Chapter 8

Theorem 8.1

Proposition 1.1

Proof. ■

Theorem 8.2

Proposition 1.2

Proof. ∎

Theorem 8.3

Proposition 1.3

If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β

Proof. ■

Theorem 8.4

Proposition 1.4

Proof. ■

Theorem 8.5

Proposition 1.5

Proof. ∎

Theorem 8.6

Proposition 1.6

Proof. ∎

Theorem 8.7

Proposition 1.7

Proof. ■

Theorem 8.8

Proposition 1.8

Proof. ■

Theorem 8.9

Proposition 1.9

Proof. ∎

Theorem 8.10

Proposition 1.10

Proof. ■

Theorem 8.11

Proposition 1.11

Proof. ■

Theorem 8.12

Proposition 1.12

Proof. ■

Theorem 8.13

Proposition 1.13

Proof. ■

Theorem 8.14

Proposition 1.14

Suppose $\phi \in L^1$, and $\int \phi(x)dx = a$.

(a) If $f \in L^p$, $p \in [1, +\infty]$, then $f * \phi_t \to af$ in the L^p norm as $t \to 0$.

(b) If f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.

(c) If $f \in L^\infty$ and f is continuous on an open set U, then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$

Proof of Part A. First, the convolution $f * \phi_t$ is in L^p by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int\limits_{y \in \mathbb{R}^n} f(x-y)t^{-n}\phi(t^{-1}y)dy - \int\limits_{y \in \mathbb{R}^n} f(x)\phi(y)dy \tag{1}$$

Now apply Theorem 2.44, with $y \mapsto y/t$, and denote this invertible map by $T \in GL(n, \mathbb{R})$, so that $|\det(T)| = t^{-n}$, then $y = T(y)t$ for every $t > 0$. It follows that

$$\begin{aligned}(f * \phi_t)(x) &= |\det(T)| \cdot \int\limits_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int\limits_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int\limits_{z \in \mathbb{R}^n} \tau_{tz} f(x)\phi(z)dz \end{aligned} \tag{2}$$

Next, $a = \int \phi$ so $af = \int f(x)\phi(z)dz$. Using Equations (1) and (2) we get

$$(f * \phi_t - af)(x) = \int\limits_{z \in \mathbb{R}^n} (\tau_{tz} f - f)\phi(z)dz \tag{3}$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

> The norm of an integral is less than the integral of the norm.

to Equation (3), and

$$\|f * \phi_t - af\|_p \le \int_{z \in \mathbb{R}^n} \|(\tau_{tz} f - f)\phi(z)\|_p dz \tag{4}$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every $z \in \mathbb{R}^{n'}$,

$$\left\|(\tau_{tz}f - f)\phi(z)\right\|_p = \left(\int_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left(2\|f\|_p\right) < +\infty$$

   Since $\|\phi\|_1 < +\infty$, $|\phi(z)| < +\infty$ almost everywhere.

2. Next, to show $z \mapsto \left\|\phi(z)(\tau_{tz}f - f)\right\|_p$ is in $L^1\mathbb{R}^n, z$. Reuse the last estimate in the previous bullet point, and

$$\left\|\phi(z)(\tau_{tz}f - f)\right\|_p \leq |\phi(z)| \left(2\|f\|_p\right)$$

   Taking the integral in L^+ with respect to z, we get

$$\left\| \left(\left\|\phi(z)(\tau_{tz}f - f)\right\|_p \right) \right\|_1 < +\infty$$

   so both assumptions are satisfied.

Therefore Equation (4) holds. Next, fix any sequence of $t_n > 0$ with $t_n \to 0$. The Dominated Convergence Theorem gives, since $|\phi(z)|\left\|\tau_{t_n z}f - f\right\|_p$ is dominated by $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned}
\lim_{n\to\infty} \int_{z \in \mathbb{R}^n} \left\|\tau_{t_n z}f - f\right\|_p |\phi(z)| dz &= \int_{z \in \mathbb{R}^n} \lim_{n\to\infty} \left\|\tau_{t_n z}f - f\right\|_p |\phi(z)| dz \\
&= \int_{z \in \mathbb{R}^n} 0 dz \\
&= 0
\end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the L^p norm for $p \in [1, +\infty)$. So almost every $z \in \mathbb{R}^n$ (since again, $|\phi(z)|$ can be infinite on a null set),

$$\left\|\tau_{t_n z}f - f\right\|_p \to 0 \implies \left\|\tau_{t_n z}f - f\right\|_p |\phi(z)| \to 0$$

as $n \to +\infty$. It follows that

$$\lim_{n\to\infty} \left\|f * \phi_{t_n} - af\right\|_p = \lim_{n\to\infty} \left\| \int_{z \in \mathbb{R}^n} [\tau_{t_n z}f(x) - f(x)]\phi(z) dz \right\|_p = 0$$

