2018-09-20 Linear First-Order ODEs (2.2)

2.2 Linear First-Order ODEs

Modeling Pollution in a Lake

- Lake contains V L of fresh water
- Water flows in and out of the pond at the rate of r L/year
- Incoming water is polluted with $\gamma(t) = 2 + sin(2t)$ kg/L of pollutant

1. Obtain an ODE for the amount of pollutant (in kg) at time t (in years.)

   NOTE: Check units and remember to define variables!

Let $q(t)$ = amount of pollutant (in kgs) in the pond at time t (in years)

Use “rate of change = rate in $-$ rate out” principle:

rate in = $r\gamma(t)$ (kg/year)

rate out = $r \cdot \left(\frac{amount\ of\ pollutant}{amount\ of\ water}\right) = r \cdot \frac{q(t)}{V}$ (kg/year)

change of pollutant over time = rate in - rate out

$$q'(t) = r\gamma(t) - r\frac{q(t)}{V}$$

2. What assumptions are made in this model?

1. V is a constant number, r is constant.
2. The pollutant is perfectly mixed in the water coming into the lake.
3. The lake doesn’t change in shape.
4. All incoming water is 100% polluted, which excludes rain and other forms of precipitation.
5. No variation in $\gamma(t)$.

3. What is the solution of the DE?

- The DE is NOT separable, so you must use the method of the integrating factor

  Integrating Factor: $\mu(t) = \square(t) = e^{\frac{r}{V}t}$

  $$q\prime + \frac{r}{v}q = r\gamma$$

  $$e^{\frac{r}{V}t}q\prime + \frac{r}{v}e^{\frac{r}{V}t} = r(2 + sin(2t))e^{\frac{r}{V}t}$$

  $$(e^{\frac{r}{V}t} * q)\prime = r(2 + sin(2t))e^{\frac{r}{V}t}$$

  Integrate both sides

$$e^{\frac{r}{V}t} \times q = 2Ve^{\frac{r}{V}t} + r(-Ve^{\frac{r}{V}t}\frac{2Vcos(2t)-rsin(2t)}{r^2+4V^2} + C$$

$$q = 2V - rV\frac{2Vcos(2t)-rsin(2t)}{r^2+4V^2} + C/e^{\frac{r}{V}t}$$

$$e^{\frac{rt}{V}}q(t) = \int re^{\frac{rt}{V}}\ (2+\sin(2t))\,dt$$

$$e^{\frac{rt}{V}}q(t) = 2r\int e^{\frac{rt}{V}}\ + r\left(e^{\frac{rt}{V}}\ \sin(2t)\right)dt$$

$$e^{\frac{rt}{V}}q(t) = \frac{2r\left(\frac{V}{r}\right)e^{\frac{rt}{V}}\ + r\left(-Ve^{\frac{rt}{V}}\right)(2V\cos(2t)-rsin(2t))}{r^2+4V^2} + c$$

divided by $e^{\frac{rt}{V}}$ from both sides

$$q(t) = \frac{-rV(2V\cos(2t)-rsin(2t))}{r^2+4V^2} + 2V + \frac{C}{e^{\frac{rt}{V}}}$$