2018-09-18 Separable ODEs (2.1)

2.1 Separable ODEs

Q: Which of these ODEs are Separable ODEs? Why?

a. $\theta'' = \frac{-g}{l} sin(\theta)$ Second order (Not Separable.) $\frac{1}{\sin(\theta)} d^2\theta = -\frac{g}{l} dt^2$

b. $v' = -g$ Yes. $dv = -gdt$

c. $v' = \frac{-g-r}{mv}$ Yes. $\frac{1}{\left(\frac{-g-r}{mv}\right)} dv = dt$

d. $y' = \frac{-gt-r}{my+10}$ Yes. $\frac{1}{my+10} dy = (-gt-r)\, dt$

e. $y' = -gt - \frac{\gamma}{m} y + 10$ Not Separable.

[Observe if the differential equation can be written as $\frac{dy}{dx} = g(x) \cdot h(y)$ then it is separable. e.g. equation c. can be written as $v' = \frac{-g-r}{m} \cdot \frac{1}{v}$]

b:

$v' = -g$

$\frac{dv}{dt} = -g$

$dv = -g\, dt$

$\int dv = \int -g\, dt$

Let us recall the model for the altitude of a boulder thrown by a catapult:

$v'(t) = \frac{-g-r}{m \cdot v(t)}$

Initially the boulder has velocity v_0

1. What is v(t)?

$\frac{dv}{dt} = -g - \frac{\gamma}{m} v(t)$

$\frac{dv}{dt} = \frac{-(gm + \gamma v(t))}{m}$

$$\frac{1}{(gm+\gamma v(t))}dv \;=\; -\frac{1}{m}dt$$

$$\int \frac{1}{(gm+\gamma v(t))}dv \;= \int -\frac{1}{m}dt$$ (integrate both sides)

$$\frac{\ln(|\gamma v(t)+gm|)}{\gamma} \;= -\frac{1}{m}t \;+c$$

$$\ln(\gamma v\,(t)+gm) \;= -\frac{\gamma}{m}t+c$$

$$\gamma v\,(t)+gm \;= e^{-\frac{r}{m}t+c}$$

$$v\,(t) = \frac{\left(ce^{-\frac{\gamma}{m}t}-gm\right)}{\gamma};\;\; v_0 \;= \frac{(ce^0-gm)}{\gamma}\;; c = gm+\gamma v_0$$

$$v\,(t) = \frac{\left((gm+\gamma v_0)e^{-\frac{\gamma}{m}t}-gm\right)}{\gamma}$$

2. What is y(t)?

$$v\,(t) = y'\,(t)$$

$$y\,(t) = \int v\,(t)\,dt$$

$$y\,(t) = \int \frac{\left((gm+\gamma v_0)e^{-\frac{y}{m}t}-gm\right)}{\gamma}\,dt$$

$$y\,(t) = \frac{-m(\gamma v_o+mg)e^{\frac{-\gamma t}{m}}}{\gamma^2} - \frac{mgt}{\gamma}+c$$

Note: $\frac{m(rv_0+mg)}{r^2}$ in above equation is the C value, found by using initial value y(0)=0.

$$0 = \frac{-m(\gamma v_0+mg)}{\gamma^2}+c$$

$$y\,(t) = \frac{-m(\gamma v_0+gm)e^{-\frac{\gamma}{m}t}}{\gamma^2} \;-\frac{gm}{\gamma}t \;+\frac{m(\gamma v_0+mg)}{\gamma^2}$$