Assorted Proofs in Analysis

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Contents

1 Sets and Functions

1.1 Theorem 1.1

WTS. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We can use a chain of equivalences. Suppose that both members are not empty.

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \land x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \land x \in B) \text{ or } (x \in A \land x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Now suppose one of the two members are empty. Then if the other member was not empty, it would imply that the original member was not empty, and this means that the two sets must be equal. $\square$

1.2 Theorem 1.2

WTS. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Define $W = (A \setminus B) \cup (B \setminus A)$, then we will apply Q1a, and deMorgan's Theorem.

$$\begin{aligned} W^c &= (A^c \cup B) \cap (B^c \cup A) \\ &= ((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A) \\ &= (A^c \cap B^c) \cup (A \cap B) \\ &= (A \cup B)^c \cup (A \cap B) \\ &= [(A \cup B) \setminus (A \cap B)]^c \end{aligned}$$

Taking complements on both sides finishes the proof. $\square$

1.3 Theorem 1.3

WTS. $f : A \to B$ is a function, and $B_1, B_2 \subseteq B$. Show that f^{-1} preserves unions, intersections and complements.

Lemma 1.1. f^{-1} preserves unions.

Proof. Fix two subsets $B_1, B_2 \subseteq B$, then

$$\begin{aligned} f^{-1}(B_1 \cup B_2) &= \{x \in A, f(x) \in B_1 \cup B_2\} \\ &= \{x \in A, f(x) \in B_1 \text{ or } f(x) \in B_2\} \\ &= \{x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)\} \\ &= f^{-1}(B_1) \cup f^{-1}(B_2) \end{aligned}$$

□

Lemma 1.2. f^{-1} preserves complements.

Proof. For every $E \subseteq B$,

$$\begin{aligned} f^{-1}(B \setminus E) &= \{x \in A : f(x) \in B \setminus E\} \\ &= \{x \in A, f(x) \in E^c\} \\ &= A \setminus f^{-1}(E) \end{aligned}$$

□

Lemma 1.3. f^{-1} preserves intersections.

Proof. Now we wish to prove that f^{-1} preserves intersections as well, for every pair of subsets, $B_1, B_2 \subseteq B$. Write their intersection as $(B_1^c \cup B_2^c)^c$, apply Lemma (1.1), and take complements (then apply Lemma (1.2))

$$\begin{aligned} f^{-1}\left((B_1^c \cup B_2^c)^c\right) &= (f^{-1}(B_1^c) \cup f^{-1}(B_2^c))^c \\ &= f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

□

1.4 Theorem 1.4

WTS. $f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. For any two sets $A_1, A_2 \subseteq A$,

$$\begin{aligned} f(A_1 \cup A_2) &= \{f(x) : x \in A_1 \cup A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } x \in A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } f(x) : x \in A_2\} \\ &= f(A_1) \cup f(A_2) \end{aligned}$$

$\square$

Corollary 1.3.1. The direct image is monotonic. For every $E_1 \subseteq E_2 \subseteq A$, then $f(E_1) \subseteq f(E_2) \subseteq B$.

Proof. Apply Theorem 1.4 to sets $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$, then $f(E_2) = f(E_2 \setminus E_1) \cup f(E_2 \cap E_1)$ implies that $f(E_1) \subseteq f(E_2)$. $\square$

1.5 Theorem 1.5

WTS. Subset relations. The following are equivalent.

1. $A \subseteq B$
2. $A \cap B = A$
3. $A \cup B = B$
4. $A \subseteq B^c$
5. $A \setminus B = \varnothing$

Proof. $\square$

1.6 Theorem 1.6

WTS. $f(f^{-1}B) = B$ if f is a surjection, and $f^{-1}(f(B)) = B$ if f is an injection.

We split this problem into two parts. We begin with the first assertion. Write $R = \{f(x) : x \in A\}$.

Lemma 1.4. For every function $f : X \to Y$, $f(f^{-1}(B)) \subseteq B$.

Proof. Use Q5a) onto the disjoint sets $f^{-1}(B \cap R)$ and $f^{-1}(B \cap R^c)$, then

$$f^{-1}(B) = f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)$$

Now $f^{-1}(B \cap R^c)$ must be empty, since no $x \in A$ satisfies $f(x) \in B \cap R^c$. Hence $f^{-1}(B) = f^{-1}(B \cap R)$.

$$\begin{aligned} f(f^{-1}(B)) &= f(f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \cup f(f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \\ &= \{f(x) : x \in f^{-1}(B \cap R)\} \\ &= \{y : y \in (B \cap R)\} \\ &= B \cap R \end{aligned}$$

Where for the second last equality we used the fact that f is always a surjection onto its range. Then $f(f^{-1}(B)) = B \cap R \subseteq B$. □

Remark. If f is a surjection, then its range $R = Y$, then $f(f^{-1}(B)) = B \cap Y = B$.

