Inquiry Based Modelling with Differential and Difference Equations

About the Document

This document is a mix of student resources, student projects, problem sets, and labs. A typical class day looks like:

1. **Preparation by students.** Students prepare for lecture by watching a short video and solving a short quiz.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in small groups on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.
5. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

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https://github.com/bigfatbernie/IBLmodellingDEs

Contents

In this section, we study some strategies to model problems mathematically in an effective manner.

Lesson 1

Defining Problem Statement

Objectives

- The first step in Mathematical modelling is to define the problem
- A good way to do this is to figure out what is the “mathematical object” we are looking for at the end of the process
- The second step is to create a mind map of the problem. This is a structured way to brainstorm possible solutions and their requirements.

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step A. Defining the problem

The first step is to define the problem we want to solve or improve.

To do this, we should start from the end! We need to decide on what kind of mathematical object we will use in the end to show that we solved, or at least improved, the problem we were tasked with.

Once this is done, we can define the problem mathematically.

EXAMPLE

Your team was tasked with optimizing the layout of an airport.

You decided that in the end, to show that the team did find a good layout fo an airport, they will show

- $T =$ the total time (in minutes) necessary by the average person to walk from their airport transportation (taxi, train, bus) to their gate.

Once this decision is made, the problem to solve (or improve) becomes the following:

- Minimize T

There will probably be some constraints, which will be studied in Step C.

Make the question precise, bring it into a "mathematical form".

- Choose a mathematical object best suited for the problem, e.g. a number, a geometric form, a graph, a function, an algorithm, ...

Task 1.A: Elevator problem at theBigCompany

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

> —— Forwarded Message ——
>
> Date: Mon, 16 September 2019 21:41:35 + 0000
> From: CEO <theCEO@theBigCompany.ca>
> To: Human Resources <hr@theBigCompany.ca>
> Subject: they're still late !?&!
>
> Hey Shophika!
>
> I still get complaints about staff being late, some by 15 minutes.
> With the staff we have, that's about one salary lost.
> Again the bottleneck of the elevators seems to be the problem.
> Can you suggest solutions?
>
> Thanks, the CEO

Notes/Misconceptions

- Students will start discussing how to solve the problem
- This question deals with what will happen **after** solving the problem
- The goal of this question is to think about how to best tell a "mathematically-challenged" CEO that you solved the problem

Scenario 1: With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.

Task. What mathematical object would you use to convince the CEO that you have solved or improved the problem?

1 For each part, what "mathematical object" would you use to communicate that you have solved or improved the problem? Then define the problem mathematically.

1.1 Help the city of Toronto choose the best recycling centre.

1.2 Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

1.3 Create a mathematical model to rank roller coasters according to thrill factor.

1.4 Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

1.5 The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1. Is it optimal?

1.6 Is it better to buy or rent?

(a) Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

(b) Does the criteria you used to evaluate the previous question change if the question is whether to buy a bicycle or use Bike Share Toronto?

Figure 1: Extension plans for Toronto subway line.

Step B. Building a mind map

A mind map is a tool to visually outline and organize ideas. Typically a key idea is the centre of a mind map and associated ideas are added to create a diagram that shows the flow of ideas. In Figure 2, we focus on the definition of "best", with three possible definitions branching off to be further explored. From here, we can focus our attention on one of the three branches at a time.

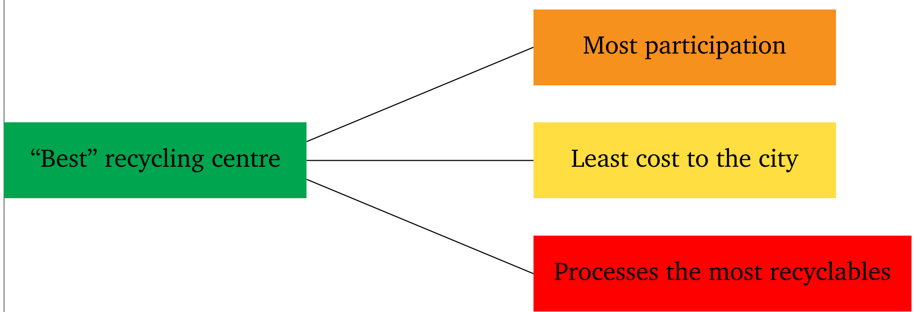


Figure 2: An example of a simple mind map.

