z-Transforms

Chapter
7

In the study of discrete-time signal and systems, we have thus far considered the time-domain and the frequency domain. The z-domain gives us a third representation. All three domains are related to each other.

A special feature of the z-transform is that for the signals and system of interest to us, all of the analysis will be in terms of ratios of polynomials. Working with these polynomials is relatively straight forward.

Definition of the z-Transform

- Given a finite length signal $x[n]$, the z-transform is defined as

$$\boxed{X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^k} \tag{7.1}$$

  where the sequence support interval is $[0, N]$, and z is any complex number

- This transformation produces a new representation of $x[n]$ denoted $X(z)$

- Returning to the original sequence (*inverse z-transform*) $x[n]$ requires finding the coefficient associated with the nth power of z^{-1}

- Formally transforming from the time/sequence/n-domain to the z-domain is represented as

$$n\text{-Domain} \overset{z}{\leftrightarrow} z\text{-Domain}$$

$$x[n] = \sum_{k=0}^{N} x[k]\delta[n-k] \overset{z}{\leftrightarrow} X(z) = \sum_{k=0}^{N} x[k]z^{-k}$$

- A sequence and its z-transform are said to form a *z-transform pair* and are denoted

$$x[n] \overset{z}{\leftrightarrow} X(z) \tag{7.2}$$

  – In the sequence or n-domain the independent variable is n

  – In the z-domain the independent variable is z

Example: $x[n] = \delta[n-n_0]$

- Using the definition

$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} \delta[k-n_0]z^{-k} = z^{-n_0}$$

- Thus,

$$\delta[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}$$

Example: $x[n] = 2\delta[n] + 3\delta[n-1] + 5\delta[n-2] + 2\delta[n-3]$

- By inspection we find that

$$X(z) = 2 + 3z^{-1} + 5z^{-2} + 2z^{-3}$$

Example: $X(z) = 4 - 5z^{-2} + z^{-3} - 2z^{-4}$

- By inspection we find that

$$x[n] = 4\delta[n] - 5\delta[n-2] + \delta[n-3] - 2\delta[n-4]$$

- What can we do with the z-transform that is useful?

The *z*-Transform and Linear Systems

- The z-transform is particularly useful in the analysis and design of LTI systems

The *z*-Transform of an FIR Filter

- We know that for any LTI system with input $x[n]$ and impulse response $h[n]$, the output is

$$y[n] = x[n] * h[n] \tag{7.3}$$

- We are interested in the z-transform of $h[n]$, where for an FIR filter

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{7.4}$$

- To motivate this, consider the input

$$x[n] = z^n, -\infty < n < \infty \tag{7.5}$$

- The output $y[n]$ is

$$\begin{aligned} y[n] &= \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k z^{n-k} \\ &= \sum_{k=0}^{M} b_k z^n z^{-k} = \left(\sum_{k=0}^{M} b_k z^{-k}\right) z^n \end{aligned} \tag{7.6}$$

- The term in parenthesis is the *z*-transform of $h[n]$, also known as the *system function* of the FIR filter

- Like $H(e^{j\omega})$ was defined in Lecture 5, we define the system function as

$$\boxed{H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}} \tag{7.7}$$

- The *z*-transform pair we have just established is

$$h[n] \overset{z}{\leftrightarrow} H(z)$$

$$\sum_{k=0}^{M} b_k \delta[n-k] \overset{z}{\leftrightarrow} \sum_{k=0}^{M} b_k z^{-k}$$

- Another result, similar to the frequency response result, is

$$y[n] = h[n] * z^n = H(z) z^n \tag{7.8}$$

  – Note if $z = e^{j\hat{\omega}}$, we in fact have the frequency response result of Chapter 6

- The system function is an Mth degree polynomial in complex variable z
- As with any polynomial, it will have M roots or *zeros*, that is there are M values z_0 such that $H(z_0) = 0$
  – These M zeros completely define the polynomial to within a gain constant (scale factor), i.e.,

$$\begin{aligned} H(z) &= b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \\ &= (1 - z_1 z^{-1})(1 - z_2 z^{-1})\cdots(1 - z_M z^{-1}) \\ &= \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{z^M} \end{aligned}$$

  where z_k, $k = 1, \ldots, M$ denote the zeros

Example: Find the Zeros of

$$h[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$$

- The z-transform is

$$\begin{aligned} H(z) &= 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2} \\ &= \left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right) \\ &= \left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right) / z^2 \end{aligned}$$

- The zeros of $H(z)$ are -1/2 and +1/3
- The difference equation

$$y[n] = 6x[n] + x[n-1] - x[n-2]$$

has the same zeros, but a different scale factor;

proof:

Properties of the *z*-Transform

- The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses

The Superposition (Linearity) Property

$$\boxed{ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z)} \tag{7.9}$$

proof

$$\begin{aligned} X(z) &= \sum_{n=0} (ax_1[n] + bx_2[n])z^{-1} \\ &= a\sum_{n=0}^{N} x_1[n]z^{-1} + b\sum_{n=0}^{N} x_2[n]z^{-1} \\ &= aX_1(z) + bX_2(z) \end{aligned}$$

The Time-Delay Property

$$x[n-1] \overset{z}{\leftrightarrow} z^{-1}X(z) \tag{7.10}$$

and

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0}X(z) \tag{7.11}$$

proof: Consider

$$X(z) = \alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_N z^{-N}$$

then

$$\begin{aligned} x[n] &= \sum_{k=0}^{N} \alpha_k \delta[n-k] \\ &= \alpha_0\delta[n] + \alpha_1\delta[n-1] + \cdots + \alpha_N\delta[n-N] \end{aligned}$$

Let

$$\begin{aligned} Y(z) &= z^{-1}X(z) \\ &= \alpha_0 z^{-1} + \alpha_1 z^{-2} + \cdots + \alpha_N z^{-N-1} \end{aligned}$$

so

$$\begin{aligned} y[n] &= \alpha_0\delta[n-1] + \alpha_1\delta[n-2] + \cdots + \alpha_N\delta[n-N-1] \\ &= x[n-1] \end{aligned}$$

Similarly

$$\begin{aligned} Y(z) &= z^{-n_0}X(z) \\ \Rightarrow y[n] &= x[n-n_0] \end{aligned}$$

A General z-Transform Formula

- We have seen that for a sequence $x[n]$ having support interval $0 \le n \le N$ the z-transform is

$$X(z) = \sum_{n=0}^{N} x[n]z^{-n} \tag{7.12}$$

- This definition extends for doubly infinite sequences having support interval $-\infty \le n \le \infty$ to

