Inquiry Based Modelling with Differential and Difference Equations

About the Document

This document is a mix of student resources, student projects, problem sets, and labs. A typical class day looks like:

1. **Preparation by students.** Students prepare for lecture by watching a short video and solving a short quiz.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in small groups on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.
5. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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https://github.com/bigfatbernie/IBLmodellingDEs

Contents

In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

A. Define the problem

B. Build a mind map

C. Make assumptions

D. Construct a model

E. Analysis of the model

Defining Problem Statement

Objectives

- The first step in Mathematical modelling is to define the problem
- A good way to do this is to figure out what is the "mathematical object" we are looking for at the end of the process
- The second step is to create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step A. Defining the problem

The first step is to define the problem we want to solve or improve.

To do this, we should start from the end! We need to decide on what kind of mathematical object we will use in the end to show that we solved, or at least improved, the problem we were tasked with.

Once this is done, we can define the problem mathematically.

EXAMPLE

Your team was tasked with optimizing the layout of an airport.

You decided that in the end, to show that the team did find a good layout fo an airport, they will show

- T = the total time (in minutes) necessary by the average person to walk from their airport transportation (taxi, train, bus) to their gate.

Once this decision is made, the problem to solve (or improve) becomes the following:

- Minimize T

There will probably be some constraints, which will be studied in Step C.

Make the question precise, bring it into a "mathematical form".

- Choose a mathematical object best suited for the problem, e.g. a number, a geometric form, a graph, a function, an algorithm, ...

Task 1.A: Elevator problem at theBigCompany

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

> ——— Forwarded Message ———
>
> Date: Mon, 16 September 2019 21:41:35 + 0000
> From: CEO <theCEO@theBigCompany.ca>
> To: Human Resources <hr@theBigCompany.ca>
> Subject: they're still late !?&!
>
> Hey Shophika!
>
> I still get complaints about staff being late, some by 15 minutes.
> With the staff we have, that's about one salary lost.
> Again the bottleneck of the elevators seems to be the problem.
> Can you suggest solutions?
>
> Thanks, the CEO

Notes/Misconceptions

- Students will start discussing how to solve the problem
- This question deals with what will happen **after** solving the problem
- The goal of this question is to think about how to best tell a "mathematically-challenged" CEO that you solved the problem

Scenario 1: With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.

Task. What mathematical object would you use to convince the CEO that you have solved or improved the problem?

1 For each part, what “mathematical object” would you use to communicate that you have solved or improved the problem? Then define the problem mathematically.

1.1 Help the city of Toronto choose the best recycling centre.

1.2 Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

1.3 Create a mathematical model to rank roller coasters according to thrill factor.

1.4 Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should “best” mean?

1.5 The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1. Is it optimal?

1.6 Is it better to buy or rent?

(a) Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

(b) Does the criteria you used to evaluate the previous question change if the question is whether to buy a bicycle or use Bike Share Toronto?

Figure 1: Extension plans for Toronto subway line.

Step B. Building a mind map

A mind map is a tool to visually outline and organize ideas. Typically a key idea is the centre of a mind map and associated ideas are added to create a diagram that shows the flow of ideas. In Figure 2, we focus on the definition of "best", with three possible definitions branching off to be further explored. From here, we can focus our attention on one of the three branches at a time.

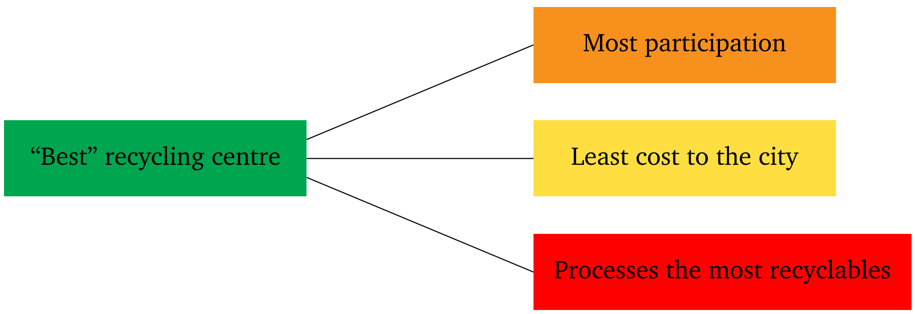

Figure 2: An example of a simple mind map.

Let's think about the least-cost option first. We probably can't determine how much any recycling program costs without knowing more about the recycling program, so a good place to start is to ask the question "What kinds of recycling programs exist?" If we aren't familiar with different types of recycling, we might need to do some research to see what kinds of programs exist.

