2018-11-15 Loan Repayment (2.1.2)

2.1.2 Loan Repayment

You just took a loan to buy a car. You'll need to make fixed payments every period, and the bank will charge an interest on the amount you still owe every period.

- D_k = amount of money owed to the bank after k periods
- $p\%$ = annual interest rate
- α = length of a payment/compounding period (in years)
- R = payment amount per period

1. Find an equation relating D_{k+1} with D_k.

**Rule for now: start with D_0 , D_1 , D_2 , . . . , D_k until you find the pattern

Defining α: α is not the length of a payment, $\alpha = \frac{1}{number\ of\ periods}$

Ex. monthly payments : α = $\frac{1}{12}$

Ex. period is 3 months : α = $\frac{1}{4}$

$D_{k+1} = D_k(1 + \frac{p\alpha}{100}) - R$- Payment at end of the month

$D_{k+1} = (D_k - R)(1 + \frac{p\alpha}{100})$- Payment at beginning of the month

2. Calculate D_1, D_2, D_3...in terms of D_0 until you find a pattern. What is D_k?

Note: Geometric series: $\sum_{n=0}^{k-1} ar^n = a(\frac{1-r^n}{1-r}),\ r \neq 1$

$$D_1 = D_0(1 + \frac{p\alpha}{100}) - R$$

$$\begin{aligned} D_2 &= (D_0(1 + \frac{p\alpha}{100}) - R)(1 + \frac{p\alpha}{100}) - R \\ &= D_0(1 + \frac{p\alpha}{100})^2 - R(1 + \frac{p\alpha}{100}) - R \end{aligned}$$

$$D_3 = D_0(1 + \frac{p\alpha}{100})^3 - R((1 + \frac{p\alpha}{100})^2 + (1 + \frac{p\alpha}{100}) + 1)$$

$$\begin{aligned} D_k &= D_0(1 + \frac{p\alpha}{100})^k - R((1 + \frac{p\alpha}{100})^{k-1} + (1 + \frac{p\alpha}{100})^{k-2} + \ldots + 1) \\ &= D_0(1 + \frac{p\alpha}{100})^k - R\sum_{n=0}^{k-1}(1 + \frac{p\alpha}{100})^n \end{aligned}$$

We can simplify this:

$$D_k = D_0(1 + \frac{p\alpha}{100})^k - R(\frac{1-(1+\frac{p\alpha}{100})^k}{1-(1+\frac{p\alpha}{100})})$$

$$D_k = D_0(1+\frac{p\alpha}{100})^k + \frac{100R}{p\alpha}(1-(1+\frac{p\alpha}{100})^k)$$

$$D_k = (D_0 - \frac{100R}{p\alpha})(1+\frac{p\alpha}{100})^k + \frac{100R}{p\alpha}$$

$$D_1 = (D_0 - R)(1+\frac{p\alpha}{100})$$

$$D_2 = ((D_0 - R)(1+\frac{p\alpha}{100}) - R)(1+\frac{p\alpha}{100})$$
$$= (D_0 - R)(1+\frac{p\alpha}{100})^2 - R(1+\frac{p\alpha}{100})$$

$$D_3 = [[(D_0 - R)(1+\frac{p\alpha}{100})^2 - R(1+\frac{p\alpha}{100})] - R](1+\frac{p\alpha}{100})$$
$$= (D_0 - R)(1+\frac{p\alpha}{100})^3 - R(1+\frac{p\alpha}{100})^2 - R(1+\frac{p\alpha}{100})$$

$$D_k = (D_0 - R)(1+\frac{p\alpha}{100})^k - R((1+\frac{p\alpha}{100})^{k-1} + (1+\frac{p\alpha}{100})^{k-2} + \ldots (1+\frac{p\alpha}{100}) + 1)$$
$$= (D_0 - R)(1+\frac{p\alpha}{100})^k - R\sum_{n=0}^{k-1}(1+\frac{p\alpha}{100})^n + R$$
$$= (D_0 - R)(1+\frac{p\alpha}{100})^k - R(\frac{1-(1+\frac{p\alpha}{100})^k}{1-(1+\frac{p\alpha}{100})}) + R$$

$$D_k = (D_0 - R)(1+\frac{p\alpha}{100})^k + \frac{100R}{p\alpha}(1-(1+\frac{p\alpha}{100})^k) + R$$

Conclusion. The results of D_k depends on whether the payment is made at the end or at the beginning of the month.