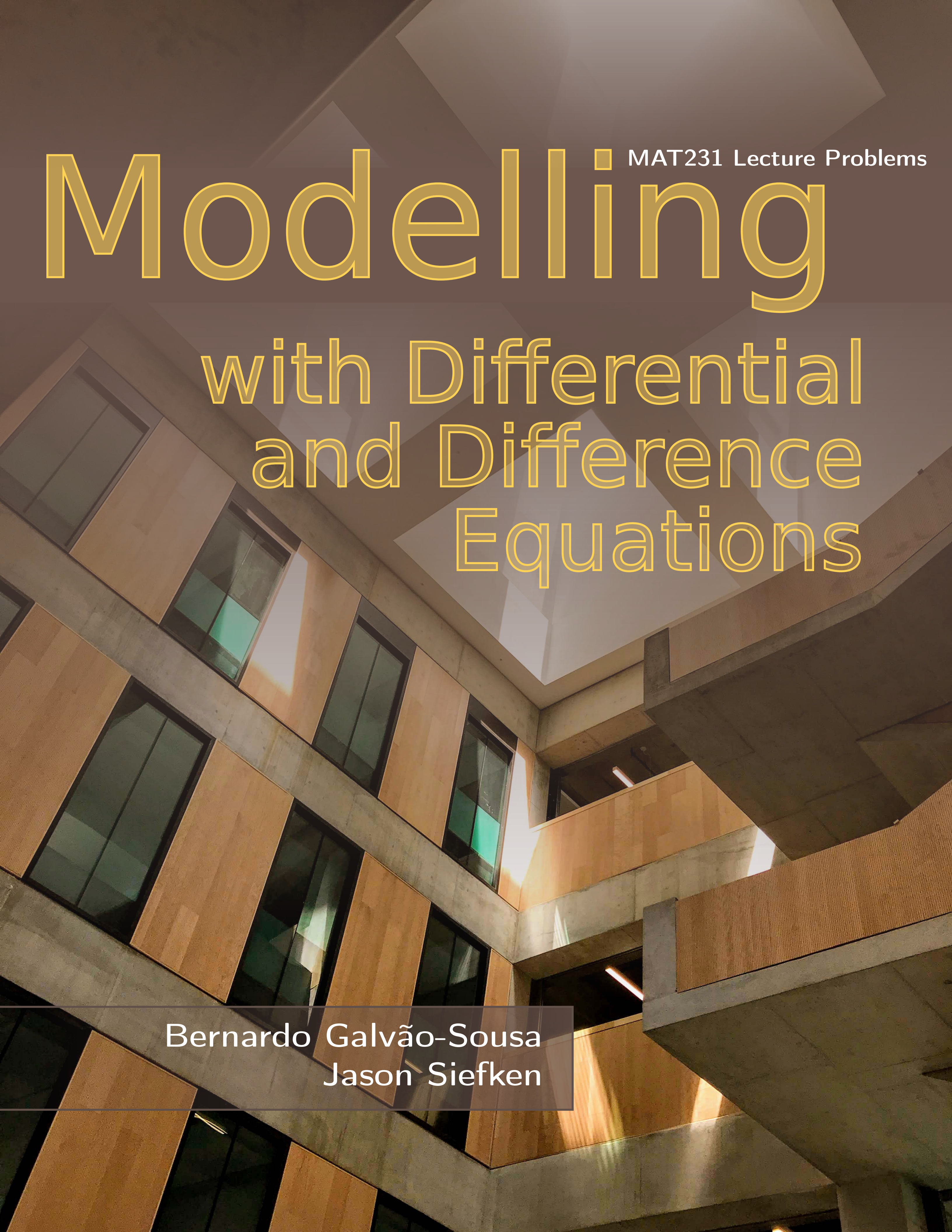

MAT231 Lecture Problems

Modelling with Differential and Difference Equations

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Inquiry Based Modelling with Differential and Difference Equations

For the student

This book is your introductory guide to mathematical modelling and modelling with differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling with differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle through you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.
5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.
2. **Work through the core exercises.**
3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.
4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

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Included in this text, in chapter 1, are expositions adapted from the handbook "Math Modeling: Getting Started and Getting Solutions" by K. M. Bliss, K. R. Fowler, and B. J. Gallizzo, published by SIAM in 2014 `https://m3challenge.siam.org/resources/modeling-handbook`.

