2018-09-24 First-Order Linear ODEs (2.2)

2.2 First Order ODEs - More on Integrating Factor

1. Calculate $(sin(x)f(x))'$

$(sin(x)f(x))' = sin(x)f'(x) + cos(x)f(x)$ (product rule)

2. Find the general solution of $sin(x)y' + cos(x)y = \sqrt{x}$

To use Integrating Factor, the equation's format must be $y' + \bigcirc y = \bigcirc$.

However, notice that the LHS of the equation looks like a product rule (similar to question 1), so calculating the Integrating Factor is unnecessary.

From Q1, using the product rule,

$(sin(x)y)' = x^{\frac{1}{2}}$

$sin(x)y = \int x^{\frac{1}{2}}\, dx$

$y = \frac{\frac{2}{3}x^{\frac{3}{2}} + C}{sin(x)}$

3. What is the integrating factor for $y' + \frac{cos(x)}{sin(x)}y = \frac{\sqrt{x}}{sin(x)}$?

Integrating factor:

$\mu(x) = e^{\int \frac{cos(x)}{sin(x)} dx} = e^{ln|sin(x)|} = |sin(x)|$

3.1. What happens when you multiply the ODE by the integrating factor?

$\mu(x) = |sin(x)|$

if x belongs to $[k\pi, (k+1)\pi]$ for $k = 0, 1, 2...$, then the differential equation is $sin(x)y' + cos(x)y = \sqrt{x}$ (the same as the equation in Q2)

Note: When getting rid of the absolute value sign, it does not matter whether sin(x) or -sin(x) is obtained. This is because the whole equation is multiplied by the integrating factor; if negative sin(x) was used instead of positive sin(x), it would be "canceled out" on both sides.

Try some exercises at home.

Attempted Solution:

4. Find the solution of $\begin{cases} y' - \frac{y}{2(x+4)} = \frac{1}{2(x+4)} \\ y(0) = -5 \end{cases}$

$$\int \left(\frac{1}{\sqrt{|x+4|}}y\right)' = \int \frac{1}{2(|x+4|)^{\frac{3}{2}}}$$

$$\frac{1}{\sqrt{|x+4|}}y = -\frac{1}{\sqrt{|x+4|}} + C$$

$$y = \frac{-\frac{1}{\sqrt{|x+4|}}+C}{\frac{1}{\sqrt{|x+4|}}}$$

$$General\ Solution : y = -1 + C\sqrt{|x+4|}$$

$$Substitute\ in\ the\ initial\ condition :$$
$$y(0) = -5 = -1 + C\sqrt{0+4}$$
$$-4 = 2C$$
$$C = -2$$
$$Solution : y = -1 - 2\sqrt{|x+4|}$$

4.1 What is the integrating factor?

$$p(x) = e^{-\frac{1}{2}\int \frac{1}{x+4}dx} = e^{-\frac{1}{2}ln|x+4|} = (e^{ln|x+4|})^{-\frac{1}{2}} = \frac{1}{\sqrt{|x+4|}}$$

4.2 What is the domain of the solution?

$\{x \,\epsilon\, \mathbb{R},\ x\ \neq -4\}$ **Incorrect. Why?**

5. Find the solution of $\begin{cases} y' - \frac{y}{2(x+4)} = \frac{1}{2(x+4)} \\ y(-5) = -5 \end{cases}$

$$General\ Solution : y = -1 + C\sqrt{|x+4|}$$
$$Substitute\ in\ the\ initial\ condition :$$
$$y(-5) = -5 = -1 + C\sqrt{|-5+4|}$$
$$C = -4$$
$$Solution : y = -1 - 4\sqrt{2|x+4|}$$

