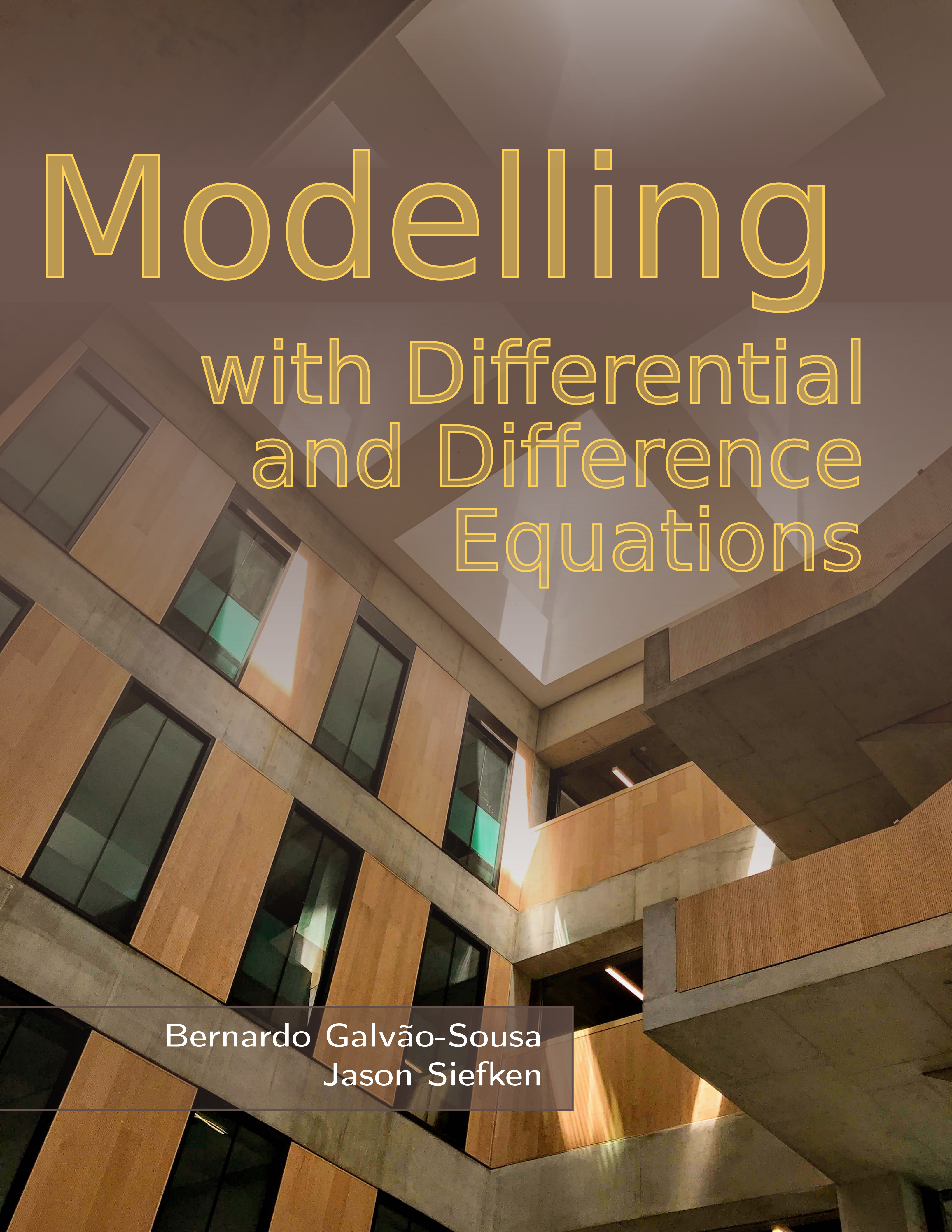

Modelling with Differential and Difference Equations

Bernardo Galvão-Sousa
Jason Siefken

Inquiry Based Modelling with Differential and Difference Equations

For the student

This book is your introductory guide to mathematical modelling and to differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling and differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle though you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a linear algebra concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.

2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.

3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.

5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.

2. **Work through the core exercises.**

3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.

4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

License

Included in this text, in chapter 1, are expositions adapted from the handbook "Math Modeling: Getting Started and Getting Solutions" by K. M. Bliss, K. R. Fowler, and B. J. Gallizzo, published by SIAM in 2014 `https://m3challenge.siam.org/resources/modeling-handbook`.

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/bigfatbernie/IBLModellingDEs/`

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Stephanie Orfano ◦ Yvan Saint-Aubin ◦ Sarah Shujah ◦ Graeme Slaght ◦

Contents

In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analysis of the model
6. Writing a report