Lemma 1.5. For every function $f : X \to Y$, $A \subseteq f^{-1}(f(A))$.

Proof. Write $f^{-1}(f(A))$ as the disjoint union of $A \cap f^{-1}(f(A))$ and $A^c \cap f^{-1}(f(A))$. Then, we shall show that $f^{-1}(f(A)) = A$. For every $x \in A$,

$$\begin{aligned} f(x) \in f(A) \wedge x \in A &\iff x \in f^{-1}(f(A)) \wedge x \in A \\ &\iff x \in A \cap \left(f^{-1}(f(A))\right) \end{aligned}$$

Hence $A \cap f^{-1}(f(A)) = A$, and $A \subseteq f^{-1}(f(A))$

Remark. If f is a injection, then for every $x \in A^c$, $f(x) \notin f(A)$, then $A^c \cap f^{-1}(f(A)) = \varnothing$, and

$$f^{-1}(f(A)) = \left[A \cap f^{-1}(f(A))\right] \cup \left[A^c \cap f^{-1}(f(A))\right] = A$$

□

1.7 Theorem 1.7

WTS. For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|_A + f|_{A^c}$, and Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restriction of f onto A and A^c are bijections onto their direct images, then f is a bijection.

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where we shall omit the trivial case of them both belonging to the same A or A^c. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by assumption $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f|_{A^c}(A^c)$. So $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in f_A(A)$ or $y \in f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets are bijections, there exists a corresponding $x \in X$ which will satisfy. This completes the proof. $\square$

1.8 Theorem 1.8

WTS. Let $f \in B^A$ satisfy the hypothesis of the previous Theorem 1.7, so that $(f|_A)^{-1}$ and $(f|_{A^c})^{-1}$ both exist, and $f|_A(A) \cap f|_{A^c}(A^c) = \varnothing$, then $f^{-1} = (f|_A)^{-1} + (f|_{A^c})^{-1} = (f^{-1})|_{B_1} + (f^{-1})|_{B_2}$, where $f|_A(A) = B_1$, and $f|_{A^c}(A^c) = B_2$.

Proof. Since B_1 and B_2 are disjoint, then fix any $y \in Y$. Without loss of generality, let us assume that $y \in B_1$. Then, $f^{-1}(y) = (f^{-1})|_{B_1}(y) = (f|_A)^{-1}(y)$. This inverse is indeed well defined, since $f|_A$ is a bijection onto its range, then there exists a unique $x \in A$ such that applying f on both sides yield

$$f\left((f|_A)^{-1}(y)\right) = f \circ (f|_A)^{-1}(y) = y$$

In the same manner, fix an $x \in A$ such that $f(x) = f|_A(x) \in B_1$, then applying $(f|_A)^{-1}$ on both sides

$$(f|_A)^{-1}(f(x)) = (f|_A)^{-1} \circ f(x) = x$$

Therefore the inverse of f can be written piecewise on two disjoint domains as follows.

$$f^{-1} = f^{-1}|_{B_1} + f^{-1}_{B_2}$$

□

Remark. We adopt a slight abuse of notation with the 'restrictions' onto f, but they should be interpreted as piecewise functions. $f|_A + f|_{A^c}$ is equal to $f|_A\chi_A + f|_{A^c}\chi_{A^c}$ where χ is the indicator function.

1.9 Theorem 1.9

WTS. If $a|(b+c)$, and $\gcd(b,c)=1$, then

$$\gcd(a,b)=\gcd(a,c)=1$$

Proof. Suppose $\gcd(a,b)\geq 1$, then there exists some $y\geq 2$ such that $y|a$ and $y|b$, $\square$

1.10 Theorem 1.10

WTS.

Proof. $\square$

1.11 Theorem 1.11

WTS.

Proof. □

1.12 Theorem 1.12

WTS.

Proof. $\square$

1.13 Theorem 1.13

WTS.

Proof. $\square$

1.14 Theorem 1.14

WTS.

Proof. $\square$

1.15 Theorem 1.15

WTS.

Proof. □

1.16 Theorem 1.16

WTS.

Proof. $\square$

1.17 Theorem 1.17

WTS.

Proof. $\square$

1.18 Theorem 1.18

WTS.

Proof. $\square$

2 The Real Numbers

2.1 Triangle Inequality

WTS. Prove the Triangle Inequality with $n \geq 2$.

