Instructor Guide

Modelling with Differential and Difference Equations

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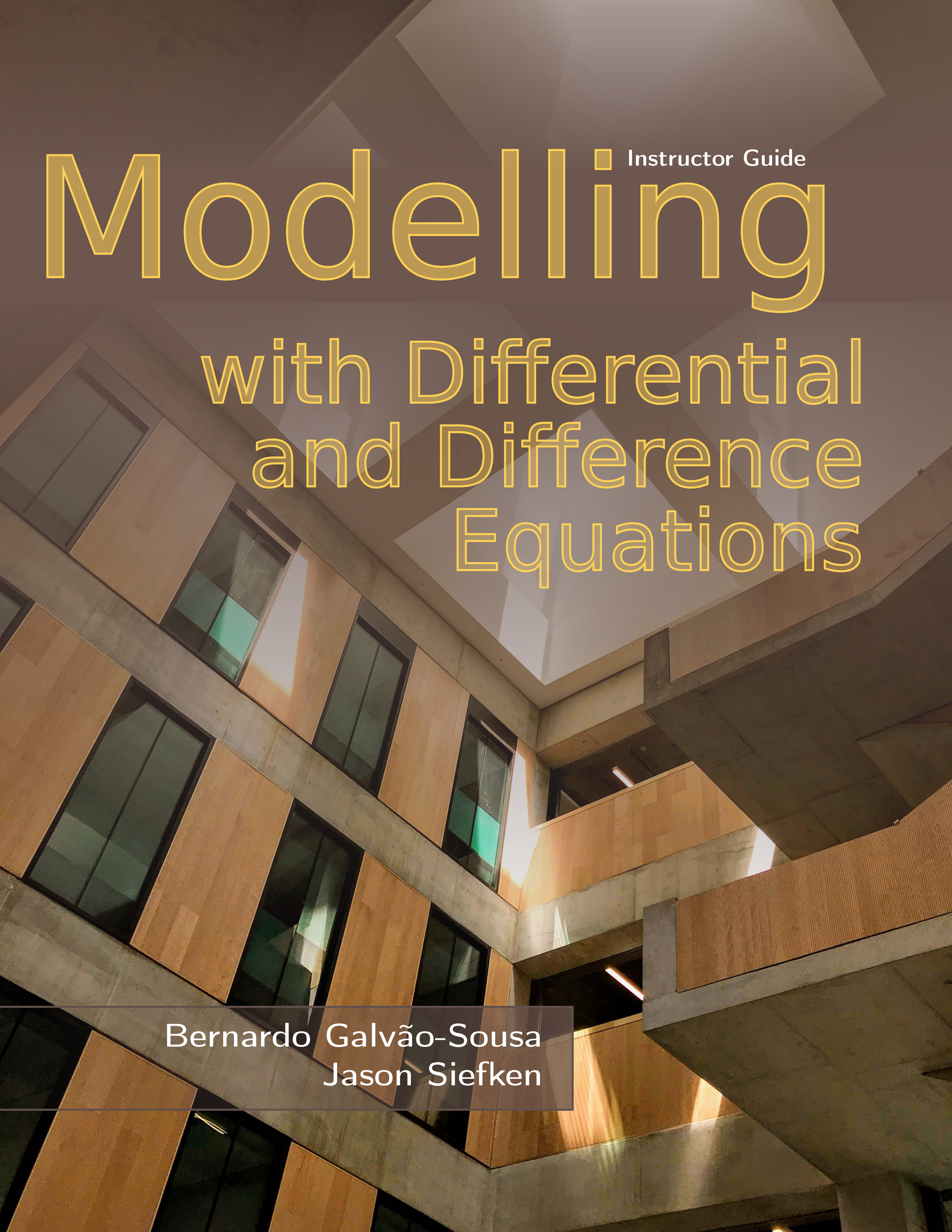

Inquiry Based Modelling with Differential and Difference Equations

For the student

This book is your introductory guide to mathematical modelling and modelling with differential and difference equations. It is divided into *modules*, and each module is further divided into *exposition*, *practice problems*, and *core exercises*.

The *exposition* is easy to find—it's the text that starts each module and explains the big ideas of modelling and differential or difference equations. The *practice problems* immediately follow the exposition and are there so you can practice with concepts you've learned. Following the practice problems are the *core exercises*. The core exercises build up, through examples, the concepts discussed in the exposition.

To optimally learn from this text, you should:

- Start each module by reading through the *exposition* to get familiar with the main ideas. In most modules, there are some videos to help you further understand these ideas, you should watch them after reading through the exposition.
- Work through the *core exercises* to develop an understanding and intuition behind the main ideas and their subtleties.
- Re-read the *exposition* and identify which concepts each core exercise connects with.
- Work through the *practice problems*. These will serve as a check on whether you've understood the main ideas well enough to apply them.

The core exercises. Most (but not all) core exercises will be worked through during lecture time, and there is space for you to work provided after each of the core exercises. The point of the core exercises is to develop the main ideas of modelling with differential or difference equations by exploring examples. When working on core exercises, think "it's the journey that matters not the destination". The answers are not the point! If you're struggling, keep with it. The concepts you struggle through you remember well, and if you look up the answer, you're likely to forget just a few minutes later.

Contributing to the book. Did you find an error? Do you have a better way to explain a concept? Please, contribute to this book! This book is open-source, and we welcome contributions and improvements. To contribute to/fix part of this book, make a *Pull Request* or open an *Issue* at `https://github.com/bigfatbernie/IBLModellingDEs`. If you contribute, you'll get your name added to the contributor list.

For the instructor

This book is designed for a one-semester introductory modelling course focusing on differential and difference equations (MAT231 at the University of Toronto).

Each module contains exposition about a subject, practice problems (for students to work on by themselves), and core exercises (for students to work on with your guidance). Modules group related concepts, but the modules have been designed to facilitate learning modelling rather than to serve as a reference.

Using the book. This book has been designed for use in large active-learning classrooms driven by a *think, pair-share*/small-group-discussion format. Specifically, the *core exercises* (these are the problems which aren't labeled "Practice Problems" and for which space is provided to write answers) are designed for use during class time.

A typical class day looks like:

1. **Student pre-reading.** Before class, students will read through the relevant module.
2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.
3. **Students work on problems.** Students work individually or in pairs/small groups on the prescribed core exercise. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.
4. **Instructor intervention.** When most students have successfully solved the problem, the instructor refocuses the class by providing an explanation or soliciting explanations from students. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to miss the point!).

   If students are having trouble, the instructor can give hints and additional guidance to ensure students' struggle is productive.
5. **Repeat step 3.**

Using this format, students are thinking (and happily so) most of the class. Further, after struggling with a question, students are especially primed to hear the insights of the instructor.

Conceptual lean. The *core exercises* are geared towards concepts instead of computation, though some core exercises focus on simple computation. They also have a modelling lean. Learning algorithms for solving differential and difference equations is devalued to make room for modelling and analysis of equations and solutions.

Specifically lacking are exercises focusing on the mechanical skills of algorithmic solving of differential and difference equations. Students must practice these skills, but they require little instructor intervention and so can be learned outside of lecture (which is why core exercises don't focus on these skills).

How to prepare. Running an active-learning classroom is less scripted than lecturing. The largest challenges are: (i) understanding where students are at, (ii) figuring out what to do given the current understanding of the students, and (iii) timing.

To prepare for a class day, you should:

1. **Strategize about learning objectives.** Figure out what the point of the day's lesson is and brain storm some examples that would illustrate that point.
2. **Work through the core exercises.**
3. **Reflect.** Reflect on how each core exercise addresses the day's goals. Compare with the examples you brainstormed and prepare follow-up questions that you can use in class to test for understanding.
4. **Schedule.** Write timestamps next to each core exercise indicating at what minute you hope to start each exercise. Give more time for the exercises that you judge as foundational, and be prepared to triage. It's appropriate to leave exercises or parts of exercises for homework, but change the order of exercises at your peril—they really do build on each other.

A typical 50 minute class is enough to get through 1–3 core exercises (depending on the difficulty), and class observations show that class time is split 50/50 between students working and instructor explanations.

License

Included in this text, in chapter 1, are expositions adapted from the handbook "Math Modeling: Getting Started and Getting Solutions" by K. M. Bliss, K. R. Fowler, and B. J. Gallizzo, published by SIAM in 2014 `https://m3challenge.siam.org/resources/modeling-handbook`.

Contributing. You can report errors in the book or contribute to the book by filing an *Issue* or a *Pull Request* on the book's GitHub page: `https://github.com/bigfatbernie/IBLModellingDEs/`

Contributors

This book is a collaborative effort. The following people have contributed to its creation:

◦ Stephanie Orfano ◦ Yvan Saint-Aubin ◦ Sarah Shujah ◦ Graeme Slaght ◦

In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analyze the model
6. Write a report