In this chapter, we follow the approach of Bliss, Fowler, and Galluzzo from

```
Math Modeling: Getting Started and Getting Solutions, K. M. Bliss,
K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014

  https://m3challenge.siam.org/resources/modeling-handbook
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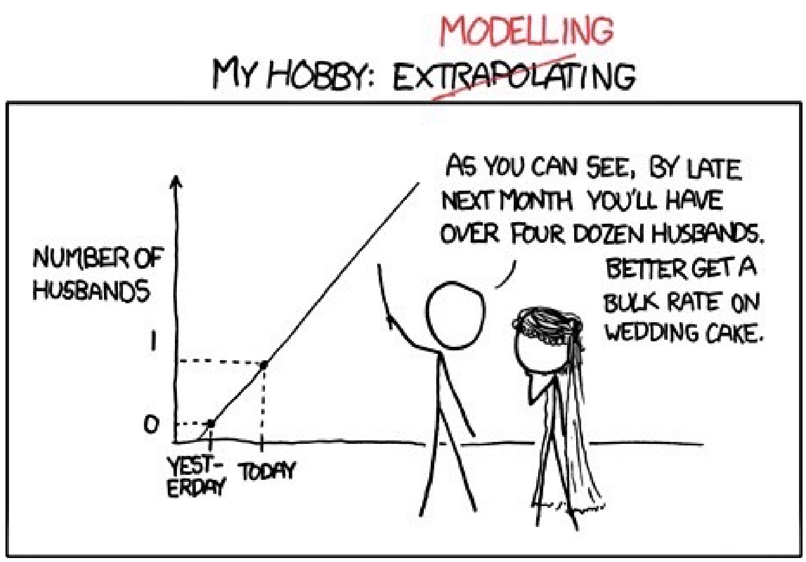


(image from xkcd - comic #605)

Defining the Problem

In this module you will learn

- how to define a problem mathematically.

The first step is to define the problem we want to solve.

To do this, we should start from the end!

We need to decide on what kind of mathematical object we will use in the end to show that we solved the problem we were tasked with.

Once this is done, we can define the problem mathematically.

Example. Your team was tasked with optimizing the layout of an airport.

The team decided to define:

- $T =$ the total time (in minutes) necessary by the average person to walk from their airport transportation (taxi, train, bus) to their gate, disregarding the time spent in security or immigration.

At the end of the project, to show that the team did find a good layout for the airport, the team will show that the new layout reduces the value of T.

Once this decision is made, the problem to solve (or improve) becomes clear:

- Minimize T

There will probably be some constraints, which will be studied in Module 4.

Practice Problems

1 For each part, what "mathematical object" would you use to communicate that you have solved or improved the problem? Then define the problem mathematically.

(a) Help the city of Toronto choose the best recycling system.

(b) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

(c) Create a mathematical model to rank roller coasters according to thrill factor.

(d) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

(e) Is it better to buy or rent?

i. Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

ii. Does the criteria you used to evaluate the previous question change if the question is whether to buy a bicycle or use Bike Share Toronto?

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their “elevator problem”.

This is the email you received:

——— Forwarded Message ———

Date: Mon, 16 September 2019 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they’re still late !?&!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that’s about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

Teamwork.
With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.

2 The mayor of Toronto wants to extend the subway line with a new **orange line** as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

Building a mind map

In this module you will learn

- How to create a mindmap.

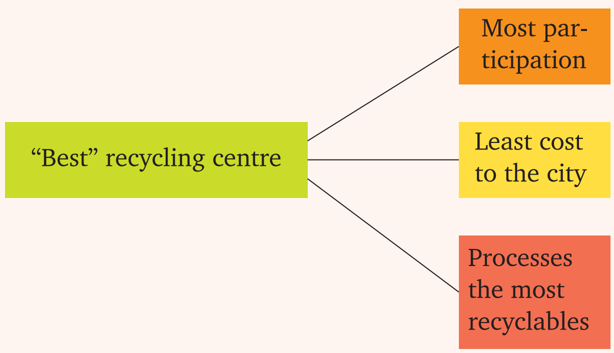
A mind map is a tool to visually outline and organize ideas. Typically a key idea is the centre of a mind map and associated ideas are added to create a diagram that shows the flow of ideas.

Example.

Let us focus on the question: "What is the best recycling system for Toronto?"
Then we can think of many different definitions for what the word "best" means:

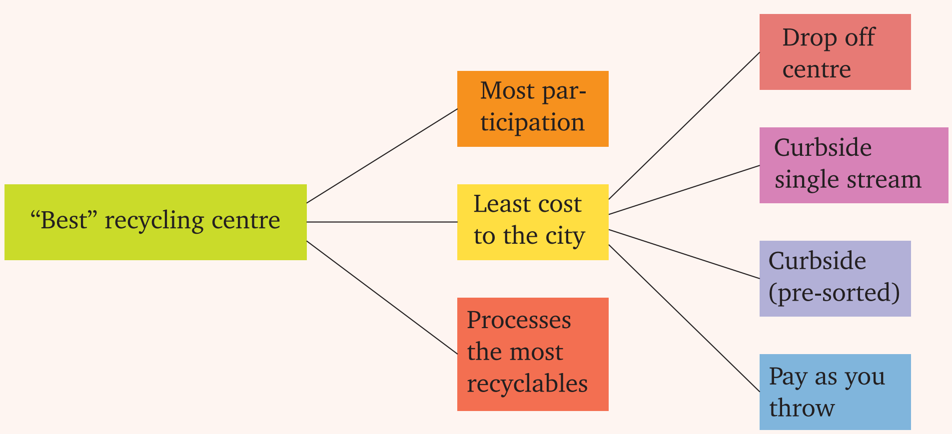
- The system that gets the most participation from the population, which can be measured by the fraction of the Toronto households participating in recycling;
- The system that costs the least amount of money for the city. How can this be measured?
- The system that processes the most amount of recyclables.

In the figure below, we focus on the definition of "best", with these three possible definitions branching off to be further explored.



From here, we can focus our attention on one of the branches at a time.

Let's think about the least-cost option first.
We probably can't determine how much any recycling program costs without knowing more about the recycling program, so a good place to start is to ask the question "What kinds of recycling programs exist?" If we aren't familiar with different types of recycling, we might need to do some research to see what kinds of programs exist.
A possible next step on your mind map for the least-cost approach could be the one shown below.



Important. There is free online software to help creating a mind map. One such is FreeMind (http://freemind.sourceforge.net).

For more details on creating a mind map, check the book:

`Math Modeling: Getting Started and Getting Solutions, K. M. Bliss, K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014`

`https://m3challenge.siam.org/resources/modeling-handbook`

Practice Problems

1 Expand the mind map from the example above by focusing on the other two approaches:

(a) Most participation

(b) Processes the most recyclables

2 For each part, create a mind map. Focus on the same approach you had for question 1 from Module .

(a) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

(b) Create a mathematical model to rank roller coasters according to thrill factor.

(c) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should “best” mean?

(d) The mayor of Toronto wants to extend the subway line with a new blue line as in core exercise 2. Is it optimal?

(e) Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $R =$ the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can R be minimized?

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to optimally control the traffic lights.

4.1 What "mathematical" meaning can we give to the word optimal?

4.2 Create a mind map for this problem.

Making assumptions

In this module you will learn

- that we need to make assumptions to be able to create a model
- how to strike a balance between accuracy and solvability

Real problems are complex, so when modelling a real problem mathematically, we must make some assumptions.

The assumptions that we make will affect the problem we are solving and its difficulty, so we need to strike a balance between:

- accuracy – the fewer assumption the better, and
- solvability – the more assumptions the better.

Many assumptions follow naturally when building a mind map.

When figuring which assumptions to make, keep in mind the key-factors of the problem and find data when available (usually online). If not available, measure data when possible, and if it's not possible, make a reasonable assumption on what the data might look like.

Another thing to keep in mind are **time constraints**. Whether in a class, test, or working in a project, there will be deadlines. Your assumptions should take time constraints into consideration.

Example.

Let us revisit the example of the previous module about the "best" recycling centre.
For this example, imagine that the team decided on focusing their attention on the least cost to the city through building drop off centres.

For this, we need to find out how many people would make use of the drop-off centres (termed "likelihood of participation").

The two extremes would be to assume that the 100% of the people near a recycling centre would use it or that none would use it. Neither of these seems like a reasonable assumption, so what would be a better assumption?

Maybe the best idea is do some investigation and see if there has been any successful research on participation rates in drop-off centres.

The team found a study that had been done in Ohio that estimated that about 15% of households participated in drop-off centre recycling, and made an assumption that this rate would hold in every city across the U.S..

One might ask if it is safe to assume that across the U.S. 15% of households will participate in drop-off centre recycling if it is available. Is it true that residents of Arizona will behave the same way residents of Ohio do? Certainly some cities would garner a participation rate much higher than 15%, while other cities would have a significantly lower participation rate. In fact, what are the chances that any city would actually have a participation rate of exactly 15%?
In some sense, one might say that assigning one participation rate to every city across the U.S. is a ridiculous assumption.

In response to that line of thinking, remember two things:

- First, remember that ***one must make assumptions in order to make a model***. It is not practical or feasible to poll every citizen of every city to determine who will bring recyclables to a drop off centre. If we had to rely on data with that level of certainty at every juncture of the modelling process, we would never get any work done.

  It's practical and important to make reasonable assumptions when we cannot find data.

- Second, you are developing a model that is intended to help one understand some complex behaviour or assist in making a complex decision. It is not likely to predict the exact outcome of a situation, only to help provide insight and predict likely outcomes. When you ***provide a list of your assumptions***, you've done your part to inform anyone who might use your model. They can decide whether they think your assumption is or is not appropriate to model the behaviour they are interested in predicting.

Practice Problems

1 For each part, you are required to make an estimate for some quantity. Make assumptions and justify them in order to solve the problem.

(a) What is the number of piano players in Toronto? **(Fermi problem)**

(b) How many linear km of roads are there in Toronto?

(c) How much salt the city of Toronto needs for its roads during the Winter?

(d) The skating season in Canada is shortening: What are the key-factors determining its length?

5 Consider the elevator problem from question 1.

We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie (in Montréal).
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Focus on a **few** parameters and variables. State hypotheses.

6

6.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

6.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

6.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Construct a model

In this module you will learn

- how to build a model based on the previous steps

This is the part of the modelling where we connect all that we have done so far: the problem we defined, the mind map, the assumptions, and all the variables and parameters in a mathematical model to answer the "mathematical" problem defined in Step A.

This usually means writing down mathematical equations, constructing a graph, analyzing a geometric figure, or do some statistical analysis.

Example. Your team is tasked with finding the best recycling centre (we looked at this example in Step B) and your team has chosen to minimize the cost to the city by using drop off centres.

As part of modelling process, your team has made the following assumptions/measurements:

- People would be willing to pay \$2.29 to recycle per month or \$0.53 per week
- People would make bi-weekly trips to the centre
- Gasoline costs around \$1.26 per litre
- On average a passenger car needs 10 litres per hundred kilometres

This means that the (one-way) distance people are willing to travel every week to the drop-off centre is

$$d = \frac{1}{4.3 \text{ trips/month}} \cdot \frac{\$2.29/\text{month}}{(\$1.26/\text{L}) \cdot (0.1 \text{ L / km})} = 4.2 \text{ km/trip}.$$

This should help us figure out the best way to place the drop-off centres:
The Mathematical model might look like this

- Maximize (number of people within a 4.2 km radius of a drop-off centre)
- subject to a certain number of drop-off centres (given by the city budget)

Sometimes, the mathematical tools necessary to tackle the problem are clear, but often they are not. In those cases it may be helpful to analyze some simple cases.

Practice Problems

1 For each part, create a model to answer the question. Remember all the previous steps.

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

7 With the same details as before in 5, write down a mathematical model for this problem.

Model Assessment

In this module you will learn

- how to analyze a model to check whether it makes sense

At this point, you have defined a problem statement, and a mind map to help you decide how to approach the problem. You have made assumptions and made note of them and justified them. You finally created a model to solve the problem.

The next step is to analyze the model.

There are two types of analysis:

Superficial assessment. Are the units correct? Are the variables and parameters of a reasonable magnitude? Does it behave as expected? Does it make sense?

In-depth assessment. Once the superficial assessment is verified, we need to understand the model at a deeper level.

What are the model's strengths? What are its weaknesses?

When you change the inputs of the model, how do the outputs change? This is called sensitivity analysis.

Next is a simple example adapted from [**?**].

Example. Modelling the flu

History of the project:

- Split population into two classes: ***infected*** and ***not infected***
- Assume that each infected person infects R number of non infected people every b days
- Define $I(n) =$ number of infected people after n days
- The two previous points imply $\quad I(n \cdot b) = R \cdot I(n)$
- We can then conclude that $\quad I(nb) = (1+R)^n I(0)$ (why?)

After plotting the resulting function $I(n)$ (click or follow the QR code on the right), we can assess our model:

Strengths:

- After two days ($b = 2$), there are 6 infected people, so it is following our assumption
- The number of infected people increases faster and faster as expected
- The disease spreads at a constant rate. Also on Desmos, check the infection rate $\dfrac{I(n+b)}{I(n)}$
- We could find an explicit formula for the number of infected individuals $I(n)$

Weaknesses:

- The model is too simple, so it doesn't model the spread of the flu accurately
- The model an exponential rate of infection, which is not possible for very long
- The model predicts that eventually the disease will spread to everyone
- The model assumes that there are only two types of people: infected and susceptible. Do people recover from the disease?

After assessing the model, if time allows, it is important to re-think the model and the assumptions made.

Module 5: Model Assessment

Practice Problems

1 Assess the models created in question **??**:

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t = \boxed{\substack{\text{loading time on}\\ \text{ground floor}}} + \boxed{\substack{\text{time of flight}\\ \text{ground} \to 33}} + \boxed{\substack{\text{time of flight}\\ 33 \to \text{ground}}} + \boxed{\substack{\text{stop time to}\\ \text{6 of the 11 floors}}} = 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes

Your task is to assess this model. Be ready to report on your assessment.

Putting it all together

In this module you will learn

- how to put all that you have done together into a well structured report

This is the final stage of the modelling project.

By now, you have started with a mathematically defined problem, with some assumptions, and you have created a mind map to help you navigate the problem. You have also constructed a model and assessed it to make sure it is sound.

All that we have left is to put all this work together into the form of a report.

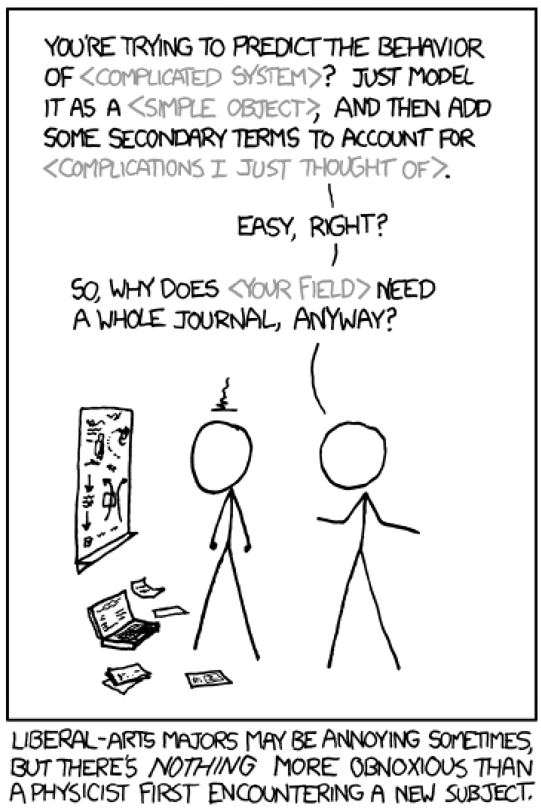
The report should consist of two parts:

1. **Summary.** Should be at most one page long, and contain a statement of the problem, a brief description of the methods chose to solve it, and some final results and a conclusion. In this part of the report, you should keep mathematical symbols to a minimum, so the reader gets an idea of what to expect in the remainder of the report without getting bogged down in unfamiliar mathematics.
2. **In-depth report.** This is where the details go in. It should start with an introduction to the problem assuming that the reader is not aware of it. It should then be structured according to the steps we did before:
   - Optionally, you can include a mind map with a description of how it guided the whole process
   - Assumptions and variables in the model
   - The model described in detail
   - The solution process
   - The assessment of the model
   - A conclusion, with a description of the results

Example. You can find the report from the winning team of the 2019 M_3C challenge in appendix 6.3.

Module 6: Putting it all together

Chapter 2 – First-Order Differential Equations



(image from xkcd - comic #793)

Introduction to Differential Equations

In this module you will learn

- what is a differential equation
- the different types of differential equations

Differential Equation. A ***differential equation*** is an equation involving an unknown function and one or more of its derivatives.

Among differential equations, there are lots of types, that require different approaches, so we need to classify them.

Types of Differential Equations. There are two main types of differential equations:

- ***Ordinary differential equations***, usually denoted as ODEs – when the unknown function is a function of one variable;
- ***Partial differential equations***, usually denoted as PDEs – when the unknown function is a a function of several variables.

In this book, we are going to focus only on ordinary differential equations.
Among ordinary differential equations, we distinguish them according to:

- ***order***: the order of a differential equation is the order of the highest derivative present in the differential equation;
- ***linear*** vs ***nonlinear***: A differential equation $F(t, y, y', \dots, y^{(n)}) = 0$ is called ***linear*** if F is a linear function of $y, y', \dots, y^{(n)}$. Linear ODEs have the form

$$a_0(t)y(t) + a_1(t)y'(t) + \cdots + a_n(t)y^{(n)}(t) = g(t).$$

All other differential equations are called ***nonlinear***.

Roughly, to check whether an ODE is **linear**, we need to check that:

- The unknown y and its derivatives appear with exponent 1;
- The unknown y and its derivatives do not multiply by each other;
- The unknown y and its derivatives are not the objects of other functions – there are no occurrences of things like $\sin(y)$ or $e^{y'}$, $\ln(y'')$, $\sqrt{y^{(3)}}$, etc.

In general, when tackling a differential equation, linear ODEs are easier to solve and study than nonlinear.

In the following chapters, observe how the methods and theory for linear ODEs is much more developed. Nonlinear ODEs are usually tackled on a case-by-case basis, and there is no theory that applies to a class of nonlinear ODEs.

Fortunately, many important problems are modelled by linear equations.

A common approach to nonlinear problems is to "transform" them into a linear problem. This means that the new linear problem is easier to study, but will be an approximation of the original problem, and often that approximation is only reasonable within some restricted conditions.

Example. Consider the nonlinear ODE

$$y' = -\sin(y).$$

This is a nonlinear ODE. However, by Taylor's Theorem, we can approximate the function $\sin(y)$ by y, as long as $|y|$ is very small.
So we can say that the solution of the original solution is very close to the solution of

$$y' = -y,$$

as long as $|y|$ is very small.

Module 7: Introduction to Differential Equations

Solutions of Differential Equations

In this module you will learn

- what is a solution of a differential equation
- the difference between a solution and an integral curve

Assume that we have found a differential equation that models a situation. Often the goal is to figure out what happens, so we usually attempt to either solve the differential equation and obtain a solution or to find an approximation for the solution.

In this module, we will discuss solutions in more detail.

Solution. Given a differential equation, a ***solution*** is a differentiable function that satisfies the differential equation.

Example. Consider the differential equation

$$t\frac{du}{dt} = u + t^2\cos(t).$$

Then the function

$$u(t) = t\sin(t)$$

is a solution, because

$$t\frac{du}{dt} = t\big(\sin(t) + t\cos(t)\big) = t\sin(t) + t^2\cos(t) = u + t^2\cos(t).$$

Integral curve. We can represent all the solutions geometrically as an infinite family of curves. These curves are called ***integral curves***.

Example. Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

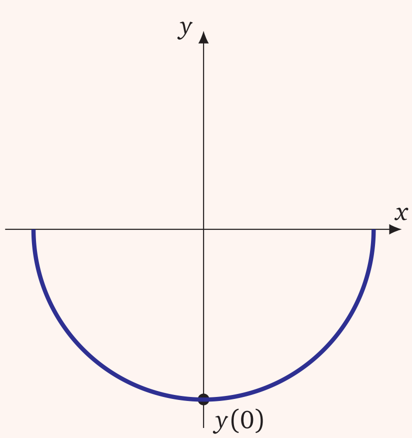
Then, we can check that curves of the form $x^2 + y^2 = C$ satisfy this differential equation.

This gives us the solution

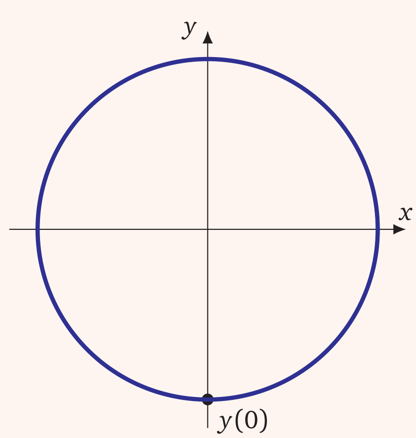
$$y(x) = -\sqrt{9 - x^2}.$$

However, the integral curve for this initial-value problem is the curve

$$x^2 + y^2 = 9$$



Solution of the initial-value problem



Integral curve for the initial-value problem

Practice Problems

1 Check that curves of the form $x^2 + y^2 = C$ satisfy the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

2 Is the piecewise-defined function

$$y(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$?

3 Consider the differential equation

$$y^{(4)} - 8y^{3)} + 26y'' - 40y' + 25y = 0.$$

(a) Is $y = 4e^{2x}\sin(x)$ a solution?

(b) Is $y = -8xe^{2x}\cos(x)$ a solution?

(c) For the two function above, if they are solutions, what are initial conditions of the form

$y(0) =$
$y'(0) =$
$y''(0) =$
$y'''(0) =$

that the solution satisfies?

4 Consider the functions

$$\begin{array}{ll} f(x) = 3x + x^2 & g(x) = e^{-7x} \\ h(x) = \sin(x) & j(x) = \sqrt{x} \\ k(x) = 8e^{3x} & \ell(x) = -2\cos(x) \end{array}$$

Match each differential to one or more functions which are solutions.

(a) $y' = 3y$

(b) $y'' + 9y' + 14y = 0$

(c) $y'' + y = 0$

(d) $2x^2y'' + 3xy' = y$

5 Consider the differential equation $u' = -2(u - 10)$.

(a) Check that the curves of the form $u = 10 + Ce^{-2t}$ satisfy the differential equation.

(b) Sketch one solution of the differential equation.

(c) Sketch all the integral curves for the differential equation.

(d) What is the difference between a solution passing through the point $(1, 20)$ and an integral curve passing through the same point?

6 Consider the differential equation $y'\big(3y^2 - 1\big) = 1$.

(a) Check that the curves of the form $y^3 - y = x + C$ satisfy the differential equation.

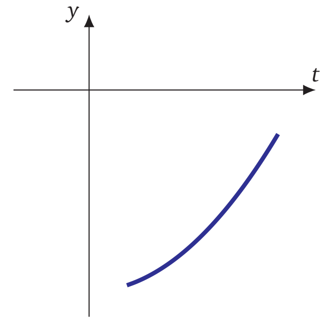
(b) Sketch the solution of the differential equation that passes through $(1, 1)$.

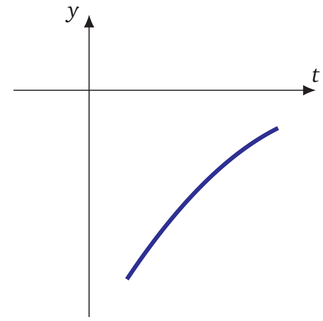
(c) Sketch the integral curve for the differential equation that passes through $(1, 1)$.

(d) What is the difference between a solution passing through the point $(1, 1)$ and an integral curve passing through the same point?

(e) Repeat (b)–(d) with the points $(1, 0)$ and $(1, -1)$ instead of $(1, 1)$.

7 Consider the ODE $y'(t) = \big(y(t)\big)^2$. One of these two graphs **cannot** describe the solution. Which one?



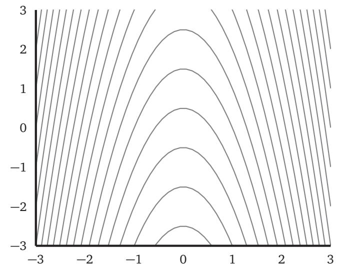


8 We seek a first-order ordinary differential equation $y' = f(\mathbf{y})$ whose solutions satisfy

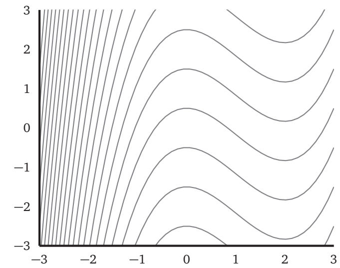
$$\begin{cases} y(x) \text{ is concave up if } y < 1 \\ y(x) \text{ is concave down if } y > 1 \end{cases}$$

Write down or graph a function $f(y)$ that would produce such solutions.

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



A



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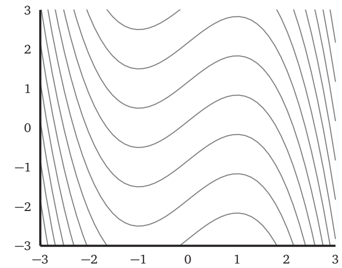

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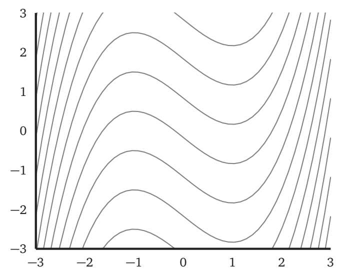

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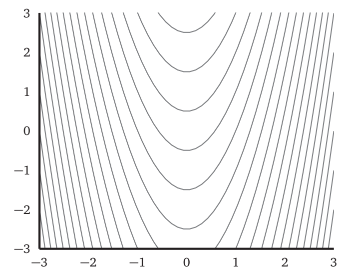
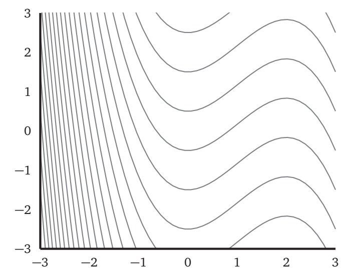


E



F

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph an $\boldsymbol{f(x)}$ that would produce such solutions.

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be positive

11.2 $y(t)$ must always be negative

11.3 $y(t)$ must always be decreasing

11.4 $y(t)$ must always be increasing

12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

Slope Fields

In this module you will learn

- what is a slope field
- how to sketch a slope field
- to interpret a slope field

As we saw in the previous module, once we have found a differential equation that models a situation, we often want to figure out what happens to the solution.

In this module, we will focus on getting an idea of the solutions and integral curves using what is called a **slope field**.

Slope field. Consider the equation $y' = f(x, y)$. If we evaluate $f(x, y)$ over a rectangular grid of points, and we draw an arrow at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all the arrows is called a ***slope field***.

We can sketch Slope Fields with Wolfram Alpha.
For a differential equation $\dfrac{dy}{dx} = f(x, y)$, we need to input

- Vector Field: $(1, f(x, y))$.

`http://www.wolframalpha.com/input/?i=slope+field`

Example. Let us take an example from the previous module.

Consider the initial-value problem

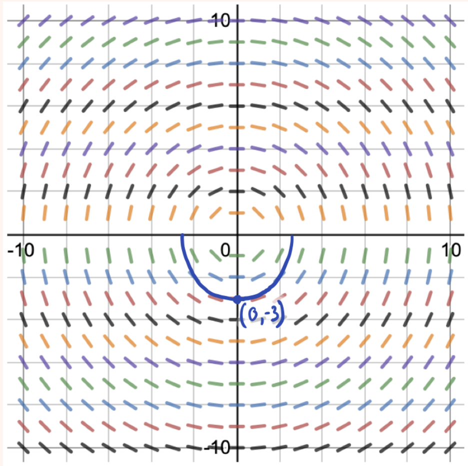
$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

We can use this definition to sketch the slope field for the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

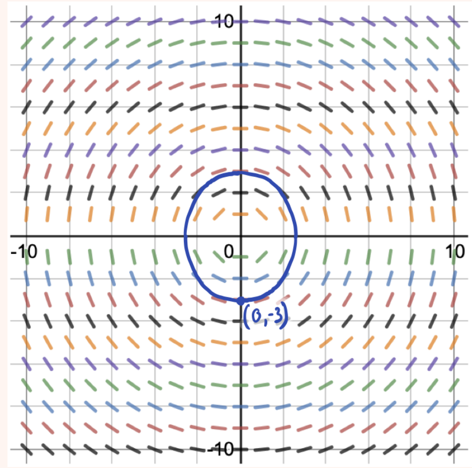
We now sketch this slope field with Desmos:

`https://www.desmos.com/calculator/scmz6ps0or`

Now notice that the arrows have the slope of a solution. This means that solutions will be tangent to the arrows, so we can *roughly* trace the solution by following the arrows.
Below, we did just that starting with the point $(0, -3)$.



approximated solution



approximated integral curve

Important. Remember that this gives us only an approximation of the solution and integral curve. From the approximation, we can tell that the solution seems circular, but we still need to show that it is so.

Video.

- `https://youtu.be/MI2xCwBekX4`
- `https://youtu.be/8Amgakx5aII`

Practice Problems

1 Use Wolfram Alpha, Desmos, or another software to sketch the slope field for the following differential equations. Then roughly trace different solutions.

(a) $y' = 2y - x$

(b) $y' = xy$

(c) $y' = \cos(y)$

(d) $y' = \frac{1}{2} + \cos(y)$

(e) $y' = 1 + \cos(y)$

(f) $y' = 2 + \cos(y)$

(g) $y' = \sin(xy)$

(h) $y' = \tan(x + y)$

2 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where

$$f(x,y) = \begin{cases} -x & \text{if } x < 1 \\ y & \text{if } x \geq 1 \end{cases}$$

3 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where the function $f(x, y)$ satisfies all of the following properties:

(a) $f(x, y)$ is continuous

(b) $f(x, y) > 0$ when $x > 1$ and $y > 1$

