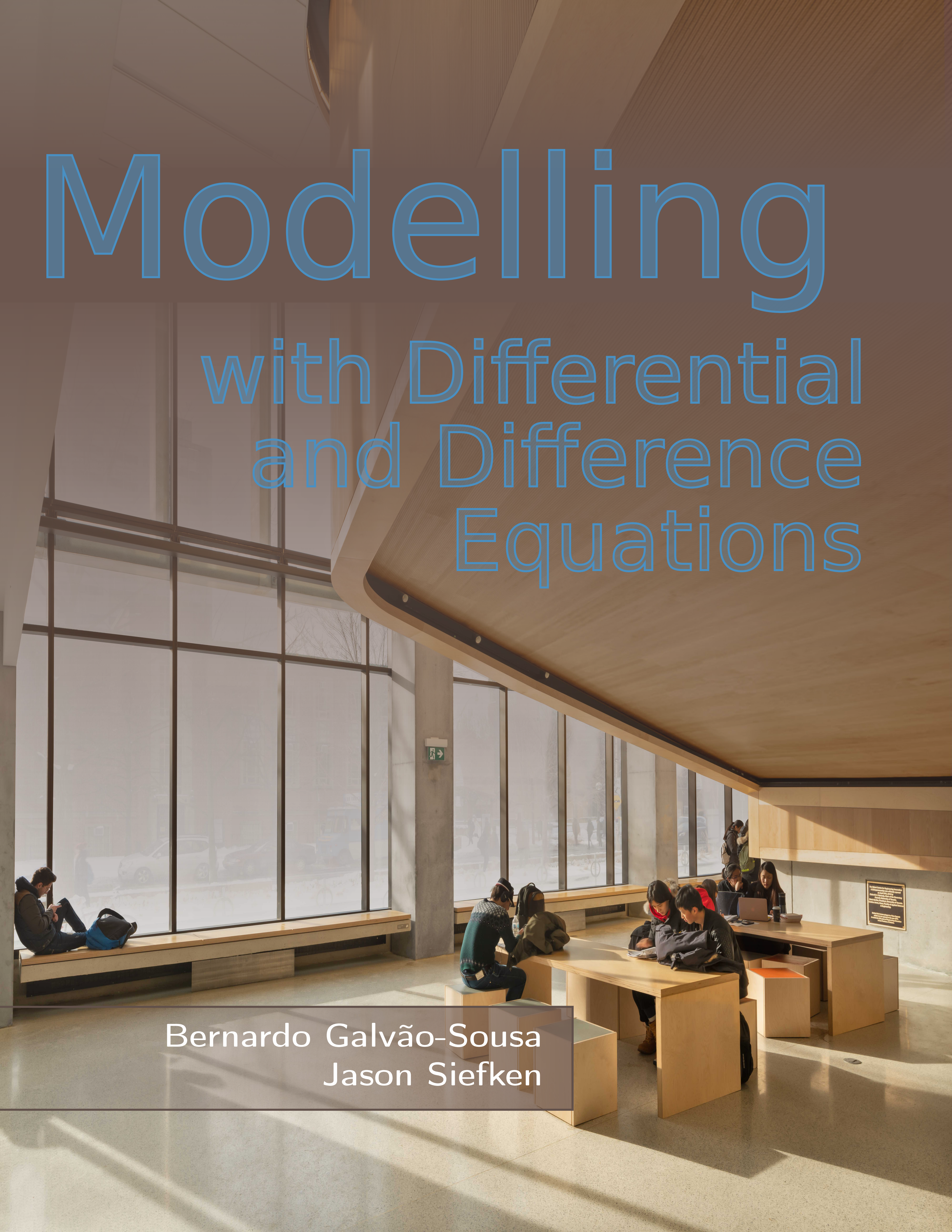


Modelling with Differential and Difference Equations

Bernardo Galvão-Sousa
Jason Siefken

Inquiry Based Modelling with Differential and Difference Equations

About the Document

This document is a mix of student resources, student projects, problem sets, and labs. A typical class day looks like:

1. **Preparation by students.** Students prepare for lecture by watching a short video and solving a short quiz.

2. **Introduction by instructor.** This may involve giving a broader context for the day's topics, or answering questions.

3. **Students work on problems.** Students work individually or in small groups on the prescribed problem. During this time the instructor moves around the room addressing questions that students may have and giving one-on-one coaching.

4. **Instructor intervention.** If most students have successfully solved the problem, the instructor regroups the class by providing a concise explanation so that everyone is ready to move to the next concept. This is also time for the instructor to ensure that everyone has understood the main point of the exercise (since it is sometimes easy to do some computation while being oblivious to the larger context).

   If students are having trouble, the instructor can give hints to the group, and additional guidance to ensure the students don't get frustrated to the point of giving up.

5. **Repeat step 2.**

Using this format, students are working (and happily so) most of the class. Further, they are especially primed to hear the insights of the instructor, having already invested substantially into each problem.

This problem-set is geared towards concepts instead of computation, though some problems focus on simple computation.

Contents

Mathematical Modelling

In this section, we study some strategies to model problems mathematically in an effective manner.

We also provide a structure to modelling problems by breaking them in small parts:

1. Define the problem
2. Build a mind map
3. Make assumptions
4. Construct a model
5. Analysis of the model
6. Writing a report

Defining the Problem

In this module you will learn

- how to define a problem mathematically.

The first step is to define the problem we want to solve.

To do this, we should start from the end!

We need to decide on what kind of mathematical object we will use in the end to show that we solved the problem we were tasked with.

Once this is done, we can define the problem mathematically.

Example. Your team was tasked with optimizing the layout of an airport.

The team decided to define:
- $T =$ the total time (in minutes) necessary by the average person to walk from their airport transportation (taxi, train, bus) to their gate, disregarding the time spent in security or immigration.

At the end of the project, to show that the team did find a good layout for the airport, the team will show that the new layout reduces the value of T.

Once this decision is made, the problem to solve (or improve) becomes clear:
- Minimize T

There will probably be some constraints, which will be studied in Module 4.

Practice Problems

1 For each part, what "mathematical object" would you use to communicate that you have solved or improved the problem? Then define the problem mathematically.

(a) Help the city of Toronto choose the best recycling system.

(b) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

(c) Create a mathematical model to rank roller coasters according to thrill factor.

(d) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

(e) Is it better to buy or rent?

i. Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

ii. Does the criteria you used to evaluate the previous question change if the question is whether to buy a bicycle or use Bike Share Toronto?

1 Elevator problem at theBigCompany

You are hired by theBigCompany to help with their “elevator problem”.

This is the email you received:

——— Forwarded Message ———

Date: Mon, 16 September 2019 21:41:35 + 0000
From: CEO <theCEO@theBigCompany.ca>
To: Human Resources <hr@theBigCompany.ca>
Subject: they’re still late !?&!

Hey Shophika!

I still get complaints about staff being late, some by 15 minutes.
With the staff we have, that’s about one salary lost.
Again the bottleneck of the elevators seems to be the problem.
Can you suggest solutions?

Thanks, the CEO

What mathematical object would you use to convince the CEO that you have solved or improved the problem?

Teamwork.
With your team, you must decide on one answer and be prepared to report on your decision and the reason for your choice.

2 The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1.

Figure 1: Extension plans for Toronto subway line.

2.1 What "mathematical object" would you use to communicate that to the Mayor that this line is optimal (or sub optimal) ?

2.2 Define the problem mathematically.

Building a mind map

In this module you will learn

- How to create a mindmap.

Mind Map

A mind map is a tool to visually outline and organize ideas. Typically a key idea is the centre of a mind map and associated ideas are added to create a diagram that shows the flow of ideas.

Example.

Let us focus on the question: "What is the best recycling system for Toronto?"
Then we can think of many different definitions for what the word "best" means:

- The system that gets the most participation from the population, which can be measured by the fraction of the Toronto households participating in recycling;
- The system that costs the least amount of money for the city. How can this be measured?
- The system that processes the most amount of recyclables.

In Figure 2, we focus on the definition of "best", with these three possible definitions branching off to be further explored.