Let's think about the least-cost option first. We probably can't determine how much any recycling program costs without knowing more about the recycling program, so a good place to start is to ask the question "What kinds of recycling programs exist?" If we aren't familiar with different types of recycling, we might need to do some research to see what kinds of programs exist.

A possible next step on your mind map for the least-cost approach could be the one shown in Figure 3.

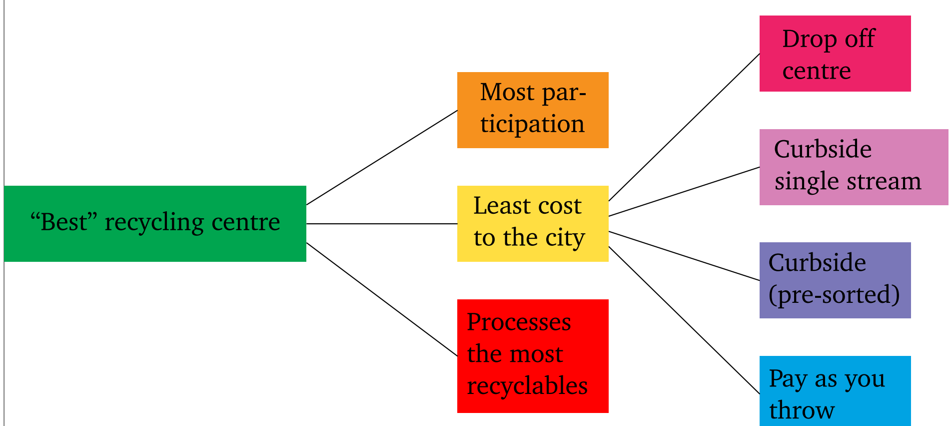


Figure 3: Next step of a mind map.

Software. There is free online software to help creating a mind map. One such is FreeMind.

Notes/Misconceptions

- Students usually come up with more complicated variations:
  - Money spent on late employees' salaries
  - sum of time in minutes that employees are late counting only employees that are at most 15 minutes late
- Stick with R, a simple first approach

FreeMind

Task 1.B: Elevator problem at theBigCompany

Scenario 2: Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $R =$ the sum in minutes by which every employee is late.

Employees that are on time count for 0 minutes.

Task. Create a mind map for the question: How can R be minimized?

2 For each part, create a mind map. Focus on the same approach you had for question 1.

2.1 Help the city of Toronto choose the best recycling centre.

2.2 Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

2.3 Create a mathematical model to rank roller coasters according to thrill factor.

2.4 Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should “best” mean?

2.5 The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1. Is it optimal?

2.6 Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

Lesson 2

Making Assumptions

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step C. Making Assumptions

Notes/Misconceptions

This is horrible and definitely needs to be worked on!

Lesson 3

Defining Variables

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Step D. Parameters or Variables?

DEFINITION

Variables and Parameters

A ***variable*** represents a model state, and may change during simulation.

A ***parameter*** is commonly used to describe objects statically. A ***parameter*** is normally a constant in a single simulation, and is changed only when you need to adjust your model behaviour.

NOTE

(from Wikipedia) The quantities appearing in the equations we classify into variables and parameters. The distinction between these is not always clear cut, and it frequently depends on the context in which the variables appear.

Usually a model is designed to explain the relationships that exist among quantities which can be measured independently in an experiment; these are the ***variables*** of the model.

To formulate these relationships, however, one frequently introduces "constants" which stand for inherent properties of nature (or of the materials and equipment used in a given experiment). These are the ***parameters***.

The choice of question in the previous lesson should determine the *dependent* variable.