Lemma 2.1. The Triangle Inequality, for every $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

Proof. Notice that for every $a, b \in \mathbb{R}$

- $-|a| \leq a \leq +|a|$
- $-|b| \leq b \leq +|b|$

Add two inequalities together, then we get

$$-(|a| + |b|) \leq (a + b) \leq +(|a| + |b|)$$

Since $|a| + |b| \geq 0$, we get $|a + b| \leq |a| + |b|$ as desired. Because for every $c \geq 0$,

$$\begin{aligned} c \geq |a| &\iff c \geq \max\{-a, +a\} \\ &\iff c \geq a \text{ and } c \geq -a \\ &\iff c \geq a \text{ and } -c \leq a \\ &\iff -c \leq a \leq +c \end{aligned}$$

$\square$

Main proof for Triangle Inequality

Proof. For every x_1, x_2 in $\mathbb{R}$,

$$\left|\sum_{j=1}^{2} x_j\right| \leq \sum_{j=1}^{2} |x_j|$$

This proves the non trivial base case, if $n = 1$ then obviously $|x_1| \leq |x_1|$. We now suppose that

$$\left|\sum_{j=1}^{k} x_j\right| \leq \sum_{j=1}^{k} |x_j|$$

Then write

$$\left|\sum_{j=1}^{k+1} x_j\right| = \left|\left(\sum_{j=1}^{k} x_j\right) + (x_{k+1})\right|$$

Applying the second estimate to the right member gives us

$$\left|\sum_{j=1}^{k+1} x_j\right| \leq \left|\sum_{j=1}^{k} x_j\right| + |x_{k+1}| = \sum_{j=1}^{k+1} |x_j|$$

This completes the proof. □

2.2 Lemma

Lemma 2.2. If A is a non-empty bounded above subset of $\mathbb{R}$, then for every $c \geq 0, c(\sup A) = \sup cA$

Proof. Let $s = \sup(A)$, then

$$cA = \{cx : x \in A\}$$

Then for every $x \in A$

$$x \leq s \implies cx \leq cs \implies cA \leq cs$$

If $c = 0$, then the equality is trivial since $cA = \{0\}$, if not, for every $\varepsilon/c > 0$, there exists an $x \in A$ such that

$$s - \frac{\varepsilon}{c} < x \implies cx - \varepsilon < cx$$

This establishes the Lemma. □

2.3 Lemma

Lemma 2.3. If A is a non-empty bounded below subset of $\mathbb{R}$, then for every $c \geq 0, c(\inf A) = \inf cA$.

Proof. Let $w = \inf(A)$, then

$$cA = \{cx : x \in A\}$$

Then for every $x \in A$

$$w \leq x \implies cw \leq cx \implies cw \leq cA$$

If $c = 0$, then the equality is trivial since $cA = \{0\}$, if not, for every $\varepsilon/c > 0$, there exists an $x \in A$ such that

$$x \leq w + \frac{\varepsilon}{c} \implies cx \leq cw + \varepsilon$$

This establishes Lemma 2.3. □

2.4 Lemma

Lemma 2.4. If A is a non-empty bounded above subset of $\mathbb{R}$, then $(-1)(\sup A) = \inf(-1)A$.

Proof. Let $s = \sup(A)$, then

$$(-1)A = \{-x : x \in A\}$$

Evidently, for every $x \in A$

$$x \leq s \implies -s \leq -x \implies -s \leq (-1)A$$

Now fix any $\varepsilon > 0$, then there exists an $x \in A$ such that

$$s - \varepsilon \leq x \implies (-1)(s - \varepsilon) \geq -x \implies (-x) \leq (-s) + \varepsilon$$

Thus establishes $\inf(-1)A = (-1)\sup(A)$. □

Corollary 2.4.1. $(-1)\inf(A) = \sup(-1)A$. The proof is trivial just replace A by $(-1)A$.

2.5 Lemma

Lemma 2.5. If A and B are non-empty bounded above subsets of $\mathbb{R}$, then $\sup A + \sup B = \sup(A + B)$

Proof. Define $s = \sup A$ and $t = \sup B$, then for every $(a, b) \in A \times B$

$$a \leq s,\ b \leq t \implies a + b \leq s + t \implies A + B \leq s + t$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq s - \varepsilon/2,\ b \leq t - \varepsilon/2 \implies s + t - \varepsilon \leq a + b$$

Therefore $\sup(A + B) = \sup(A) + \sup(B)$. □

2.6 Lemma

Lemma 2.6. If A and B are non-empty bounded below subsets of $\mathbb{R}$, then $\inf A + \inf B = \inf(A + B)$

Proof. Define $w = \inf A$ and $q = \inf B$, then for every $(a, b) \in A \times B$

$$w \leq a,\ q \leq b \implies w + q \leq a + b \implies w + q \leq A + B$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq w + \varepsilon/2,\ b \leq q + \varepsilon/2 \implies a + b \leq w + q + \varepsilon$$

Therefore $\inf(A + B) = \inf(A) + \inf(B)$. $\square$

2.7 Lemma

Lemma 2.7. If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. $\square$

Remark. We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.

2.8 Super Triangle Inequality

Lemma 2.8 (Super Triangle Inequality). For any $x, y, z \in X$, where X denotes a metric space then,

$$\Bigg| d(x,z) - d(y,z) \Bigg| \leq d(x,y)$$

Proof. From the regular Triangle Inequality,

$$d(x,z) \leq d(x,y) + d(y,z)$$
$$d(y,z) \leq d(x,y) + d(x,z)$$

The proof is complete upon isolating $d(x,y)$ in both equations. $\square$

2.9 Lemma

Lemma 2.9. Let $\{x_n\}$, and $\{y_n\}$ be sequences of reals, and if $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$.

