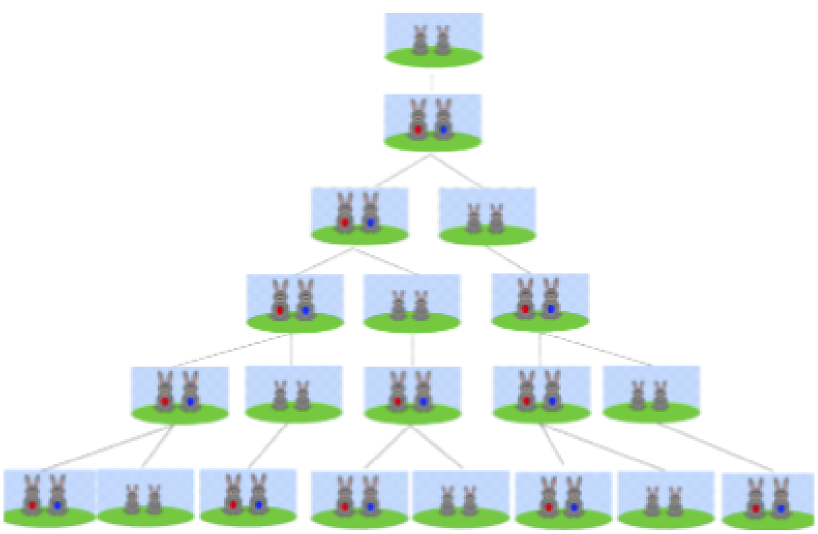
2018-11-29 Simple Rabbit Populations

2.2.* Rabbit Populations

Consider the following rabbit population:

- **Rabbits live forever**
- **They are babies for 1 season (don't reproduce)**
- **They become mature adults after 1 season**
- **A pair of mature adults has 1 pair of baby rabbits each season**
- **At the beginning there is 1 pair of baby rabbits**

1. Model the rabbit population: Difference equation and conditions



r_k = number of pairs of rabbits at the beginning of season k

Method 1:

Assume babies mature at the end of the season. This means that the babies that just turned into adults won't have babies until the next season.

r_0 = 1 (1 baby pair)

r_1 = 1 (1 adult pair)

r_2 = 2 (1 adult pair, 1 baby pair)

r_3 = 3 (2 adult pairs, 1 baby pair)

r_4 = 5 (3 adult pairs, 2 baby pairs)

r_5 = 8 (5 adult pairs, 3 baby pairs)

r_6 = 13 (8 adult pairs, 5 baby pairs)

Method 2:

Assume babies mature at the beginning of the season.

r_0 = 1 (babies)

r_1 = 2 (1 adult pair, 1 baby pair)

r_2 = 3 (2 adult pairs, 1 baby pair)

r_3 = 5 (3 adult pairs, 2 baby pairs)

r_4 = 8 (5 adult pairs, 3 baby pairs)

r_5 =13 (8 adult pairs, 5 baby pairs)

Population mimics the Fibonacci Sequence.

$\mathbf{r_k = r_{k-1} + r_{k-2}}$, for k>2

Since it is a second order difference equation, we also need to know the two initial conditions:

$\mathbf{r_0 = 1}$

$\mathbf{r_1 = 1}$

Note: This is a Second Order Difference Equation. Find the difference between the highest and lowest subscripts k, and k-2 to get the order of the difference equation. We also need 2 initial conditions to be able to express explicitly.

2. Show that the Difference equation found follows from the “rules” above:

Note: Just like how a differential equation/ graph for a population going extinct is only valid up to a certain point, sequences do not always follow the rules of a situation forever (may it only works until the 20th term). We need to prove that the Fibonacci sequence is true for any season and follows the rules.

		babies	adults
Season k:	r_k =	r_{k-2}	r_{k-1}
Season k - 1:		r_{k-1}	$adults_{k-1}$
Season k - 2:		r_{k-2}	

Explanation for highlighted connections:

- All the rabbits from r_{k-2}, whether they are adults or babies, are adults in r_{k-1}. Babies become adults in the next season, and adults live forever so they are still adults in the next season.
- Same logic to prove that all the rabbits in r_{k-1} become the adults of r_k. All the babies become adults, and all the adults remain adults.
- The amount of babies that exist in r_k is equal to the adults of r_{k-1} as only adults can reproduce. 1 pair of adults produces 1 baby. Adults of r_{k-1} has been proven to be all the rabbits of r_{k-2} (first bullet point)

This proves that the sequence is defined by $r_k = r_{k-1} + r_{k-2}$, same as the Fibonacci Sequence.

Another way to look at the problem is with the equation $r_k = 2*r_{k-2} + (r_{k-1}-r_{k-2})$

$2*r_{k-2}$ represents $r_{k-2} + r_{k-2}$. The first r_{k-2} is the population pair count 2 seasons ago (who must all be adults by now) and the second r_{k-2} is all the baby pairs who have been birthed by all the adults, $2*r_{k-2}$. Then $(r_{k-1}-r_{k-2})$ represents the population difference between 2 seasons ago and 1 season ago, also known as all the babies who did not give birth because they were busy becoming adults.

This equation also simplifies to $r_k = r_{k-2} + r_{k-1}$ which is the same answer corroborated above. The r_{k-2} represents the number of baby rabbits for that season while r_{k-1} represents the number of adult/mature rabbits for that season. For example, for $r_k = r_{k-2} + r_{k-1} \Rightarrow r_4 = r_{4-2} + r_{4-1} \Rightarrow r_4 = r_2 + r_3 \Rightarrow r_4 = 2 + 3$, and in r_4, there are 2 pairs of baby rabbits and 3 pairs of adult rabbits. Therefore, r_{k-2} represents the number of baby rabbits and r_{k-1} represents the number of adult/mature rabbits. The total number of rabbits for the kth season is just the sum of these two values.

3. Find an explicit formula for the solution

Assume the solution is an exponential r^k

$$r^k = r^{k-1} + r^{k-2}$$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Substituting the initial conditions:

- $F_0 = 1$
- $F_1 = 1$

$$F_0 = a + b = 1$$

$$F_1 = a\left(\frac{1+\sqrt{5}}{2}\right) + b\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

We get $a = \frac{\sqrt{5}+5}{10}$ and $b = \frac{5-\sqrt{5}}{10}$

$$F_n = \left(\frac{5+\sqrt{5}}{10}\right) \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right) \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n$$