2018-11-19 Loan Repayment (2.1.2)

2.1.2 Loan Repayment

You just took a loan to buy a car. You'll need to make fixed payments every period, and the bank will charge an interest on the amount you still owe every period.

- D_k = amount of money owed to the bank after k periods
- $p\%$ = annual interest rate
- α = length of a payment/compounding period (in years)
- R = payment amount per period

1. Find an equation relating D_{k+1} with D_k.

**Rule for now: start with D_0 , D_1 , D_2 , ... , D_k until you find the pattern

Defining α: α is not the length of a payment, $\alpha = \frac{1}{number\ of\ periods}$

Ex. monthly payments : α = $\frac{1}{12}$

Ex. period is 3 months : α = $\frac{1}{4}$

$D_{k+1} = D_k(1+\frac{p\alpha}{100}) - R$- Payment at end of the month (Interest is applied before making payment)

$D_{k+1} = (D_k - R)(1+\frac{p\alpha}{100})$- Payment at beginning of the month (Payment is made before interest is applied)

2. Calculate D_1, D_2, D_3...in terms of D_0 until you find a pattern. What is D_k?

$D_1 = D_0(1+\frac{p\alpha}{100}) - R$

$D_2 = (D_0(1+\frac{p\alpha}{100}) - R)(1+\frac{p\alpha}{100}) - R = D_0(1+\frac{p\alpha}{100})^2 - R(1+\frac{p\alpha}{100}) - R$

$D_3 = D_0(1+\frac{p\alpha}{100})^3 - R((1+\frac{p\alpha}{100})^2 + (1+\frac{p\alpha}{100}) + 1)$

$D_k = D_0(1+\frac{p\alpha}{100})^k - R((1+\frac{p\alpha}{100})^{k-1} + (1+\frac{p\alpha}{100})^{k-2} + \ldots + 1) = D_0(1+\frac{p\alpha}{100})^k - R \sum_{n=1}^{k}(1+\frac{p\alpha}{100})^{k-n}$

$D_1 = (D_0 - R)(1+\frac{p\alpha}{100})$

$D_2 = ((D_0 - R)(1+\frac{p\alpha}{100}) - R)(1+\frac{p\alpha}{100}) = D_0(1+\frac{p\alpha}{100})^2 - R((1+\frac{p\alpha}{100})^2 + (1+\frac{p\alpha}{100}))$

$D_3 = D_0(1+\frac{p\alpha}{100})^2 - R((1+\frac{p\alpha}{100})^2 + (1+\frac{p\alpha}{100}) + 1)$

$D_k = D_0(1+\frac{p\alpha}{100})^k - R\sum_{n=0}^{k}(1+\frac{p\alpha}{100})^{k-n}$ OR

Using geometric series:

$\sum_{m=0}^{k-1} r^m = \frac{1-r^k}{1-r}$

$then, D_k = D_0(1+\frac{p\alpha}{100})^k + \frac{R}{\frac{p\alpha}{100}}[1-(1+\frac{p\alpha}{100})^m]$

<u>Conclusion.</u> we can see that no matter equation we use, the result of D_k is the same.

3. What is an equilibrium solution D_{eq}?

To get an equilibrium solution D_{eq}, D_k must be constant. This means $D_0 = D_1 = D_2 = D_3$ and so on, and $D_{k+1} = D_k$ for all values of k.