Proof. Fix an $\varepsilon > 0$, cut our ε into half, so that $\{x_n\}$, and $\{y_n\}$ lies in a half-ε ball about their limits eventually. Then, take the triangle inequality, since the above estimate holds eventually, $|x_n + y_n - (x + y)| < \varepsilon$ eventually as well. $\square$

2.10 Lemma

Lemma 2.10. Let $\{x_n\}$, and $\{y_n\}$ be sequences of reals, and if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.

Proof. Since $\{x_n\}$ and $\{y_n\}$ are convergent sequences, they must be bounded. So there exists some $M_1, M_2 \in \mathbb{N}^+$ with $|y| \leq M_1$ and $|x_n| \leq M_2$. Next,

$$\begin{aligned} |x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq M_2|y_n - y| + M_1|x_n - x| \end{aligned}$$

Now fix an $\varepsilon > 0$, and there exists N_1 and N_2 so large that

$$\begin{aligned} |y_n - y| &< \varepsilon M_2^{-1} \\ |x_n - x| &< \varepsilon M_1^{-1} \end{aligned}$$

for all $n \geq N_1 + N_2$. Therefore $|x_n y_n - xy| < \varepsilon$ eventually, and this completes the proof. $\square$

2.11 Lemma

Lemma 2.11. If $\{x_n\} \in \mathbb{R}$, $x_n \to x \neq 0$, then $(x_n)^{-1} \to x^{-1}$.

Proof. If $x_n \to x \neq 0$, then x_n lies in a ball about x of radius $|x|2^{-1}$ eventually, so $x_n x \neq 0$. Moreover, using the fact that x_n lies in said ball, $x_n x \geq 0$ eventually, and

$$\begin{aligned}
|x_n - x| \leq |x|2^{-1} &\iff |x_n x - x^2| \leq |x|^2 2^{-1} \\
&\iff \Big| |x_n x| - |x|^2 \Big| \leq |x|^2 2^{-1} \\
&\iff |x|^2 2^{-1} \leq |x_n x| \leq 3|x|^2 2^{-1} \\
&\implies |x_n x|^{-1} \leq (|x^2|2^{-1})^{-1} = M^{-1}
\end{aligned}$$

Now for every $M\varepsilon > 0$, the following must hold eventually,

$$|x_n - x|M^{-1} < \varepsilon \implies \frac{|x_n - x|}{|x_n x|} \leq \frac{|x_n - x|}{M} < \varepsilon$$

Therefore,

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| < \varepsilon \to 0$$

□

2.12 Lemma

Lemma 2.12. If P is a proposition on the space of all sequences (onto, into? what is the right word) X, denoted by

$$\Omega = \{x_n : \mathbb{N} \to X\}$$

And if

- $P(\forall n \geq 0) = \{x_n \in \Omega,\ \forall n \geq 0, P(x_n)\}$
- $P(\forall^\infty n) = \{x_n \in \Omega,\ \exists N,\ \forall n \geq N, P(x_n)\}$
- $P(\exists^\infty n) = \{x_n \in \Omega,\ \forall N,\ \exists n \geq N, P(x_n)\}$

Then

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \subseteq P(\exists^\infty n) \subseteq \Omega$$

Proof. Suppose that $x_n \in P(\forall n \geq 0)$, then $P(x_n)$ eventually is trivial, so

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n)$$

Now fix $x_n \in P(\forall^\infty n)$, this induces some $N \in \mathbb{N}$ such that for every $n \geq N$ means that $P(x_n)$. To show that $P(x_n)$ frequently, notice for every $M \in \mathbb{N}$ we can choose some $n = M + N$ such that $P(x_n)$ holds, so

$$P(\forall^\infty n) \subseteq P(\exists^\infty n)$$

The last inclusion is obvious as all three are subsets of Ω. □

2.13 Every convergent real sequence is bounded

Lemma 2.13. Every convergent sequence in $\mathbb{R}$ is bounded.

Proof. Fix $\{x_n\}_{n\geq 1} \to x$, also fix $\varepsilon = 1 > 0$, then there exists some $N \geq 0$ so large with

$$d(x_n, x) \leq 1 \quad \forall n \geq N$$

Then for every $n \geq N$ we have

$$d(x_n, 0) \leq d(x_n, x) + d(x, 0) \leq 1 + d(x, 0)$$

Now for every $n \geq 1$, obviously $d(x_n, 0) \leq 1 + d(x, 0) + \sum_{k\leq N} d(x_k, 0)$, and this establishes the Lemma. □

2.14 Lemma

Lemma 2.14. If $\{x_n\}$ is a sequence in $\mathbb{R}$, and if $x_{n+1}/x_n < 0$ eventually, and if $|x_n| \to a > 0$, then x_n diverges.

