2018-10-23 Systems of ODEs with Complex Eigenvalues (3.4)

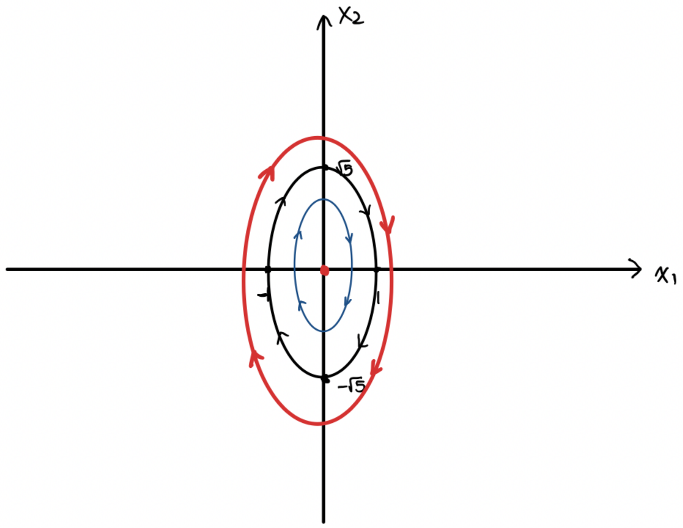
Continue from the last lecture.

3. Sketch some solutions in the phase plane

$$x\,(t) = A\begin{pmatrix}\cos(\sqrt{5}t)\\ -\sqrt{5}\sin(\sqrt{5}t)\end{pmatrix} + B\begin{pmatrix}\sin(\sqrt{5}t)\\ \sqrt{5}\cos(\sqrt{5}t)\end{pmatrix}$$

Hint: If the constants are A, B, consider A = 1, B = 0.

$$x\,(t) = \begin{pmatrix}\cos(\sqrt{5}t)\\ -\sqrt{5}\sin(\sqrt{5}t)\end{pmatrix}$$



It is ellipse shaped and centre stable.

NOTE : If we sketch A = 1/2, B=0, it will be the same shape i.e an ellipse but just smaller. Similarly, if we sketch A = 3/2, B=0, we will get a bigger ellipse.

<u>Today's lecture</u>

Consider some damping: k=5, r=2

$$x' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} x$$

1. Find one solution $\overrightarrow{x_1(t)}$
2. Write the solution in the form: $\overrightarrow{x_1(t)}$ = $\overrightarrow{u(t)}$+ i $\overrightarrow{v(t)}$ where $\overrightarrow{u(t)}$ and $\overrightarrow{v(t)}$ are real-valued.

   And general solution is of the for $\overrightarrow{x_1(t)}$ = A$\overrightarrow{u(t)}$+ B $\overrightarrow{v(t)}$

Solution:

$$\begin{pmatrix} 0-\lambda & 1 \\ -5 & -2-\lambda \end{pmatrix} = (-\lambda)(-2-\lambda)+5$$

$$\det = \lambda^2+2\lambda+5 = 0,\ (\lambda+1)^2+4=0,\ \lambda+1=\pm\sqrt{-4}$$

$$\lambda=-1\pm 2i$$

$$\lambda_1 = 2i-1,\ \begin{pmatrix} 0-2i+1 & 1 \\ -5 & -2-2i+1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1-2i & 1 \\ -5 & -1-2i \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$(1-2i)\cdot y_1+y_2=0$$

$$\lambda_1 = 2i-1,\ V_1 = \begin{bmatrix} 1 \\ 2i-1 \end{bmatrix}$$

$$x_1(t) = \begin{bmatrix} 1 \\ 2i-1 \end{bmatrix} e^{(2i-1)t}$$

$$= \begin{bmatrix} 1 \\ 2i-1 \end{bmatrix} e^{-t}(\cos 2t + i \sin 2t) = \begin{bmatrix} \cos 2t + i \sin 2t \\ 2i\cos 2t - 2\sin 2t - \cos 2t - i\sin 2t \end{bmatrix} e^{-t}$$

$$= \left(\begin{bmatrix} \cos 2t \\ -2\sin 2t - \cos 2t \end{bmatrix} + i\begin{bmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix}\right) e^{-t}$$

$$x(t) = A\begin{bmatrix} \cos 2t \\ -2\sin 2t - \cos 2t \end{bmatrix} e^{-t} + B\begin{bmatrix} \sin 2t \\ 2\cos 2t - \sin 2t \end{bmatrix} e^{-t}$$