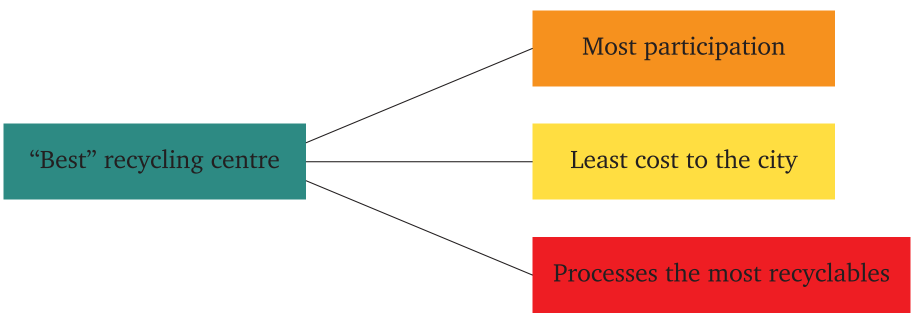


Figure 2: An example of a simple mind map.

From here, we can focus our attention on one of the branches at a time.

Example.

Let's think about the least-cost option first.
We probably can't determine how much any recycling program costs without knowing more about the recycling program, so a good place to start is to ask the question "What kinds of recycling programs exist?"
If we aren't familiar with different types of recycling, we might need to do some research to see what kinds of programs exist.
A possible next step on your mind map for the least-cost approach could be the one shown in Figure 3.

Important. There is free online software to help creating a mind map. One such is FreeMind (http://freemind.sourceforge.net).

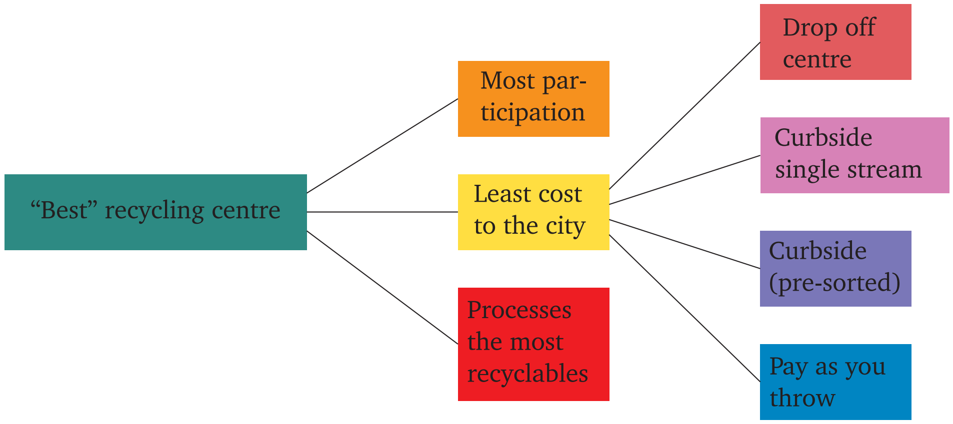


Figure 3: Next step of a mind map.

Practice Problems

1 Expand the mind map from figure 3 by focusing on the other two approaches:

 (a) Most participation

 (b) Processes the most recyclables

2 For each part, create a mind map. Focus on the same approach you had for question 1 from Module .

 (a) Help the Canadian Institute of Health Information (CIHI) estimate how significant the outbreak of illnesses will be in the coming year in Canada.

 (b) Create a mathematical model to rank roller coasters according to thrill factor.

 (c) Gas stations offer different prices for gas. I would like to create an app that finds the best gas station to go to. What should "best" mean?

 (d) The mayor of Toronto wants to extend the subway line with a new blue line as in Figure 1. Is it optimal?

 (e) Is it better to buy a car or rent Zipcar, Enterprise Carshare, or Car2go?

3 Consider the elevator problem from question 1.

Your team decides that the mathematical object you will use to show the CEO that you solved or improved the problem is

- $R =$ the sum in minutes by which every employee is late.

Note that employees that are on time count for 0 minutes (not a negative amount of minutes).

Create a mind map for the question: How can R be minimized?

4 The city of Toronto decided to tear down the Gardiner expressway. While the demolition is taking place, several key arteries are closed and many intersections are bottled. At peak times, a police officer is often posted at this intersection to optimally control the traffic lights.

4.1 What "mathematical" meaning can we give to the word optimal?

4.2 Create a mind map for this problem.

Making assumptions

In this module you will learn

- that we need to make assumptions to be able to create a model
- how to strike a balance between accuracy and solvability

Real problems are complex, so when modelling a real problem mathematically, we must make some assumptions.

The assumptions that we make will affect the problem we are solving and its difficulty, so we need to strike a balance between:

- accuracy – the fewer assumption the better, and
- solvability – the more assumptions the better.

Many assumptions follow naturally when building a mind map.

When figuring which assumption to make, keep in mind the key-factors of the problem and find data when available (usually online). If not available, measure data when possible, and if it's not possible, make a reasonable assumption on what the data might look like.

Another thing to keep in mind are *time constraints*. Whether in a class, test, or working in a project, there will be deadlines. Your assumptions should take time constraints into consideration.

Example.

AN EXAMPLE, PROBABLY BASED ON THE RECYCLING.

Practice Problems

1 For each part, you are required to make an estimate for some quantity. Make assumptions and justify them in order to solve the problem.

(a) What is the number of piano players in Toronto? *(Fermi problem)*

(b) How many linear km of roads are there in Toronto?

(c) How much salt the city of Toronto needs for its roads during the Winter?

(d) The skating season in Canada is shortening: What are the key-factors determining its length?

Module 3: Making assumptions

5 Consider the elevator problem from question 1.

We now give you some technical details about theBigCompany:

- The company occupies the floors 30–33 of the building Place Ville-Marie (in Montréal).
- Personnel is distributed in the following way:
  - 350 employees in floor 30,
  - 350 employees in floor 31,
  - 250 employees in floor 32,
  - 150 employees in floor 33.

Note. Even though these details are fictional, the numbers respect the building code.

Focus on a **few** parameters and variables. State hypotheses.

6

6.1 With your team, decide on what kind of information you would need to have to be able to solve this problem.

6.2 Find the relevant information about the elevators (search the internet, by experimentation). Check the reliability of the data you found.

6.3 For the relevant information that you cannot obtain, make assumptions. These assumptions should be reasonable and you should be able to justify them.

Construct a model

In this module you will learn

- how to build a model based on the previous steps

This is the part of the modelling where we connect all that we have done so far: the problem we defined, the mind map, the assumptions, and all the variables and parameters in a mathematical model to answer the "mathematical" problem defined in Step A.

This usually means writing down mathematical equations, constructing a graph, analyzing a geometric figure, or do some statistical analysis.

Example. Your team is tasked with finding the best recycling centre (we looked at this example in Step B) and your team has chosen to minimize the cost to the city by using drop off centres.

As part of modelling process, your team has made the following assumptions/measurements:

- People would be willing to pay \$2.29 to recycle per month or \$0.53 per week
- People would make bi-weekly trips to the centre
- Gasoline costs around \$1.26 per litre
- On average a passenger car needs 10 litres per hundred kilometres

This means that the (one-way) distance people are willing to travel every week to the drop-off centre is

$$d = \frac{1}{4.3 \text{ trips/month}} \cdot \frac{\$2.29/\text{month}}{(\$1.26/\text{L}) \cdot (0.1 \text{ L / km})} = 4.2 \text{ km/trip}.$$

This should help us figure out the best way to place the drop-off centres:
The Mathematical model might look like this

- Maximize (number of people within a 4.2 km radius of a drop-off centre)
- subject to a certain number of drop-off centres (given by the city budget)

Sometimes, the mathematical tools necessary to tackle the problem are clear, but often they are not. In those cases it may be helpful to analyze some simple cases.

Practice Problems

1 For each part, create a model to answer the question. Remember all the previous steps.

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

7 With the same details as before in 5, write down a mathematical model for this problem.

Model Assessment

In this module you will learn

- how to analyze a model to check whether it makes sense

At this point, you have defined a problem statement, and a mind map to help you decide how to approach the problem. You have made assumptions and made note of them and justified them. You finally created a model to solve the problem.

The next step is to analyze the model.

There are two types of analysis:

Superficial assessment. Are the units correct? Are the variables and parameters of a reasonable magnitude? Does it behave as expected? Does it make sense?

In-depth assessment. Once the superficial assessment is verified, we need to understand the model at a deeper level.

What are the model's strengths? What are its weaknesses?

When you change the inputs of the model, how do the outputs change? This is called *s*ensitivity analysis.

Next is a simple example adapted from [**?**].

Example. Modelling the flu

History of the project:

- Split population into two classes: *infected* and *not infected*
- Assume that each infected person infects R number of non infected people every b days
- Define $I(n) =$ number of infected people after n days
- The two previous points imply $\quad I(n \cdot b) = R \cdot I(n)$
- We can then conclude that $\quad I(nb) = (1+R)^n I(0)$ (why?)

