2018-10-15 Systems of 2 ODEs (3.2)

3.2 Systems of Two Linear ODEs

2. If we include a fixed amount of "harvesting" every year, what is the new system of ODEs?

Assumption: "harvesting rate" is 11 lions and 11 cheetahs per year. "Harvesting" could be defined as decreasing the population of lions and cheetahs by a fixed amount each year (for ex. by taking them and bringing them to the zoo).
Let:
k = growth rate of lions (if the lions are left alone, how would the population grow?)
a = growth rate of cheetahs (if the cheetahs are left alone, how would the population grow?)
m = constant describing how cheetahs (hunting food/competing against lions) affect the lion population. This is not the growth rate of the cheetah population.
b = constant describing how lions (hunting food/competing against cheetahs) affect the cheetah population. This is not the growth rate of the lion population.

A) $l' = k(l-11) - m(c-11)$, $c' = a(c-11) - b(l-11)$

- Incorrect because we want it to work even if we start off with l or c less than 11

B) $l' = k(l-11) - mc$, $c' = a(c-11) - bl$

- Incorrect. If lion population minus 11, then cheetahs population should also minus 11)

C) $l' = kl - mc - 11$

$c' = ac - bl - 11$

- Correct. This incorporates the 11 fixed amount of harvesting for the year no matter how many l or c we start with

Note: "harvesting" is independent of how lions/cheetahs will grow

Possible questions: i) Doesn't C disregard the number of animals harvested from the other species?

Answer: No. Keep in mind this is a SYSTEM of 2 equations where the two equations affect each other. **The number of animals harvested from the other species is accounted for by the term present in the equation**

- i.e. in $l' = kl - mc - 11$, the presence of "c" accounts for the harvest of cheetahs.
- Keep in mind that c is a function (and not a constant) that contains terms that account for harvesting.

ii) Doesn't C assume the that the number harvested plays no role in growth as it's not multiplied by the growth constant?

Answer: No. The two equations are $\frac{dl}{dt}$ and $\frac{dc}{dt}$ (they contain TIME), so the **equations only "capture" what's happening at a particular instant of time**.

If we were to integrate to get l and c, we would have the term $11t$. So, the term that is multiplied by the growth constant (l or c) contains a term within it that accounts for the loss due to harvesting in real time.

Example:

$l' = 3$ --> $l = 3t + l_o$

After 1 day

$l(\frac{1}{365}) = \frac{3}{365} + l_o$

Remember that it is a continuous function.

3. Define $p(t) = \binom{l(t)}{c(t)}$. Write the system in matrix form.

$p'(t) = \begin{pmatrix} k & -m \\ -b & a \end{pmatrix} p(t) - \binom{11}{11}$

Note:

1. The -b and a's order. k and -b refer to the l term, while -m and a refer to the c term.

2. $p'(t) = \begin{pmatrix} k & -m \\ -b & a \end{pmatrix} p(t) - \binom{11}{11}$ is not separable because it cannot be divided by a vector.

It is autonomous because $l' = kl - mc - 11$, $c' = ac - bl - 11$ do not depend on t (or has t on the right side). However, if $\begin{pmatrix} k & -m \\ -b & a \end{pmatrix}$ and $\binom{11}{11}$ are pure numbers, $p'(t) = \begin{pmatrix} k & -m \\ -b & a \end{pmatrix} p(t) - \binom{11}{11}$ can be separated by dividing both sides by the entire term on the right side.

This is why we want to now find the equilibrium solution.

Consider the system $\frac{dP}{dt} = \begin{pmatrix} 1 & -\frac{1}{6} \\ -\frac{1}{2} & 1 \end{pmatrix} p + \binom{-11}{-11}$

4. What is the equilibrium solution?

Equilibrium solution --> $\frac{dP}{dt} = 0$

$0 = \begin{pmatrix} 1 & -\frac{1}{6} \\ -\frac{1}{2} & 1 \end{pmatrix} p + \binom{-11}{-11}$

System of two equations:

$\binom{p_1 - \frac{1}{6}p_2 - 11 = 0}{-\frac{1}{2}p_1 + p_2 - 11 = 0} \implies \binom{p_1 - \frac{1}{6}p_2 - 11 = 0}{-p_1 + 2p_2 - 22 = 0} \implies \frac{11}{6}p_2 - 33 = 0$

$\frac{11}{6}p_2 = 33 \implies 11p_2 = 33\,(6) \implies p_2 = 18$

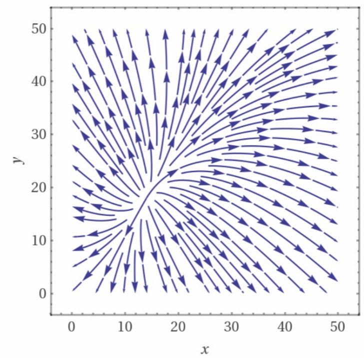
Substitute value of p_2 back into one of the two equations:

$-\frac{1}{2}p_1 + p_2 - 11 = 0 \implies -\frac{1}{2}p_1 + 18 - 11 = 0 \implies -\frac{1}{2}p_1 = -7 \implies p_1 = 14$

$\therefore P_{eq} = \binom{14}{18}$

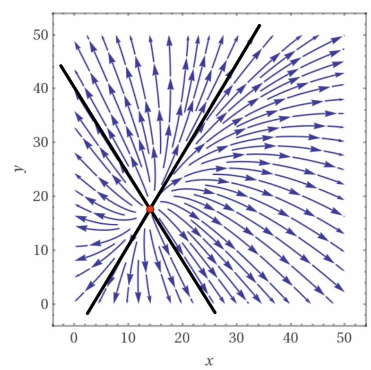
What does this mean?
If there are 14 lions and 18 cheetahs, the population would remain constant (or the same) since the equilibrium of the system is 0 (or rate of change is 0).

Given the stream plot, what are the eigenvalues and eigenvectors?



In the stream plot, the lines that are completely straight throughout demonstrate this property.

The eigenvectors are both stretched, so their eigenvalues are positive.



Note on the graph:

l(t) is the x-axis, c(t) is y-axis

When there is an equilibrium solution of the system of DEs, the slope stream plot tends to oscillate around that specific point.

In this example, since the equilibrium solution is $\binom{14}{18}$,the given graph shows an oscillation at a center of point (14,18) which matches our equilibrium solution.

This is an unstable equilibrium point because all of the arrows on the stream plot point away from the equilibrium point.