# Chapter 8

# Theorem 8.1

**Proposition 1.1**

*Proof.* ∎

# Theorem 8.2

> **Proposition 1.2**

*Proof.*                                                                                    ∎

# Theorem 8.3

> **Proposition 1.3**
>
> If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices $\alpha$, $\beta$

*Proof.*                                                                ■

# Theorem 8.4

> **Proposition 1.4**

*Proof.* ∎

# Theorem 8.5

**Proposition 1.5**

*Proof.* ∎

# Theorem 8.6

> **Proposition 1.6**

*Proof.* ∎

# Theorem 8.7

> **Proposition 1.7**

*Proof.* ∎

# Theorem 8.8

**Proposition 1.8**

*Proof.* ∎

# Theorem 8.9

> **Proposition 1.9**

*Proof.*                                                                                    ∎

# Theorem 8.10

> **Proposition 1.10**

*Proof.* ∎

# Theorem 8.11

> **Proposition 1.11**

*Proof.* ∎

# Theorem 8.12

> **Proposition 1.12**

*Proof.* ∎

# Theorem 8.13

> **Proposition 1.13**

*Proof.* ∎

# Theorem 8.14

> **Proposition 1.14**
>
> Suppose $\phi \in L^1$, and $\int \phi(x)dx = a$.
>
> (a) If $f \in L^p$, $p \in [1, +\infty]$, then $f * \phi_t \to af$ in the $L^p$ norm as $t \to 0$.
>
> (b) If $f$ is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.
>
> (c) If $f \in L^\infty$ and $f$ is continuous on an open set $U$, then $f * \phi_t \to af$ uniformly on compact subsets of $U$ as $t \to 0$

*Proof of Part A.* First, the convolution $f * \phi_t$ is in $L^p$ by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int\limits_{y \in \mathbb{R}^n} f(x-y)t^{-n}\phi(t^{-1}y)dy - \int\limits_{y \in \mathbb{R}^n} f(x)\phi(y)dy \qquad (1)$$

Now apply Theorem 2.44, with $y \mapsto y/t$, and denote this invertible map by $T \in GL(n, \mathbb{R})$, so that $|\det(T)| = t^{-n}$, then $y = T(y)t$ for every $t > 0$. It follows that

$$(f * \phi_t)(x) = |\det(T)| \cdot \int\limits_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy$$

$$= \int\limits_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz$$

$$= \int\limits_{z \in \mathbb{R}^n} \tau_{tz}f(x)\phi(z)dz \qquad (2)$$

Next, $a = \int \phi$ so $af = \int f(x)\phi(z)dz$. Using Equations (1) and (2) we get

$$(f * \phi_t - af)(x) = \int\limits_{z \in \mathbb{R}^n} (\tau_{tz}f - f)\phi(z)dz \qquad (3)$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

> The norm of an integral is less than the integral of the norm.

to Equation (3), and

$$\|f * \phi_t - af\|_p \leq \int\limits_{z \in \mathbb{R}^n} \|(\tau_{tz}f - f)\phi(z)\|_p dz \qquad (4)$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every $z \in \mathbb{R}^{n'}$,

$$\|(\tau_{tz}f - f)\phi(z)\|_p = \left( \int\limits_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left( 2\|f\|_p \right) < +\infty$$

Since $\|\phi\|_1 < +\infty$, $|\phi(z)| < +\infty$ almost everywhere.

2. Next, to show $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$ is in $L^1\mathbb{R}^n, z$. Reuse the last estimate in the previous bullet point, and

$$\|\phi(z)(\tau_{tz}f - f)\|_p \leq |\phi(z)| \left(2\|f\|_p\right)$$

Taking the integral in $L^+$ with respect to $z$, we get

$$\left\| \left( \|\phi(z)(\tau_{tz}f - f)\|_p \right) \right\|_1 < +\infty$$

so both assumptions are satisfied.

Therefore Equation (4) holds. Next, fix any sequence of $t_n > 0$ with $t_n \to 0$. The Dominated Convergence Theorem gives, since $|\phi(z)|\|\tau_{t_nz}f - f\|_p$ is dominated by $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\lim_{n\to\infty} \int_{z\in\mathbb{R}^n} \|\tau_{t_nz}f - f\|_p |\phi(z)|dz = \int_{z\in\mathbb{R}^n} \lim_{n\to\infty} \|\tau_{t_nz}f - f\|_p |\phi(z)|dz$$

$$= \int_{z\in\mathbb{R}^n} 0 dz$$

$$= 0$$

The second last equality is from Lemma 8.4, as translation is continuous in the $L^p$ norm for $p \in [1, +\infty)$. So almost every $z \in \mathbb{R}^n$ (since again, $|\phi(z)|$ can be infinite on a null set),

$$\|\tau_{t_nz}f - f\|_p \to 0 \implies \|\tau_{t_nz}f - f\|_p|\phi(z)| \to 0$$

as $n \to +\infty$. It follows that

$$\lim_{n\to\infty} \|f * \phi_{t_n} - af\|_p = \lim_{n\to\infty} \left\| \int_{z\in\mathbb{R}^n} [\tau_{t_nz}f(x) - f(x)]\phi(z)dz \right\|_p = 0$$

