2018-11-22 Gambler's Ruin (2.1.3)

2.1.3

Pre-lecture:

Fibonacci Sequence: 1 1 2 3 5 8 13 21

n= 0 1 2 3 4 5 6 7

$F_0 = 1$

$F_1 = 1$

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 2$$

Idea: $F_n = r^n$, $r \neq 0$

$$r^k = r^{k-1} + r^{k-2}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Substituting the initial conditions:

- $F_0 = 1$
- $F_1 = 1$

$$F_0 = a + b = 1$$

$$F_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

We get $a = \frac{\sqrt{5}+5}{10}$ and $b = \frac{5-\sqrt{5}}{10}$

$$F_n = \left(\frac{5+\sqrt{5}}{10}\right) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Consider the previous two problems:

$S_k = \mu S_k \rightarrow S_x = aq^k$

$D_{k+1} = \mu D_k - R \rightarrow D_k = bq^k + C$

$q^k = e^{k\ln(q)} = e^{rk} \rightarrow e^{rt}$

Difference equations

Differential equations

For the problem:

$$qp_{n+1} - p_n + (1-q)\,p_{n-1} = 0$$
$$p_0 = 1 \; and \; p_w = 0$$

4) Find the values of r that solve the Difference equation

$$qr^{n+1} - r^n + (1-q)\,r^{n-1} = 0$$
$$\frac{qr^{n+1}}{r^{n+1}} - \frac{r^n}{r^{n+1}} + \frac{(1-q)r^{n-1}}{r^{n+1}} = 0$$
$$q - r^{-1} + (1-q) = 0$$
$$qr^2 - r + (1-q) = 0$$

$$r = \frac{1 \pm \sqrt{1-4q(1-q)}}{2q} = \frac{1 \pm \sqrt{4q^2-4q+1}}{2q} = \frac{1 \pm \sqrt{(2q-1)^2}}{2q}$$

$$r = \frac{1 \pm (2q-1)}{2q}$$

$$r_1 = 1 \; and \; r_2 = \frac{1-q}{q}$$

5) Assume that q does not equal ½ . We obtain two solutions $P_n = R_1^n$ and $P_n = R_2^n$. Obtain a general solution

$P_n = a(\frac{1}{q} - 1)^n + b(1)^n$

**if q = ½ then the two R values are equal

$P_n = a(1)^n + b*n(1)^n$

6) Find the solution by matching the extra two conditions.

$p_n = a(\frac{1}{q} - 1)^n + b(1)^n$

$$Equation\,1: \; p_0 = a\left(\frac{1}{q} - 1\right)^0 + b(1)^0 = a + b = 1$$

$$Equation\,2: \; p_w = a\left(\frac{1}{q} - 1\right)^w + b(1)^w = a\left(\frac{1}{q} - 1\right)^w + b = 0$$

Subtract Equation 1 from Equation 2:

$$a - a\left(\frac{1-q}{q}\right)^w = 1$$
$$a\left(1 - \left(\frac{1-q}{q}\right)^w\right) = 1$$
$$a = \frac{1}{1-\left(\frac{1-q}{q}\right)^w}$$
$$a + b = 1$$
$$b = 1 - \frac{1}{1-\left(\frac{1-q}{q}\right)^w}$$

$$p_n = \left(\frac{1}{1-\left(\frac{1-q}{q}\right)^w}\right)\left(\frac{1-q}{q}\right)^n + \left(1 - \left(\frac{1}{1-\left(\frac{1-q}{q}\right)^w}\right)\right)$$

7) Assume W=200 and q = 0.47. If you start with $190, how likely are you to go bankrupt (if q = $\frac{1}{2}$ the equation is different)

**P_n is the probability the gambler will go bankrupt

Plugging into equation above:

$$p_{190} = \left(\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}\right)\left(\frac{1-0.47}{0.47}\right)^{190} + \left(1 - \left(\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}\right)\right)$$

P_{190}= 0.69924

8) How much money do you need to start with to have a 50-50 chance of winning?

Plugging into the equation found in 6) and solve for n:

$$p_n = 0.5 = \left(\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}\right)\left(\frac{1-0.47}{0.47}\right)^n + \left(1 - \left(\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}\right)\right)$$

$$\ln\left(\frac{-0.5+\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}}{\frac{1}{1-\left(\frac{1-0.47}{0.47}\right)^{200}}}\right) = nln\left(\frac{1-0.47}{0.47}\right)$$

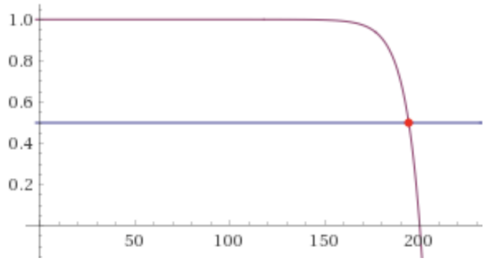
$$n = 194.2307$$

You would need to start with $194.23 to have a 50-50 chance of winning.

Plotting $p_n\ vs.\ n$:

x-axis is the amount of money you are starting with

y-axis is the probability of going bankrupted



As we can see from the graph, for most of the time, the probability of going bankrupted is very high (close to 1). Therefore, we should not gamble.

What if q = 1/2 ?

$p_n = a(1)^n + b\mathbf{n}(1)^n$, just like in DEs

using $p_0=1$, $p_W=0$,

$p_n = 1 - n/W$

Notice if n=W, the probability is 0. Also, this is the case we intuitively think of: the closer our intial \$n is to \$W, the less likely we are to go bankrupt. So our intuition is only correct if the chances of winning/losing are 50/50.