Proof. Using the fact that $|x_n| \to a$, and fix $\varepsilon = a/2 > 0$, then

$$\Big||x_n| - a\Big| < a/2 \iff a/2 < |x_n| < 3a/2$$

Using the fact that $x_{n+1}/x_n < 0$ eventually,

- either $-x_{n+1} = |x_{n+1}|$,
- or $|-x_n + x_{n+1}| = |x_n| + |x_{n+1}|$,

We have,

$$\begin{aligned} d(x_n, x_{n+1}) &= \Big|x_n - x_{n+1}\Big| \\ &= |x_n| + |x_{n+1}| \\ &> a/2 + a/2 \\ &> a \end{aligned} \tag{1}$$

Now suppose that $x_n \to x$ for some $x \in \mathbb{R}$, then for any $\varepsilon = a/2 > 0$, we must have

$$d(x_n, x_{n+1}) \le d(x_n, x) + d(x_{n+1}, x)$$

Using Equation (1), we get

$$a < a/2 + d(x_{n+1}, x) \implies a/2 < d(x_{n+1}, x) < a/2$$

Therefore $x_n \not\to x$, and this completes the proof. □

2.15 sup, inf of A, B when A subset of B

WTS. If $A \subseteq B \subseteq \mathbb{R}$, then $\sup(A) \le \sup(B)$, and $\inf(A) \ge \inf(B)$.

Proof. If we allow for the $\sup$ and $\inf$ of A and B to take on symbols in the extended reals. Then, $\sup(B)$ is an upper-bound for A and $\inf(B)$ is a lower-bound for A, therefore

$$\sup(A) \le \sup(B), \quad \inf(A) \ge \inf(B)$$

$\square$

2.16 sup, inf with unbounded sets

WTS. Suppose that $A \subseteq \mathbb{R}$, and $\sup(A) = +\infty$, then for every $M \in \mathbb{R}$, there exists an $x \in A$ with $x > A$. Conversely, if $B \subseteq \mathbb{R}$, and $\inf(B) = -\infty$, then there exists an $x \in B$ with $x < M$ for an arbitrary $M \in \mathbb{R}$.

Proof. Take the contrapositive of both statements and apply the definitions of the $\sup$ and the $\inf$. $\square$

2.17 Every element in A is less than every element in B

WTS. If A, B are non-empty subsets of $\mathbb{R}$,

$$\sup A \leq \inf B \iff \forall a \in A, \forall b \in B, a \leq b$$

Proof. Suppose that $\sup A \leq \inf B$, then for every $a \in A$, and we can safely assume that both $\sup A$ and $\inf B$ are finite (see remark),

$$a \leq \sup A \leq \inf B$$

so that a is a lower bound for B, but this is equivalent to saying that $a \leq b$ for every $b \in B$.

Now suppose that for every $a, b \in A, B$, $a \leq b$. Then every single $b \in B$ is an upper bound for the set A, therefore

$$\forall b \in B, \sup A \leq b \implies \sup A \leq \inf B$$

where the last estimate is due to $\sup A$ being a lower bound for B. $\square$

Remark. If $\sup A = +\infty$, then $\inf B = +\infty$, this can only happen if $B = \varnothing$, so $\sup A = +\infty$ is impossible, so is $\inf B = -\infty$. (We assume that A and B are subsets of $\mathbb{R}$, and not of $\overline{\mathbb{R}}$).

Further, if $\sup A = -\infty$, then either $A = \{-\infty\}$ which is not a subset of $\mathbb{R}$, or $A = \varnothing$, which is again impossible.

2.18 Tail behaviour of a sequence

WTS. Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let $x \in \mathbb{R}$. Fix any $\varepsilon > 0$, and eventually for $n \geq N$, $m \geq M$, $M, N \in \mathbb{N}^+$,

$$x_n \in V_\varepsilon(x) \iff E_m \subseteq V_\varepsilon(x)$$

Proof. Suppose $x_n \in V_\varepsilon(x)$ eventually, so there exists an $N \in \mathbb{N}^+$ such that

$$x_{n\geq N} \in V_\varepsilon(x) \implies E_N \subseteq V_\varepsilon(x)$$

and $E_n \subseteq E_N \subseteq V_\varepsilon(x)$ for every $n \geq N$, so $E_m \subseteq V_\varepsilon(x)$ eventually.

If $E_m \subseteq V_\varepsilon(x)$ eventually, then there exists an M such that $E_M \subseteq V_\varepsilon(x)$, and for every $n \geq M$

$$x_n \in E_M \subseteq V_\varepsilon(x)$$

so that $x_n \in V_\varepsilon(x)$ eventually. $\square$

2.19 Eventually, frequently, and subsequences

WTS. Let P be a subset of $\mathbb{R}$, and if $\{x_n\}$ is any sequence in $\mathbb{R}$, then

(a) If $x_n \in P$ eventually, if x_{n_k} is a subsequence of x_n, then $x_{n_k} \in P$ eventually as $k \to \infty$.