Since the sequence $t_n \to 0$ is arbitrary, we conclude that the function $t \mapsto \|f * \phi_t - af\|_p$ has a limit of 0 as $t \to 0$. ∎

*Proof of Part B.* Suppose $f \in \mathrm{UBC}(\mathbb{R}^n)$, so that $f$ is uniformly continuous and bounded. We wish to show $f * \phi_t \to af$ uniformly as $t \to 0$. In symbols,

$$g : t \mapsto \|f * \phi_t - af\|_u, g \to 0, \text{ as } t \to 0$$

The convolution between $f$ and $\phi_t$ makes sense at every $x \in \mathbb{R}^n$, as

$$\int |\tau_y f(x)||\phi(y)|dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (3), we get

$$\|f * \phi_t - af\|_u = \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz} f - f) \cdot \phi(z) dz \right|$$

$$\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz$$

$$\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz$$

$$= \int_{z \in \mathbb{R}^n} \|\tau_{tz} f - f\|_u \cdot |\phi(z)| dz \tag{5}$$

the last equality is a simple consequence of the monotonicity of the integral in $L^+$, indeed. For every $x \in \mathbb{R}^n$, the following holds pointwise for almost every $z$

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u$$

Apply the Dominated Theorem to the right member of (5), noting that it is dominated by $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$ as we have done for Part A of the proof. Since this holds for every sequence $t_n \to 0$, the proof is complete. ∎

*Proof of Part C.* Next, suppose that $f \in L^\infty$, and $f \in C(U)$, where $U$ is open in $\mathbb{R}^n$. We claim that

$$f * \phi_t \to af$$

within the topology of uniform convergence on compact subsets of $U$. So that for every compact $K \subseteq U$

$$\sup_{x \in K} \left| f * \phi_t - af \right| \to 0, \text{ as } t \to 0$$

First, a small technical Lemma.

---

**Lemma 1.1**

If $\phi \in L^1(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists a compact $E \subseteq \mathbb{R}^n$, with

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

---

*Proof.* Assume that $\phi \geq 0$, if not, replace $\phi$ by $|\phi|$. Since $C_c(\mathbb{R}^n)$ is dense in $L^1$ for every $\varepsilon 2^{-1} > 0$ there exists some $\psi \in C_c(\mathbb{R}^n)$ with $\|\psi - \phi\|_1 < \varepsilon^{-1}$, and denote the compact support of $\psi$ by $E = \text{supp}(\psi)$, then

$$\|\,|\psi| - |\phi|\,\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume $\psi \geq 0$ as well, perhaps by relabelling $\psi$ by $|\psi|$. Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in $L^+$. The Triangle Inequality in $L^1$ gives

$$\|\chi_{E^c}\phi\|_1 = \|\phi - \chi_E\phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E\phi\|_1 < \varepsilon$$

∎

Back to the main proof of Part C, fix any $\varepsilon > 0$, then by Lemma 1.1, $\phi$ induces some compact $E$ with $\|\chi_{E^c}\phi\|_1 < +\varepsilon$. By Lemma 8.4, $\chi_K f \in \mathrm{C}_c(\mathbb{R}^n) \subseteq \mathrm{UBC}(\mathbb{R}^n)$. Uniform continuity of $\chi_K f$ gives us the continuity of translations. Now for the same $\varepsilon > 0$, there exists $r > 0$, for every $w \in \mathbb{R}^n$,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \tag{6}$$

Since $E$ is compact, it is bounded, and let $t$ be a small positive number such that for every $z \in E$,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a a $t$, namely $t = r 2^{-1}(1 + \sup_{z \in E} |z|)^{-1}$. And for this $t > 0$, it follows that for every $z \in E$,

$$\sup_{x \in K} |\tau_{tz} f - f| < +\varepsilon$$

Since this holds for every $z \in E$, we write

$$\sup_{x \in K, z \in E} |\tau_{tz} f - f| < +\varepsilon$$

And

$$|\phi(z)|\left[\sup_{x \in K, z \in E} |\tau_{tz} f - f|\right] < |\phi(z)|\varepsilon$$

