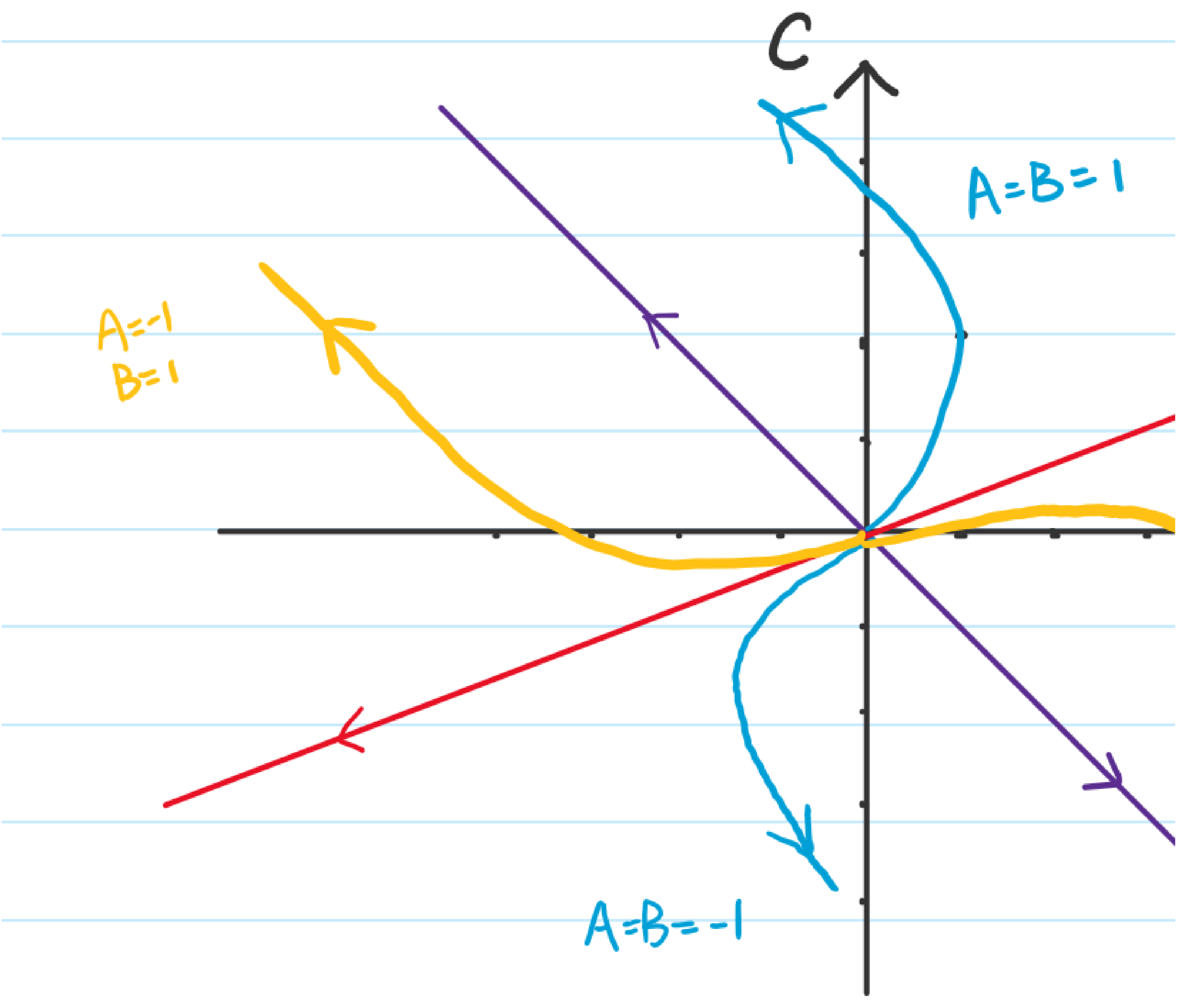
- Connect the plotted points. Note: A solution will start parallel to first eigenvector (that has an eigenvalue with a smaller magnitude) and end parallel to the other (that has an eigenvalue with a larger magnitude)

d. Plot more solution curves

- The eigenvectors create new 4 sections in the graph. It is enough to plot the easy solutions since **all solutions within a section will have the same behavior**.

e. Determine the direction of the solution curves

- The direction can be found by looking at the behaviour of the solution curves as t approaches infinity.



*Note: If |A| and |B| increase, the graph will grow closer to the straight lines. ($\binom{2}{1}$ and $\binom{-1}{1}$)