2018-12-03 Complicated Rabbit Populations

Topic: Rabbit Populations (Continued)

Now consider complicated rabbits:

Season 0	Born	Doesn't Reproduce
Season 1	Baby	Doesn't Reproduce
Season 2	Young Adult	Reproduces
Season 3	Adult	Reproduces
Season 4	Old	Doesn't Reproduce
Season 5	Dies	

We started with 1 pair of newborn rabbits.

Question 4:

Model the rabbit population: Define sequences, difference equation and conditions.

Method 1:

Let r_k = pairs of rabbits alive at season k.

Question 4:

$P_0 = 1\ (1\ newborn)$
$P_1 = 1\ (1\ baby)$
$P_2 = 2\ (1\ newborn + 1\ young\ adult)$
$P_3 = 3\ (1\ newborn + 1\ baby\ +\ 1\ middle\ age)$
$P_4 = 4\ (1\ newborn + 1\ young\ adult + 1\ baby\ +\ 1\ old)$
$P_5 = 5\ (2\ newborn + 1\ young\ adult + 1\ middle\ age\ +\ 1\ baby)$
$P_6 = 7\ (2\ newborn + 1\ young\ adult + 1\ middle\ age\ +\ 2\ baby\ + 1\ old)$
$P_7 = 9\ (3\ newborn + 2\ young\ adult + 1\ middle\ age\ +\ 2\ baby\ +\ 1\ old)$
$P_8 = 12\ (4\ newborn + 2\ young\ adult + 2\ middle\ age\ +\ 3\ baby\ +\ 1\ old)$
$P_9 = 16\ (5\ newborn + 3\ young\ adult + 2\ middle\ age\ +\ 4\ baby\ +\ 2\ old)$

Thus, the sequence is:
$P_n = P_{n-2} + P_{n-3} \quad (for\ n \geq 5)$

Method 2 (Matrix):

$$r_{k+1} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} * r_k \text{ where } r_k = \begin{bmatrix} n_k \\ b_k \\ y_k \\ a_k \\ o_k \end{bmatrix}$$

where, n_k = newborn at season k
b_k = baby at season k
y_k = young adult at season k
a_k = adult at season k
o_k = old at season k

Row 1 of the matrix represents the birth of newborn from young adult and adult during season 1-2 and season 2-3.

1s in the following rows represent the aging of rabbits.

Question 5: find an explicit formula for the solution.

$r_{k+1} = M \cdot r_k$

$r_k = M^k \cdot r_0$

Proof:

Using given initial condition, we find that $r_1 = M \cdot r_0$

so $r_2 = M \cdot r_1 = M \cdot (M \cdot r_0) = M^2 \cdot r_0$

$r_3 = M \cdot (M \cdot (M \cdot r_0)) = M^3 r_0$

If we look for sole of the form

$\lambda\, is\ the\ eigenvalue\ of\ M,\ V\ is\ the\ eigenvector\ of\ M\ for\ \lambda.$

$r_k = V \cdot \lambda^k$

$r_{k+1} = V \cdot \lambda^{k+1} = M \cdot V \cdot \lambda^k$

Divide both side by λ^k

$V\lambda = MV$