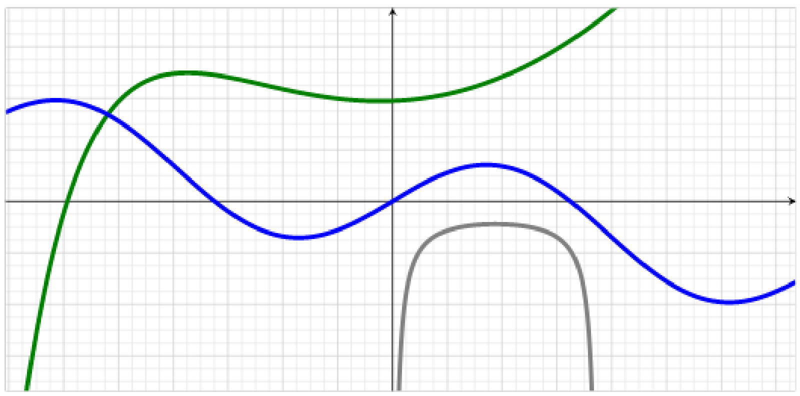
2018-09-27 Linear vs Nonlinear ODEs (2.4)

Continuing the problem

$y' = f(t,y)$ **where** $f(t,y)$ **and** $\frac{\partial t}{\partial y}(t,y)$ **are continuous for all t, y.**

- (graph shown)
- 

- **Could this be the graph of 3 solutions with 3 different initial conditions?**
  - No, this can not be - The theorem implies that there is a unique solution, which means that for any given point on the graph, only 1 solution curve can pass through it. Meaning: none of the solutions of a DE can intersect (the green and the blue solutions cannot "co-exist") In this graph, two of the solutions intersect at a point and therefore when the initial value is set to be the intersection, they will not be unique.

Q. **Consider the problem**

$$y' = f(t,y)$$

where $f(t,y)$ **and** $\frac{\partial t}{\partial y}(t,y)$ **are continuous for all t,y**

- **Assume that** $y = \frac{1}{t}$ **is a solution for all t>0**
- **Assume that** $y = -e^{-t}$ **is a solution for all t.**

(i) Let $f(t) = y = \phi(t)$ **be the solution of this ODE with the initial condition y(1) = 1/2. Calculate** $lim_{t\to\infty} f(t)$.

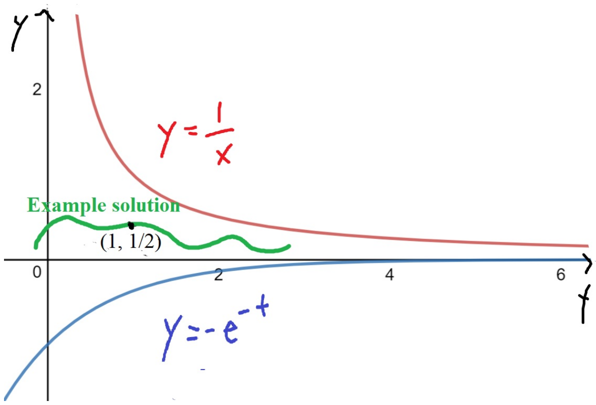
- We can conclude that the solution through (1, 1/2) also goes to 0 if (1, 1/2) is bewtween the curves $g(t) = \frac{1}{t}$ and $h(t) = -e^{-t}$:

Recall SQUEEZE THEOREM:

$$h(t) \le f(t) \le g(t)$$

$$\Rightarrow lim_{t\to\ a} h(t) \le lim_{t\to\ a} f(t) \le lim_{t\to\ a} g(t)$$

- If we plug in the same t: then
- $h(1) \leq f(1) \leq g(1)$
- $= -\frac{1}{e} \leq \frac{1}{2} \leq 1$ is a true relationship
- We know that $lim_{t\to\infty}\frac{1}{t} = 0$ and $lim_{t\to\infty} - e^{-t} = 0$ (as shown below).
- $\Rightarrow 0 \leq lim_{t\to\infty} f(t) \leq 0$
- $\Rightarrow lim_{t\to\infty} f(t) = 0$



Existence and Uniqueness THEOREM: (Linear DE's only)