In this chapter, we follow the approach of Bliss, Fowler, and Galluzzo from

```
Math Modeling: Getting Started and Getting Solutions, K. M. Bliss,
K. R. Fowler, and B. J. Galluzzo, SIAM, Philadelphia, 2014

  https://m3challenge.siam.org/resources/modeling-handbook
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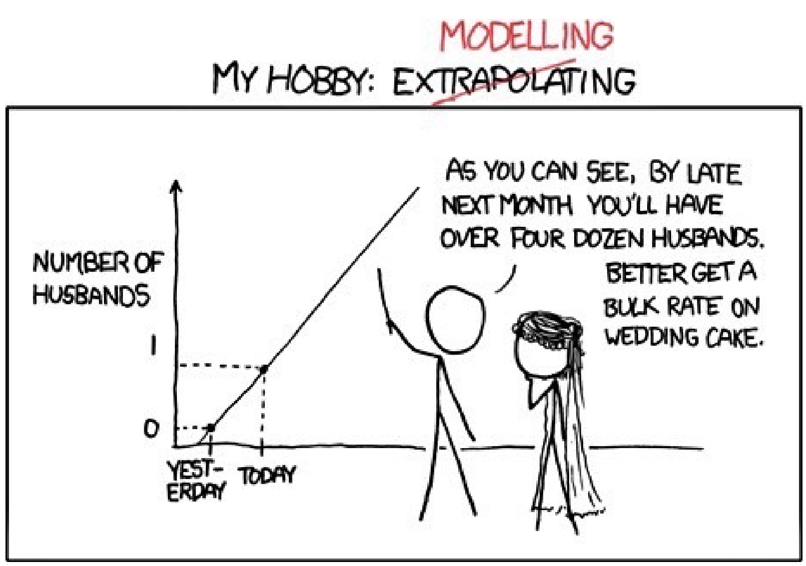

(image adapted from xkcd - comic #605)

Lesson 1

Defining Problem Statement

Textbook

- Module 1

Objectives

- The first step in Mathematical modelling is to define the problem
- A good way to do this is to figure out what is the "mathematical object" we are looking for at the end of the process

Motivation

Students have heard the words "Model" and "Modelling", yet they don't have a good idea what it means.

The goal of this first chapter is to establish a common procedure to approach all (or at least most) modelling tasks.

The very first step in the procedure is to *define the problem* in a clear and Mathematical way.

Notes/Misconceptions

- Students usually want to start thinking of ways to solve the problem.
- They need to learn to tackle a modelling task step-by-step.

Preparation for Class

- Read textbook

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

https://m3challenge.siam.org/resources/modeling-handbook

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their "elevator problem".

This is the email you received:

> —— Forwarded Message ——
>
> **Date:** Monday, 7 September 2020 21:41:35 + 0000
> **From:** CEO <theCEO@theBigCompany.ca>
> **To:** Human Resources <hr@theBigCompany.ca>
> **Subject:** they're still late!
>
> Hey Shophika!
>
> I still get complaints about staff being late, some by 15 minutes.
> With the staff we have, that's about one salary lost.
> Again the bottleneck of the elevators seems to be the problem.
> Can you suggest solutions?
>
> Thanks, the CEO

Make the question precise, bring it into a "mathematical form".

- Choose a mathematical object best suited for the problem, e.g. a number, a geometric form, a graph, a function, an algorithm, ...

Notes/Misconceptions

- Students will start discussing how to solve the problem
- This question deals with what will happen **after** solving the problem
- The goal of this question is to think about how to best tell a "mathematically-challenged" CEO that you solved the problem
- Student teamwork: "With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice."

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

> **Teamwork.**
> With your team, you must decide on ***one*** answer and be prepared to report on your decision and the reason for your choice.

2 The mayor of Toronto wants to extend the subway line with a new orange line as in the figure below.

(Map taken from Wikimedia Commons created by Craftwerker)

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

Lesson 2

Defining Problem Statement

Textbook

- Module 2

Objectives

- The second step in Mathematical modelling is to construct a representation of how the team will be attempting to solve the problem.
- Create a mind map of the problem. This is a structured way to brainstorm possible solutions, requirements, other objects that are related, etc.

Motivation

In this step, students are supposed to brainstorm and relate the problem at hand with everything that is affected or can be affected by it.

The idea is to get a simple visual representation of possible solutions, without all the details.

This is a fundamental step in modelling.

Preparation for Class

- Read textbook

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

https://m3challenge.siam.org/resources/modeling-handbook

3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- T = the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can T be minimized?

Notes/Misconceptions

- Students usually come up with more complicated variations:
  - Money spent on late employees' salaries
  - sum of time in minutes that employees are late counting only employees that are at most 15 minutes late
- Stick with T, a simple first approach

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to *optimally* control the traffic lights.

4.1 What mathematical meaning can we give to the word optimal in this circumstance?

4.2 Create a mind map for this problem.

Lesson 3

Making Assumptions

Textbook

- Module 3

Objectives

- The third step is to decide on a path to the solution and start making assumptions.
- This is a difficult balance between:
  - Accuracy, but difficult to analyze/solve;
  - Simple, easy to analyze/solve but not very accurate.
- Make sure assumptions and conditions of the modelling are clearly mentioned to the future reader/user of the model

Motivation

Students often make assumptions explicitly and implicitly, but they often keep them out of their notes. In the end they forget to include them in the final report.

It is imperative that they include their assumptions in the final report of their model.

Moreover, students should make an effort to find out the implicit assumptions and the conditions that to the model that their assumptions require.

Assumptions.
Include the assumptions in the model's final report.

Preparation for Class

- Read textbook

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading
https://m3challenge.siam.org/resources/modeling-handbook

5 Consider the elevator problem from core exercise 1.
We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Hint. Focus on a **few** parameters and variables.

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
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To: Human Resources <hr@theBigCompany.ca>
Subject: they're still late!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

5.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

5.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

5.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Notes/Misconceptions

- Students usually have trouble starting.
- They usually agree that they have to figure out how elevators work, so you can prompt them to be more specific.
- In the end they should come up with questions like these:
  - How fast are the elevators?
  - How much time do elevators take in each floor?
  - How many floors do elevators stop on their way up?
  - How many people fit in the elevator?
  - Should we consider elevator failures?

6 How much would it cost to make a bridge between Toronto and the U.S.?

Lesson 4

Construct a model

Textbook

- Module 4

Objectives

- The fourth step is to use the mind map created in step 2, the assumptions from step 3, and assemble everything into one model.
- The model is not the solution to the problem: it is the framework to solve the problem.

Motivation

Students usually think that the solution is the model that they need.

Emphasize that students should see the example in the textbook to get an idea of what a model looks like.

Notes/Misconceptions

- Model $\neq$ Solution

Preparation for Class

- Read textbook
- Ask students to prepare the core exercise 7 before class.
- In class, they should combine their mind map(s) and assumptions from the previous lessons and come up with an idea of a model to discuss with their classmates in lecture.

Notes/Misconceptions

There won't be enough time in class to finish the question if students don't prepare in advance.

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

https://m3challenge.siam.org/resources/modeling-handbook

7 Recall the core exercise 5.

- The company occupies the floors 30–33 of the building Place Ville-Marie in Montréal.
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Write down a mathematical model for this problem.

Teamwork.
Each team should have *one* model and be prepared to present it to the class.

——— Forwarded Message ———