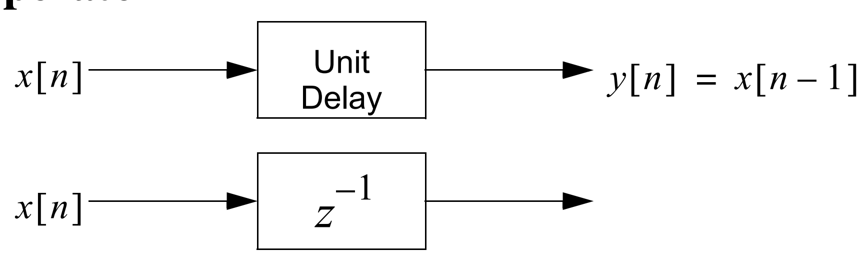
$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{7.13}$$

  – There will be discussion of this case in next Lecture when we deal with infinite impulse response (IIR) filters

The z-Transform as an Operator

The z-transform can be considered as an operator.

Unit-Delay Operator



- In the case of the unit delay, we observe that

$$y[n] = \underset{\substack{\uparrow \\ \text{unit delay operator}}}{z^{-1}}\{x[n]\} = x[n-1] \tag{7.14}$$

which is motivated by the fact that $Y(z) = z^{-1}X(z)$

- Similarly, the filter

$$y[n] = x[n] - x[n-1]$$

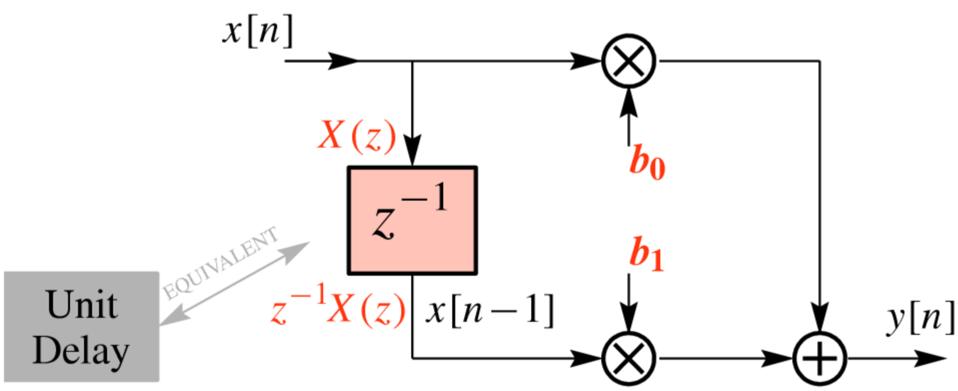
can be viewed as the operator

$$y[n] = (1 - z^{-1})\{x[n]\} = x[n] - x[n-1]$$

since

$$Y(z) = X(z) - z^{-1}X(z) = (1 - z^{-1})X(z)$$

<u>Example</u>: Two-Tap FIR



- Using the operator convention, we can write by inspection that

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$
$$y[n] = b_0 x[n] + b_1 x[n-1]$$

Convolution and the *z*-Transform

- The impulse response of the unity delay system is

$$h[n] = \delta[n-1]$$

and the system output written in terms of a convolution is

$$y[n] = x[n] * \delta[n-1] = x[n-1]$$

- The system function (z-transform of $h[n]$) is

$$H(z) = z^{-1}$$

and by the previous unit delay analysis,

$$Y(z) = z^{-1}X(z)$$

- We observe that

$$\boxed{Y(z) = H(z)X(z)} \tag{7.15}$$

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proof: $\mathcal{Z}(y[n]) = Y(z)$

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k] \tag{7.16}$$

We now take the z-transform of both sides of (7.16) using superposition and the general delay property

$$\begin{aligned} Y(z) &= \sum_{k=0}^{M} h[k](z^{-k}X(z)) \\ &= \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z) \end{aligned} \tag{7.17}$$

- Note: For the case of $x[n]$ a finite duration sequence, $X(z)$ is a polynomial, and $H(z)X(z)$ is a product of polynomials in z^{-1}

Example: Convolving Finite Duration Sequences

- Suppose that

$$\begin{aligned} x[n] &= 2\delta[n] - 3\delta[n-2] + 4\delta[n-3] \\ h[n] &= \delta[n] + 2\delta[n-1] + \delta[n-2] \end{aligned}$$

- We wish to find $y[n]$ by first finding $Y(z)$
- We begin by z-transforming each of the sequences

$$\begin{aligned} X(z) &= 2 - 3z^{-2} + 4z^{-3} \\ H(z) &= 1 + 2z^{-1} + z^{-2} \end{aligned}$$

- We find $Y(z)$ by direct multiplication

$$\begin{aligned} Y(z) &= (2 - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} + z^{-2}) \\ &= 2 + 4z^{-1} - z^{-2} - 2z^{-3} + 5z^{-4} + 4z^{-5} \end{aligned}$$

- We find $y[n]$ using the delay property on each of the terms of $Y(z)$

$$\begin{aligned} y[n] = {} & 2\delta[n] + 4\delta[n-1] - \delta[n-2] \\ & - 2\delta[n-3] + 5\delta[n-4] + 4\delta[n-5] \end{aligned}$$

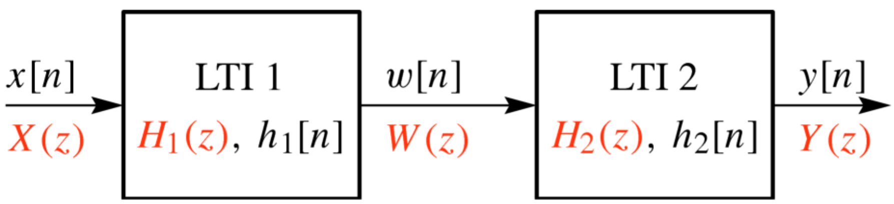
Convolve directly?