Since the sequence $t_n \to 0$ is arbitrary, we conclude that the function $t \mapsto \left\|f * \phi_t - af\right\|_p$ has a limit of 0 as $t \to 0$. ■

Proof of Part B. Suppose $f \in \mathrm{UBC}(\mathbb{R}^n)$, so that f is uniformly continuous and bounded. We wish to show $f * \phi_t \to af$ uniformly as $t \to 0$. In symbols,

$$g : t \mapsto \left\|f * \phi_t - af\right\|_u, g \to 0, \text{as } t \to 0$$

The convolution between f and ϕ_t makes sense at every $x \in \mathbb{R}^n$, as

$$\int |\tau_y f(x)||\phi(y)|dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (3), we get

$$\begin{aligned}
\|f * \phi_t - af\|_u &= \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz} f - f) \cdot \phi(z) dz \right| \\
&\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\
&\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\
&= \int_{z \in \mathbb{R}^n} \|\tau_{tz} f - f\|_u \cdot |\phi(z)| dz
\end{aligned} \tag{5}$$

the last equality is a simple consequence of the monotonicity of the integral in L^+, indeed. For every $x \in \mathbb{R}^n$, the following holds pointwise for almost every z

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u$$

Apply the Dominated Theorem to the right member of (5), noting that it is dominated by $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$ as we have done for Part A of the proof. Since this holds for every sequence $t_n \to 0$, the proof is complete. ∎

Proof of Part C. Next, suppose that $f \in L^\infty$, and $f \in C(U)$, where U is open in $\mathbb{R}^n$. We claim that

$$f * \phi_t \to af$$

within the topology of uniform convergence on compact subsets of U. So that for every compact $K \subseteq U$

$$\sup_{x \in K} \left| f * \phi_t - af \right| \to 0, \text{ as } t \to 0$$

First, a small technical Lemma.

Lemma 1.1

If $\phi \in L^1(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists a compact $E \subseteq \mathbb{R}^n$, with

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

Proof. Assume that $\phi \geq 0$, if not, replace ϕ by $|\phi|$. Since $\mathrm{C_c}(\mathbb{R}^n)$ is dense in L^1 for every $\varepsilon 2^{-1} > 0$ there exists some $\psi \in \mathrm{C}_c(\mathbb{R}^n)$ with $\|\psi - \phi\|_1 < \varepsilon^{-1}$, and denote the compact support of ψ by $E = \mathbf{supp}\,(\psi)$, then

$$\big\||\psi| - |\phi|\big\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume $\psi \geq 0$ as well, perhaps by relabelling ψ by $|\psi|$. Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in L^+. The Triangle Inequality in L^1 gives

$$\|\chi_{E^c}\phi\|_1 = \|\phi - \chi_E\phi\|_1 = \|\phi(1-\chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E\phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any $\varepsilon > 0$, then by Lemma 1.1, ϕ induces some compact E with $\|\chi_{E^c}\phi\|_1 < +\varepsilon$. By Lemma 8.4, $\chi_K f \in \mathrm{C}_c(\mathbb{R}^n) \subseteq \mathrm{UBC}(\mathbb{R}^n)$. Uniform continuity of $\chi_K f$ gives us the continuity of translations. Now for the same $\varepsilon > 0$, there exists $r > 0$, for every $w \in \mathbb{R}^n$,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \tag{6}$$

Since E is compact, it is bounded, and let t be a small positive number such that for every $z \in E$,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a a t, namely $t = r2^{-1}(1 + \sup_{z\in E}|z|)^{-1}$. And for this $t > 0$, it follows that for every $z \in E$,

$$\sup_{x \in K} \left|\tau_{tz} f - f\right| < +\varepsilon$$

Since this holds for every $z \in E$, we write

$$\sup_{x \in K, z \in E} \left|\tau_{tz} f - f\right| < +\varepsilon$$

And

$$|\phi(z)|\left[\sup_{x \in K, z \in E} |\tau_{tz} f - f|\right] < |\phi(z)|\varepsilon$$

Monotonicity in $L^+(E, z)$ reads, for every $x \in K$,

$$\int\limits_{z \in E} |\phi(z)(\tau_{tz} f - f)|dz \leq \int\limits_{z \in E} |\phi|\varepsilon dz = \varepsilon\|\chi_E\phi\|_1 \leq \varepsilon\|\phi\|_1$$