Date: Monday, 7 September 2020 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
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With the staff we have, that's about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

Create a model.

1. Students should join in teams of 4 and come up with *one model* for the team.
2. The model should include:
   - Definition of the problem
   - Mind map
   - Assumptions and conditions
   - Clearly defined path to solve the problem
3. A few teams present their model for everyone else.
4. Full class brainstorm about each model:
   - Does it include all the parts specified?
   - Is it solvable?
   - Could we make some extra assumptions to make it simpler?
   - Is it accurate?
   - Could we change something to make it more accurate without sacrificing simplicity?

Notes/Misconceptions

- Solution should not be included!

Lesson 5

Model Assessment

Textbook

- Module 5

Objectives

- After creating a model, students should make an analysis of the model
- The analysis is meant to test the model as well as obtain some of its consequences
- If there is a solution to the model, here is where it can be found and analyzed

Motivation

Once a model is created, one must check if the model solves the problem it is meant to solve. There are different types of assessment for a model:

- Check some known cases to see if it works as expected;
- Check some extreme cases to see if it works as expected;
- Check some implications of the model to make sure they are reasonable;
- Check that the assumptions made are reasonable for the problem at hand;
- If possible, use approximation methods to estimate the solution;
- If possible, find the solution and analyze it.

Preparation for Class

- Read textbook

Extra Reading

Math Modelling: Getting started and getting solutions, Bliss-Fowler-Galluzzo

Extra Reading

`https://m3challenge.siam.org/resources/modeling-handbook`

8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t \quad = \quad \boxed{\text{loading time on ground floor}} + \boxed{\text{time of flight ground} \to 33} + \boxed{\text{time of flight } 33 \to \text{ground}} + \boxed{\text{stop time to 6 of the 11 floors}} \quad = \quad 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes
- Accumulated late time $= \boldsymbol{T} = 180 \cdot 20 \cdot 8 + 74 \cdot 20 \cdot 8 = 40\,640$ seconds $= 11\text{h}18\text{m}$

Your task is to assess this model. Be ready to report on your assessment.

Teamwork.
Each team should have ***one*** assessment and be prepared to present it to the class.

Notes/Misconceptions

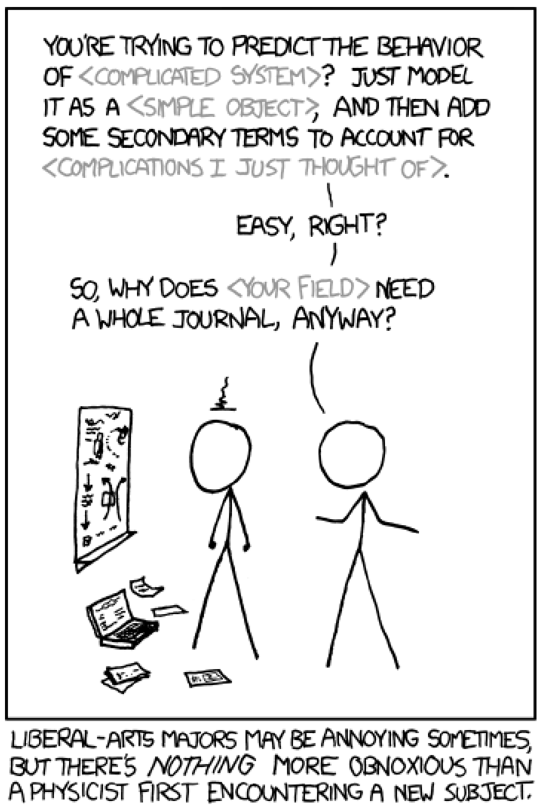
Some questions to guide the students:
- What are the strengths of this model?
- What are the weaknesses of this model?
- Is the result around what you expected?

In case students don't realize that something is wrong:
- People start arriving 30 minutes before the starting time, so *almost everybody will be on time?*
- Assume that the CEO of the BigCompany is right: people are arriving late! What's wrong with the model?
- Which assumptions should be relaxed? Or checked?
- If one needs to be replaced, by what?
- Do we need extra assumptions? Which?

First make sure model works. Then try to find a solution.

Notice that the model doesn't attempt to find a solution to the question.

If there weren't any problems with this model, we could then start asking other questions:
- How can we get people in their office faster?
- How will each idea affect the estimate?
- Will they cost money to the company?



(image from xkcd - comic #793)

Lesson 6

Solutions of Differential Equations

Textbook

- Modules 7, 8

Objectives

- Identifying the order of a differential equation
- Identifying a linear vs nonlinear differential equation
- Knowing how to check if a function is a solution of a differential equation

Motivation

This is an introduction to differential equations.

Students have different levels of experience with differential equations. We want to establish a common notation.

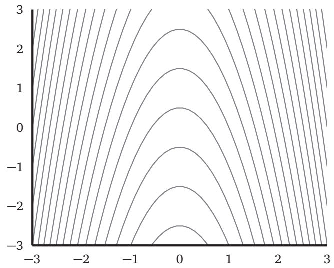
Preparation for Class

- Students should bring an ODE of their choice to class with some information about it
- Read textbook modules.

Using pre-class ODEs.

- At the start of class, put about 8 student ODEs on the board
- Get students to identify them: order + linearity
- Get some info about the ODEs to the whole class

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



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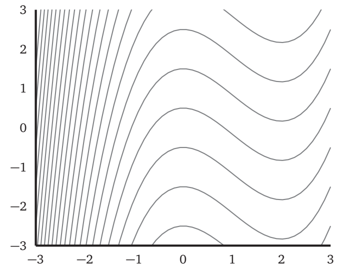

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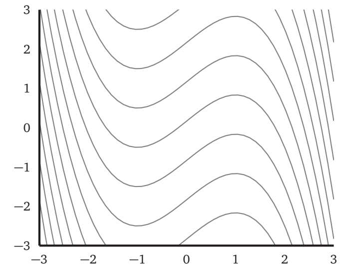

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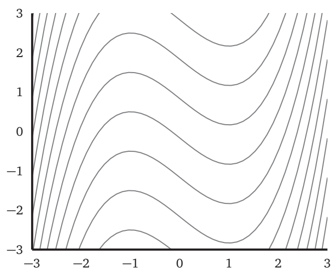

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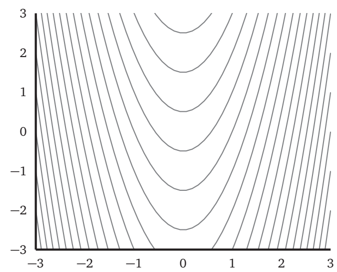

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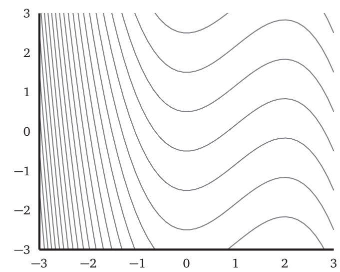


F

Without solving.

- Students should try to answer this question without solving the differential equation.
- Check properties of the ODE:
  - What does $x^2 - 1 = 0$ mean for the solution?
  - When is $y' > 0$? What does that mean for the solution?
  - When is $y' < 0$? What does that mean for the solution?

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph a function $f(x)$ that would produce such solutions.

Get students to figure out this core exercise in two different ways:

- by solving the ODE
- without solving the ODE

Quick exercise

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be decreasing

11.2 $y(t)$ must always be increasing

11.3 $y(t)$ must always be positive

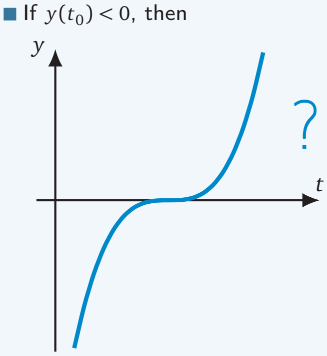
11.4 $y(t)$ must always be negative

11.5 $y(t)$ must never change sign.

Stronger students.

The 5[th] part is for stronger students to think about while they wait for the others to finish.
Will be addressed later in module 13 (Properties of solutions).
They can see that:

- If $y(t_0) > 0$, then $y(t) > 0$ for $t > t_0$
- If $y(t_0) < 0$, then



12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

Similar to some practice problems. Skip if the other exercises take too long.

Lesson 7

Slope Fields

Textbook

- Module 9

Objectives

- Sketch a slope field
- Use technology to create a slope field: WolframAlpha, Desmos, Geogebra, etc.
- Interpret a slope field
- Deduce properties of slope field from the ODE
- Deduce properties of solution from slope field

Motivation

ODEs are often difficult to solve, so we need tools to be able to interpret their solutions and deduce properties without having to solve them.

Slope fields are such a tool.

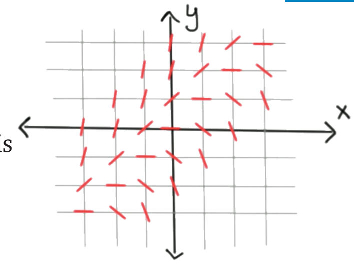
Slope fields are not meant to be sketched by hand, so they shouldn't be asked to do that, except at the beginning to learn how they are sketched so they can understand them better.

Slope fields should be used to help interpret solutions of ODEs.

Preparation for Class

- Read textbook
- Watch first video (second is optional)
- Solve the core exercise 13

13 Consider the slope field from the first video of the module.



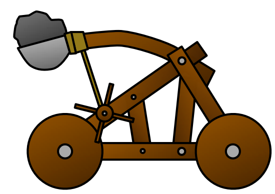
13.1 If $y(0) = 5$, then estimate $y(-7)$.

13.2 If $y(0) = a$, then $y(x) > 0$ for all $x > 0$. For which values of a is this statement true?

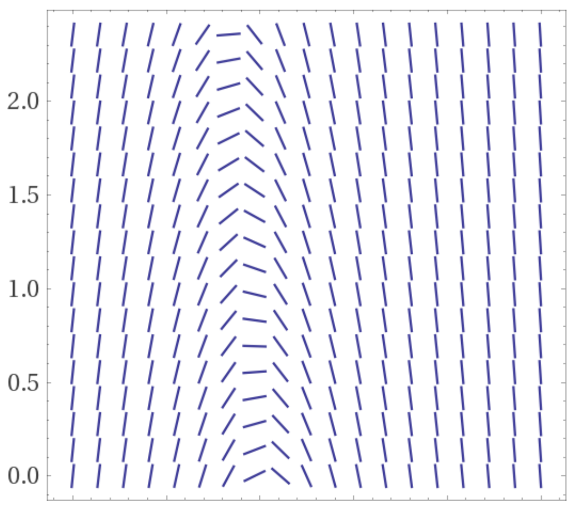
Pre-Class exercise

In class, quickly solve it.

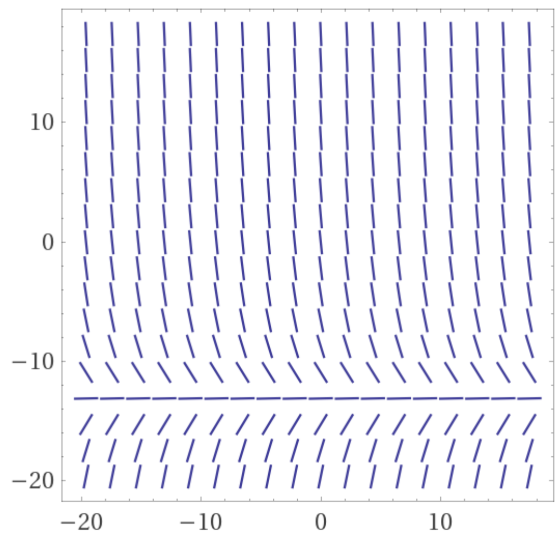
14 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

(These slope fields were created using WolframAlpha)

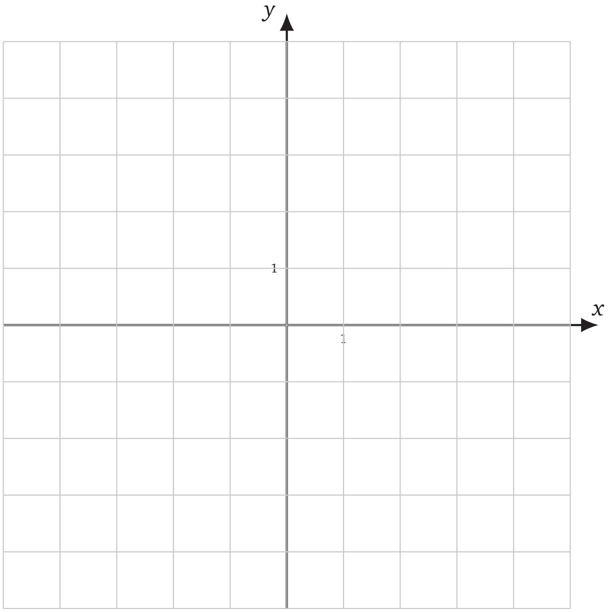
14.1 On the slope field, sketch a *possible* solution.

14.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

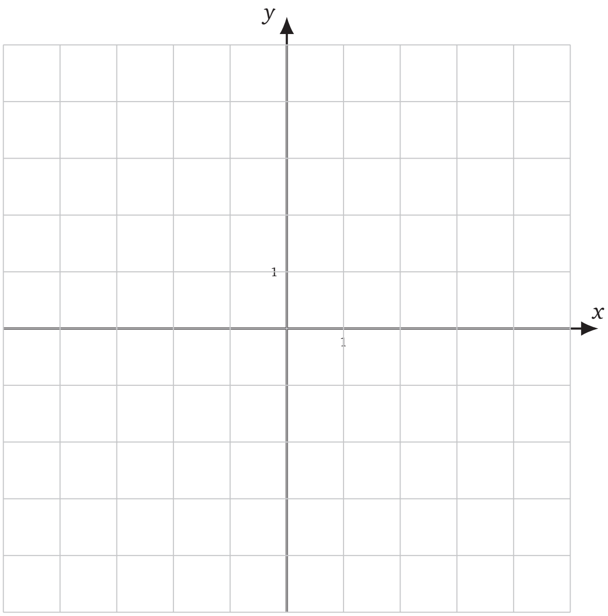
- Students should think about the initial conditions. What is a possible value for $y(0)$? What is a possible value for $y'(0)$?