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/bigfatbernie/IBLModellingDEs/`

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Stephanie Orfano ◦ Yvan Saint-Aubin ◦ Sarah Shujah ◦ Graeme Slaght ◦

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they're still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

2 The mayor of Toronto wants to extend the subway line with a new **orange line** as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $T =$ the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can T be minimized?

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to *optimally* control the traffic lights.

4.1 What mathematical meaning can we give to the word optimal in this circumstance?

4.2 Create a mind map for this problem.

5 Consider the elevator problem from core exercise 1.
We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Hint. Focus on a **few** parameters and variables.

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Thanks, the CEO

5.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

5.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

5.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

6 How much would it cost to make a bridge between Toronto and the U.S.?

7 Recall the core exercise 5.

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Write down a mathematical model for this problem.

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Can you suggest solutions?

Thanks, the CEO

8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

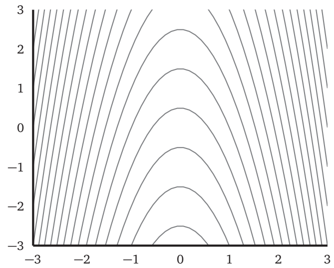
- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\begin{array}{c}\text{loading time on}\\ \text{ground floor}\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ \text{ground} \to 33\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ 33 \to \text{ground}\end{array}} + \boxed{\begin{array}{c}\text{stop time to}\\ \text{6 of the 11 floors}\end{array}} \quad = \quad 106 \text{ s}$$

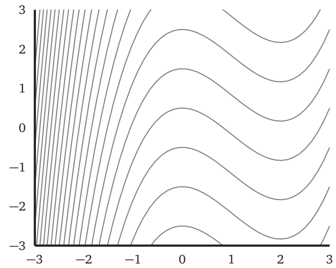
- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes
- Accumulated late time $= \boldsymbol{T} = 180 \cdot 20 \cdot 8 + 74 \cdot 20 \cdot 8 = 40\,640$ seconds $=$ 11h18m

Your task is to assess this model. Be ready to report on your assessment.

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



A



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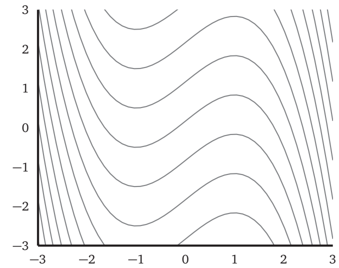

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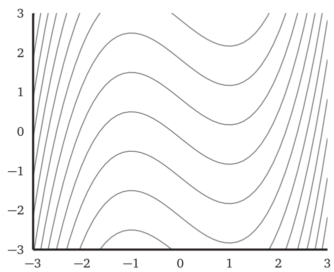

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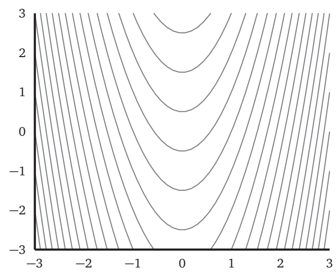

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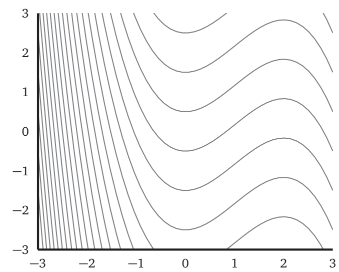


F

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph an $\boldsymbol{f(x)}$ that would produce such solutions.