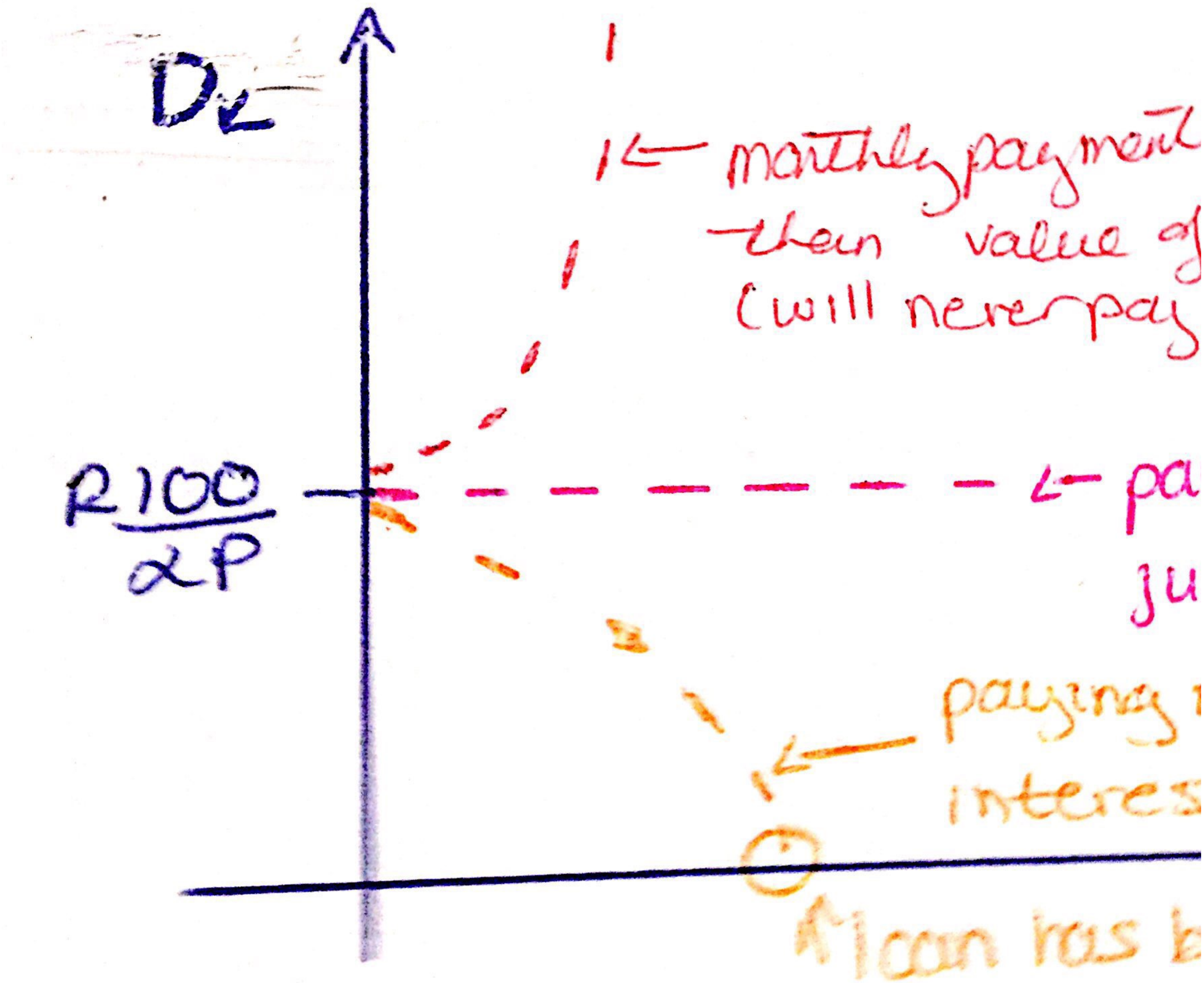
$$D_{eq} = D_{k+1} = D_k$$
$$D_{k+1} = D_k\left(1+\frac{p\alpha}{100}\right) - R$$
$$D_k = D_k\left(1+\frac{p\alpha}{100}\right) - R$$
$$\frac{D_k}{D_k} = \frac{D_k}{D_k}\left(1+\frac{p\alpha}{100}\right) - \frac{R}{D_k}$$
$$1 = 1 + \frac{p\alpha}{100} - \frac{R}{D_k}$$
$$\frac{R}{D_k} = \frac{p\alpha}{100}$$
$$D_k = D_{eq} = R \cdot \frac{100}{p\alpha}$$

The equilibrium solution is when $D_{eq} = R \cdot \frac{100}{p\alpha}$. If we owe an amount of money equal to $R \cdot \frac{100}{p\alpha}$, the amount will always remain constant and we will never pay off the loan.

$RearrangeSolution: D_k = (D_0 - R\frac{p\alpha}{100})(1+\frac{p\alpha}{100})^k + \frac{R}{\frac{p\alpha}{100}}$

$Notice, = const * (1+\frac{p\alpha}{100})^k + equilibriumSoln$

4. Sketch a Graph for some possible outcomes of Dk



Graph can be drawn by studying the solution.
$red : D_0 > equilibrium, and Orange : D_0 < equilibrium$

increase in D_0, the farther the orange line will go before meeting the x-axis (meaning it takes longer to pay back loan)

Extra homework problems:

$$D_k = \left(D_0 - \frac{100R}{p\alpha}\right)\left(1 + \frac{p\alpha}{100}\right)^k + \frac{100R}{p\alpha}$$

5. If $D_0 = \$20,000.00,\ p = 20\%,\ \alpha = \frac{1}{12}$, then what is the monthly payment R so that the loan will be paid off in 5 years?

$$k = 5 \times 12 = 60,\ D_{60} = 0 = \left(20000 - \frac{100}{20\cdot\frac{1}{12}}R\right)\cdot\left(1 + \frac{20\cdot\frac{1}{12}}{100}\right)^{60} + \frac{100}{20\cdot\frac{1}{12}}R$$

$$R \approx \$530$$

6. If the monthly payment is R = $1000.00, how many periods does it take to payoff the loan?

$$D_k = 0 = \left(20000 - \frac{100\cdot1000}{20\cdot\frac{1}{12}}\right)\cdot\left(1 + \frac{20\cdot\frac{1}{12}}{100}\right)^k + \frac{100\cdot1000}{20\cdot\frac{1}{12}}$$

$$k = 24.53\,months\ \approx 2.044\,years$$