- What does the second slope field tell us? The equilibrium in the slope field for $y'(t)$ is called *terminal velocity*.
- Sketch a possible solution again, but for $t \in [0, 30]$.

15 Sketch the slope field for the following differential equations.

15.1 $y' = x$



15.2 $y' = y^2$

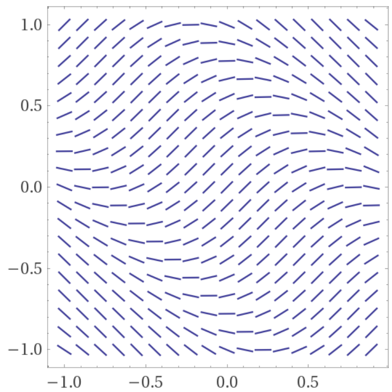


Symmetry.

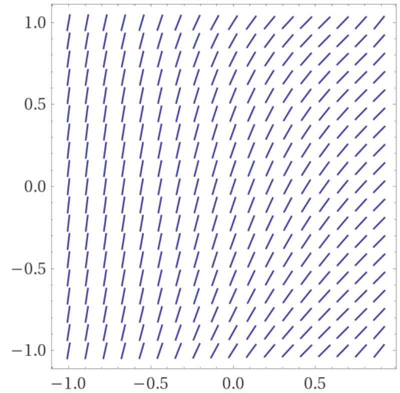
- The goal is not to be very accurate, but to capture the symmetry of each of these slope fields.
- Which property of the slope field allowed you to sketch it more quickly?

Students should be able to justify their choices .

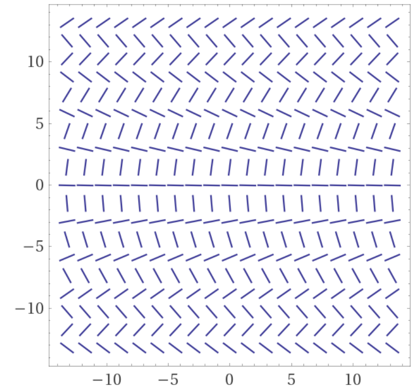
16 Consider the following slope fields:



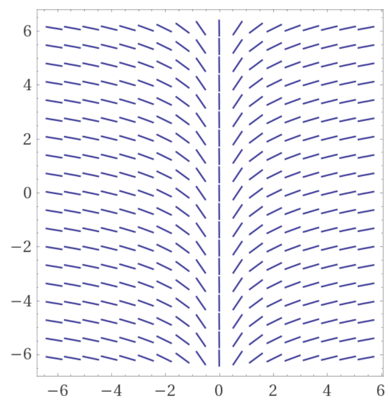
(A)



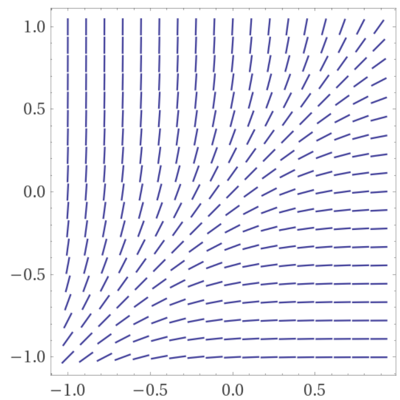
(B)



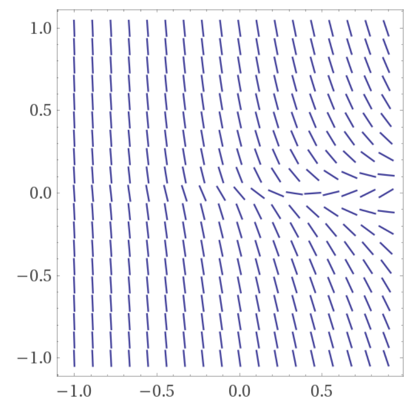
(C)



(D)



(E)



(F)

(These slope fields were created using WolframAlpha)

16.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

16.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

16.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

16.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

16.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

16.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \big(m(x,y)\big)^2$?

Lesson 8

Approximating Solutions

Textbook

- Module 10

Objectives

- Know the idea of Euler's method
- Use Euler's method
- Be aware of the limitations of Euler's method
- Deduce some properties of Euler's method

Motivation

After sketching slope fields and observing that they can be used to get an idea of the solution, the idea to use slope fields to rigorously define an approximation of the solution should come naturally.

Euler's method is just that approximation method.

This is probably one of the most important tools for an Engineer or a Physicist studying Differential equations, since the ODEs that often arise naturally from real problems are too complicated to solve rigorously and approximating the solution might be the only way.

Preparation for Class

- Read textbook
- Watch first video (second is optional – algebraic approach of approximating the derivative)
- Solve the core exercise 17

17 Consider the initial-value problem

$$\begin{cases} y' = -\sin(x) + \frac{y}{20} \\ y(-10) = 2 \end{cases}$$

The solution satisfies $y(10) = \frac{20\sin(10)+400\cos(10)-2e(-401-10\sin(10)+200\cos(10))}{401} \approx 6.7738406\ldots$.

17.1 Using some software, approximate the solution at $x = 10$ for different values of Δx.

17.2 Calculate the error between the solution and the approximation at $x = 10$ for the different values of Δx.

17.3 Plot the error. Is it decreasing as Δx decreases? Does it decrease linearly / quadratically / cubicly as Δx decreases?

18 Consider the differential equation

$$y' = y - 2.$$

18.1 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,3)$.

18.2 Find the solution of the differential equation with the same initial condition.

18.3 Use Euler's Method to find an approximation of the solution of this differential equation that passes through the point $(0,1)$.

18.4 Find the solution of the differential equation with the same initial condition.

18.5 Compare the approximations with the actual solutions. Is there a property of the Euler's Method that you can infer?

18.6 Explain in words why the Method satisfies that property.

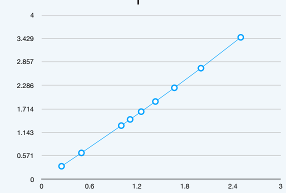
19 Which differential equations will be approximated perfectly using Euler's Method?

Pre-class exercise

In class, quickly solve it.

https://www.desmos.com/calculator/ebfn5vpudr

Δx	Error
2.5	3.4552
2	2.7050
1.67	2.2269
1.43	1.8937
1.25	1.6475
1.11	1.4581
1	1.3078
0.5	0.6439
0.25	0.3193



- Linear with Δx: $E \approx C\Delta x$.

- The goal is to have student's recognize that the Euler approximation "curves slower" than the actual solution.
- Students can explain in words why that is the case using the way the approximations are generated.

- For .2 and .4:
  - If students learned how to solve ODEs before, then fine!
  - If students didn't learn, then tell them to use WolframAlpha: `solve y'=y-2, y(0)=3`

Only if there is time.

- The question is purposefully ambiguous. What do we mean by approximated perfectly?
- Ex: The IVP $y' = \text{sign}(t)$ (assuming $\text{sign}(0) = 1$) with $y(-5) = 5$ has solution $y = |t|$ and it is captured with Euler's method if $\Delta t = \frac{5}{k}$ for any $k \in \mathbb{N}$.

- Once students discuss, they'll find ODE's of the form $y' = c$ for any constant.
- Prompt them to find other types. Show them the example above only after they tried for a bit. Then, let them revise their Conjecture.

- Ex 2: $y' = f(y+t)$ with $y(0) = 0$ and $f(z) = \lfloor z \rfloor$ is approximated perfectly if $\Delta t = 1$, but not if Δt takes any other value.

Lesson 9

Modelling with Differential Equations

Textbook

- Module 11

Objectives

- Start modelling physical quantities
- Follow the procedure from chapter 1 when creating a model

Motivation

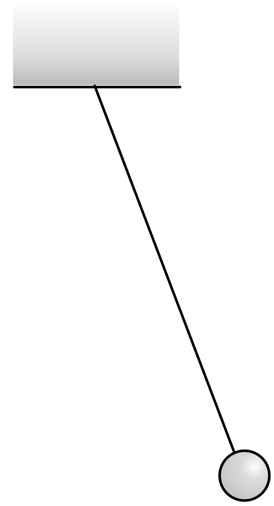
This is one of the main goals of this course.

Students should be given the opportunity to create models in class. They should be encouraged to follow the procedure from chapter 1, as it will improve their models.

Preparation for Class

- Read textbook
- Read the core exercise 20 and solve steps 1 (define problem) and 2 (create a mind map)

20 A pendulum is swinging side to side. We want to model its movement.



20.1 Define the problem. Which function(s) do we want to find in the end?

20.2 Build a mind map.

20.3 Make assumptions. Remember to use your mind map to help structure the problem.

20.4 Construct a model. You should end up with one (or more) differential equations.

Remember that there are some Physics principles that can help you (e.g. Newton's 2$^{\text{nd}}$ Law, Conservation of Energy, Linear Momentum, and Angular Momentum, Rate of Change is Rate in — Rate out).

20.5 Assess your model:

(a) Find one test that your model passes.

(b) Find one test that your model fails.

Students will mostly likely identify the goal as finding the position of the ball $\vec{r}(t) = \big(x(t), y(t)\big)$. That's fine!
Later, in Step 3, try to guide the students to recognize the following:

- Rope is massless (negligible)
- Rope doesn't bend (negligible)
- So can assume that the rope is rigid. How does that affect the position of the ball?
- No friction (negligible)
- **Important:** Students always focus on string tension. One can consider it, but it all cancels out. It's one of the exercises of the module (above). For the lecture, don't consider tension.

Then on Step 4, guide students to recognize that they actually only need to find a model for the angle, because the position of the ball really only depends on the angle $\theta(t) : \vec{r}\big(\theta(t)\big)$

Students should finish it at home.

Lesson 10

Modelling with Differential Equations

Textbook

- Module 11

Objectives

- Model an open ended problem
- Follow the procedure from chapter 1 when creating a model

Motivation

Students may be used to some models from physics classes or from Calculus classes.

They should practice developing more open ended models.

Preparation for Class

- Read textbook
- Read the core exercise 21 and solve steps 1 (define problem) and 2 (create a mind map)

21 Model the spreading of a rumour through the students of a school.

Lesson 11

Solvable Types of Differential Equations

Textbook

- Module 12

Objectives

- Identify a Separable ODE
- Know how to solve a Separable ODE
- Identify a first-order linear ODE
- Know how to solve a first-order linear ODE: Method of the Integrating Factor

Motivation

This is the traditional class on differential equations.