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be positive

11.2 $y(t)$ must always be negative

11.3 $y(t)$ must always be decreasing

11.4 $y(t)$ must always be increasing

12 Consider the differential equation $2xy' = y$.

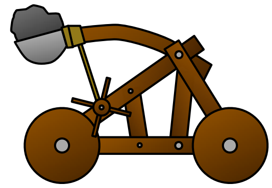
12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

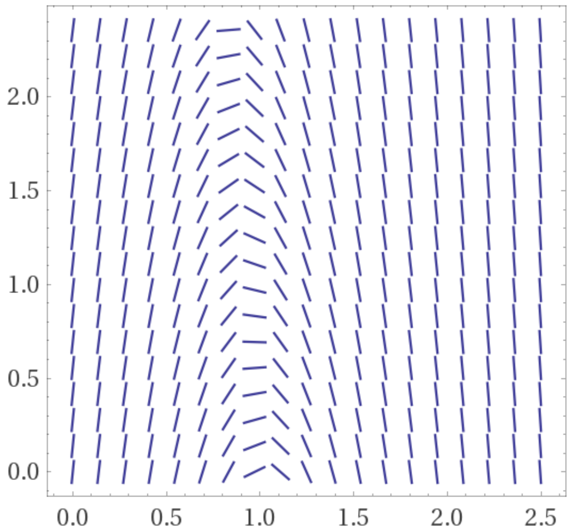
12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

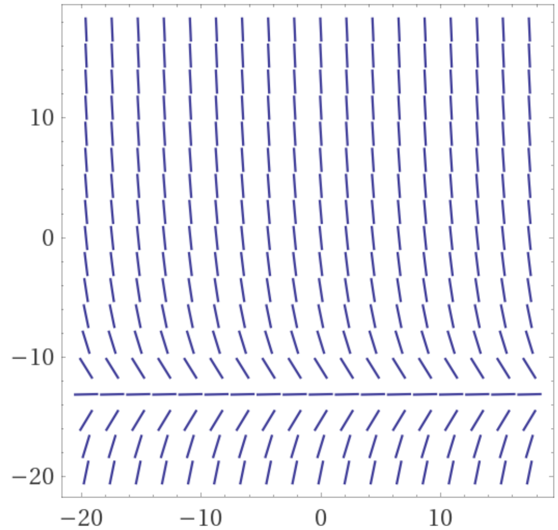
13 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

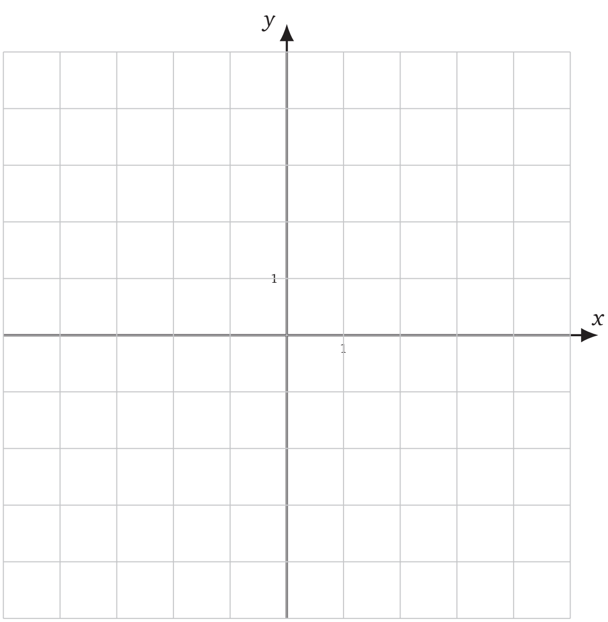
(These slope fields were created using WolframAlpha)

13.1 On the slope field, sketch a *possible* solution.

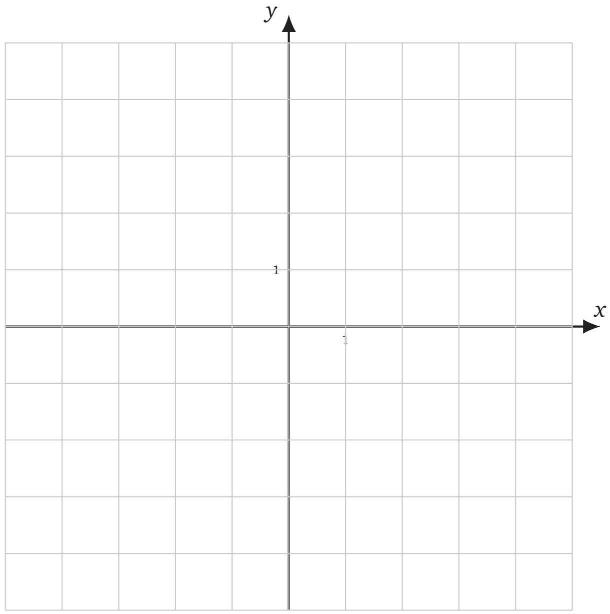
13.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