Task 1.D: Elevator problem at theBigCompany

We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie (in Montréal).
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Tasks. Focus on a **few** parameters and variables. State hypotheses.

1. With your team, decide on what kind of information you would need to have to be able to solve this problem.
2. Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.
3. For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

3 For each part, you are required to make an estimate for some quantity. Make assumptions and justify them in order to solve the problem.

3.1 What is the number of piano players in Toronto? *(Fermi problem)*

3.2 How many linear km of roads are there in Toronto?

3.3 How much salt the city of Toronto needs for its roads during the Winter?

3.4 The skating season in Canada is shortening: What are the key-factors determining its length?

Lesson 4

Building Solutions

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Lesson 5

Model Assessment

Objectives

- Bla bla bla

Motivation

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Lesson 6

Putting it all together

Textbook

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Objectives

- Bla bla bla

Motivation

Lesson 7

Basic Models

Textbook

Objectives

- Bla bla bla

Motivation

Lesson 8

Direction Fields

Objectives

- Bla bla bla

Motivation

Discrete Models

Linear Algebra

Lesson 9

Linear Combinations

Textbook

Section 1.1

Objectives

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Relate points an vectors and be able to interpret a point as a vector and a vector as a point.
- Solve simple equations involving vectors.

Motivation

Students have differing levels of experience with vectors. We want to establish a common notation for vectors and use vector notation along with algebra to solve simple questions. E.g., "How can I get to location A given that I can only walk parallel to the lines $y = 4x$ and $y = -x$?"

We will use column vector notation and the idea of equating coordinates in order to solve problems.

Notes/Misconceptions

- We will use the language *component of $\vec{v}$ in the direction $\vec{u}$* in the future and it will be a *vector*. For this reason, try to refer to the entries of a column vector as *coordinates* or *entries* instead of components.
- Though we will almost exclusively use column vector notation in this course, students should be able to parse questions phrased in terms of row vectors.

Task 1.1: The Magic Carpet Ride

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario One: The Maiden Voyage

Your Uncle Cramer suggests that your first adventure should be to go visit the wise man, Old Man Gauss. Uncle Cramer tells you that Old Man Gauss lives in a cabin that is 107 miles East and 64 miles North of your home.

Task:

Investigate whether or not you can use the hover board and the magic carpet to get to Gauss's cabin. If so, how? If it is not possible to get to the cabin with these modes of transportation, why is that the case?

Hands-on experience with vectors as displacements.

- Internalize vectors as geometric objects representing displacements.
- Use column vector notation to write vectors.
- Use pre-existing knowledge of algebra to answer vector questions.

Notes/Misconceptions

- There are many ways to solve this problem. Some students might start with equations. After they use their equations to solve the problem, make them draw a picture and come up with a graphical solution.
- When the students start coming up with vector equations, give them the vocabulary of *linear combinations* and *column vector notation*.

Lesson 10

Linear Combinations

Textbook

Section 1.2

Objectives

- Set up and solve vector equations $a\vec{v} + b\vec{u} = \vec{w}$. The solving method may be ad hoc.
- Use set notation and set operations/relations $\cup$, $\cap$, $\in$, $\subseteq$.
- Translate between set-builder notation and words in multiple ways.

Motivation

We revisit questions about linear combinations more formally and generate a need for algebra. The algebra we do to solve vector equations will become algorithmic when we learn row reduction, but at the moment, any method is fine.

As we talk about more complex objects, we need precise ways to talk about groups of vectors. I.e., we need sets and set-builder notation. This preview of set-builder notation will take some of difficulty away when we define span as a set of vectors.

In this course we will be using formal and precise language. Part of this lesson is that there are multiple correct ways (and multiple incorrect ways) to use formal language. Gone are the days of "there's only one right answer and it is 4"!

Notes/Misconceptions

You will have a mix of MAT135/136 and MAT137 students. The MAT137 students will be doing logic and sets in their class. The MAT135 students won't. Make sure not to leave them behind!