Even though the focus of the course is not on recipes for solving differential equations, when creating a model that involves a differential equation and analyzing it, it is necessary to be aware of some of the basic methods for solving them.

Students should practice these methods more than the limited time devoted to them in lecture. Even though there aren't many practice problems, it is easy to find exercises for this module in any differential equations textbook.

Preparation for Class

- Read textbook
- Watch first video for each method

22 Decide whether the following differential equations are separable, first-order linear, both, or neither. If they are of one of the solvable types, solve it.

22.1 $\theta''(t) = \frac{g}{L}\sin\big(\theta(t)\big)$

22.2 $P'(t) = rP(t)\left(1 - \frac{P(t)}{K}\right)$

22.3 $v'(t) = -g - \frac{\gamma}{m}v(t)$

22.4 $y'(t) = -gt - \frac{g}{m}y(t) + 10$

23 23.1 Calculate $\big(\sin(x)f(x)\big)'$.

23.2 Find the general solution of

$$\sin(x)y' + \cos(x)y = \sqrt{x}.$$

23.3 What is the integrating factor for the differential equation

$$y' + \frac{\cos(x)}{\sin(x)}y = \frac{\sqrt{x}}{\sin(x)}$$

- First, ask students to identify all the ODEs
- Then, students can start solving them one by one

Notes/Misconceptions

- Make sure there is time for next core exercise. Skip some ODEs if necessary.

Idea of Integrating Factor

By recognizing that the left-hand side is the result of a product rule, it is much easier to find the solution.

Lesson 12

Properties of Differential Equations

Textbook

- Module 13

Objectives

- Identify which Theorem to use for a given ODE
- Know the Existence and Uniqueness Theorems
- Deduce properties of the solutions from the Theorems

Motivation

This is a little glimpse into the theory of Differential Equations.

Knowing whether a differential equation has solutions, and whether there is one unique solution is important on its own right.

But even from a more applied point-of-view, these Theorems offer consequences to the behaviour of solutions of differential equations that make them very important.

One of these properties is the fact that in the region in the (t, y)-plane where the Theorems' conditions are satisfied, solutions cannot intersect.

Preparation for Class

- Read textbook
- Watch first video: about the Theorem for Nonlinear ODEs
- Watch the second video: short example video
- Solve the core exercise 24

24 Consider the example in the video:

$$\begin{cases} x\dfrac{dy}{dx} = y \\ y(0) = b \end{cases}$$

Without solving, but only according to the Existence and Uniqueness Theorem, what can we conclude?

(a) We can conclude that there is a unique solution.

(b) We can conclude that if $b = 0$ there are many solutions, but if $b \neq 0$, then there are no solutions.

(c) We can conclude that there are many solutions.

(d) We can conclude that there are no solutions.

(e) We can't conclude anything.

Pre-Class exercise

In class, quickly solve it.

25 For the following initial-value problems, answer the following questions:

(a) Is there a unique solution?

(b) Without solving, what is its domain?

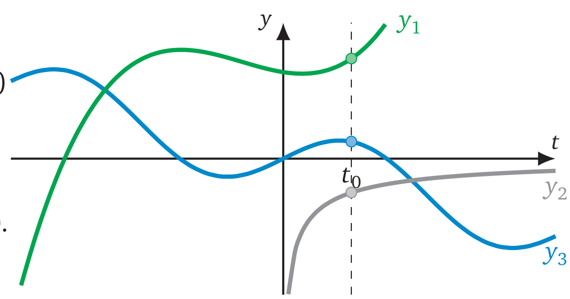
25.1 $y' = t + \frac{y}{t-\pi}$ with $y(1) = 1$

25.2 $y' = t + \sqrt{y-\pi}$ with $y(1) = 1$

25.3 $y' = \sqrt{4-(t^2+y^2)}$ with $y(1) = 1$

26 Consider a differential equation $y' = f(t, y)$ where

- $f(t, y)$ is continuous for all $t, y \in \mathbb{R}$;
- $\frac{\partial f}{\partial y}(t, y)$ is continuous for all $t \in \mathbb{R}, y > 0$.



26.1 Can **green** $y1$ and **blue** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.2 Can **green** $y1$ and **gray** $y2$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.3 Can **gray** $y2$ and **blue** $y3$ be two solutions of the same differential equation above with two different initial conditions? Why?

26.4 Based on the answers to the three parts above, write a Corollary to the Existence and Uniqueness Theorems.

Intersecting solutions?

The goal is for students to understand why solutions cannot touch *IF* the Theorem is valid at the intersection point.

- $y1$, y_3 not ok!
- $y1$, y_2 ok!
- $y2$, y_3 ok!

27 The initial-value problem

$$\begin{cases} y' = -\dfrac{x}{y} \\ y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}. \end{cases}$$

has the solutions

$$y_1(x) = \cos\big(\arcsin(x)\big) \qquad \text{and} \qquad y_2(x) = \sqrt{1-x^2} \quad .$$

27.1 Does the problem satisfy the conditions of one of the Existence and Uniqueness Theorems?

27.2 What can you conclude?

Existence and Uniqueness Thms

The goal is for the students to realize that if the Theorem's conditions hold, then its result must also hold, even though it might not look like it.
So if the Theorem says that the solution is unique and we see two solutions, then they must be the same function (at least for x near $\frac{1}{2}$).

Hint. "When you have eliminated the impossible, whatever remains, however improbable, must be the truth." – Sherlock Holmes

Lesson 13

Autonomous Differential Equations

Textbook

- Module 14

Objectives

- Identify an Autonomous ODE
- Find and plot equilibrium solutions
- Sketch solutions of an autonomous ODE without solving it
- Classify equilibrium solutions

Motivation

Even though nonlinear ODEs are in general very hard to study, Autonomous ODEs, which are often nonlinear, are an exception.

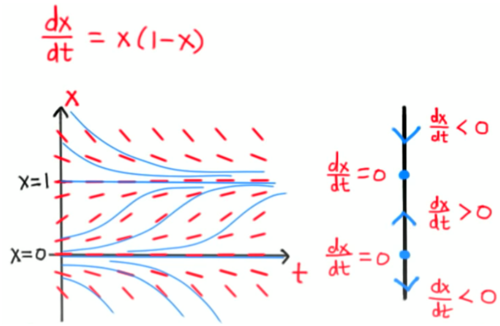
These are ODEs of the form $y' = f(y)$, which can usually be thoroughly studied without having to find their solution.

One of the key ways that we use to study Autonomous equations is to find equilibrium points and classify them as stable / unstable. This will be extended to systems of ODEs.

Preparation for Class

- Read textbook
- Watch the video
- Solve the core exercise 28

28 Consider the ODE from the video in the module.



Select all the initial conditions that yield a decreasing solution.

(a) $x(-2) = \sqrt{2}$

(b) $x(20\,000) = 0.000000001$

(c) $x(5) = \pi$

(d) $x(0) = \frac{1}{2}$

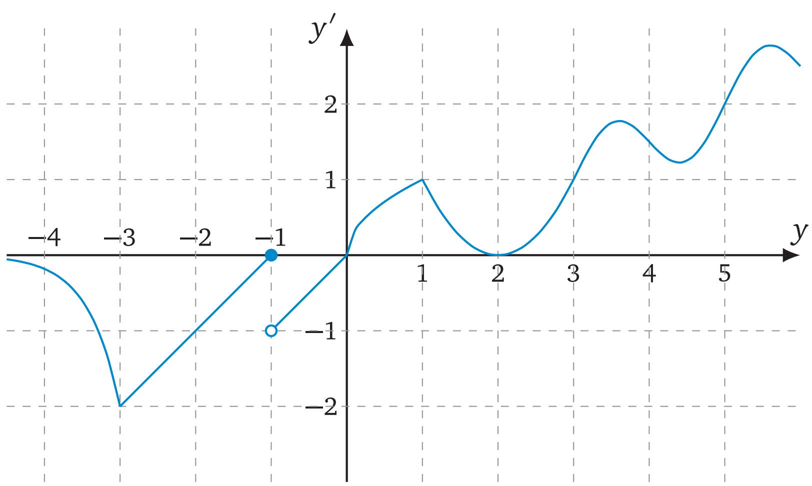
(e) $x(3) = -\frac{1}{2}$

(f) $x(1000) = \frac{1}{e}$

Pre-Class exercise

In class, quickly solve it.

29 Consider the differential equation $y' = f(y)$ where $f(y)$ is given by the following graph:



- *1–3*: Students come up with a definition of stable, unstable, semi-stable equilibrium points.
- *4*: Sketch solutions (do (a),(b) in class)
- *5–6*: Find properties of solutions
- *7*: Leave as HW

29.1 What are the equilibrium points?

29.2 Which equilibrium solutions are stable, unstable, or semi-stable?

29.3 Write a definition for a **stable**, **unstable**, and **semi-stable** equilibrium point.

29.4 Roughly, sketch a solution satisfying:

(a) $y(0) = 2.5$.

(b) $y(0) = -\frac{1}{4}$.

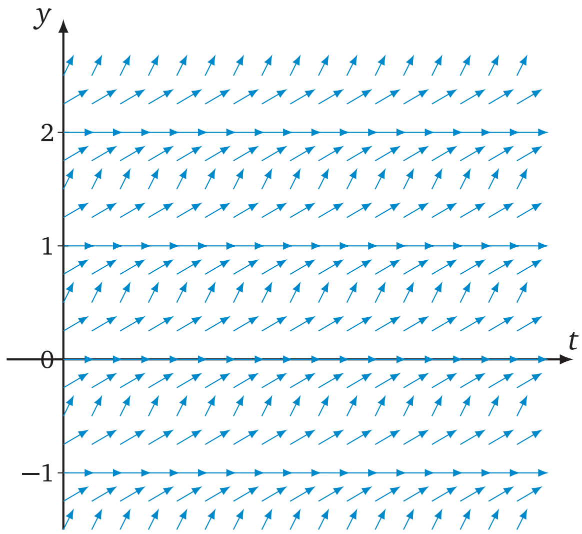
(c) $y(1) = \frac{1}{4}$.

29.5 If $y(0) = 2$, then $y(t) =$

29.6 If $y(0) = \frac{1}{2}$, then $\lim_{t\to\infty} y(t) =$

29.7 If $y(0) = -2$, then $\max_{t\in[0,\infty)} y(t) =$

30 Consider a differential equation $y' = f(t,y)$ with the following slope field.



30.1 What are the equilibrium solutions of the ODE?

30.2 Directly on the direction field above, sketch the solution of the problem

$$\begin{cases} y' = f(t,y) \\ y(0) = \dfrac{1}{4} \end{cases}$$

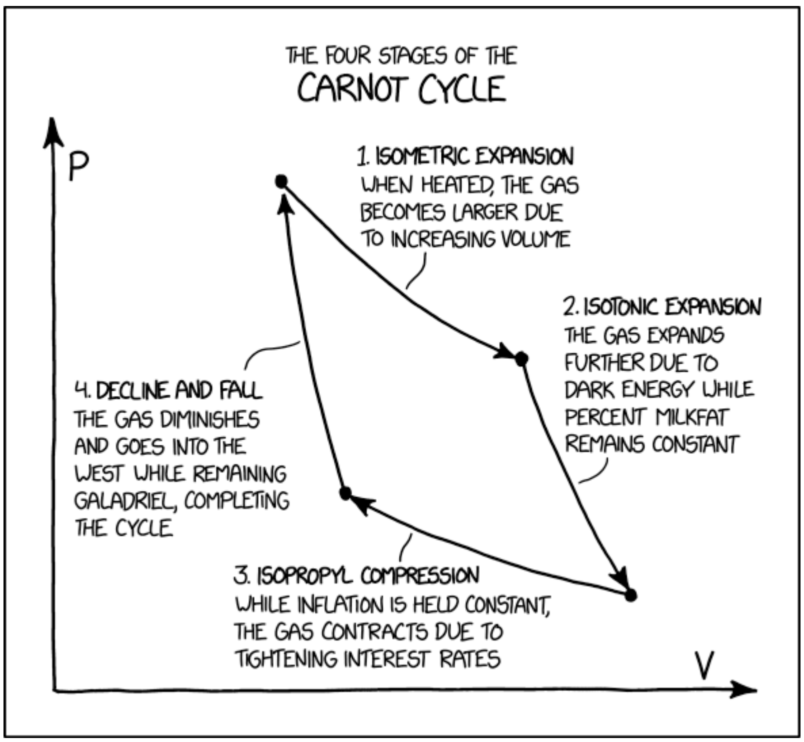
30.3 From the direction field above, circle the correct type(s) of this ODE? Justify your answer.

(a) separable.
(b) of first-order and linear.
(c) autonomous.
(d) none of the other options.

30.4 Assume that $y = g(t)$ and $y = h(t)$ are two solutions of the differential equation with $g(0) < h(0)$, then

(select all the possible options)

(a) $g(3) < h(3)$ (b) $g(3) = h(3)$ (c) $g(3) > h(3)$



(image from xkcd - comic #2063)

Lesson 14

Modelling Two Quantities I

Textbook

- Module 15

Objectives

- Model with two interconnected quantities
- Follow the step-by-step procedure to create a model