14 Sketch the slope field for the following differential equations.

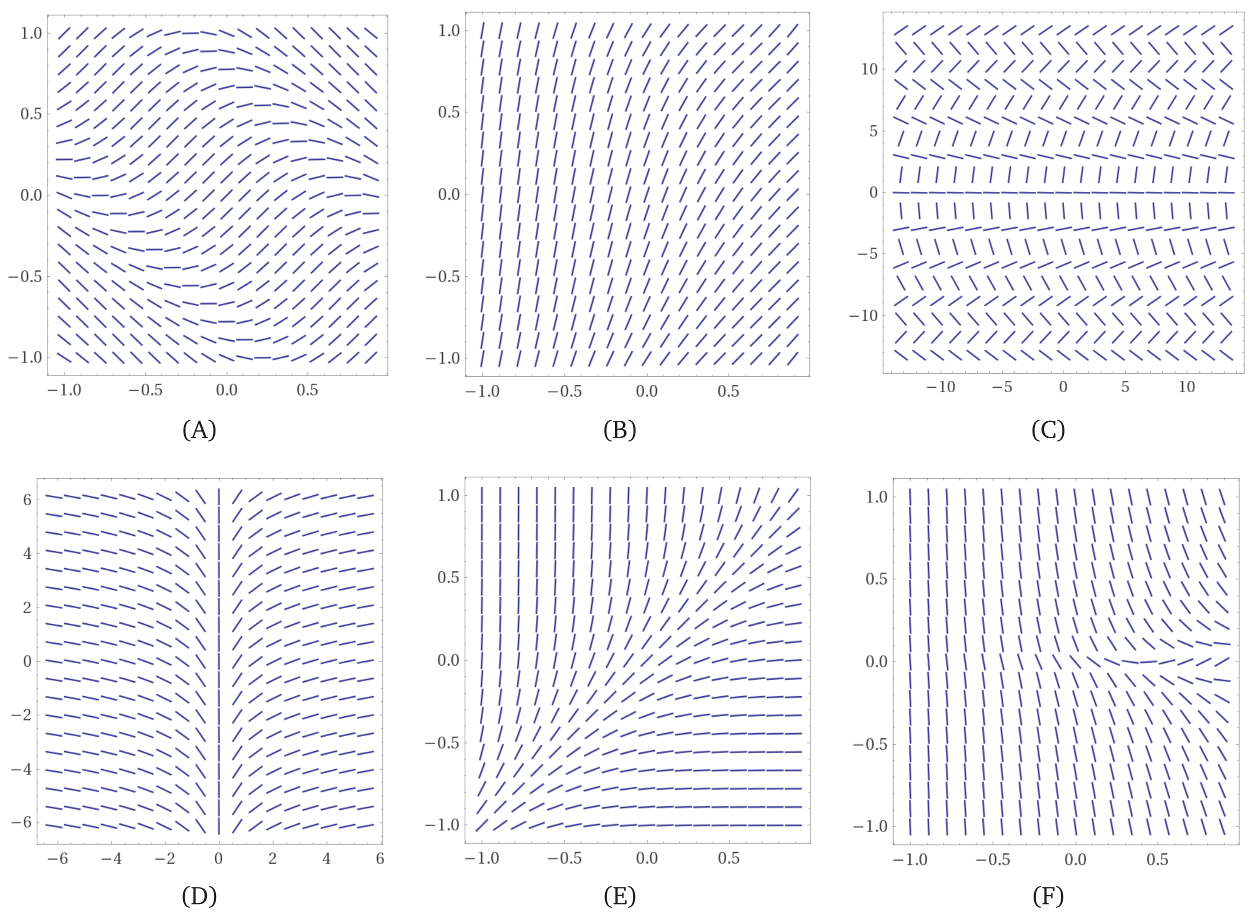
14.1 $y' = x$



14.2 $y' = y^2$



15 Consider the following slope fields:



(These slope fields were created using WolframAlpha)

15.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

15.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

15.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

15.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

15.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

15.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \left(m(x, y)\right)^2$?