Since this holds for every $x \in \mathbb{R}^n$,

$$\sup_{x \in K}\left\{\int_{z \in E} |\phi(z)| \cdot |\tau_{tz} f - f| dz\right\} \leq \varepsilon\|\phi\|_1 \tag{7}$$

Next, notice for every t, z, we have

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds $z \in E^c$ a.e,

$$|\phi(z)| \cdot |\tau_{tz} f - f| \leq |\phi(z)| \cdot 2\|f\|_u$$

Taking the integral, and applying the condition we imposed on E from Lemma (1.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz} f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz} f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \tag{8}$$

Combining Equations (7) and (8). Applying the additivity of the supremum (of $x \in K$), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz} f - f) dz + \int_{E^c} |\phi(z)|(\tau_{tz} f - f) dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to $\sup_{x \in K} |f * \phi_t - af|$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of Part C. ■

Theorem 8.15

Proposition 1.15

If $|\phi(x)| \leq C(1+|x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1,+\infty)$, then

$$f * \phi_t \to af$$

pointwise for every x in the Lebesgue set of f,

$$\mathcal{L}_f = \left\{x \in \mathbb{R}^n, \quad \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{y \in B(r,x)} |f(x) - f(y)| dy = 0\right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of f, and several pointwise estimates that will be of use.

Lemma 1.2

If $\phi : \mathbb{R}^n \to \mathbb{C}$, and

$$|\phi(x)| \leq C(1+|x|)^{n-\varepsilon}, \varepsilon > 0 \tag{9}$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

Proof of 1.2. If $x \neq 0$, then

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some B^c as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot (1+|x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in L^+, and applying Theorem 2.44, with $T : x \mapsto t^{-1}$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ∎

Lemma 1.3

If $f:\mathbb{R}^n \to \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.

Proof of 1.3. Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \to 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k,x))}\int_{y\in B(r_k,x)} |f(x)-f(y)|dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k,x)\setminus\{x\}$ with

$$|f(x)-f(y)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such y_k exists, and by monotonicity,

$$\frac{1}{m(B(r_k,x))}\int\limits_{y\in B(r_k,x)} |f(x)-f(y)|dy < \frac{1}{m(B(r_k,x))}\int\limits_{y\in B(r_k,x)} \varepsilon_0 dy = \varepsilon_0$$

So choose y_k as above, and it is clear that $y_k \to x$ as $k \to \infty$, but $f(y_k) \not\to f(x)$. Therefore f is not continuous at x. ■

Lemma 1.4

If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with

$$r \leq \eta \implies \int_{|y|<r} |f(x-y)-f(x)|dy \leq \delta \cdot r^n$$

Proof of 1.4. We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{10}$$

where $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$. By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_B(x/r)dx \\ &= |\det(T)|^{-1}\int \chi_B(x)dx \\ &= r^n m(B(1)) \end{aligned}$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta/m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r,x))}\int\limits_{y\in B(r,x)} |f(x)-f(y)|dy \leq \varepsilon$$

By translation invariance of m,

$$m(B(r,x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $|-1|$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x-y)| dy = \int_{y \in B(r,x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x-y,x) < r &\iff d(-y,0) < r \\ &\iff d(y,0) < r \end{aligned}$$

hence

$$\chi_{B(r,x)}(x-y) = \chi_{B(r,0)}(y)$$

■

Lemma 1.5

Let $A_j = \left\{|y| \in [2^{-j}\eta, 2^{1-j}\eta)\right\}$, and if Equation (9) holds for ϕ then ϕ_t satisfies

$$|\phi_t| \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)} \tag{11}$$

on A_j for every $t > 0$, where $\alpha = t^{-1}\eta$ for some $\eta > 0$.

Moreover, if $A_0 = \left\{|y| < 2^{-K}\eta\right\}$, where $K \geq 0$, then

$$|\phi_t(y)| \leq C \cdot t^{-n} \tag{12}$$

on A_0

Proof of 1.5. Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (9) establishes the first claim.

The second claim follows from Equation (9),

$$|\phi_t(y)| \le C \cdot t^{-n}(1+|t^{-1}y|)^{-(n+\varepsilon)} \le C \cdot t^{-n}$$

■

Main Proof of Theorem 8.15. The outline of the proof is as follows,

1. $|\phi| \le C \cdot (1+|x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and
2. $f \in L^p$ for $p \in [1, +\infty)$,
3. for any $x \in \mathcal{L}_f$, we wish to show

   $$|f * \phi_t - af|(x) \to 0, \quad \text{as } t \to 0$$

4. To prove this, we fix some $\beta > 0$ and show that

   $$|f * \phi_t - af|(x) < \beta$$

   since β is arbitrary, the proof will be complete.
5. By Lemma 1.4, for every $\delta > 0$ there exists a $\eta > 0$ where $r \le \eta$ implies

   $$\int_{|y|<r} |f(x) - f(x-y)| dy \le \delta \cdot r^n$$

   and using the L^1 inequality,

   $$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x-y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\le \int |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int\limits_{|y|<\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy + \int\limits_{|y|\ge\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let $\delta = \beta(2A)^{-1}$, where

   $$A = 2^n \cdot C \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

   we make the claim that this choice of δ will give us $I_1 < \beta/2$
7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for t sufficiently small, and applying the Triangle Inequality finishes the proof.

