Chapter A: Review of Topology

Set Operations

This section is meant for reference.

Proposition 1.1: Direct and Inverse Images of Maps

Let $f : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are sets. If $A \subseteq \mathbf{X}$, $B \subseteq \mathbf{Y}$, and $\{E_\alpha\}$ is an indexed collection of subsets of $\mathbf{X}$, $\{G_\beta\}$ is an indexed collection of subsets of $\mathbf{Y}$, then

Direct images

$$f\left(\bigcap E_\alpha\right) \subseteq \bigcap f(E_\alpha) \quad \text{equality if injective} \tag{1}$$

$$f\left(\bigcup E_\alpha\right) = \bigcup f(E_\alpha) \tag{2}$$

Estimates

$$f\left(f^{-1}(B)\right) \subseteq B \quad \text{equality if surjective} \tag{3}$$

$$A \subseteq f^{-1}(f(A)) \quad \text{equality if injective} \tag{4}$$

Inverse images

$$f^{-1}\left(\bigcup G_\beta\right) = \bigcup f^{-1}(G_\beta) \tag{5}$$

$$f^{-1}\left(\bigcap G_\beta\right) = \bigcap f^{-1}(G_\beta) \tag{6}$$

$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c \tag{7}$$

Proposition 1.2: Composition of Maps

Let $h = g \circ f$, we assume this composition is well defined.

- If h is a surjection, then g is a surjection,
- If h is an injection, then f is an injection.

Proof. Take the contrapositive. ■

Proposition 1.3: Left and Right inverses

Let $F : \mathbf{X} \to \mathbf{Y}$,

- F is surjective if and only if there exists right inverse $G : \mathbf{Y} \to \mathbf{X}$,

$$F \circ G = \mathrm{id}_{\mathbf{Y}}$$

if $A \subseteq \mathbf{X}$,

$$G^{-1}(A) \subseteq F(A)$$

- F is injective if and only if there exists a left inverse $H : F(\mathbf{X}) \to \mathbf{X}$

$$H \circ F = \mathrm{id}_{\mathbf{X}}$$

and if $B \subseteq \mathbf{Y}$,

$$F^{-1}(B) \subseteq H(B)$$

Topological Spaces

This section will roughly follow Munkres text on General Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered proper by the subsequent section.

Definition 1.1: Topology

Let $\mathbf{X}$ be a non-empty set. A topology $\mathcal{T}$ on $\mathbf{X}$, sometimes denoted by $\mathcal{T}_{\mathbf{X}}$ is a family of subsets of $\mathbf{X}$,

- $\{\varnothing, \mathbf{X}\} \subseteq \mathcal{T}$,
- If U_1 and U_2 are elements of $\mathcal{T}$, so is their intersection.
- If $\{U_\alpha\}$ is an arbitrary family of sets in $\mathcal{T}$, their union is also contained in $\mathcal{T}$ as an element.

We call the elements of $\mathcal{T}$ open sets. The complements of elements in $\mathcal{T}$ are closed sets.

Basis of a Topology

Definition 1.2: Basis of a topology

A basis $\mathbb{B}$ is a family of subsets of $\mathbf{X}$, that satisfies:

- Every $x \in \mathbf{X}$ belongs (as an element) in some $V \in \mathbb{B}$.
- If B_1 and B_2 are basis elements, such that their intersection is non-empty. Then every $x \in B_1 \cap B_2$ induces a $B_3 \in \mathbb{B}$ with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in x.

If $\mathbb{B}$ is a basis, it 'generates' a topology $\mathcal{T}$ through

$$\mathcal{T} = \left\{ U \subseteq \mathbf{X},\ \forall x \in U,\ x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \tag{8}$$

Notice this is equivalent to $\mathcal{T}$ is the collection of all unions of basis elements in $\mathbb{B}$.

Proposition 1.4

Let $\mathbb{B}$ be a basis as defined in Definition 1.2, then $\mathcal{T}$ as defined in Equation (8) is a valid topology on $\mathbf{X}$. And every member of $\mathcal{T}$ is and is precisely the union of elements in $\mathbb{B}$.

