2018-09-17 Modelling with ODEs (2.3)

Modelling a Simple Pendulum

Assumptions:

- No friction
- Rigid rope
- Massless rope

$\vec{r}(t)$ = position of the mass at time t

We know , ma = F = Gravity + Tension

$$m\vec{r}''(t) = -mg + (-T\vec{r}(t))$$

*We cannot add scalars and vectors, so we convert '-mg' to a vector

(Tension is negative because it always acts opposite to the position vector)

$\hat{j}$ = (0, 1)

Idea. Use the angle to define position.
Then

1. $\vec{r}\,(t) = (L\sin\theta\,(t)\,,\ -L\cos\theta\,(t))$ Note: θ is a function of time
2. Newton's 2nd law implies
    a. x-component <=> $-\sin(\theta\,(t))\theta'^2 + \cos(\theta\,(t))\theta'' = 0$
    b. y-component <=> $L\cos(\theta\,(t))\theta'^2 + L\theta'' \cdot \sin(\theta\,(t)) = -g$
3. Notes
    a. Ignore tension
    b. At home, redo with tension and compare
    ma = F = Gravity

    $$m\vec{r}''(t) = -mg\hat{j}$$

    Hint: θ is a function of t.

    $$\frac{(L\cdot\cos\theta(t)\cdot\theta'')}{\tan\theta(t)} + L\cdot\sin\theta\,(t)\cdot\theta'' = -g$$

$$\frac{L\cdot\cos^2(\theta(t))}{\sin\theta(t)}\theta'' \;+\; L\cdot\sin\theta\,(t)\,\theta'' \;=-g$$

$$L\cdot\theta'' \;=\; -g\cdot\sin\theta\,(t)$$

This is the same result that we get in the video where they used conservation of energy to calculate θ.

MODELLING A PENDULUM WITH TENSION FORCE (3B)

The equation that is being used to model the position of the pendulum:

$$m\vec{r}''(t) = -mg\vec{j} + (-T\vec{r}(t))$$

Let T = constant tension

The position of the pendulum in x-direction:

$$-mL\sin(\theta\,(t))(\theta')^2 \;+\; mL\theta''\cos(\theta\,(t)) = TL\sin\theta\,(t)$$

Isolate the term $(\theta')^2$:

$$(\theta')^2 = \frac{(\theta'')\cos\theta(t)}{\sin(\theta(t))} - \frac{T}{m}$$

The position of the pendulum in y-direction:

$$L\cos(\theta\,(t))(\theta')^2 \;+\; L\sin(\theta\,(t))\theta'' \;=\; -g \;-\; \frac{TL\cos(\theta(t))}{m}$$

Substitute the term $(\theta')^2 = \frac{(\theta'')\cos\theta(t)}{\sin(\theta(t))} - \frac{T}{m}$

$$L\cos(\theta\,(t))\left[\frac{(\theta'')\cos\theta(t)}{\sin(\theta(t))} - \frac{T}{m}\right] \;+\; L\sin(\theta\,(t))\theta'' \;=\; -g \;-\; \frac{TL\cos(\theta(t))}{m}$$

Multiply both sides by $\sin(\theta\,(t))$*m :

$$mL\theta''\left(\cos^2(\theta\,(t)) + \sin^2(\theta\,(t))\right) \;=\; -mgsin\,(\theta\,(t)) - TL\cos(\theta\,(t))\sin(\theta\,(t)) + TL\cos(\theta\,(t))\sin(\theta$$

The last two terms cancel out.

Therefore,

$$\theta'' \;=\; -\frac{gsin(\theta(t))}{L}$$

DERIVING THE EQUATION OF MOTION (using energy)

The kinetic energy of a simple pendulum is:

$$K = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt} \right)^2$$ $K = \frac{1}{2} \mathrm{mL}^2 \left(\frac{\mathrm{d}}{\mathrm{dt}} \right)^2$

The potential energy of the pendulum is:

$$U = mgL \left(1 - \cos\theta \right)$$ $\mathrm{U} = \mathrm{mgL} \left(1 - \cos \right)$

The total energy of the pendulum is therefore:

$$E_T = K + U$$ $\mathrm{E_T} = \mathrm{K} + \mathrm{U}$

$$E_T = \frac{1}{2} m L^2 \left(\frac{d\theta}{dt} \right)^2 + mgL \left(1 - \cos\theta \right)$$ $\mathrm{E_T} = \frac{1}{2} \mathrm{mL}^2 \left(\frac{\mathrm{d}}{\mathrm{dt}} \right)^2 + \mathrm{mgL} \left(1 - \cos \right)$

The total energy of the system is constant, therefore:

$$\frac{dE}{dt} = 0$$ $\frac{\mathrm{dE}}{\mathrm{dt}} = 0$

Taking the derivative of the total energy with respect to ' should allow us to rearrage for the equation of motion of a pendulum,

$$0 = mL^2 \cdot \theta' \cdot \theta'' \ + mgL \sin\theta \cdot \theta'$$

Which finally gives us,

$$\theta''\left(t\right) = \frac{-g sin\theta}{L}$$