Motivation

Again, this is the main goal of this course: learn how to create models.

In this class, the goal is to study how to model quantities that affect each other (and are affected by external factors).

After the students create a model, they should get used to making sure the model is "good", i.e., using some software look at some approximation of solutions to see the consequences of the model.

Preparation for Class

- Read textbook
- Read the core exercise 31 and solve steps 1 (define problem) and 2 (create a mind map)

31 We want to model two competing populations, like cheetahs and lions: they don't hunt each other, but they hunt the same prey.

31.1 Create a model for these two populations.

31.2 Using Desmos or WolframAlpha, create a slope field in the plane where the horizontal axis is one population and the vertical one is the other.

31.3 Using the slope field, deduce some properties of your model and discuss how closely it matches what you expect from these populations.

31.4 Extend the model to include a population of antelopes.

Pre-class exercise.
Students bring their mind map and the definition of the functions they need to find.

Stress that students should follow the step-by-step approach from chapter 1. *.4* only if there is time. Tell the students to "go nuts" and include everything that relates.

Lesson 15

Modelling Two Quantities II

Textbook

- Module 15

Objectives

- Model with two interconnected quantities
- Follow the step-by-step procedure to create a model

Motivation

This class students work on a model that involves a little more calculus (for the cheetah's movement) and also is more *open ended* on how the cheetah will follow the antelope, but mainly on how the antelope will try to escape the cheetah.

Preparation for Class

- Read textbook
- Read the core exercise 32 and solve steps 1 (define problem) and 2 (create a mind map)

32 A cheetah is chasing an antelope. We want a model of their positions as they run.

Pre-class exercise.

Students bring their mind map and the definition of the functions they need to find.

This exercise is not required to do in lecture.

Be careful with assumptions! A very general model will be very hard to study.

Allow some brainstorming and try to create a structure for this problem:

- Positions seen from above (xy−plane).
- Only need $x_a(t), y_a(t)$ and $x_c(t), y_c(t)$
- Focus on the cheetah: where is she heading to?
- For the antelope, students need to come up with an escape strategy
- Model will be nonlinear!

Lesson 16

Systems of two linear ODEs with constant coefficients I

Textbook

- Module 16

Objectives

- Know how to find eigenvalues and eigenvectors of a matrix
- Know how to find a solution from an eigenvalue and an eigenvector
- Know how to write the general solution to a system of ODEs
- Know how to find the constants given an initial condition

Motivation

Linear Algebra strikes again!!! It will be very important to know how to solve a linear system of equations (although we'll focus on 2×2 systems only) and how to find eigenvalues and eigenvectors.

This is the more computational part of the course.

Even though it's not the focus of the course, students need to know how solutions are found. This will give some insight for the analysis of a system of ODEs later.

We start with problems involving two real distinct eigenvalues.

Preparation for Class

- Review Linear Algebra (Appendix **??**)
- Read textbook: Two real distinct eigenvalues
- Watch video
- Solve the core exercise 33.1.

33 Consider a cheetah-lion inspired problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{r}.$$

33.1 Find the two solutions $\vec{r}_1, \vec{r}_2$.

33.2 Is $\vec{r}_1(t) + \vec{r}_2(t)$ a solution?

33.3 Is $\vec{r}_1(t) - \vec{r}_2(t)$ a solution?

33.4 Is $2\vec{r}_1(t) + 3\vec{r}_2(t)$ a solution?

33.5 What is the general solution?

33.6 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

Notes/Misconceptions

Superposition Principle.

After .5, can introduce the principle of superposition:

- If $\vec{r}_1$ and $\vec{r}_2$ are solutions of a linear homogeneous (system of) ODE(s), then $\vec{r} = c_1\vec{r}_1 + c_2\vec{r}_2$ is also a solution for any constants c_1, c_2.

Notes/Misconceptions

If there is more time.

Start working on the core exercise for the next lesson.

Lesson 17

Systems of two linear ODEs with constant coefficients II

Textbook

- Module 16

Objectives

- Know how to find complex eigenvalues and eigenvectors of a matrix
- Know how to find a complex solution from an eigenvalue and an eigenvector
- Know how to re-write the solution using Euler's Formula and only real numbers

Motivation

We now add some complex numbers into the mix! Although students only need a very (Very) superficial knowledge of complex numbers.

We study problems involving two complex eigenvalues or one repeated real eigenvalue.

The third case: One repeated real eigenvalue will not be worked on in lecture. Students should learn it by themselves and practice it.

Preparation for Class

- Read textbook: Two complex eigenvalues
- Watch corresponding video
- Solve the core exercise 34.

34 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r}.$$

34.1 Find the general solution.

34.2 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$?

35 Consider a problem:

$$\frac{d\vec{r}}{dt} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{r} - \begin{bmatrix} 9 \\ 4 \end{bmatrix}.$$

35.1 Find the equilibrium solution.

35.2 Find the general solution.

35.3 Find the solution that satisfies $\vec{r}(0) = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$?

Skip .2 and start working on next core exercise

Non-Homogeneous Problem

Students don't know how to solve it yet:

- Equilibrium solution $\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)$
- Show phase portrait using WolframAlpha `https://uoft.me/modelling-sys-nonhom`
- Ask students about properties of the phase portrait (*Goal*: solutions revolve around the equilibrium point)
- Redefine centre: $\vec{r} = \vec{eq} + \vec{p}$. What system foes $\vec{p}$ solve?
- ...

There are practice problems about this.

Lesson 18

Phase Portraits I

Textbook

- Module 17

Objectives

- Interpret a phase portrait
- Sketch a phase portrait

Motivation

After learning how to create a model involving a system of ODEs and how to find the solution, we turn our attention to how to represent the solutions.

The phase portrait is a compact way to represent all possible solutions of a system of two ODEs.

Through sketching phase portraits, we are also learning the different possible behaviours of solutions of systems of two ODEs.

We start with systems of two ODEs with two real distinct eigenvalues. It usually takes a while to sketch the first phase portrait, so the first lesson is dedicated to sketching only one.

Preparation for Class

- Read textbook
- Watch first video
- Solve the core exercise 36.

36 Consider the following model for cheetah's and lions, where

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}$$

The general solution is:

$$\vec{p}(t) = c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

36.1 Without computing them, what are the eigenvalues and eigenvectors of the matrix?

36.2 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = 0$.

36.3 Sketch the graph of the solution with $c_1 = 0$ and $c_2 = \pm 1$.

36.4 When one constant is set to 0, what is the shape of the graph? Is it always like that? Can you prove it?

36.5 Sketch the graph of the solution with $c_1 = \pm 1$ and $c_2 = \pm 1$.

36.6 Provide an interpretation of the different types of solutions.

Pre-class exercise

Unstable Saddle Point

At the end, let the students know that the equilibrium is called *saddle point* and it is *unstable*, because solutions go away from it.