After plotting the resulting function $I(n)$ (click or follow the QR code on the right), we can assess our model:

Strengths:

- After two days ($b = 2$), there are 6 infected people, so it is following our assumption
- The number of infected people increases faster and faster as expected
- The disease spreads at a constant rate. Also on Desmos, check the infection rate $\dfrac{I(n+b)}{I(n)}$
- We could find an explicit formula for the number of infected individuals $I(n)$

Weaknesses:

- The model is too simple, so it doesn't model the spread of the flu accurately
- The model an exponential rate of infection, which is not possible for very long
- The model predicts that eventually the disease will spread to everyone
- The model assumes that there are only two types of people: infected and susceptible. Do people recover from the disease?

After assessing the model, if time allows, it is important to re-think the model and the assumptions made.

Practice Problems

1 Assess the models created in question **??**:

(a) You want to open a piano store in Toronto, where should you open it?

(b) There was a big snow storm in Toronto and the roads need cleaning. How should the city deploy its snow plowers?

(c) The city of Toronto wants to deactivate the Pickering nuclear power plant in favour of renewable power sources. What is the best way to create the same amount of electricity using only renewable sources in the GTA?

(d) Loblaws wants to start an online food delivery service. How should they do it?

(e) The city airport (YTZ) built a tunnel to access the island airport from the city. Before that, they used a ferry. Was building the tunnel a good decision?

8 Continuing on the elevator problem, let us think of this model for the problem.

Facts:

- Loading time of people at ground floor = 20 s
- Speed of uninterrupted ascent/descent = 1.5 floors/s
- Stop time at a floor = 7 s
- Number of elevators serving floors 30–33 = 8
  (these elevators serve floors 23-33 = 11 floors)
- Maximal capacity of elevators = 25 people

Assumptions:

- Personnel that should start at time t, arrive uniformly in the interval $[t-30, t-5]$ in minutes
- First arrived, first served
- During morning rush hour, elevators don't stop on the way down
- Elevators stop only at half the floors they serve
- Elevator failures are neglected
- Mean number of people per floor is equal to the mean number of people per floor of the BigCompany
- Elevators are filled, in average, to 80% of their capacity

Model:

- Mean number of people per floor $= d = \dfrac{350+350+250+150}{4} = 275$ people / floor
- Number of people on floors served by elevators (11 floors) $= N = d \cdot 11 = 3025$ people
- Time Δt of one trip

$$\Delta t = \boxed{\begin{array}{c}\text{loading time on}\\ \text{ground floor}\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ \text{ground} \to 33\end{array}} + \boxed{\begin{array}{c}\text{time of flight}\\ 33 \to \text{ground}\end{array}} + \boxed{\begin{array}{c}\text{stop time to}\\ \text{6 of the 11 floors}\end{array}} = 106 \text{ s}$$

- Number of trips necessary per elevator $= n = \dfrac{3025}{20 \cdot 8} \approx 19$ trips
- Time necessary to carry the staff of the BigCompany $= \boldsymbol{t} = \dfrac{19 \cdot 106}{60} = 33$ minutes

Your task is to assess this model. Be ready to report on your assessment.

Putting it all together

In this module you will learn

- how to put all that you have done together into a well structured report

This is the final stage of the modelling project.

By now, you have started with a mathematically defined problem, with some assumptions, and you have created a mind map to help you navigate the problem. You have also constructed a model and assessed it to make sure it is sound.

All that we have left is to put all this work together into the form of a report.

The report should consist of two parts:

1. **Summary.** Should be at most one page long, and contain a statement of the problem, a brief description of the methods chose to solve it, and some final results and a conclusion. In this part of the report, you should keep mathematical symbols to a minimum, so the reader gets an idea of what to expect in the remainder of the report without getting bogged down in unfamiliar mathematics.
2. **In-depth report.** This is where the details go in. It should start with an introduction to the problem assuming that the reader is not aware of it. It should then be structured according to the steps we did before:
   - Optionally, you can include a mind map with a description of how it guided the whole process
   - Assumptions and variables in the model
   - The model described in detail
   - The solution process
   - The assessment of the model
   - A conclusion, with a description of the results

Example. You can find the report from the winning team of the 2019 M_3C competition in appendix 6.1.

Definition

Textbook

Objectives

- Bla bla bla

Motivation

Solutions

In this module you will learn

- the difference between a solution and an integral curve

Assume that we have found a differential equation that models a situation. Often the goal is to figure out what happens, so we usually attempt to either solve the differential equation and obtain a solution or to find an approximation for the solution.

In this module, we will discuss solutions in more detail.

Solution. Given a differential equation, a ***solution*** is a differentiable function that satisfies the differential equation.

Example. Consider the differential equation

$$t\frac{du}{dt} = u + t^2\cos(t).$$

Then the function

$$u(t) = t\sin(t)$$

is a solution, because

$$t\frac{du}{dt} = t\big(\sin(t) + t\cos(t)\big) = t\sin(t) + t^2\cos(t) = u + t^2\cos(t).$$

Integral curve. We can represent all the solutions geometrically as an infinite family of curves. These curves are called **integral curves**.

Example. Consider the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

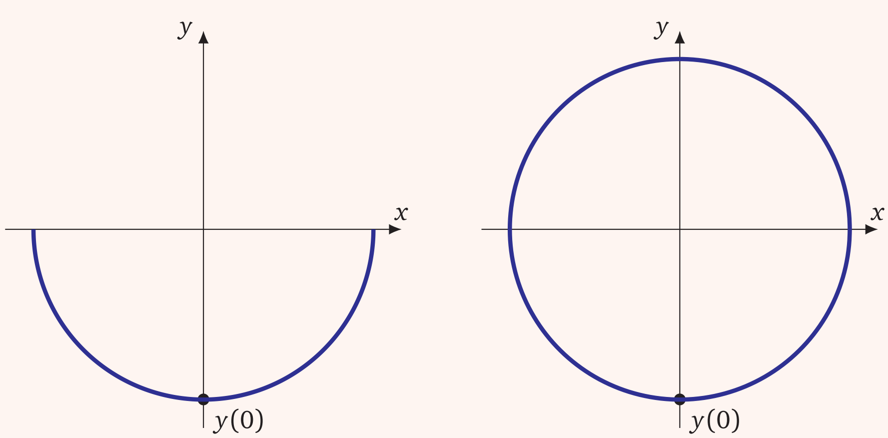
Then, we can check that curves of the form $x^2 + y^2 = C$ satisfy this differential equation.

This gives us the solution

$$y(x) = -\sqrt{9 - x^2}.$$

However, the integral curve for this initial-value problem is the curve

$$x^2 + y^2 = 9$$



Solution of the initial-value problem   Integral curve for the initial-value problem

Practice Problems

1 Check that curves of the form $x^2 + y^2 = C$ satisfy the differential equation $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

2 Is the piecewise-defined function

$$y(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$?

3 Consider the differential equation

$$y^{(4)} - 8y^{3)} + 26y'' - 40y' + 25y = 0.$$

(a) Is $y = 4e^{2x}\sin(x)$ a solution?

(b) Is $y = -8xe^{2x}\cos(x)$ a solution?

(c) For the two function above, if they are solutions, what are initial conditions of the form

$y(0) =$

$y'(0) =$

$y''(0) =$

$y'''(0) =$

that the solution satisfies?

4 Consider the functions

$$\begin{array}{ll} f(x) = 3x + x^2 & g(x) = e^{-7x} \\ h(x) = \sin(x) & j(x) = \sqrt{x} \\ k(x) = 8e^{3x} & \ell(x) = -2\cos(x) \end{array}$$

Match each differential to one or more functions which are solutions.

(a) $y' = 3y$

(b) $y'' + 9y' + 14y = 0$

(c) $y'' + y = 0$

(d) $2x^2y'' + 3xy' = y$

5 Consider the differential equation $u' = -2(u - 10)$.

(a) Check that the curves of the form $u = 10 + Ce^{-2t}$ satisfy the differential equation.

(b) Sketch one solution of the differential equation.

(c) Sketch all the integral curves for the differential equation.

(d) What is the difference between a solution passing through the point $(1, 20)$ and an integral curve passing through the same point?

6 Consider the differential equation $y'\big(3y^2 - 1\big) = 1$.

(a) Check that the curves of the form $y^3 - y = x + C$ satisfy the differential equation.