(c) $f(x, y) < 0$ when $x < -1$ and $y < -1$

(d) $f(x, y)$ depends only on x when $x < -1$ and $y > 1$

(e) $f(x, y)$ depends only on y when $x > 1$ and $y < -1$

4 (a) On the slope field from the previous problem, show that there must exist a smooth continuous curve with horizontal lines.

(b) Show that the curve divides the (x, y) plane in two parts.

5 Consider a differential equation

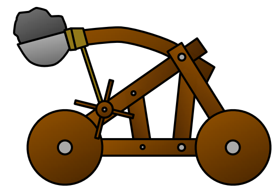
$$y' = f(x, y)$$

where the solutions satisfy

$$\lim_{x \to \infty} y(x) = 1.$$

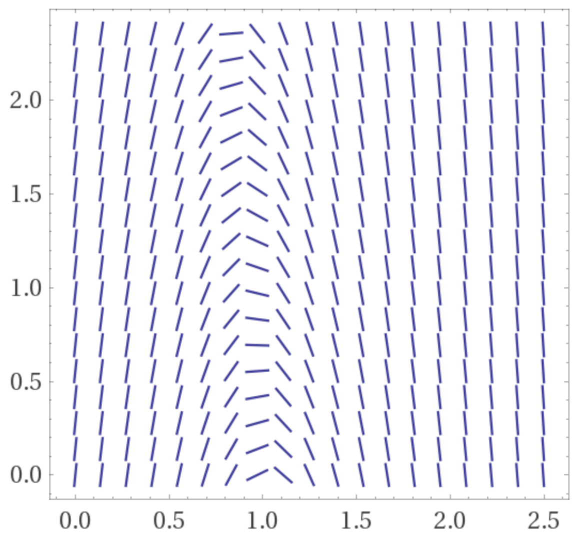
(a) What property must the slope field satisfy?

(b) Sketch a possible slope field for this differential equation.

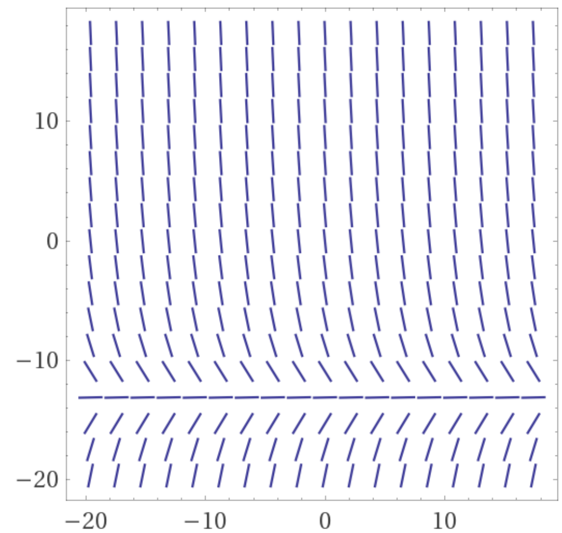


13 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.

Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

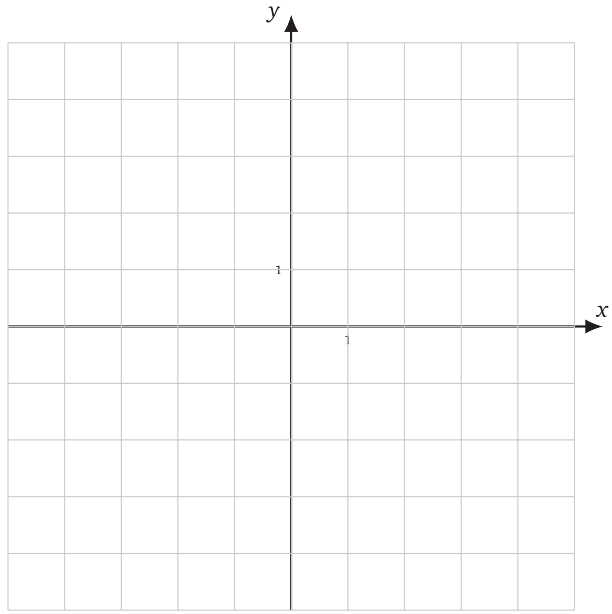
(These slope fields were created using WolframAlpha)

13.1 On the slope field, sketch a *possible* solution.

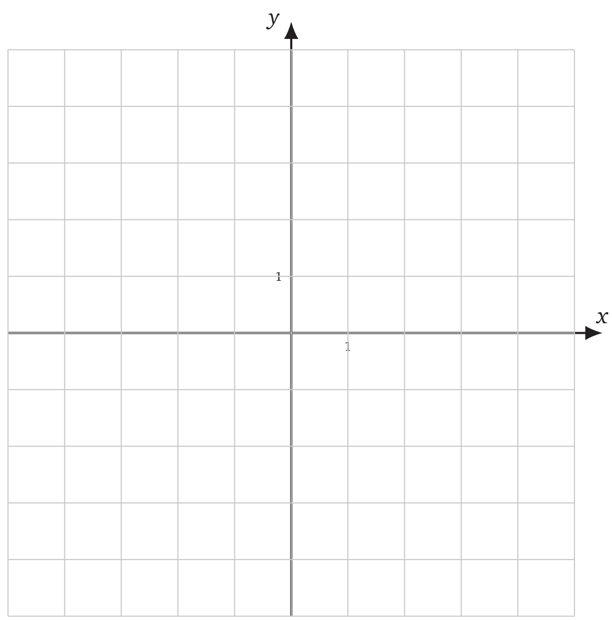
13.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

14 Sketch the slope field for the following differential equations.

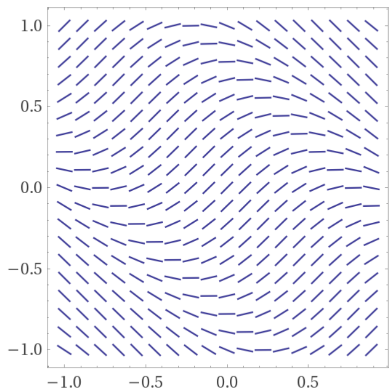
14.1 $y' = x$



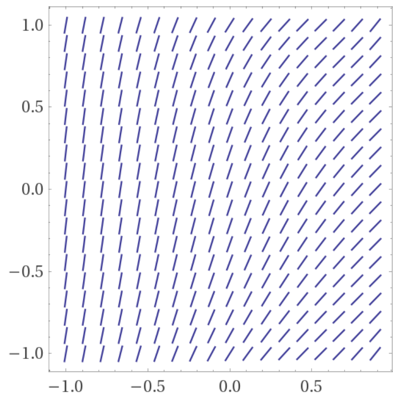
14.2 $y' = y^2$



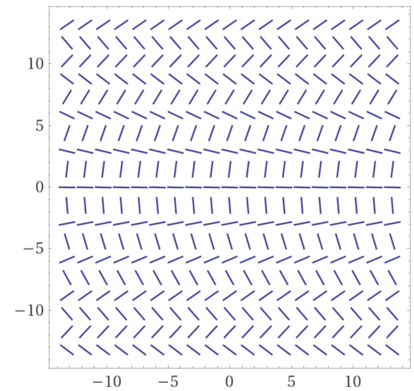
15 Consider the following slope fields:



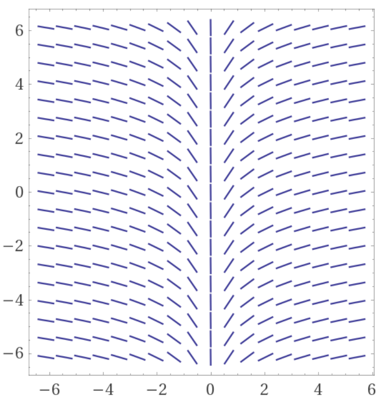
(A)



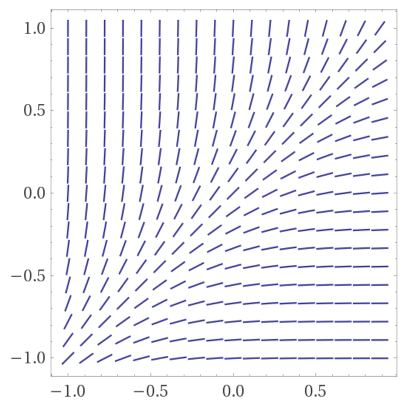
(B)



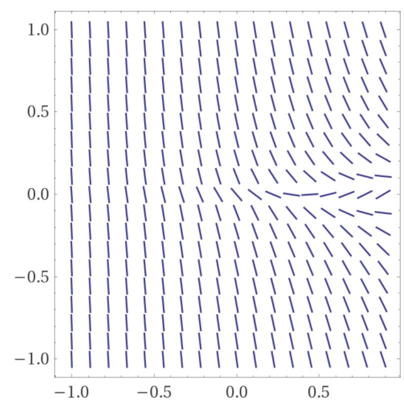
(C)



(D)



(E)



(F)

(These slope fields were created using WolframAlpha)

15.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

15.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

15.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

15.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

15.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

15.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \big(m(x,y)\big)^2$?

Approximating Solutions

In this module you will learn

- to approximate the solutions of differential equations

We just learned to sketch a slope field and how to use it to sketch a rough approximation of a solution of a differential equation.

The method of "following the arrows" of a slope field, when formalized mathematically is called **Euler's Method**.

So let us start with an initial-value problem

$$\begin{cases} y'(t) = f\big(t, y(t)\big) \\ y(0) = y_0 \end{cases}$$

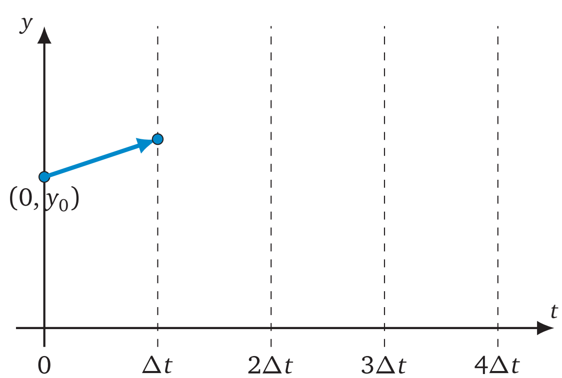
The idea is to follow the directions given by the differential equation, so we know that

- $y(0) = y_0$
- $y'(0) = f(0, y_0)$

This means that we have a starting point $(0, y_0)$. We still need to decide the distance that we want to follow the arrow:

- smaller distance: more accurate approximation, but will take more calculations
- longer distance: less accurate approximation, but will take fewer calculations

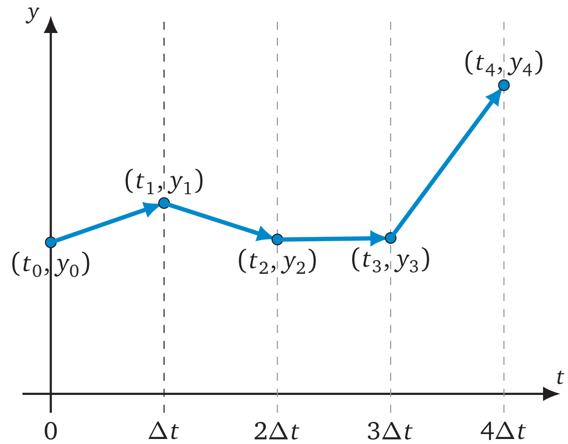
The typical way to decide is to set a parameter Δt, that measures the distance we will travel in the t-axis.



This way we find our second point $(\Delta t, y_1)$ where:

$$\frac{y_1 - y_0}{\Delta t} = \text{ slope of the arrow } = f(0, y_0) \quad \Rightarrow \quad y_1 = y_0 + f(0, y_0)\Delta t$$

We continue in this way to find more points (t_i, y_i):



Euler's Method. Let $y'(t) = f(t, y)$ be a first-order differential equation. The ***Euler approximation*** to the initial value problem $y'(t) = (f(t, y)$ and $y(t_0) = y_0$ with step size Δt is the sequence of points (t_i, y_i) given by (t_0, y_0) if $i = 0$ and

- $t_i = t_{i-1} + \Delta t$
- $y_i = y_{i-1} + f\big(t_{i-1}, y_{i-1}\big)\Delta t$.

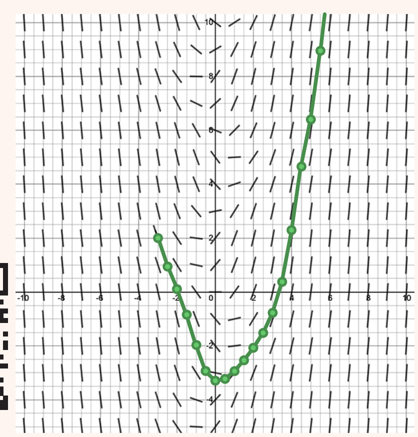
The method used to generate (t_i, y_i) is called ***Euler's Method***.

Example. Consider the initial-value problem

$$\begin{cases} y'(t) = \sin(y) + t \\ y(-3) = 2 \end{cases}$$

Then, we can follow Euler's Method with $h = 0.5$ to obtain:

- $y_0 = 2$
- $y_1 = 2 + \frac{1}{2}\big(\sin(2) - 3\big) \approx 0.95$
- $y_2 = 0.95 + \frac{1}{2}\big(\sin(0.95) - 2.5\big) \approx 0.1$
- $y_3 = 0.1 + \frac{1}{2}\big(\sin(0.1) - 2\big) \approx -0.85$



Here is the link to the desmos graph:

- https://www.desmos.com/calculator/kkgj5jhggd

Video.

- `https://youtu.be/q87L9R9v274`
- `https://youtu.be/g3Xw1r7QGOE`
- Euler's Method helping to take a person to the Moon

Practice Problems

1 For the following initial-value problems, approximate their solution with different values of Δt and compare with their exact solutions.

(a) $y' = -y + 5 + t,\ y(0) = 4, 5, 6$

(b) $y' = y + 5 - t,\ y(0) = -4$

(c) $y' = (t - y)\sin(y),\ y(0) = -1$

(d) $y' = \dfrac{y + 3t}{1 + t^2},\ y(0) = -1, 1$

Hint. Write a computer program that does the approximation for you.

2 Consider the differential equation

$$y' = -\frac{x}{y}.$$

(a) Sketch a slope field for this differential equation.

(b) Use Euler's Method to approximate the solution for some values of Δx and for some initial conditions.

(c) Does Euler Method do a good job approximating the solution?

3 In this module, we derived Euler's Method. One of the main steps was obtaining the equation

$$\frac{y_1 - y_0}{\Delta t} = \text{slope of the arrow.}$$

In Euler's Method, we used the slope at the beginning of the arrow. We can derive a new Method where we use the slope at the end of the arrow.

(a) Find a formula and the algorithm for this new method.

(b) Use this method with to approximate the solution of $y' = -y + 5 + t,\ y(0) = 4, 5, 6$ and compare the results with your results from question 1.

(c) Which of these two methods gives a better approximation?

(d) In your opinion, which of these two methods is better? Why?

4 Consider an initial-value problem with solution $y(t)$. If we want to find an approximation for $t \in [0, T]$, we define the error of the approximation $\{y_i^{\Delta t}\}$ by

$$E(\Delta t) = \left| y(T) - y_N^{\Delta t} \right|, \qquad \text{(E)}$$

where $T = N\Delta t$.

(a) For the initial-value problems from the previous question, study what happens when the value of Δt decreases.

(b) What do you expect to happen as Δt converges to 0?

(c) Estimate how fast Euler's method converges. Find a value of p such that

$$E(\Delta t) \le C(\Delta t)^p,$$

where the constant C changes for each ODE, but doesn't change if you keep the same ODE but change only the value of Δt.

5 Using Euler's Method with a step size of $\Delta t = 0.05$, and keeping only three digits throughout your computations, determine the approximations at $T = 0.2, 0.3, 0.4$ for each of the following initial-value problems.

(a) $y' = -y + 5 + t,\ y(0) = 4$

(b) $y' = y + 5 - t,\ y(0) = -4$

Compare the results with what you obtained for problem 1. Where do the differences come from?

6 Round-off errors become important when the value of N is very large, which happens if we want a very accurate approximation. This means that the actual error (E) of the approximation has two components:

$$E(\Delta t) = f(\Delta t) + g(\Delta t),$$

where

- $\lim_{\Delta t \to 0^+} f(\Delta t) = 0$ (approximation error)
- $\lim_{\Delta t \to \infty} f(\Delta t) = \infty$ (approximation error)
- $\lim_{\Delta t \to 0^+} g(\Delta t) = \infty$ (round-off error)
- $\lim_{\Delta t \to \infty} f(\Delta t) = 0$ (round-off error)

Answer the following questions and justify your answers based on these ideas.

(a) Justify why the four limits above make sense.

(b) Does the approximation converge to the solution as $\Delta t \to 0$?

(c) Is there an optimal Δt that gives the best possible approximation?

16 Consider the differential equation

$$y' = y - 2.$$

16.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 3)$.

16.2 Find the solution of the differential equation with the same initial condition.

16.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0, 1)$.

16.4 Find the solution of the differential equation with the same initial condition.

16.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

16.6 Explain in words why the Method satisfies that property.

17 Which differential equations will be approximated perfectly using Euler's Method?

Modelling with Differential Equations

In this module you will learn

- how to start modelling a physical phenomenon into a differential equation

We started by studying some mathematical modelling in chapter 1. Then, we just used mathematical tools that we learned before.

We now want to focus on mathematical models that arise from physical applications. These will often take the form of one or more differential equations.

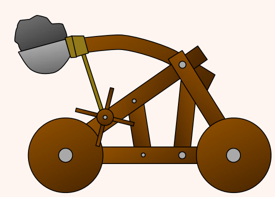
The modelling of the situation will develop in a similar way.

Step 1. Defining the problem

As before, we should start by thinking about what our ultimate goal is. Once we settle on a goal, we define it as the function we want to study.

Example.

In this module, we are going to think about the catapult problem from Module 9 - Slope Fields.



A catapult throws a projectile into the air.
Our goal is to track the height (in metres) of the projectile from the ground.
This means that we have a goal: to find the height of the projectile at every moment in time after it is launched.

This means that we define

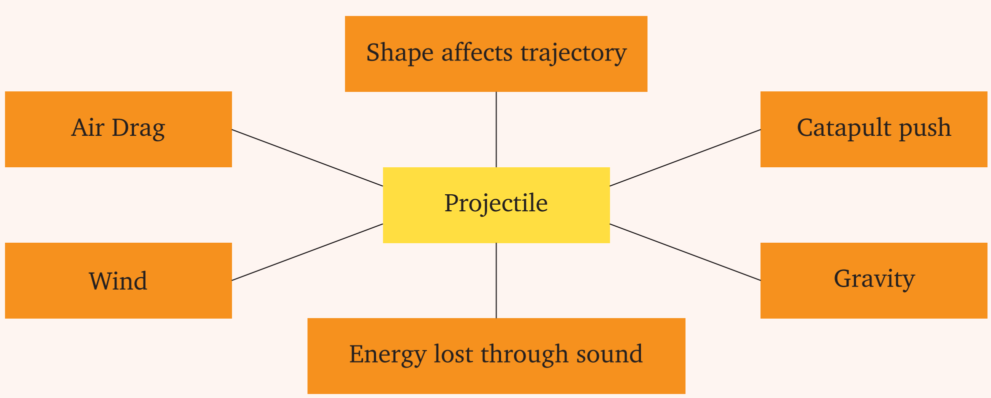
- $y(t) =$ height of the projectile, in metres, t seconds after it was launched from the catapult.

Step 2. Building a mind map

A mind map will help us identify the notions that we want to include in our model.

Example.

In the catapult example, since we decided to study the projectile's height, we need to find everything that affects its height.



We can include more layers to these topics if we want.

Step 3. Make assumptions

This is a fundamental step in any modelling endeavour. The real world is too complicated, so we make assumptions that simplify our model.

This has two main consequences:

1. It makes our model simpler and easier to study;
2. It creates constraints on our model: it is only valid under certain conditions.

Example.

Let us discuss the topics included in the mind map above:

- Catapult Push – the catapult pushes on the projectile for a small period of time when $t < 0$. If we are considering only $t \geq 0$, then this will likely provide us with some starting conditions for the projectile
- Gravity – The height of the projectile is affected by gravity. We have a choice to make:
  - assume that the Earth is flat and gravity is constantly accelerating the projectile downwards;
  - assume that the Earth is spherical and gravity is constantly accelerating the projectile towards the centre of the Earth;
  - assume that the Earth is spherical and gravity is a force accelerating the projectile towards the centre of the Earth with a magnitude that decreases with the square of the distance to the centre of the Earth
  - or other more complicated and more accurate models.
- Air Drag – air is making it hard for the projectile to move forward. We have another choice to make:
  - assume that the air drag is a force that accelerates the projectile in the direction opposite to its movement and with magnitude proportional to its speed;
  - assume that the air drag is a force that accelerates the projectile in the direction opposite to its movement and with magnitude proportional to the square of its speed;
  - or other more complicated and more accurate models.

I will leave it to you to think about the remaining three topics in the mind map.

We now need to make a decision about what to assume.

To keep this model simple, let us assume the following:

1. The projectile's height will stay within a small range: $y(t) \in [0, 100]$. Is this reasonable for a catapult?

   This means that we can consider the first of the gravitational models above: define gravitational acceleration as a constant $-g$.

2. The projectile will not move very fast, so we can approximate the air drag to be directly proportional to the speed: define air drag acceleration as $\pm\gamma v$, where $\gamma > 0$ is a constant that depends on the projectile and v is the velocity of the projectile. Which sign should we have?

3. Again, the projectile will not move very fast, so we can approximate the air drag to use only the vertical speed of the projectile: define air drag acceleration as $\pm\gamma v_y$, where $\gamma > 0$ is a constant that depends on the projectile and v_y is the vertical velocity of the projectile. Which sign should we have?

4. The shape of the projectile will affect air drag in the form of the constant $\gamma > 0$.

5. Assume that for a medieval catapult (as in the drawing above), the other components are negligible.

We come out of this step with some conditions for the validity of our model and some new constants and terms to use in our model.

Step 4. Construct a model

This is the part where we put together the last three steps into one (or a system of) differential equations. This should not be a difficult part if the last three steps were completed carefully.

Example.

Summary of Steps 1–3:

- **Goal:** study $y(t) =$ height of a projectile in metres, t seconds after being released from a catapult
- **Forces:**
  - Gravity: constant acceleration $-g$
  - Air Drag: acceleration $\pm\gamma y'(t)$
- **Conditions:**
  - $y(t) \in [0, 100]$
  - Air drag should really by quadratic, but in this example we will consider this as an academic case.

So the model we end up is:

$$F_y = ^{\text{vertical component}}_{\text{of force}} = -g \pm \gamma y'.$$

Now we bring a little bit of a Physics class into here: Newton's 2nd Law states that $F = ma$, so we obtain the model

$$my''(t) = -g \pm \gamma y'(t).$$

Step 5. Model Assessment

We just found a differential equation (model) for our situation. It is now time to test it to make sure that it behaves correctly.

For this step, we need to obtain a solution of the differential equation, either by solving it mathematically and finding a formula for the solution, or by approximating the solution numerically (see Module **??**).

Then we need to check if the differential in one of several ways:

- We can test it empirically: make an experiment and compare the results of the experiment with the results of the model
- We can test it mathematically: change the parameters and the initial conditions to make sure that we know how the model should behave and test some qualitative aspects of the model

Important. Even if the model passes all the tests, it might still not be correct.
Also, if it fails one test, it might mean that the model is incorrect, or that it has some limitations that are more subtle and we hadn't thought about them.

Example. We have found the following model:

- $y(t) =$ height of a projectile in metres, t seconds after being released from a catapult
- It satisfies:

  $$my''(t) = -g + \gamma y'(t).$$

  (note that I chose the + sign for the air drag component)
- Constraints:
  - $y(t) \in [0, 100]$;
  - $\gamma > 0$ is the drag constant: more air drag for larger values of γ;

This differential equation tells us what the second derivative, $y''(t)$, of $y(t)$ is given the first derivative $y'(t)$. This means that to start solving the problem, we need to know what the initial values for $y'(t)$ and $y(t)$ are.

Need to know the starting conditions:

- $y(t_0) = y_0$;

- $y'(t_0) = v_0$.

For this example, consider a situation where:

- $g, \gamma > 0$ can take any value.
- $y(0) = 0$;
- $y'(0) = g/\gamma > 0$;

The projectile is being catapulted from the ground with a positive velocity, so we expect it to go up for a while and then come back down to the ground.
What happens is

$$y''(0) = -g + \gamma y'(0) = 0,$$

so the initial acceleration is 0, which means that the velocity is not changing.
The result is a function with constant velocity equal to its initial velocity:

- $y(t) = \frac{g}{\gamma} t$.

This means that the height of the projectile keeps increasing, so the ***projectile never falls back to the ground***!

This means that there is a problem with our differential equation:

- Is the model incorrect?
- Is there a limitation on the initial velocity that we were not aware of?

We must check our process again and correct it.

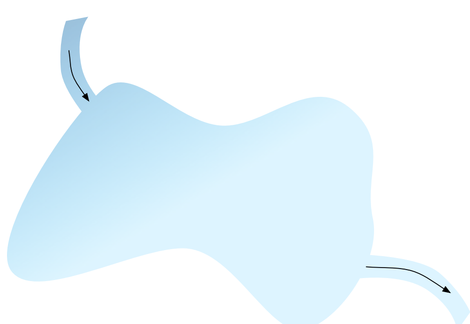
Step 6. Putting it all together in a report

We're not going to elaborate much on this step. For more on the subject, please check Module 8.

Video.

- `https://youtu.be/njg8xwMviGQ`
- `https://youtu.be/nKDsJB8iwb0`

Practice Problems



1 Model the pollution in a lake where water flows in and out at the same rate and incoming water is polluted with $2 + \sin(2t)$ kg/L of pollutant, where t is measured in years.

2 Construct a model for a population with a rate of growth proportional to its current size.

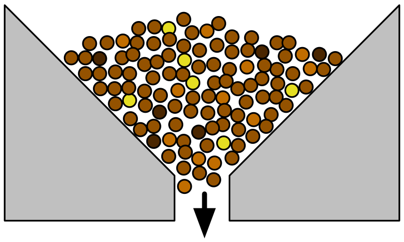
3 Find a model for a population that grows proportionally to its current size but with a variable proportion constant. This variable proportion constant should guarantee the following properties for the population:

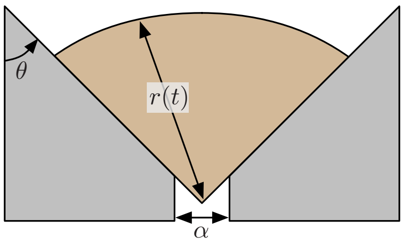
- If the population is too large, then it should decrease;
- If the population is small, then it should increase.

4 Improve the previous model by considering also a **survivability threshold**: if the population is below this value, it should decrease and eventually become extinct.

5 Consider two competing populations, like cheetahs ($c(t)$) and lions ($\ell(t)$): two populations that do not hunt each other, but compete for the same food (prey). Create a model for these two populations that captures how the competition for food affects them.
Hint. It might be helpful to think about how one population would grow in the absence of the other;

and how one population is affected by the competition of the other.





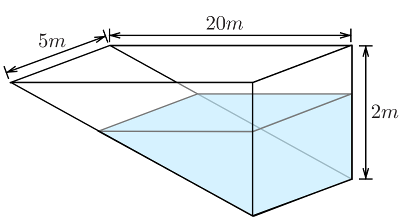
6 People are in a stadium watching cricket match. When the match is over, people leave the stadium.

(a) Model the way people leave the stadium.

To help you with this task, use the fact that in this situation, people behave like a fluid according to Torricelli's Law:

The area of the region occupied by the fans decreases proportionally to the square root of the radius and also proportionally to the size of the exit.

(b) How do the parameters θ and α affect the total time it will take for the stadium to empty?



7 Consider the pool in the figure. The goal is to track the amount of chlorine in the water for one Summer month. At the beginning of the month, the pool is full and contains 150g of chlorine uniformly mixed in the water. Consider evaporation and rain. To make the model simpler, assume that water evaporates with the chlorine.

8 After solving the core exercise 18 below, we find a property of this model.

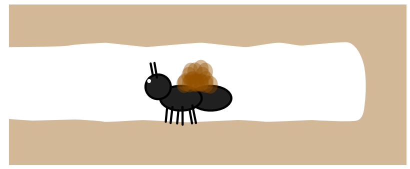
(a) The constants g and L (length of the string) appear only has $\frac{g}{L}$. What does this imply?

(b) We are sending a mission to the Moon and we need to know how a 1m long pendulum behaves on the Moon. To test it, we need to build on Earth a pendulum that behaves in the same way. How long should the length of the string be on Earth?

9 After solving the core exercise 18 below, construct a model for the same problem considering string tension.

(a) Show that you obtain the same model that you get while disregarding tension.

(b) Explain why this makes sense.



10 An ant queen, known affectionately as Aunty Ant, is commissioning a construction assessment for a new tunnel. Aunty Ant's worker ants only know one way to construct a tunnel: they grab some dirt in their pincers, walk the dirt out of the tunnel, deposit it, and then return to grab more dirt.

Prepare a report which uses differential equations to address the following construction scenarios. Include a description of how you modelled the scenario and a graph of tunnel-length vs. worktime. Also make sure to define any variables and constants you are using.

(a) One tireless worker is assigned dig the tunnel. The worker walks the same speed whether she is carrying dirt or not.

(b) One tireless worker is assigned to dig the tunnel, but she can walk twice as fast when she is not carrying dirt as when she is carrying dirt.

(c) Aunty Ant really wants the tunnel to progress linearly after the first day of construction (that is, the graph of tunnel-depth vs. time after the first day should be a straight line). She will give you full control over how many workers are devoted to the tunnel at any given time.

(d) (Optional) A single ant is assigned to dig the tunnel, but she gets fatigued the farther she walks. Her speed after walking a total distance of k units is $1/k$.

11 The alien world of Robotron is inhabited by billions of tiny nanobots. These nanobots all share a common source of power, and their speed is directly proportional to the total amount of energy shared among all the nanobots.

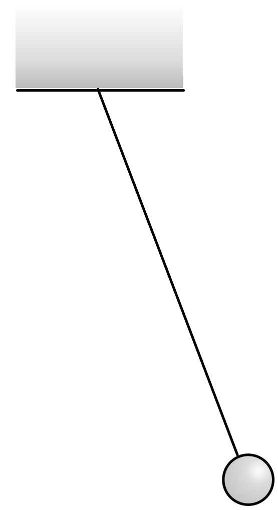
One day the nanobots decide to beam their energy into space. They all form lines, march to the edge of their colony, and send a tiny portion of their shared energy into space. Since the nanobots are very polite, after an individual nanobot has sent its energy into space, it moves aside and lets the next nanobot take a turn.

(a) Suppose the nanobots live in a tube with an opening at only one end. Come up with a differential equation to model the amount of energy left in the nanobot colony over time.

(b) How does your model change if the nanobots live in a disk where energy can be launched from anywhere on the perimeter? What about a sphere?

(c) Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference between the object's temperature and the ambient (outside) temperature. Does this law relate to your model for the nanobots? If so, how?

18 A pendulum is swinging side to side. We want to model its movement.



18.1 Define the problem. Which function(s) do we want to find in the end?

18.2 Build a mind map.

18.3 Make assumptions. Remember to use your mind map to help structure the problem.

18.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2nd Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in — Rate out).

18.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.

19 Model the spreading of a rumour through the students of a school.

Solvable Types of ODEs

In this module you will learn

- to identify specific types of differential equations that can be solved rigorously
- how to solve these types of differential equations

We just learned how to model a situation and end up with a differential equation. We will now focus on solving differential equations.

There are a few different techniques that depend on the differential equation.

12.1. Separable Differential Equations

Separable ODE. A differential equation is called ***separable*** if it has the form

$$g(y)y'(t) = h(t),$$

that is if we can separate all the y's into the left-hand side and the all the t's into the right-hand side of the equation. Observe that the y's on the left hand side must all be multiplied by $y'(t)$.

Method of solution. The idea to solve this type of DEs is simple:

1. Integrate both sides with respect to t:

$$\int g(y)y'(t)\,dt = \int h(t)\,dt$$

2. Change variables on the left-hand side to $u = y(t)$, so $du = y'(t)dt$ and we get

$$\int g(u)\,du = \int h(t)\,dt.$$

3. Solve both integrals and we obtain a solution, usually in implicit form:

$$G(u) = H(t) + C.$$

4. To finish, recall that $u = y(t)$, so we obtain

$$G\big(y(t)\big) = H(t) + C.$$

Important. Observe that the solution is given in implicit form. In general, when using this technique, the solution $y(t)$ will be given in implicit form, so there is still some work ahead to find an explicit formula for $y(t)$.

Example. The shape $y(x)$ of a free falling chain under its own weight, called a catenary, satisfies the differential equation:

$$y''(x) = \frac{1}{a}\sqrt{1 + \big(y'(x)\big)^2}.$$

It doesn't seem to be a **separable equation**, but if we can define $z(x) = y'(x)$, which satisfies

$$z'(x) = \frac{1}{a}\sqrt{1 + \big(z(x)\big)^2} \qquad \Leftrightarrow \qquad \frac{1}{\sqrt{1 + \big(z(x)\big)^2}}z'(x) = \frac{1}{a}.$$

This is now clearly in the form of a ***separable ODE***.

We can solve it using the method described above: we need to solve

$$\int \frac{1}{\sqrt{1+z^2}}\,dz \quad = \quad \int \frac{1}{a}\,dx \quad = \quad \frac{x}{a} + C_1$$

The integral on the left can be solved using a hyperbolic substitution $z = \sinh u$:

$$\int \frac{1}{\sqrt{1+z^2}}\,dz = \int 1\,du = u = \text{arcsinh z}.$$

This means that the solution satisfies

$$\operatorname{arcsinh} z = \frac{x}{a} + C_1 \qquad \Leftrightarrow \qquad z = \sinh\left(\frac{x}{a} + C_1\right).$$

Now recall that $z(x) = y'(x)$, so we need to integrate $z(x)$ to obtain the catenary curve $y(x)$:

$$y(x) = \int z(x)\,dx = a\cosh\left(\frac{x}{a} + C_1\right) + C_2.$$

To find C_1 and C_2, we use the fact that $y'(0) = 0$:

$$y(x) = a\cosh\frac{x}{a} + C_2.$$

(the constant C_2 moves the curve up or down, so it doesn't change the shape).

Video.

- https://youtu.be/txtFH89HwOA
- https://youtu.be/8xG_Xg6X2MQ
- https://youtu.be/ZE1Agfkhr28

12.2. First-Order Linear Differential Equations

First-Order Linear ODE. A differential equation is called ***first-order linear*** if it has the form

$$y'(t) + p(t)y(t) = f(t),$$

that is if we can separate all the y's into the left-hand side and the all the t's into the right-hand side of the equation. Observe that the y's on the left hand side must all be multiplied by $y'(t)$.

The idea to solve this type of DEs is to transform it into the result of a product rule.

Example. Consider the following **first-order linear ODE**

$$t^2\frac{dy}{dt}+2ty=\sin t.$$

Observe that the left-hand side of the DE is the result of the product rule:

$$\frac{d}{dt}\left[t^2y\right]=\sin t.$$

So we can integrate both sides with respect to t to obtain

$$t^2y=-\cos t+C \quad\Leftrightarrow\quad y=-\frac{\cos t}{t^2}+\frac{C}{t^2}.$$

Now let us look at another example, where the left-hand side of the ODE is not in the form of the result of a product rule, but can be transformed into one.

Example. Consider the **first-order linear ODE**

$$\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{3}e^{\frac{t}{3}}. \tag{$\star$}$$

Again, the "trick" is to look at this equation and realize that the left-hand side can look like the result of the product rule. It's not obvious that this can be done (yet!), but if we multiply the whole ODE by the function