For the interpretation question, when one population hits zero, it is extinct, so the graph doesn't make sense.
We can interpret that if a population becomes extinct, then the other will behave as it would without competitors: grow exponentially fast!

Lesson 19

Phase Portraits II

Textbook

- Module 17

Objectives

- Interpret a phase portrait
- Sketch a phase portrait

Motivation

There are a few more different behaviours, so we study some more cases.

We start with a non-homogeneous case and then study some other cases of phase portraits with two real distinct eigenvalues.

Preparation for Class

- Solve the core exercise 37.

37 Let us expand the model from the previous exercise to:

$$\vec{p}(t) = \begin{bmatrix} \ell(t) = \text{population of lions} \\ c(t) = \text{population of cheetahs} \end{bmatrix}$$

which satisfies

$$\frac{d\vec{p}}{dt} = \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix}\vec{p} + \begin{bmatrix} -10 \\ 50 \end{bmatrix}.$$

The extra term corresponds to the effect of harvesting 10 lions and bringing in 50 cheetahs every year to the reserve.

The general solution is:

$$\vec{p}(t) = \begin{bmatrix} 20 \\ 10 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} e^{(1-\sqrt{3})t} + c_2 \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix} e^{(1+\sqrt{3})t}.$$

37.1 Sketch the phase portrait.

37.2 Provide an interpretation of the different types of solutions.

Pre-class exercise

- Solve quickly in class.
- Get students to compare their results with the previous core exercise.
- How does the interpretation of the population dynamics change?

38 For each of the following general solutions, sketch the phase portrait.

38.1 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{5t}$.

38.2 $\vec{r}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-5t}$.

At the end, let the students know what these equilibria are called:

- *source* and it is *unstable*, because solutions go away from it.
- *sink* and it is *asymptotically stable*, because solutions converge to it.

If there is time, students can think about:

- Given a matrix A, which part of A indicates whether the equilibrium is stable / unstable? Which part indicates whether it's a sink/source vs spiral sink/source?

Lesson 20

Analysis of Models with Systems

Textbook

- Module 18

Objectives

- Deduce properties of solutions of systems of ODEs using different approaches

Motivation

Analysis of a system of ODEs is important.

It's not always possible to find the solution of a model, but it's usually possible to sketch a rough phase portrait, to approximate the solution, or to just deduce some properties of the solutions.

Preparation for Class

- Read textbook
- Solve the core exercise 39.1

39 Consider the following model for the sales from a designer clothing brand:

- $x_1(t) =$ purchases by "common mortals" (CM) at time t in years since the beginning of 2015.
- $x_2(t) =$ purchases by "famous people" (FP) at time t.

Our model is based on the following two principles:

(P_1) CM will buy more items from the brand when CM or FP buy more.

(P_2) FP will buy less when CM buy them, but will buy more when FP buy it.

The model we considered is:

$$\vec{x}'(t) = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \vec{x}(t)$$

39.1 Suppose that at the beginning only CM buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

39.2 Suppose that at the beginning only FP buy this brand. Describe how $x_1(t)$ and $x_2(t)$ evolve as $t > 0$.

39.3 What conditions on the constants a, b, c, d will guarantee that the solutions will spiral? In that case, is it a spiral source or spiral sink? Is it clockwise or counterclockwise?

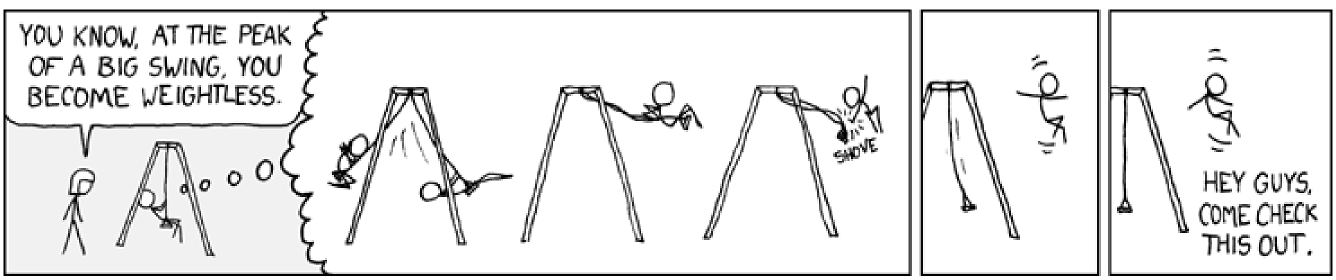
39.4 Are there constants $a, b, c, d > 0$, such that the solution $\vec{x}$ is periodic?

39.5 Consider the constants $a = b = c = d = 1$. Assume that initially CM were buying $c_0 > 0$ items and FP were buying $f_0 > 0$ items. What will happen to $x_1(t)$ and $x_2(t)$ as $t \to \infty$? Explain the results in terms of the evolution of purchases from CM and FP.

39.6 Consider the constants $a = b = c = d = 1$. If $c_0 = 10$, $f_0 = 100$, then at what time will FP stop buying items? And at what time will FP be buying the maximum number of items?

If there isn't enough time to finish:

- Take an extra lecture if there is enough to do
- Can leave .6 as a practice problem



(image from xkcd - comic #226)

Lesson 21

Modelling with Second-Order ODEs I

Objectives

- Bla

Motivation

40 Here are some facts about laptop keys:

(da) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.

(di) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.

(fo) On average, a person exerts the force of 42 N with one finger on a key.

(gr) Gravity is much weaker than the spring that keeps the key in place.

(hl) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").

(lo) Keys last 50 million presses on average.

(ve) Keys can only move vertically.

40.1 Model a laptop keypress.

40.2 What happens if the damping system of the key is broken? What happens if the damping system is too strong? How strong should the damping system be?

40.3 What happens to the key when the spring breaks?

.1 should be very quick, since a very (very) similar example was solved in the module.

Lesson 22

Modelling with Second-Order ODEs II

Objectives

- Bla

Motivation

41 Model a ball rolling down a ramp.

Ball rolling

Different approach depending on students.

Students need some challenge and have time:

- Ramp $y = f(x)$ makes it simpler
- Need projection (Linear Algebra) to find gravity force along the ramp at $(x_0, y_0) = \big(x_0, f(x_0)\big)$:

$$\ell = (0, -mg)\cdot(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{\sqrt{1+k^2}}$$

  where $k = f'(x_0)$. So gravity force along the ramp is:

$$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

- Yields second-order ODE

Weaker students with less time:

- Give the formula for gravity force along the ramp:

$$\vec{F}_g = \ell(1,k)\frac{1}{\sqrt{1+k^2}} = -\frac{mgk}{1+k^2}(1,k)$$

Follow-up question:

- Ramp is $y = (x-1)^2$
- Get second-order ODE for ball position
- Will the ball always move to the right? Justify with the ODE.
- Approximate near the bottom of the ramp: $y' \approx 0 \Leftrightarrow \sqrt{1+(y')^2} \approx 1$ and solve the simpler ODE.
- When is this approximation valid? (when ball oscillates back and forth near the bottom)

Lesson 23

Second-Order Linear ODEs with Constant Coefficients I

Objectives

- Bla

Motivation

42 Consider the ODE $y''(t) - 9y(t) = f(t)$.

42.1 Find a complementary solution.

42.2 Find a particular solution for $f(t) = 14e^{-4t}$.

42.3 Find a particular solution for $f(t) = 9e^{-3t}$.

42.4 Find a particular solution for $f(t) = 10\cos(t)$.

Lesson 24

Second-Order Linear ODEs with Constant Coefficients II

Objectives

- Bla

Motivation

43 Consider the ODE $y''(t) - 2y'(t) + 5y(t) = f(t)$.

43.1 Find a complementary solution.

43.2 Find a particular solution for $f(t) = \sin(2t)e^t$.

43.3 Find a particular solution for $f(t) = (4t+2)\sin(2t)e^t$.

44 Consider the ODE $y'' + 3y' = 3t$.

44.1 Find the complementary solution.

44.2 Find a particular solution.

44.3 Find the solution that also satisfies

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

Lesson 25

Analysis of Models with Higher Order ODEs I

Objectives

- Bla

Motivation

45 Consider the second-order ODE:

$$y''(t) - 3y(t) = t\big(2 + \sin(t)\big).$$

45.1 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) > 0$ for $t \geq 0$.

45.2 Assume that $y(0) = a < 0$ and $y'(0) = b$. Give an example of a, b such that $y(t)$ is increasing for $t \geq 0$.

45.3 Assume that $y(0) = 0$ and $y'(0) = b$. Which values of b guarantee that $y(t) < 0$ for all $t > 0$.

Without solution