Task 1.2: The Magic Carpet Ride, Hide and Seek

You are a young traveler, leaving home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix}3\\1\end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 miles East and 1 mile North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix}1\\2\end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 mile East and 2 miles North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Address an existential question involving vectors: "Is it possible to find a linear combination that does. . . ?"

The goal of this problem is to
- Formalize geometric questions using the language of vectors.
- Find both geometric and algebraic arguments to support the same conclusion.
- Establish what a "negative multiple" of a vector should be.

Notes/Misconceptions

- Both *yes* and *no* are valid answers to this question depending on whether you are allowed to go backwards. Establish that "negative" multiples of a vector mean traveling backwards along that vector.
- This problem can be solved with algebra by finding a formula for the coefficients for an arbitrary position or with geometry, with arguments eventually hinging on the fact that non-parallel lines do not intersect.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \ : \ \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \,|\, \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}.$$

$$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}.$$

$$\mathbb{R} = \{\text{real numbers}\}.$$

$$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}.$$

4

4.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$. True

(b) $1.5 \in \{1,2,3\}$. False

(c) $4 \in \{1,2,3\}$. False

(d) "b"$\in \{x : x \text{ is an English letter}\}$. True

(e) "ò"$\in \{x : x \text{ is an English letter}\}$. False

(f) $\{1,2\} \subseteq \{1,2,3\}$. True

(g) For some $a \in \{1,2,3\}$, $a \geq 3$. True

(h) For any $a \in \{1,2,3\}$, $a \geq 3$. False

(i) $1 \subseteq \{1,2,3\}$. False

(j) $\{1,2,3\} = \{x \in \mathbb{R} : 1 \leq x \leq 3\}$. False

(k) $\{1,2,3\} = \{x \in \mathbb{Z} : 1 \leq x \leq 3\}$. True

Practice reading sets and set-builder notation.

The goal of this problem is to

- Become familiar with $\in$, $\subseteq$, and $=$ in the context of sets.
- Distinguish between $\in$ and $\subseteq$.
- Use quantifiers with sets.

Notes/Misconceptions

- Most are easy up through (h).
- Make students "fix" (i) so it becomes true.
- (j) and (k) are an opportunity to use the definition of set equality. Students don't realize that $=$'s has a definition.

5

Write the following in set-builder notation

5.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

$$\{x \in \mathbb{R} : x > \sqrt{2}\}.$$

5.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

$$\left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \text{ with } a = 2b\right\} \text{ or } \left\{\vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} 2t \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$$

$$\text{or } \left\{\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 : a = 2b\right\}.$$

Practice writing sets using set-builder notation.

The goal of this problem is to

- Express English descriptions using math notation.
- Recognize there is more than one correct way to write formal math.
- Preview vector form of a line.

Notes/Misconceptions

- There are multiple correct ways to write each of these sets. It's a good opportunity to get man correct and incorrect sets up on the board for discussing.
- Don't worry about the geometry of B. That's coming in a later problem.

DEFINITION

Unions & Intersections

Two common set operations are ***unions*** and ***intersections***. Let X and Y be sets.

$$\text{(union)}\quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)}\quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

6

Let $X = \{1,2,3\}$ and $Y = \{2,3,4,5\}$ and $Z = \{4,5,6\}$. Compute

6.1 $X \cup Y$ $\{1,2,3,4,5\}$

6.2 $X \cap Y$ $\{2,3\}$

6.3 $X \cup Y \cup Z$ $\{1,2,3,4,5,6\}$

6.4 $X \cap Y \cap Z$ $\emptyset = \{\}$

Apply the definition of $\cup$ and $\cap$.

Notes/Misconceptions

- It's not important to emphasize that $\cup$ and $\cap$ are binary operations but we ask for $X \cup Y \cup Z$ without parenthesis. Students won't worry if you don't bring it up.
- It won't be clear to them how to write the empty set. Some will write $\{\emptyset\}$. Make sure this comes out.