- This section has established the very important result that polynomial multiplication can be used to replace sequence convolution, when we work in the z-domain, i.e.,

$$\boxed{\begin{array}{c}\text{z-Transform Convolution Theorem} \\ y[n] = h[n]*x[n] \overset{z}{\leftrightarrow} H(z)X(z) = Y(z)\end{array}}$$

Cascading Systems

- We have seen cascading of systems in the time-domain and the frequency domain, we now consider the z-domain



- We know from the convolution theorem that

$$W(z) = H_1(z)X(z)$$

- It also follows that

$$Y(z) = H_2(z)W(z)$$

  so by substitution

$$\begin{aligned} Y(z) &= [H_2(z)H_1(z)]X(z) \\ &= [H_1(z)H_2(z)]X(z) \end{aligned} \tag{7.18}$$

- In summary, when we cascade two LTI systems, we arrive at the cascade impulse response as a cascade of impulse responses in the time-domain and a product of the z-transforms in the z-domain

$$\boxed{h[n] = h_1[n] * h_2[n] \overset{z}{\leftrightarrow} H_1(z)H_2(z) = H(z)}$$

Factoring z-Polynomials

- Multiplying z-transforms creates a cascade system, so factoring must create subsystems

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Example: $H(z) = 1 + 3z^{-1} - 2z^{-2} + z^{-3}$

- Since $H(z)$ is a third-order polynomial, we should be able to factor it into a first degree and second degree polynomial
- We can use the MATLAB function `roots()` to assist us

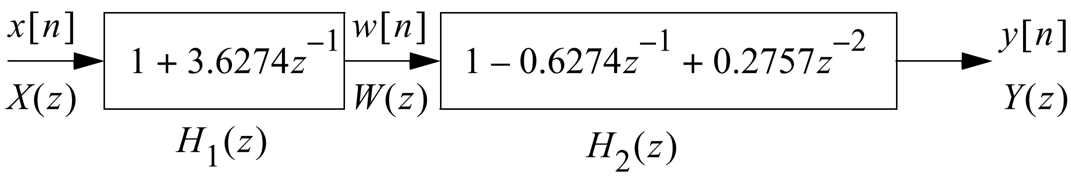
```
>> p = roots([1 3 -2 1])

p = -3.6274
     0.3137 + 0.4211i
     0.3137 - 0.4211i

>> conv([1 -p(2)],[1 -p(3)])

ans =  1.0000    -0.6274      0.2757 - 0.0000i
```

- With one real root, the logical factoring is to create two polynomials as follows

$$H_1(z) = 1 + 3.6274z^{-1}$$

$$H_2(z) = (1 - (0.3137 + j0.4211)z^{-1})$$
$$(1 - (0.3137 - j0.4211)z^{-1})$$
$$= 1 - 0.6274z^{-1} + 0.2757z^{-2}$$

- The cascade system is thus:



- As a check we can multiply the polynomials

```
>> conv([1 -p(1)],conv([1 -p(2)],[1 -p(3)]))

ans = 1.0000, 3.0000, -2.0000-0.0000i, 1.0000-0.0000i
```

- The difference equations for each subsystem are

$$w[n] = x[n] + 3.6274x[n-1]$$
$$y[n] = w[n] - 0.6274w[n-1] + 0.2757w[n-2]$$

Deconvolution/Inverse Filtering

- In a two subsystems cascade can the second system undo the action of the first subsystem?
- For the output to equal the input we need $H(z) = 1$
- We thus desire

$$H_1(z)H_2(z) = 1 \text{ or } H_2(z) = \frac{1}{H_1(z)}$$

<u>Example</u>: $H_1(z) = 1 - az^{-1}, |a| < 1$

- The inverse filter is

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - az^{-1}}$$

- This is no longer an FIR filter, it is an infinite impulse response (IIR) filter, which is the topic of next Chapter
- We can approximate $H_2(z)$ as an FIR filter via long division

$$\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + \cdots \\
\hline
1 - az^{-1} & 1 \\
 & \underline{1 - az^{-1}} \\
 & \quad az^{-1} \\
 & \quad \underline{az^{-1} - a^2z^{-2}} \\
 & \qquad\quad a^2z^{-2} \\
 & \qquad\quad \underline{a^2z^{-2} - a^3z^{-3}} \\
 & \qquad\qquad\qquad a^3z^{-3}
\end{array}$$

- An $M+1$ term approximation is

$$H_2(z) = \sum_{k=0}^{M} a^k z^{-k}$$

–

Relationship Between the z-Domain and the Frequency Domain

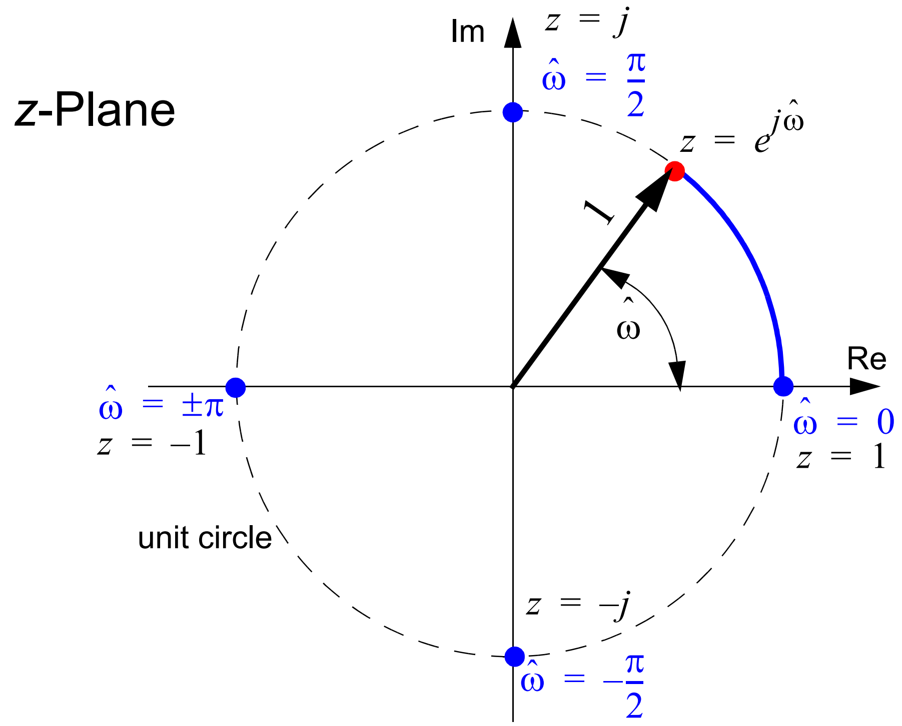
$\hat{\omega}$ - Domain		z - Domain
$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$	versus	$H(z) = \sum_{k=0}^{M} b_k z^{-k}$

- Comparing the above we see that the connection is setting $z = e^{j\hat{\omega}}$ in $H(z)$, i.e.,

$$\boxed{H(e^{j\hat{\omega}}) = H(z)\big|_{z = e^{j\hat{\omega}}}} \tag{7.19}$$

The z-Plane and the Unit Circle

- If we consider the z-plane, we see that $H(e^{j\hat{\omega}})$ corresponds to evaluating $H(z)$ on the unit circle



