2018-11-12 Method of Undetermined Coefficients (4.5)

Pre-lecture:

Solving non-homogeneous DEs: ay''(t) + by'(t) + cy(t) = g(t)

E.g. Given y' - 2y = sin(x), y(0) = 0

Let y' - 2y = 0 -> r-2 = 0 -> r = 2 -> $y_h = ce^{2x}$

Let y = asin(x) + bcos(x), y' = acos(x) - bsin(x)

Plug in y' - 2y = sin(x), we get acos(x) - bsin(x) - 2(asin(x) + bcos(x)) = sin(x)

(-b - 2a - 1)sin(x) = (-a + 2b)cos(x)

- -b - 2a - 1 = 0
- -a + 2b = 0

a=-2/5 , b=-1/5

$y_p = -\frac{2}{5}\sin(x) - \frac{1}{5}\cos(x)$

$y = ce^{2x} - \frac{2}{5}\sin(x) - \frac{1}{5}\cos(x)$

Substitute initial conditions:

0=c-1/5 -> c=1/5

$y = \frac{1}{5}e^{2x} - \frac{2}{5}\sin(x) - \frac{1}{5}\cos(x)$

Using equation $y'' + 3y' = 3t$

11. What is the particular solution $y_p\,(t)$?

$Y(t) = (At + B)t$ (this works)

$Y(t) = At + B$ => $3A = 3t$ (does not work)

$y'' + 3y' = 3t$

Using $Y\,(t) = (At + B)\,t$:

$Y\,(t) = At^2 + Bt$

$Y'(t) = 2At + B$

$Y''(t) = 2A$

Substitute $Y(t)$, $Y'(t)$, $Y''(t)$ into $y'' + 3y' = 3t$

$2A + 3(2At + B) = 3t \implies 2A + 6At + B = 3t \implies 6At = 3t \implies A = \frac{1}{2}$

$2A + 3B = 0 \implies B = -\frac{1}{3}$

$\therefore Y_p(t) = \frac{1}{2}t^2 - \frac{1}{3}t$

Initial Conditions

$Y(0) = 0$

$Y'(0) = 0$

13. What is the general solution $y_g(t)$?

Given two initial conditions and two constants to solve for, solve the system of equations.

$Y(0) = 0 \implies c_1 + c_2 = 0 \implies c_1 = -c_2$

$Y'(0) = 0\,,\ Y'(t) = -3c_2 e^{-3t} + t - \frac{1}{3}$

$0 = -3c_2 - \frac{1}{3} \implies \frac{1}{3} = -3c_2 \implies c_2 = -\frac{1}{9}$

$c_1 = -c_2 \implies c_1 = \frac{1}{9}$

$\therefore Y_g(t) = -\frac{1}{9}e^{-3t} + \frac{1}{2}t^2 - \frac{1}{3}t + \frac{1}{9}$

Given the equation $y'''' - 4y''' + 10y'' - 12y' + 5 = te^t + t^2\cos(t) - (2t+1)\,e^t\sin(t)$

14. What is the particular solution $y_p(t)$? (Don't find the constants)

Hint: $x^4 - 4x^3 + 10x^2 - 12x + 5 = \left[(x-1)^2\right]\left[(x-1)^2 + 4\right]$

$Y(t) = (At + B)\,e^t + \left(Ct^2 + Dt + E\right)(\cos(t)) + \left(Ft^2 + Gt + H\right)(\sin(t)) + (Jt + K)\left(e^t\right)(\sin(t))$
$+ (Lt + M)\left(e^t\right)(\cos(t))$

NOTE: The $At+B$ term matches the $c_1e^t+tc_2e^t$ term of the general solution. Multiply the $At+B$ term by t

$$Y(t)=t(At+B)e^t+\left(Ct^2+Dt+E\right)(\cos(t))+\left(Ft^2+Gt+H\right)(\sin(t))+(Jt+K)\left(e^t\right)(\sin($$
$$+(Lt+M)\left(e^t\right)(\cos(t))$$

NOTE: The $(At+B)te^t$ term matches the tc_2e^t term of the general solution. Multiply the $(At+B)t$ term by t

$$Y(t)=t^2(At+B)e^t+\left(Ct^2+Dt+E\right)(\cos(t))+\left(Ft^2+Gt+H\right)(\sin(t))+(Jt+K)\left(e^t\right)(\sin$$
$$+(Lt+M)\left(e^t\right)(\cos(t))$$

NOTE: The $(Jt+K)\left(e^t\right)(\sin(t))$ and $(Lt+M)\left(e^t\right)(\cos(t))$ terms match the $c_3e^t\cos(2t)$ and $c_4e^t\sin(2t)$ terms of the general solution. Multiply these terms by t

$$\therefore Y_p(t)=t^2(At+B)e^t+\left(Ct^2+Dt+E\right)(\cos(t))+\left(Ft^2+Gt+H\right)(\sin(t))$$
$$+t(Jt+K)\left(e^t\right)(\sin(t))+t(Lt+M)\left(e^t\right)(\cos(t))$$

15. What is the general solution y(t)?

Using the hint,

$$(r-1)^2=0 \implies r=1\ ,\ 1$$

$$(r-1)^2-4=0 \implies r^2-2r+1+4=0 \implies r^2-2r+5=0$$

$$r=\frac{2\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} \implies r=\frac{2\pm\sqrt{16i}}{2} \implies r=1\pm 2i$$

$$\therefore Y_g(t)=c_1e^t+c_2te^t+c_3e^t\cos(2t)+c_4e^t\sin(2t)$$