Monotonicity in $L^+(E, z)$ reads, for every $x \in K$,

$$\int_{z \in E} |\phi(z)(\tau_{tz} f - f)| dz \leq \int_{z \in E} |\phi| \varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every $x \in \mathbb{R}^n$,

$$\sup_{x \in K}\left\{\int_{z \in E} |\phi(z)| \cdot |\tau_{tz} f - f| dz\right\} \leq \varepsilon \|\phi\|_1 \tag{7}$$

Next, notice for every $t, z$, we have

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds $z \in E^c$ a.e,

$$|\phi(z)| \cdot |\tau_{tz} f - f| \leq |\phi(z)| \cdot 2\|f\|_u$$

Taking the integral, and applying the condition we imposed on $E$ from Lemma (1.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz} f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz}f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \tag{8}$$

Combining Equations (7) and (8). Applying the additivity of the supremum (of $x \in K$), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz}f - f) dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f) dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to $\sup_{x \in K} |f * \phi_t - af|$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of Part C. ∎

# Theorem 8.15

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**Proposition 1.15**

If $|\phi(x)| \le C(1 + |x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1, +\infty)$, then

$$f * \phi_t \to af$$

pointwise for every $x$ in the Lebesgue set of $f$,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \quad \lim_{r \to 0} \frac{1}{m(B(r,x))} \int_{y \in B(r,x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

---

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of $f$, and several pointwise estimates that will be of use.

---

**Lemma 1.2**

If $\phi : \mathbb{R}^n \to \mathbb{C}$, and

$$|\phi(x)| \le C(1 + |x|)^{n-\varepsilon}, \, \varepsilon > 0 \tag{9}$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

---

*Proof of 1.2.* If $x \ne 0$, then

$$|\phi| \le C \cdot (1 + |x|)^{-(n+\varepsilon)} \le C \cdot |x|^{-(n+\varepsilon)}$$

on some $B^c$ as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \le C \cdot (1 + |x|)^{-(n+\varepsilon)} \le C \cdot (1 + |x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1} x)|$$

taking the integral in $L^+$, and applying Theorem 2.44, with $T : x \mapsto t^{-1}$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ∎

---

> **Lemma 1.3**
>
> If $f : \mathbb{R}^n \to \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.

*Proof of 1.3.* Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \to 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k, x) \setminus \{x\}$ with

$$|f(x) - f(y)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such $y_k$ exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose $y_k$ as above, and it is clear that $y_k \to x$ as $k \to \infty$, but $f(y_k) \not\to f(x)$. Therefore $f$ is not continuous at $x$. ∎

> **Lemma 1.4**
>
> If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with
>
> $$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

*Proof of 1.4.* We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{10}$$

where $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$. By Theorem 2.44,

$$m(B(r)) = \int \chi_B(x/r) dx$$

$$= |\det(T)|^{-1} \int \chi_B(x) dx$$

$$= r^n m(B(1))$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta/m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of $m$,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $|-1|$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int\limits_{|y| \in B(r)} |f(x) - f(x-y)|dy = \int\limits_{y \in B(r,x)} |f(x) - f(y)|dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$d(x-y,x) < r \iff d(-y,0) < r$$
$$\iff d(y,0) < r$$

hence

$$\chi_{B(r,x)}(x-y) = \chi_{B(r,0)}(y)$$

∎

---

**Lemma 1.5**

Let $A_j = \left\{ |y| \in [2^{-j}\eta,\, 2^{1-j}\eta) \right\}$, and if Equation (9) holds for $\phi$ then $\phi_t$ satisfies

$$|\phi_t| \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)} \tag{11}$$

on $A_j$ for every $t > 0$, where $\alpha = t^{-1}\eta$ for some $\eta > 0$.

Moreover, if $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$, where $K \geq 0$, then

$$|\phi_t(y)| \leq C \cdot t^{-n} \tag{12}$$

on $A_0$

---

*Proof of 1.5.* Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t,\, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha,\, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (9) establishes the first claim.

The second claim follows from Equation (9),

$$|\phi_t(y)| \leq C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

∎

*Main Proof of Theorem 8.15.* The outline of the proof is as follows,

---

1. $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and

2. $f \in L^p$ for $p \in [1, +\infty)$,

3. for any $x \in \mathcal{L}_f$, we wish to show

$$|f * \phi_t - af|(x) \to 0, \quad \text{as } t \to 0$$

4. To prove this, we fix some $\beta > 0$ and show that

$$|f * \phi_t - af|(x) < \beta$$

since $\beta$ is arbitrary, the proof will be complete.