- From this interpretation we also can see why $H(e^{j\hat{\omega}})$ is periodic with period 2π
  - As $\hat{\omega}$ increases it continues to sweep around the unit circle over and over again

The Zeros and Poles of $H(z)$

- Consider

$$H(z) = 1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} \tag{7.20}$$

where we have assumed that $b_0 = 1$

- Factoring $H(z)$ results in

$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1})(1 - z_3 z^{-1}) \tag{7.21}$$

- Multiplying by z^3/z^3 allows to write $H(z)$ in terms of positive powers of z

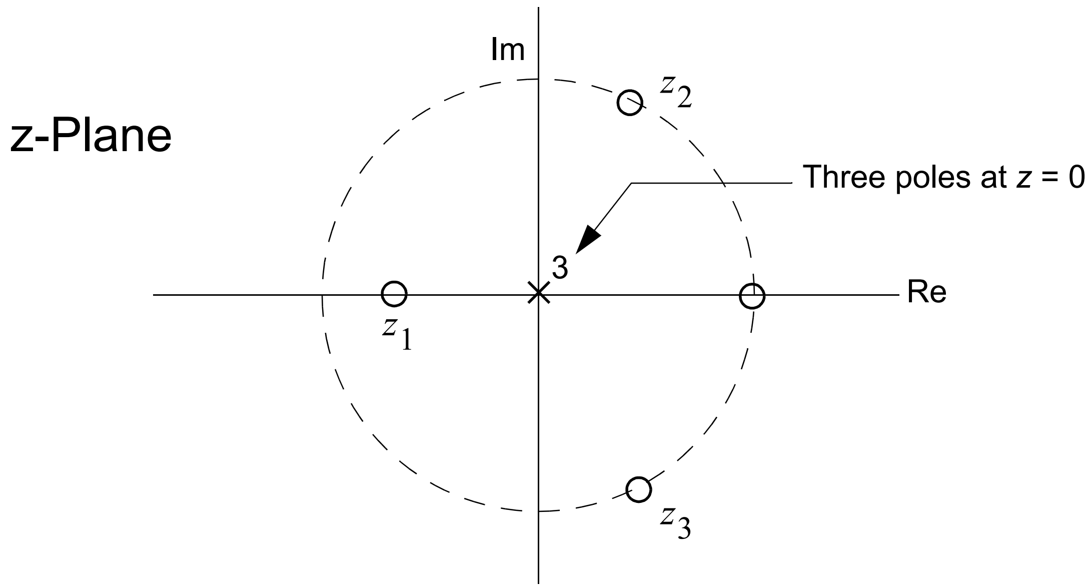
$$\begin{aligned} H(z) &= \frac{z^3 + b_1 z^2 + b_2 z^1 + b_3 z^0}{z^3} \\ &= \frac{(z - z_1)(z - z_2)(z - z_3)}{z^3} \end{aligned} \tag{7.22}$$

numerator

- The *zeros* are the locations where $H(z) = 0$, i.e., z_1, z_2, z_3
- The *poles* are where $H(z) \to \infty$, i.e., $z \to 0$
- <u>Note</u> that the poles and zeros only determine $H(z)$ to within a constant; recall the example on page 7-5

denominator

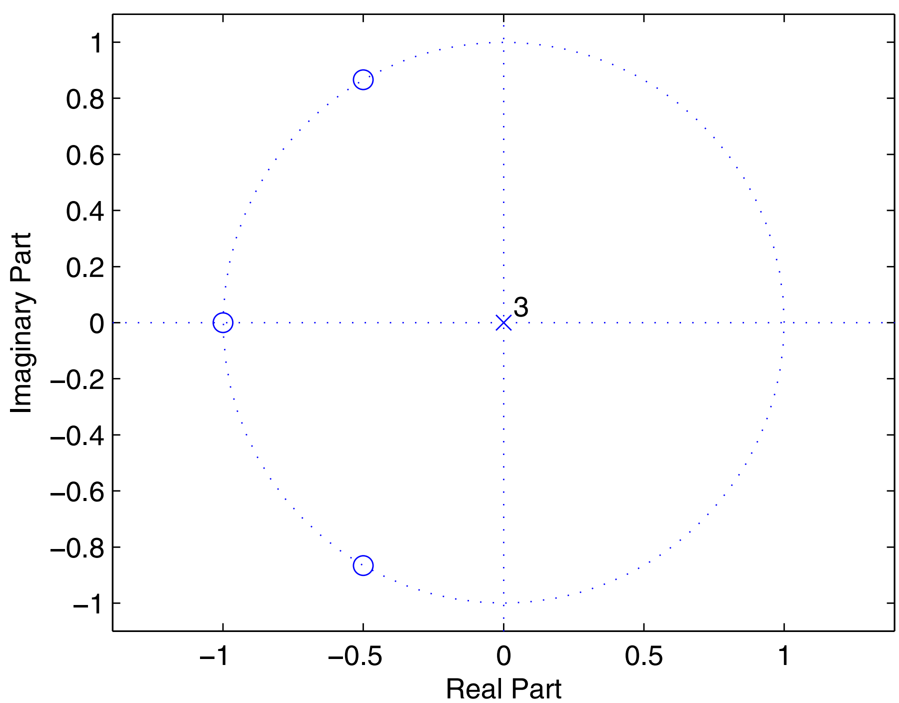
- A *pole-zero plot* displays the pole and zero locations in the *z*-plane



Example: $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

- MATLAB has a function that supports the creation of a pole-zero plot given the system function coefficients

```
>> zplane([1 2 2 1],1)
```



The Significance of the Zeros of $H(z)$

- The difference equation is the actual time domain means for calculating the filter output for a given filter input
- The difference equation coefficients are the polynomial coefficients in $H(z)$
- For $x[n] = z_0^n$ we know that

$$y[n] = H(z_0)z_0^n, \tag{7.23}$$

so in particular if z_0 is one of the zeros of $H(z)$, $H(z_0) = 0$ and the output $y[n] = 0$

- <u>If a zero lies on the unit circle</u> then the output will be zero for a sinusoidal input of the form

$$x[n] = z_0^n = (e^{j\hat{\omega}_0})^n = e^{j\hat{\omega}_0 n} \tag{7.24}$$

where $\hat{\omega}_0$ is the angle of the zero relative to the real axis, which is also the frequency of the corresponding complex sinusoid; why?

