Difference Equations

2.1.1 Compound Interest

2.1.2 Loan Repayment

2.1.3 Gambler's Ruin

2.2.2 Exponential Population Growth

2.2.3 Average Lifespan

2.2.$\star$ Rabbit Populations

2.2.4 Nonlinear Population Models

2.1.3 Gambler's Ruin

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Each round, one of two things happens:

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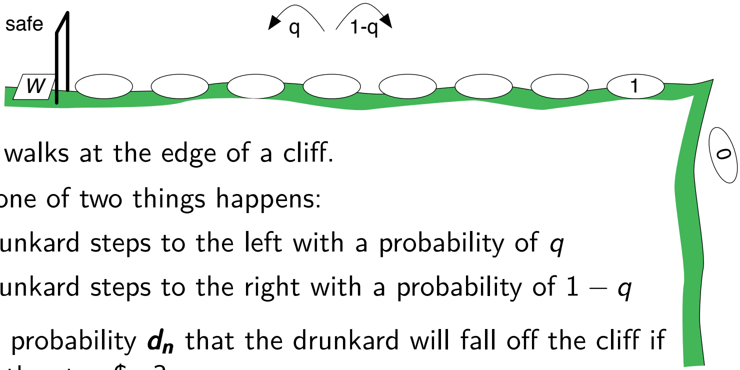
1. How does d_n compare to p_n?

2.1.3 Drunkard's fall



A drunkard walks at the edge of a cliff.

Each step, one of two things happens:

- The drunkard steps to the left with a probability of q
- The drunkard steps to the right with a probability of $1 - q$

What is the probability $\boldsymbol{d_n}$ that the drunkard will fall off the cliff if he starts at the step $\$n$?

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2. Set up the problem in terms of p_n:

describe all aspects of the game

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The problem we want to solve is

$$\begin{cases} qp_{n+1} - p_n + (1-q)p_{n-1} = 0 \\ p_0 = 1 \quad \text{and} \quad p_W = 0 \end{cases}$$

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Consider the previous two problems:

$$S_{k+1} = \mu S_k$$
$$D_{k+1} = \mu D_k - R$$

3. What did solutions look like? What kind of "functions"?

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We're looking for exponential solutions: $p_n = r^n$.

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4. Find values of r that solve the Difference equation.
5. Assume that $q \neq \frac{1}{2}$. We obtained two solutions $p_n = r_1^n$ and $p_n = r_2^n$. Obtain a general solution.

   Hint. What happens when we add two solutions? What happens when we multiply a solution by a number?
6. Find the solution by matching the extra two conditions.

7. Assume $W = 200$ and $q = 0.47$. If you start with \$190, how likely are you to go bankrupt?

8. How much money do you need to start to have a $50 - 50$ chance of winning?

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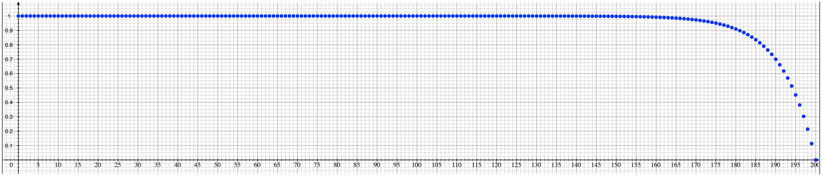

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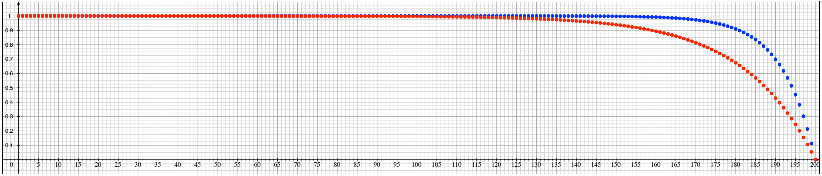

American (blue) vs European (red) roulettes

Extra Q. What is q for each type of roulette when betting red/black?

9 What is the solution with $q = \frac{1}{2}$?

$$\begin{cases} qp_{n+1} - p_n + (1-q)p_{n-1} = 0 \\ p_0 = 1 \quad \text{and} \quad p_W = 0 \end{cases}$$

Hint 1. Remember $p_n = r^n$ yields $r = 1$ or $r = \frac{1-q}{q}$
Hint 2. How do you deal with a repeated value for r?

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