Proof. Every point in $\mathbf{X}$ belongs in some basis element, so $\mathbf{X} \in \mathcal{T}$, so does $\varnothing$. Next, if U_1 and U_2 are in $\mathcal{T}$, then

$$\begin{cases} x \in U_1 \looparrowright x \in B_1 \subseteq U_1 \\ x \in U_2 \looparrowright x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some $B_3 \in \mathbb{B}$, so $\mathcal{T}$ is closed under finite intersections (perhaps after a standard induction argument).

If $\{U_\alpha\} \subseteq \mathcal{T}$, and x belongs in the union of all U_α, then $x \in B_\alpha \subseteq U_\alpha$, which is a subset of the entire union. So the union over U_α is again contained in $\mathcal{T}$, and $\mathcal{T}$ is a topology on $\mathbf{X}$.

It is worth noting that $\mathbb{B} \subseteq \mathcal{T}$. Finally, if $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} B_x$$

where B_x is the basis element taken to satisfy $x \in B_x \subseteq U$. Every point in U is included in some B_x, and hence is included in the union. For the reverse inclusion, notice the union of subsets of U is again a subset of U.

Now, if $E \subseteq \mathbf{X}$ is the union of basis elements in $\mathbb{B}$, if E is non-empty, then every point $x \in E$ belongs in some B_x. Recycling the previous argument, and we see that E is open in $\mathcal{T}$. If E is empty, we define the 'union' of no sets as the empty set. So $\mathcal{T}$ is precisely the collection of all unions of basis elements $\mathbb{B}$. ■

We are now in a position to compare the relative 'fineness' of topologies.

Definition 1.3: Fineness of topologies

If $\mathcal{T}'$ and $\mathcal{T}$ are both topologies on some non-empty set $\mathbf{X}$. We say $\mathcal{T}'$ is finer than $\mathcal{T}$, or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if

$$\mathcal{T}' \supseteq \mathcal{T}$$

Proposition 1.5

If $\mathbb{B}$ and $\mathbb{B}'$ are bases for $\mathcal{T}'$ and $\mathcal{T}$, the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$,
- If B is an arbitrary basis element in $\mathbb{B}$, then every point $x \in B$ induces a basis element in $\mathbb{B}'$ with
$$x \in \mathbb{B}' \subseteq \mathbb{B}$$

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Notice $\mathbb{B} \subseteq \mathcal{T}'$ as well. By Equation (8), each $x \in B$ induces a $B' \in \mathbb{B}'$
$$x \in B' \subseteq B$$
Conversely, fix any open set $U \in \mathcal{T}$, and for each $x \in U$,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 1.2 tells us U is open in $\mathcal{T}'$. ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily) the second, is satisfied in Definition 1.2

Definition 1.4: Sub-basis of a topology

A sub-basis $S \in \mathbb{P}(\mathbf{X})$ is a family of subsets of $\mathbf{X}$ that satisfies one property. Any point x in $\mathbf{X}$ belongs to at least one member of S.

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

Proposition 1.6

Let S be a sub-basis of $\mathbf{X}$, then the collection of all finite intersections of S forms a basis $\mathbb{B}$ of $\mathbf{X}$.

Proof. Every point in $\mathbf{X}$ lies in some element of S, hence in some element of $\mathbb{B}$. The second basis property is immediate, since $\mathbb{B}$ is closed under finite intersections. ■

Product Topology

We will start with products of a finite collection of topological spaces.

Definition 1.5: Finite Product of Topological Spaces

Let $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$ and $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$ be topological spaces. The product topology (denoted by $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$) on $X \times Y$ is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X}\times\mathbf{Y}} = \left\{ U \times V,\ (U, V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \tag{9}$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

Proposition 1.7

If $\mathbb{B}_{\mathbf{X}}$ and $\mathbb{B}_{\mathbf{Y}}$ are bases for $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{T}_{\mathbf{Y}}$, then the product topology (as described in Definition 1.5) is also generated by

$$\mathcal{M} = \left\{ U \times V,\ (U, V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \tag{10}$$

Proof. We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by $\mathcal{M}$ in Equation (10) by $\mathcal{T}_{\mathcal{M}}$.