$$e^{\frac{t}{2}},$$

then we obtain

$$e^{\frac{t}{2}}\frac{dy}{dt}+\frac{1}{2}e^{\frac{t}{2}}y=\frac{1}{3}e^{\frac{t}{2}}e^{\frac{t}{3}}$$

and now the left-hand side is the result of a product rule

$$\frac{d}{dt}\left[e^{\frac{t}{2}}y\right]=\frac{1}{3}e^{\frac{5}{6}t}.$$

We integrate both sides to obtain

$$e^{\frac{t}{2}}y=\frac{1}{3}\frac{6}{5}e^{\frac{5}{6}t}+c$$

thus

$$y=\frac{2}{5}e^{\frac{t}{3}}+ce^{-\frac{t}{2}}.$$

This last example required us to come up with a function to multiply the ODE so that it becomes of the right form: with a left-hand side that is the result of the product rule.

This function is called the **integrating factor**.

Let us now see how we can find this function in more detail.

Example. Consider the same ODE ($\star$):

$$\frac{dy}{dt}+\frac{1}{2}y=\frac{1}{3}e^{\frac{t}{3}}. \tag{$\star$}$$

So we multiply both sides of the equation with an unknown function $\mu(t)$, called the **integrating factor**:

$$\mu(t)\frac{dy}{dt}+\frac{1}{2}\mu(t)y=\frac{1}{3}\mu(t)e^{\frac{t}{3}}. \tag{\#}$$

And we find which $\mu(t)$ makes the left-hand side equal to the product rule:

$$\frac{d}{dt}\left[\mu(t)y\right]=\mu(t)\frac{dy}{dt}+\frac{d\mu(t)}{dt}y,$$

Module 12: Solvable Types of ODEs

and this needs to equal the left-hand side:

$$\mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y = \mu(t)\frac{dy}{dt} + \frac{1}{2}\mu(t)y \qquad \Leftrightarrow \qquad \mu'(t) = \frac{1}{2}\mu(t).$$

We now need to solve this equation for $\mu(t)$. Fortunately, this is a **separable ODE**:

$$\begin{aligned} \mu'(t) = \frac{1}{2}\mu(t) \quad &\Leftrightarrow \quad \frac{\mu'(t)}{\mu(t)} = \frac{1}{2} \\ &\Leftrightarrow \quad \ln|\mu(t)| = \frac{t}{2} + A \\ &\Leftrightarrow \quad \mu(t) = ae^{\frac{t}{2}}, \end{aligned}$$

where $a = e^A$.

We say that the function $\mu(t) = e^{\frac{t}{2}}$ is an **integrating factor** for the equation $(\star)$. Observe that we chose $a = 1$ $(A = 0)$, because we only need one function $\mu(t)$ that satisfies our condition $\mu' = \frac{1}{2}\mu$, we don't need to find all possible solutions.

After finding the integrating factor $\mu(t)$, the rest of the solution is the same as in the previous example.

Now that we have a good idea of the method needed to solve these ODEs, let us tackle the general equation.

Method of solution. This method is also known as the **Method of the Integrating Factor**.

1. Multiply both sides by $\mu(t)$, the integrating factor:

$$\mu(t)\frac{dy}{dt} + p(t)\mu(t)y = \mu(t)g(t).$$

   Note that we don't know what this function is yet. So it is just a placeholder for a function we will find next.

2. Find function $\mu(t)$ which satisfies

$$\mu'(t) = p(t)\mu(t).$$

   This is a **separable ODE**, so we can solve it:

$$\mu(t) = Ae^{\int p(t)\,dt}.$$

   We only need one function $\mu(t)$, not the general one, so we take $A = 1$ to get

$$\mu(t) = e^{\int p(t)\,dt}.$$

3. Observe that $\mu(t)$ satisfies

$$\mu(t)p(t) = \mu'(t),$$

   so we use this in the equation:

$$\frac{d}{dt}\big[\mu(t)y\big] = \mu(t)g(t).$$

4. Integrate the equation:

$$\mu(t)y = \int \mu(t)g(t)\,dt + c,$$

   which means that the solution is

$$y = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right],$$

   where

$$\mu(t) = e^{\int p(t)\,dt}.$$

Important. Observe that the solution is given in explicit form. This is always the case with this type of ODEs.
Also, be careful to add the integration constant as soon as you integrate, so that in the end you will have a term $\frac{c}{\mu(t)}$.

Video.

- https://youtu.be/ezhi3E_bdvk
- https://youtu.be/VdD26Iy4Bkk
- https://youtu.be/GIp0cHNK7eQ

Practice Problems

1 Solve the differential equation $\ln(t)y' + \frac{1}{t}y = 3$.

2 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one type, solve it.

(a) $(t^2+4)y'(t) = \frac{2t}{y^2}$

(b) $\frac{1}{t^2}y'(t) = 2$

(c) $\frac{dy}{dx} = \sqrt{y}(x+1)^2$

(d) $y'(t) = t + y$

(e) $y'(t) = t + y^2$

(f) $y'(t) = \frac{t}{y}$

(g) $y'(t) = -\frac{1}{t}y$

(h) $y'(t) = 1 - 4t - \frac{5}{t}y$

(i) $y'(t) = 5t - 2ty$

(j) $y'(t) = 2 + \cos^2(y)$

(k) $e^{-t}y'(t) - e^{-t}y = 3e^{2t}$

3 Decide whether the following statements are true or false. Give an explanation or a counterexample.

(a) There are differential equations that are both separable and first-order linear.

(b) There are differential equations that are separable, but are not first-order linear.

(c) There are differential equations that are first-order linear, but not separable.

(d) There are first-order differential equations that are neither separable nor linear.

(e) All first-order linear differential equations have solutions defined in the whole real line.

4 Consider the differential equation

$$y' - \frac{y}{2(x+4)} = \frac{1}{2(x+4)}$$

(a) Find the general solution.

(b) Find the solution with initial condition $y(0) = -5$.

(c) What is the domain of the previous solution?

(d) Find the solution with initial condition $y(-5) = -5$.

5 Even though the following differential equation is not linear, find its general solution:

$$2\ln(x)e^{2y}y'(x) + \frac{e^{2y}}{x} = 4x^3.$$

20 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

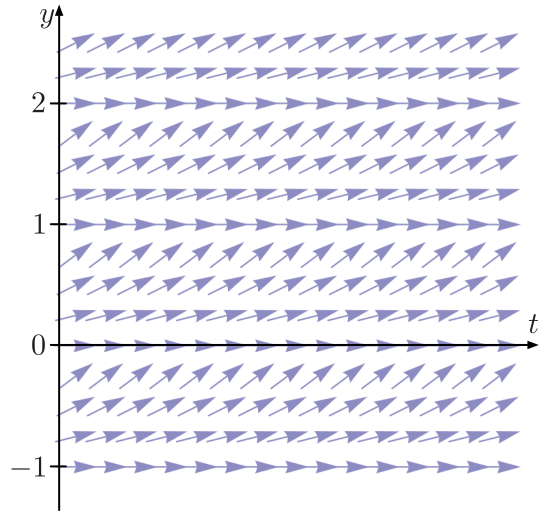
20.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

20.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

20.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

20.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

21 Consider a differential equation $y' = f(t, y)$ with the following slope field.



21.1 What are the equilibrium solutions of the ODE?

21.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t, y) \\ y(0) = \dfrac{1}{4} \end{cases}$$

21.3 From the direction field above, what is the type(s) of this ODE? Justify your answer.

(a) separable.

(b) of first-order and linear.

(c) autonomous.

(d) none of the other options.

21.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then (select all the possible options)

(a) $g(3) < h(3)$ **(b)** $g(3) = h(3)$ **(c)** $g(3) > h(3)$

22 22.1 Calculate $\left(\sin(x)f(x)\right)'$.

22.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

22.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

Properties of Differential Equations

In this module you will learn

- to find some properties of solutions without the need to find a solution or approximating it
- an existence and uniqueness of solution theorem

Until now we studied problems where there was one unique solution. Is this true for every problem?

- There are DEs with no solutions, e.g. $(y')^2 = -1$ or $\sin(y') = 2$.

So if a problem has a solution, is it always unique?

- This is also not true. For example: $ty' = 2y$ with $y(0) = 0$.
  Check that

$$\begin{aligned} y &= 0 && \text{is a solution} \\ y &= t^2 && \text{is also a solution} \end{aligned}$$

It is important (not just to mathematicians) to know whether a problem has solutions or not before trying to solve it. It is also important to know whether there is one unique solution or multiple solutions.

So for **linear differential equations** we have the following theorem.

Theorem. Let p and g be continuous functions in an open interval $I = (a, b)$ containing the point t_0. Then there exists a unique function $y = \phi(t)$ that satisfies

$$\begin{aligned} & y' + p(t)y = g(t) \qquad \text{for each } t \in I, \\ & y(t_0) = y_0, \end{aligned}$$

for any $y_0 \in \mathbb{R}$.

Example. Consider the initial-value problem

$$\begin{cases} y' + \frac{1}{\sin(t)} y = e^t \\ y(1) = 2 \end{cases}$$

We can see that

- $p(t) = \frac{1}{\sin(t)}$, which is continuous for $t \in (0, \pi)$ and $t_0 = 1$ is included in this interval;
- $g(t) = e^t$ is continuous for all values of t.

So we can conclude, from the Theorem, that there is a unique solution $y(t)$ defined for $t \in (0, \pi)$.

Example. We can see why on the previous example $ty' = 2y$, this Theorem doesn't apply. To use the Theorem, we need to write this equation as

$$y' - \frac{2}{t} y = 0,$$

and the function $p(t) = -\frac{2}{t}$ is not continuous at 0.

Example. The equation $y' = \frac{2}{3\sqrt[3]{x}}$ with the condition $y(0) = 0$ has a unique solution:

$$y = x^{\frac{2}{3}}.$$

So even though $g(t) = \frac{2}{3\sqrt[3]{x}}$ is not continuous at 0, the DE still has a unique solution.

The previous Theorem is very restrictive – it only applies to some very particular differential equations.

Below, we state another Theorem that applies to a much broader range of differential equations.

Theorem. Let the functions $f(t,y)$ and $\frac{\partial f}{y \partial y}$ be continuous in some rectangle $|t-t_0| \leq a$ and $|y-y_0| \leq b$ for $a,b>0$.
Then, in some interval (t_0-h, t_0+h), there is a unique solution $y=\phi(t)$ of the IVP

$$y' = f(t,y)$$
$$y(t_0) = y_0.$$

Furthermore, $h \geq \min\{a, b/M\}$ where $M = \max \left|f(t,y)\right|$.

Partial derivative. Consider a function $f(t,y)$. Then its ***partial derivative with respect to*** y at the point (t_0, y_0), denoted by $\frac{\partial f}{\partial y}(t_0, y_0)$ is $g'(y_0)$, the derivative of the function $g(y) = f(t_0, y)$ at the point y_0.
Roughly, assume that the variable $t = t_0$ is a fixed number and take the derivative on the variable y.

You should spend some time comparing these two Theorems.

Observe that the last Theorem gives a much weaker result when the differential equation is linear.

Example. Consider the IVP

$$\begin{cases} y' = y^2 \\ y(0) = 3. \end{cases}$$

This problem is nonlinear, so we need to use the second Theorem. To apply, compute

$$f(x,y) = y^2$$
$$\frac{\partial f}{\partial y}(x,y) = 2y,$$

which are continuous for all $x, y \in \mathbb{R}$.
The previous Theorem guarantees that a solution exists and is unique in some interval around $x = 0$.
Even though the rectangle spans the whole space of x and y, it doesn't mean that the solution exists for all x. The extra part of the Theorem, guarantees that the solution exists for $t < h$ where $h = \frac{1}{b}$ (because $M = b^2$). Since $b \geq y_0$, we know that a solution exists for $t < \frac{1}{y_0} = \frac{1}{3}$.
In fact, this is a separable ODE, so we can find its solution:

$$y(x) = \frac{1}{\frac{1}{3} - x},$$

which is defined only for $x < \frac{1}{3}$.

This kind of Theorems are called **Existence and Uniqueness Theorems**.

Video.

- https://youtu.be/53BPf9JrFcU
- https://youtu.be/GV1gFLZ7V18

Practice Problems

1 For the following initial-value problems, answer the following questions:

(i) Is there a unique solution?

(ii) Without solving, what is its domain?

(a) $y' + y = t$ with $y(0) = 0$.

(b) $y' + \dfrac{1}{e^t} y = t$ with $y(0) = 0$.

(c) $y' + \dfrac{1}{e^t - 2} y = t$ with $y(0) = 0$.

(d) $y' + \ln(t) y = t$ with $y(e) = 1$.

(e) $y' + \dfrac{1}{1+t^2} y = \tan(t)$ with $y(0) = 0$.

(f) $y' + \dfrac{1}{1+t^2} y = \tan(2t)$ with $y(\pi) = 0$.

(g) $y' = \dfrac{1}{1+\sin(t)} y - \tan(t)$ with $y(0) = 0$.

(h) $y' = \dfrac{1}{1+\sin(t)} y - \tan(t)$ with $y(t_0) = 0$.

(i) $y' = \dfrac{1}{1+\sin(t)} y^2 - \tan(t)$ with $y(t_0) = 0$.

(j) $y' + \ln(y) = t$ with $y(e) = 1$.

(k) $y' = \dfrac{ty}{1+y}$ with $y(0) = 0$.

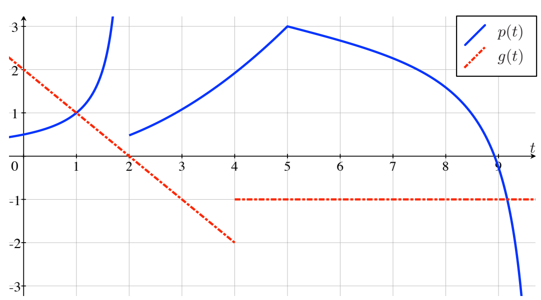
(l) $(t + y^2) y' = ty$ with $y(-1) = 1$.

(m) $y' = \dfrac{t \sin(y)}{y}$ with $y(1) = 0$.

2 Consider the problem

$$y' + p(t) y = g(t) \qquad \text{with} \qquad y(t_0) = y_0,$$

where $p(t)$ and $g(t)$ are graphed below



(a) Is there a unique solution satisfying $y(3) = 2$? If so, what is its domain?

(b) Is there a unique solution satisfying $y(t_0) = -1$ for which values of t_0? If so, what is the domain of these solutions?

3 Consider the problem

$$y' = f(t, y)$$

where $f(t, y)$ and $\frac{\partial f}{\partial y}(t, y)$ are continuous for all t, y.

- Assume that $y = \frac{1}{t}$ is a solution for $t > 0$
- Assume that $y = -e^{-t}$ is a solution for all t

Let $y = \phi(t)$ be the solution of this ODE with the initial condition $y(1) = \frac{1}{2}$.
Calculate $\lim_{t \to +\infty} y(t)$.

4 Consider the initial-value problem:

$$\begin{cases} y' = \ln(t+2) y + \frac{1}{t-3} \\ y(0) = 0 \end{cases}$$

(a) Is this ODE linear or nonlinear?

(b) Show that this problem has a unique solution.

(c) Use the Existence and Uniqueness Theorem for **Linear** ODEs. What is the domain of the solution?

(d) Use the Existence and Uniqueness Theorem for **Nonlinear** ODEs. What is the domain of the solution?

(e) Compare both Theorems.

5 Consider the initial-value problem:

$$\begin{cases} y' = \ln(t+2) y + \frac{1}{t-3} \\ y(0) = 0 \end{cases}$$

(a) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Linear** ODEs.

(b) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Nonlinear** ODEs. Simplify the conditions.

(c) Compare the conditions of both theorems.

6 Consider the initial-value problem:

$$\begin{cases} y' + p(t) y = g(t) \\ y(0) = 0 \end{cases}$$

(a) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Linear** ODEs.

(b) State the conditions to be able to apply the Existence and Uniqueness Theorem for **Nonlinear** ODEs. Simplify the conditions.

(c) Compare the conditions of both theorems.

23 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

23.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

23.2 $y' = t + \sqrt{y - \pi}$ with $y(1) = 1$

23.3 $y' = \sqrt{4 - (t^2 + y^2)}$ with $y(1) = 1$

24 The initial-value problem

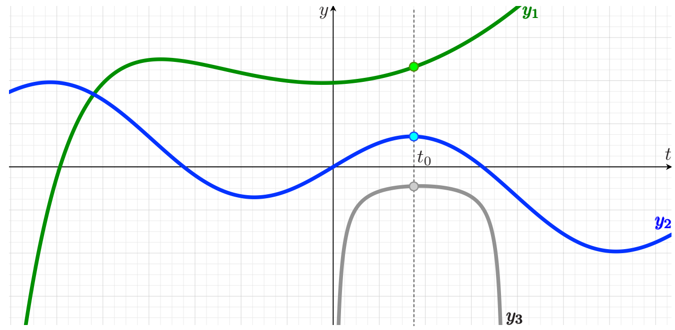
$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

$$y_1(x) = \cos\big(\arcsin(x)\big) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

24.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

24.2 What can you conclude?



25 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all t, y;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all t, y.

25.1 Can **green** $y1$ and **blue** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.2 Can **green** $y1$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.3 Can **blue** $y2$ and **gray** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

25.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

Autonomous Differential Equations

In this module you will learn

- what is an autonomous differential equation
- how to obtain some properties of solutions of autonomous differential equations without solving them

In this module, we focus on another type of differential equations. The ultimate goal of this module is to learn that with some creativity and observation of the differential equation, it is possible to study solutions without actually solving them.

We start by defining autonomous equations.

Autonomous differential equations. A first-oder DE is called ***autonomous*** if it has the form

$$y' = f(y).$$

These are basically ODEs where the rate of change does not depend on time, meaning that the nature of the ODE always stays the same.

Observe that autonomous ODEs are also Separable ODEs.

So let us look at an autonomous ODE and think what happens when $f(y_0) = 0$?

Then if the solution is unique (what are the conditions that will guarantee that?), then if $y(t_0) = y_0$, that means that

$$y'(t_0) = f(y_0) = 0.$$

So we can find one immediate solution:

$$y(t) = y_0,$$

a constant solution. Since the solution is unique, that must be **the** solution.

This is a property of autonomous ODEs:

Equilibrium points. Consider an autonomous ODE $y'(f(y)$. The zeros of the function f are called ***critical points***. They can also be called ***equilibrium*** or ***stationary*** points.

Important. Consider an autonomous ODE $y' = f(y)$ and let c be a zero of f, i.e. $f(c) = 0$. Then the constant function $y(t) = c$ is a solution of the ODE, called an ***equilibrium solution***.

In an ODE where solutions are unique, these equilibrium solutions are extremely important, as they give bounds for all other solutions.

Example. Consider the autonomous ODE

$$y' = \sin(2y).$$

The ***equilibrium solutions*** for this ODE are

$$y = k\pi,$$

for all values $k \in \mathbb{Z}$.
That means that even without solving, we can infer that the solution passing through $y(0) = 1$, must satisfy

$$y(t) \in (0, \pi),$$

for all t.

Equilibrium solutions are even more important. In fact, we can also study what happens between equilibrium solutions without having to actually solve the ODE.

If the function $f(y)$ is continuous, then its sign cannot change between equilibrium points, so the solutions will be monotonic between equilibrium solutions.

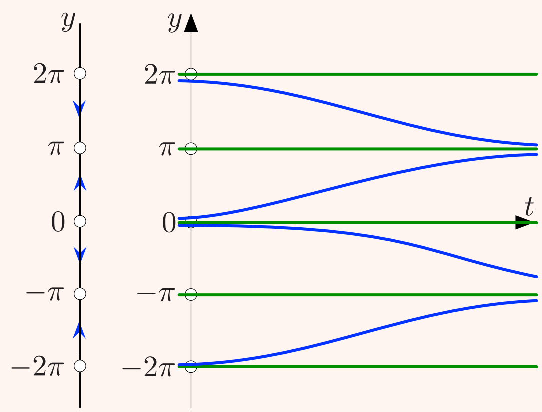
Example. Consider the same ODE:

$$y' = \sin(2y).$$

We can study whether solutions will be increasing or decreasing by studying the function $f(y)$.

y	$\cdots$	-2π		$-\pi$		0		π		2π	$\cdots$
$y' = \sin(2y)$		0	$+$	0	$-$	0	$+$	0	$-$	0	
$y(t)$	$\cdots$	c	$\nearrow$	c	$\searrow$	c	$\nearrow$	0	$\searrow$	c	$\cdots$

In a graph, we have



We can infer that the graphs will approach the constant solutions without touching because the derivative y' will become smaller and smaller the more they approach the equilibria. We also know that solutions cannot touch each other.

There is also a distinction that we make about equilibrium points that helps us understand the behaviour of solutions. We will study this distinction in the core exercises.

Population Models. The fact that these differential equations keep the same rate of change independently of time, makes them an ideal candidate when studying populations.

Video.

- `https://youtu.be/swt-let4pCI`

Practice Problems

1 Show that all autonomous differential equations are separable.

2 Consider the ODE $y' = \dfrac{1}{y}$. Then which of the statements below are true or false and justify your choice.

(a) The solutions always stay positive.

(b) The solutions always stay negative.

(c) The solutions never change sign.

(d) The solutions always change sign.

(e) Some solutions change sign, and some don't.

3 Sketch a slope field for the autonomous ODE

$$y' = \cos(y).$$

4 Sketch a slope field for the autonomous ODE

$$y' = \tan(y).$$

5 What is a common property of all slope fields of autonomous ODEs?

6 Consider the ODE

$$y' = \operatorname{sign}(y) = \begin{cases} 1 & \text{if } y > 0 \\ 0 & \text{if } y = 0 \\ -1 & \text{if } y < 0 \end{cases}.$$

(a) What are the equilibrium solutions?

(b) Find two solutions that satisfy $y(0) = 0$.

(c) Are there solutions that satisfy $\lim_{t\to\infty} y(t) = \pi$?

(d) What are the possible limits of solutions as $t \to \infty$?

7 Consider a function $f(y)$ such that

- $f(1) = 0$;
- f' is continuous for all y;
- $f'(1) < 0$.

(a) Show that there is an open interval (a, b) satisfying $1 \in (a, b)$ and $f'(y) < 0$ for all $y \in (a, b)$.

(b) Show that
- $f(y) > 0$ if $y \in (a, 1)$,
- $f(y) < 0$ if $y \in (1, b)$.

(c) Consider the initial-value problem $y' = f(y)$ with $y(0) = y_0$. Show that this problem has a unique solution.

(d) Show that if $y_0 \in (a, b)$, then $\lim_{t\to\infty} y(t) = 1$.

(e) Write a Theorem about equilibrium points based on the results of this question.

8 Consider a function $f(y)$ such that

- $f(2) = 0$;
- f' is continuous for all y;
- $f'(2) > 0$.

Complete a study similar to question 7 for this function f.

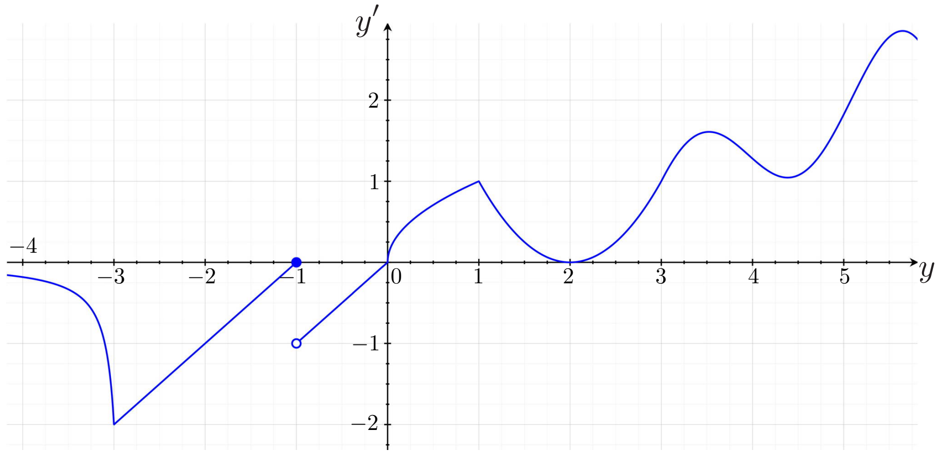
9 Consider an autonomous ODE $y' = f(y)$ with two stable equilibrium solutions $y = 1$ and $y = 2$ and where f is continuous.

(a) Show that there must exist another equilibrium point $c \in (1, 2)$.

(b) Show that if $f(y) \not\equiv 0$ in $(1, 2)$, then there must exist an equilibrium point $c \in (1, 2)$ that is either semi-stable or unstable.

Module 14: Autonomous Differential Equations

26 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



26.1 What are the equilibrium points?

26.2 Which equilibrium solutions are stable, unstable, or semi-stable?

26.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

26.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

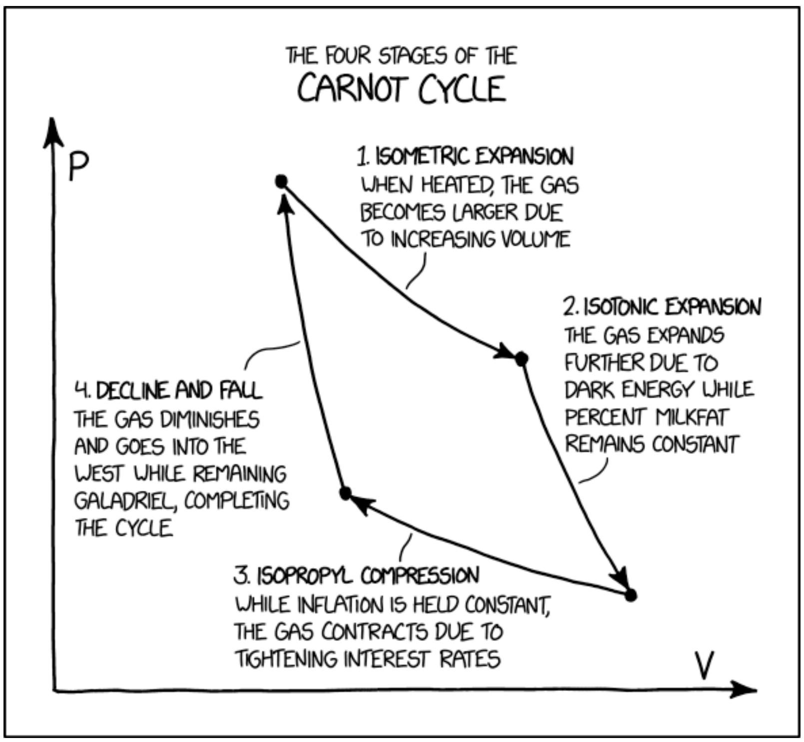
(b) $y(0) = -\frac{1}{4}$.

(c) $y(1) = \frac{1}{4}$.

26.5 If $y(0) = 2$, then $y(t) =$

26.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

26.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$



(image from xkcd - comic #2063)

Modelling Two Quantities

In this module you will learn

- how to model two or more inter-dependent quantities using systems

Often, when modelling something, we are faced with two or more quantities that depend on each other. This means that one equation is not enough, so we need to learn how to deal with a system of equations.

Just like we did in module 25, we will follow the step by step procedure developed in chapter 1.

Step 1. Define the problem

Example. We want to model two interacting populations, like the populations of bears and salmon in a specific natural park.

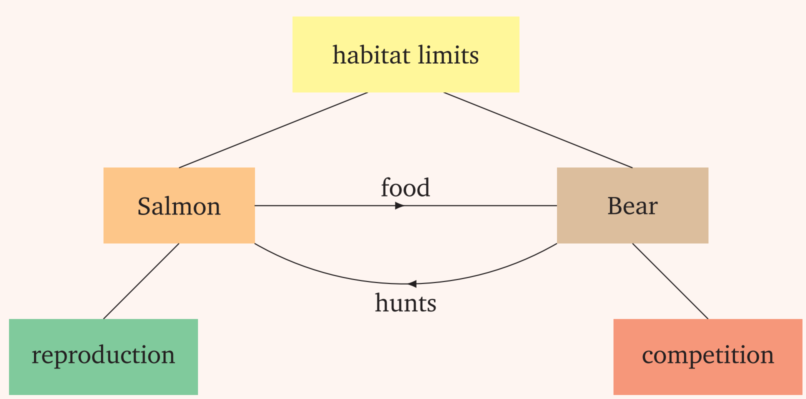
The first step is to decide on what we want to find at the end of the process. In this case, we want to know the number of individuals in each population and how they change as time passes. So we define:

- $b(t)$ = number of bears in the natural park at time t;
- $s(t)$ = number of salmon in the natural park at time t.

Step 2. Build a mind map

Example. We start with both species in the centre:

We can start brainstorming about the things that affect these populations:



Step 3. Make assumptions

Example. In this step, we discuss which of the boxes in the mind map we want to actually consider in our model, and which assumptions we need to make to consider them.

Let us start with how these species interact with each other:

1. Salmon provide food for bears: the bear population profits from each encounter with salmon. How does each bear-salmon encounter affect the bear population?
2. Bears hunt salmon: the salmon population is likely to decrease with each encounter with a bear. How

does each bear-salmon encounter affect the salmon population?

These two components are essential in our model, so we need to include them. It still leaves some freedom on how to do this.

There are other elements that we might want to include in our model:

3. Salmon reproduction: in the absence of predators and under ideal conditions, salmon should grow according to the Malthusian model, i.e. the rate of growth is proportional to the number of salmon;

4. Bear competition: bears are mainly predators, so without salmon, their numbers will decrease, also according to the Malthusian model;

5. Habitat limits: these species live in habitats that have limited resources, so we can consider a carrying capacity for each species.

To make the model simpler, we will ***ignore habitat limits***. This means that this model will not be accurate if the populations become very large.

Step 4. Construct a model

Example. We will start with our populations:

- $b(t)$
- $s(t)$

and we will start adding components to each of these one by one.

For the first two items, we need to estimate the number of encounters salmon-bear. We assume that the number of encounters is proportional to the number of all possible encounters: $b(t)s(t)$.

1. Salmon provide food for bears: for every possible salmon-bear encounter, there is a probability that a bear actually encounters a salmon, and then there is a chance that the bear will catch the salmon. Each catch improves the possibility that the bear population will increase. All these put together means that this factor should increase the bear growth rate by $a\,b(t)s(t)$, where the constant a needs to be found.

2. Bears hunt salmon: similarly to the previous item, for every possible encounter, there is a probability that the bear actually encounters a salmon, and then there is a chance that the bear will catch the salmon. Every catch will decrease the salmon population, so the salmon growth rate will decrease by $c\,b(t)s(t)$, there the constant b needs to be found.

Right now we have the following model:

$$\begin{cases} b'(t) = a\,b(t)s(t) + \cdots \\ s'(t) = -c\,b(t)s(t) + \cdots \end{cases}$$

We continue with the other elements:

3. Salmon reproduction: this was explained before and should contribute to the salmon growth rate with the term $ds(t)$.

4. Bear competition: this was also explained above and should contribute to the bear growth rate with the term $-eb(t)$.

5. Habitat limits: we decided to ignore this.

We have the model:

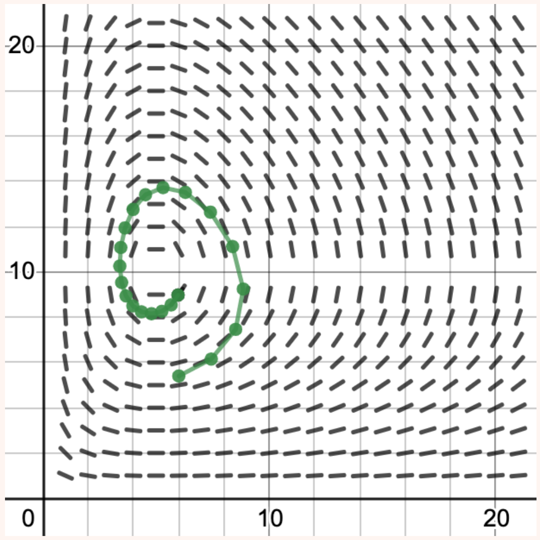
$$\begin{cases} b'(t) = a\,b(t)s(t) - e\,b(t) \\ s'(t) = -c\,b(t)s(t) + d\,s(t) \end{cases}$$

To find the constants a, c, d, e, we would probably need to go back to Step 3 and make further assumptions related to the way that we are measuring them.

Step 5. Model assessment

Example.

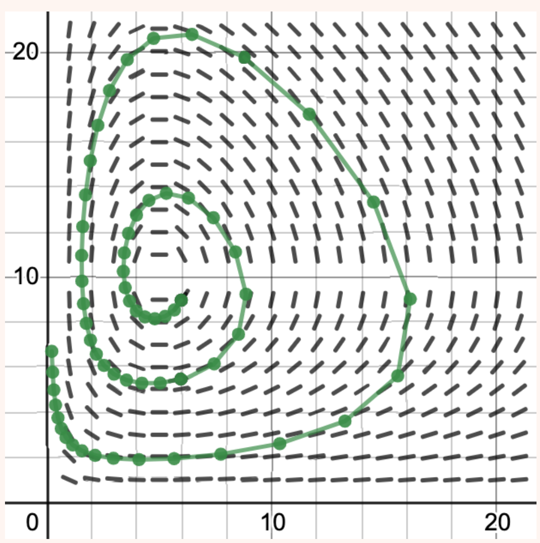
We can do several things here. I'll let you brainstorm and think of ways you can assess this model. One of the things that we can do is approximate its solutions using Euler's Method discussed in Module 15. Let us assume, for this example the constants: $a = 1$, $e = 10$, $c = 1$, $d = 5$ and a time step $\Delta t = 0.5$ and we assumed an initial population of $b(0) = 6$ and $s(0) = 9$. Then we obtain the graph below:



- https://www.desmos.com/calculator/zywspwstwk

The $x-$axis is the bear population while the $y-$axis is the salmon population. Each dot gives gives an approximation of the populations $\Delta t = 0.5$ time units after the previous approximation.

From this approximation, we can say infer that this model creates a population cycle, but it seems to spiral outwards:



- Is having a population cycle a feature that our model should have?
- Is the spiralling outwards a feature we want in our model?
- Is the spiralling a feature of the model or the approximation? If it's from the approximation, how does the model behave?

Module 15: Modelling Two Quantities

There are lots of other tools to create slope fields and approximate solutions of systems of ODEs.

- GeoGebra approximation of the same model, called the Lotka-Volterra model:

  `https://www.geogebra.org/m/KqNV7eHB`

- WolframAlpha slope field of the same model:

  `https://uoft.me/modelling-sys-wa`

- WolframAlpha stream plot of the same model:

  `https://uoft.me/modelling-sys-wa2`

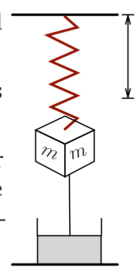
Step 6. Putting it all together in a report

We'll skip this part here.

Practice Problems

1 Create a model for two cooperating populations, like sharks and remoras.

2 We have a spring attached to a mass and with a dashpot.



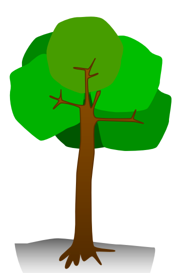
(a) Model the position of the mass as time changes.