Lesson 11

Visualizing Sets, Formal Language of Linear Combinations

Textbook

Section 1.2

Objectives

- Draw pictures of formally-described subsets of $\mathbb{R}^2$.
- Graphically represent $\cup$ and $\cap$ for subsets of $\mathbb{R}^2$.
- Graphically represent linear combinations and then come up with algebraic arguments to support graphical intuition.

Motivation

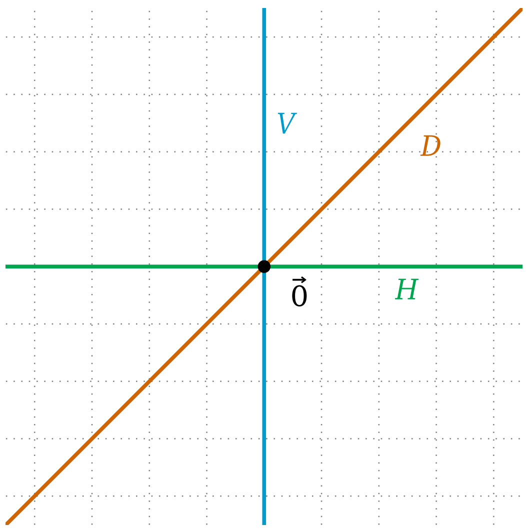
We want to build a bridge between the formal language of linear combinations and set-builder notation and geometric intuition. Where as last time the focus was on formal language, this time the focus is on linking geometry to formal descriptions.

7 Draw the following subsets of $\mathbb{R}^2$.

7.1 $V = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.2 $H = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.

7.3 $D = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for some } t \in \mathbb{R}\right\}$.



7.4 $N = \left\{\vec{x} \in \mathbb{R}^2 : \vec{x} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for all } t \in \mathbb{R}\right\}$. $N = \{\}$.

7.5 $V \cup H$. $V \cup H$ looks like a "+" going through the origin.

7.6 $V \cap H$. $V \cap H = \{\vec{0}\}$ is just the origin.

7.7 Does $V \cup H = \mathbb{R}^2$?

No. $V \cup H$ does not contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ while $\mathbb{R}^2$ does contain $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Vector Combinations

DEFINITION — Linear Combination

A ***linear combination*** of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a vector

$$\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n.$$

The scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the ***coefficients*** of the linear combination.

8 Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\vec{w} = 2\vec{v}_1 + \vec{v}_2$.

8.1 Write $\vec{w}$ as a column vector. When $\vec{w}$ is written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, what are the coefficients of $\vec{v}_1$ and $\vec{v}_2$?

$\vec{w} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$; the coefficients are $(2, 1)$.

8.2 Is $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\vec{v}_1 + 0\vec{v}_2$.

8.3 Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2$.

8.4 Is $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$ a linear combination of $\vec{v}_1$ and $\vec{v}_2$? Yes. $\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 2\vec{v}_1 + 2\vec{v}_2$.

Visualize sets of vectors.

The goal of this problem is to
- Apply set-builder notation in the context of vectors.
- Distinguish between "for all" and "for some" in set builder notation.
- Practice unions and intersections.
- Practice thinking about set equality.

Notes/Misconceptions
- 1–3 will be easy.
- Have a discussion about when you should draw vectors as arrows vs. as points.
- 4 gets at a subtle point that will come up again when we define span.
- Many will miss 7. Writing a proof for this is good practice.

Practice linear combinations.

The goal of this problem is to
- Practice using the formal term *linear combination*.
- Foreshadow span.

Notes/Misconceptions
- In 2, the question should arise: "Is $3\vec{v}_1$ a linear combination of $\vec{v}_1$ *and* $\vec{v}_2$?" Address this.
- Refer to the magic carpet ride for 5. You don't need to do a full proof.

8.5 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$ and $\vec{v}_2$?