Consider the problem

$$y'(t) + p(t)y = g(t) \;\; with \;\; y(t_0) = y_0$$

Theorem (Existence and Uniqueness for Linear DEs)

If

- $p(t)$ *and* $g(t)$ *are continuous in* (a, b)
- $t_0 \in (a, b)$

Then

- *There is one unique solution* $y = \phi(t)$ *defined for* $t \in (a, b)$

Theorem (Existence and Uniqueness for Nonlinear DEs)

If

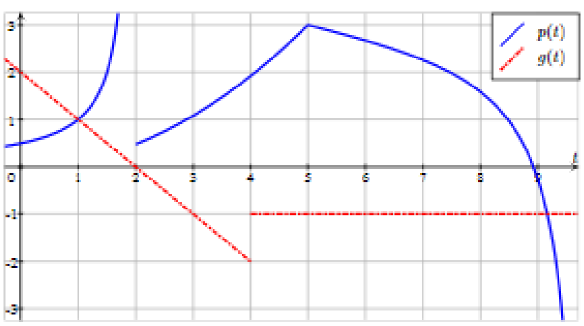
- $f(t, y)$ *and* $\frac{\partial f}{\partial y}(t, y)$ *are continuous near* (t_0, y_0)

Then

- *there is one unique solution* $y = \phi(t)$ *defined for* t *near* t_0.

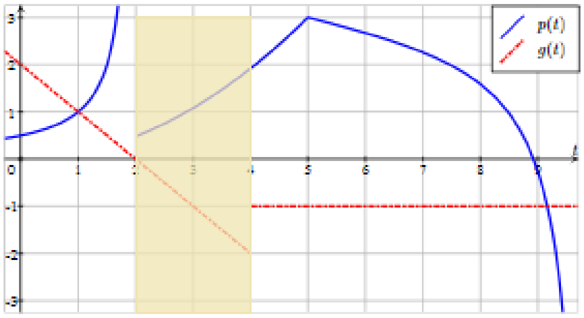
Linear Theorem	Non Linear Theorem
• Don't need to take partial derivative	• Need to take partial derivative
• Defined for $t \in (a, b)$ • The whole range is continuous and defined	• Technically, also defined an interval "near t" (check proof) • Only tells us about a little bit of the graph, we don't know about the rest
• Specific to Linear DEs	• More general, can use for both!
• $y' = g(t) - p(t)y = f(t, y)$ $\implies \frac{\partial f}{\partial y} = -p(t)$ • if g(t) and p(t) are continuous , **then y' is continuous** • if p(t) is continuous, then $\frac{\partial f}{\partial y}$ **is continuous** ◦ **Recall: CONTINUITY** ▪ $lim_{t \to a} f(t) = f(a)$	• y' must exist and be continuous • $\frac{\partial f}{\partial y}$ must exist and be continuous

Q. **Consider the graph**



(i) There exists a unique solution satisfying y(3) = 2 defined for what interval?

- ANS: $t \in (2, 4)$
  - Given (3,2) , $t_0 = 3$. At t=3, p(t) is continuous from (2 to +∞) and g(t) is continuous from (-∞ to 4) ; the intersection of these intervals is $t \in (2, 4)$
    - Tip: start at t=3 and move to the left until either p(t) or g(t) breaks to get the starting interval; move right for the ending interval
    - Any interval within (2,4) would also work, but (2,4) is the best we can do.

- 

(ii) There exists a unique solution satisfying y(t_0) = -1 defined for

At point t=5, there is a sharp change of the line. The function is continuous at that point cause you can draw the function without letting pen leaving the paper, but the function is not differentiable at that point.

- ANS: $t_0 \in (-\infty, 2)\ and\ (4,\ +\infty)$ and (2,4). Little tip: "and" can also be represented using "U" for union.
  - Given y(t_0) = -1 .
    - On the left side: At y=-1, p(t) is continuous from (-∞ to 2) and g(t) is continuous from (-∞ to 4) ; the intersection of these intervals is $t \in (-\infty,\ 2)$.
    - Both g(t) and p(t) are continuous within the interval (2,4).
    - On the right side: At y=-1, p(t) is continuous from (2 to +∞) and g(t) is continuous from (4 to +∞) ; the intersection of these intervals is $t \in (4,\ +\infty)$.