Assorted Proofs in Analysis

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1 Sets

1.1 Theorem 1.1

WTS. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We can use a chain of equivalences. Suppose that both members are not empty.

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \wedge x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \wedge x \in B) \text{ or } (x \in A \wedge x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Now suppose one of the two members are empty. Then if the other member was not empty, it would imply that the original member was not empty, and this means that the two sets must be equal. □

1.2 Theorem 1.2

WTS. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Define $W = (A \setminus B) \cup (B \setminus A)$, then we will apply Q1a, and deMorgan's Theorem.

$$\begin{aligned}
W^c &= (A^c \cup B) \cap (B^c \cup A) \\
&= ((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A) \\
&= (A^c \cap B^c) \cup (A \cap B) \\
&= (A \cup B)^c \cup (A \cap B) \\
&= [(A \cup B) \setminus (A \cap B)]^c
\end{aligned}$$

Taking complements on both sides finishes the proof. □

1.3 Theorem 1.3

WTS. $f : A \to B$ is a function, and $B_1, B_2 \subseteq B$. Show that f^{-1} preserves unions, intersections and complements.

Lemma 1.1. f^{-1} preserves unions.

Proof. Fix two subsets $B_1, B_2 \subseteq B$, then

$$\begin{aligned} f^{-1}(B_1 \cup B_2) &= \{x \in A, f(x) \in B_1 \cup B_2\} \\ &= \{x \in A, f(x) \in B_1 \text{ or } f(x) \in B_2\} \\ &= \{x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)\} \\ &= f^{-1}(B_1) \cup f^{-1}(B_2) \end{aligned}$$

□

Lemma 1.2. f^{-1} preserves complements.

Proof. For every $E \subseteq B$,

$$\begin{aligned} f^{-1}(B \setminus E) &= \{x \in A : f(x) \in B \setminus E\} \\ &= \{x \in A, f(x) \in E^c\} \\ &= A \setminus f^{-1}(E) \end{aligned}$$

□

Lemma 1.3. f^{-1} preserves intersections.

Proof. Now we wish to prove that f^{-1} preserves intersections as well, for every pair of subsets, $B_1, B_2 \subseteq B$. Write their intersection as $(B_1^c \cup B_2^c)^c$, apply Lemma (1.1), and take complements (then apply Lemma (1.2))

$$\begin{aligned} f^{-1}\left((B_1^c \cup B_2^c)^c\right) &= (f^{-1}(B_1^c) \cup f^{-1}(B_2^c))^c \\ &= f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

□

1.4 Theorem 1.4

WTS. $f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Proof. For any two sets $A_1, A_2 \subseteq A$,

$$\begin{aligned} f(A_1 \cup A_2) &= \{f(x) : x \in A_1 \cup A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } x \in A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } f(x) : x \in A_2\} \\ &= f(A_1) \cup f(A_2) \end{aligned}$$

$\square$

Corollary 1.3.1. The direct image is monotonic. For every $E_1 \subseteq E_2 \subseteq A$, then $f(E_1) \subseteq f(E_2) \subseteq B$.

Proof. Apply Theorem Theorem ? to sets $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$, then $f(E_2) = f(E_2 \setminus E_1) \cup f(E_2 \cap E_1)$ implies that $f(E_1) \subseteq f(E_2)$. $\square$

1.5 Theorem 1.5

WTS. Subset relations. The following are equivalent.

1. $A \subseteq B$
2. $A \cap B = A$
3. $A \cup B = B$
4. $A \subseteq B^c$
5. $A \setminus B = \varnothing$

Proof. $\square$

1.6 Theorem 1.6

WTS. $f(f^{-1}B) = B$ if f is a surjection, and $f^{-1}(f(B)) = B$ if f is an injection.

We split this problem into two parts. We begin with the first assertion. Write $R = \{f(x) : x \in A\}$.

Lemma 1.4. For every function $f : X \to Y$, $f(f^{-1}(B)) \subseteq B$.