Since $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$, if $U \times V \in \mathcal{M}$ as in Equation (10), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 1.5, $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ is finer than $\mathcal{T}_{\mathcal{M}}$.

Fix any set $U \times V \in \mathbb{B}_{\mathbf{X}\times\mathbf{Y}}$, and if $(p, q) \in U \times V$, each coordinate induces basis elements from $\mathbb{B}_{\mathbf{X}}$ and By, more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p, q) \in \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 1.5, $\mathcal{T}_{\mathcal{M}}$ is finer than $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ and $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}} = \mathcal{T}_{\mathcal{M}}$. ■

Continuity

Definition 1.6: Continuous maps $C(\mathbf{X},\, \mathbf{Y})$

Let f be a map from $\mathbf{X}$ to $\mathbf{Y}$. It is called *continuous* if $f^{-1}(U)$ is open in $\mathbf{X}$ for every open set U in $\mathbf{Y}$. We denote the set of continuous functions from $\mathbf{X}$ to $\mathbf{Y}$ by $C(\mathbf{X},\, \mathbf{Y})$.

Proposition 1.8: Continuity preserving operations

The composition of continuous functions is again continuous, and the product of continuous functions is again continuous.

Proof. Suppose $f \in C(\mathbf{X},\, \mathbf{Y})$ and $g \in C(\mathbf{Y},\, \mathbf{Z})$. Fix an open set $U \subseteq \mathbf{Z}$. Then $g^{-1}(U)$ is open in $\mathbf{Y}$, hence

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \quad \text{is open in } \mathbf{X}$$

Next, let $\{f_\alpha\}_{\alpha \in A}$ be a collection of continuous functions, where each $f_\alpha \in C(\mathbf{X}_\alpha,\, \mathbf{Y}_\alpha)$. Let us write

$$\mathbf{X} \triangleq \prod \mathbf{X}_\alpha \quad \text{and} \quad \mathbf{Y} \triangleq \prod \mathbf{Y}_\alpha$$

and the projection maps:

$$\pi^{\mathbf{X}}_\alpha : \mathbf{X} \to \mathbf{X}_\alpha, \quad \text{For every } x \in \mathbf{X},\ \pi^{\mathbf{X}}_\alpha(x) = x(\alpha) \in \mathbf{X}_\alpha$$

similarly for $\pi^{\mathbf{Y}}_\alpha : \mathbf{Y} \to \mathbf{Y}_\alpha$. The product function $F = \prod f_\alpha$, is defined through its behaviour 'on each coordinate'

$$\pi^{\mathbf{Y}}_\alpha \circ F = f_\alpha \circ \pi^{\mathbf{X}}_\alpha \tag{11}$$

A function $F : \mathbf{X} \to \prod \mathbf{Y}_\alpha$ is continuous iff $\pi^{\mathbf{Y}}_\alpha \circ F$ is continuous for each $\alpha \in A$. By Equation (11), it is clear that each $\pi^{\mathbf{Y}}_\alpha \circ F$ is continuous, since the right member is the composition of two continuous functions, which is again continuous by the first part of this proof, therefore F is continuous. ■

Definition 1.7: Open/Closed Maps

Let $f : \mathbf{X} \to \mathbf{Y}$ be a map (not necessarily continuous), it is called *open* (resp. *closed*) if for every open (resp. closed) set $E \subseteq \mathbf{X}$, $f(E)$ is open (resp. closed).

Clearly, the composition of open (resp. closed) maps is again open (resp. closed).

Quotient Topology

Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If $\{X_\alpha\}_{\alpha \in A}$ is a family of topological spaces which are ______, then $\prod X_\alpha$ is ______. Replace ______ with:

1. Hausdorff, (Folland)
2. Regular,
3. Connected, (Munkres chp23, exercise 10)
4. First countable, if A is countable,
5. Second countable, if A is countable,
6. Compact (Tynchonoff's Theorem, Folland)

Proposition 1.9: Product of Closed sets again Closed

The product of closed sets is again closed. More concretely, if $\{E_\alpha\}_{\alpha \in A}$ is a family of sets such that $E_\alpha \subseteq \mathbf{X}_\alpha$, then

$$\prod \overline{E_\alpha} = \overline{\prod E_\alpha}$$

Connectedness

Definition 1.8: Connectedness

A topological space $\mathbf{X}$ is connected if U and V are disjoint open subsets whose union is $\mathbf{X}$, then at least one of U or V is empty.