A possible next step on your mind map for the least-cost approach could be the one shown in Figure 3.

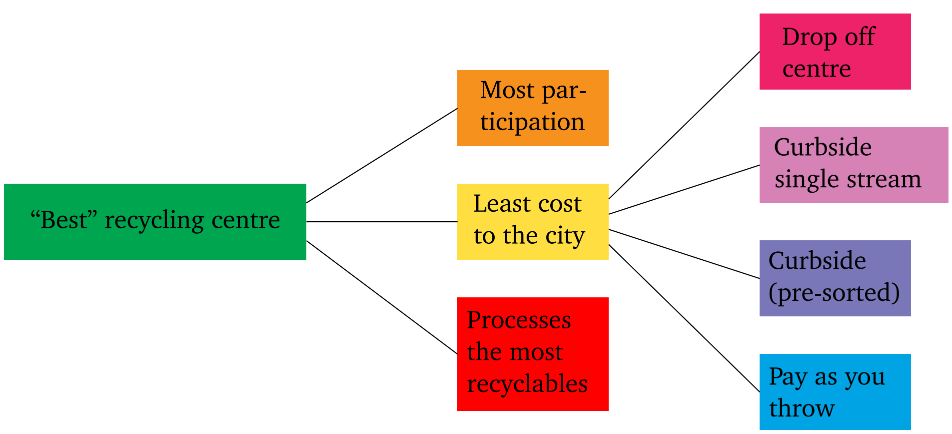

Figure 3: Next step of a mind map.

Software

There is free online software to help creating a mind map. One such is FreeMind.

Task 1.B: Elevator problem at theBigCompany

Scenario 2: Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $R =$ the sum in minutes by which every employee is late.

Employees that are on time count for 0 minutes.

Task. Create a mind map for the question: How can R be minimized?

Notes/Misconceptions

- Students usually come up with more complicated variations:
  - Money spent on late employees' salaries
  - sum of time in minutes that employees are late counting only employees that are at most 15 minutes late
- Stick with R, a simple first approach

2 For each part, create a mind map. Focus on the same approach you had for question 1.

2.1 Help the city of Toronto choose the best recycling centre.

2.2 Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

2.3 Create a mathematical model to rank roller coasters according to thrill factor.

2.4 Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

2.5 The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1. Is it optimal?

2.6 Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

Module 2

Making Assumptions

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

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Step C. Making Assumptions

Real problems are complex, so when modelling a real problem mathematically, we must make some assumptions.

The assumptions that we make will affect the problem we are solving and its difficulty, so we need to strike a balance between:

- accuracy – the fewer assumption the better, and
- solvability – the more assumptions the better.

Many assumptions follow naturally when building a mind map.

When building a mind map, keep track of the assumptions necessary for each step.

When figuring which assumption to make, keep in mind the key-factors of the problem and find data when available (usually online), if not available, measure data when possible, and if it's not possible, make a reasonable assumption on what the data might look like.

Remember to justify all your assumptions.

Another thing to keep in mind are *time constraints*. Whether in a class, test, or working in a project, there will be deadlines. Your assumptions should take time constraints into consideration.

Task 1.C: Elevator problem at theBigCompany

We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie (in Montréal).
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Tasks. Focus on a **few** parameters and variables. State hypotheses.

1. With your team, decide on what kind of information you would need to have to be able to solve this problem.
2. Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.
3. For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Notes/Misconceptions

- Students usually have trouble starting.
- They usually agree that they have to figure out how elevators work, so you can prompt them to be more specific.
- In the end they should come up with questions like these:
  - How fast are the elevators?
  - How much time do elevators take in each floor?
  - How many floors do elevators stop on their way up?
  - How many people fit in the elevator?
  - Should we consider elevator failures?

3 For each part, you are required to make an estimate for some quantity. Make assumptions and justify them in order to solve the problem.

3.1 What is the number of piano players in Toronto? *(Fermi problem)*

3.2 How many linear km of roads are there in Toronto?

3.3 How much salt the city of Toronto needs for its roads during the Winter?

3.4 The skating season in Canada is shortening: What are the key-factors determining its length?

Building Solutions

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step D. Construction of a Model

This is the part of the modelling where we connect all that we have done so far: the problem we defined, the mind map, the assumptions, and all the variables and parameters in a mathematical model to answer the "mathematical" problem defined in Step A.

This usually means writing down mathematical equations, constructing a graph, analyzing a geometric figure, or do some statistical analysis.

EXAMPLE

Your team is tasked with finding the best recycling centre (we looked at this example in Step B) and your team has chosen to minimize the cost to the city by using drop off centres.