Proof. Use Q5a) onto the disjoint sets $f^{-1}(B \cap R)$ and $f^{-1}(B \cap R^c)$, then

$$f^{-1}(B) = f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)$$

Now $f^{-1}(B \cap R^c)$ must be empty, since no $x \in A$ satisfies $f(x) \in B \cap R^c$. Hence $f^{-1}(B) = f^{-1}(B \cap R)$.

$$\begin{aligned} f(f^{-1}(B)) &= f(f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \cup f(f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \\ &= \{f(x) : x \in f^{-1}(B \cap R)\} \\ &= \{y : y \in (B \cap R)\} \\ &= B \cap R \end{aligned}$$

Where for the second last equality we used the fact that f is always a surjection onto its range. Then $f(f^{-1}(B)) = B \cap R \subseteq B$. □

Remark. If f is a surjection, then its range $R = Y$, then $f(f^{-1}(B)) = B \cap Y = B$.

Lemma 1.5. For every function $f : X \to Y$, $A \subseteq f^{-1}(f(A))$.

Proof. Write $f^{-1}(f(A))$ as the disjoint union of $A \cap f^{-1}(f(A))$ and $A^c \cap f^{-1}(f(A))$. Then, we shall show that $f^{-1}(f(A)) = A$. For every $x \in A$,

$$\begin{aligned} f(x) \in f(A) \wedge x \in A &\iff x \in f^{-1}(f(A)) \wedge x \in A \\ &\iff x \in A \cap \left(f^{-1}(f(A))\right) \end{aligned}$$

Hence $A \cap f^{-1}(f(A)) = A$, and $A \subseteq f^{-1}(f(A))$

Remark. If f is a injection, then for every $x \in A^c$, $f(x) \notin f(A)$, then $A^c \cap f^{-1}(f(A)) = \varnothing$, and

$$f^{-1}(f(A)) = \left[A \cap f^{-1}(f(A))\right] \cup \left[A^c \cap f^{-1}(f(A))\right] = A$$

□

1.7 Theorem 1.7

WTS.

Proof. $\square$

1.8 Theorem 1.8

WTS.

Proof. $\square$

1.9 Theorem 1.9

WTS. If $a|(b+c)$, and $\gcd(b,c)=1$, then

$$\gcd(a,b)=\gcd(a,c)=1$$

Proof. Suppose $\gcd(a,b)\geq 1$, then there exists some $y\geq 2$ such that $y|a$ and $y|b$, $\square$

1.10 Theorem 1.10

WTS.

Proof. ∎

1.11 Theorem 1.11

WTS.

Proof. □

1.12 Theorem 1.12

WTS.

Proof. $\square$

1.13 Theorem 1.13

WTS.

Proof. $\square$

1.14 Theorem 1.14

WTS.

Proof. $\square$

1.15 Theorem 1.15

WTS.

Proof. ∎

1.16 Theorem 1.16

WTS.

Proof. □

1.17 Theorem 1.17

WTS.

Proof. □

1.18 Theorem 1.18

WTS.

Proof. $\square$

2 Functions

2.1 Triangle Inequality

WTS. Prove the Triangle Inequality with $n \geq 2$.

Lemma 2.1. The Triangle Inequality, for every $a, b \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

Proof. Notice that for every $a, b \in \mathbb{R}$

- $-|a| \leq a \leq +|a|$
- $-|b| \leq b \leq +|b|$

Add two inequalities together, then we get

$$-\left(|a| + |b|\right) \leq (a + b) \leq +\left(|a| + |b|\right)$$

Since $|a| + |b| \geq 0$, we get $|a + b| \leq |a| + |b|$ as desired. Because for every $c \geq 0$,

$$\begin{aligned} c \geq |a| &\iff c \geq \max\{-a, +a\} \\ &\iff c \geq a \text{ and } c \geq -a \\ &\iff c \geq a \text{ and } -c \leq a \\ &\iff -c \leq a \leq +c \end{aligned}$$