$$y[n] = \left(H(z)\Big|_{z = e^{j\hat{\omega}_0}} \right) e^{j\hat{\omega}_0 n} = 0 \tag{7.25}$$

Nulling Filters

- The special case of zeros on the unit circle allows a filter to *null*/block/*annihilate* complex sinusoids that enter the filter at frequencies corresponding to the angles the zeros make with respect to the real axis in the z-plane

- The nulling property extends to real sinusoids since they are composed of two complex sinusoids at $\pm\hat{\omega}_0$, and zeros not on the real axis will always occur in conjugate pairs if the filter coefficients are real
- This nulling/annihilating property is useful in rejecting unwanted jamming and interference signals in communications and radar applications

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<u>Example</u>: $H(z) = 1 - 2\cos(\hat{\omega}_0)z^{-1} + z^{-2}$, $x[n] = \cos(\hat{\omega}_0 n)$

- Factoring $H(z)$ we find that

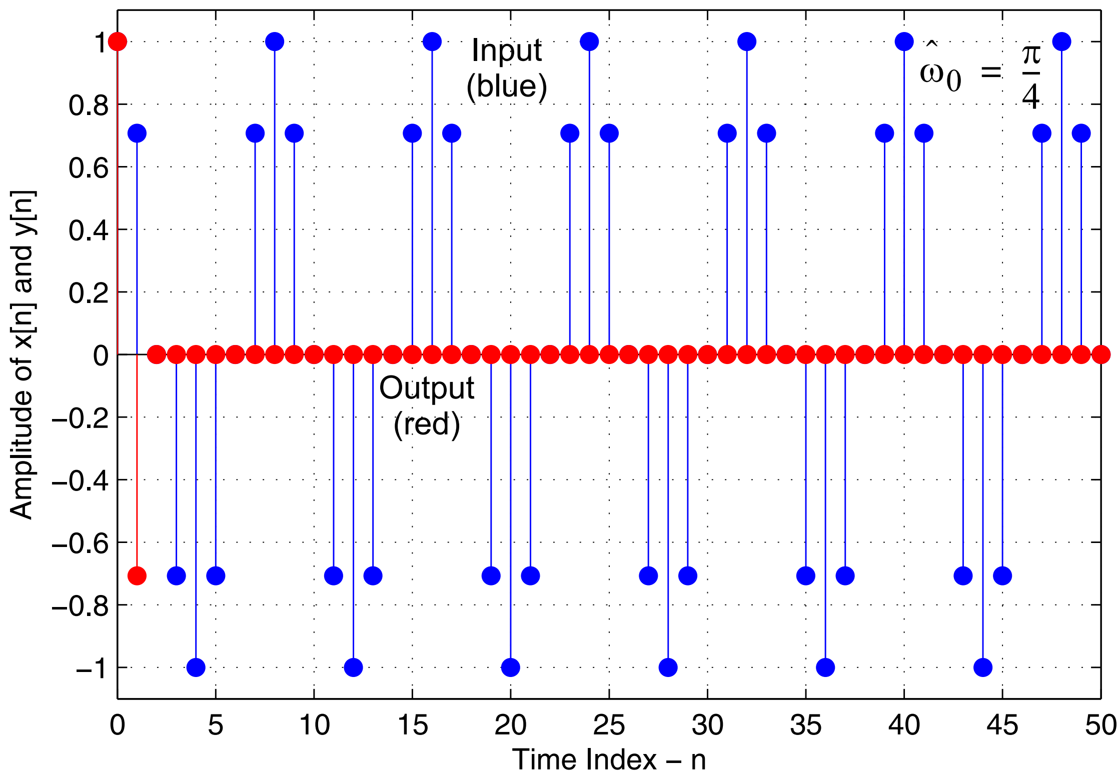
$$H(z) = \Big(1 - \underbrace{e^{j\hat{\omega}_0}}_{z_1} z^{-1}\Big)\Big(1 - \underbrace{e^{-j\hat{\omega}_0}}_{z_2} z^{-1}\Big)$$

- Expanding $x[n]$ we see that

$$x[n] = \frac{1}{2}e^{-j\hat{\omega}_0 n} + \frac{1}{2}e^{j\hat{\omega}_0 n}$$

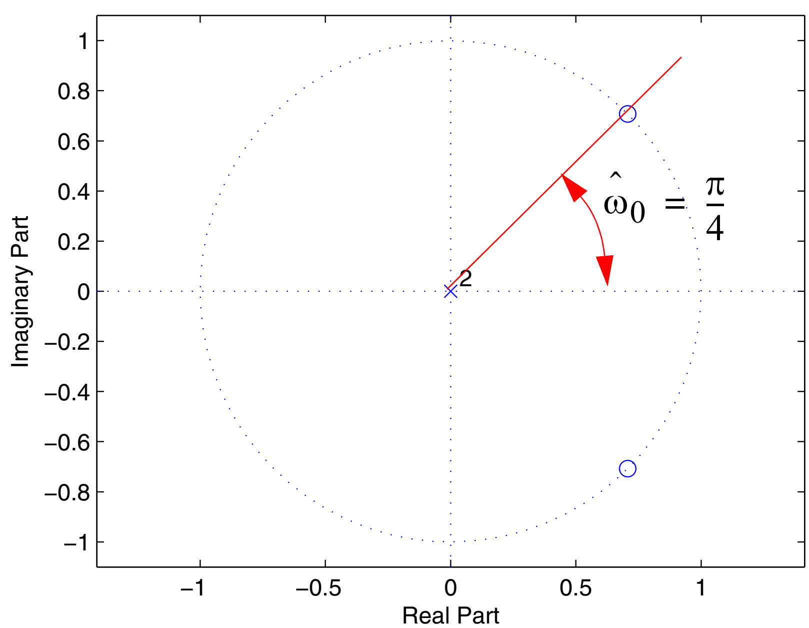
- The nulling action of $H(z)$ at $\pm\hat{\omega}_0$ will remove the signal from the filter output
- We can set up a simple simulation in MATLAB to verify this

```
>> n = 0:100;
>> w0 = pi/4;
>> x = cos(w0*n);
>> y = filter([1 -2*cos(w0) 1],1,x);
>> stem(n,x,'filled')
>> hold
Current plot held
>> stem(n,y,'filled','r')
>> axis([0 50 -1.1 1.1]); grid
```



- Since the input is applied at $n = 0$, we see a small transient while the filter settles to the final output, which in this case is zero

```
>> zplane([1 -2*cos(w0) 1],1)% check the pole-zero plot
```