(b) Obtain a system of two first-order ODEs. Remember to explain how the new functions relate to the spring-mass-dashpot system.

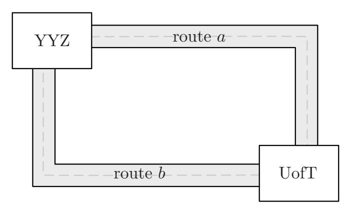
3 Model a vehicle with a special engine that provides an acceleration to the car proportional to the fuel left.

4 Imagine two twin babies and model their crying volume. Assume that they naturally become tired and stop crying if alone, but they cry more if the other twin is crying.

5 Create a simplified model for a tree, considering the height of the tree and its leaf area and how they affect each other.



6 Create a model on how a student's confidence in her own ability affects her learning/knowledge of a subject. Remember the Ebbinghaus' "forgetting curve".



7 Imagine that there are two ways to travel from UofT to Toronto's Pearson airport (YYZ). Both paths take the same time if there is no traffic. You want to direct people on the fastest path. Create a model for choosing the fastest path.

8 Create a model for the sales of a specific brand of sneakers. The goal is to capture the influence of famous people and non-famous people on each other's purchases.

9 Create a model on how the population and the cost of living in Toronto affect each other.

27 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

27.1 Create a model for these two populations.

27.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

27.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

27.4 Extend the model to include a population of antelopes.

28 A cheetah is chasing an antelope. We want a model of their positions as they run.

Systems of two linear ODEs with constant coefficients

In this module you will learn

- how to solve systems of two linear first-order ODEs with constant coefficients

First, a system of two first-order ODEs has the form:

$$\begin{cases} x'(t) = f\big(t, x(t), y(t)\big) \\ y'(t) = g\big(t, x(t), y(t)\big) \end{cases}$$

where the functions f and g are continuous and have continuous derivatives.

This system could be nonlinear, so we are only considering linear systems with constant coefficients, which means that they have a very specific form:

$$\begin{cases} x'(t) = ax(t) + by(t) + e \\ y'(t) = cx(t) + dy(t) + f \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} e \\ f \end{bmatrix} \quad \Leftrightarrow \quad \vec{r}\,'(t) = A\vec{r}(t) + \vec{b}$$

where

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad , \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad and \quad \vec{b} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

The unknown functions we are trying to find is $\vec{r}(t)$.

Homogeneous Systems. These are systems of the form above with $\vec{b} = \vec{0}$.

This means that we want to find all functions $\vec{r}(t)$ that satisfy

$$\vec{r}\,'(t) = A\vec{r}(t).$$

Example. Let us start with an example of the same problem where $\vec{r}(t)$ is a "one-dimensional" vector, a scalar function $u(t)$, and the matrix A is a "one-dimensional matrix", a constant a.

We want to solve the problem

$$u'(t) = a \cdot u(t).$$

We have seen how to solve these kind of problems before. The solutions are

$$u(t) = ce^{at},$$

where c can be any constant.

In our two-dimensional case, it is a little more complicated. We can't just write e^{At} where A is a matrix (this expression can make sense, but we would have to find out what is the exponential of a matrix).

So we can use the example above to make an **educated guess**: the solution should look like an exponential:

$$\vec{r}(t) = \vec{c}\,e^{\lambda t},$$

where $\vec{c}$ is a constant vector.

If our guess is correct, to find $\vec{r}(t)$, we only need to find λ and $\vec{c}$.

Let us see what happens when we use this guess and plug it into the system of ODEs:

$$\begin{aligned} \vec{r}\,'(t) = A\vec{r}(t) \quad &\Leftrightarrow \quad \vec{c}\lambda e^{\lambda t} = A\vec{c}e^{\lambda t} \\ &\Leftrightarrow \quad \vec{c}\lambda = A\vec{c}. \end{aligned}$$

This is a problem you have seen before – and eigenvalue-eigenvector problem:

- λ can be any eigenvalue of the matrix A
- $\vec{c}$ can be any eigenvector of A associated with λ

This means that we might have multiple choices for eigenvalues and eigenvectors, or even that eigenvalues and eigenvectors involve complex numbers. Let us split our study of possible solutions in three cases.

3.1 Two real and distinct eigenvalues

Example. Consider the problem

$$\vec{r}'(t) = \begin{bmatrix} 10 & 18 \\ -6 & -11 \end{bmatrix} \vec{r}(t).$$

Then, the eigenvalues and eigenvectors of the matrix are

- $\lambda_1 = -2$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$
- $\lambda_2 = 1$ with eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

This means that we found two solutions:

$$\vec{r}_1(t) = \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-2t} \quad \text{and} \quad \vec{r}_2(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}.$$

Then, we can show that

$$\vec{r}(t) = c_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$$

is also a solution of the problem for any constants c_1 and c_2.
In fact, we can show that this formula captures all possible solutions for this problem.

Video.

- `https://youtu.be/YUjdyKhWt6E`

3.2 Two complex eigenvalues

We actually don't need to know a lot about complex numbers to be able to understand how to solve this case. The results about complex values that are necessary to know will be included in the box below.

Complex numbers.

- A complex number is a number of the form $z = a + ib$ where i is called the imaginary constant and satisfies $i^2 = -1$.
- Given a complex number $z = a + ib$, we call $\bar{z} = a - ib$ its complex conjugate. It satisfies:

$$z \cdot \bar{z} = a^2 + b^2 = |z|^2.$$

- If a matrix has real components and two complex eigenvalues, then the eigenvalues are complex conjugates of each other. Moreover, the eigenvectors are also complex conjugates of each other.
- Euler's Formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Example. Consider the problem

$$\vec{r}'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{r}(t).$$

Then, the eigenvalues and eigenvectors of the matrix are

- $\lambda_1 = 1+i$ with eigenvector $\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$
- $\lambda_2 = 1-i$ with eigenvector $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

This means that we found two solutions:

$$\vec{r}_1(t) = \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} \quad \text{and} \quad \vec{r}_2(t) = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t}.$$

Then, all solutions of this system of ODEs can be expressed as

$$\vec{r}(t) = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t}.$$

There is a problem with the form of these solutions: they involve complex numbers!
Imagine that we start with a problem where we have two (real) quantities that interact with each other through this system of differential equations. Then we expect these quantities to measure in real numbers, not complex.

- This means that we expect ***the imaginary part of this solutions to cancel out***.

So let us manipulate this formula using Euler's formula and see if we can re-write in such a way that doesn't involve complex numbers.

We have:

$$\begin{aligned} e^{(1+i)t} &= e^t \cdot e^{it} = e^t\big(\cos(t) + i\sin(t)\big) \\ e^{(1-i)t} &= e^t \cdot e^{-it} = e^t\big(\cos(t) - i\sin(t)\big) \end{aligned}$$

So our solution expands to:

$$\vec{r}(t) = c_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^t\big(\cos(t) + i\sin(t)\big) + c_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^t\big(\cos(t) - i\sin(t)\big).$$

We can now manipulate this expression:

$$\begin{aligned} \vec{r}(t) &= e^t \begin{bmatrix} -ic_1\big(\cos(t)+i\sin(t)\big) + ic_2\big(\cos(t)-i\sin(t)\big) \\ c_1\big(\cos(t)+i\sin(t)\big) + c_2\big(\cos(t)-i\sin(t)\big) \end{bmatrix} \\ &= e^t \begin{bmatrix} (c_1+c_2)\sin(t) - i(c_1-c_2)\cos(t) \\ (c_1+c_2)\cos(t) + i(c_1-c_2)\sin(t) \end{bmatrix} \\ &= e^t \left((c_1+c_2)\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + i(c_1-c_2)\begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} \right) \end{aligned}$$

So now we do something that might look like a "cheating". We define:

$$a_1 = c_1 + c_2 \quad \text{and} \quad a_2 = i(c_1 - c_2).$$

Then the solution is

$$\vec{r}(t) = a_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} e^t + a_2 \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix} e^t.$$

This last form doesn't include any complex numbers and is equivalent to the previous form.

It may look like the final solution above still includes complex numbers in the constants a_1 and a_2. To convince yourself that this is not the case, solve the following exercise.

Find the unique solution of

$$\vec{r}'(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{r}(t) \qquad \text{with} \qquad \vec{r}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Find the constants c_1, c_2 and then the constants a_1, a_2. Which ones are complex and which ones are real?

Video.

- `https://youtu.be/TRVS5Wo9LoM`

3.3 One real repeated eigenvalue

Example. Consider the problem

$$\vec{r}'(t) = \begin{bmatrix} 5 & 0 \\ 1 & 5 \end{bmatrix} \vec{r}(t).$$

Then, there is only one eigenvalue with one eigenvector

- $\lambda_1 = 5$ with eigenvector $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which yield a solution $\vec{r}_1(t) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$.

This is a ***problem*** because we need two solutions to put together and obtain two constants, as in the two previous cases.

To convince yourself that it is a problem, try solving the problem above with initial conditions

$$\vec{r}(0) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}.$$

What about with initial conditions

$$\vec{r}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad ?$$

This means that we ware missing one solution – that will enable us to solve the problem for any initial conditions.

Let us re-write the original problem in a different form by letting

$$\vec{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Then we have

$$\begin{cases} x'(t) = 5x(t) \\ y'(t) = x(t) + 5y(t) \end{cases}$$

These are two ODEs, but we can solve the first and then tackle the second one. We obtain

$$\begin{cases} x(t) = c_2 e^{5t} \\ y(t) = c_1 e^{5t} + c_2 t e^{5t} \end{cases} \quad \Leftrightarrow \quad \vec{r}(t) = \begin{bmatrix} c_2 \\ c_1 + c_2 t \end{bmatrix} e^{5t} \quad \Leftrightarrow \quad \vec{r}(t) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ t \end{bmatrix} e^{5t}$$

Observe that the solution we found has the form:

$$\vec{r}(t) = \underbrace{c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}}_{\text{solution } \vec{r}_1(t)} + c_2 \left(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{new vector } \vec{w}} e^{5t} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}_1} t e^{5t} \right).$$

So we can make another **educated guess** that the solution we were missing has the form:

$$\vec{r}_2(t) = (\vec{w} + \vec{v}_1 t) e^{\lambda t}.$$

With this form in mind, we can plug it into the system of ODEs $\vec{r}\,'(t) = A\vec{r}(t)$, which has exactly one eigenvalue λ, to get:

$$\lambda \vec{w} e^{\lambda t} + \lambda \vec{v}_1 t e^{\lambda t} + \vec{v}_1 e^{\lambda t} = A\vec{w} e^{\lambda t} + A\vec{v}_1 t e^{\lambda t}$$

which is equivalent to:

$$\lambda \vec{w} + \underbrace{\lambda \vec{v}_1}_{=A\vec{v}_1} t + \vec{v}_1 = A\vec{w} + A\vec{v}_1 t \quad \Longleftrightarrow \quad (\lambda I - A)\vec{w} = \vec{v}_1$$

Since at this point we already know λ and $\vec{v}_1$, we can now find $\vec{w}$ in a similar way used to find the eigenvector $\vec{v}_1$. The vector $\vec{w}$ is called a **generalized eigenvector** of A associated with the eigenvalue λ.

Video.

- `https://youtu.be/hCShTLmeZN4`

Practice Problems

1 Find the general solution of the problem $\vec{r}\,'(t) = A\vec{r}(t)$ for the following matrices:

(a) $A = \begin{bmatrix} -7 & 6 \\ -9 & 8 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 22 & 24 \\ -15 & -16 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$;

(d) $A = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$;

(e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$;

(f) $A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$;

(g) $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$;

(h) $A = \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$;

(i) $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$;

(j) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$;

(k) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$;

(l) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;

2 For each of the problems in the previous exercise, find the solution that satisfies the initial conditions:

(i) $\vec{r}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

(ii) $\vec{r}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$;

(iii) $\vec{r}(1) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

3 Consider the problem $\vec{r}\,'(t) = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \vec{r}(t) + \begin{bmatrix} -2 \\ 11 \end{bmatrix}$.

(a) Show that $\vec{e}(t) = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is a solution of this problem.

(b) Find the general solution of

$$\vec{u}\,'(t) = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \vec{u}(t).$$

(c) Let $\vec{r}(t) = \vec{u}(t) + \vec{e}(t)$. Show that this is a solution of the original problem.

(d) Let $\vec{r}_1(t)$ and $\vec{r}_2(t)$ be two solutions of the original problem.

i. Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

ii. Is $3\vec{r}_1(t)$ a solution?

iii. Write a result on how one can safely combine solutions of non-homogeneous problems.

4 Consider the problem $\vec{r}\,'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \vec{r}(t) + \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

(a) Observe that this system is an autonomous system of ODEs. What is the equilibrium solution?

(b) Let the equilibrium solution solution you just found be called $\vec{e}$. Consider $\vec{u}(t) = \vec{r}(t) - \vec{e}$, where $\vec{r}(t)$ is the solution of the original problem. Show that

$$\vec{u}\,'(t) = A\vec{u}(t).$$

(c) Find $\vec{u}(t)$.

(d) Find $\vec{r}(t)$.

(e) Write a procedure to solve any problem of the form
$$\vec{r}(t) = A\vec{r}(t) + \vec{b}.$$

5 Consider the problem $\vec{r}'(t) = A\vec{r}(t)$. Assume that we have two solutions $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

(a) Show that $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$ is a solution also.

(b) Show that $\vec{r}(t) = 2\vec{r}_1(t) - 3\vec{r}_2(t)$ is a solution also.

(c) Find all possible solutions of the problem.

6 Consider the problem $\vec{r}'(t) = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \vec{r}(t)$.

(a) Find the solution that satisfies the initial condition $\vec{r}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Call it $\vec{u}(t)$.

(b) Find the solution that satisfies the initial condition $\vec{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Call it $\vec{v}(t)$.

(c) Define the matrix function
$$\Phi(t) = \big[\vec{u}(t) \mid \vec{v}(t)\big] = \begin{bmatrix} u_1(t) & v_1(t) \\ u_2(t) & v_2(t) \end{bmatrix}.$$
Show that $\vec{r}(t) = \Phi(t)\vec{r}_0$ is a solution of the original system of ODEs. Which initial condition does it satisfy?

(d) Write a result relating $\Phi(t)$ to the solution of initial-value problems.

7 Consider a system of ODEs $\vec{r}'(t) = A\vec{r}(t)$ with two solutions $\vec{r}_1(t)$ and $\vec{r}_2(t)$.
We want to study the conditions that are necessary on the solutions $\vec{r}_1$ and $\vec{r}_2$ to guarantee that we can solve any initial-value problem.

(a) What is the general solution for this problem?

(b) If the initial condition is $\vec{r}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then what are the conditions on $\vec{r}_1, \vec{r}_2$?

(c) If the initial condition is $\vec{r}(0) = \vec{r}_0$, then what are the conditions on $\vec{r}_1, \vec{r}_2$?

8 Consider a system of ODEs $\vec{r}'(t) = A\vec{r}(t)$ with two solutions $\vec{r}_1(t), \vec{r}_2(t)$.
Let $R(t)$ be the matrix $R(t) = \big[\vec{r}_1(t) \mid \vec{r}_2(t)\big]$ and let $W(t) = \det R(t)$.

(a) Show that $W(t)$ is a solution of $W' = (a_{11} + a_{22})W$.

(b) Solve the ODE above to obtain an expression for $W(t)$.

(c) Show that $W(t)$ is either identically zero, or it's never zero.

(d) Use this result to simplify your answer to problem 7(c).

29 Consider a cheetah-lion inspired problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{r}.$$

29.1 Find the two solutions $\vec{r}_1, \vec{r}_2$.

29.2 Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

29.3 Is $\vec{r}_1(t) - \vec{r}_2(t)$ a solution?

29.4 Is $2\vec{r}_1(t) + 3\vec{r}_2(t)$ a solution?

29.5 What is the general solution?

29.6 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

30 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r}.$$

30.1 Find the general solution.

30.2 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

31 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix} \vec{r} - \begin{bmatrix} 5 \\ 10 \end{bmatrix}.$$

31.1 Find the equilibrium solution.

31.2 Find the general solution.

31.3 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$?

Phase Portraits

In this module you will learn

- how to sketch a phase portrait for a linear system of ODEs with constant coefficients
- how to use a phase portrait to deduce properties of solutions

When we solve a system of two ODEs, we obtain two functions $x(t)$ and $y(t)$, so when we want to graph solutions, we have a problem:

- Should we sketch each of these functions separately?
- Should we sketch them together?
- Should we sketch the path as if a particle is moving with coordinates $x(t)$ and $y(t)$?

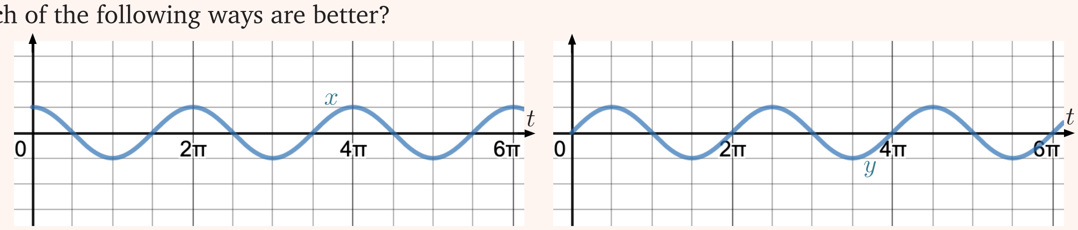
Example. Consider the initial-value problem

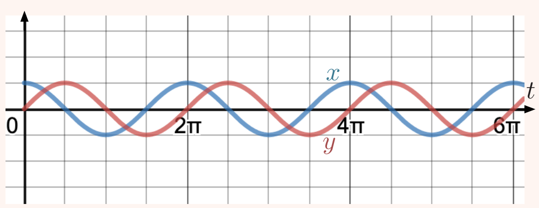
$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{r} \quad \text{with} \quad \vec{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

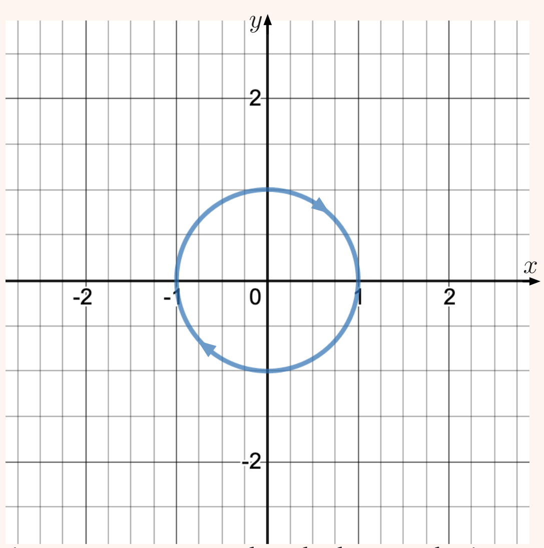
which has the solution

$$\vec{r}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}.$$

Which of the following ways are better?

- 

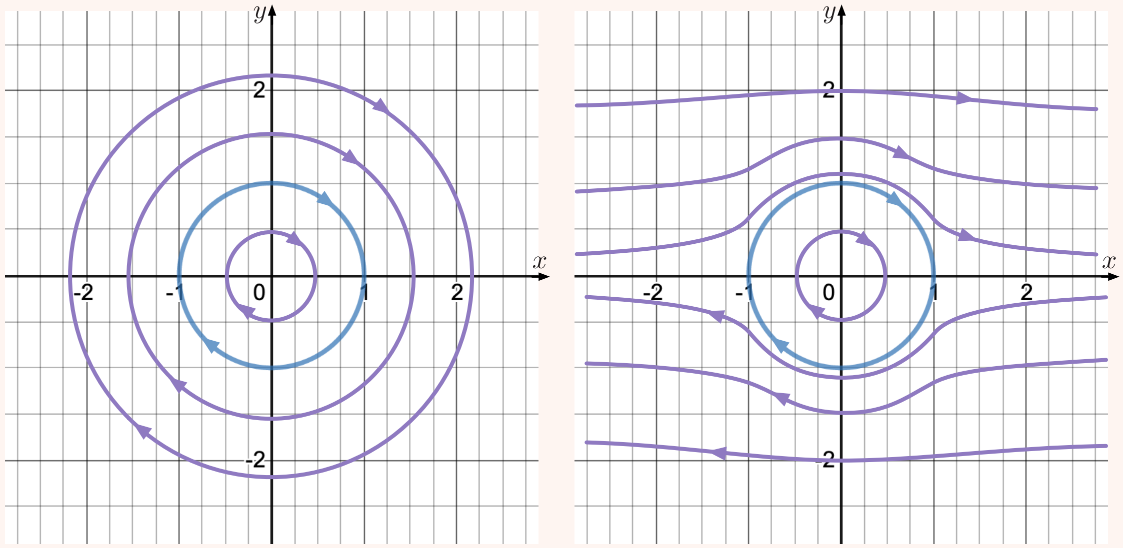
- 

- 

There is no correct answer, but the last graph gives more information on how the two quantities interact with each other, so we will focus on that type of graph.

The graphs in the example above are graphs of one specific solution. A phase portrait gives an idea of all possible solutions.

Example. The last graph of the example is part of which phase portrait?



A phase portrait gives a good idea of how all solutions behave.

Sketching phase portraits for systems of two first-order linear ODEs is important because it gives us insight on how the two components affect each other for all solutions.

Example. Consider the problem

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} -2 & 3 \\ -3 & -2 \end{bmatrix} \vec{r},$$

where the matrix A has the eigenvalues and eigenvectors:

- Eigenvalues $\lambda_\pm = -2 \pm 3i$ with eigenvectors $v_\pm = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$

This means that the general solution has the form

- $\vec{r}(t) = a_1 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(-2+3i)t} + a_2 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(-2-3i)t}$

or

- $\vec{r}(t) = c_1 \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -\cos(3t) \\ \sin(3t) \end{bmatrix} e^{-2t}$

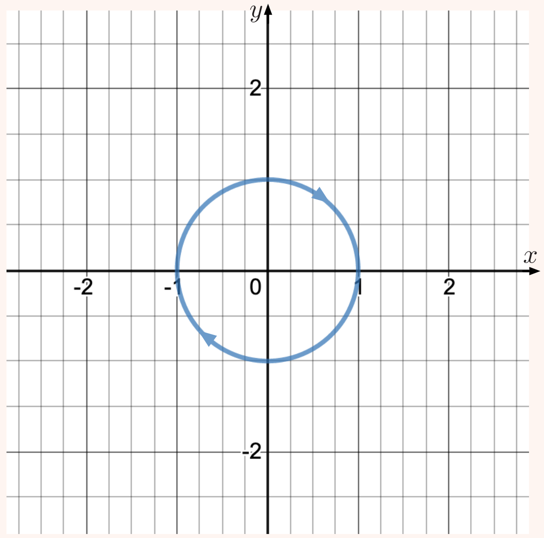
We can use the general form and start sketching some solutions.

- Let $c_1 = 1$ and $c_2 = 0$ and we obtain

$$\vec{r}(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix} e^{-2t}.$$

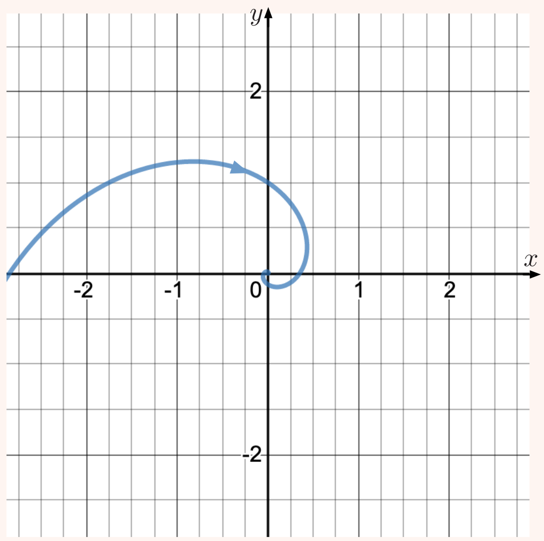
To sketch this solution, let us start by ignoring the term e^{-2t}.

So we want to sketch $\vec{r}(t) = \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix}$:



The path is going in circles in the clockwise direction.

By multiplying the solution by e^{-2t}, which starts at 1 when $t = 0$ and keeps decreasing towards 0 as t increases, we are creating a graph that keeps revolving around the origin as it converges towards the origin, yielding a spiral.

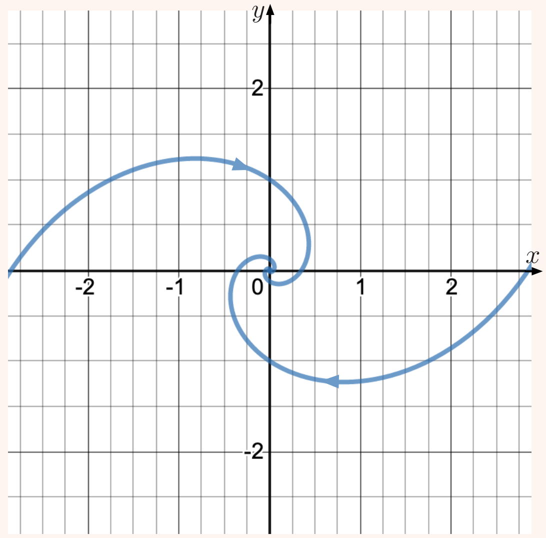


In this graph, we also include the graph for $t < 0$.

- Let $c_1 = -1$ and $c_2 = 0$ and we obtain

$$\vec{r}(t) = \begin{bmatrix} -\sin(3t) \\ -\cos(3t) \end{bmatrix} e^{-2t}$$

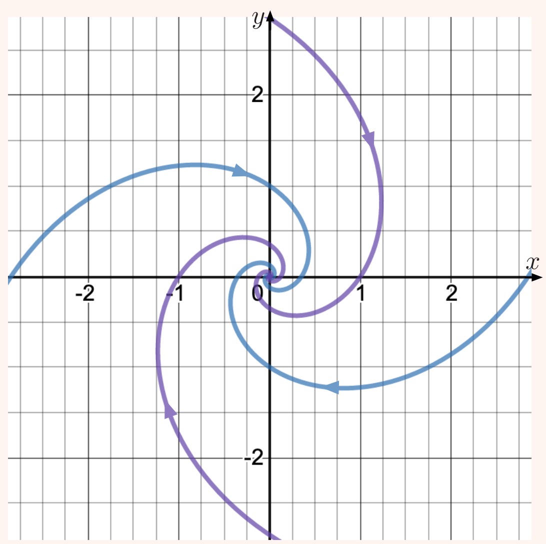
We add this graph to the previous one.



- Let $c_1 = 0$ and $c_2 = \pm 1$ and we obtain

$$\vec{r}(t) = \begin{bmatrix} \mp\cos(3t) \\ \pm\sin(3t) \end{bmatrix} e^{-2t}.$$

And add these to the graph:



Sometimes, we need some solutions of the type $c_1 = \pm 1$ and $c_2 = \pm 1$ to get some different types of solutions, but we'll let you discover that on the core exercises.
These four solutions seem to give a good idea of all possible solutions: clockwise spirals converging to the origin.

Also observe that $\vec{r}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so this system has an equilibrium solution.
This kind of equilibrium is called a ***spiral sink*** and it is ***asymptotically stable***. This means that it is a spiral and it converges to the equilibrium (the origin).

Video.

- https://youtu.be/nyI_JPDrJ_I

■ https://youtu.be/dpbRUQ-5YWc

Practice Problems

1 For each matrix from practice problem 1 from Module 17, sketch its phase portrait and label them as asymptotically stable or unstable. The system of ODEs is $\vec{r}'(t) = A\vec{r}(t)$ for the following matrices:

(a) $A = \begin{bmatrix} -7 & 6 \\ -9 & 8 \end{bmatrix}$;

(b) $A = \begin{bmatrix} 22 & 24 \\ -15 & -16 \end{bmatrix}$;

(c) $A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ This is called a centre, which is stable, but not asymptotically stable. Can you tell why?

(d) $A = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}$;

(e) $A = \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$;

(f) $A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$;

(g) $A = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}$ This is called an improper node.

(h) $A = \begin{bmatrix} -4 & -6 \\ 2 & 3 \end{bmatrix}$;

(i) $A = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$;

(j) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$; This is called a proper node.

(k) $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$;

(l) $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$;

2 Consider a system of ODEs $\vec{r}'(t) = A\vec{r}(t)$. For each part, give an example of eigenvalues and eigenvectors of A that would yield the required phase portrait:

(a) Spiral sink (asymptotically stable);

(b) Spiral source (unstable);

(c) Centre (stable);

(d) Sink node (asymptotically stable);

(e) Source node (unstable);

(f) Saddle point (unstable);

(g) Improper node (stable);

(h) Improper node (unstable);

(i) Proper node (stable);

(j) Proper node (unstable);

3 Consider the system of ODEs

$$\vec{r}'(t) = \begin{bmatrix} 1 & 1 \\ k & 1 \end{bmatrix} \vec{r}(t).$$

This system of ODEs changes behaviour depending on the parameter k.

(a) Label the behaviour of the system for different values of k.

(b) We call the $k^\star$ the critical value of k when the behaviour is different for $k < k^\star$ and for $k > k^\star$. For the critical value of k, sketch the phase portrait.

4 Consider the system of ODEs

$$\vec{r}'(t) = \begin{bmatrix} 0 & 1 \\ -4 & -k \end{bmatrix} \vec{r}(t).$$

This system of ODEs changes behaviour depending on the parameter k.

(a) Label the behaviour of the system for different values of k.

(b) We call the $k^\star$ the critical value of k when the behaviour is different for $k < k^\star$ and for $k > k^\star$. For the critical value of k, sketch the phase portrait.

(c) Which kinds of behaviours could be critical values?

5 Consider the system of ODEs

$$\vec{r}'(t) = A\vec{r}(t).$$

Let $T = \operatorname{trace}(A) = a_{11} + a_{22}$, $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$, and $\Delta = T^2 - 4D$.

(a) Show that the equilibrium solution is a saddle point if $D < 0$.

(b) Show that the equilibrium solution is a spiral if $\Delta < 0$ and $T \neq 0$.

(c) Show that the equilibrium solution is a centre if $T = 0$ and $D > 0$.

(d) When is the equilibrium point a node?

(e) Show that the equilibrium solution is asymptotically stable if $T < 0$ and $D > 0$.

(f) Show that the equilibrium solution is unstable if $T > 0$ and $D < 0$.

32 Consider the following model for cheetah's and lions, where

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

The general solution is:

$$\vec{p}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

32.1 Without computing them, what are the eigenvalues and eigenvectors of the matrix?

32.2 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = 0$.

32.3 Sketch the graph of the solution with $c_1 = 0$ and $c_2 = \pm 1$.

32.4 When one constant is set to 0, what is the shape of the graph? Is it always like that? Can you prove it?

32.5 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = \pm 1$.

32.6 Provide an interpretation of the different types of solutions.

33 Let us expand the model from the previous exercise to:

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \vec{p} + \begin{bmatrix} -10 \\ 50 \end{bmatrix} \vec{p}.$$

The extra term corresponds to the effect of harvesting 10 lions and bringing in 50 cheetahs every year to the reserve.

The general solution is:

$$\vec{p}(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

33.1 Sketch the phase portrait.

33.2 Provide an interpretation of the different types of solutions.

34 For each of the following general solutions, sketch the phase portrait.

34.1 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}$.

34.2 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}$.

Analysis of Models with Systems

In this module you will learn

- different ways to analyze models with several differential equations

In this chapter, we have learned how to create models involving systems of ODEs and how to solve some special types of systems of ODEs.

