1 Chapter B

1.1 B1 Signal Sizes

We begin with mathematical definitions.

Definition. Energy of a signal $x(t) \iff \mathbf{X(f)}$ is a non-negative quantity,

$$E_x = \int |x|^2 dt = \|x\|_2^2$$
$$E_x = \int \mathbf{|X|^2 df} = \mathbf{\|X\|_2^2}$$

It has units of J.

Definition. Power Spectral Density of a signal $x(t) \iff \mathbf{X(f)}$ is a function,

$$P_{\mathbf{X}} : \Omega \to \mathbb{R}, \quad P_{\mathbf{X}}(f) = \lim_{d\to\infty} \frac{|\mathbf{X}_d(f)|^2}{d}$$

where $\mathbf{X}_d(f) = \mathcal{F}\left\{x(t)\chi_{[-\frac{d}{2},+\frac{d}{2}]}\right\}$. $P_{\mathbf{X}}$ has units W/Hz. Other names are:

- Power Spectrum, and
- Signal Power distribution in the frequency domain.

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Definition. Power of a signal $x(t) \iff \mathbf{X(f)}$ is a quantity

$$P_x = \lim_{d\to\infty} d^{-1} \int |x(t)|^2 \chi_{[-\frac{d}{2},+\frac{d}{2}]} dt$$

$$P_x = \int P_{\mathbf{X}} df = \int \lim_{d\to\infty} \frac{|\mathbf{X}_d(f)|^2}{d} df$$

For a sinusoid in $x(t) = A\cos(2\pi f_c t + \theta)$, $P_x = A/2$. It has units of W.

Remark. *One should not conflate Power with Power Spectral Density, one is a quantity and the other is a function.*

An alternate characterisation of the Power of a signal would be to use the average function for some $|x(t)|^2 \in L^1_{loc}$

$$P_x = \lim_{r \to \infty} A_r(x(0)) = \frac{1}{m(B(r,0))} \int_{B(r,0)} |x(t)|^2 dt$$

Definition. The absolute bandwidth B of a signal is

$$\mu(\mathrm{supp}\,(S(f)) \cap \mathbb{R}^+) = \mu\left(\bigcap\{[a,b],\, 0 \leq a \leq b,\, \mathrm{supp}\,(S(f) \subseteq [a,b])\}\right)$$

Definition. If the smallest closed interval that contains $\mathrm{supp}\,(S(f)) \cap \mathbb{R}^+$ is given by $[f_1, f_2]$, then the center frequency f_0 of $s(t)$ is

$$f_0 = (f_2 - f_1)/2$$

Definition. A bandlimited signal is one with compact support in the frequency domain. In symbols, $\mathbf{X}(f) \in C_c(\Omega)$.

Definition. A strictly bandlimited, real-valued signal is called a baseband signal if and only if $f_0 = 0$.

What to Remember.

1.2 B2 LTI Channel

Fix an LTI-channel, with transfer function $H(f)$, where f is given in Hertz. We are concerned mostly about its response to sinusoids, so we require the Fourier Transform only. Below are the relationships between an input $\mathbf{X}(f)$ and $\mathbf{Y}(f)$.

1. Amplitude relationship (in the time-domain and frequency domain)

$$y(t) = h(t) * x(t)$$

$$\mathbf{Y}(f) = \mathbf{H}(f) \cdot \mathbf{X}(f)$$

2. Energy Spectral Densities (in the frequency domain)

$$|\mathbf{Y}(f)|^2 = |\mathbf{H}(f)|^2 \cdot |\mathbf{X}(f)|^2$$

3. Power Spectral Densities (in the frequency domain)

$$P_{\mathbf{y}}(f) = |\mathbf{H}(f)|^2 \cdot P_{\mathbf{X}}(f)$$

We will hereinafter refer to the transfer function of the LTI channel as 'the LTI channel $\mathbf{H}(f)$' and use those two terms interchangably. We state some characterizations of the transfer function $\mathbf{H}(f)$:

1. Polar decomposition

$$\mathbf{H}(f) = |\mathbf{H}(f)| e^{j\theta(f)}, \quad \theta(f) := \angle(\mathbf{H}(f))$$

2. Gain of the channel, or the magnitude response $|\mathbf{H}(f)|$,

3. Phase of the channel, or the phase response $\theta(f)$,

4. Group delay at frequency f,

$$\tau(f) := \frac{1}{2\pi} \frac{d\theta(f)}{df}$$

5. Relative channel magnitude attenuation at frequency f,

$$|\mathbf{H}(f)|_{dB}^{-1} = -20 \log_{10} |\mathbf{H}(f)|$$

1.2.1 Channel Response Equations

Let us agree to make the following definitions

- $s(t)$: transmitted signal,
- $r(t)$: received signal,
- $\mathbf{H}(f)$: transfer function of LTI channel
- $h(t)$: normalized impulse response to channel
- $i(t)$: additive interference
- $n(t)$: additive noise
- a: attenuation (assumed to be a constant), in addition to channel attenuation
- $r_0(t)$ interference, noise free received component