Graphical Relation Between z and $\hat{\omega}$

- When we make the substitution $z = e^{j\hat{\omega}}$ in $H(z)$ we know that we are evaluating the z-transform on the unit circle and thus obtain the frequency response
- If we plot say $|H(z)|$ over the entire z-plane we can visualize how cutting out the response on just the unit circle, gives us the frequency response magnitude

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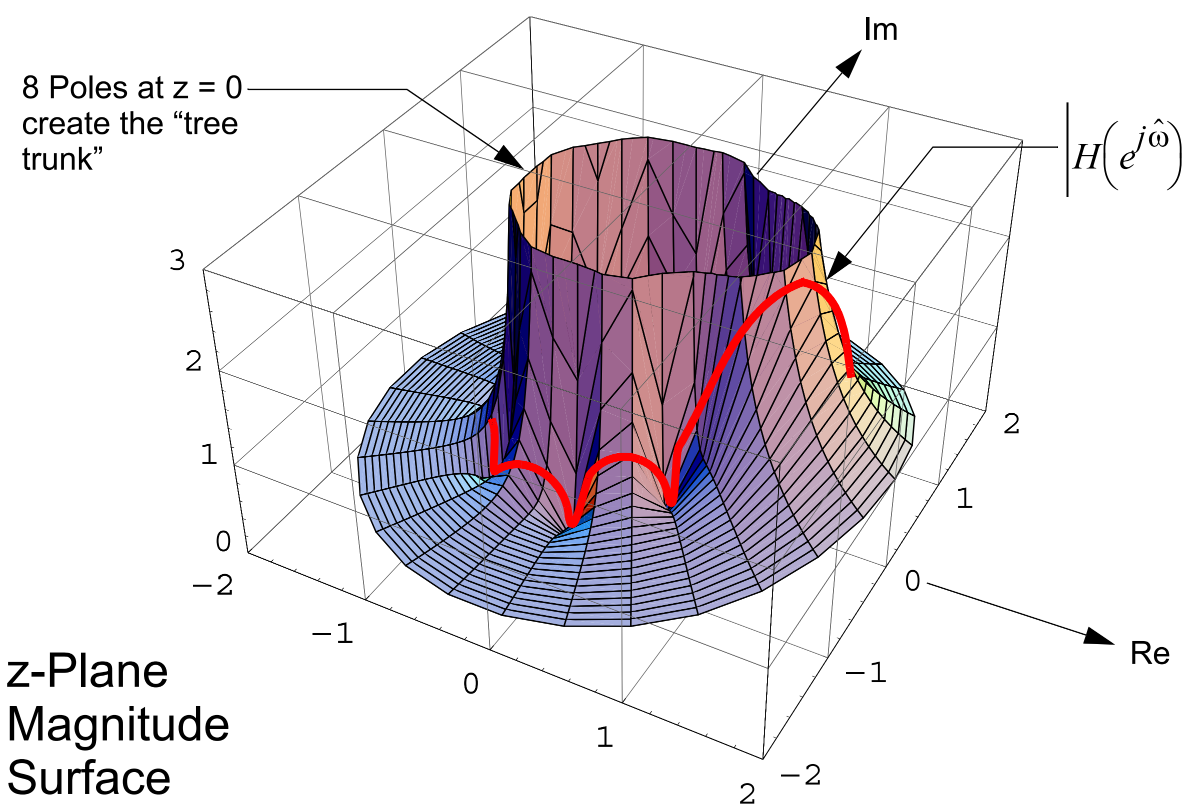
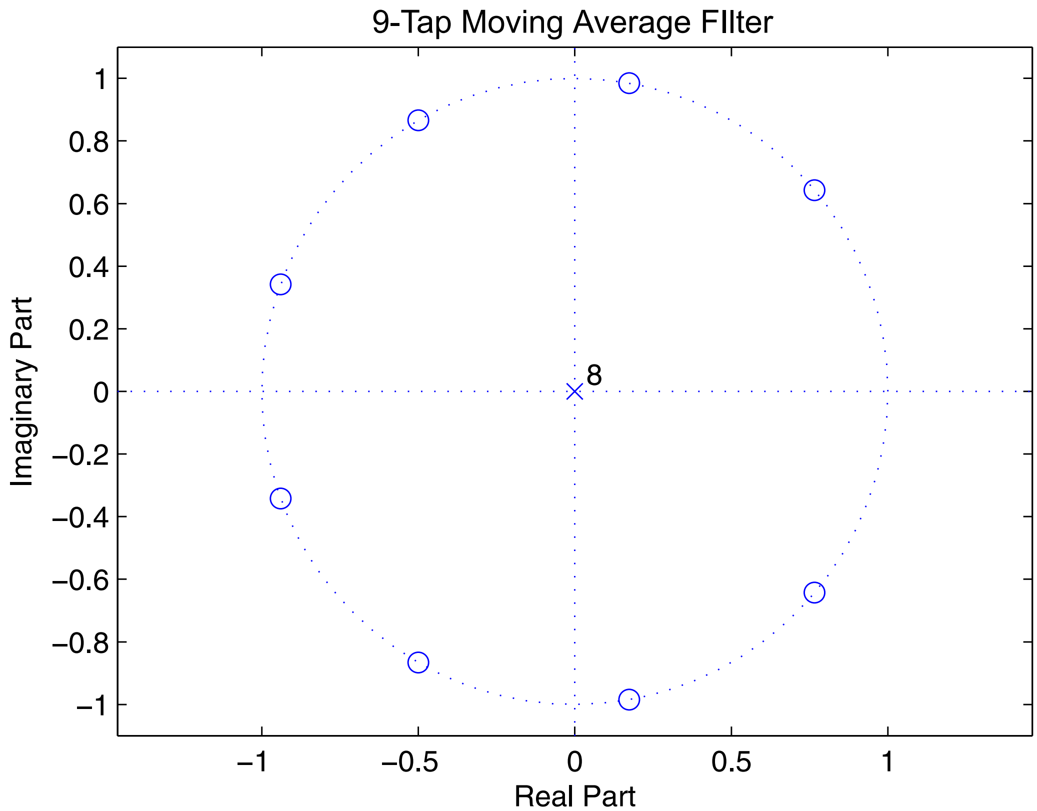
<u>Example</u>: $L = 9$ Moving Average Filter (9 taps/8th-order)

