2018-11-13 Compound Interest (2.1.1)

Difference Equation

Pre-lecture

Sequences defined recursively:

- Some initial terms defined e.g. $a_1 = 1$
- Later terms defined recursively by earlier terms e.g. $a_k = a_{k-1} + 3,\ k > 1$

Finding an explicit formula from a recursive sequence:

Given $a_1 = 1$ and $a_k = a_{k-1} + 3,\ k > 1$

$$a_1 = 1 \ \ a_2 = 1 + 3 \ \ a_3 = (1+3) + 3 \ \ a_4 = (1+3+3) + 3 \ \ a_k = 1 + 3\,(k-1)$$

Verification:

1+3(k-1)=1+3(k-1-1)+3 -> 1+3(k-1)=1+3(k-1-1+1) $\surd$

2.1.1 Compound Interest

We put a certain amount of money in a savings bank account with an annual interest rate of p% and compounded at regular periods of α (in years).

1. If the interest is compounded monthly, what is α?

What is α if the interest is compounded every 3 months?

Let S_k = amount of money in the bank account after k periods.

ANS. $\alpha \ = \frac{1}{12}\ (1\,month)\ \ or\ \alpha\ = \frac{3}{12}\ (3\,months)$

2. Find an equation relating S_{k+1} and S_k.

ANS. $S_{k+1} = S_k\left(1 + \frac{p\alpha}{100}\right)$

3. Calculate S_1, S_2, S_3 in terms of S_0.

ANS.

$$S_1 = S_0\left(1 + \frac{p\alpha}{100}\right)$$

$$S_2 = S_0\left(1+\frac{p\alpha}{100}\right)^2$$

$$S_3 = S_0\left(1+\frac{p\alpha}{100}\right)^3$$

4. Can you find a pattern for S_k?

ANS. $S_k = S_0\left(1+\frac{p\alpha}{100}\right)^k$

Note: The effective interest rate p_{eff} is the annual rate that gives the same amount of money at the end of the year as if it was compounded at periods of α at the rate p%. Therefore, the p_{eff} (relative to the conversion period) is given by:

$$1+p_{\text{eff}} = \left(1+\frac{\alpha p}{100}\right)^{1/\alpha}$$

Note: for the cases we discussed in the class, we assume that we always pay the loan monthly

Extra homework slides:

The annual rate is $p\%$, but the interest is compounded.

5. If the interest was compounded annually, how much money should there be after one year?
6. After 1 year with a monthly compounded interest, is there more or less money than the one found for 5?
7. If each period is α long (in years), how many periods are there in a year?
8. How much money is there after one year?

The **effective interest rate $p_{\text{eff}}\%$** is the annual rate that gives the same amount of money at the end of the year as if it was compounded in periods of α at the rate $p\%$.

9. What is $p_{\text{eff}}\%$?

5. $\alpha = 1,\ S_1 = S_0\left(1+\frac{p}{100}\right)$

6. $if\,\alpha = \frac{1}{12}\,,\ B_{12} = B_0\left[1+\frac{1}{12}\left(\frac{p}{100}\right)\right]^{12},\ where\ B_k\ is\ the\ balance\ after\ k\ months$

In comparison to the equation in 5, more frequent compouns yields more money. (Imagine p =1 and compare results, also since exponentials grow faster)

7. $\alpha = \frac{years}{period}\ ,\ therefore\ periods\ per\ year\ = \frac{1}{\alpha}$

8. $S_1 = S_0\left(1+\frac{p_{eff}}{100}\right)^{\frac{1}{\alpha}}$, $Generally,\ S_n = S_0\left(1+\frac{p_{eff}}{100}\right)^{\frac{1}{\alpha}n}$, $where\ n = periods$

9.

$$[1 + yearly\ interest\ rate]^{per\ year} = 1 + p_{eff} = \left(1+\frac{p\alpha}{100}\right)^{\frac{1}{\alpha}} = \left[1+\frac{(interest\ rate\ per\ year)}{(periods\ per\ year)}\right]^{per\ period}$$

$$p_{eff} = \left(1+\frac{p\alpha}{100}\right)^{\frac{1}{\alpha}} - 1$$