16 Consider the differential equation

$$y' = y - 2.$$

16.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 3)$.

16.2 Find the solution of the differential equation with the same initial condition.

16.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 1)$.

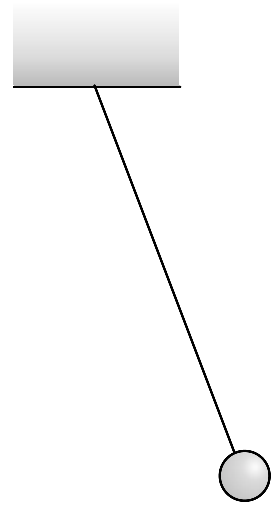
16.4 Find the solution of the differential equation with the same initial condition.

16.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

16.6 Explain in words why the Method satisfies that property.

17 Which differential equations will be approximated perfectly using Euler's Method?

18 A pendulum is swinging side to side. We want to model its movement.



18.1 Define the problem. Which function(s) do we want to find in the end?

18.2 Build a mind map.

18.3 Make assumptions. Remember to use your mind map to help structure the problem.

18.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2nd Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in — Rate out).

18.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.

19 Model the spreading of a rumour through the students of a school.

20 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

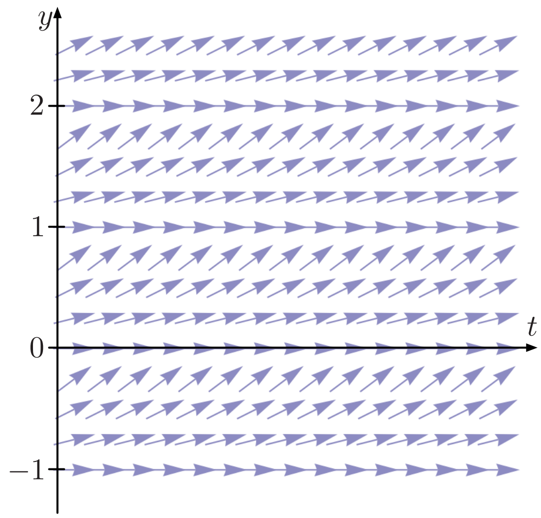
20.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

20.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

20.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

20.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

21 Consider a differential equation $y' = f(t, y)$ with the following slope field.



21.1 What are the equilibrium solutions of the ODE?

21.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \frac{1}{4} \end{cases}$$

21.3 From the direction field above, what is the type(s) of this ODE? Justify your answer.

(a) separable. **(c)** autonomous.

(b) of first-order and linear. **(d)** none of the other options.

21.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then

(select all the possible options)

(a) $g(3) < h(3)$ **(b)** $g(3) = h(3)$ **(c)** $g(3) > h(3)$

22 22.1 Calculate $\big(\sin(x)f(x)\big)'$.

22.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

22.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

23 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

23.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

23.2 $y' = t + \sqrt{y-\pi}$ with $y(1) = 1$

23.3 $y' = \sqrt{4-(t^2+y^2)}$ with $y(1) = 1$

24 The initial-value problem

$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

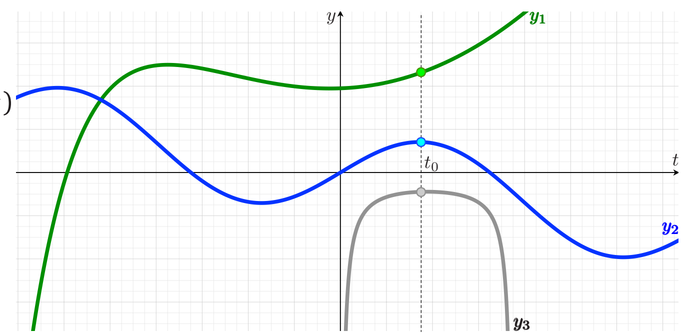
$$y_1(x) = \cos\big(\arcsin(x)\big) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

24.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

24.2 What can you conclude?

25 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all t, y;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all t, y.