(b) If x_{n_k} is a subsequence of x_n, and if $x_{n_k} \in P$ eventually, then $x_n \in P$ frequently.

Proof of Part A. Suppose $x_n \in P$ eventually, so there exists a large $N \in \mathbb{N}^+$, and fix a subsequence x_{n_k} of x_n, then for every $k \geq N \implies n_k \geq k \geq N$, we have $x_{n_k} \in P$, so $x_{n_k} \in P$ eventually.

□

Proof of Part B. Suppose $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, and that $x_{n_k} \in P$ eventually, so there exists a large K such that $k \geq K \implies x_{n_k} \in P$. Then for every $N \in \mathbb{N}^+$ we can find an $m \geq N + n_K$ such that $m = n_k$ for some $k \geq 1$ (so that $x_m = x_{n_k}$), and $x_m \in P$. So $x_n \in P$ frequently. □

2.20 Closed and Totally Bounded equivalent to Heine Borel

Let (X, d) be a metric space, and $E \subseteq X$.

WTS. If E is closed and totally bounded, then every sequence in E has a convergent subsequence.

Proof of Part A. Fix a sequence $\{x_n\} \subseteq E$, and if E is totally bounded, there exists a finite cover of open balls of radius 2^{-1}. One of these covers must contain infinitely many points of x_n.

- Choose one ball of radius 2^{-1} that contains infinitely many x_n, and call it B_1, and define
$$N_1 = \{n \in \mathbb{N}^+,\, x_n \in B_1\}$$
- Suppose that $B_1, \ldots B_{k-1}$ are balls of radii $2^{-1}, \ldots, 2^{1-k}$ that contain infinitely many x_n in $E, \ldots E \cap B_{k-2}$, then choose B_k as follows. If a finite family of open balls of radius 2^{-k} covers E, it must cover $E \cap B^{k-1}$. And one of these balls, must contain infinitely many points of $\{x_n\} \cap E \cap B^{k-1}$. Otherwise $E \cap B^{k-1}$ would only contain finitely many points in $\{x_n\}$, a contradiction.
- Choose B_k accordingly, and denote $N_k = \{n \in \mathbb{N}^+,\, x_n \in B_k\}$. Where each N_j is an infinite set.
- Use the Well Ordering Property of the naturals to obtain our subsequence in k such that for $k = 1$
$$n_1 = \text{least}\left\{n \in N_1\right\}$$
- A simple induction on k yields
$$n_k = \text{least}\left\{n \in N_k \setminus \bigcup_{j=1}^{k-1} N_j\right\}$$
- Clearly, $n_1 < n_2 < \ldots$. Furthermore, suppose that c_1, c_2 are centers of balls of radii r_1, r_2 in an arbitrary metric space. We claim that if $r_1 + r_2 < d(c_1, c_2)$, then $V_{r_1}(c_1) \cap V_{r_2}(c_2) = \varnothing$. This simple proof is left as an exercise. We now take the contrapositive of the previous statement. Since each B_k contains points in B_{k-1}, if $B_k = V_{r_k}(c_k)$, we must have
$$d(c_k, c_{k-1}) \leq r_k + r_{k-1}$$

- Now let j and k be arbitrary numbers in $\mathbb{N}^+$. Then

  $$d(x_{n_j}, x_{n_k}) \leq d(c_j, c_k) + d(x_{n_j}, c_j) + d(x_{n_k}, c_k)$$

  A moment's thought will show that, if $j < k$

  $$d(x_{n_j}, x_{n_k}) \leq \left(\sum_{m=j}^{k} 2^{-m} \right) + 2^{-j} + 2^{-k} \leq 2^{2-j}$$

- and for every $\varepsilon > 0$ there exists a K so large that $\varepsilon < 2^{2-K}$, and for every $k > j > K$ we have

  $$d(x_{n_j}, x_{n_k}) \leq 2^{2-K} < \varepsilon$$

  The subsequence is Cauchy, and converges to some limit in E.

□

WTS. If every sequence in E has a convergent subsequence, then E is closed and totally bounded.

Proof of Part B. Suppose E is not closed, then there exists a Cauchy sequence in $\{x_n\}$ that has no limit in E, and suppose that it has a convergent subsequence in $x_{n_k} \to x \in E$. For every $\varepsilon/2 > 0$, if we agree to define the tail of the sequence, $E_m = \{x_{n_m},\, m \in \mathbb{N}^+\}$, then

$$E_m \subseteq V_{\varepsilon/2}(x)$$

But $\{x_n\}$ is Cauchy, so $d(x_j, x_k) < \varepsilon/2$ eventually, and choose $j, k \geq n_m$ and $k = n_{m'}$ for some $m' \geq m$,

$$d(x_j, x) \leq d(x_j, x_k) + d(x_k, x) < \varepsilon$$

So E must be closed. Let us now assume E is not totally bounded, so there exists an $r > 0$ where no finite covering of E exists. And we can construct a sequence that has no convergent subsequence.

