2018-11-08 Method of Undetermined Coefficients (4.5)

Method of undetermined coefficients works when g(t) is one of these:(only works for functions that doesn't change type when differentiate it)

e^{rt} (exponentials)

$sin(t)$ or $cos(t)$ (trig)

$ct^p + \ldots + bt^a$ (Polynomial)

Example 1: $y'' - 4y = 10e^{3t}$

4. **What is the particular solution y(t)?**
(Since y and its second derivative y" has subtraction equals to an exponential equation, meaning differentiation does not change its type, we guessed y should be in exponential form)

- We pick $y_p(t) = Ce^{3t}$ for any constant C

$$y_p'(t) = 3Ce^{3t}$$

$$y_p''(t) = 9Ce^{3t}$$

$$9Ce^{3t} - 4\left(Ce^{3t}\right) = 10e^{3t}$$

$$5Ce^{3t} = 10e^{3t}$$

Compare coefficients: 5C = 10, therefore C = 2

$$\therefore y_p(t) = 2e^{3t}$$

5. **What is the general solution y(t)?**

$$y(t) = y_c(t) + y_p(t)$$

$$\therefore y(t) = C_1 e^{2t} + C_2 e^{-2t} + 2e^{3t}$$

- y(0) = 0; C_1 + C_2 = -2
- **NOTE:** you can only solve for the contants C_1, C_2, once you have the full general solution. If you try to solve the constants by only using $y_c(t)$, you would get different constants than if you included the

particular solution $y_p\left(t\right)$.
- without $y_p\left(t\right)$ = $2e^{3t}$, $y\left(0\right) = C_1 + C_2 = 0$
- with $y_p\left(t\right)$ = $2e^{3t}$, $y\left(0\right) \ = \ C_1 + C_2 + 2 = 0 \Longrightarrow -2 = C_1 + C_2$

As shown, you would get different constants if you solve the general solution before and after adding the particular solution $y_p\left(t\right)$.

6. **What is the complementary solution y(t)?**

$$y'' \ - \ 4y \ = \ 0$$

$$y = e^{rt}, \ y' = re^{rt}, \ y'' = r^2 e^{rt}$$

$$(r^2 - 4) = 0$$

$$r = +2, -2$$

using formula: $c_1 e^{r_1 t} + c_2 e^{r_2 t}$, plugging in r values, we get the complementary solution y(t).

$$y_c\left(t\right) = c_1 e^{2t} + c_2 e^{-2t}$$

- **NOTE:** terms of the complementary solution will always be equal to 0 no matter what C1, C2, are

Example 2: $y'' - 4y \ = \ -e^{2t}$

7. **What is the particular solution y(t)?**

Try $y_p = -Ae^{2t}$:

$y_p = -Ae^{2t}$

$y_p'' = -4Ae^{2t} \, y_p' = -2Ae^{2t}$

$$y'' - 4y \ = \ -e^{2t}$$

$$-4Ae^{2t} - 4\left(-Ae^{2t}\right) = -e^{2t}$$

$$-4Ae^{2t} + 4Ae^{2t} \ = -e^{2t}$$

$$0 = -e^{2t}$$

(this doesn't make any sense, so $y_p = -Ae^{2t}$ is not the particular solution)

Note: (If your initial guess was an exponential without the extra t term, it would have cancelled out). This is because $\left(c_1 e^{2t}\right)'' - 4\left(c_1 e^{2t}\right)$ is ALWAYS = 0, regardless of what value for c_1 is inputted. (Always solve for complementary solution first and compare with particular solution)

Now we should try $y_p\left(t\right) = Ate^{2t}$:

$$y' = Ae^{2t}(2t + 1)$$

$$y'' = Ae^{2t}\left(4t+4\right)$$

plugging into $y'' - 4y = -e^{2t}$ and solving for A:

$$A = -\tfrac{1}{4}$$

=>

$$y_p\left(t\right) = -\tfrac{1}{4}te^{2t}$$

8. **What is the general solution y(t)?**

$$y\left(t\right) = c_1 e^{2t} + c_2 e^{-2t} - \tfrac{1}{4}te^{2t}$$

NOTE: Why should we multiply $y_p\left(t\right)$ by t?
To understand if we should multiply the predicted y_p(t) by t without trying to solve for not having them cancel out, look for the y_p(t) you are giving. If it is already in the complementary solution, you should multiply particular solution by t at the first sight. In this case, we have $C_1 e^{2t}$ in $y_c\left(t\right)$ and $-e^{2t}$ on the right side, so $y_p = -Ae^{2t}$ cannot be the particular solution and we multiply the particular solution by t to obtain $y_p\left(t\right) = Ate^{2t}$.

Example 3: $y'' + y' - 6y = \sin(t)$

9. **What is the particular solution yp(t)?**

$$y(t) = Asin(t) + Bcos(t)$$

$$y'\left(t\right) = A\cos(t) - B\sin(t)$$

$$y''\left(t\right) \ = \ -A\sin(t) - B\cos(t)$$

NOTE: If we only use y(t) = Asin(t), it will not work as we get both sin and cos terms when we differentiate once and twice. So, we use both sin and cos (they always come together, this should be your first attempt)

For reference, first attempt:

$y\left(t\right) = A\sin(t)$

$y'\left(t\right) = A\cos(t)$

$y''\left(t\right) = -A\sin(t)$

- $A\sin(t) + A\cos(t) - 6\sin(t) = \sin(t)$

$A\cos(t) = 6\sin(t)$ (Not possible to solve)

Overview of Method of Undetermined Coefficients: ay'' + by' + cy = g(t)

1) Find the general solution of the corresponding homogeneous equation

2) Make sure g(t) belongs to the class of functions listed above

3) If $g(t) = g_1(t) + g_2(t) + ... + g_n(t)$, then form n sub-problems, each containing only one of these terms. Find the particular solution for each of these sub-problems and the particular solution for the full non-homogeneous equation is the sum of all the particular solutions of each sub-problem.

4) The general solution for the non-homogeneous equation is the sum of the general solution for the homogeneous equation (1) and the particular solution (3).