As part of modelling process, your team has made the following assumptions/measurements:

- People would be willing to pay \$2.29 to recycle per month or \$0.53 per week
- People would make bi-weekly trips to the centre
- Gasoline costs around \$1.26 per litre
- On average a passenger car needs 10 litres per hundred kilometres

This means that the (one-way) distance people are willing to travel every week to the drop-off centre is

$$d = \frac{1}{4.3 \text{ trips/month}} \cdot \frac{\$2.29\text{/month}}{(\$1.26\text{/L}) \cdot (0.1 \text{ L / km})} = 4.2 \text{ km/trip}.$$

This should help us figure out the best way to place the drop-off centres:

The Mathematical model might look like this

- Maximize (number of people within a 4.2 km radius of a drop-off centre)
- subject to a certain number of drop-off centres (given by the city budget)

Sometimes, the mathematical tools necessary to tackle the problem are clear, but often they are not. In those cases it may be helpful to analyze some simple cases.

Task 1.D: Elevator problem at theBigCompany

With the same details as before in step C, write down a mathematical model for this problem.

4 For each part, create a model to answer the question. Remember all the previous steps.

4.1 You want to open a piano store in Toronto, where should you open it?

4.2 There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

4.3 The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

4.4 Loblaws wants to start an online food delivery service. How should they do it?

4.5 The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

Module 4

Model Assessment

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step E. Analysis of the Model

At this point, you have defined a problem statement, and a mind map to help you decide how to approach the problem. You have made assumptions and made note of them and justified them. You finally created a model to solve the problem.

The next step is to analyze the model.

There are two types of analysis:

Superficial assessment. Are the units correct? Are the variables and parameters of a reasonable magnitude? Does it behave as expected? Does it make sense?

In-depth assessment. Once the superficial assessment is verified, we need to understand the model at a deeper level.

What are the model's strengths? What are its weaknesses?

When you change the inputs of the model, how do the outputs change? This is called *s*ensitivity analysis.

Next is a simple example adapted from [**?**].

Desmos Graph

Modelling the flu

EXAMPLE

History of the project:

- Split population into two classes: ***infected*** and ***not infected***
- Assume that each infected person infects R number of non infected people every b days
- Define $I(n) =$ number of infected people after n days
- The two previous points imply $\quad I(n \cdot b) = R \cdot I(n)$
- We can then conclude that $\quad I(nb) = (1+R)^n I(0)$ (why?)

After plotting the resulting function $I(n)$ (click or follow the QR code on the right), we can assess our model:

Strengths:

- After two days ($b = 2$), there are 6 infected people, so it is following our assumption
- The number of infected people increases faster and faster as expected
- The disease spreads at a constant rate. Also on Desmos, check the infection rate $\dfrac{I(n+b)}{I(n)}$
- We could find an explicit formula for the number of infected individuals $I(n)$

Weaknesses:

- The model is too simple, so it doesn't model the spread of the flu accurately
- The model an exponential rate of infection, which is not possible for very long
- The model predicts that eventually the disease will spread to everyone
- The model assumes that there are only two types of people: infected and susceptible. Do people recover from the disease?

After assessing the model, if time allows, it is important to re-think the model and the assumptions made.

Task 1.E: Elevator problem at theBigCompany

Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floow of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\text{loading time on ground floor}} + \boxed{\text{time of flight ground} \to 33} + \boxed{\text{time of flight } 33 \to \text{ground}} + \boxed{\text{stop time to 6 of the 11 floors}} \quad = \quad 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes

Your task is to assess this model. Be ready to report on your assessment.

Notes/Misconceptions

Some questions to guide the students:

- What are the strengths?
- What are the weaknesses?
- Is the result around what you expected?

In case students don't realize that something is wrong:

- People start arriving 30 minutes before the starting time, so *everybody will be on time?*
- Assume that the CEO of the BigCompany is right: people are arriving late! What's wrong with the model?
- Which assumptions should be relaxed? Or checked?
- If one needs to be replaced, by what?

5 Assess the models created in question 4:

5.1 You want to open a piano store in Toronto, where should you open it?

5.2 There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

5.3 The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

5.4 Loblaws wants to start an online food delivery service. How should they do it?

5.5 The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

Putting it all together

Textbook

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Objectives

- Bla bla bla

Motivation

Step F. Writing a report

Definitions

Textbook

Objectives

- Bla bla bla

Motivation

Module 7

Solutions

Objectives

- Bla bla bla

Motivation

Module 8

Slope Fields

Objectives

- Bla bla bla

Motivation

Higher-Order Models

Discrete Models