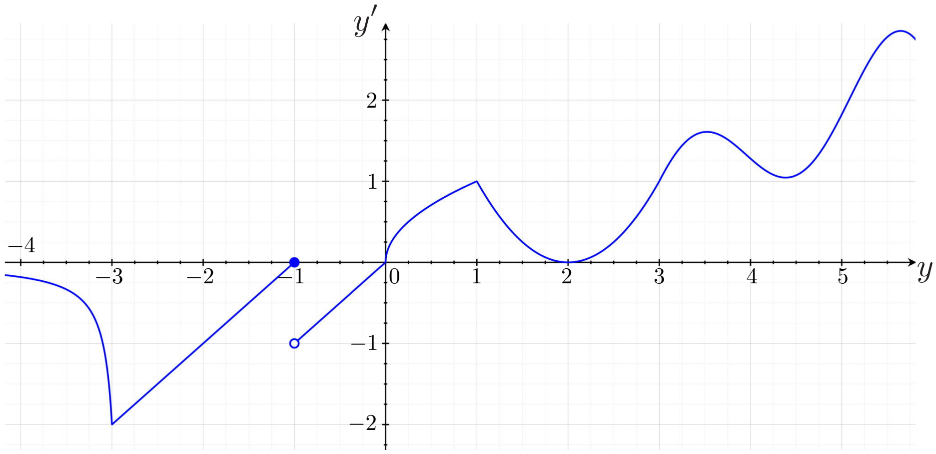
25.1 Can **green** $y1$ and **blue** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.2 Can **green** $y1$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.3 Can **blue** $y2$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

26 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



26.1 What are the equilibrium points?

26.2 Which equilibrium solutions are stable, unstable, or semi-stable?

26.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

26.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

(b) $y(0) = -\frac{1}{4}$.

(c) $y(1) = \frac{1}{4}$.

26.5 If $y(0) = 2$, then $y(t) =$

26.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

26.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$

27 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

27.1 Create a model for these two populations.

27.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

27.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

27.4 Extend the model to include a population of antelopes.

28 A cheetah is chasing an antelope. We want a model of their positions as they run.

29 Consider a cheetah-lion inspired problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{r}.$$

29.1 Find the two solutions $\vec{r}_1, \vec{r}_2$.

29.2 Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

29.3 Is $\vec{r}_1(t) - \vec{r}_2(t)$ a solution?

29.4 Is $2\vec{r}_1(t) + 3\vec{r}_2(t)$ a solution?

29.5 What is the general solution?

29.6 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

30 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r}.$$

30.1 Find the general solution.

30.2 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

31 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \vec{r} - \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

31.1 Find the equilibrium solution.

31.2 Find the general solution.

31.3 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

32 Consider the following model for cheetah's and lions, where

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

The general solution is:

$$\vec{p}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

32.1 Without computing them, what are the eigenvalues and eigenvectors of the matrix?

32.2 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = 0$.

32.3 Sketch the graph of the solution with $c_1 = 0$ and $c_2 = \pm 1$.

32.4 When one constant is set to 0, what is the shape of the graph? Is it always like that? Can you prove it?

32.5 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = \pm 1$.

32.6 Provide an interpretation of the different types of solutions.

33 Let us expand the model from the previous exercise to:

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \vec{p} + \begin{bmatrix} -10 \\ 50 \end{bmatrix} \vec{p}.$$

The extra term corresponds to the effect of harvesting 10 lions and bringing in 50 cheetahs every year to the reserve.