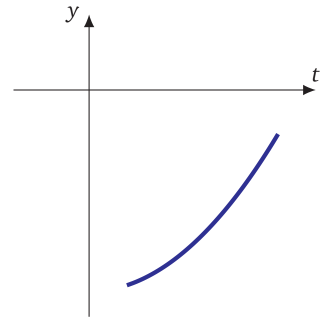
(b) Sketch the solution of the differential equation that passes through $(1, 1)$.

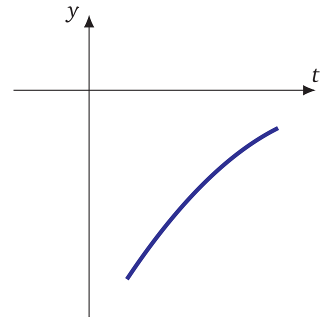
(c) Sketch the integral curve for the differential equation that passes through $(1, 1)$.

(d) What is the difference between a solution passing through the point $(1, 1)$ and an integral curve passing through the same point?

(e) Repeat (b)–(d) with the points $(1, 0)$ and $(1, -1)$ instead of $(1, 1)$.

7 Consider the ODE $y'(t) = \big(y(t)\big)^2$. One of these two graphs **cannot** describe the solution. Which one?



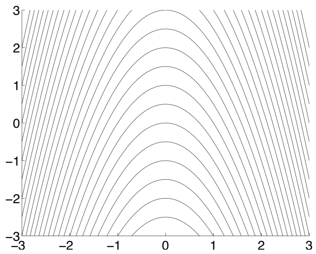


8 We seek a first-order ordinary differential equation $y' = f(\boldsymbol{y})$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is concave up if } y < 1 \\ y(x) \text{ is concave down if } y > 1 \end{cases}$$

Write down or graph a function $f(y)$ that would produce such solutions.

9 Which of these shows solutions of $y' = (x-1)(x+1) = x^2 - 1$?



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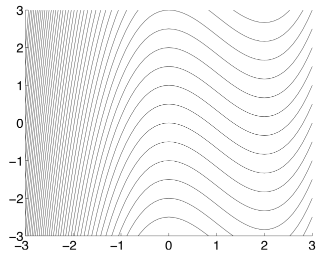

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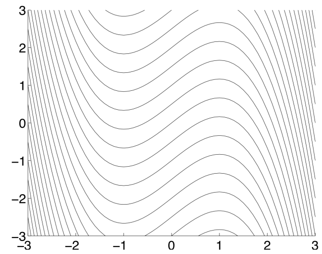

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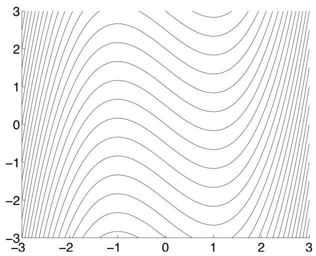

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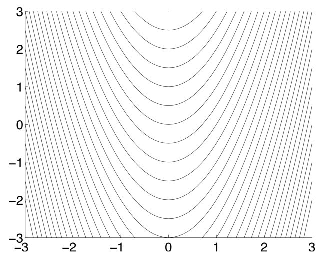

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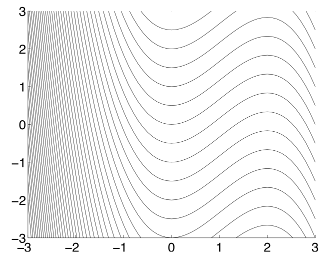


F

10 We seek a first-order ordinary differential equation $y' = f(x)$ whose solutions satisfy

$$\begin{cases} y(x) \text{ is increasing if } x < 2 \\ y(x) \text{ is decreasing if } 2 < x < 4 \\ y(x) \text{ is increasing if } x > 4 \end{cases}$$

Write down or graph an $\boldsymbol{f(x)}$ that would produce such solutions.

11 Consider the ODE $y'(t) = \big(y(t)\big)^2$. Which of the following is true?

11.1 $y(t)$ must always be positive

11.2 $y(t)$ must always be negative

11.3 $y(t)$ must always be decreasing

11.4 $y(t)$ must always be increasing

12 Consider the differential equation $2xy' = y$.

12.1 Check that the curves of the form $y^2 + Cx = 0$ satisfy the differential equation.

12.2 Sketch one solution of the differential equation.

12.3 Sketch all the integral curves for the differential equation.

12.4 What is the difference between a solution passing through the point $(1, -1)$ and an integral curve passing through the same point?

Slope Fields

In this module you will learn

- what is a slope field
- how to sketch a slope field
- to interpret a slope field

As we saw in the previous module, once we have found a differential equation that models a situation, we often want to figure out what happens to the solution.

In this module, we will focus on getting an idea of the solutions and integral curves using what is called a **slope field**.

Slope field. Consider the equation $y' = f(x, y)$. If we evaluate $f(x, y)$ over a rectangular grid of points, and we draw an arrow at each point (x, y) of the grid with slope $f(x, y)$, then the collection of all the arrows is called a ***slope field***.

We can sketch Slope Fields with Wolfram Alpha.
For a differential equation $\frac{dy}{dx} = f(x, y)$, we need to input

- Vector Field: $(1, f(x, y))$.

http://www.wolframalpha.com/input/?i=slope+field

Example. Let us take an example from the previous module.

Consider the initial-value problem

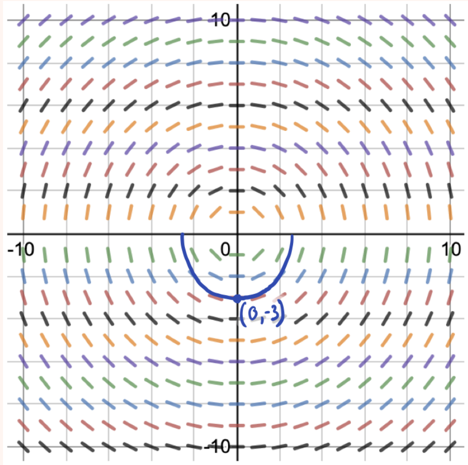
$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{x}{y} \\ y(0) = -3 \end{cases}$$

We can use this definition to sketch the slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

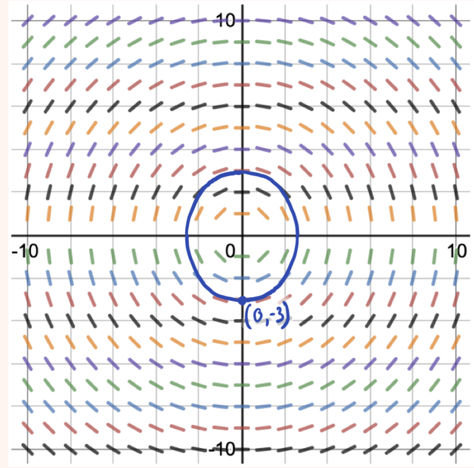
We now sketch this slope field with Desmos:

https://www.desmos.com/calculator/scmz6ps0or

Now notice that the arrows have the slope of a solution. This means that solutions will be tangent to the arrows, so we can *roughly* trace the solution by following the arrows.
Below, we did just that starting with the point $(0, -3)$.



approximated solution



approximated integral curve

Important. Remember that this gives us only an approximation of the solution and integral curve. From the approximation, we can tell that the solution seems circular, but we still need to show that it is so.

Video.

- https://youtu.be/MI2xCwBekX4
- https://youtu.be/8Amgakx5aII

Practice Problems

1 Use Wolfram Alpha, Desmos, or another software to sketch the slope field for the following differential equations. Then roughly trace different solutions.

(a) $y' = 2y - x$

(b) $y' = xy$

(c) $y' = \cos(y)$

(d) $y' = \frac{1}{2} + \cos(y)$

(e) $y' = 1 + \cos(y)$

(f) $y' = 2 + \cos(y)$

(g) $y' = \sin(xy)$

(h) $y' = \tan(x + y)$

2 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where

$$f(x, y) = \begin{cases} -x & \text{if } x < 1 \\ y & \text{if } x \geq 1 \end{cases}$$

3 Sketch a slope field for the following differential equation

$$y' = f(x, y)$$

where the function $f(x, y)$ satisfies all of the following properties:

(a) $f(x, y)$ is continuous

(b) $f(x, y) > 0$ when $x > 1$ and $y > 1$

(c) $f(x, y) < 0$ when $x < -1$ and $y < -1$

(d) $f(x, y)$ depends only on x when $x < -1$ and $y > 1$

(e) $f(x, y)$ depends only on y when $x > 1$ and $y < -1$

4 (a) On the slope field from the previous problem, show that there must exist a smooth continuous curve with horizontal lines.