Let η be as above, and for $t>0$ and suppose we can find a $K \in \mathbb{N}^+$ with

$$2^K \le \eta/t \le 2^{K+1} \tag{13}$$

and define $\alpha = \eta/t$ for convenience.

Notice for any $K \ge 1$, the interval $[0,1)$ can be partitioned in the following manner

$$[0,1) = [0,2^{-K}) \cup \left(\bigcup_{j=1}^{K} [2^{-j},2^{1-j})\right)$$

and let us define

$$A_j = \left\{|y| \in [2^{-j}\eta, 2^{1-j}\eta)\right\}, \quad A_0 = \left\{|y| \in [0,2^{-K}\eta)\right\}$$

If no such K exists, then let $A_j = \varnothing$ and set $A_0 = \{|y| \in [0,\eta)\}$. The disjoint union of all $A_{j\ge 0}$ is the open ball $\{|y| \in [0,\eta)\}$. By Lemma 1.5 and Lemma 1.4 each $j \ge 0$,

$$\begin{aligned}
I_1 &= \sum_{j=0}^{K} \int_{y\in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\
&\le Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} \int_{y\in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\
&\le Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n
\end{aligned}$$

The left member reads,

$$\begin{aligned}
Ct^{-n}\delta(2^{-K}\eta)^n &\le C\delta\alpha^n 2^{-Kn} \\
&\le C\delta 2^{n(K+1)}2^{-Kn} \\
&= C\delta 2^n
\end{aligned}$$

and termwise for the right,

$$\begin{aligned}
Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^{\varepsilon} \cdot 2^{j\varepsilon+n}\eta^{-\varepsilon} \\
&= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon}
\end{aligned}$$

Summing over the geometric series,

$$\begin{aligned}
\sum_{j=1}^{K} 2^{j\varepsilon} &= 2^{\varepsilon} \sum_{j=0}^{K-1} 2^{j\varepsilon} \\
&= \frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1}
\end{aligned}$$

using the estimate for α in Equation (13)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned}(C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^{\varepsilon} - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^{\varepsilon}}{2^{\varepsilon} - 1}\end{aligned}$$

Finally, $I_1 \leq (C\delta 2^n)\left[\dfrac{2^{\varepsilon}}{2^{\varepsilon}-1} + 1\right]$, and by Step 6, $I_1 \leq \beta/2$.

Obtaining an estimate for I_2 is another laborious entreprise. Let us define $W = \{|y| \geq \eta\}$, and

- By Holder's Inequality,

  $$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

  where q is the conjugate exponent to p. Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \to 0$ as $t \to 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,

  $$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

  then $\|\chi_W \cdot \phi_t\|_\infty \leq C t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C t^{\varepsilon} \eta^{-(n+\varepsilon)}$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned}\|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q\cdot(n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q\cdot(n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q\cdot(n+\varepsilon)} dr \\ &= \frac{C^q t^{\varepsilon q}}{n - q\cdot(n+\varepsilon)} r^{n-q\cdot(n+\varepsilon))}\Big]_\eta^\infty \\ &= \frac{C^q t^{\varepsilon q}}{q\cdot(n+\varepsilon) - n} \eta^{n-q\cdot(n+\varepsilon)} \\ \|\chi_W \cdot \phi_t\|_q &= \left[\frac{C}{(q\cdot(n+\varepsilon)-n)^{1/q}} \left(\eta^{n-q\cdot(n+\varepsilon)}\right)^{1/q}\right] t^{\varepsilon} \\ &= C_3(q) t^{\varepsilon}\end{aligned}$$

- Find a t sufficiently small so that

$$t^{\varepsilon} < \min\left\{\beta(4C_3(1)|f(x)|)^{-1}, \beta(4C_3(q)\|f\|_p)^{-1}, \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1}\right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \to 0$.

■

Theorem 8.16

Proposition 1.16

See Theorem 8.15

Proof. ■

Theorem 8.17

Proposition 1.17

Proof. ■

Theorem 8.18

Proposition 1.18

Proof. ∎

Theorem 8.19

Proposition 1.19

Proof. ■

Theorem 8.20

Proposition 1.20

Proof. ■