The general solution is:

$$\vec{p}(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

33.1 Sketch the phase portrait.

33.2 Provide an interpretation of the different types of solutions.

34 For each of the following general solutions, sketch the phase portrait.

34.1 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}.$

34.2 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}.$

35 Consider the following model for the sales from a designer clothing brand:

- $x_1(t)$ = purchases by "common mortals" (CM) at time t in years since the beginning of 2015.
- $x_2(t)$ = purchases by "famous people" (FP) at time t.

Our model is based on the following two principles:

(P_1) CM will buy more items from the brand when CM or FP buy more.

(P_2) FP will buy less when CM buy them, but will buy more when FP buy it.

The model we considered is:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \vec{x}(t)$$

35.1 Suppose that at the beginning only CM buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

35.2 Suppose that at the beginning only FP buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

35.3 What conditions on the constants a, b, c, d will guarantee that the solutions will spiral? In that case, is it a spiral source or spiral sink? Is it clockwise or counterclockwise?

35.4 Are there constants $a, b, c, d > 0$, such that the solution $\vec{x}$ is periodic?

35.5 Consider the constants $a = b = c = d = 1$. Assume that initially CM were buying $c_0 > 0$ items and FP were buying $f_0 > 0$ items. What will happen to $x_1(t)$ and $x_2(t)$ as $t \to \infty$? Explain the results in terms of the evolution of purchases from CM and FP.

35.6 Consider the constants $a = b = c = d = 1$. If $c_0 = 10$, $f_0 = 100$, then at what time will FP stop buying items? And at what time will FP be buying the maximum number of items?

36 Here are some facts about laptop keys:

(da) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.

(di) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.

(fo) On average, a person exerts the force of 42 N with one finger on a key.

(gr) Gravity is much weaker than the spring that keeps the key in place.

(hl) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").

(lo) Keys last 50 million presses on average.

(ve) Keys can only move vertically.

36.1 Model a laptop keypress.

36.2 What happens if the damping system of the key is broken? What happens if the damping system is too strong? How strong should the damping system be?

36.3 What happens to the key when the spring breaks?

37 Consider the ODE $y''(t)-9y(t)=f(t)$.

37.1 Find a complementary solution.

37.2 Find a particular solution for $f(t)=14e^{-4t}$.

37.3 Find a particular solution for $f(t)=9e^{-3t}$.

37.4 Find a particular solution for $f(t)=10\cos(t)$.

38 Consider the ODE $y''(t)-2y'(t)+5y(t)=f(t)$.

38.1 Find a complementary solution.

38.2 Find a particular solution for $f(t)=\sin(2t)e^{t}$.

38.3 Find a particular solution for $f(t)=(4t+2)\sin(2t)e^{t}$.

39 Consider the ODE $y''+3y'=3t$.

39.1 Find the complementary solution.

39.2 Find a particular solution.

39.3 Find the solution that also satisfies

$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

40 Consider the second-order ODE:

$$y''(t) - 3y(t) = t\big(2 + \sin(t)\big).$$

40.1 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) > 0$ for $t \geq 0$.

40.2 Assume that $y(0) = a < 0$ and $y'(0) = b$. Give an example of a, b such that $y(t)$ is increasing for $t \geq 0$.

40.3 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) < 0$ for all $t > 0$.

41 Consider the second-order ODE:

$$\begin{cases} y''(t) + 4y(t) = f(t) \\ y(0) = y_0 \\ y'(0) = 0 \end{cases}$$

41.1 Let $f(t) = 0$ and $y_0 = 1$. Sketch the solution.

41.2 Let $f(t) = 396\cos(20t)$ and $y_0 = 0$. Sketch the solution.

41.3 Let $f(t) = -4\sin(2t)$ and $y_0 = 1$. Sketch the solution.

41.4 Let $f(t) = 0.39\cos(1.9t)$ and $y_0 = 2$. Sketch the solution.

Hint. $\cos(at) + \cos(bt) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}t\right)$

42 Consider the difference equation

$$u_{k+1} = 6u_k - 9u_{k-1}$$

42.1 Find the solution that satisfies:

$$\begin{cases} u_0 = 1 \\ u_1 = 3 \end{cases}$$

42.2 Find the solution that satisfies:

$$\begin{cases} u_0 = 1 \\ u_1 = 4 \end{cases}$$

43 Consider a difference equation that has solutions $u_k = r^k$ for $r = 2$ and $r = 3$.

We also have the conditions:

$$u_0 = 0 = 7 \quad \text{and} \quad u_1 = 6.$$

What is u_{22}?