No. $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\vec{v}_1 - \frac{1}{2}\vec{v}_2$. Therefore

$$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = a(\tfrac{1}{2}\vec{v}_1 + \tfrac{1}{2}\vec{v}_2) + b(\tfrac{1}{2}\vec{v}_1 - \tfrac{1}{2}\vec{v}_2) = (\tfrac{a+b}{2})\vec{v}_1 + (\tfrac{a-b}{2})\vec{v}_2.$$

Therefore any vector in $\mathbb{R}^2$ can be written as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

8.6 Can you find a vector in $\mathbb{R}^2$ that isn't a linear combination of $\vec{v}_1$?

Yes. All linear combinations of $\vec{v}_1$ have equal x and y coordinates, therefore $\vec{w} = \begin{bmatrix}2\\1\end{bmatrix}$ is not a linear combination of $\vec{v}_1$.

9 Recall the *Magic Carpet Ride* task where the hover board could travel in the direction $\vec{h} = \begin{bmatrix}3\\1\end{bmatrix}$ and the magic carpet could move in the direction $\vec{m} = \begin{bmatrix}1\\2\end{bmatrix}$.

Practice formal writing.

Notes/Misconceptions

- Make everyone *write*. They will think they can do it, but they will find it hard if they try.

9.1 Rephrase the sentence *"Gauss can be reached using just the magic carpet and the hover board"* using formal mathematical language.

Gauss's location can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

9.2 Rephrase the sentence *"There is nowhere Gauss can hide where he is inaccessible by magic carpet and hover board"* using formal mathematical language.

Every vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{m}$ and $\vec{h}$.

9.3 Rephrase the sentence "$\mathbb{R}^2$ *is the set of all linear combinations of* $\vec{h}$ *and* $\vec{m}$" using formal mathematical language.

$\mathbb{R}^2 = \{\vec{v} : \vec{v} = t\vec{m} + s\vec{h} \text{ for some } t, s \in \mathbb{R}\}$.

Lesson 12

Restricted Linear Combinations, Lines

Textbook

Section 1.2

Objectives

- Read and digest a new definition.
- Use pictures to explore a new concept.
- Convert from an equation-representation of a line to a set-representation.

Motivation

Part of doing math in the world is reading and understanding other people's definitions. Most students will not have heard of non-negative linear combinations or convex linear combinations. This is a chance for them to read and try to understand these formal definitions. They will need to draw pictures to get an intuition about what these concepts mean.

These concepts are useful in their own right, and in particular, convex linear combinations can be used to describe line segments. Adding these definitions to a student's toolbox serves the goal of *being able to describe the world with mathematics*.

To that end, we start working with lines. Lines are something students have used since grade school, but they worked with them in $y = mx + b$ form which is only applicable in $\mathbb{R}^2$. We want to convert this representation into vector form and set-based descriptions which apply to all dimensions.

DEFINITION

Non-negative & Convex Linear Combinations

The linear combination $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$ is called a ***non-negative*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ if $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$.

If $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, then $\vec{w}$ is called a ***convex*** linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

10 Let

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{d} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \vec{e} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

10.1 Out of $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$, and $\vec{e}$, which vectors are

(a) linear combinations of $\vec{a}$ and $\vec{b}$? All of them, since any vector in $\mathbb{R}^2$ can be written as a linear combination of $\vec{a}$ and $\vec{b}$.

(b) non-negative linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$.

(c) convex linear combinations of $\vec{a}$ and $\vec{b}$? $\vec{a}$, $\vec{b}$, $\vec{c}$.

10.2 If possible, find two vectors $\vec{u}$ and $\vec{v}$ so that

(a) $\vec{a}$ and $\vec{c}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{b}$ is not.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{c}$.

(b) $\vec{a}$ and $\vec{e}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$.

Let $\vec{u} = \vec{a}$ and $\vec{v} = \vec{e}$.

(c) $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$ but $\vec{d}$ is not.

Impossible. If $\vec{a}$ and $\vec{b}$ are non-negative linear combinations of $\vec{u}$ and $\vec{v}$, then every non-negative linear combination of $\vec{a}$ and $\vec{b}$ is also a non-negative linear combination of $\vec{u}$ and $\vec{v}$. And, we already concluded that $\vec{d}$ is a non-negative linear combination of $\vec{a}$ and $\vec{b}$.