Once we create a model that involves a system of ODEs, the ultimate goal is not to solve the system of ODEs, but to be able to understand how the situation will proceed. Solving the system of ODEs is often a large step in that direction, but it is more important to be able to take the solution and knowing how to interpret in light of the original situation.

Sometimes, when we cannot find an explicit formula for the solution, it is still possible to study the system to find some properties and behaviours of the solutions.

In this module, we'll study one example using two different methods.

Example. The goal here is not the modelling but the analysis of the model, so we will quickly explain the model.

We are going to model population versus cost of living in Toronto.
Consider the following functions

- $p(t) =$ Population of Toronto (GTA) in millions at time t in years since the beginning of 2015.
- $c(t) =$ Cost of living in Toronto (in thousands of dollars) at time t.
- Define a vector $\vec{x}(t) = \begin{bmatrix} p(t) \\ c(t) \end{bmatrix}$.

These two factors are related according to the following properties:

- In the absence of any migration, the population will decrease proportionally to the cost of living (with constant a);
- There are always people moving into Toronto independently of its current population or cost of living (with constant b)
- In the absence of any other factors, the cost of living; increases proportionally to the population (with constant d)
- In the absence of any other factors, the cost of living; increases proportionally to the cost of living due to inflation (with constant e);
- The city is always expanding, so the cost of living is always decreasing independently of its current population or cost of living (with constant f).

The constants a, b, d, e, f are all positive.

Our model is:

$$\vec{x}'(t) = \begin{bmatrix} 0 & -a \\ d & e \end{bmatrix} \vec{x}(t) + \begin{bmatrix} b \\ -f \end{bmatrix}$$

Qualitative evolution of quantities

We can try to figure out how these quantities, $p(t)$ and $c(t)$, are going to increase or decrease.

As an academic example, let us imagine that initially $p(0) = c(0) = 0$.

Then, at $t = 0$, we have

$$p'(0) = b > 0 \quad \text{and} \quad c'(0) = -f < 0.$$

This means that $p(t)$ is increasing while $c(t)$ wants to decrease.

Here we need to make sure that everything still makes sense: since it doesn't make sense to have a negative cost of living (government incentives to move into the city?!), we need to disregard our system and assume that $c(t)$ will continue constant while $c'(t) < 0$.

We then have:

t	0							$+\infty$
p	0	$\nearrow$						
c	0	$\rightarrow$	0					

While $c(t) = 0$, we have

$$p'(t) = b > 0 \quad \text{and} \quad c'(t) = d\,p(t) - f.$$

This means that $p(t)$ is increasing with constant slope (linearly) until $c'(t_1) = 0$. We can figure out when this will happen:

$$0 = c'(t_1) = d\,p(t_1) - f \quad \Leftrightarrow \quad p(t_1) = \frac{f}{d}.$$

So we continue our table:

t	0		t_1					$+\infty$
p	0	$\nearrow$	$\frac{f}{d}$					
c	0	$\rightarrow$	0					

What happens after t_1?

Consider $t > t_1$ slightly after t_1. Then

$$\begin{cases} p'(t) = -ac(t) + b > 0 & \text{still positive because } c(t) \text{ is very small, but the slope is decreasing} \\ c'(t) = dp(t) + ec(t) - f > 0 & \text{increasing quickly as both } p \text{ and } c \text{ increase} \end{cases}$$

t	0		t_1					$+\infty$
p	0	$\nearrow$	$\frac{f}{d}$	$\nearrow$				
c	0	$\rightarrow$	0	$\nearrow$				

At a certain time t_2, the population will stop increasing. Let us find out when this happens:

$$0 = p'(t_2) = -ac(t_2) + b > 0 \quad \Leftrightarrow \quad c(t_2) = \frac{b}{a}.$$

t	0		t_1		t_2			$+\infty$
p	0	$\nearrow$	$\frac{f}{d}$	$\nearrow$	$\rightarrow$			
c	0	$\rightarrow$	0	$\nearrow$	$\frac{b}{a}$			

After this point we have $t > t_2$ slightly after t_2:

$$\begin{cases} p'(t) = -ac(t) + b < 0 & \text{decreasing rapidly while } c'(t) > 0 \\ c'(t) = dp(t) + ec(t) - f > 0 & \text{still increasing quickly until } p(t) = 0 \end{cases}$$

We expect that at some point $p(t_3) = 0$. From that point on we have

$$\begin{cases} p'(t_3) = -ac(t_3) + b < 0 & \text{we need to ignore the model at this point and keep } p \text{ constant} \\ c'(t_3) = ec(t_3) - f > 0 & \text{still increasing exponentially} \end{cases}$$

So this is our final table:

t	0		t_1		t_2		t_3	$+\infty$
p	0	↗	$\frac{f}{d}$	↗	→	↘	0	→
c	0	→	0	↗	$\frac{b}{a}$	↗		↗

Observe that to do this analysis, we didn't need to know how to solve the system of ODEs.

Finding the equilibrium point(s)

This is often easy to find, and by using the intuition we gained while learning to sketch phase portraits, this can give us a lot of insight about the solutions.

Let us find the equilibrium point:

$$\begin{cases} 0 = p'(t) = -ac(t) + b \\ 0 = c'(t) = dp(t) + ec(t) - f \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} c(t) = \dfrac{b}{a} \\ p(t) = \dfrac{af - be}{ad} \end{cases}$$

Observe that if the population and cost of living are at these levels, then they will remain constant.

This also informs us that the disastrous scenario on the first analysis, where the population all left the city, might have been caused by the stating position.

Properties of the system

We can look for other properties of the system of ODEs.

Based on the two analyses above, we can ask the following question:

- Is there a value for the cost of living such that if it is above that, then eventually all the population will leave the city?

We know that

$$p'(t) = -ac(t) + b < 0 \quad \Leftrightarrow \quad c(t) > \frac{b}{a}.$$

So as long as the cost of living is above $\frac{b}{a}$, then the population will continue to decrease.

Observe that depending on the constants d, e, f, we could still have

$$c'(t) = dp(t) + e\frac{b}{a} - f < 0,$$

so that we could end up with a cycle around the equilibrium we found before.

Practice Problems

1 Consider the model for student learning:

- $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
- $x_1 =$ student confidence in his/her own abilities ($x_1 \in [0, 1]$)
- $x_2 =$ student knowledge measured in IQ past 100
- $\vec{x}'(t) = \begin{bmatrix} a & b \\ c & -d \end{bmatrix} \vec{x}(t) + \begin{bmatrix} -e \\ 0 \end{bmatrix}$
- Constants $a, b, c, d, e > 0$.

(a) What is the equilibrium solution $\vec{x}_e$?

(b) If tests are harder, then d is larger. How does that affect the equilibrium confidence and knowledge of students?

(c) Is the equilibrium solution stable?

(d) Assume $a = 1, b = c = 2, d = 3, e = 0$. As $t \to +\infty$, what are the possible outcomes for $\vec{x}(t)$ Explain the meaning for the students.

(e) Assume $a = 1, b = c = 2, d = 3, e = 0$. Some solutions satisfy $\lim_{t\to+\infty} \begin{bmatrix} c(t) \\ k(t) \end{bmatrix} = \begin{bmatrix} +\infty \\ +\infty \end{bmatrix}$.
Show on a graph which initial conditions $\vec{x}(0) = \begin{bmatrix} c(0) \\ k(0) \end{bmatrix}$ guarantee this limit?

(f) If the tests become harder, i.e., d increases, then is that good or bad for students?.

2 Consider the model for a tree:

- $\vec{x}(t) = \begin{bmatrix} \ell(t) \\ h(t) \end{bmatrix}$
- $\ell(t) =$ area of leafs on the tree
- $h(t) =$ height of the tree
- $\vec{x}'(t) = \begin{bmatrix} a & -b \\ c & -d \end{bmatrix} \vec{x}(t)$
- Constants $a, b, c, d > 0$.

(a) Is it possible to have the tree growing taller and taller forever while the leaf area remains bounded?

(b) What would happen to the tree if the area of leafs is proportional to the height squared (not square root)?

(c) If $ad = bc$, explain what happens to the tree as $t \to \infty$.

3 Consider the model of a car:

- $\vec{c}(t) = \begin{bmatrix} v(t) \\ f(t) \end{bmatrix}$
- $v(t) =$ speed of the car
- $f(t) =$ amount of fuel in the car's tank
- $\vec{c}'(t) = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \vec{c}(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

(a) What is the equilibrium solution $\vec{c}_{eq}$? What is the meaning of your result?

(b) If the car runs out of fuel at 300 m/s, then describe what happens to the car.

(c) Describe what happens to the car when it starts at rest with a full tank of 300 L.

(d) If the car attains its maximum velocity when there are still 300 L of fuel left, what was the car's maximum velocity?

4 Consider the model for crying babies:

- $\vec{c}(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$
- $a(t) =$ volume of baby A's cries in dB
- $b(t) =$ volume of baby B's cries in dB
- $\vec{c}'(t) = \begin{bmatrix} -\alpha & \beta \\ \beta & -\alpha \end{bmatrix} \vec{c}(t)$
- Constants $\alpha, \beta > 0$.

The constants α and β are 1 and 2. Does it make a difference which is 1 and which is 2?

35 Consider the following model for the sales from a designer clothing brand:

- $x_1(t) =$ purchases by "common mortals" (CM) at time t in years since the beginning of 2015.
- $x_2(t) =$ purchases by "famous people" (FP) at time t.

Our model is based on the following two principles:

(P_1) CM will buy more items from the brand when CM or FP buy more.

(P_2) FP will buy less when CM buy them, but will buy more when FP buy it.

The model we considered is:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \vec{x}(t)$$

35.1 Suppose that at the beginning only CM buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

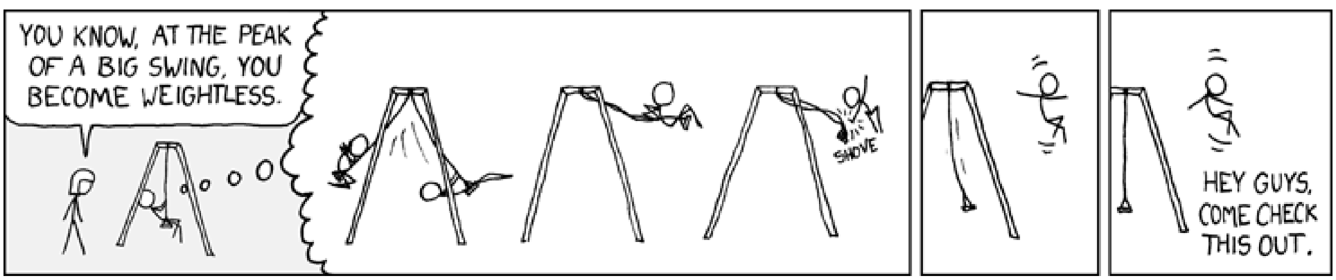
35.2 Suppose that at the beginning only FP buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

35.3 What conditions on the constants a, b, c, d will guarantee that the solutions will spiral? In that case, is it a spiral source or spiral sink? Is it clockwise or counterclockwise?

35.4 Are there constants $a, b, c, d > 0$, such that the solution $\vec{x}$ is periodic?

35.5 Consider the constants $a = b = c = d = 1$. Assume that initially CM were buying $c_0 > 0$ items and FP were buying $f_0 > 0$ items. What will happen to $x_1(t)$ and $x_2(t)$ as $t \to \infty$? Explain the results in terms of the evolution of purchases from CM and FP.

35.6 Consider the constants $a = b = c = d = 1$. If $c_0 = 10$, $f_0 = 100$, then at what time will FP stop buying items? And at what time will FP be buying the maximum number of items?



(image from xkcd - comic #226)

Modelling with Second-Order ODEs

In this module you will learn

- how to model physical phenomena to obtain second-order ODEs

Whenever we model the movement of objects, we often find ourselves using **Newton's Second Law of motion**:

Newton's Second Law of Motion. $F = m \cdot a$, where a is the acceleration of the object, m is its mass, and F is the net force acting on the object.

Because this "Law" includes the acceleration of the object, and we know that

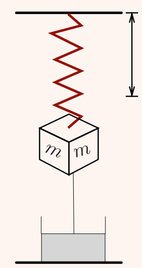
$$\text{acceleration} = a = \frac{d\,(\text{velocity})}{dt} = \frac{dv}{dt} = \frac{d^2\,(\text{position})}{dt^2} = \frac{d^2 r}{dt^2},$$

we will often end up with a Second-Order ODE.

Just like we did in module 25, we will follow the step by step procedure developed in chapter 1.

Step 1. Define the problem

Example.



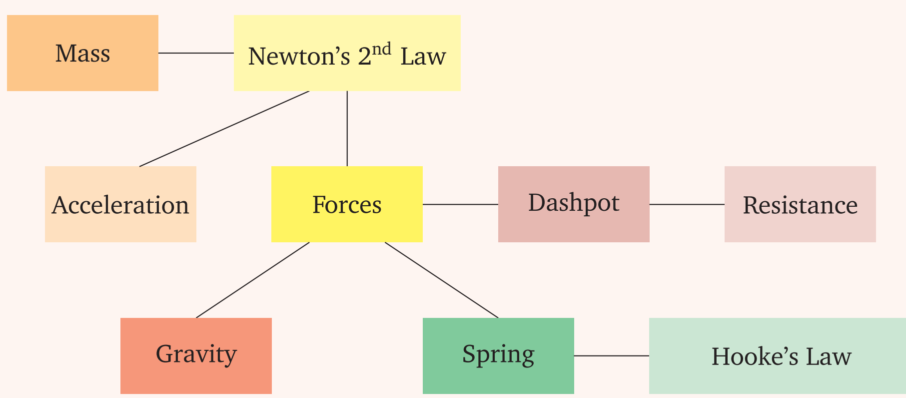
We want to model the position of an object attached to the end of a spring.

The first step is to decide on what we want to find at the end of the process. So we define:

- $y(t)$ = the vertical position of the mass, where $y = 0$ is the position of the mass at rest.

Step 2. Build a mind map

Example. We start with the mass and then we brainstorm about the things that affect the mass:



Step 3. Make assumptions

Example. In this step, we discuss the mind map we created and how we plan to address each of the boxes, or only some of the boxes. This will involve making assumptions and providing an explanation to the assumptions we make.

1. As we described before the example, the plan is to use Newton's Second Law of Motion to describe

the motion of the mass. This involves three quantities:

- mass: we assume that this is known to the modeller;
- acceleration: as we mentioned above, this is directly related to the position of the object. We have $y''(t) =$ acceleration, as long as we are assuming that the object is moving only vertically;
- forces: we need to find all the forces acting on the object and add them.

The forces acting on the object need to be discussed separately:

- Gravity: we will go on a limb here and say that the force of the spring is much larger, so we will ignore this force;
- Spring: the force of the spring that acts on the mass follows Hooke's Law that says that the force is proportional to the extension/contraction of the spring. The constant of proportionality depends on the spring and we assume that it is known;
- Dashpot: the dashpot provides resistance. We will assume that it provides linear resistance to movement: the force is proportional to the velocity, with a proportionality constant that depends on the dashpot and is assumed to be known to the modeller.

Step 4. Construct a model

Example. We will start with Newton's Second Law of motion:

- $my''(t) = F(t)$

and we will add the different forces one by one:

- Spring: the force of the spring is $-ky(t)$; (you should check that the sign makes sense)
- Dashpot: the force of the dashpot is $-\gamma y'(t)$. (you should also check the sign of this term)

Right now we have the following model:

$$my''(t) = -ky(t) - \gamma y'(t).$$

Step 5. Model assessment

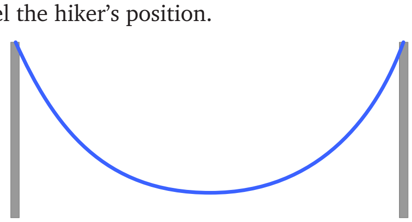
We'll skip this part here, but you should try to develop some tests to check the validity of the model we came up with. Specifically, the fact that we ignored gravity should be checked to make sure that it doesn't affect our model too much.

Step 6. Putting it all together in a report

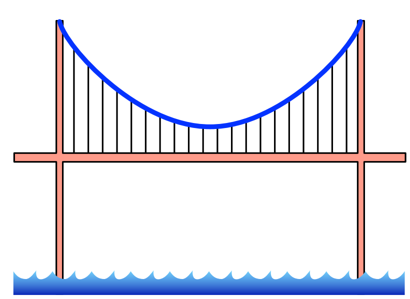
We'll skip this part here.

Practice Problems

1 Consider a mountain with shape $y = f(x)$ a hiker who is climbing down the mountain with horizontal position $x(t)$. She starts at a peak of the mountain at $x_0 = 0$. As she climbs down the mountain, she notices that from her point-of-view, the rate of change of the slope of the mountain is decreasing linearly with time. Model the hiker's position.

2 

3 Model the shape of a rope hanging between two poles.



4 Model the shape of the cables of a suspension bridge.

5 Imagine a cylinder floating vertically partially submerged in a lake. Model the position of its top.

6 Start with the Law of Conservation of Energy and as-

sume a conservative force. Then show that you obtain Newton's Second Law of motion.

36 Here are some facts about laptop keys:

- (da) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.
- (di) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.
- (fo) On average, a person exerts the force of 42 N with one finger on a key.
- (gr) Gravity is much weaker than the spring that keeps the key in place.
- (hl) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").
- (lo) Keys last 50 million presses on average.
- (ve) Keys can only move vertically.

36.1 Model a laptop keypress.

36.2 What happens if the damping system of the key is broken? What happens if the damping system is too strong? How strong should the damping system be?

36.3 What happens to the key when the spring breaks?

Second-Order Linear ODEs with Constant Coefficients

In this module you will learn

- how to solve this type of ODEs

In this module we will learn how to solve a specific type of Second-Order ODEs: linear second-order ODEs with constant coefficients. These equations have the form

$$ay''(t) + by'(t) + cy(t) = f(t).$$

4.1 Homogeneous ODEs

These are ODEs above with $f(t) \equiv 0$. So we are trying to solve

$$ay''(t) + by'(t) + cy(t) = 0.$$

The main idea to solve these problems is the same as for systems: making an **educated guess** that the solution should look like an exponential:

$$y(t) = e^{rt},$$

and we need to find which values of r yield solutions.

We do that by plugging this formula for $y(t)$ into the ODE:

- $y'(t) = re^{rt}$
- $y''(t) = r^2 e^{rt}$

We get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \quad \Longleftrightarrow \quad ar^2 + br + c = 0$$

and we know how to solve this:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

That means that we have three possible cases.

Two real distinct roots. When $b^2 - 4ac > 0$, we have two possible values for r that are real numbers: r_1 and r_2.

Then, similarly to what we did with systems of ODEs, we obtain two solutions

$$y_1(t) = e^{r_1 t} \quad \text{and} \quad y_2(t) = e^{r_2 t},$$

and the general solution is

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Video.

- `https://youtu.be/_8fcT95JV34`
- `https://youtu.be/nE_OnX8ulHA`

- https://youtu.be/v1xKZ0rGsVc

Two complex roots. When $b^2 - 4ac < 0$, we have two possible values for r, but they are complex values:

$$r_\pm = a \pm ib.$$

What are the value of a and b?

Then we have two solutions

$$y_+(t) = e^{(a+ib)t} \quad \text{and} \quad y_-(t) = e^{(a-ib)t},$$

and the general solution is

$$y(t) = a_1 e^{(a+ib)t} + a_2 e^{(a-ib)t}.$$

Just like we did with systems with complex eigenvalues, we prefer to write the solutions without complex numbers, so we expand it using Euler's formula to get

$$\begin{aligned} y(t) &= a_1 e^{(a+ib)t} + a_2 e^{(a-ib)t} \\ &= a_1 e^{at} e^{ibt} + a_2 e^{at} e^{-ibt} \\ &= a_1 e^{at}\big(\cos(bt) + i\sin(bt)\big) + a_2 e^{at}\big(\cos(bt) - i\sin(bt)\big) \\ &= (a_1 + a_2)\cos(bt)e^{at} + i(a_1 - a_2)\sin(bt)e^{at} \\ &= c_1 \cos(bt)e^{at} + c_2 \sin(bt)e^{at} \end{aligned}$$

How do c_1 and c_2 depend on a_1, a_2?

So another way to write the general solution is

$$y(t) = c_1 \cos(bt)e^{at} + c_2 \sin(bt)e^{at}.$$

Video.

- https://youtu.be/DOR16GMPtjM?t=396

One real repeated root. When $b^2 - 4ac = 0$, then we are left with only one value for $r = -\frac{b}{2a}$.

We then have one solution

$$y_1(t) = e^{-\frac{b}{2a}t}.$$

Example. Consider the ODE

$$y''(t) + 2y'(t) + y(t) = 0.$$

To find the general solution, we assume that the solutions have the form $y(t) = e^{rt}$, which means that r must satisfy

$$r^2 + 2r + 1 = 0 \quad \Leftrightarrow \quad r = -1,$$

so $y_1(t) = c_1 e^{-t}$.

Now can we solve this ODE with the following initial conditions?

- $y(0) = 2$ and $y'(0) = -2$.
- $y(0) = 2$ and $y'(0) = 1$.

This previous example, should give a good idea on why having one value for r means that we are missing something. We need to find a second solution $y_2(t)$.

If we want to find all the divisors of 42, and we already know that $d_1 = 2$ is a divisor, then we can use the divisor d_1 we know to write

$$d_1 \cdot x = 42 \quad \Leftrightarrow \quad 2x = 42 \quad \Leftrightarrow \quad x = 21,$$

where x is the product of all the other divisors.
We used the divisor we knew d_1 to obtain a simpler problem for the other divisors.

Reduction of Order. The idea here is the same. We use the solution we found to try to obtain a simpler ODE for the other solution:

$$y(t) = y_1(t) \cdot u(t),$$

where $y(t)$ is the solution we are still missing, $y_1(t)$ is the solution we already found, and $u(t)$ is a function. If we find $u(t)$, then we find $y(t)$. We hope that the function $u(t)$ satisfies a simpler problem.

To do that, we need to plug the formula above for $y(t)$ into the original ODE.

You should do these calculations yourself. Remember to use the product rule and to be careful not to make any mistakes.
Also remember that we know the value of r

We obtain

$$u''(t) = 0 \quad \Leftrightarrow \quad u(t) = c_1 + c_2 t.$$

This means that we found

$$y(t) = (c_1 + c_2 t)e^{rt} \quad \Leftrightarrow \quad y(t) = \underbrace{c_1 e^{rt}}_{\substack{\text{previous} \\ \text{solution } y_1(t)}} + c_2 t e^{rt}.$$

The general solution is this

$$y(t) = c_1 e^{rt} + c_2 t e^{rt},$$

where $r = -\frac{b}{2a}$.

Video.

- https://youtu.be/DOR16GMPtjM

4.2 Non-Homogeneous ODEs

We are trying to solve

$$ay''(t) + by'(t) + cy(t) = f(t),$$

where $f(t)$ is a known function.

If $u(t)$ is the general solution of
$$ay''(t)+by'(t)+cy(t)=0,$$
and $v(t)$ satisfies
$$ay''(t)+by'(t)+cy(t)=f(t),$$
then $y(t)=u(t)+v(t)$ gives the general solution of
$$ay''(t)+by'(t)+cy(t)=f(t).$$
This is a practice problem at the end of this module.

This means that to solve this ODE, we split the general solution into two parts
$$y(t)=y_c(t)+y_p(t),$$
where

- $y_c(t)$ is called the **complementary solution** and it is the general solution of the corresponding homogeneous ODE. It is solved using the technique we studied above.
- $y_p(t)$ is called the **particular solution** and it is one function that satisfies the original ODE.

Important. It may seem strange that to solve the original ODE, we need its solution, but what we are trying to do is find ***all possible solutions*** of the original ODE.
To find all possible solutions of the original ODE, we require two things:
- One solution of the original ODE: $y_p(t)$,
- and all possible solution of the homogeneous ODE: $y_c(t)$.

We already know how to find the complementary solution, so we will focus our attention on finding one particular solution.

Method of Undetermined Coefficients. As you probably have gotten used to by now, this is a method of educated guess-and-check.

Let us look at the equation from a different point-of-view
$$ay''(t)+by'(t)+cy(t)=f(t)$$
$$\begin{array}{c}\text{linear combination of}\\ \text{function and derivatives}\end{array}=f(t)$$
and remember that some functions don't change much when differentiated:

- Exponentials $y=ce^{rt}$ don't change their form after differentiation $y'=cre^{rt}=de^{rt}$. They even keep the same exponential term.
- Polynomials don't change their form either: their derivative is also a polynomial, with lower degree.
- Cosines and Sines alternate between one and the other, so functions of the form $y=c_1\sin(rt)+c_2\cos(rt)$ don't change after differentiation.

This means that, if $f(t)$ is one of these types of function, then $y(t)$ must be of the same form.

Example. Find a particular solution for the ODE
$$y''-4y=10e^{3t}=(\text{constant})\cdot(\text{exponential of } 3t).$$
Our candidate is
$$y_p(t)=Ae^{3t}.$$
Now we need to find the constant A by plugging it into the ODE:
$$9Ae^{3t}-4\cdot Ae^{3t}=10e^{3t}\quad\Leftrightarrow\quad A=2,$$
so $y_p(t)=2e^{3t}$ is a particular solution.

Example. Find a particular solution for the ODE

$$y'' - 4y = 3t^2 + 2t = (\text{polynomial of degree 2}).$$

Our candidate is

$$y_p(t) = At^2 + Bt + C.$$

Now we need to find the constants A, B, C by plugging the formula for y_p into the ODE:

$$2A - 4At^2 - 4Bt - 4C = 3t^2 + 2t \quad \Leftrightarrow \quad \begin{cases} A = -\frac{3}{4} \\ B = -\frac{1}{2} \\ C = \frac{A}{2} = -\frac{3}{8}. \end{cases}$$

so $y_p(t) = -\frac{3}{4}t^2 - \frac{t}{2} - \frac{3}{8}$ is a particular solution.

There are some more details to deal with when using this method that will be addressed in the core exercises.

Video.

- `https://youtu.be/CjZ0TfPnWVU`
- `https://youtu.be/ubdSxJ2nmVk`
- `https://youtu.be/YRvqem1n0nQ`

Practice Problems

1 Find the complementary and particular solutions for the following ODEs

(a) $y'' - 2y' - 3y = 3e^{2t}$

(b) $y'' - 2y' - 3y = -3te^{-t}$

(c) $y'' - 9y = t^2e^{-3t} - 6$

(d) $y'' + 2y' - 8y = e^{-t} - 2e^{t}$

(e) $y'' - y' - 6y = \sin(t)$

(f) $y'' - y' - 6y = \sin(t) + 3e^{3t}$

(g) $y'' + 4y = (2t + 1)\sin(t) + 4\cos(2t)$

(h) $y'' + y = \cos(2t) + t^3$

(i) $y'' - y' - 2y = t\cos(t) - t\sin(t)$

(j) $y'' + 5y' + 6y = 2e^{-2t}$

2 What is the form of the particular solution for the ODE

$$\begin{aligned} &y^{(6)} + y^{(5)} - 5y^{(4)} + 31y''' - 176y'' + 220y' \\ &= (3t-1)e^{2t} + t^3e^{-5t}\sin(3t) + (4t^2 - 2t)e^{-2t}\sin(3t), \end{aligned}$$

knowing that

$$\begin{aligned} &x^6 + x^5 - 5x^4 + 31x^3 - 176x^2 + 220x \\ &\quad = \big((x^2+2)+9\big) * (x-2)^2 * x * (x+5) \quad ? \end{aligned}$$

3 What is the form of the particular solution for the ODE

$$\begin{aligned} &y'''' - 4y''' + 10y'' - 12y' + 5y \\ &\quad = te^t + t^2\cos(2t) - (2t+1)e^t\sin(t), \end{aligned}$$

knowing that

$$\begin{aligned} &x^4 - 4x^3 + 10x^2 - 12x + 5 \\ &\quad = (x-1)^2\big((x-1)^2 + 4\big) \quad ? \end{aligned}$$

4 Consider the ODE

$$t^2y'' + ty' - 9y = 0,$$

and a solution $y_1(t) = t^3$.

(a) Use the reduction of order technique to deduce the general solution to this problem.

Hint. You should find a second-order ODE for $u(t)$ without the term $u(t)$. So define $v(t) = u'(t)$ and solve the first-order ODE for $v(t)$.

(b) Find the solution with initial conditions $y(1) = 1$ and $y'(1) = -3$.

(c) Find the solution with initial conditions $y(1) = 1$ and $y'(1) = 3$.

(d) Find the solution with initial conditions $y(1) = 1$ and $y'(1) = 0$.

5 Consider the ODE

$$t^2 y'' - 3ty' + 4y = 0$$

and a solution $y_1(t) = t^2$.

(a) Use the reduction of order technique to deduce the general solution to this problem.

(b) Find the solution with initial conditions $y(1) = 1$ and $y'(1) = 2$.

(c) Find the solution with initial conditions $y(1) = 0$ and $y'(1) = 1$.

6 Consider the ODE $ay'' + by' + cy = f(t)$, with complementary solution $y_c(t) = c_1 y_1(t) + c_2 y_2(t)$ and particular solution $y_p(t)$.

Consider also the initial conditions $y(0) = y_0$ and $y'(0) = v_0$.

Show that there exist constants c_1, c_2 such that $y(t) = y_c(t) + y_p(t)$ solves the ODE with these initial conditions.

37 Consider the ODE $y''(t) - 9y(t) = f(t)$.

37.1 Find a complementary solution.

37.2 Find a particular solution for $f(t) = 14e^{-4t}$.

37.3 Find a particular solution for $f(t) = 9e^{-3t}$.

37.4 Find a particular solution for $f(t) = 10\cos(t)$.

38 Consider the ODE $y''(t) - 2y'(t) + 5y(t) = f(t)$.

38.1 Find a complementary solution.

38.2 Find a particular solution for $f(t) = \sin(2t)e^t$.

38.3 Find a particular solution for $f(t) = (4t+2)\sin(2t)e^t$.

39 Consider the ODE $y'' + 3y' = 3t$.

39.1 Find the complementary solution.

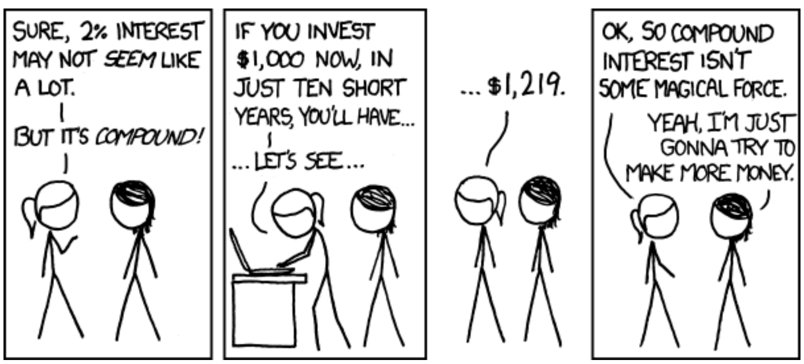
39.2 Find a particular solution.

39.3 Find the solution that also satisfies

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Analysis of Models with Higher Order ODEs

40



(image from xkcd - comic #947)

Introduction to Difference Equations

In this module you will learn

- what is a difference equation
- the different types of difference equations

Difference Equation. A ***difference equation*** is an equation involving an unknown sequence and a recursive relation between different terms of that sequence.

Example.

1. $u_{k+1} = u_k + u_{k-1}$
2. $x_k = 2x_{k-1}$

Among difference equations, there are lots of types, that require different approaches, so we need to classify them.