See Folland Exercise 4.10 for more properties.

Definition 1.9: Path-connectedness

A topological space $\mathbf{X}$ is path-connected if for any two pair of points x, $y \in \mathbf{X}$. There exists a continuous function $f : [a, b] \to \mathbf{X}$, with $f(a) = x$ and $f(b) = y$.

Definition 1.10: Connected component

The connected components of $\mathbf{X}$ is the family of equivalence classes on $\mathbf{X}$, where $x \sim y$ if there is a connected subspace of $\mathbf{X}$ that contains both of them.

Proposition 1.10

Continuous functions map connected spaces to connected spaces (in the subspace topology).

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $f : \mathbf{X} \to \mathbf{Y}$ be continuous. If $f(\mathbf{X})$ is disconnected, then we can find U and V, open and disjoint in $\mathcal{T}_{f(\mathbf{X})}$ such that

$$U \cup V = f(\mathbf{X}) \implies f^{-1}(U) \cup f^{-1}(V) = \mathbf{X}$$

where $f^{-1}(f(\mathbf{X})) = \mathbf{X}$. Both $f^{-1}(U)$ and $f^{-1}(V)$ are open, non-empty, and are pairwise disjoint. So $\mathbf{X}$ is separated. ■

Proposition 1.11

Let $(\mathbf{X}_\alpha, \mathcal{T}_\alpha)$ be a family of connected topological spaces indexed by $\alpha \in A$. Then $\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected in the product topology.

Proof. We will attempt the contrapositive. Suppose $\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected, then ■

Interiors and closures

Definition 1.11: Interior of a set

A^o is defined to be the largest open subset of A,

$$A^o = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

Corollary 1.1

The union of subsets of A is again a subset of A, therefore Corollary 1.1 implies $A^o \subseteq A$ for any $A \subseteq \mathbf{X}$.

Definition 1.12: Closure of a set

and $\overline{A}$ is the smallest closed superset of A,

$$\overline{A} = \bigcap_{\substack{K \text{ closed}, \\ A \subseteq K}} K$$

Proposition 1.12

The complement of the closure is the interior of the complement, or equivalently: $(\overline{A})^c = A^{co}$

Proof. Taking complements, and the substitution $U = K^c$ reads

$$\begin{aligned}(\overline{A})^c &= \left[\bigcap_{\substack{K \text{ closed}, \\ A \subseteq K}} K\right]^c \\ &= \bigcup_{\substack{K \text{ closed}, \\ K^c \subseteq A^c}} K^c \\ &= \bigcap_{\substack{U \text{ open}, \\ U \subseteq A^c}} U \\ &= A^{co}\end{aligned}$$

∎

Remark 1.1

Personally, I remember this as pushing the complement inside and flipping the bar to a c!

Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

Definition 1.13: Neighbourhood (not necessarily open)

A neighbourhood of $x \in \mathbf{X}$ is a set $U \subseteq \mathbf{X}$ where $x \in U^o$. The set of neighbourhoods for a point $x \in \mathbf{X}$ will sometimes be denoted by $\mathcal{N}(()x)$.

Proposition 1.13: Characterization of the interior

If $W = \Big\{x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A\Big\}$, then $W = A^o$.

Proof. If $x \in A^o$, then A is a neighbourhood of x, and $A \subseteq A$, so $x \in W$. Conversely, if x is a member of W, it has a neighbourhood $U \subseteq A$ (not necessarily open). By monotonicity of the interior,

$$x \in U^o \subseteq A^o$$

and $x \in A^o$. ■

It is easy to see that A is open $\iff$ $A^o = A$ $\iff$ A is a neighbourhood of itself.