- Here we have

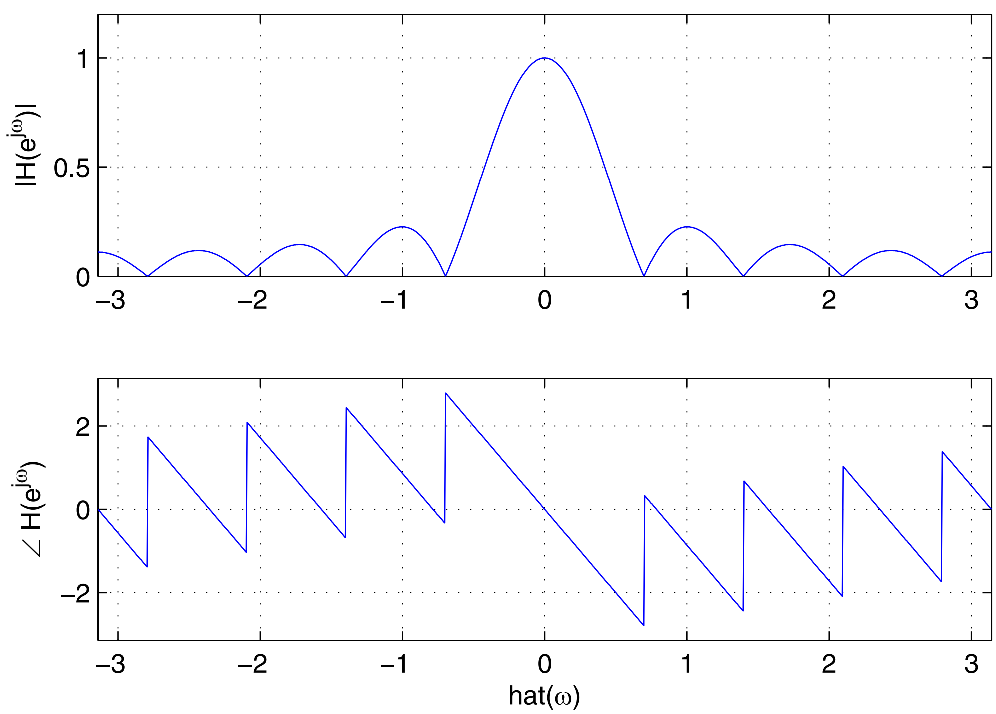
$$H(z) = \frac{1}{9}\sum_{k=0}^{9-1} z^{-k} = \frac{1}{9}\prod_{k=1}^{8}(1 - e^{-j2\pi k/9}z^{-1})$$



```
>> zplane([ones(1,9)]/9,1)
```




```
>> w = -pi:(pi/500):pi;
>> H = freqz([ones(1,9)/9],1,w);
```



Useful Filters

The L-Point Moving Average Filter

- The L-point moving average (running sum) filter has

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \tag{7.26}$$

and system function (z-transform of the impulse response)

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} \tag{7.27}$$

- The sum in (7.27) can be simplified using the geometric series sum formula

$$H(z) = \frac{1}{L}\sum_{k=0}^{L-1} z^{-k} = \frac{1}{L}\cdot\frac{1-z^{-L}}{1-z^{-1}} = \frac{1}{L}\cdot\frac{z^{L}-1}{z^{L-1}(z-1)} \tag{7.28}$$

- Notice that the zeros of $H(z)$ are determined by the roots of the equation

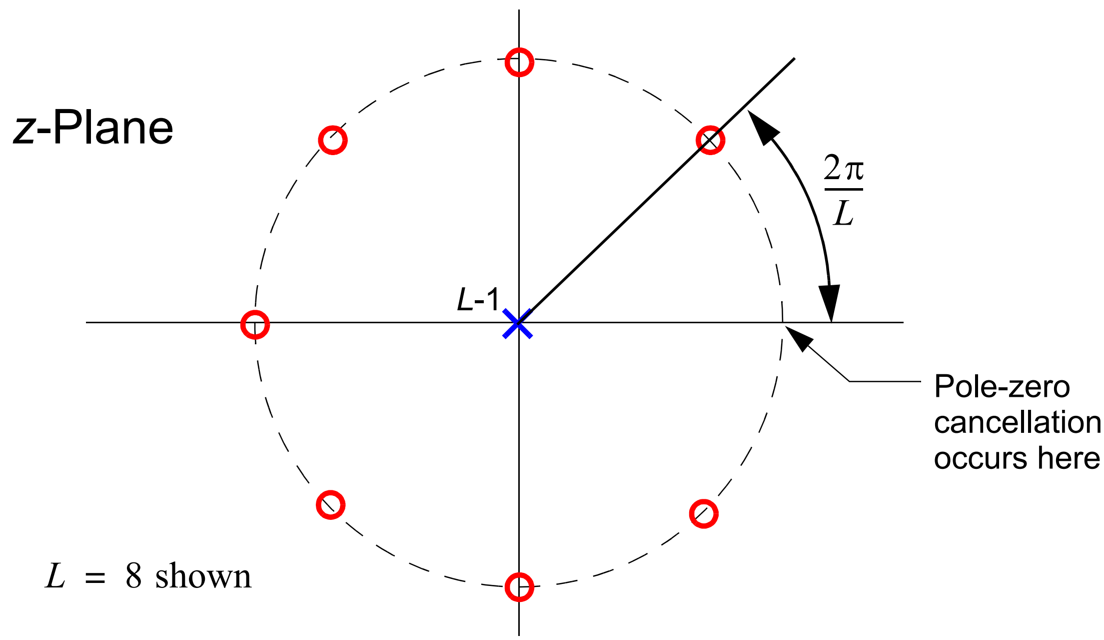
$$z^{L} - 1 = 0 \Rightarrow z^{L} = 1 \tag{7.29}$$

- The roots of this equation can be found by noting that $e^{j2\pi k} = 1$ for k any integer, thus the roots of (7.29) (zeros of (7.28)) are

