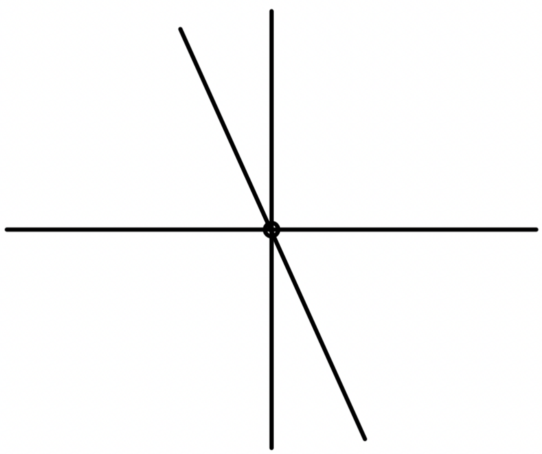
2018-10-29 Systems of ODEs with Repeated Eigenvalues (3.5) + Summary of Stability

$$x(t) = A\binom{1}{-2}e^{-2t} + B\left[\binom{1}{-1} + \binom{1}{-2}t\right]e^{-2t}$$

7. Sketch the solutions for A = +/-1 and B=0 in phase plane



NOTE: Initial point: (1, -2). Values converge to 0 so direction is towards 0.

8. Sketch the solutions for A=0, B=+/-1 in the phase plane (e=2)

$t = 0,\ x(0) = \pm\binom{1}{-1}$

- $t = \frac{1}{2},\ x\left(\frac{1}{2}\right) = \pm\left[\binom{1}{-1} + \binom{\frac{1}{2}}{-1}\right]e^{-1} = \pm\frac{1}{2}\binom{\frac{3}{2}}{-2} = \pm\binom{\frac{3}{4}}{-1}$

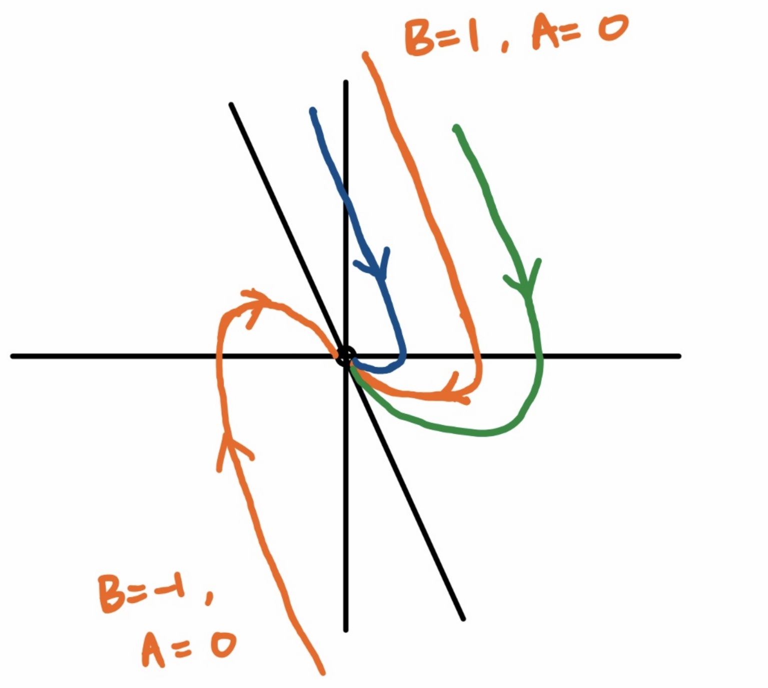
- $t = 1,\ x(1) = \pm\left[\binom{1}{-1} + \binom{1}{-2}\right]e^{-2} = \pm\frac{1}{4}\binom{2}{-3} = \pm\binom{\frac{1}{2}}{-\frac{3}{4}}$

- $t = 2,\ x(2) = \pm\left[\binom{1}{-1} + \binom{2}{4}\right]e^{-4} = \pm - \frac{1}{16}\binom{3}{-5} = \pm\binom{\frac{3}{16}}{-\frac{5}{16}}$

Note: Orange Graph is the case with condition B =1 , A=0;

Green Graph is the case wit condition B>1, A =0;

Navy Graph is the case with condition B<1, A=0.



(The "A part" is a straight line because it just stretch of vector(1,-2) ; however, since the "B part is multiplied by t, it is no longer just a stretch as t changes.)

Asymptotically stable.

Solutions approach (0, 0) but never reach it.

9. find the limit of x(t):

- when t approach to ∞
  - converges to 0
- when t approach to -∞
  - has the same slope as the actual eigen vector

- is x(t) asymptotically stable?

Yes, it is asymptotically stable.(the solutions approach to 0.)

-Compare x(t) and eclipse sketched from previous classes:

- eclipse is stable but not asymptotically stable.(Its solutions never try to approach to 0)
- x(t) is asymptotically stable.(Its solutions always try to approach to 0).

