2018-10-11 Systems of two ODEs (3.2)

3.2 Systems of two ODEs

Systems of linear ODEs: $x^{->'} + p(t)\, x^{->} = g^{->}(t)$

- p(t) is a matrix with functions of t as components.
- $g^{->}(t)$ is a vector-valued function.

We will start with simpler systems of the form: $x^{->'} = Ax^{->} + b^{->}$

Educated guess: $x^{->}(t) = v^{->} e^{rt}$

- $v^{->}$ is a constant vector.
- r is a real number and is an eigenvalue of the matrix A

From video:

Linear Systems: Matrix Method

Solve the system using the matrix method:

- $x' = 6x + 5y$
- $y' = x + 2y$

$$\binom{x'}{y'} = \begin{pmatrix} 6 & 5 \\ 1 & 2 \end{pmatrix} \binom{x}{y}$$

Solve for eigenvalues:

$$\begin{vmatrix} 6-\lambda & 5 \\ 1 & 2-\lambda \end{vmatrix} = 0,\ (6-\lambda)(2-\lambda) - 5 = 0,\ \lambda^2 - 8\lambda + 7 = 0,\ \lambda_1 = 1,\ \lambda_2 = 7$$

Solve for eigenvectors:

$\lambda_1 = 1$, $\begin{pmatrix} 6-1 & 5 \\ 1 & 2-1 \end{pmatrix}\binom{a_1}{a_2} = \binom{0}{0}$, $\begin{pmatrix} 5 & 5 \\ 1 & 1 \end{pmatrix}\binom{a_1}{a_2} = \binom{0}{0}$, $a_1 + a_2 = 0$, $v_1^{->} = \binom{1}{-1}$

$\lambda_2 = 7$, $\begin{pmatrix} 6-7 & 5 \\ 1 & 2-7 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \binom{0}{0}$, $\begin{pmatrix} -1 & 5 \\ 1 & -5 \end{pmatrix}\binom{a_1}{a_2} = \binom{0}{0}$, $-a_1 + 5a_2 = 0$, $v_2^{->} = \binom{5}{1}$

Solution to the original system: $x^{->} = c_1 v_1^{->} e^{\lambda_1 t} + c_2 v_2^{->} e^{\lambda_2 t}$ = $c_1 \binom{1}{-1} e^t + c_2 \binom{5}{1} e^{7t}$

Consider two competing populations:

- **Lions l(t)**
- **Cheetahs c(t)**

Properties of their populations:

- In the absence of cheetahs,
  - a. Logistic Growth: $l'(t) \propto l\left(1 - \frac{l}{k}\right)$ or $l(l-k)$
  - b. Simpler: $l'(t) \propto l$
- If there are a lot of cheetahs,
  - a. $l'(t) \leq 0$ or
  - b. $l'(t) \propto l - c$
  - How about $l'(t) \propto \frac{1}{c}$? We do not choose it because the population of cheetahs, c, will always be positive => $\frac{1}{c}$ will always be positive => $l'(t)$will always be positive => the population of lions will always increase, which is not realistic.
  - How about $l'(t) \propto l - c'$? We do not choose it because the population of cheetahs, c, is large does not imply c' is large. The population of cheetahs can get very large and still increasing.
- In the absence of lions, c'(t) $\propto$ c
- If there are a lot of lions, c'(t) $\propto$ c-l

1. Obtain a DE for l(t) and one for c(t).

- For l(t):
  - $\frac{dl}{dt} = kl(t) - mc(t) = (k-m)l + m(l-c)$, k and m are different growth constants. We choose this formula instead of the one below because of the different constraints in the growth rates. This formula is actually simpler and allows us to analyze more, since it's possible that k=m.
  - How about $\frac{dl}{dt} = k(l-c) = kl - kc$? We do not choose it because lions and cheetahs are different, so they have different growth rates.Having k and m makes the equation flexible because k could be equal to m which is when $\frac{dl}{dt} = kl(t) - mc(t)$ = $\frac{dl}{dt} = k(l-c) = kl - kc$

- For c(t):
  - $\frac{dc}{dt} = ac - bl$, a and b are two arbitrary growth constants. (a and b can be the same)

*Note: a may not be equal to k and b not equal to m because it might take more cheetahs to decrease the population of lions and fewer lions to decrease the population of cheetahs.

2. If we include a fixed amount of “harvesting” every year, what is the new system of ODEs?

Suppose the fixed amount of "harvesting" is 11 for both lions and cheetahs.

$\{l' = kl - mc$

$\{c' = ac - bl$

Options:

a)

$\{l' = k(l-11) - m(c-11)$

$\{c' = a(c-11) - b(m-11)$

Expand: $\{l' = kl - 11k - mc + 11m$; The units would not match: (11 cheetahs)*(cheetah population) / per year ?? NO!

b)

$\{l' = k(l-11) - mc$

$\{c' = a(c-11) - bl$

We have to remove 11 from both populations in each rate of change

c)

$\{l' = kl - mc - 11$

$\{c' = ac - bl - 11$

YES! Shows we are removing 11 lions per year, and 11 cheetahs per year

we obtain $l' = kl - mc - 11$ **.**

Note: we assume continuous removal rate. (ie: at t=$\frac{1}{365}$ we have l + $\frac{1}{365}$ lions)

Lecture ended.