(d) $\vec{a}$, $\vec{c}$, and $\vec{d}$ are convex linear combinations of $\vec{u}$ and $\vec{v}$.

Impossible. Convex linear combinations all lie on the same line segment, but $\vec{a}$, $\vec{c}$, and $\vec{d}$ are not collinear.

Otherwise, explain why it's not possible.

Geometric meaning of *non-negative* and *convex* linear combinations.

The goal of this problem is to
- Read and apply the definition of non-negative and convex linear combinations.
- Gain geometric intuition for non-negative and convex linear combinations.
- Learn how to describe line segments using convex linear combinations.

Notes/Misconceptions
- This question is about reading and applying; emphasize that before they start.
- The geometry won't be obvious. Ask them to *draw* specific linear combinations (e.g., $(1/2, 1/2)$) to get an idea.
- They know $\vec{a}$ and $\vec{b}$ span all vectors from problem 8.
- In part 1, they will forget $\vec{a}$ and $\vec{b}$ are linear combinations of themselves.
- Part 2 (b) highlights a degeneracy that will come up again when discussing linear independence and dependence. Explain how the picture for non-negative linear combinations almost always looks one way, but this case is an exception.

Lines and Planes

11 Let L be the set of points $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 1$.

11.1 Describe L using set-builder notation.

$$\left\{ \vec{v} \in \mathbb{R}^2 : \vec{v} = \begin{bmatrix} t \\ 2t+1 \end{bmatrix} \text{ for some } t \in \mathbb{R} \right\}$$

$$\text{or } \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = 2x + 1 \right\} \text{ or } \left\{ \begin{bmatrix} t \\ 2t+1 \end{bmatrix} \in \mathbb{R}^2 : t \in \mathbb{R} \right\}$$

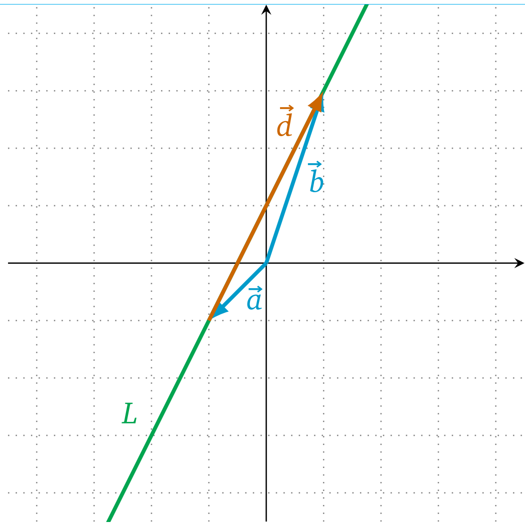
11.2 Draw L as a subset of $\mathbb{R}^2$.

11.3 Add the vectors $\vec{a} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\vec{d} = \vec{b} - \vec{a}$ to your drawing.

Link prior knowledge to new notation/concepts.

The goal of this problem is to
- Convert between $y = mx + b$ form of a line and the set-builder definition of the same line.
- Think about lines in terms of vectors rather than equations.

Notes/Misconceptions
- This question is foreshadowing for vector form of a line.
- In part 3, some will draw $\vec{d}$ from the origin and some will draw it on the line. Both are fine, but make sure they understand that $\vec{d} \notin L$ by the end of part 4.



11.4 Is $\vec{d} \in L$? Explain.

No. $\vec{d} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and so its entries don't satisfy $y = 2x + 1$.

11.5 For which $t \in \mathbb{R}$ is it true that $\vec{a} + t\vec{d} \in L$? Explain using your picture.

$\vec{a} + t\vec{d} \in L$ for any $t \in \mathbb{R}$. We can see this because if we start at the vector $\vec{a}$ and the displace by $t\vec{d}$, we will always be on the line L.