(b) Show that the curve divides the (x, y) plane in two parts.

5 Consider a differential equation

$$y' = f(x, y)$$

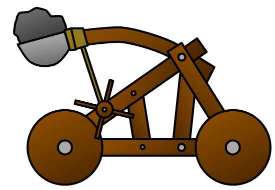
where the solutions satisfy

$$\lim_{x \to \infty} y(x) = 1.$$

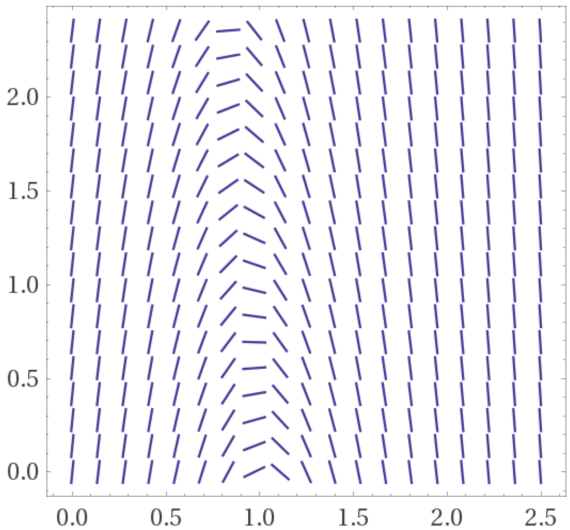
(a) What property must the slope field satisfy?

(b) Sketch a possible slope field for this differential equation.

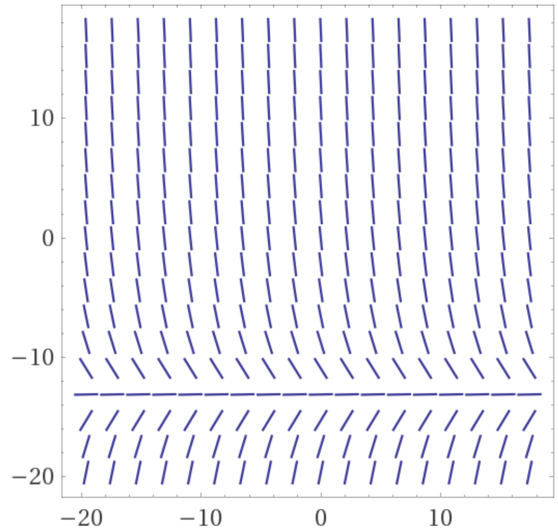
13 A catapult throws a projectile into the air and we track the height (in metres) of the projectile from the ground as a function $y(t)$, where t is the time (in seconds) that elapsed since the object was launched from the catapult.



Then, the slope fields for $y(t)$ and $y'(t)$ are shown below:



Slope field for $y(t)$



Slope field for $y'(t)$

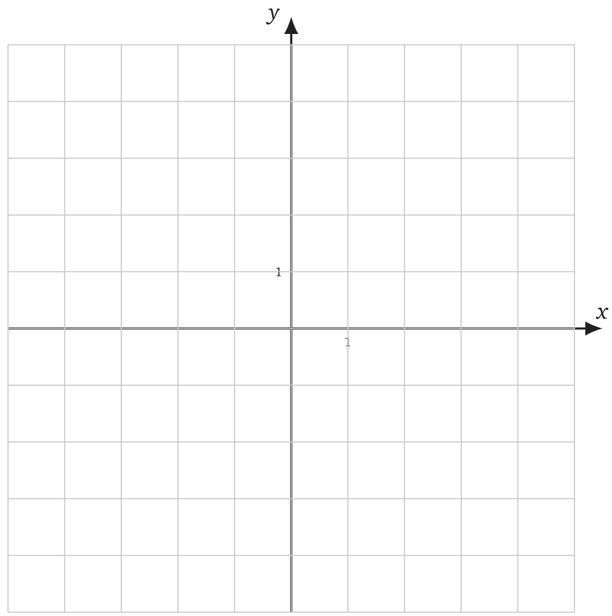
(These slope fields were created using WolframAlpha)

13.1 On the slope field, sketch a *possible* solution.

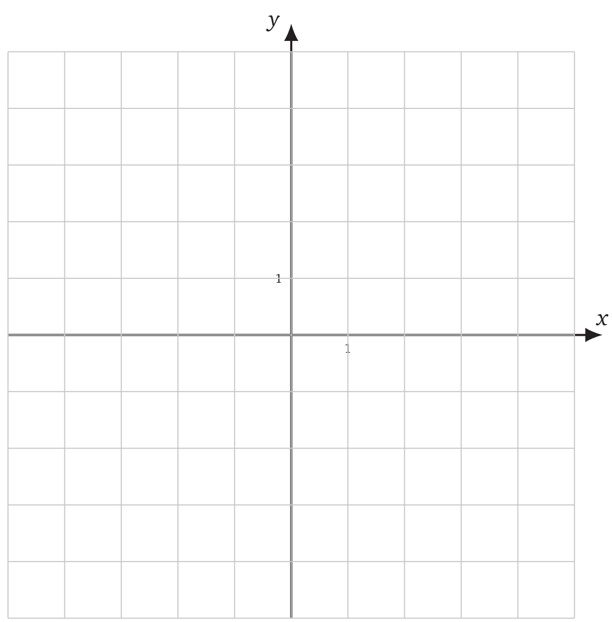
13.2 Consider the graph of $y(t)$. Does it form a parabola? Justify your answer.

14 Sketch the slope field for the following differential equations.

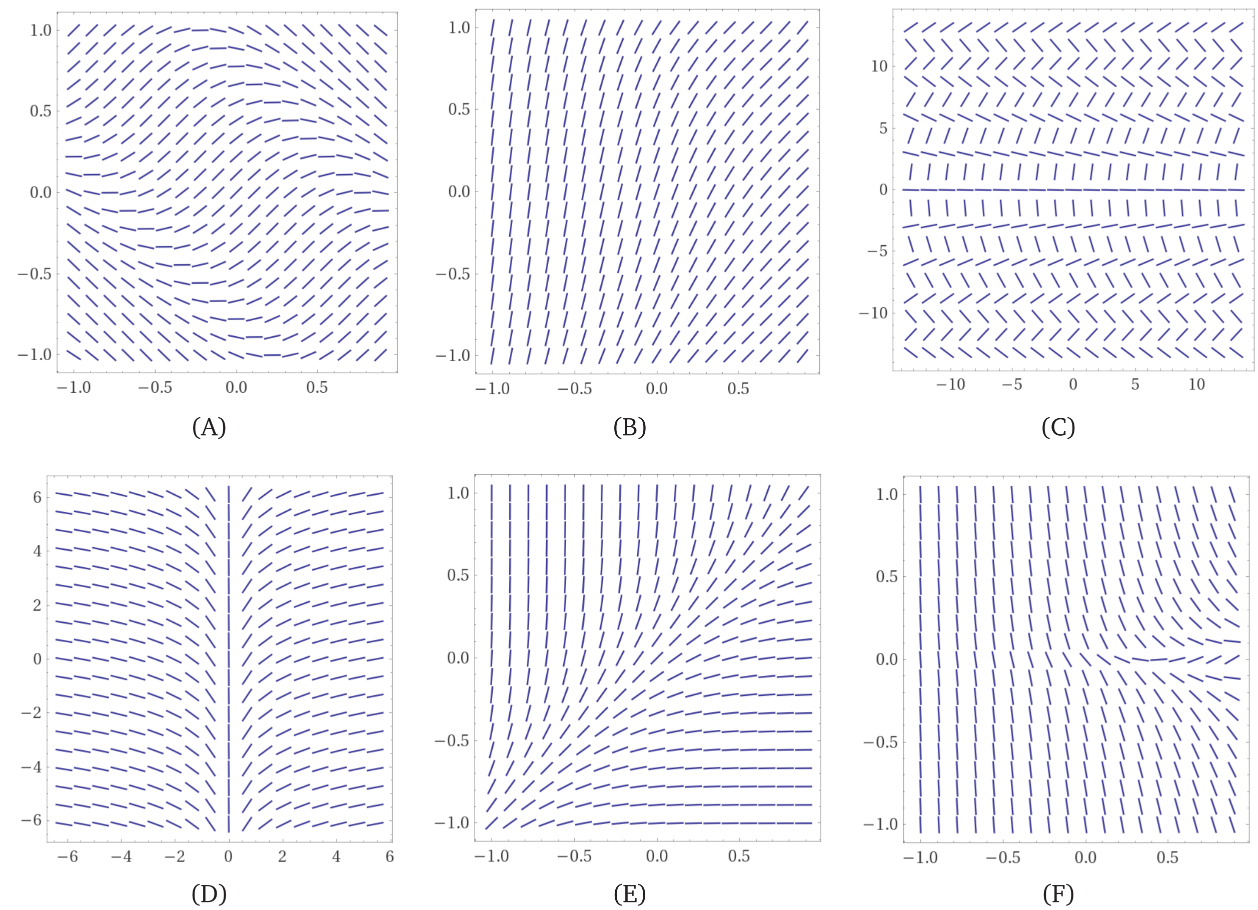
14.1 $y' = x$



14.2 $y' = y^2$



15 Consider the following slope fields:



(These slope fields were created using WolframAlpha)