Equations in the time-domain

$$r(t) = r_0(t) + i(t) + n(t) \tag{1}$$

$$r_0(t) = a^{-1}h(t) * s(t) \tag{2}$$

Likewise in the frequency domain

$$\mathbf{R}(f) = \mathbf{R}_0(f) + \mathbf{I}(f) + \mathbf{N}(f) \tag{3}$$

$$\mathbf{R}_0(f) = a^{-1}\mathbf{H}(f) \cdot \mathbf{S}(f) \tag{4}$$

Combining the two equations, we get

$$r(t) = a^{-1}ht(t) * s(t) + i(t) + n(t) \tag{5}$$

$$\mathbf{R}(f) = a^{-1}\mathbf{H}(f) \cdot \mathbf{S}(f) + \mathbf{I}(f) + \mathbf{N}(f) \tag{6}$$

1.2.2 Constant Gain and Group Delay

Let us consider a real-valued, strictly bandlimited, signal $s(t)$, and a channel $\mathbf{H}(f)$. Suppose that $[f_1, f_2]$ is a closed interval that contains supp $(S(f)) \cap \mathbb{R}^+$, and $|\mathbf{H}(f)| = 1$, and $\tau(f) = \tau_0$ on $[f_1, f_2]$. Then,

$$\tau(f) = \tau_0 = \frac{1}{2\pi}\frac{d}{df}\theta(f) \implies \theta(f) = 2\pi\tau_0 f$$

Writing $\mathbf{H}(f)$ in polar form yields

$$\mathbf{R}_0(f) = a^{-1}|\mathbf{H}(f)|e^{j2\pi\tau_0 f}S(f)$$

Therefore, the interference, noise-free component only suffers from constant attenuation without distortion. If the group delay were not a constant on $[f_1, f_2]$, then a distorted version of $s(t)$ will be received.

$$r_0(t) = a^{-1}s(t + \tau_0)$$

1.2.3 Equalizer Function

If indeed that $\mathbf{H}(f)$ has a non-constant group-delay on $[f_1, f-2]$, then we can 'undo' the channel response by applying another LTI filter in the form of $\mathbf{Q}(f)$, define

$$\mathbf{Q}(f) = \frac{bae^{j2\pi f\tau'}}{H(f)}$$

Notice the original transfer function $\mathbf{H}(f)$ at the denominator, and the attenuation is undone by a gain factor of a. An ideal equalizer only introduces additional delay in τ' (see the numerator). Applying one $\mathbf{Q}(f)$ after $\mathbf{H}(f)$, we get the new effective transfer function

$$\mathbf{QH}(f) = be^{j2\pi f\tau')} \implies s'(t) = bs(t + \tau')$$

Which just delays the signal by τ' and applies a gain of b.

1.2.4 Noise and Filtering

The ideal bandpass filter has constant group delay of d_F (delay of the filter) over $B = [f_1, f_2]$, and its amplitude response is just the indicator function on B.

$$|\mathbf{B}_F(f)| = \chi_B$$

If we apply $\mathbf{B}_F(f)$ onto $\mathbf{N}(f)$ and $\mathbf{I}(f)$, we can filter out the out of band interference and noise.

Consider the following equations onto the noise-interference free component:

$$\mathbf{R}_{oF}(f) = \mathbf{R}_o(f)\mathbf{B}_F(f) = a^{-1}\mathbf{R}_0(f)e^{j2\pi f d_F} \tag{7}$$

$$r_{oF}(t) = a^{-1}r_o(t + d_F) \tag{8}$$

The effects on the $\mathbf{N}(f)$ and $\mathbf{I}(f)$ hold no surprises,

- $\mathbf{N}_F(f)$ vanishes outside $[f_1, f_2]$
- $\mathbf{I}_F(f)$ vanishes outside $[f_1, f_2]$

What to Remember.

1.3 B3 Analog Modulation

We will discuss three main modulation techniques, namely

1. Amplitude Modulation (AM),
2. Phase Modulation (PM), and
3. Frequency Modulation (FM)

Let us agree to define $m(t) \iff \mathbf{S}(f)$ as some real-valued, strictly bandlimited, baseband signal. And our carrier wave $A_c \cos 2\pi f_c t$ at carrier frequency f_cHz.

Definition. Coherent Demodulation is when the demodulator requires a reference signal which has exactly the same frequency and phase as the carrier signal.