5. By Lemma 1.4, for every $\delta > 0$ there exists a $\eta > 0$ where $r \leq \eta$ implies

$$\int_{|y|<r} |f(x) - f(x-y)| dy \leq \delta \cdot r^n$$

and using the $L^1$ inequality,

$$
\begin{aligned}
|f * \phi_t - af|(x) &= \left| \int [f(x-y) - f(x)] \cdot \phi_t(y) dy \right| \\
&\leq \int |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \\
&= \int_{|y|<\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy + \int_{|y|\geq\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy \\
&= I_1 + I_2
\end{aligned}
$$

6. Let $\delta = \beta(2A)^{-1}$, where

$$A = 2^n \cdot C \left[ \frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

we make the claim that this choice of $\delta$ will give us $I_1 < \beta/2$

7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for $t$ sufficiently small, and applying the Triangle Inequality finishes the proof.

Let $\eta$ be as above, and for $t > 0$ and suppose we can find a $K \in \mathbb{N}^+$ with

$$2^K \leq \eta/t \leq 2^{K+1} \tag{13}$$

and define $\alpha = \eta/t$ for convenience.

Notice for any $K \geq 1$, the interval $[0, 1)$ can be partitioned in the following manner

$$[0, 1) = [0, 2^{-K}) \cup \left( \bigcup_{j=1}^{K} [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta,\, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such $K$ exists, then let $A_j = \varnothing$ and set $A_0 = \{|y| \in [0, \eta)\}$. The disjoint union of all $A_{j \geq 0}$ is the open ball $\{|y| \in [0, \eta)\}$. By Lemma 1.5 and Lemma 1.4 each $j \geq 0$,

$$I_1 = \sum_{j=0}^{K} \int_{y \in A_j} |f(x-y) - f(y)||\phi_t(y)| dy$$

$$\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} \int_{y \in A_j} |f(x-y) - f(y)||\phi_t(y)| dy$$

$$\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n$$

The left member reads,

$$\begin{aligned} Ct^{-n}\delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)}2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^\varepsilon \cdot 2^{j\varepsilon + n}\eta^{-\varepsilon} \\ &= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\sum_{j=1}^{K} 2^{j\varepsilon} = 2^\varepsilon \sum_{j=0}^{K-1} 2^{j\varepsilon}$$

$$= \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1}$$

using the estimate for $\alpha$ in Equation (13)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$(C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} \leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^\varepsilon - 1}$$

$$\leq (C\delta 2^n) \cdot \frac{2^\varepsilon}{2^\varepsilon - 1}$$

Finally, $I_1 \le (C\delta 2^n) \left[ \dfrac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$, and by Step 6, $I_1 \le \beta/2$.

Obtaining an estimate for $I_2$ is another laborious entreprise. Let us define $W = \{|y| \ge \eta\}$, and

- By Holder's Inequality,

$$I_2 \le \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

  where $q$ is the conjugate exponent to $p$. Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \to 0$ as $t \to 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,
$$y \in W \iff |y| \ge \eta \iff |t^{-1}y| \ge \alpha$$
  then $\|\chi_W \cdot \phi_t\|_\infty \le Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \le Ct^\varepsilon \eta^{-(n+\varepsilon)}$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\|\chi_W \cdot \phi_t\|_q^q = t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q \cdot (n+\varepsilon)} dy$$

$$= C^q \cdot t^{\varepsilon q} \int_{|y| \ge \eta} |y|^{-q \cdot (n+\varepsilon)} dy$$

$$= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \ge \eta} r^{n-1} \cdot r^{-q \cdot (n+\varepsilon)} dr$$

$$= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n+\varepsilon)} r^{n - q \cdot (n+\varepsilon))} \Bigg]_\eta^\infty$$

$$= \frac{C^q t^{\varepsilon q}}{q \cdot (n+\varepsilon) - n} \eta^{n - q \cdot (n+\varepsilon)}$$

$$\|\chi_W \cdot \phi_t\|_q = \left[ \frac{C}{(q \cdot (n+\varepsilon) - n)^{1/q}} \left( \eta^{n - q \cdot (n+\varepsilon)} \right)^{1/q} \right] t^\varepsilon$$

$$= C_3(q) t^\varepsilon$$

- Find a $t$ sufficiently small so that

$$t^\varepsilon < \min \left\{ \beta(4C_3(1)|f(x)|)^{-1},\ \beta(4C_3(q)\|f\|_p)^{-1},\ \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \to 0$.

∎

# Theorem 8.16

> **Proposition 1.16**
>
> See Theorem 8.15

*Proof.* ∎

# Theorem 8.17

> **Proposition 1.17**

*Proof.* ∎

# Theorem 8.18

> **Proposition 1.18**

*Proof.* ∎

# Theorem 8.19

> **Proposition 1.19**

*Proof.* ∎

# Theorem 8.20

> **Proposition 1.20**

*Proof.*                                                                ∎