15.1 Which slope field(s) corresponds to a differential equation of the form $y' = f(x)$?

15.2 Which slope field(s) corresponds to a differential equation of the form $y' = g(y)$?

15.3 Which slope field(s) corresponds to a differential equation of the form $y' = h(x+y)$?

15.4 Which slope field(s) corresponds to a differential equation of the form $y' = \kappa(x-y)$?

15.5 Which slope field(s) corresponds to a differential equation of the form $y' = 1 + \big(\ell(x,y)\big)^2$?

15.6 Which slope field(s) corresponds to a differential equation of the form $y' = 1 - \big(m(x,y)\big)^2$?

Numerical Methods

Objectives

- Bla bla bla

Motivation

Separable ODEs

Textbook

Objectives

- Bla bla bla

Motivation

Module 11: Separable ODEs

First-Order Linear ODEs

Textbook

Objectives

- Bla bla bla

Motivation

6 2019 M_3C competition report from the winning team

In the folowing pages you can find an abridged version of the full report. The full report can be found at http://uoft.me/modelling-app-report.

MathWorks Math Modeling Challenge 2019

High Technology High School–

Team #12038 Lincroft, New Jersey

Coach: Raymond Eng

Students: Eric Chai, Gustav Hansen, Emily Jiang, Kyle Lui, Jason Yan

MathWorks Math Modeling Challenge Champions

$20,000 Team Prize

MathWorks Math Modeling Challenge

***Note: This cover sheet has been added by SIAM to identify the winning team after judging was completed. Any identifying information other than team # on a MathWorks Math Modeling Challenge submission is a rules violation.

***Note: This paper underwent a light edit by SIAM staff prior to posting.

Substance Use and Abuse

Executive Summary

In recent years, substance abuse has intensified to an alarming degree in the United States. In particular, the rise of vaping, a new form of nicotine consumption, is dangerously exposing drug abuse to a new generation. With the need to understand how substance use spreads and impacts individuals differently, our team seeks to provide a report with mathematically founded insights on this prevalent issue.

We first strove to predict the spread of nicotine use due to both vaping and cigarettes over the next decade. By comparing the spread of nicotine use to an infectious disease, we modified the SIRS epidemiology model to create our adapted SIRI model in which individuals are divided into four compartments: infected (drug users), recovered (users who quit drugs), susceptible (potential drug users), and nonsusceptible (those who will never use drugs). People progress from susceptible to infected to recovered, but may relapse into their old habits, causing them to re-enter the infected population. Birth and death rates of our designated population were modeled with linear equations. We solved a system of differential equations to determine e-cigarette and cigarette use in 2029: 26.63% of the American population will vape and 6.45% will smoke cigarettes. These results align with the expectation that vaping will increase in popularity while cigarette smoking will decline.

Substance abuse is associated with numerous social factors and personal attributes. We incorporated those determinants to create a second mathematical model that computes the probability that an individual will use nicotine, marijuana, alcohol, and unprescribed opioids. A binary multivariate logistic model was used to assess the effects of age, gender, ethnicity, income, parental status, friendship, opinion about school, overall health, weapon possession, and bullying on substance use. To demonstrate our model, we coded and executed a Monte Carlo simulation that created 300 high school seniors with varying attributes. We found that 46.3% of the students would use nicotine, 17.3% would use marijuana, 66.0% would use alcohol, and 0.0% would use opiates.

Substance use has far-reaching implications in personal and societal spheres. It is crucial to rank substances based on their overall impact in order to assess necessary government action regarding drug abuse. To address this issue, we developed a robust metric to rank the effects of nicotine, marijuana, alcohol, and opioid abuse. Our model and ranking considers physical harm, dependence, social harm, and economic impact of the drugs. The former three factors were measured on a scale of 0 to 3 based on psychiatrist surveys. Then economic impact was defined as GDP loss from the decrease in life expectancy caused by drug abuse. After applying risk factors obtained from the amount of people that use each drug, the four substances were ranked. From highest to lowest individual impact, the ranking was opioids, alcohol, cigarettes, and marijuana. From highest to lowest total societal impact, the ranking was alcohol, cigarette, marijuana, and opioids.

The repercussions of substance abuse are reverberating and remain with an individual for life. However, drugs not only severely affect the user but also cause extensive societal harm. Increased understanding of the projected spread and impact of substance abuse, as well as the underlying factors that lead to poor judgment, are needed to optimize measures to restrict consumption. Ultimately, we believe that our models provide novel insight into the nationwide issue of substance use and abuse.

1 Introduction

This section delineates the components of the modeling problem and their objectives. Global assumptions applying to the entire modeling process are also listed.

1.1 Restatement of the Problem

The problem we are tasked with addressing is as follows:

1. Build a mathematical model that predicts the spread of nicotine use due to vaping over the next 10 years. Analyze how this growth compares to that of cigarettes.

2. Create a model that simulates the likelihood that a given individual will use a given substance, accounting for social influence, characteristic traits, and properties of the drug itself. Demonstrate the model by predicting how many students among a class of 300 high school seniors with varying characteristics will use nicotine, marijuana, alcohol, and unprescribed opioids.

3. Develop a metric for the impact of substance use, considering both financial and nonfinancial factors. Use the metric to rank the substances listed in Part II.

1.2 Global Assumptions

1. *The current drug scene remains constant.* We assume that there will be no radical changes in the recreational drug industry, such as new drugs or drug products. This assumption is imperative because attempting to account for unpredictable and volatile factors would make model development virtually impossible.

2. *All vapes count as e-cigarettes.* Some people distinguish between e-cigarettes and vaping. For the purposes of this model, e-cigarettes and vapes will be considered synonymous.

3. *People respond honestly to surveys.* Our model is dependent on survey results to calculate weight constants. Because we have no way of determining the accuracy of the survey responses, we will assume that they are accurate and without bias for simplicity.

2 Part 1: Darth Vapor

First commercialized in 2003, electronic cigarettes have become an increasingly popular product among youth [1]. Although they are advertised as safer alternatives to traditional cigarettes, e-cigarettes contain high doses of nicotine and have introduced a new generation to tobacco products. This section outlines a mathematical model for predicting the change in nicotine use in the United States due to vaping compared to the change due to cigarettes.

2.1 Assumptions

1. *Nicotine use can be modeled as an infectious disease.* Like an epidemic, nicotine use is prevalent and contagious, reflected in the surge in popularity of smoking due to peer pressure, advertisements, and social media. Additionally, the U.S. Surgeon General declared youth vaping a nationwide epidemic in 2018 [2].

2. *Individuals can smoke from age 11 until death.* Peak years for first trying nicotine products is 6th or 7th grade [3].

3. *Rate of entry into pre-adolescence in the U.S. is 0.00103.* [4] Our model defines "birth" as reaching an age at which substance use becomes possible—around 11 years. Thus, we assumed the current birth rate to be constant for the past 11 years, assuming no children die before they turn 11. The current birth rate is 1.03 people/month/person.

4. *Death rate in the U.S. is constant and equal to 0.0007 people per month per person.*[4] Our model assumes that individuals have the capacity to use drugs until their death.

5. *Individuals can only start smoking due to influence from other smokers.* To model substance use as an infectious disease, we must assume that susceptible individuals can become infected only from contact with the already infected. This assumption is valid because peer influence and social media presence are the driving factors behind the popularity of smoking [5].