$\square$

Main proof for Triangle Inequality

Proof. For every x_1, x_2 in $\mathbb{R}$,

$$\left| \sum_{j=1}^{2} x_j \right| \leq \sum_{j=1}^{2} |x_j|$$

This proves the non trivial base case, if $n = 1$ then obviously $|x_1| \leq |x_1|$. We now suppose that

$$\left| \sum_{j=1}^{k} x_j \right| \leq \sum_{j=1}^{k} |x_j|$$

Then write

$$\left|\sum_{j=1}^{k+1} x_j\right| = \left|\left(\sum_{j=1}^{k} x_j\right) + (x_{k+1})\right|$$

Applying the second estimate to the right member gives us

$$\left|\sum_{j=1}^{k+1} x_j\right| \leq \left|\sum_{j=1}^{k} x_j\right| + |x_{k+1}| = \sum_{j=1}^{k+1} |x_j|$$

This completes the proof. $\square$

2.2 Theorem 2.2

WTS.

Proof. $\square$

2.3 Theorem 2.3

WTS.

Proof. □

2.4 Theorem 2.4

WTS.

Proof. ∎

2.5 Theorem 2.5

WTS.

Proof. $\square$

2.6 Theorem 2.6

WTS.

Proof. $\square$

2.7 Theorem 2.7

WTS.

Proof. □

2.8 Theorem 2.8

WTS.

Proof. $\square$

2.9 Theorem 2.9

WTS.

Proof. □

2.10 Theorem 2.10

WTS.

Proof. $\square$

2.11 Theorem 2.11

WTS.

Proof. $\square$

2.12 Theorem 2.12

WTS.

Proof. □

2.13 Theorem 2.13

WTS.

Proof. $\square$

2.14 Theorem 2.14

WTS.

Proof. $\square$

2.15 Theorem 2.15

WTS.

Proof. $\square$

2.16 Theorem 2.16

WTS.

Proof. □

2.17 Theorem 2.17

WTS.

Proof. $\square$

2.18 Theorem 2.18

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Proof. $\square$

2.19 Theorem 2.19

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Proof. $\square$

2.20 Theorem 2.20

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Proof. $\square$

2.21 Theorem 2.21

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Proof. $\square$

2.22 Theorem 2.22

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Proof. $\square$

2.23 Theorem 2.23

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Proof. $\square$

2.24 Theorem 2.24

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Proof. $\square$

2.25 Theorem 2.25

WTS.

Proof. $\square$

2.26 Theorem 2.26

WTS.

Proof. □

2.27 Theorem 2.27

WTS.

Proof. $\square$

2.28 Theorem 2.28

WTS.

Proof. □

2.29 Theorem 2.29

WTS.

Proof. ∎

2.30 Theorem 2.30

WTS.

Proof. $\square$

2.31 Theorem 2.31

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Proof. $\square$

2.32 Theorem 2.32

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2.33 Theorem 2.33

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2.34 Theorem 2.34

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2.35 Theorem 2.35

WTS.

Proof. $\square$

2.36 Theorem 2.36

WTS.

Proof. □

2.37 Theorem 2.37

WTS.

Proof. ∎

2.38 Theorem 2.38

WTS.

Proof. □

2.39 Theorem 2.39

WTS.

Proof. ∎

3 The Real Numbers

3.1 Theorem 3.1

WTS.

Proof. $\square$

3.2 Theorem 3.2

WTS.

Proof. □

3.3 Theorem 3.3

WTS.

Proof. $\square$

3.4 Theorem 3.4

WTS.

Proof. $\square$

3.5 Theorem 3.5

WTS.

Proof. $\square$

3.6 Theorem 3.6

WTS.

Proof. $\square$

3.7 Theorem 3.7

WTS.

Proof. ∎

3.8 Theorem 3.8

WTS.

Proof. $\square$

3.9 Theorem 3.9

WTS.

Proof. □

3.10 Theorem 3.10

WTS.

Proof. $\square$

3.11 Theorem 3.11

WTS.

Proof. $\square$

3.12 Theorem 3.12

WTS.

Proof. $\square$

3.13 Theorem 3.13

WTS.

Proof. $\square$