# 2018-09-13 Modelling with ODEs (2.3)

## 2.3 Modelling with ODEs

The equations that describe the boulder's altitude are

- $my'' = -mg - \gamma y'$
- $y(0) = 0, y'(0) = 10$

The Solution is $y\left(t\right) = -\frac{mg}{\Upsilon}t - \left(\frac{m^2 g}{\Upsilon^2} + \frac{10m}{\Upsilon}\right)\left(e^{-\frac{\Upsilon}{m}t} - 1\right)$

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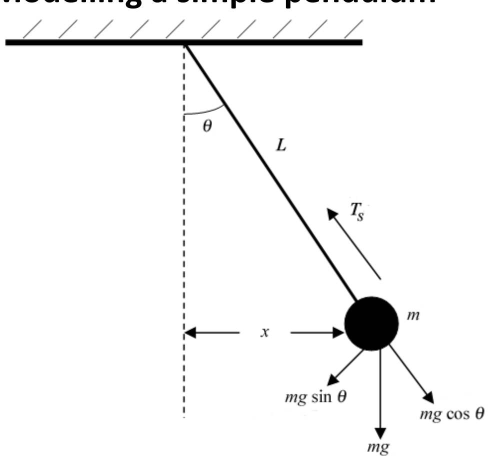
1. **Can you think of ways/experiments to measure (gamma) and m based on using this formula?**
    a. Measure $y(t_1) = y_1$
    b. Throw the boulder
        You can measure the drag coefficient by this.
        i. Use terminal velocity $y\prime(t) = 0$

## When modelling

- Define Variables (write units if you know)
    - Example: h(t) = height of boulder t seconds after launch
- Start with Basic Principles
- Models are Approximations -- Require Assumptions (difficult part of modelling)

## Modelling a simple pendulum




- Which basic principle should we use?
    - We need to know
        - $Force = Gravity + Tension$
            - Gravity = -mg
            - Tension = T
    - Newton's 2nd law ($F = ma$)

$F = -mg + (-T)$

- Going to be made up of gravity acting on the mass and the tension in the rope

- Assume
  - Massless rope
  - Rigid rope - string is always at a constant length L, no elastic energy
  - No friction from either the air or the hinge or the rope
  - No air resistance
  - Small angles
- Define variables
  - The force of T is in the direction of the string, we need to represent this. If we used $T cos\theta$ or $-T sin\theta$ , we would have the problem of introducing another variable, $\theta$, to find! Instead, use vectors:
    - $\vec{r}(t)$= position of the mass t seconds after it is launched
    - $F = -mg\hat{\jmath} + (-T\vec{r})$
      - Note: We cannot add a scalar and vector together! Therefore, the gravity component includes $\hat{\jmath}$ to make it a vector.
      - Acceleration = $\vec{v}'$ = $\vec{r}''$

**Other Examples of Modelling with First Order Differential Equations (not covered in lecture)**

1. **Mixing**

   i.e. Stir Tank Reaction: Adding salt to a tank of water, and at the same time, draining salt water out of the tank.
   - Define variables:
     - t = time (min), m(t) = mass of salt in the water (g), V = volume of water in the tank (L), R = flow rate of water (L/min), and C = salt concentration of water (g/L)
   - Basic principle:
     - $C = \frac{m}{V} \left( concentration = \frac{mass}{volume} \right)$
     - $\frac{dm}{dt} = rate\ in - rate\ out$
   - Constructing the model
     - Assume the inflow rate of water is the same as the outflow rate of water.
     - Assume the volume of water in the tank is constant at all times.
     - Model: $\frac{dm}{dt} = rate\ in - rate\ out = RC_{in} - RC_{out} = RC_{in} - R\frac{m(t)}{V}$
     - If given initial mass of salt in the tank, then can solve differential equation above for m(t) to find mass of salt in the tank at any time t.
2. **Paying Off a Loan**

   i.e. Borrowing money from the bank and making monthly payments to payback the bank.
   - t = time (years), S(t) = balance on the bank loan at any time t (\$), r = annual interest rate (%/year), k = monthly payment rate (\$/month)

- Assume annual interest rate is constant for all years.
- Model: $\frac{dS}{dt} = rS(t) - 12k$, k is multiplied by 12 to account for 12 months in a year.

3. **Population Growth**
   - Exponential growth model: $\frac{dP}{dt} = rP(t)$
   - Logistic growth model: $\frac{dP}{dt} = rP(t)\left(1 - \frac{P(t)}{K}\right)$
   - r = population growth rate, P(t) = population at time t, K = carrying capacity of the population
   - Logistic growth model is a better model with better approximations than exponential growth model because population will not grow indefinitely in real life.
4. **Newton's Law of Cooling**
   - k = transmission coefficient, rate of heat exchange between the object and its surroundings, measured in $(time)^{-1}$,
     u(t) = temperature of object at any time t (Kelvin, Celsius, or Fahrenheit), T = ambient temperature/temperature of surroundings (Kelvin, Celsius, or Fahrenheit)
   - Model: $\frac{du}{dt} = -k(u(t) - T)$
5. **Escape Velocity**
   - $v_e$ = escape velocity, the least initial velocity for which the object launched from the ground will not return to the Earth (m/s), R = radius of the earth (m), x = distance above sea level (m), g = acceleration due to gravity ($\frac{m}{s^2}$), k = a constant, w(x) = gravitational force acting on the mass (N), m = mass (kg), x+R = altitude/distance from the center of the Earth (m)
   - Assume there is no air resistance.
   - $w(x) = -\frac{k}{(x+R)^2},\ and\ w(0) = -mg\ at\ sea\ level,\ so\ k = mgR^2$
   - Since there are no other forces acting on the object:
   - $w(x) = F_{net} = ma = m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2},\ where\ \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$
   - Model: $v\frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}$
   - Solving the model, it is found that $v_e = \sqrt{2gR}$, and the escape velocity on Earth is about 11.1 km/s
   - Full derivation can be found in section 2.2 of the textbook (second edition)