6. *Individuals are either not susceptible to, susceptible to, infected by, or recovered from substance abuse.* As in the SIR epidemiology model, we assume that people are either unwilling to smoke (not susceptible), open to smoking (susceptible), regular smokers (infected), or past smokers who have quit (recovered).

7. *The infection rate is constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the infection rate will also not drastically change.

8. *The percentage of susceptible people will stay constant over time.* Because we are assuming that the drug industry does not drastically change, it is reasonable to assume that the number of people susceptible to it will also not drastically change.

9. *Nobody starts as recovered.* At the start of the model, we do not consider any individuals to be former smokers who have quit.

10. *The recovery and relapse constant for cigarette and e-cigarette users are the same.* The two contain similar amounts of nicotine, which acts as the addictive agent. Thus, the recovery and relapse constants are assumed to be the same.

2.2 Model Development

The surge in popularity of conventional cigarettes in the mid-20th century, as well as the current boom of vaping among American youth, is comparable to the spread of an infectious disease during an epidemic. As stated in assumption 1, we model nicotine use as a disease because it rapidly spreads as a result of interpersonal communications (in-person peer pressure to try a drug as well as social media prevalence); additionally, substance use is a condition from which individuals can recover (by quitting smoking).

Our model is a derivation of the SIRS epidemiological model, a technique used to map the spread of infectious diseases such as influenza. We also consider birth and death rate, since population naturally changes over time. The model separates individuals in a population into four categories: NS for Not Susceptible, S for Susceptible, I for Infected, and R for Recovered. At the start of the model, individuals are either in NS, S, or I, since nobody starts off as recovered. While those in NS remain there permanently, individuals in S can move to I, who can then move to R.

The additional S in SIRS represents the possibility of returning to the Susceptible compartment—in this case, a regular user quitting but relapsing. However, we modified the classic SIRS model by recognizing that a relapsing individual would re-enter the Infected category rather than Susceptible, since they will once again become smokers rather than people merely open to smoking. Thus, we renamed the traditional epidemiology model as SIRI to represent this adjustment. Figure 2.2.1 diagrams the aforementioned movement of individuals between categories, while Table 2.2.1 defines and details values for variables and constants used in the SIRI model for both e-cigarette and cigarette smoking.

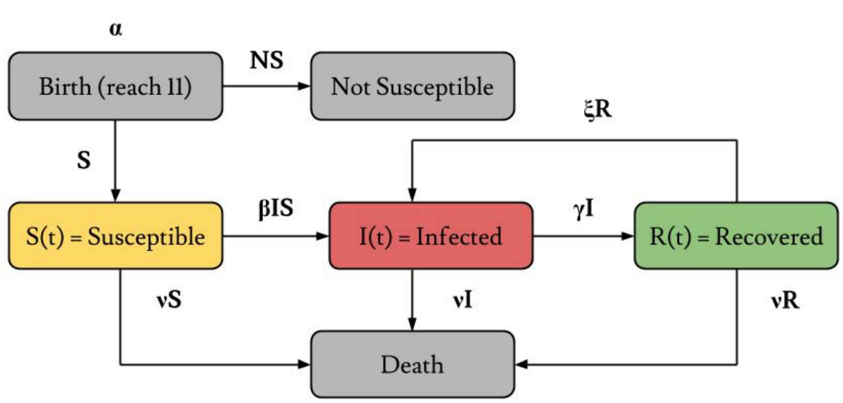


Figure 2.2.1: Diagram of the SIRI Model for Spread of Nicotine Use

2.2.1 Parameters in SIRI Model

Proportion of infected people (I_0). The total number of people that currently vape is approximately 10.8 million [6]. Dividing by the total population of America, 325.7 million

[7], results in an I_0 value of 0.0332 for e-cigarettes. The total number of people that currently smoke cigarettes is approximately 34.3 million [8], resulting in an I_0 value of 0.1053.

Proportion of recovered people (R_0). As per assumption 9, without loss of generality, R_0 was assumed to be 0 at time = 0.

Proportion of susceptible people (S_0). Because I, R, and S are proportions of the total population, their sums must add to 1. Thus, $S_0 = 1 - R - I$, resulting in 0.9667 for e-cigarettes and 0.8947 for cigarettes.

Susceptibility (S). A 2016 Surgeon General report stated that 32% of people are considered susceptible to e-cigarette use [5], while a 2012 report stated that 20% of people are susceptible to cigarettes, which correspond to the S values [9].

Infection constant (β). This was determined based on responses to the survey question "If one of your best friends offered you a cigarette, would you smoke it?" For e-cigarettes, the chance of infection was taken from a 2016 U.S. Surgeon General report that indicated that 18% of young adults responded "yes" to the question [5]. For cigarettes, we obtained β by adding the percentages of the responses "Definitely Yes" and "Probably Yes," from the 2014 National Survey on Drug Use and Health, to get 0.3%, which represented the infection constant [10].

Recovery constant (γ). In a given year, around 40% of smokers attempt to quit [11]. Therefore, in a month, $1.40^{1/12} = 1.0284$ recover, so the recovery rate is 0.0284.

Relapse constant (ξ). In a given year, approximately 6% of attempts to quit smoking succeed and 94% of attempts failed and the person relapsed [12]. Therefore, in a month, $1.94^{1/12} = 1.0568$ fail, so the relapse constant is 0.0568.

Infection rate (y_{inf}). In accordance with assumption 4, we assume that people will only start smoking if they are influenced by a current smoker. In other words, a susceptible person can only become infected if they come into contact with an infected person, which occurs at a rate proportional to $I \cdot S$. The infection constant β represents the likelihood that a susceptible person becomes infected when influenced by a smoker. Thus, infection rate is as follows:

$$y_{inf} = \beta \cdot I \cdot S \tag{1}$$

Recovery rate (y_{rec}). Unlike infection rate, the recovery rate is dependent only on the average probability of an individual quitting. The recovery constant γ multiplied by the proportion of people that currently are infected gives the recovery rate:

$$y_{rec} = \gamma \cdot I \tag{2}$$

Relapse rate (y_{rel}). The relapse rate is dependant only on the average probability of an individual relapsing. The relapse constant is much higher than the infection rate, which is logical because an individual who was previously a regular smoker will be more likely to succumb to the addictive cycle again [12]. Designating ξ as the relapse constant, relapse rate is given by

$$y_{rel} = \xi \cdot R \tag{3}$$

Birth rate (α). The birth rate, as defined by assumption 3, is 1.03 people/month/person.

Death rate (μ). From assumption 4, the death rate is assumed to be constant and equal to 0.0007 people per month per person. Therefore, the number of people dead for each category will be the death rate multiplied by the proportion of the people in each category.

$$\mu_S = v \cdot S \tag{4}$$

$$\mu_I = v \cdot I \tag{5}$$

$$\mu_R = v \cdot R \tag{6}$$

Table 2.2.1 Variables and Constants of SIRI Model for E-Cigarettes and Cigarettes

Variable	Definition	E-Cigarette Values	Cigarette Values
I	Proportion of infected people	$I_0 = 0.0332$	$I_0 = 0.1053$
R	Proportion of recovered people	$R_0 = 0$	$R_0 = 0$
S	Proportion of susceptible people	$S_0 = 0.9667$	$S_0 = 0.8947$
N	Proportion of total individuals in SIR cycle	$N_0 = 0.32$	$N_0 = 0.20$
α	Birth rate	0.00103	0.00103
β	Infection constant	0.18	0.003
γ	Recovery constant	0.0284	0.0284
ξ	Relapse constant	0.0568	0.0568
μ	Death rate	0.0007	0.0007

2.2.2 Differential Equations for SIRI Model

The change in each of the dependent variables S, I, and R is equal to the sum of the input of the respective category minus the sum of its output, as diagrammed by the arrows entering and leaving each box in Figure 2.2.1. Thus, our SIRI model is summarized by the set of ordinary differential equations below:

$$\frac{dS}{dt} = \alpha - \beta \cdot I \cdot S - \mu \cdot S \tag{7}$$

$$\frac{dI}{dt} = B \cdot I \cdot S - \gamma \cdot I + \xi \cdot R - \mu \cdot I \tag{8}$$

$$\frac{dR}{dt} = \gamma \cdot I - \xi \cdot R - \mu \cdot R \tag{9}$$

2.3 Results

With the SIRI model established, we utilized it to predict the change in nicotine use due to e-cigarettes and cigarettes in the next decade. We coded and executed a Python program to solve the system of differential equations, with appropriate constants for each product, and graph the proportion of compartments over time. Figures 2.3.1 and 2.3.2 graph the proportion of the total population falling under each of the SIR categories for both tobacco products, respectively, over a 10-year time period. Table 2.3.1 enumerates the proportion of the population that is susceptible, infected, and recovered for vaping and cigarettes in 2029.