44 Let us expand on the economic example above.

We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

Even though we call $p\%$ the annual interest rate, because it is compounded during the year, at the end of the year the effective annual interest rate $p_{\text{eff}}\%$ is actually higher.

Calculate the effective interest rate $p_{\text{eff}}\%$.

45 Given a population with

- $\mu =$ probability that an individual will die between two seasons.

45.1 Define the following quantity

- $P(k) =$probability that an individual born at season 0 is alive at the beginning of season k.

Find a model for $P(k)$.

45.2 What is the probability of the individual dying at age k?

45.3 What is the average lifespan of an individual in this population?

46 Consider a population of special rabbits. Once a pair of rabbits is born, they grow and one year later they are still immature. But two years after they are born they give birth to another pair of rabbits.

Model this population of rabbits.

47 Consider another population of rabbits. This is the lifecycle of a pair of rabbits:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (2 pairs of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Model this population of rabbits.

48 Consider the following difference equation:

$$u_{k+1} = a(u_k - b)$$

48.1 What is the equilibrium solution?

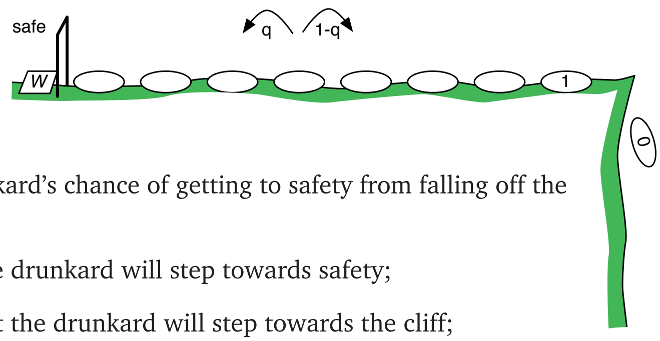
48.2 Are there periodic solutions? I.e. satisfying

- $v_0 = v_2 = v_4, v_5 = \cdots$
- $v_1 = v_3 = v_5 = v_7 = \cdots$

48.3 What happens to the solutions for different values of a?

48.4 What happens to the solutions for different values of b?

49 Consider a drunkard that is walking randomly near a cliff.



Consider this model for the drunkard's chance of getting to safety from falling off the cliff:

- q is the probability that the drunkard will step towards safety;
- $1-q$ is the probability that the drunkard will step towards the cliff;
- p_n = probability that the drunkard will get to safety if he is in step number n;
- The drunkard will stop moving if he gets to safety (step W) or if he falls out of the cliff (step 0);
- $p_n = qp_{n+1} + (1-q)p_{n-1}$.

49.1 Is p_n increasing or decreasing?

49.2 What is p_0? What is p_W?

49.3 Let $q = \frac{1}{2}$. What is $p_{W/2}$? Is p_n symmetric around $n = \frac{w}{2}$?

49.4 Let $q > \frac{1}{2}$. Is $p_{W/2} > \frac{1}{2}$? Is $p_{W/2} < \frac{1}{2}$?

49.5 How do solutions for $q = \alpha$ and $q = 1-\alpha$ compare?

50 Consider a population of rabbits with the following lifecycle:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pair of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Consider the definitions:

- We start with 1 pair of newborn rabbits in year 0;
- r_n = number of pairs of rabbits alive during year n;
- i_k = number of immature pairs;
- y_k = number of young adult pairs;
- a_k = number of adult pairs;
- o_k = number of old pairs.

50.1 Show that $b_k = b_{k-2} + b_{k-3}$.

50.2 Show that $y_{k+1} = o_k + o_{k+1}$.

50.3 Show that $r_n = r_{n-2} + r_{n-3}$.