1.3.1 DSB-LC AM

Transmitted Signal

$$s_{LC}(t) = A_c\Big(1 + km(t)\Big)\cos(2\pi f_c t) \tag{9}$$

$$\mathbf{S}_{LC}(f) = \frac{A_c}{2}\Big[\delta(f - f_c) + \delta(f + f_c)\Big] + \frac{kA_c}{2}\Big[\mathbf{M}(f - f_c) + \mathbf{M}(f + f_c)\Big] \tag{10}$$

Modulation Considerations

- k is chosen such that $1 + km(t) \geq 0$ for all $t \geq 0$ to prevent phase reversal,
- AM Modulation index: $\phi = -km_{min} \leq 1$,
- Percentage Modulation: 100ϕ,
- Under/Over-modulation $\phi < 1$ or $\phi > 1$

Power efficiency

- The unmodulated carrier component has power

$$P_c = \frac{A_c^2}{2}$$

- The information signal power,

$$P_s = \frac{(kA_c)^2}{2} \int |m(t)|^2 dt$$

- The required power,

$$P_t = P_c + P_s$$

- It can be shown that assuming that $m(t)$ is a sinusoid, then the information signal power is bounded above by

$$P_s \leq \frac{1}{3} P_t$$

(Waste power bad!)

What to Remember (DSB-LC AM)**.**

1. Simple and Robust
2. Envelope Detection, does not require coherent demodulation.
3. Bandwidth: $m(t)$ has bandwidth W,then $s_{LC}(t)$ will require $2W$ bandwidth,
4. Low POWER efficiency, because of unmodulated carrier component
5. Bandwidth Overlapping: Require $W \ll f_c$.
6. Within AWGN channel, provides better SNR than DSB-SC, SSB-SC.

1.3.2 DSB-SC AM

Transmitted Signal

$$s_{SC}(t) = A_c k m(t) \cos(2\pi f_c t) \tag{11}$$

$$\mathbf{S}_{SC}(f) = \frac{kA_c}{2} \Big[\mathbf{M}(f - f_c) + \mathbf{M}(f + f_c) \Big] \tag{12}$$

What to Remember (DSB-SC AM).

1. Not as simple as DSB-LC
2. Requires coherent demodulation. Complicated set up.
3. Bandwidth: $2W$, same as DSB-LC
4. Power Efficiency: Higher than DSB-LC

1.3.3 SSB-SC AM

Single-sideband, suppressed carrier. We either choose Upper or Lower side bands (away and towards the origin), because of hermitian symmetry of $\mathbf{S}(f)$.

- Bandwidth: W, improved,
- Hard to realize the phase splitter (unit-step in frequency domain) at baseband, because of the discontinuity at $f = 0$.
- Demodulation is even more complex than DSB-SC

1.3.4 FM

What to Remember.

1. $s_{FM}(t) = A_c \cos(\theta(t))$, with $\theta(t)$ being the 'phase' of the transmitted signal
$$\theta(t) = 2\pi\left(f_c t + k\int_{-\infty}^{t} m(x)dx\right)$$
The instantaneous frequency is therefore
$$\frac{d}{dt}\theta(t) = 2\pi(f_c + km(t))$$
2. Phase proportional to integral of $m(t)$.
3. Peak Frequency Deviation: $\Delta f = k\|m(t)\|_\infty$,
4. FM Index: $\beta = \Delta f/W$, W is the bandwidth of $m(t)$
5. Carson's Rule: required bandwidth for FM $B_{FM} \approx 2(1+\beta)W$

6. FM requires much larger BW than AM,
7. Increasing β increases required BW, and improves $SNR_{out} = SNR_{in}[3\beta^2(1+\beta)/2]$.
8. AM radio systems operate at much lower BW than FM. 500-1700kHz compared to 88-108MHz.

1.3.5 PM

What to Remember.

1. Transmitted Signal

$$s_{PM}(t) = A_c \cos\left(2\pi f_c t + km(t)\right)$$

2. Instant Frequency is proportional to $\frac{d}{dt}m(t)$.
3. Phase proportional to $m(t)$.

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1.4 B4 Digital Modulation

1.5 B5 ISI and AWGN

What to Remember.

1. Nyquist Criterion for Digital Modulation.

$$\varepsilon = \frac{f_s}{B} \leq 1$$

   Where f_s is the symbol rate transmitted over a passband bandwidth B, without ISI.

2. Shannon's Theorem for AWGN bitrate

$$\varepsilon_{max} = \log_2(1 + SNR)$$

3. Effective bitrate R_b (bits per sec),

$$R_b = f_s \log_2 |\mathcal{A}|, \quad |\mathcal{A}| \text{ size of alphabet}$$