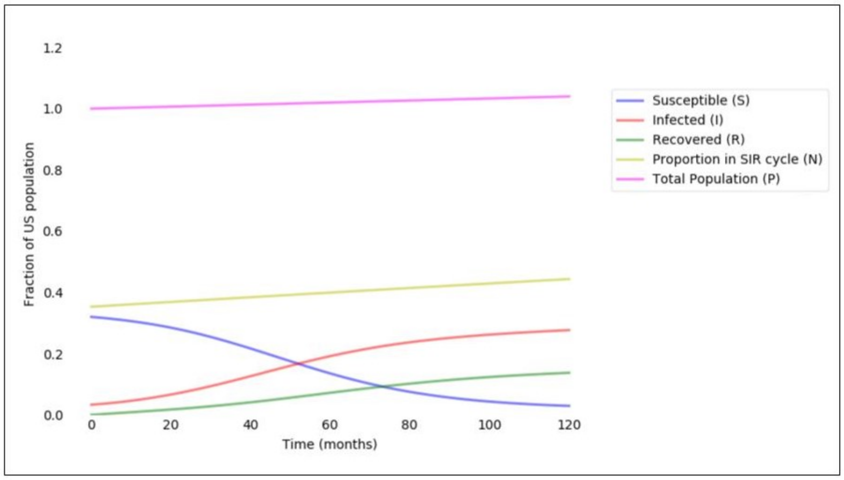


Figure 2.3.1: Graph of SIRI Compartments for E-Cigarettes over Ten Years

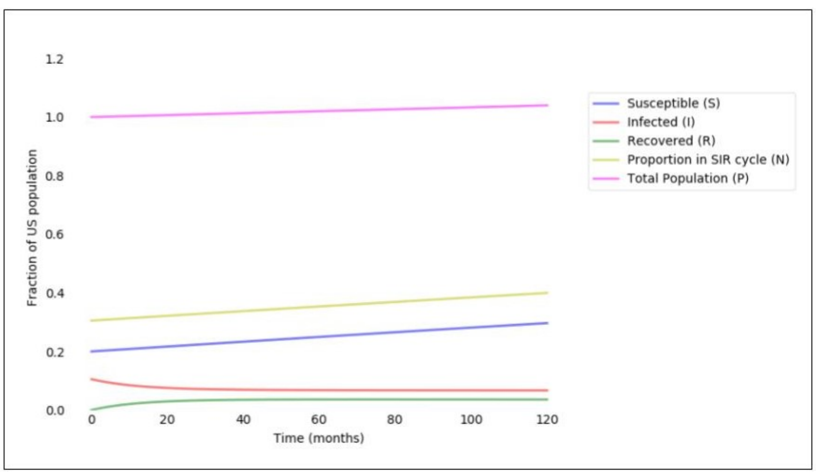


Figure 2.3.2: Graph of SIRI Compartments for Cigarettes over Ten Years

Table 2.3.1 SIR Distribution of 2029 Population for E-Cigarettes and Cigarettes

	Susceptible	Infected	Recovered
E-Cigarettes	2.82%	26.63%	13.21%
Cigarettes	28.53%	6.45%	3.45%

Our model concludes that in 2029, 26.63% of the population will use e-cigarettes, while 6.45% will use cigarettes. This disparity is consistent with previously researched trends, which suggest that as e-cigarettes gain popularity amongst teens, regular cigarettes decrease in popularity [13].

2.4 Sensitivity Analysis

Table 2.4.1 shows the sensitivity analysis for our SIRI model based on an independent increase and decrease of 10% of the infection constant β, recovery constant γ, and relapse constant ξ.

Table 2.4.1 Sensitivity Analysis for Part I

Constant	% Change in Constant	% Change in Vaping (I)	% Change in Cigarette Use (I)
β	10%	1.014%	0.6202%
β	-10%	-1.615%	-0.6202%
γ	10%	-3.492%	-3.566%
γ	-10%	4.018%	3.721%
ξ	10%	3.098%	3.367%
ξ	-10%	-3.496%	-3.905%

Positive changes in the infection or relapse constants resulted in positive changes in the percentage of infected people for both vaping and cigarette use. This is consistent with our predictions because the rate of infection for susceptible and recovered people is increasing. In contrast, a positive change in recovery constant resulted in a decrease in percent infected because the rate at which people are leaving the infected population is increasing.

2.5 Strengths and Weaknesses

Our model is resilient to small changes and outputs sensible results. As demonstrated in the sensitivity analysis, a 10% change in each of the infection, recovery, and relapse constants accounts for less than 5% change in final vaping and cigarette use after a decade. Changes in the model's output due to shifts are consistent with expected trends as well. SIRS is also an established mathematical modeling technique that we adapted to fit our own aims, lending credence to the validity of our model. Additionally, our model is comprehensive, accounting for many contributing factors such as population change, nonsusceptible individuals, and the possibility of relapse for smokers who have attempted to quit.

The model's weaknesses lie in its inability to account for the introduction of new forms of drugs or rapid changes in popularity of existing forms, as stated in global assumption 1. Specifically, a surge in use of a particular drug would likely impact vaping and cigarette use in unforeseen ways that our model will not accurately predict. Furthermore, our model does not consider the association between vaping and cigarette use, and how the growth or decline of one product would influence the other. This is unrealistic because the popularity of e-cigarettes among youth has led many to smoke traditional cigarettes and prompted cigarette smokers to transition to vaping [13]; however, the opposite effects of these two phenomena can reasonably counterbalance each other.

5 Conclusion

5.1 Further Studies

Our first model does not currently account for the introduction of new drugs in the industry, which would greatly impact the change in usage for pre-existing substances. Taking these market changes into account would greatly strengthen our model. The second model used survey data from 2005–2006. The resulting model fits well for this time period, but requires more recent data to reflect recent trends. Applying the same modeling approach for 2019 would create a more accurate model that is applicable to today. Finally, the third model is heavily based on the personal opinions of psychiatrists. Recreating the model to account for each factor with independent methods would greatly complicate the model, but make it more flexible for incorporating newer drugs into our ranking.

5.2 Summary

The first model focuses on comparing the percent of e-cigarette users versus cigarette users in the next ten years. The SIRS epidemic model was used as the basis for ours. People were split into four main categories: infected (those that used drugs), recovered (those that quit using drugs), susceptible (those that may use drugs in the future), and non-susceptible (those that will never use drugs). Birth rate and death rate were both modeled with linear equations. Simultaneous differential equations were solved to determine the number of "infected" people in 2029. According to our model, 26.63% of the American population will vape in 2029 and 6.45% will smoke cigarettes. The results correspond with observed increasing popularity of e-cigarettes and decreasing popularity of regular cigarettes.

The second model determines the probability of a student using nicotine, marijuana, alcohol, and opioids and applies itself to a randomly generated sample of 300 high school seniors. A binary multivariate logistic regression was used to create the model based on an HBSC survey. A machine learning algorithm using an L2 regression was used to calculate the weights and bias in our logistic model. Using a Monte Carlo simulation, 300 random seniors were created based on response frequencies to each of questions necessary for our model. Running this sample of high school seniors through our model, we found 46.33% would use nicotine, 17.33% would use marijuana, 66.00% would use alcohol, and 0.00% would use opiates.

The third and final model focuses on ranking nicotine, marijuana, alcohol, and opioids based on their financial and nonfinancial effects. Factors were analyzed in four main categories: physical harm, dependence, social harm, and economic impact. These factors were further split into 2–3 subcategories each that were each assigned scores on a scale from 0.0 to 3.0 based on expert surveys. To calculate the impact of drugs on GDP, the average annual GDP per person was multiplied by the average decrease in life as a result of using drugs. The impact of drugs on GDP was then rescaled from 0.0 to 3.0 to make them comparable to the other factors. Each of the four main categories was averaged for a total harm score for each of the four drugs. The total harm score was multiplied by a risk factor based on the number of people that used each drug to obtain a final score for each drug that could be used for ranking purposes. This model showed that opioids had the greatest substance harm per person, but since relatively few people use opiods, it had a lower total detriment score. Marijuana had the lowest substance harm per person and the second lowest total impact. Alcohol had the highest total impact, while cigarettes had the second highest because of the great number of people using these substances.

6 References

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