2018-11-06 Second-Order ODEs - Linear Homogeneous (4.3)

4.3 Linear Homogeneous

7. What happens to the key is critically damped?

$$y'' = -9y - 6y'$$

- **c. Which ODE does v(t) satisfy?**
  - $y(t) = c_1 e^{-3t} v(t)$
  - $y'(t) = -c_1 3e^{-3t} v(t) + c_1 e^{-3t} v'(t)$
  - $y''(t) = c_1 \left[9e^{-3t} v(t) - 3v'(t) e^{-3t} + v''(t) e^{-3t} + v'(t) e^{-3t}(-3)\right]$
  - Plug $y(t), y'(t), y''(t)$, into $y'' = -9y - 6y'$
  - We get ODE for v(t): $c_1 v''(t) e^{-3t} = 0$
  - So $v''(t)$ has to be zero, $v''(t) = 0$
  - $v'(t)$ = constant = c_2
- **d. Find v(t). Find y(t).**
  - Given $v''(t) = 0$ and v'(t) = c_2, we integrate again $v'(t)$ and obtain $v(t) = c_2 t + c_3$
  - Therefore, $y(t) = c_1 e^{-3t}(c_2 t + c_3)$ = $c_1 c_2 t e^{-3t} + c_1 c_3 e^{-3t}$ = $a_1 t e^{-3t} + a_2 e^{-3t},\ a_1 = c_1 c_2\ and\ a_2 = c_1 c_3$
  - We can see from the above $y(t)$ that $a_1 t e^{-3t}$ is our new solution while $a_2 e^{-3t}$ is the first solution we had found

4.5 Method of Undetermined Coefficients

Pre-lecture:

Sometimes we need to solve problems in the form:

$$ay''(t) + by'(t) + cy(t) = g(t)$$

for some function $g(t)$.
Below is an example on how to solve a problem in this form. First, we solve for the homogeneous part of the solution, $y_h(x)$.

$$y' - 2y = x + 2$$

$$y' - 2y = 0$$

$$y' = 2y$$

$$y = c_1 e^{2x}$$

Next, we solve for the particular solution component, $y_p(x)$, of the solution. Let $y = ax + b$, so $y' = a$.

$$y' - 2y = x + 2$$

$$a - 2(ax + b) = x + 2$$

$$(-2a)x + (a - 2b) = x + 2$$

Look at the coefficient for the components with x: $-2a = 1, \therefore a = -\frac{1}{2}$.
Then, solve for b: $a - 2b = 2 \implies -\frac{1}{2} - 2b = 2, \therefore b = -\frac{5}{4}$. Write the general solution:

$$y = c_1 e^{2x} + \left(-\frac{1}{2}x - \frac{5}{4}\right)$$

where $y_h(x) = c_1 e^{2x}$ and $y_p(x) = \left(-\frac{1}{2}x - \frac{5}{4}\right)$. For the homogeneous solution component, $y_h' - 2y_h = 0$ and for the particular solution component, $y_p' - 2y_p = x + 2$.

Lecture:

Solve ODEs of the type: $ay''(t) + by'(t) + cy(t) = e^{\sin(t)}$

Let,

- $x(t)$ satisfies: $ax''(t) + bx'(t) + cx(t) = 0$
  - All solutions (Textbook 4.3). Also known as **complementary solution**.
    - $x(t) = \{e^{rt}\ or\ \ e^{at}\sin(t)\ or\ e^{at}\cos(t)\ or\ te^{rt}$
- $z(t)$ satisfies: $az''(t) + bz'(t) + cz(t) =\ e^{\sin(t)}$
  - One solution z(t) (Textbook 4.5). Also known as **particular solution**.

1. Then $y(t) = x(t) + z(t)$ satisfies which ODE?

$ay''(t) + by'(t) + cy(t) = e^{\sin(t)}$

Solve ODEs of the type: $ay''(t) + by'(t) + cy(t) = g(t)$

Idea: Some functions don't change much when we take derivatives.

2. Think of functions that don't change type when differentiated.

- 0
- Exponentials, e^t
- Trigonometry, $sin(t)$ or $cos(t)$
- Polynomial, $ct^p + \ldots + bt + a$

3. To solve,

$2y''(t) - 4y'(t) + 4y(t)$ = polynomial

We need $y(t)$ to be what kind of function?