6. Find the general solution of $2ln(x)e^{2y}y' + \frac{e^{2y}}{x} = 4x^3$

$f(x)\ is\ ln(x)\ and\ g(x)\ is\ e^{2y}$

$\left(\ln(x)e^{2y}\right)' = 2\ln(x)e^{2y}y' + \frac{1}{x}e^{2y}$

$\int \left(\ln(x)e^{2y}\right)'dx = \int 4x^3dx$

$ln(x)e^{2y} = x^4 + C$

$e^{2y} = \frac{x^4+C}{ln(x)}$

$General\ Solution: y = \frac{1}{2}ln\left(\frac{x^4+C}{ln(x)}\right)$

2.3 Linear vs Nonlinear ODEs

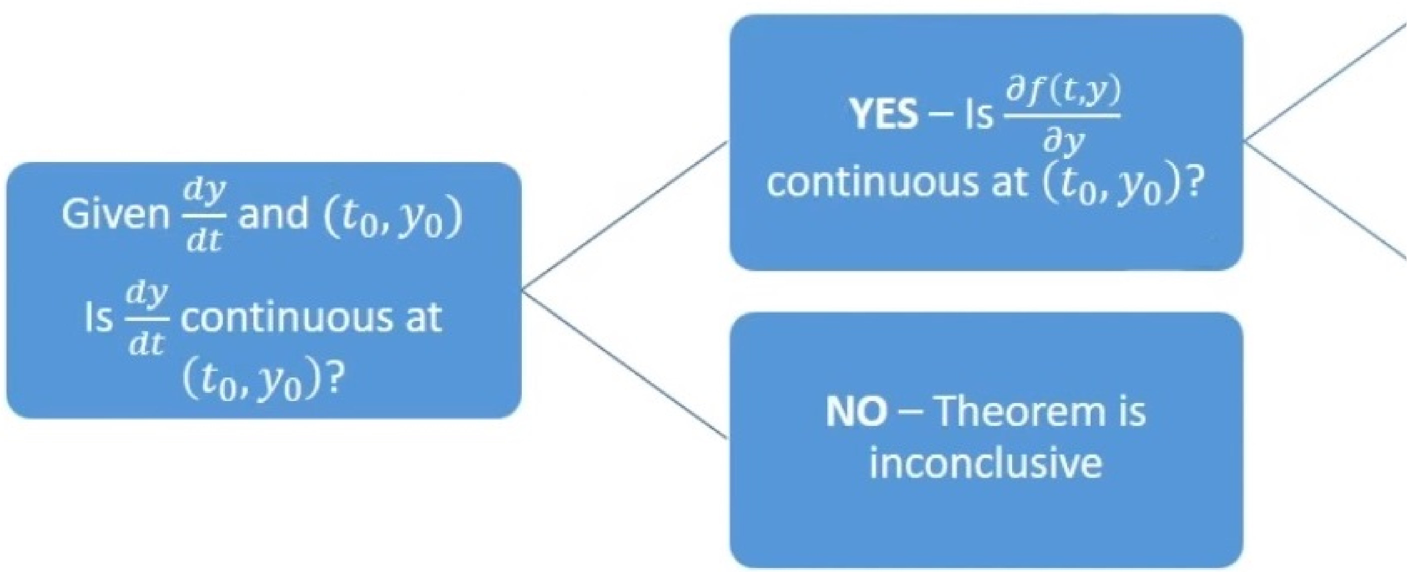
Consider the problem $y' = f(t, y)$ **with** $y(t_o) = y_o$

Theorem: Existence and Uniqueness for nonlinear DEs:

If $f(t, y)$ **and** $\frac{\delta f}{\delta y}(t, y)$ **are continuous near the initial value** (t_o, y_o)

Then, there is one unique solution $y = f(t)$ **defined for** t **near** t_o

Note: If either of the above conditions is not satisfied, then the theorem does not tell us anything ("we don't know"). In addition, it is possible for the existence part of the theorem to be satisfied (existence - if $f(t, y)$ **is continuous near** (t_o, y_o)) but not uniqueness (uniqueness - if $\frac{\delta f}{\delta y}(t, y)$ **is continuous near** (t_o, y_o)).



Why is it important to check for existence and uniqueness?

1. Saves time; don't waste time trying to solve a DE that may not actually have a solution!
2. Many of our models only make sense if there is one solution, otherwise some of our methods may not be valid (ex: Would the boulder model make sense if there are 2 trajectories given the same initial velocity and angle?)

Q. Consider the problem

$y' = t + \sqrt{y - \pi}$

$y(1) = 1$

1. Is there a unique solution?

Look for when the derivative is undefined or where things go wrong.

Notice that for a valid square root operation:

$y - \pi \geq 0$

$\Rightarrow y \geq \pi$

The given point (1,1) does not satisfy this restriction. Therefore, the derivative is not continuous around the given point and fails the theorem. Hence we cannot conclude uniqueness.

2. Without solving, what is its domain?

Tip: Look for where the derivative is undefined or won't work

Note that: $f(t, y) = t + \sqrt{y - \pi}$ **and** $\frac{\partial f(t,y)}{\partial y} = \frac{1}{2\sqrt{y-\pi}}$

So, the DE is defined when $y > \pi$ and $t \in (-\infty, \infty)$

According to the Theorem, what is the domain of the solution?

Notes not from lecture:

Partial Derivatives

Choose a variable that will vary and set all others as constants:

For example: $f(x, y) = \frac{dy}{dx} = x^2 y + \sin(y)$

$\frac{\partial f}{\partial x} = 2xy$ (let x vary, take y is const)

$\frac{\partial f}{\partial y} = x^2 + \cos(y)$ (let y vary, take x as const)

More existence and uniqueness

If we can conclude that a solution exists and is unique for a given point, then there is exactly 1 solution curve that can pass through that point. Meaning that none of the solutions of the DE will intersect.