1. Since there are infinitely many balls of radius ε that cover E, let $\{B_\alpha\}$ be the family of such balls, where $\alpha \in A$.

2. Since we can always find infinitely many balls whose centers are at least 2ε apart (by invoking the axiom of countable choice). We can construct a sequence $\{x_n\}$ such that $d(x_n, x_{n+k}) > k\varepsilon > 0$.

3. The above sequence can have no subsequential limit in E.

□

2.21 Eventual Behaviour of Sequences

WTS. Let $\{x_n\}$ be a sequence in an arbitrary space X. We define the m-tail of the sequence,

$$E_m = \{x_n,\, n \geq m\}$$

If $A \subseteq X$ is any set, the following are equivalent.

(a) $x_n \in A$ eventually,

(b) $E_m \subseteq A$ eventually, (as $m \to \infty$),

(c) $A^c \cap E_m = \varnothing$ eventually, (as $m \to \infty$),

(d) $A^c \cap \{x_n\}$ is finite,

(e) it is false that $x_n \in A^c$ frequently,

(f) no subsequence x_{n_k} of x_n can lie in A^c eventually, (as $k \to \infty$),

(g) every subsequence of x_n can be found frequently in A

Proof. Suppose (a) holds, then $\{x_{n \geq N}\} \subseteq A$. So $E_N \subseteq A$, and for every $m \geq N$, $E_m \subseteq E_N \subseteq A$, so (a) $\implies$ (b).

Suppose (b) holds, then

$$E_m \subseteq A \iff A^c \cap E_m = \varnothing, \qquad \text{eventually}$$

Hence (c) follows.

To show (c) $\implies$ (d), we assume (d) is false. So $A^c \cap \{x_n\}$ is infinite, and denote

$$\mathcal{N} = \left\{ n \in \mathbb{N}^+,\, x_n \in A^c \right\}$$

is an unbounded set. Now choose any $m \in \mathbb{N}^+$, so this m must not be an upper-bound of $\mathcal{N}$ (otherwise $\mathcal{N}$ would be bounded above, and therefore finite). For this m, there exists an $n > m'$, where $n \in \mathcal{N}$, with

$$x_n \in A^c \cap E_m \implies A^c \cap E_m \neq \varnothing$$

This holds for every m (we have proven a negation that is stronger than the negation of (c)), and (c) is invalid. Therefore (c) $\implies$ (d).

Suppose now (d) holds. Since $A^c \cap \{x_n\}$ is finite, there exists an $N \in \mathbb{N}^+$ where

$$N = \max\left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\} + 1$$

for every $n \geq N$, we have $x_n \notin A^c$. So $x_n \notin A^c$ eventually $\iff$ the claim that x_n is in A^c frequently is false, and (e) follows.

Now suppose (e), unboxing the quantifiers, reads

$$\neg\left(\forall N \in \mathbb{N}^+,\, \exists n \geq N,\, x_n \in A^c\right) \iff \exists N \in \mathbb{N}^+,\, \forall n \geq N,\, x_n \in A$$

The right member is equivalent to claim (a).

To show (f) is indeed equivalent with the rest. Suppose claim (d) does not hold. So $A^c \cap \{x_n\}$ is infinite. Let $\mathcal{K} = \{n \in \mathbb{N}^+,\, x_n \in A^c\}$ is an infinite set of natural numbers, and is therefore unbounded above. Following the argument within (c) $\implies$ (d), we can construct an increasing sequence of naturals $n_1 < n_2 < \ldots$ such that $n_k \in \mathcal{K}$, and

$$\{x_{n_k}\} \subseteq A^c$$

This proves $\neg$(d) $\implies$ $\neg$(f). To show the converse, suppose that $x_{n_k} \in A^c$ eventually, then the set of naturals (also denoted by $\mathcal{K}$),

$$\mathcal{K} = \left\{k \in \mathbb{N}^+,\, x_{n_k} \in A^c\right\}$$

is an infinite set, so (d) is false.

Lastly, to show $(f) \iff (g)$, we unbox the quantifiers

$$\begin{aligned}
(g) &\iff \forall \{x_{n_k}\} \subseteq x_n, \neg\left(\exists K \in \mathbb{N}^+,\, \forall k \geq K,\, x_{n_k} \in A^c\right) \\
&\iff \forall \{x_{n_k}\} \subseteq x_n, \forall k \in \mathbb{N}^+,\, \exists k \geq K,\, x_{n_k} \in A^c \\
&\iff (f)
\end{aligned}$$

This completes the proof. $\square$

2.22 Frequent Behaviour of Sequences

WTS. Let $\{x_n\}$ be a sequence in an arbitrary space X. Let E_m be the m-tail of the sequence as usual. If $A \subseteq X$ is any set, the following are equivalent.