Types of Differential Equations. Just like with differential equations, the main way we distinguish difference equations is according to:

- ***order***: the order of a difference equation is the difference between the highest and the smallest terms of the sequence present in the difference equation;
- ***linear*** vs ***nonlinear***: A difference equation $F\big(k, u_k, u_{k-1}, \ldots, u_{k-n}\big) = 0$ is called ***linear*** if F is a linear function of $u_k, u_{k-1}, \ldots, u_{k-n}$. Linear difference equations have the form

$$a_0(k)u_k + a_1(k)u_{k-1} + \cdots + a_n(k)u_{k-n} = b(k).$$

All other differential equations are called ***nonlinear***.

Roughly, to check whether a difference equation is **linear**, we need to check that:

- The unknown u_k and its other terms appear with exponent 1;
- The unknown u_k and its other terms do not multiply by each other;
- The unknown u_k and its other terms are not the objects of other functions – there are no occurrences of things like $\sin(u_k)$ or $e^{u_{k-4}}$, $\ln(u_{k+1})$, $\sqrt{u_{k-1}}$, etc.

Example.

1. The difference equation $u_k = 2u_{k-2}$ is linear and second-order,because $k-(k-2) = 2$.
2. The difference equation $u_{k+1} = u_k^2 + u_{k-2}$ is nonlinear and third-order, because $(k+1)-(k-2) = 3$.

Similarly to differential equations, linear difference equations are, in general, easier to study and their theory is much more developed.

Module 22: Introduction to Difference Equations

Solving Difference Equations

In this module you will learn

- how to solve some types of difference equations

Let us start with a technique that is very simple and useful, although because it is so simple, it requires some ingenuity to pull off in some cases.

23.1. Expanding to find a pattern

We'll start with an example.

Example. Consider the initial-value problem

$$\begin{cases} u_{k+1} = \frac{3}{2}u_k & \text{for } k \geq 0 \\ u_0 = 5 \end{cases}$$

Then we can start calculating:

- $u_1 = \frac{3}{2}u_0 = 7.5$
- $u_2 = \frac{3}{2}u_1 = 11.25$
- $u_3 = \frac{3}{2}u_2 = 16.875$
- $u_4 = \frac{3}{2}u_3 = 25.3125$
- $u_5 = \frac{3}{2}u_4 = 37.96875$
- $\vdots$

As you can notice, it's not particularly easy to find a pattern in these numbers.
The problem is that we ***over-simplified***. The trick with this technique is to simplify without over-simplifying.
Let's calculate again:

- $u_1 = \frac{3}{2}u_0 = \frac{3}{2} \cdot 5$
- $u_2 = \frac{3}{2}u_1 = \frac{3}{2} \cdot \frac{3}{2} \cdot 5 = \left(\frac{3}{2}\right)^2 \cdot 5$
- $u_3 = \frac{3}{2}u_2 = \frac{3}{2} \cdot \left(\frac{3}{2}\right)^2 \cdot 5 = \left(\frac{3}{2}\right)^3 \cdot 5$
- $u_4 = \frac{3}{2}u_3 = \frac{3}{2} \cdot \left(\frac{3}{2}\right)^3 \cdot 5 = \left(\frac{3}{2}\right)^4 \cdot 5$
- $u_5 = \frac{3}{2}u_4 = \frac{3}{2} \cdot \left(\frac{3}{2}\right)^4 \cdot 5 = \left(\frac{3}{2}\right)^5 \cdot 5$
- $\vdots$

Now the pattern should be clear:

$$u_k = \left(\frac{3}{2}\right)^k \cdot 5.$$

To show that this is indeed the solution, we need to use Mathematical Induction (see appendix 6.2) to prove it.

The main idea of this technique is to calculate the the terms of the fraction one by one in terms of the initial data.

This is a technique that requires practice, as it is often difficult to judge which parts to simplify and which parts to expand to make sure the pattern emerges clearly.

Video.

- https://youtu.be/O0cUAjOXmFc

23.2. Educated Guessing

This is the technique we used several times in the book already. We used it with systems of differential equations and with second-order differential equations.

Observe that in the last example, the solution was an exponential, as was the case with differential equations.

Example. Consider the Fibonacci sequence:

$$\begin{cases} f_{k+1} = f_k + f_{k-1} \\ f_0 = 0 \\ f_1 = 1 \end{cases}$$

We want to find a formula for f_k. To do that, let us assume that the sequence is an exponential. So we can assume that

$$f_k = r^k,$$

for some value of r.

Let us now use this form of f_k into the difference equation to obtain:

$$r_{k+1} = r_k + r_{k-1},$$

which can be simplified by dividing by r^{k-1}:

$$r^2 = r + 1 \quad \Leftrightarrow \quad r^2 - r - 1 = 0.$$

This is a quadratic equation that we can solve:

$$r_\pm = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

So we have two values of r that seem to work.

That is similar to what we had when solving second-order ODEs (and this is a second-order difference equation). In that case, the solution turned out to be a linear combination of the two solutions found:

$$f_k = c_1 r_-^k + c_2 r_+^k = c_1 \left(\frac{1-\sqrt{5}}{2}\right)^k + c_2 \left(\frac{1+\sqrt{5}}{2}\right)^k.$$

Now we need to find c_1 and c_2 using the initial data:

$$0 = c_1 + c_2 \qquad (k=0)$$

$$1 = c_1 \frac{1-\sqrt{5}}{2} + c_2 \frac{1+\sqrt{5}}{2} \qquad (k=1)$$

This yields:

$$c_1 = -\frac{1}{\sqrt{5}}$$

$$c_2 = \frac{1}{\sqrt{5}}$$

So the formula we obtain is

$$f_k = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right].$$

The idea of this technique is to assume that the solution is an exponential of the form r^k and find the values for r that solve the particular difference equation. The general solution will be a linear combination of these solutions.

Video.

- `https://youtu.be/A5tBvxDM9V4`

Practice Problems

1 Consider the initial-value problem

$$\begin{cases} x_{k+1} = 3x_k + 4 \\ x_0 = 1 \end{cases}$$

(a) Using the expand-until-you-find-the-pattern technique, find the solution of this problem.

(b) Observe that this problem has an equilibrium solution $x^\star$. What is $x^\star$?

(c) Define a new sequence $y_k = x_k - x^\star$. Which initial-value problem does ti satisfy?

(d) Find y_k.

(e) Find x_k.

2 Find the solution to the problem

$$\begin{cases} x_{k+1} = -2x_k + 3 \\ x_1 = 2 \end{cases}$$

3 When we were solving ODEs, we considered exponential solutions of the form $u_k = e^{rk}$, but above we considered $u_k = r^k$. Are these equivalent?

Consider the initial-value problem

$$\begin{cases} x_{k+1} = x_k - x_{k-1} \\ x_0 = 1 \\ x_1 = 2 \end{cases}$$

(a) Solve the problem assuming the solution is of the form $u_k = r^k$.

(b) Solve the problem assuming the solution is of the form $u_k = e^{rk}$.

(c) What can you conclude?

4 Find the solution to the problem

$$\begin{cases} x_{k+1} = -2x_k - x_{k-1} \\ x_0 = 1 \\ x_1 = 2 \end{cases}$$

41 Consider the difference equation

$$u_{k+1} = 6u_k - 9u_{k-1}$$

41.1 Find the solution that satisfies:

$$\begin{cases} u_0 = 1 \\ u_1 = 3 \end{cases}$$

41.2 Find the solution that satisfies:

$$\begin{cases} u_0 = 1 \\ u_1 = 4 \end{cases}$$

42 Consider a difference equation that has solutions $u_k = r^k$ for $r = 2$ and $r = 3$.

We also have the conditions:

$$u_0 = 0 = 7 \quad \text{and} \quad u_1 = 6.$$

What is u_{22}?

Modelling with Difference Equations

In this module you will learn

- when to model a quantity using a difference equation instead of a differential equation
- different ways to create a model using difference equations

In all the modelling scenarios of chapters 2, 3, and 4, we dealt with continuously changing quantities, and so the appropriate way to model these was through differential equations.

However, not everything changes continuously, somethings change at specific time intervals. For those quantities, differential equations are not the best tool and we turn to difference equations.

This module will be divided in several parts depending on the type of situation.

24.1. Economic Models

Economic quantities don't often change continuously but in bursts. That's what happens to a savings account as in the example below, or to the stock prices, or to the balance left on a mortgage. Economic quantities are usually modelled by difference equations.

Example. We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

How does the balance in the savings account change over time?

Step 1. The goal is to model the balance on the account, so define

- $b(t) =$ balance on the savings account at time t.

Notice that the balance on the bank account doesn't change continuously. The balance doesn't change at all and when the compounding period passes, the bank adds the interest into the account.
So the balance only changes at each compounding period, so we can change our goal to define

- $b_n =$ balance on the savings account after n compounding periods.

Step 2. Create a mind map.
We will skip this part for this example. You should create a mind map of everything that affects the savings account.

Step 3. Let us make the following assumptions:

- We make one initial deposit into the account at time $n = 0$.
- We don't make any more withdrawals or deposits.
- The only way the savings account balance changes is through the interest, which is the interest rate $p\%$ of the current balance.

Step 4. We create the following model

$$b_{n+1} = b_n + \alpha \frac{p}{100} b_n = \left(1 + \alpha \frac{p}{100}\right) b_n.$$

24.2. Probability Models

There are several circumstances that involve probabilities that can be modelled using difference equations.

Below is an example of one such circumstance.

Example.

A gambler plays a game at a casino. The game is played one round at a time.
Each round, one of two things happens:

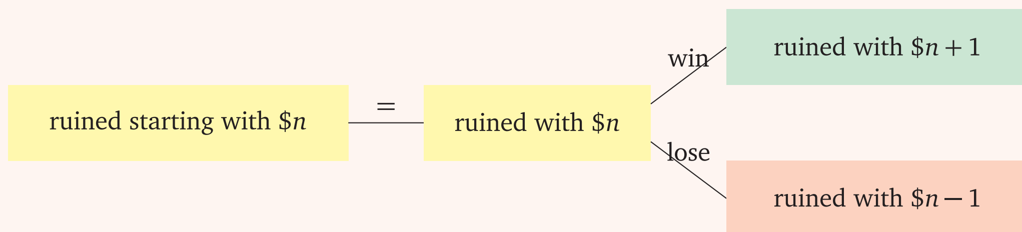
- The gambler wins \$1 with a probability of q
- The gambler loses \$1 with a probability of $1-q$

The gambler will stop playing only if

- The gambler is ruined (bankrupt)
- The gambler reaches $\$W$.

What is the probability $\boldsymbol{p_n}$ that the player will be ruined if he starts gambling with $\$n$?

Step 2. Mind map.



The two boxes on the left are very important. The crucial idea is to realize that it doesn't matter when the gambler has $\$n$. If s/he has $\$n$ at two different points in time, then the probability of becoming ruined is the same.

This mind map, shows us that we can relate p_n, p_{n+1} and p_{n-1}.

The rest of the modelling will be left as a practice problem.

Video.

- `https://youtu.be/Rr2iSKlengg`

24.3. Population Models

We have modelled populations using differential equations. Populations can be modelled using both differential or difference equations. Which kind of equations to use depends on the goal of the model and the assumptions that we make.

Below we'll see an example of a population model using difference equations.

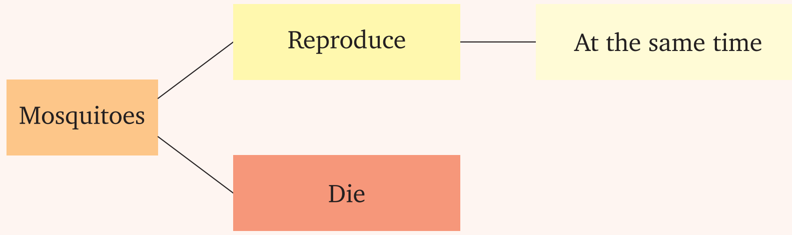
Example.

Model a population of mosquitoes, who reproduce at specific times of the year.

Step 1. The goal is to model the population, so we define

- $p(t) =$ population of mosquitoes at time t.

Step 2. We create a mind map for this problem.



Step 3. Given that the mosquito population all reproduces at the same time, we don't need to track the population at all times t.
So we can assume that the mosquito population doesn't change (much) between seasons, and we change our objective function from $p(t)$ to p_n:

- p_n = population of mosquitoes at the beginning of season n.

We understand that mosquitoes die in between seasons, but in this model, we only count the deaths at the beginning of each season.

The next assumption is that the number of nymphs (baby mosquitoes) is proportional to the number of mosquitoes in the beginning of the season.
Similarly, the number of deaths is proportional to the number of mosquitoes in the beginning of the season.
Also observe that mosquitoes only live for one season, which means that the proportionality constant $\mu > 1$.

Step 4. So our model is

- p_n = population of mosquitoes at the beginning of season n.
- $p_{n+1} = rp_n - \mu p_n = (r-\mu)p_n$;
- r = the average number of nymphs per per mosquito per season;
- μ = the average number of deaths per mosquito per season;
- $\mu > 1$, which means that each mosquito itself dies (at the end of the seasons if not earlier), but also some of its nymphs will die.

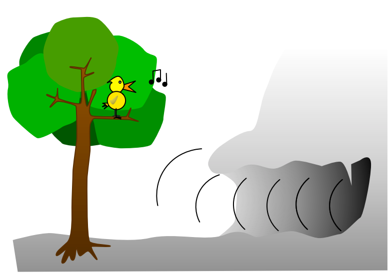
Video.

- `https://youtu.be/qmm9GPhA1MY`

Practice Problems

1 Create a model for the following situation:

(a) You just took a loan to buy a car. You'll need to make fixed payments every period, and the bank will charge an interest on the amount you still owe every period.



(b) A bird is chirping to find a mate. Unfortunately it is standing next to a cave which echoes its chirps. Consider the following:

(P_1) The bird chirps once every minute;

(P_2) The maximum volume the bird can chirp is M dB;

(P_3) If it hears a chirp, then it chirps at a volume proportional to the volume of the chirp it heard times the difference between the maximum volume it is capable and the volume heard with constant $A\ \frac{1}{dB}$;

(c) IBM just developed an new software that they wish to charge for usage. In this program, there is a parameter n that you can choose to change how it performs:

- n^2 = number of operations it takes to run the program;
- The profit IBM will make is $-\ln(\text{error})$ in Canadian dollars (negative means that IBM has to pay a penalty).

Model the profit that IBM makes. Remember to consider all sources of costs.

(d) Let us study Engineering students at the University of Toronto. Find a model for the number of undergraduate students in the Engineering school at the University of Toronto and how they change from year to year.

(e) You are working for Canada Revenue Agency and the queue in the IRS complaints section is getting too large and lengthy. One way to solve this would be to stop collecting taxes, but that's not possible, so you are tasked with modelling the queue.

Model a queue on a typical weekday afternoon minute by minute.

2 Read the example above about the gambler's ruin. Finish creating a model for it.

43 Let us expand on the economic example above.

We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

Even though we call $p\%$ the annual interest rate, because it is compounded during the year, at the end of the year the effective annual interest rate $p_{\text{eff}}\%$ is actually higher.

Calculate the effective interest rate $p_{\text{eff}}\%$.

44 Given a population with

- $\mu =$ probability that an individual will die between two seasons.

44.1 Define the following quantity

- $P(k)$ =probability that an individual born at season 0 is alive at the beginning of season k.

Find a model for $P(k)$.

44.2 What is the probability of the individual dying at age k?

44.3 What is the average lifespan of an individual in this population?

45 Consider a population of special rabbits. Once a pair of rabbits is born, they grow and one year later they are still immature. But two years after they are born they give birth to another pair of rabbits.

Model this population of rabbits.

46 Consider another population of rabbits. This is the lifecycle of a pair of rabbits:

(year 1) Born

(year 2) Immature (no babies)

(year 3) Young Adult (2 pairs of babies)

(year 4) Adult (1 pair of babies)

(year 5) Old (no babies)

(year 6) Die

Model this population of rabbits.

Analysis of Difference Equations

s module you will learn

- some ways to analyze models with difference equations

Module 25: Analysis of Difference Equations

47 Core Exercise with several parts

47.1 Part 1

47.2 Part 2

48 One more core exercise

6.1 Linear Algebra Review

Algebra of Solving Systems of 2 Linear Equations

We can write a linear system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

into matrix form

$$\mathbf{A}\vec{x} = \vec{b},$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We can solve a system like this one in several different ways.

Example. Solve the system

$$\begin{aligned} 3x_1 + 2x_2 &= 7 \\ 2x_1 + 3x_2 &= 8. \end{aligned}$$

Solution by substitution.. We can write

$$x_2 = \frac{7-3x_1}{2},$$

and use this on the second equation

$$2x_1 + 3\frac{7-3x_1}{2} = 8 \quad \Leftrightarrow \quad 4x_1 + 21 - 9x_1 = 16 \quad \Leftrightarrow \quad -5x_1 = -5 \quad \Leftrightarrow \quad x_1 = 1$$

Then re-use the first equation we obtained to get $x_2 = 2$.

Solution by Cramer's rule.
Using the same method of substitution on the general system, we obtain

$$a_{12}x_2 = b_1 - a_{11}x_1,$$

and we use this into the second equation (after multiplying by a_{12})

$$a_{12}a_{21}x_1 = a_{22}b_1 - a_{11}a_{22}x_1 = a_{12}b_2$$

This implies

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Then we use this to obtain

$$x_2 = \frac{a_{11}b_2 - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

Important. This implies that there is a unique solution of the system if and only if

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

Solution by inverse matrix. A matrix is **invertible** or **nonsingular** iff $\mathbf{A}^{-1}$ exists iff $\det(\mathbf{A}) \neq 0$.
If the matrix $\mathbf{A}$ is invertible, then we can write

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

We can now use this to solve the system of equations:

$$\begin{aligned} \mathbf{A}\vec{x} &= \vec{b} \\ \mathbf{A}^{-1}\mathbf{A}\vec{x} &= \mathbf{A}^{-1}\vec{b} \\ \mathbf{I}\vec{x} &= \mathbf{A}^{-1}\vec{b} \\ \vec{x} &= \mathbf{A}^{-1}\vec{b} \end{aligned}$$

Homogeneous Systems.
A system of equations is called ***homogeneous*** if $\vec{x} = \vec{0}$ is a solution, which means that $\vec{b} = 0$:

$$\mathbf{A}\vec{x} = \vec{0}.$$

Otherwise, it is called ***nonhomogeneous***.

Eigenvalues and Eigenvectors

We can think of the matrix multiplication $\vec{y} = \mathbf{A}\vec{x}$ as a mapping or transformation: given a vector $\vec{x}$ it transforms it into a different vector $\vec{y}$.

In many applications, it is important to know which vectors $\vec{x}$ are transformed into multiples of themselves.

These vectors satisfy the property

$$\mathbf{A}\vec{x} = \lambda\vec{x} \qquad \Leftrightarrow \qquad (\mathbf{A} - \lambda\mathbf{I})\vec{x} = \vec{0}.$$

One such vector is $\vec{x} = \vec{0}$. But that's not very interesting. We want to look for nonzero vectors that satisfy this property.

These vectors are called **eigenvectors** and the corresponding λ is called an **eigenvalue**.

Important. The second formulation above implies that the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular, otherwise the unique solution would be $\vec{x} = \vec{0}$. So that implies that

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \qquad \textbf{(characteristic equation)}$$

Example. Let us find the eigenvalues and eigenvectors for

$$\mathbf{A} = \begin{pmatrix} 1 & 8 \\ 4 & 5 \end{pmatrix}.$$

First we solve the characteristic equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 8 \\ 4 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - 32 = 0 \qquad \Leftrightarrow \qquad \lambda^2 - 6\lambda - 27 = 0$$

which implies that

$$\lambda = 3 \pm \sqrt{9+27} = 3 \pm 6$$

Eigenvalue $\lambda_1 = -3$. To find the eigenvector, we write its equation

$$(\mathbf{A} - \lambda_1\mathbf{I})\vec{x} = \vec{0} \qquad \Leftrightarrow \qquad \begin{pmatrix} 4 & 8 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that

$$4x_1 + 8x_2 = 0 \qquad \Leftrightarrow \qquad x_1 = -2x_2.$$

So one eigenvector for this eigenvalue is

$$\vec{x}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Eigenvalue $\lambda_2 = 9$. To find the eigenvector, we write its equation

$$\begin{pmatrix} -8 & 8 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies that

$$-8x_1 + 8x_2 = 0 \qquad \Longleftrightarrow \qquad x_1 = x_2.$$

So one eigenvector for this eigenvalue is

$$\vec{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Theorem. Let $\mathbf{A}$ have real or complex eigenvalues λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$ and let the corresponding eigenvectors be

$$\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \qquad \text{and} \qquad \vec{x}_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.$$

If $\mathbf{X}$ is the matrix with columns taken from the eigenvectors:

$$\mathbf{X} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},$$

then

$$\det(\mathbf{X}) \neq 0.$$

6.2 Mathematical Induction Review

Mathematical induction is a very powerful tool for proving results. It allows us to prove generalized results. Its shortcoming is that you already have to suspect what the solution is. It then allows you to prove it.

The Principle of Mathematical Induction.
Assume that $\mathbf{P}(1), \mathbf{P}(2), \mathbf{P}(3), \ldots$ is an infinite sequence of mathematical statements.

If	**If**
(a) $\mathbf{P}(1)$ is true,	first domino falls
and	**and**
(b) for any k, $\mathbf{P}(k)$ implies $\mathbf{P}(k+1)$	if a domino falls, then the next one falls
then	**then**
all the statements in the sequence are true	all dominoes fall!

Example. For any $n \in \mathbb{N}$, $1+3+5+\cdots+(2n-1) = n^2$.

Let us prove this formula using Mathematical Induction.

Proof. If $n=1$, we get $1 = 1^2$, which is true – $\mathbf{P}(1)$ holds.
Assume that $1+3+5+\cdots+(2k-1) = k^2$ for some k. Then

$$\begin{aligned} 1+3+5+\cdots+\big(2(k+1)-1\big) &= \underbrace{1+3+5+\cdots+(2k-1)}_{=k^2 \text{ by induction hypothesis}}+(2k+1) \\ &= k^2+(2k+1) \\ &= k^2+2k+1 \\ &= (k+1)^2. \end{aligned}$$

By induction, the equality holds for all $n \in \mathbb{N}$.

Example.
For any $x \in \mathbb{R}$ with $x \geq -1$ and $n \in \mathbb{N}$, then $(1+x)^n \geq 1+nx$. (Bernoulli's Inequality)

Proof. For $n=1$, $1+x \geq 1+x$, which is true!
Assume that $(1+x)^k \geq 1+kx$. Then

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x) \geq (1+kx)(1+x) \\ &= 1+x+kx^2+kx = 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x. \end{aligned}$$

By induction, the claim holds for all $n \in \mathbb{N}$.

In the proof, we didn't use the fact that $x \geq -1$, but for $n=3$ and $x=-4$, $(1+x)^n = -27 \ngeq -11 = 1+nx$.

Where did we use the hypothesis $x \geq -1$?

Principle of Strong Mathematical Induction.
Assume that $\mathbf{P}(1)$, $\mathbf{P}(2)$, $\mathbf{P}(3), \ldots$ is an infinite sequence of mathematical statements.

If	**If**
(a) $\mathbf{P}(1)$ is true,	first domino falls
and	**and**
(b) For any k, $\mathbf{P}(1), \ldots, \mathbf{P}(k)$ implies $\mathbf{P}(k+1)$	if all previous dominos fell, then the next one falls
then	**then**
all the statements in the sequence are true	all dominoes fall!

Example.

Theorem. Every number $n \in \mathbb{N}, n \geq 2$ can be written as a product of primes (or is a prime).

Proof. Base case: $n = 2$ is a prime.
Assume that the Theorem holds for $n = 2, 3, 4, \ldots, k$ and consider $n = k + 1$.
If $k + 1$ is prime, the claim holds.
If $k + 1$ is not a prime, then it is divisible by some $2 \leq m \leq k$: $k + 1 = m \cdot \ell$ for some $2 \leq m, \ell \leq k$. By hypothesis, both m and ℓ are products of primes, hence so is $k + 1$. The Theorem follows by strong induction.

Practice Problems

1 Is the triangle inequality true for more than two numbers?

$$|x_1 + x_2 + \cdots + x_n| \overset{?}{\leq} |x_1| + |x_2| + \cdots + |x_n|$$

If it is, prove it.

2 Is the AGM inequality true for any $x_1, x_2, \ldots, x_n \geq 0$?

$$\sqrt{x_1 x_2 \cdots x_n} \overset{?}{\leq} \frac{x_1 + x_2 + \cdots + x_n}{n}$$

If it is, prove it.

3 Prove that for any $n \in \mathbb{N}$, $2^{6n} + 3^{2n-2}$ is divisible by 5.

4 How many subsets does a set S with n elements have (including S and $\emptyset$)?

5 Show that If $x_1, \ldots, x_n \in [0, 1]$ then $\displaystyle\prod_{i=1}^{n}(1 - x_i) \geq 1 - \sum_{i=1}^{n} x_i$

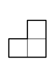
6 Can you use Mathematical induction to prove that $P(m), P(m+1), P(m+2), \ldots$ are true? If so, how?

7 Can you use Mathematical induction to prove that $P(2), P(4), P(6), P(8), \ldots$ are true? If so, how?

8 Can you use Mathematical induction to prove that $P(1), P(3), P(5), P(7), \ldots$ are true? If so, how?

9 For which $n \in \mathbb{N}$, $2^n \geq (n+1)^2$? Prove your answer.

10 Show that for even n's, $n(n^2 + 3n + 2)$ is divisible by 24.



11 Prove that for any $n \in \mathbb{N}$, a $2^n \times 2^n$ checkerboard with one single square removed has an L–tiling (i.e., can be covered with L–shapes).

12 Read the following proof:

Theorem. All horses have the same colour.

Proof. Assume that the claim holds for groups of k horses, and consider a group with $k + 1$ horses $S = \{h_1, \ldots, h_{k+1}\}$.

By hypothesis, the horses in $A = \{h_1, \ldots, h_k\}$ and $B = \{h_2, \ldots, h_{k+1}\}$ have the same colour. Since the horse h_2 is in both groups, we deduce that the colour of the horses in A must be the same as of these in B.

In conclusion, the horses in $S = A \cup B$ must have the same colour, and the claim holds by induction.

This proof is flawed. Explain how.

6.3 2019 M_3C competition report from the winning team

In the following pages you can find an abridged version of the full report. The full report can be found at https://uoft.me/modelling-app-report.

MathWorks Math Modeling Challenge 2019

High Technology High School–

Team # 12038 Lincroft, New Jersey

Coach: Raymond Eng

Students: Eric Chai, Gustav Hansen, Emily Jiang, Kyle Lui, Jason Yan

MathWorks Math Modeling Challenge Champions

$20,000 Team Prize

MathWorks Math Modeling Challenge

***Note: This cover sheet has been added by SIAM to identify the winning team after judging was completed. Any identifying information other than team # on a MathWorks Math Modeling Challenge submission is a rules violation.

***Note: This paper underwent a light edit by SIAM staff prior to posting.

Substance Use and Abuse

Executive Summary

In recent years, substance abuse has intensified to an alarming degree in the United States. In particular, the rise of vaping, a new form of nicotine consumption, is dangerously exposing drug abuse to a new generation. With the need to understand how substance use spreads and impacts individuals differently, our team seeks to provide a report with mathematically founded insights on this prevalent issue.

We first strove to predict the spread of nicotine use due to both vaping and cigarettes over the next decade. By comparing the spread of nicotine use to an infectious disease, we modified the SIRS epidemiology model to create our adapted SIRI model in which individuals are divided into four compartments: infected (drug users), recovered (users who quit drugs), susceptible (potential drug users), and nonsusceptible (those who will never use drugs). People progress from susceptible to infected to recovered, but may relapse into their old habits, causing them to re-enter the infected population. Birth and death rates of our designated population were modeled with linear equations. We solved a system of differential equations to determine e-cigarette and cigarette use in 2029: 26.63% of the American population will vape and 6.45% will smoke cigarettes. These results align with the expectation that vaping will increase in popularity while cigarette smoking will decline.