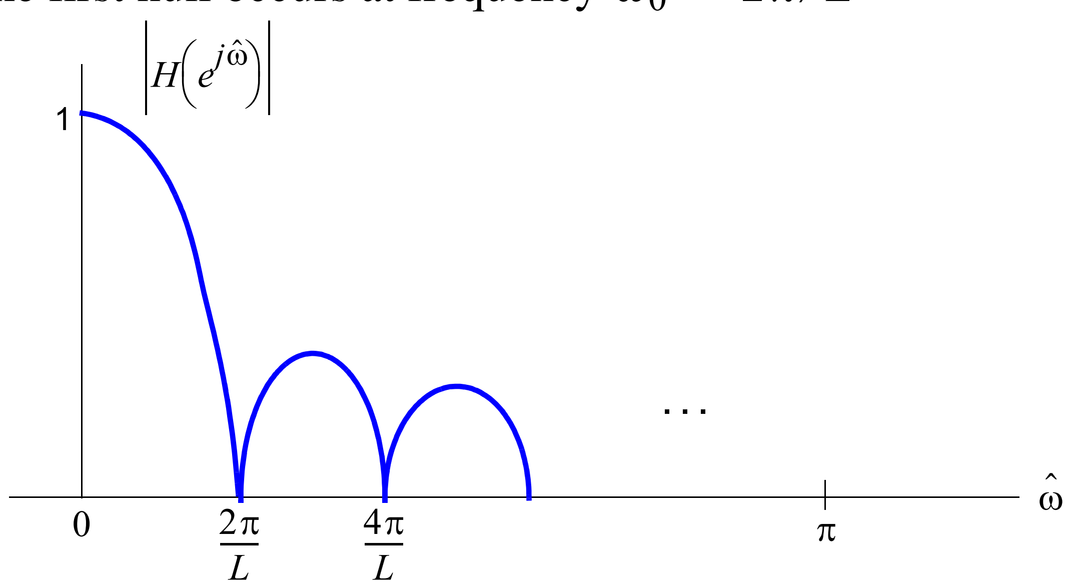
$$z_k = e^{j2\pi k/L},\ k = 0, 1, 2, \ldots, L-1 \tag{7.30}$$

- These roots are referred to as the L roots of unity

- One of the zeros sits at $z = 1$, but there is also a pole at $z = 1$, so there is a pole-zero cancellation, meaning that the pole-zero plot of $H(z)$ corresponds to the L-roots of unity, less the root at $z = 0$



- We have seen the frequency response of this filter before
- The first null occurs at frequency $\hat{\omega}_0 = 2\pi/L$



A Complex Bandpass Filter

see text

A Bandpass Filter with Real Coefficients

see text

Practical Filter Design

- Here we will use `fdatool` from the MATLAB signal processing toolbox to design an FIR filter

Properties of Linear-Phase Filters

- A class of FIR filters having symmetrical coefficients, i.e., $b_k = b_{M-k}$ for $k = 0, 1, \ldots, M$ has the property of linear phase

The Linear Phase Condition

- For a filter with symmetrical coefficients we can show that $H(e^{j\hat{\omega}})$ is of the form

$$H(e^{j\hat{\omega}}) = R(e^{j\hat{\omega}})e^{-j\omega M/2} \tag{7.31}$$

  where $R(e^{j\hat{\omega}})$ is a real function

- The fact that $R(e^{j\hat{\omega}})$ is real means that the phase of $H(e^{j\hat{\omega}})$ is a linear function of frequency plus the possibility of $\pm\pi$ phase jumps whenever $R(e^{j\hat{\omega}})$ passes through zero

Example: $H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_1 z^{-3} + b_0 z^{-4}$

- By factoring out z^{-2} we can write

$$H(z) = [b_0(z^2 + z^{-2}) + b_1(z^1 + z^{-1}) + b_2]z^{-2}$$

- We now move to the frequency response by letting $z \to e^{j\hat{\omega}}$

$$H(e^{j\hat{\omega}}) = [2b_0 \cos(2\hat{\omega}) + 2b_1 \cos(\hat{\omega}) + b_2]e^{-j\hat{\omega}4/2}$$

- Note that here we have $M = 4$, so we see that the linear phase term is indeed of the form $e^{-j\hat{\omega}M/2}$ and the real function $R(e^{j\hat{\omega}})$ is of the form

$$R(e^{j\hat{\omega}}) = b_2 + 2[b_0 \cos(2\hat{\omega}) + b_1 \cos(\hat{\omega})]$$

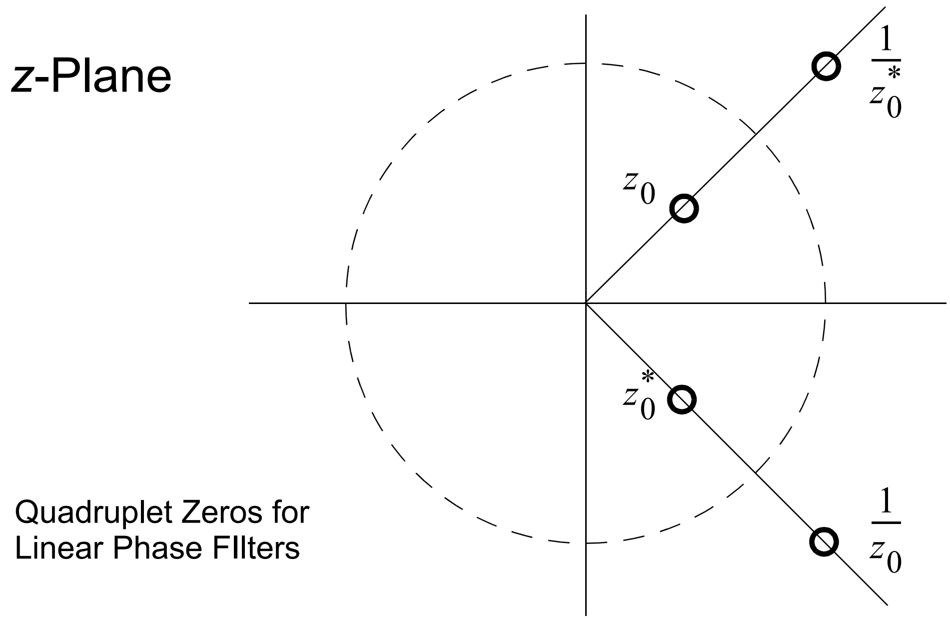
Locations of the Zeros of FIR Linear-Phase Systems

- Further study of $H(z)$ for the case of symmetric coefficients reveals that

$$H(1/z) = z^M H(z) \tag{7.32}$$

- A consequence of this condition is that for $H(z)$ having a zero at z_0 it will also have a zero at $1/z_0$
- Assuming the filter has real coefficients, complex zeros occur in conjugate pairs, so the even symmetry condition further implies that the zeros occur as quadruplets

$$\left\{ z_0, z_0^*, \frac{1}{z_0}, \frac{1}{z_0^*} \right\}$$



Example: $H(z) = 1 - 2z^{-1} + 4z^{-2} - 2z^{-3} + z^{-4}$

```
>> zplane([1 -2 4 -2 1],1)
```