The goal is to solve this without finding an expression for the solution.
For .2, the idea is to make sure that $y'' < 0$.

Lesson 26

Analysis of Models with Higher Order ODEs II

Objectives

- Bla

Motivation

46 Consider the second-order ODE:

$$\begin{cases} y''(t) + 4y(t) = f(t) \\ y(0) = y_0 \\ y'(0) = 0 \end{cases}$$

46.1 Let $f(t) = 0$ and $y_0 = 1$. Sketch the solution.

46.2 Let $f(t) = 396\cos(20t)$ and $y_0 = 0$. Sketch the solution.

46.3 Let $f(t) = -4\sin(2t)$ and $y_0 = 1$. Sketch the solution.

46.4 Let $f(t) = 0.39\cos(1.9t)$ and $y_0 = 2$. Sketch the solution.

Hint. $\cos(at) + \cos(bt) = 2\cos\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}t\right)$

Goals:

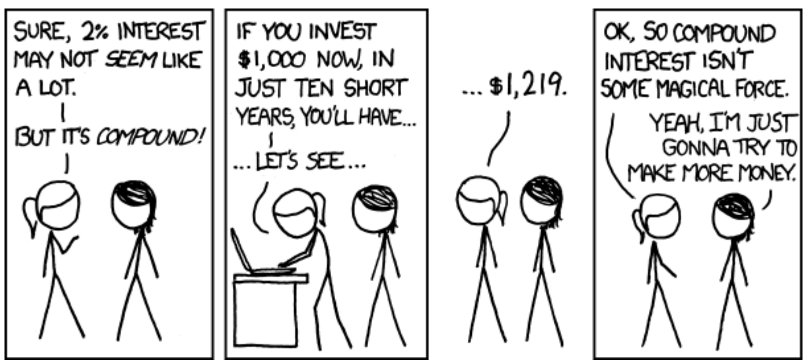
- Learn some different types of behaviour of for second-order ODEs
- Learn how to sketch trig functions combined with linear or other trig functions

.1. Complementary solution!

.2. *Adding* two trig functions: one oscillating slowly and one oscillating quickly

.3. Resonance: t times trig function

.4. Beats: *product* of two trig functions – one oscillating slowly and one oscillating quickly



(image from xkcd - comic #947)

Lesson 27

Solving Difference Equations

Objectives

- Bla

Motivation

47 Consider the difference equation

$$u_{k+1} = 6u_k - 9u_{k-1}$$

47.1 Find the solution that satisfies $u_0 = 1, u_1 = 3$.

47.2 Find the solution that satisfies $u_0 = 1, u_1 = 4$.

48 Consider a difference equation that has solutions $u_k = r^k$ for $r = 2$ and $r = 3$.

We also have the conditions $u_0 = 7$ and $u_1 = 6$.

What is u_{22}?

Lesson 28

Modelling with Difference Equations I

Objectives

- Bla

Motivation

49 Let us expand on the economic example above.

We put a certain amount of money in a savings bank account with an annual interest rate of $p\%$, and compounded at regular periods of α (in years).

Even though we call $p\%$ the annual interest rate, because it is compounded during the year, at the end of the year the effective annual interest rate $p_{\text{eff}}\%$ is actually higher.

Calculate the effective interest rate $p_{\text{eff}}\%$.

The effective annual interest rate is the interest rate with a compounding period of 1 year that gives the same result s the rate of $p\%$ compounded every α years.

Lesson 29

Modelling with Difference Equations II

Objectives

- Bla

Motivation

50 The goal of this question is to try to understand the meaning of average lifespan.

50.1 Consider a small tribe, where the people in there died at the ages:

42, 56, 46, 52, 5, 103, 47, 67, 67, 85, 57, 42, 47, 67, 46, 42, 5, 46, 57, 42.

What is the average lifespan of this tribe's population?

50.2 Consider another small tribe, where people recorded their lifespans differently. Below is a table with the percentage of the population that died at each age:

Percentage of population	2%	5%	9%	9%	16%	22%	37%
Age at death	98	82	71	66	61	53	48

What is the average lifespan of this tribe's population?

Pre-class question

The goal of this question is to prepare for calculating the average lifespan in the next page.

Not to do in lecture. Assign to students to solve at home before.

Some hints:

- individual dying during season $k \Longleftrightarrow$ lifespan $= k$ seasons
- From previous two exercises, deduce that: average lifespan = expected value of lifespan $\ell = E$:

$$E = \sum_{k=0}^{\infty} k\ell(k)$$

- $\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$ for $|r| < 1$.
- End result should be $\frac{1}{\mu}$.

51 Given a population with

- $\mu =$ probability that an individual will die between two seasons.

51.1 Define the following quantity

- $P(k) =$probability that an individual born at season 0 is alive at the beginning of season k.

Find a model for $P(k)$.

51.2 What is the probability of the individual dying during the k^{th} season?

51.3 What is the average lifespan of an individual in this population?

Lesson 30

Modelling with Difference Equations III

Objectives

- Bla

Motivation

52 Consider a population of special rabbits. Once a pair of rabbits is born, they grow and one year later they are still immature. But two years after they are born they give birth to another pair of rabbits.

Model this population of rabbits.

If there is time, students should show that the Fibonacci sequence does indeed match the number of rabbits.

53 Consider another population of rabbits. This is the lifecycle of a pair of rabbits:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pairs of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Model this population of rabbits.

Students might try to find a pattern. It is possible, but very difficult. Hint: Use a system of difference equations.

In core exercise 56, the students are asked to prove the formula.

Lesson 31

Analysis of Difference Equations I

Objectives

- Bla

Motivation

54 Consider the following difference equation:

$$u_{k+1} = a(u_k - b)$$

54.1 What is the equilibrium solution?

54.2 Are there 2-periodic solutions? I.e. satisfying

- $v_0 = v_2 = v_4 = v_6 = \cdots$
- $v_1 = v_3 = v_5 = v_7 = \cdots$
- $v_0 \neq v_1$

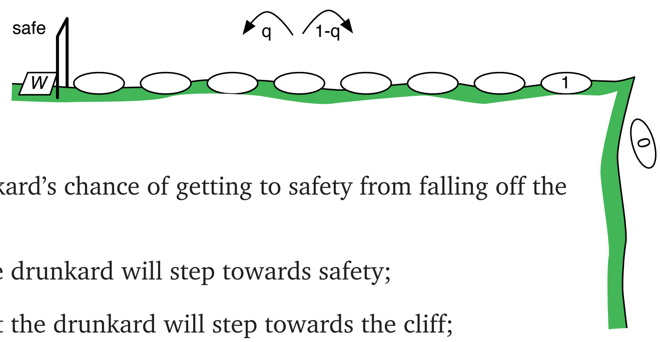
54.3 What happens to the solutions for different values of a?

54.4 What happens to the solutions for different values of b?

- In the calculations for .2, there is a step that involves a division by $(1-a^2)$, so it can only be done for $a \neq \pm 1$.
- The final result for .2 is:

$$\begin{aligned} a \neq \pm 1. &\quad \text{periodic} \Rightarrow u_0 = \frac{ab}{a-1} && \Rightarrow 1 \\ a = 1. &\quad \text{periodic} \Rightarrow b = 0 && \Rightarrow 1 \\ a = -1. &\quad \text{periodic} \Rightarrow b = 0 && \Rightarrow 2 \end{aligned}$$

55 Consider a drunkard that is walking randomly near a cliff.



Consider this model for the drunkard's chance of getting to safety from falling off the cliff:

- q is the probability that the drunkard will step towards safety;
- $1-q$ is the probability that the drunkard will step towards the cliff;
- p_n = probability that the drunkard will get to safety if he is in step number n;
- The drunkard will stop moving if he gets to safety (step W) or if he falls out of the cliff (step 0);
- $p_n = qp_{n+1} + (1-q)p_{n-1}$.

55.1 Is p_n increasing or decreasing?

55.2 What is p_0? What is p_W?

55.3 Let $q = \frac{1}{2}$. What is $p_{W/2}$? Is p_n symmetric around $n = \frac{W}{2}$?

55.4 Let $q > \frac{1}{2}$. Is $p_{W/2} > \frac{1}{2}$? Is $p_{W/2} < \frac{1}{2}$?

55.5 How do solutions for $q = \alpha$ and $q = 1-\alpha$ compare?

Question .3 is purposefully ambiguous about symmetry. What kind of symmetry is there? Is there any?

Lesson 32

Analysis of Difference Equations II

Objectives

-

Motivation

Finish core exercise 55. Then continue with rabbits.

56 Consider a population of rabbits with the following lifecycle:

(year 0) Born

(year 1) Immature (no babies)

(year 2) Young Adult (1 pair of babies)

(year 3) Adult (1 pair of babies)

(year 4) Old (no babies)

(year 5) Die

Consider the definitions:

- We start with 1 pair of newborn rabbits in year 0;
- r_n = number of pairs of rabbits alive during year n;
- i_k = number of immature pairs;
- y_k = number of young adult pairs;
- a_k = number of adult pairs;
- o_k = number of old pairs.

56.1 Show that $b_k = b_{k-2} + b_{k-3}$.

56.2 Show that $y_{k+1} = o_k + o_{k+1}$.

56.3 Show that $r_n = r_{n-2} + r_{n-3}$.

$$r_k = r_{k-1} - o_{k-1} + b_k$$

On the other hand,

$$\begin{aligned} r_{k-1} - o_{k-1} &= b_{k-2} + 2y_{k-2} + a_{k-2} \\ &= r_{k-2} + y_{k-2} - o_{k-2} \end{aligned}$$

and

$$\begin{aligned} b_k &= b_{k-2} + b_{k-3} \\ &= y_{k-3} + a_{k-3} + b_{k-3} \\ &= r_{k-3} - o_{k-3} \end{aligned}$$

So we have

$$\begin{aligned} r_k &= r_{k-2} + \cancel{y}_{k-2} - \cancel{o}_{k-2} + r_{k-3} - \cancel{o}_{k-3} \\ &= r_{k-2} + r_{k-3} \end{aligned}$$