(a) $x_n \in A$ frequently,

(b) it is false that $x_n \in A^c$ eventually,

(c) $A \cap E_m$ is infinite, for every $m \geq 1$,

(d) there exists a subsequence x_{n_k} of x_n that lies in A eventually,

Proof. Notice that (a) is equivalent to the negation of Theorem 2.21a, but with A taking the place of A^c (within Theorem 2.21).

It immediately follows that

$$(a) \iff (b) \iff (c) \iff (d)$$

and the proof is complete. □

Corollary 2.14.1. If x_n is in A eventually, then x_n lies in A frequently. Or the contrapositive: if x_n is in A^c frequently, then x_n does not lie in A eventually.

2.23 BlankTheorem

WTS.

Proof. $\square$

2.24 Theorem 2.23

WTS.

Proof. $\square$

2.25 Theorem 2.24

WTS.

Proof. $\square$

2.26 Theorem 2.25

WTS.

Proof. $\square$

2.27 Theorem 2.26

WTS.

Proof. $\square$

2.28 Theorem 2.27

WTS.

Proof. $\square$

3 Epsilon Proofs

3.1 Every quadratic curve is closed in R2

WTS. The set of points $A \subseteq \mathbb{R}^2$ is closed, where

$$A = \{(x, y),\, y = ax^2 + bx + c\}$$

Proof. It suffices to show that every accumulation point of A is in A. So fix some $(x_0, y_0) \in \operatorname{acc} A$, and a $\varepsilon > 0$, the following proof will be fussy, and we will divide it into 3 parts. By definition of $\operatorname{acc} A$, there exists some $(x_1, y_1) \in A$ such that

$$(x_0 - x_1)^2 + (y_0 - y_1)^2 < \varepsilon^2 \tag{2}$$

From Equation (2) we know that both $|x_0 - x_1|^2$ and $|y_0 - y_1|^2 < \varepsilon^2$. If otherwise, the contradiction that immediately follows is obvious. From this, we can substitute $y_1 = ax_1^2 + bx_1 + c$,

$$\Big|y_0 - (ax_1^2 + bx_1 + c)\Big|^2 < \varepsilon^2 \implies \Big|y_0 - (ax_1^2 + bx_1 + c)\Big| < \varepsilon \tag{3}$$

We multiply $|x_0 - x_1| < \varepsilon$ by $|b|$, and

$$|bx_1 - bx_0| < \varepsilon|b| \tag{4}$$

We wish to also bound $|x_1^2 - x_0^2|$ by some multiple of ε. This is a bit more tricky than the rest. Indeed, suppose that $\varepsilon < 1$, then

$$\begin{aligned}
x_1^2 - x_0^2 &= (x_1 - x_0) \cdot (x_0 + x_1) \\
|x_1^2 - x_0^2| &= |x_1 - x_0| \cdot |x_0 + x_1| \\
&\leq \varepsilon\Bigg(|x_0| + |x_1 - x_0| + |x_0 - 0|\Bigg) \\
&\leq \varepsilon(2|x_0| + \varepsilon) \\
|x_1^2 - x_0^2| &\leq \varepsilon(2|x_0| + 1)
\end{aligned}$$

Multiplying the last estimate by $|a|$ yields

$$|ax_0^2 - ax_1^2| \leq \varepsilon|a|(2|x_0| + 1) \tag{5}$$

Adding equations (5) and (4),

$$\Big|ax_1^2 + bx_1 + c - (ax_0^2 + bx_0 + c)\Big| \leq \varepsilon\Bigg(|a|(2|x_0| + 1) + |b|\Bigg) \tag{6}$$

Finally, add (6) to (3),

$$\begin{multline}|y_0 - (ax_1^2 + bx_1 + c) + ax_1^2 + bx_1 + c - \\ (ax_0^2 + bx_0 + c)| \leq \varepsilon\Bigg(|a|(2|x_0| + 1) + |b| + 1\Bigg) \tag{7}\end{multline}$$

Now for every $\delta > 0$, we can simply choose some $\varepsilon < \min\left\{1, \delta\Bigg(|a|(2|x_0| + 1) + |b| + 1\Bigg)^{-1}\right\}$ such that

$$|y_0 - (ax_0^2 + bx_0 + c)| < \delta$$

Therefore $(x_0, y_0) \in A$ and $\operatorname{acc} A \subseteq A$. This completes the proof. $\square$

3.2 Theorem 5.2

WTS.

Proof. $\square$

3.3 Theorem 5.3

WTS.

Proof. ∎

3.4 Theorem 5.4

WTS.

Proof. $\square$

3.5 Theorem 5.5

WTS.

Proof. □

3.6 Theorem 5.6

WTS.

Proof. $\square$

3.7 Theorem 5.7

WTS.

Proof. $\square$

3.8 Theorem 5.8

WTS.

Proof. ∎