2018-10-25 Systems of ODEs with Repeated Eigenvalues (3.5)

Q. Consider the same problem but with k=4

$$x' = \begin{bmatrix} 0 & 1 \\ -4 & -r \end{bmatrix} x$$

1. Find the eigenvalues of the matrix.

$$\begin{pmatrix} 0-\lambda & 1 \\ -4 & -r-\lambda \end{pmatrix} = (-\lambda)(-r-\lambda) + 4 = 0$$

$$\lambda^2 + r\lambda + 4 = 0$$

Recall: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\lambda = \frac{\left(-r \pm \sqrt{r^2 - 16}\right)}{2}$$

2. (1)What happens for r > 4 or r < -4?

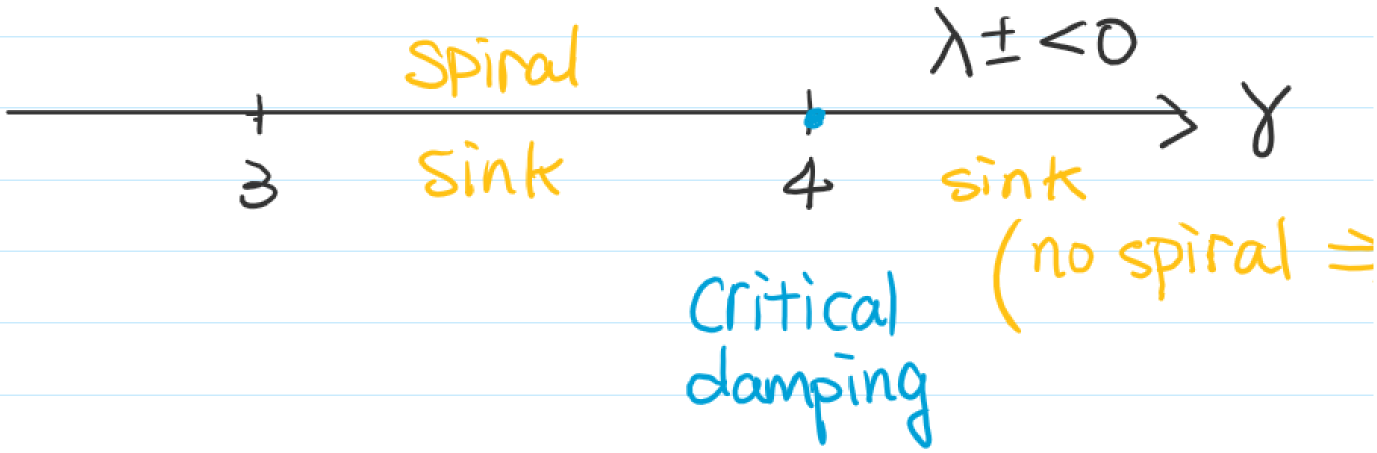
real value solution: $\lambda_{\pm}$ (Lecture Oct.21) --> over damped

(2)What happens for -4 < r < 4?

complex solution: $\lambda_{\pm}$ (Last Lecture) --> under damped

(3)What happens for r = +/- 4?

repeated solution: $\lambda = -2$ (This Lecture) --> critically damped



Q. Consider the critically damped problem with r=4

1. **Find one solution x1(t)**

$$x' = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} x$$

$$\lambda = -2,\ \begin{pmatrix} 0+2 & 1 \\ -4 & -4+2 \end{pmatrix} \binom{v_1}{v_2} = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \binom{v_1}{v_2} = 0\,,\ 2v_1 + v_2 = 0$$

$\lambda = -2,\ v = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ (eigenvalue and eigenvector)

$$x_1(t) = e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

2. Find general eigenvalue and eigenvector

$$(A - \lambda I) \begin{pmatrix} w1 \\ w2 \end{pmatrix} = \begin{pmatrix} v1 \\ v2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \binom{w_1}{w_2} = \binom{1}{-2},\ 2w_1 + w_2 = 1$$

$w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ *or* $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *or* $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ NOTE: If one row is not a multiple of the other one, you did something wrong so check your solution!

$$x_2(t) = \left[\binom{1}{-1} + \binom{1}{-2} t \right] e^{-2t}$$

generalized and eigenvector

$$x\,(t) = A\binom{1}{-2}e^{-2t} + B\left[\binom{1}{-1} + \binom{1}{-2}t\right]e^{-2t}$$

NOTE: YOU CAN ONLY CHOOSE EITHER (0, 1) OR (1, -1) FOR YOUR GENERALIZED EIGENVECTOR

NOTE: $\binom{w_1}{w_2}$ DEPENDS ON THE EIGENVECTOR CHOSEN

Summary of the steps to find the solution in case of repeated eigenvalues:

1. Find the repeated eigenvalue by solving for λ: $det(A - I\lambda) = 0$
2. Find the first eigenvector by solving for $\overrightarrow{v_1} = \binom{v_{11}}{v_{12}}$: $(A - I\lambda)\binom{v_{11}}{v_{12}} = \binom{0}{0}$
3. Find the generalized eigenvector by solving for $\overrightarrow{v_2} = \binom{v_{21}}{v_{22}}$: $(A - I\lambda)\binom{v_{21}}{v_{22}} = \binom{v_{11}}{v_{12}}$
4. Write the general solution $\overrightarrow{x(t)}$: $\overrightarrow{x(t)} = c_1 e^{\lambda t}\overrightarrow{v_1} + c_2 e^{\lambda t}(t\overrightarrow{v_1} + \overrightarrow{v_2})$