Chapter 22: Symplectic Manifold

Symplectic Tensors

Definition 1.1: Billinear forms

Let V be a vector space, a *billinear form* $\omega : V \times V \to \mathbb{R}$ is a 2-tensor on V.

Definition 1.2: Characterization of billinear forms

Let ω be a billinear form on V, it is

- *symmetric* if
$$\omega(x,y) = \omega(y,x)$$
- *skew-symmetric* or *anti-symmetric* if
$$\omega(x,y) = (-1)\omega(y,x)$$
- *alternating* if
$$\omega(x,x) = 0$$

If V is a vector space over the field F and $\mathsf{char}(F) \neq 2$, then the last two conditions are equivalent. Moreover,

- V is called an *orthogonal geometry* if ω is symmetric.
- V is called a *symplectic geometry* if ω is alternating.

Definition 1.3: Metric vector space

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

Matrices and billinear forms

Definition 1.4: Matrix of billinear form

If $B = (b_1, \dots, b_n)$ is an ordered basis for V, we define the *matrix representation of* ω by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i, b_j))$$

Proposition 1.1: Matrix induces a billinear form

Let $A = (a_{ij})$ be a matrix on V with respect to some basis $B = (b_{\underline{n}})$ it is clear that A induces a billinear form, on V through $A(x,y) = [x]_B^T A[y]_B$, where $[\cdot]_B$ denotes the canonical isomorphism $V \cong \mathbb{R}^n$ with respect to the basis B.

$$[x]_B^T A[y]_B = \begin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

for $x = x^i b_i$ and $y = y^j b_j$.

Moreover,

$$A[x]_B = \begin{bmatrix} A(b_1, x) \\ \vdots \\ A(b_n, x) \end{bmatrix}$$ is a *column* vector whose entries are given by applying x on the second coordinate

and

$$[x]_B^T A = \begin{bmatrix} A(x, b_1) & \cdots & A(x, b_n) \end{bmatrix}$$ is a *row* vector whose entries are given by applying x on the first coordinate

Let A_B be the matrix representation of ω with respect to the B, if C is another basis on V, then how do we compute A_C? The answer is simple, recall for any vector $x \in V$, $x = x_B^i b_i$ and $x = x_C^j c_j$, then

$$[x]_B = M_{C,B}[x]_C \quad \text{for some matrix of an automorphism } M_{C,B}$$

$\omega(x, y) = [x]_B^T A_B [y]_B = ([x]_C^T M_{C,B}^T) A_B (M_{C,B}[y]_C) = [x]_C^T A_C [y]_C$, then

$$M_{C,B}^T A_B M_{C,B} = A_C \tag{1}$$

We can describe this relation between the two matrices A_B and A_C by the following

Definition 1.5: Congruent matrices

Two matrices M and N are said to be *congruent*, if there exists an invertible matrix P for which

$$P^T M P = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

Proposition 1.2: Characterization of matrices using congruence

Let A_1 and A_2 be matrix representations of two billinear forms with respect to the basis B.

$$A_1 = (A_1(b_i, b_j))_{ij} \quad A_2 = (A_2(b_i, b_j))_{ij}$$

They induce the same billinear form if and only if they are congruent.

Definition 1.6: Alternate matrices

Let M be a matrix with F-coefficients, it is *alternate* if it is skew symmetric and is *hollow*; meaning it has 0s on the main diagonal. If $F = \mathbb{R}$ or $\mathrm{char}(F) \neq 2$, then alternate matrices are and are precisely the skew-symmetric matrices.

Orthogonality

For this section, (V, ω) will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

Definition 1.7: Orthogonal complements

A vector $x \in V$ is orthogonal to another vector $y \in V$, written $x \perp y$, if $\omega(x, y) = 0$.

If V is an orthogonal or symplectic geometry then $\perp$ is a symmetric relation. If E is a subset of V, we denote the *orthogonal complement of* E by

$$E^{\perp} \overset{\Delta}{=} \left\{ v \in V,\, v \perp E \right\}$$

Definition 1.8: Characterization of metric vector spaces

- A nonzero vector $x \in V$ is *isotropic*, or *null* if $\omega(x, x) = 0$
- V is *isotropic* if it contains at least one isotropic vector.
- V is *anisotropic* or *nonisotropic* if for every $x \in V$, $\omega(x, x) = 0 \implies x = 0$,
- V is *totally isotropic* (that is, symplectic if $\mathrm{char}(F) \neq 2$) if $\omega(x, x) = 0$ for every vector $x \in V$.

The first bullet point above is about vectors in V, while the others are properties of V.

- A vector $x \in V$ is called *degenerate* if $x \perp V$, that is,

$$\forall y \in V, \omega(x, y) = 0$$

- The *radical* of V, denoted by $\mathrm{rad}(V)$ is the set of all degenerate vectors in V,

$$\mathrm{rad}(V) \overset{\Delta}{=} V^{\perp}$$

- V is *singular* or *degenerate* if $\mathrm{rad}(V) \neq \{0\}$,
- V is *non-singular* or *non-degenerate* if $\mathrm{rad}(V) = \{0\}$,
- V is *totally singular*, if $\mathrm{rad}(V) = V$.

To summarize,

- V is isotropic if there exists a non-zero isotropic vector, meaning $\omega(x,x) = 0$, for some $x \neq 0$,
- V is degenerate if there exists a degenerate vector, $x \perp V$.

Proposition 1.3: Matrix invariants under congruence

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

Proof. ■

Proposition 1.4: Characterization of non-degeneracy

V is non-degenerate if and only if every matrix representation A of ω is non-singular.