3. Sketch some solutions in the phase plane.

4. Clockwise or counterclockwise?

Let A=1, B=0.

$$t=0,\ \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

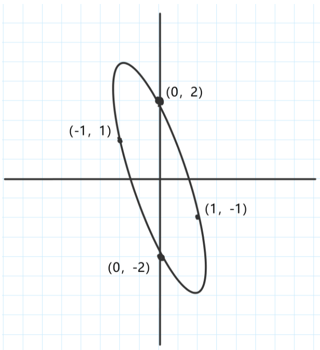
$$t=\tfrac{\pi}{4},\ \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$t=\tfrac{\pi}{2},\ \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$t=\tfrac{3\pi}{4},\ \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$t = \pi$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

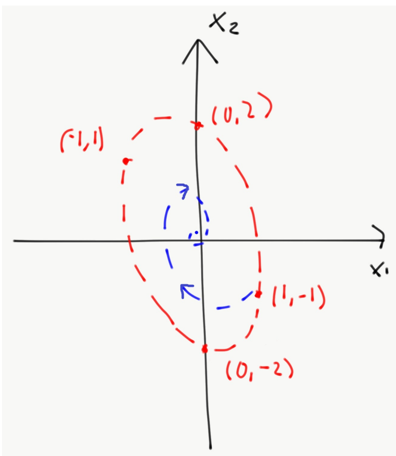
Graph where A=1, B=0, and e^{-t} **is not** taken into account:



*Note:

- Solutions do not move away or converge to (0, 0)
- Solutions cannot touch each other
- Solutions are all eclipse

Graph where A=1, B=0, and e^{-t} **is** taken into account (in blue):



Center point is **asymptotically stable**. In the long run, it is stable, since it spirals inward.

The graph is a **spiral sink**, which every point converge to zero.

The e^{-t} part contributes to the sink, and the $e^{\pm 2i}$ contributes to the spin.

The solution to this problem would be classified as asymptotically stable and a spiral sink as a spiraling pattern leads all solutions towards the origin.

When sketching first ignore exponential, then your solution will always be a circle or ellipse. After you have drawn this, think about how the exponential would impact your initial solution then include it.

Equilibrium solutions in which solutions that start "near" them move toward the equilibrium solution are called **asymptotically stable equilibrium points.** Link (http://tutorial.math.lamar.edu/Classes/DE/EquilibriumSolutions.aspx)

This is from the textbook

Spiral Points and Centers

The phase portrait in Figure 3.4.1 is typical of all two-dimensional systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ whose eigenvalues are complex with a negative real part. The origin is called a **spiral point** and is asymptotically stable because all trajectories approach it as t increases. Such a spiral point is often called a **spiral sink**. For a system whose eigenvalues have a positive real part, the trajectories are similar to those in Figure 3.4.1, but the direction of motion is away from the origin and the trajectories become unbounded. In this case, the origin is unstable and is often called a **spiral source**.

If the real part of the eigenvalues is zero, then there is no exponential factor in the solution and the trajectories neither approach the origin nor become unbounded. Instead, they repeatedly traverse a closed curve about the origin. An example of this behavior can be seen in Figure 3.4.3. In this case, the origin is called a **center** and is said to be **stable**, but not asymptotically stable. In all three cases, the direction of motion may be either clockwise, as in Example 1, or counterclockwise, depending on the elements of the coefficient matrix **A**.