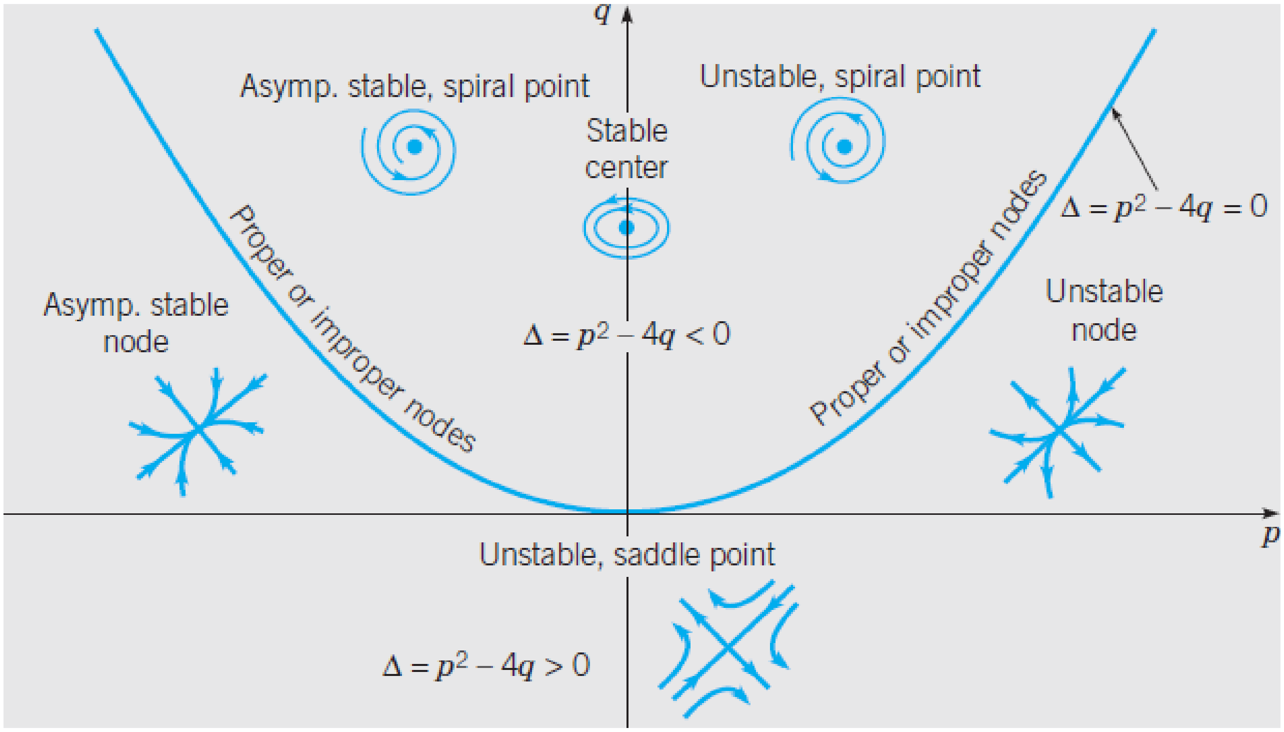


FIGURE 3.5.7 Stability diagram.

(From textbook page 190)

Recap: Stability properties of linear systems (Eigenvalues Type of Critical Point Stability)

Eigenvalues	Type of Critical Point	Stability
$\lambda 1 > \lambda 2 > 0$	Node source	Unstable
$\lambda 1 < \lambda 2 < 0$	Node sink	Asymptotically stable
$\lambda 2 < 0 < \lambda 1$	Saddle point	Unstable
$\lambda 1 = \lambda 2 > 0$	Proper or improper node	Unstable
$\lambda 1 = \lambda 2 < 0$	Proper or improper node	Asymptotically stable
$\lambda 1, \lambda 2 = \mu \pm iv$	Spiral point	
$\mu > 0$	Spiral source	Unstable
$\mu < 0$	Spiral sink	Asymptotically stable
$\lambda 1 = iv, \lambda 2 = -iv$	Center	Stable

Useful Links About Stabilities (Have Visual Demonstrations):

http://staff.www.ltu.se/~larserik/applmath/chap9en/part7.html
(http://staff.www.ltu.se/~larserik/applmath/chap9en/part7.html)

http://www.math.psu.edu/tseng/class/Math251/Phase_portrait_reference_card.pdf?fbclid=IwAR36fdCOZfjAFnqEcpHO1Rk5d0u0zfOkyusgvZw49Y8hHPWe_vWvt1PUqo8
(http://www.math.psu.edu/tseng/class/Math251/Phase_portrait_reference_card.pdf?fbclid=IwAR36fdCOZfjAFnqEcpHO1Rk5d0u0zfOkyusgvZw49Y8hHPWe_vWvt1PUqo8)

Summary:

- As $t \to \infty$, each trajectory exhibits 1 of only 3 types of behaviours:
  1. Becomes unbounded
  2. Approaches the critical point x = 0
  3. Repeatedly traverses a closed curve, corresponding to a periodic solution that surrounds the critical point
- Properties of pattern of trajectories:
  1. Only 1 trajectory passes through each point (x_0, y_0) in the phase plane, so trajectories do not cross each other
  2. The only solution passing through the origin is the equilibrium solution x = 0
  3. Other solutions appear to pass through the origin, but they actually only approach this point as $t \to +\infty$ or $t \to -\infty$
- For the set of all trajectories, 1 of 3 situation occurs:
  1. Asymptotically stable
     - As $t \to \infty$, all trajectories approach the critical point x = 0
     - Eigenvalues: real and negative or complex with negative real part
     - Origin: a nodal sink or a spiral sink
  2. Stable
     - As $t \to \infty$, all trajectories remain unbounded but do not approach the origin
     - Eigenvalues: purely imaginary
     - Origin: a center
  3. Unstable
     - As $t \to \infty$, some trajectories and possibly all except x = 0, becomes unbounded
     - Eigenvalues: at least 1 is positive or if the eigenvalues have a positive real part
     - Origin: a nodal source, a spiral source, or a saddle point