Substance abuse is associated with numerous social factors and personal attributes. We incorporated those determinants to create a second mathematical model that computes the probability that an individual will use nicotine, marijuana, alcohol, and unprescribed opioids. A binary multivariate logistic model was used to assess the effects of age, gender, ethnicity, income, parental status, friendship, opinion about school, overall health, weapon possession, and bullying on substance use. To demonstrate our model, we coded and executed a Monte Carlo simulation that created 300 high school seniors with varying attributes. We found that 46.3% of the students would use nicotine, 17.3% would use marijuana, 66.0% would use alcohol, and 0.0% would use opiates.

Substance use has far-reaching implications in personal and societal spheres. It is crucial to rank substances based on their overall impact in order to assess necessary government action regarding drug abuse. To address this issue, we developed a robust metric to rank the effects of nicotine, marijuana, alcohol, and opioid abuse. Our model and ranking considers physical harm, dependence, social harm, and economic impact of the drugs. The former three factors were measured on a scale of 0 to 3 based on psychiatrist surveys. Then economic impact was defined as GDP loss from the decrease in life expectancy caused by drug abuse. After applying risk factors obtained from the amount of people that use each drug, the four substances were ranked. From highest to lowest individual impact, the ranking was opioids, alcohol, cigarettes, and marijuana. From highest to lowest total societal impact, the ranking was alcohol, cigarette, marijuana, and opioids.

The repercussions of substance abuse are reverberating and remain with an individual for life. However, drugs not only severely affect the user but also cause extensive societal harm. Increased understanding of the projected spread and impact of substance abuse, as well as the underlying factors that lead to poor judgment, are needed to optimize measures to restrict consumption. Ultimately, we believe that our models provide novel insight into the nationwide issue of substance use and abuse.

1 Introduction

This section delineates the components of the modeling problem and their objectives. Global assumptions applying to the entire modeling process are also listed.

1.1 Restatement of the Problem

The problem we are tasked with addressing is as follows:

1. Build a mathematical model that predicts the spread of nicotine use due to vaping over the next 10 years. Analyze how this growth compares to that of cigarettes.

2. Create a model that simulates the likelihood that a given individual will use a given substance, accounting for social influence, characteristic traits, and properties of the drug itself. Demonstrate the model by predicting how many students among a class of 300 high school seniors with varying characteristics will use nicotine, marijuana, alcohol, and unprescribed opioids.

3. Develop a metric for the impact of substance use, considering both financial and nonfinancial factors. Use the metric to rank the substances listed in Part II.

1.2 Global Assumptions

1. *The current drug scene remains constant.* We assume that there will be no radical changes in the recreational drug industry, such as new drugs or drug products. This assumption is imperative because attempting to account for unpredictable and volatile factors would make model development virtually impossible.

2. *All vapes count as e-cigarettes.* Some people distinguish between e-cigarettes and vaping. For the purposes of this model, e-cigarettes and vapes will be considered synonymous.

3. *People respond honestly to surveys.* Our model is dependent on survey results to calculate weight constants. Because we have no way of determining the accuracy of the survey responses, we will assume that they are accurate and without bias for simplicity.

2 Part 1: Darth Vapor

First commercialized in 2003, electronic cigarettes have become an increasingly popular product among youth [1]. Although they are advertised as safer alternatives to traditional cigarettes, e-cigarettes contain high doses of nicotine and have introduced a new generation to tobacco products. This section outlines a mathematical model for predicting the change in nicotine use in the United States due to vaping compared to the change due to cigarettes.

2.1 Assumptions

1. *Nicotine use can be modeled as an infectious disease.* Like an epidemic, nicotine use is prevalent and contagious, reflected in the surge in popularity of smoking due to peer pressure, advertisements, and social media. Additionally, the U.S. Surgeon General declared youth vaping a nationwide epidemic in 2018 [2].

2. *Individuals can smoke from age 11 until death.* Peak years for first trying nicotine products is 6th or 7th grade [3].

3. *Rate of entry into pre-adolescence in the U.S. is 0.00103.* [4] Our model defines "birth" as reaching an age at which substance use becomes possible—around 11 years. Thus, we assumed the current birth rate to be constant for the past 11 years, assuming no children die before they turn 11. The current birth rate is 1.03 people/month/person.

4. *Death rate in the U.S. is constant and equal to 0.0007 people per month per person.*[4] Our model assumes that individuals have the capacity to use drugs until their death.

5. *Individuals can only start smoking due to influence from other smokers.* To model substance use as an infectious disease, we must assume that susceptible individuals can become infected only from contact with the already infected. This assumption is valid because peer influence and social media presence are the driving factors behind the popularity of smoking [5].

6. *Individuals are either not susceptible to, susceptible to, infected by, or recovered from substance abuse.* As in the SIR epidemiology model, we assume that people are either unwilling to smoke (not susceptible), open to smoking (susceptible), regular smokers (infected), or past smokers who have quit (recovered).

7. *The infection rate is constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the infection rate will also not drastically change.

8. *The percentage of susceptible people will stay constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the number of people susceptible to it will also not drastically change.

9. *Nobody starts as recovered.* At the start of the model, we do not consider any individuals to be former smokers who have quit.

10. *The recovery and relapse constant for cigarette and e-cigarette users are the same.* The two contain similar amounts of nicotine, which acts as the addictive agent. Thus, the recovery and relapse constants are assumed to be the same.

2.2 Model Development

The surge in popularity of conventional cigarettes in the mid-20th century, as well as the current boom of vaping among American youth, is comparable to the spread of an infectious disease during an epidemic. As stated in assumption 1, we model nicotine use as a disease because it rapidly spreads as a result of interpersonal communications (in-person peer pressure to try a drug as well as social media prevalence); additionally, substance use is a condition from which individuals can recover (by quitting smoking).

Our model is a derivation of the SIRS epidemiological model, a technique used to map the spread of infectious diseases such as influenza. We also consider birth and death rate, since population naturally changes over time. The model separates individuals in a population into four categories: NS for Not Susceptible, S for Susceptible, I for Infected, and R for Recovered. At the start of the model, individuals are either in NS, S, or I, since nobody starts off as recovered. While those in NS remain there permanently, individuals in S can move to I, who can then move to R.

The additional S in SIRS represents the possibility of returning to the Susceptible compartment—in this case, a regular user quitting but relapsing. However, we modified the classic SIRS model by recognizing that a relapsing individual would re-enter the Infected category rather than Susceptible, since they will once again become smokers rather than people merely open to smoking. Thus, we renamed the traditional epidemiology model as SIRI to represent this adjustment. Figure 2.2.1 diagrams the aforementioned movement of individuals between categories, while Table 2.2.1 defines and details values for variables and constants used in the SIRI model for both e-cigarette and cigarette smoking.

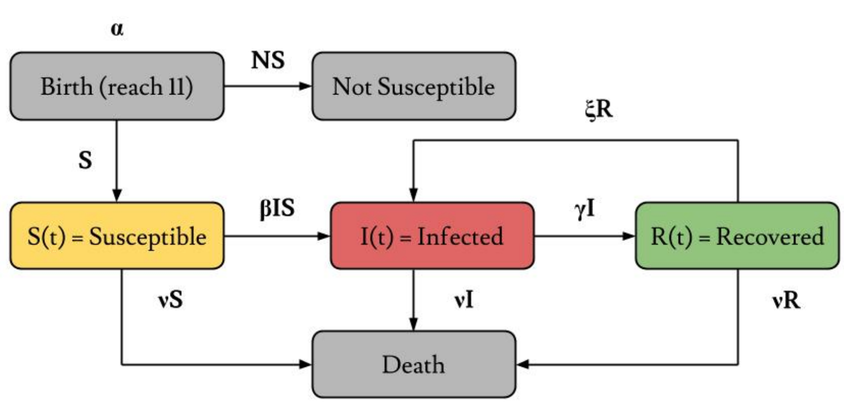


Figure 2.2.1: Diagram of the SIRI Model for Spread of Nicotine Use

2.2.1 Parameters in SIRI Model

Proportion of infected people (I_0). The total number of people that currently vape is approximately 10.8 million [6]. Dividing by the total population of America, 325.7 million

[7], results in an I_0 value of 0.0332 for e-cigarettes. The total number of people that currently smoke cigarettes is approximately 34.3 million [8], resulting in an I_0 value of 0.1053.

Proportion of recovered people (R_0). As per assumption 9, without loss of generality, R_0 was assumed to be 0 at time = 0.

Proportion of susceptible people (S_0). Because I, R, and S are proportions of the total population, their sums must add to 1. Thus, $S_0 = 1 - R - I$, resulting in 0.9667 for e-cigarettes and 0.8947 for cigarettes.

Susceptibility (S). A 2016 Surgeon General report stated that 32% of people are considered susceptible to e-cigarette use [5], while a 2012 report stated that 20% of people are susceptible to cigarettes, which correspond to the S values [9].

Infection constant (β). This was determined based on responses to the survey question "If one of your best friends offered you a cigarette, would you smoke it?" For e-cigarettes, the chance of infection was taken from a 2016 U.S. Surgeon General report that indicated that 18% of young adults responded "yes" to the question [5]. For cigarettes, we obtained β by adding the percentages of the responses "Definitely Yes" and "Probably Yes," from the 2014 National Survey on Drug Use and Health, to get 0.3%, which represented the infection constant [10].

Recovery constant (γ). In a given year, around 40% of smokers attempt to quit [11]. Therefore, in a month, $1.40^{1/12} = 1.0284$ recover, so the recovery rate is 0.0284.

Relapse constant (ξ). In a given year, approximately 6% of attempts to quit smoking succeed and 94% of attempts failed and the person relapsed [12]. Therefore, in a month, $1.94^{1/12} = 1.0568$ fail, so the relapse constant is 0.0568.

Infection rate (y_{inf}). In accordance with assumption 4, we assume that people will only start smoking if they are influenced by a current smoker. In other words, a susceptible person can only become infected if they come into contact with an infected person, which occurs at a rate proportional to $I \cdot S$. The infection constant β represents the likelihood that a susceptible person becomes infected when influenced by a smoker. Thus, infection rate is as follows:

$$y_{inf} = \beta \cdot I \cdot S \tag{1}$$

Recovery rate (y_{rec}). Unlike infection rate, the recovery rate is dependent only on the average probability of an individual quitting. The recovery constant γ multiplied by the proportion of people that currently are infected gives the recovery rate:

$$y_{rec} = \gamma \cdot I \tag{2}$$

Relapse rate (y_{rel}). The relapse rate is dependant only on the average probability of an individual relapsing. The relapse constant is much higher than the infection rate, which is logical because an individual who was previously a regular smoker will be more likely to succumb to the addictive cycle again [12]. Designating ξ as the relapse constant, relapse rate is given by

$$y_{rel} = \xi \cdot R \tag{3}$$

Birth rate (α). The birth rate, as defined by assumption 3, is 1.03 people/month/person.

Death rate (μ). From assumption 4, the death rate is assumed to be constant and equal to 0.0007 people per month per person. Therefore, the number of people dead for each category will be the death rate multiplied by the proportion of the people in each category.

$$\mu_S = v \cdot S \tag{4}$$

$$\mu_I = v \cdot I \tag{5}$$

$$\mu_R = v \cdot R \tag{6}$$

Table 2.2.1 Variables and Constants of SIRI Model for E-Cigarettes and Cigarettes

Variable	Definition	E-Cigarette Values	Cigarette Values
I	Proportion of infected people	$I_0 = 0.0332$	$I_0 = 0.1053$
R	Proportion of recovered people	$R_0 = 0$	$R_0 = 0$
S	Proportion of susceptible people	$S_0 = 0.9667$	$S_0 = 0.8947$
N	Proportion of total individuals in SIR cycle	$N_0 = 0.32$	$N_0 = 0.20$
α	Birth rate	0.00103	0.00103
β	Infection constant	0.18	0.003
γ	Recovery constant	0.0284	0.0284
ξ	Relapse constant	0.0568	0.0568
μ	Death rate	0.0007	0.0007

2.2.2 Differential Equations for SIRI Model

The change in each of the dependent variables S, I, and R is equal to the sum of the input of the respective category minus the sum of its output, as diagrammed by the arrows entering and leaving each box in Figure 2.2.1. Thus, our SIRI model is summarized by the set of ordinary differential equations below:

$$\frac{dS}{dt} = \alpha - \beta \cdot I \cdot S - \mu \cdot S \tag{7}$$

$$\frac{dI}{dt} = B \cdot I \cdot S - \gamma \cdot I + \xi \cdot R - \mu \cdot I \tag{8}$$

$$\frac{dR}{dt} = \gamma \cdot I - \xi \cdot R - \mu \cdot R \tag{9}$$

2.3 Results

With the SIRI model established, we utilized it to predict the change in nicotine use due to e-cigarettes and cigarettes in the next decade. We coded and executed a Python program to solve the system of differential equations, with appropriate constants for each product, and graph the proportion of compartments over time. Figures 2.3.1 and 2.3.2 graph the proportion of the total population falling under each of the SIR categories for both tobacco products, respectively, over a 10-year time period. Table 2.3.1 enumerates the proportion of the population that is susceptible, infected, and recovered for vaping and cigarettes in 2029.

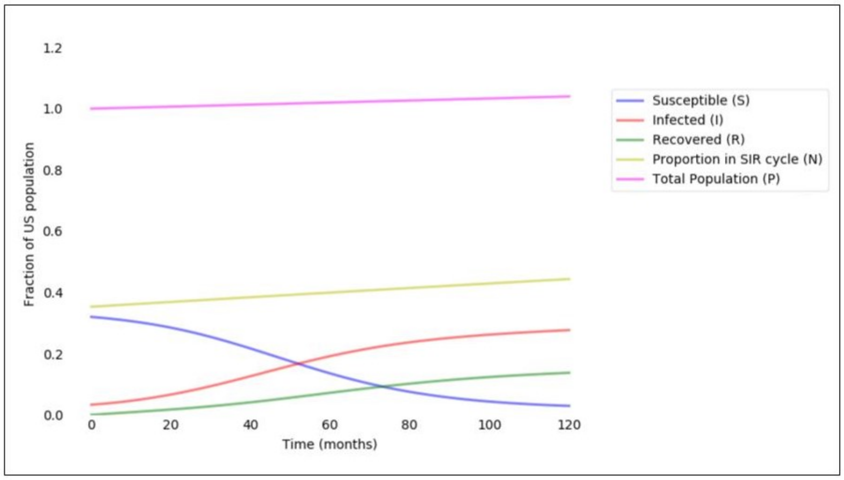


Figure 2.3.1: Graph of SIRI Compartments for E-Cigarettes over Ten Years

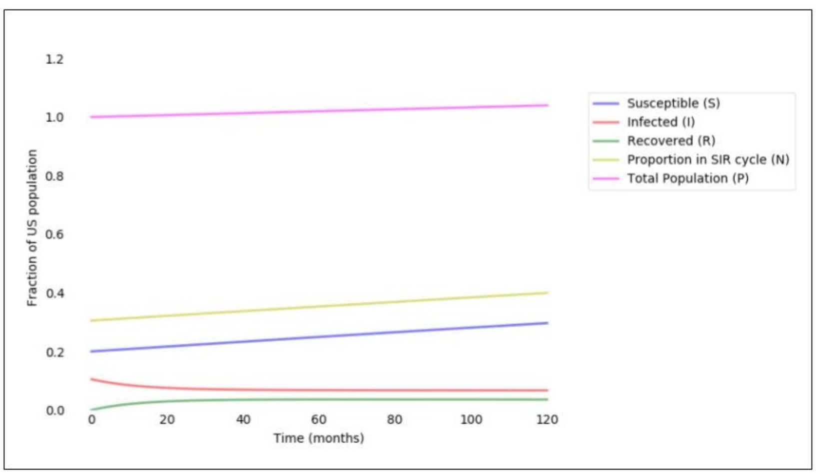


Figure 2.3.2: Graph of SIRI Compartments for Cigarettes over Ten Years

Table 2.3.1 SIR Distribution of 2029 Population for E-Cigarettes and Cigarettes

	Susceptible	Infected	Recovered
E-Cigarettes	2.82%	26.63%	13.21%
Cigarettes	28.53%	6.45%	3.45%

Our model concludes that in 2029, 26.63% of the population will use e-cigarettes, while 6.45% will use cigarettes. This disparity is consistent with previously researched trends, which suggest that as e-cigarettes gain popularity amongst teens, regular cigarettes decrease in popularity [13].

2.4 Sensitivity Analysis

Table 2.4.1 shows the sensitivity analysis for our SIRI model based on an independent increase and decrease of 10% of the infection constant β, recovery constant γ, and relapse constant ξ.

Table 2.4.1 Sensitivity Analysis for Part I

Constant	% Change in Constant	% Change in Vaping (I)	% Change in Cigarette Use (I)
β	10%	1.014%	0.6202%
β	−10%	−1.615%	−0.6202%
γ	10%	−3.492%	−3.566%
γ	−10%	4.018%	3.721%
ξ	10%	3.098%	3.367%
ξ	−10%	−3.496%	−3.905%

Positive changes in the infection or relapse constants resulted in positive changes in the percentage of infected people for both vaping and cigarette use. This is consistent with our predictions because the rate of infection for susceptible and recovered people is increasing. In contrast, a positive change in recovery constant resulted in a decrease in percent infected because the rate at which people are leaving the infected population is increasing.

2.5 Strengths and Weaknesses

Our model is resilient to small changes and outputs sensible results. As demonstrated in the sensitivity analysis, a 10% change in each of the infection, recovery, and relapse constants accounts for less than 5% change in final vaping and cigarette use after a decade. Changes in the model's output due to shifts are consistent with expected trends as well. SIRS is also an established mathematical modeling technique that we adapted to fit our own aims, lending credence to the validity of our model. Additionally, our model is comprehensive, accounting for many contributing factors such as population change, nonsusceptible individuals, and the possibility of relapse for smokers who have attempted to quit.

The model's weaknesses lie in its inability to account for the introduction of new forms of drugs or rapid changes in popularity of existing forms, as stated in global assumption 1. Specifically, a surge in use of a particular drug would likely impact vaping and cigarette use in unforeseen ways that our model will not accurately predict. Furthermore, our model does not consider the association between vaping and cigarette use, and how the growth or decline of one product would influence the other. This is unrealistic because the popularity of e-cigarettes among youth has led many to smoke traditional cigarettes and prompted cigarette smokers to transition to vaping [13]; however, the opposite effects of these two phenomena can reasonably counterbalance each other.

5 Conclusion

5.1 Further Studies

Our first model does not currently account for the introduction of new drugs in the industry, which would greatly impact the change in usage for pre-existing substances. Taking these market changes into account would greatly strengthen our model. The second model used survey data from 2005–2006. The resulting model fits well for this time period, but requires more recent data to reflect recent trends. Applying the same modeling approach for 2019 would create a more accurate model that is applicable to today. Finally, the third model is heavily based on the personal opinions of psychiatrists. Recreating the model to account for each factor with independent methods would greatly complicate the model, but make it more flexible for incorporating newer drugs into our ranking.

5.2 Summary

The first model focuses on comparing the percent of e-cigarette users versus cigarette users in the next ten years. The SIRS epidemic model was used as the basis for ours. People were split into four main categories: infected (those that used drugs), recovered (those that quit using drugs), susceptible (those that may use drugs in the future), and non-susceptible (those that will never use drugs). Birth rate and death rate were both modeled with linear equations. Simultaneous differential equations were solved to determine the number of "infected" people in 2029. According to our model, 26.63% of the American population will vape in 2029 and 6.45% will smoke cigarettes. The results correspond with observed increasing popularity of e-cigarettes and decreasing popularity of regular cigarettes.

The second model determines the probability of a student using nicotine, marijuana, alcohol, and opioids and applies itself to a randomly generated sample of 300 high school seniors. A binary multivariate logistic regression was used to create the model based on an HBSC survey. A machine learning algorithm using an L2 regression was used to calculate the weights and bias in our logistic model. Using a Monte Carlo simulation, 300 random seniors were created based on response frequencies to each of questions necessary for our model. Running this sample of high school seniors through our model, we found 46.33% would use nicotine, 17.33% would use marijuana, 66.00% would use alcohol, and 0.00% would use opiates.

The third and final model focuses on ranking nicotine, marijuana, alcohol, and opioids based on their financial and nonfinancial effects. Factors were analyzed in four main categories: physical harm, dependence, social harm, and economic impact. These factors were further split into 2–3 subcategories each that were each assigned scores on a scale from 0.0 to 3.0 based on expert surveys. To calculate the impact of drugs on GDP, the average annual GDP per person was multiplied by the average decrease in life as a result of using drugs. The impact of drugs on GDP was then rescaled from 0.0 to 3.0 to make them comparable to the other factors. Each of the four main categories was averaged for a total harm score for each of the four drugs. The total harm score was multiplied by a risk factor based on the number of people that used each drug to obtain a final score for each drug that could be used for ranking purposes. This model showed that opioids had the greatest substance harm per person, but since relatively few people use opiods, it had a lower total detriment score. Marijuana had the lowest substance harm per person and the second lowest total impact. Alcohol had the highest total impact, while cigarettes had the second highest because of the great number of people using these substances.

6 References

[1] - Historical Timeline of Electronic Cigarettes. (2018, October 18). Retrieved March 3, 2019, from http://www.casaa.org/historical-timeline-of-electronic-cigarettes/

[2] - Stein, R. (2018, December 18). Surgeon General Warns Youth Vaping Is Now An 'Epidemic'. Retrieved March 3, 2019, from https://www.npr.org/sections/health-shots/2018/12/18/677755266/surgeon-general-warns-youth-vaping-is-now-an-epidemic

[3] - Bach, L. (2018). The Path to Addiction Starts Early. Retrieved March 3, 2019, from https://www.tobaccofreekids.org/assets/factsheets/0127.pdf

[4] - Birth rate, crude (per 1,000 people). (2019). Retrieved March 3, 2019, from https://data.worldbank.org/indicator/SP.DYN.CBRT.IN?locations=US&view=chart

[5] - United States, U.S. Department of Health and Human Services, Office on Smoking and Health. (2016). Surgeon General. Retrieved March 3, 2019, from https://e-cigarettes.surgeongeneral.gov/documents/2016_SGR_Full_Report_non-508.pdf

[6] - Mirbolouk, M., Charkhchi, P., Kianoush, S., Uddin, S. I., Orimoloye, O. A., Jaber, R., . . . Blaha, M. J. (2018). Prevalence and Distribution of E-Cigarette Use Among U.S. Adults: Behavioral Risk Factor Surveillance System, 2016. Annals of Internal Medicine, 169(7), 429. doi:10.7326/m17-3440

[7] - U.S. Population (LIVE). (n.d.). Retrieved March 3, 2019, from http://www.worldometers.info/world-population/us-population/

[8] - Current Cigarette Smoking Among Adults in the United States — CDC. (n.d.). Retrieved March 3, 2019, from https://www.cdc.gov/tobacco/data_statistics/fact_sheets/adult_data/cig_smoking/index.htm

[9] - United States., Public Health Service., Office of the Surgeon General. (2012). A report of the Surgeon General: Preventing tobacco use among youth and young adults. Washington, D.C.?: Centers for Disease Control and Prevention (U.S.), Office on Smoking and Health.

[10] - Substance Abuse and Mental Health Services Administration. (2014). 2014 National Survey On Drug Use And Health [Public Use File Codebook]. Retrieved from https://samhda.s3-us-gov-west-1.amazonaws.com/s3fs-public/field-uploads-protected/studies/NSDUH-2014/NSDUH-2014-datasets/NSDUH-2014-DS0001/NSDUH-2014-DS0001-info/NSDUH-2014-DS0001-info-codebook.pdf

[11] - Borland, R., Partos, T. R., Yong, H., Cummings, K. M., & Hyland, A. (2012). How much unsuccessful quitting activity is going on among adult smokers? Data from the International Tobacco Control Four Country cohort survey. Addiction, 107(3), 673-682. doi:10.1111/j.1360-0443.2011.03685.x

[12] - Malarcher, A., & Dube, S. (2011). Quitting Smoking Among Adults — United States, 2001–2010 (United States, Centers for Disease Control and Prevention).

[13] - National Institute on Drug Abuse. (2016, February 11). Teens and E-cigarettes. Re-

trieved March 3, 2019, from https://www.drugabuse.gov/related-topics/trends-statistics/infographics/teens-e-cigarettes

[14] - Martinez, E., Kaplan, C. P., Guil, V., Gregorich, S. E., Mejia, R., & Pérez-Stable, E. J. (2006). Smoking behavior and demographic risk factors in Argentina: A population-based survey. Prevention and Control, 2(4), 187-197. doi:10.1016/j.precon.2007.07.002

[15] - Year / Grade Placement. (n.d.). Retrieved March 3, 2019, from https://www.acs-schoo com/acs-hillingdon-class-ages

[16] - United States Department of Health and Human Services, Substance Abuse & Mental Health Data Archive. (2005-2006). Health Behavior in School-Aged Children. Retrieved from https://www.icpsr.umich.edu/icpsrweb/

[17] - Monitoring the Future National Survey Results on Drug Use, 1975-2005: Secondary School Students 2005. (2005). doi:10.3998/2027.42/142406

[18] - Golub, A., Johnson, B. D., & Dunlap, E. (2006). The Growth in Marijuana Use Among American Youths During the 1990s and the Extent of Blunt Smoking. Journal of Ethnicity in Substance Abuse, 4(3-4), 1-21. doi:10.1300/j233v04n03_01

[19] - Felter, C. (2019, January 17). The U.S. Opioid Epidemic. Retrieved March 3, 2019, from https://www.cfr.org/backgrounder/us-opioid-epidemic

[20] - ACMT FAQ Nicotine. (n.d.). Retrieved March 3, 2019, fromhttps://www.acmt.net/_Library/Public_Affairs/ACMT_FAQ_Nicotine_.pdf

[21] - National Institute on Drug Abuse. (2014). Principles of Adolescent Substance Use Disorder Treatment: A Research-Based Guide. Retrieved March 3, 2019, from https://www.drugabuse.gov/publications/principles-adolescent-substance-use-disorder-treatment- research-based-guide/introduction

[22] - Nutt, D., King, L. A., Saulsbury, W., & Blakemore, C. (2007). Development of a rational scale to assess the harm of drugs of potential misuse. The Lancet, 369(9566), 1047-1053. doi:10.1016/s0140-6736(07)60464-4

[23] - COUNTRY COMPARISON :: GDP - PER CAPITA (PPP). (n.d.). Retrieved March 3, 2019, from https://www.cia.gov/library/publications/the-world-factbook/rankorder/2004rank.html

[24] - Substance Abuse and Mental Health Services Administration. (2018). Results From the 2017 National Survey on Drug Use and Health: Detailed Tables. Retrieved from https://www.samhsa.gov/data/sites/default/files/cbhsq-reports/NSDUHDetailedTabs2017/NSDUHDetailedTabs2017.pdf