Proof. Suppose V is non-degenerate, then let $B = (b_{\underline{n}})$ be a basis for V, if A is the matrix representation of ω with respect to B, let x be a non-zero vector in V, so $x \notin \mathrm{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so A is non-singular. If A' is another matrix representation with respect to another basis C, by Equation (1) A' is non-singular as well.

Conversely, if every matrix representation of ω is non-singular, let x be a non-zero vector in V, then $A[x]_B \neq 0$ is a non-zero vector so there exists some basis component $(A[x]_B)^j$ that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore V is non-degenerate. ■

Proposition 1.5: Characterisation of billinear forms from matrix representations

Let ω be a billinear form on V, if $\mathcal{M}(\omega)$ the induced matrix representation relative to any basis. Assume V is a vector space over $\mathbb{R}$, then

- it is symmetric iff $\mathcal{M}(\omega)$ symmetric as a matrix,
- it is skew-symmetric, iff alternating iff $\mathcal{M}(\omega)$ is skew-symmetric as a matrix.

Corollary 1.1: Characterisation of non-singular symplectic form

Let (V,ω) be a finite dimensional vector space over $\mathbb{R}$, equipped with a billinear form ω. (V,ω) is a non-singular symplectic vector space iff the matrix representation of ω with respect to every basis is non-singular and skew-symmetric.

Riesz Representation Theorems

Proposition 1.6

Let (V,ω) be a nonsingular metric vector space, the map $x \mapsto x \lrcorner \omega \in V^*$ defined by

$$x \lrcorner \omega = \omega(x,\cdot), \quad \text{and} \quad (x \lrcorner \omega)(y) = \omega(x,y), \quad \forall y \in V$$

is a linear isomorphism from V to V^*.

Isometries

Definition 1.9: Isometry between MVS

Let (V,ω) and (W,η) be metric vector spaces. An *isometry* $\tau \in L(V,W)$ is a linear isomorphism that preserves the billinear form.

$$\omega(u,v) = \eta(\tau u, \tau v)$$

Definition 1.10: Orthogonal, symplectic groups

Let V be a nonsingular metric vector space. If V is an orthogonal (resp. symplectic) geometry, the set of all isometries on V is called the *orthogonal (resp. symplectic) group on* V. It is a group under composition, and is denoted by $\mathcal{O}(V)$ (resp. $\mathsf{Sp}(V)$).

Hyperbolic spaces, nonsingular completions

Canonical Forms

Symplectic Manifolds

Darboux's Theorem

Proposition 1.7: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let M be a smooth manifold. Suppose $V : J \times M \to TM$ is a smooth time-varying vector field on M. Denote the time-varying flow of V by $\psi : \mathcal{E} \to M$. Let $A \in \mathcal{T}^k(M)$ be a smooth time-invariant covariant k-tensor field on M. For every $(t_1, t_0, p) \in \mathcal{E}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_1} (\psi^*_{t,t_0} A)_p = (\psi^*_{t_1,t_0}(\mathcal{L}_{V_{t_1}} A))_p \tag{2}$$

Chapter Hofer book

Definition 2.1: Symplectic vector space

Let V be a finite dimensional vector space over $\mathbb{R}$. It is a *symplectic vector space* if it admits a non-singular, antisymmetric billinear form $\omega : V \times V \to \mathbb{R}$.

$$\omega(u,v) = -\omega(v,u)$$

for $u, v \in V$. By the previous section on Riesz Representation, the linear map

$$\hat{\omega} : V \to V^*, \quad v \mapsto \omega(v,\cdot)$$

is a linear isomorphism of V onto its dual vector V^*.

We define the *standard symplectic vector space* $(\mathbb{R}^{2n}, \omega_0)$, where $n \in \mathbb{N}^+$, where

$$\omega_0(u,v) = < Ju, v > \quad J \triangleq \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where $< \cdot, \cdot >$ denotes the standard inner product on $\mathbb{R}^{2n}$.

$$\omega_0(u,v) = < Ju, v > = < u, J^T v > = u^T J^T v \tag{3}$$

$J^T = -J$ by Corollary 1.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

Definition 2.2: Symplectic linear map

Let (V, ω) be a symplectic vector space. A linear map $F \in \mathsf{Hom}(V)$ is *symplectic* if it preserves symplectic form ω. For every $u \in V$,

$$< u, v > = < Au, Av > \triangleq A^* \omega(u,v)$$

where $A^* : \mathcal{T}^*(V) \to \mathcal{T}^*(V)$ denotes the tensor pullback by precomposing any tensor S by A

$$\forall S \in \mathcal{T}^k(V), \quad A^* S(v_{\underline{k}}) \triangleq S(Av_{\underline{k}})$$

The set of linear symplectic maps on a $2n$-dimensional vector space form a group under composition. It is a Lie Group denoted by $\mathsf{Sp}(n)$.

Proposition 2.1: Symplectic Maps are Area-preserving

Let $(\mathbb{R}^{2n}, \omega_0)$ denote the standard symplectic space. If $\varphi \in \mathsf{Sp}(n)$, then $\det \varphi = 1$.

Proof. See page 4. ■

$$
\begin{aligned}
(\Lambda_{s-|\alpha|}\partial^\alpha f)\hat{} &= (1+|\zeta|^2)^{s/2-|\alpha|/2}\cdot(\partial^\alpha f)\hat{} && (4)\\
&= (1+|\zeta|^2)^{s/2-|\alpha|/2}\cdot(2\pi i\zeta)^\alpha\cdot\hat{f} && (5)\\
&= (2\pi i)^{|\alpha|}(1+|\zeta|^2)^{(s-|\alpha|)/2}\cdot|\zeta|^{|\alpha|}\cdot\hat{f} && (6)\\
&\leq |\alpha|(1+|\zeta|^2)^{s/2}\hat{f} && (7)
\end{aligned}
$$