- The first equivalence follows from:

  $$E \subseteq \mathbf{X} \implies E^o \subseteq E$$

  and if A is an open set, it is an open subset of itself, by Corollary 1.1 $A \subseteq A^o$. If $A^o = A$, then it suffices to show that A^o is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose $A^o = A$, then each $x \in A$ has a neighbourhood contained (as a subset) in A, namely A itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

  $$x \in A^o \subseteq A \implies A \subseteq A^o$$

  so A is a neighbourhood of itself. Conversely, if $A \subseteq A^o$, then $A = A^o$, since the reverse inclusion follows immediately from Corollary 1.1.

Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

Definition 1.14: Adherent point of a set

Let $A \subseteq \mathbf{X}$, $x \in \mathbf{X}$ is an adherent point of A if every neighbourhood U of x intersects A. In symbols,

$$U \cap A \neq \varnothing, \quad \forall U \in \mathcal{N}(x)$$

Proposition 1.14: Characterization of the closure

Let $A \subseteq X$, and let W be the set of adherent points of A, then $\overline{A} = W$

Proof. Suppose $x \notin W$, then there exists a neighbourhood U of x where

$$U \cap A = \varnothing \iff U \subseteq A^c$$

this is exactly the definition of the interior of A^c, so $x \in A^{co}$ and recall (from Proposition 1.12) that $(\overline{A})^c = A^{co}$, so $x \notin \overline{A}$. For the reverse inclusion, read the proof backwards, by flipping $\forall \to \exists$ within the set, and we see that

$$W^c = A^{co} = (\overline{A})^c$$

■

Dense and nowhere dense subsets

Definition 1.15: Dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is dense if $\overline{E} = \mathbf{X}$.

Definition 1.16: Nowhere dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is nowhere dense if $\overline{E}^o = \varnothing$.
This means E is dense in none of the (non-trivial) open subspaces of $\mathbf{X}$.

Proposition 1.15

E is dense in $\mathbf{X}$ iff for every non-empty, open set $U \subseteq \mathbf{X}$, $U \cap E \neq \varnothing$.

Proof of Proposition 1.15. Suppose E is dense, then $\overline{E} = \mathbf{X}$. Every point of $\mathbf{X}$ is an adherent point of E. Let $U \subseteq \mathbf{X}$ be a non-empty open set. If $x \in U$ then U is a neighbourhood of x, thus U intersects E. Conversely, suppose every non-empty open set U intersects E. Fix any point $x \in \mathbf{X}$, and any neighbourhood U of x. U has a non-empty interior (because it must contain x). But U^o is a non-empty open set, therefore $\varnothing \neq U^o \cap E \subseteq U \cap E$ ■

Proposition 1.16

Let $f : \mathbf{X} \to \mathbf{X}$ be a homeomorphism. E is nowhere dense iff $f(E)$ is nowhere dense.

Proof. Since f^{-1} is a homeomorphism, suppose $\overline{f^{-1}(E)}^{o} \neq \varnothing$, there exists a non-empty, open subset $U \subseteq \mathbf{X}$ with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\Big((\overline{f^{-1}(E)}) \cap U\Big) = f(U)$$

since f is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\Big((\overline{f^{-1}(E)}) \cap U\Big) = f(U) \tag{12}$$

and f is continuous, so $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq \mathbf{X}$. For the reverse inclusion, f is a closed map, so $f(\overline{A})$ is a closed superset of $f(A)$ so

$$f(\overline{A}) = \overline{f(A)}$$

Take $A = f^{-1}(E)$, and $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$. From eq. (12), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$ is a non-empty open subset of $\mathbf{X}$, since f is an open map, so E is not nowhere dense. The reverse implication can be proven by replacing f with f^{-1}. ■

Urysohn's Lemma

Proposition 1.17: Folland Theorem 4.14

Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that

1. $A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,
2. $\overline{U_r} \subseteq U_s$ for $r < s$, and
3. For every $r < s$, $\overline{U}_r \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N-1$, we have
$$r = \frac{k}{2^n},\ 1 \leq n \leq N-1,\ 1 \leq k \leq 2^{n-1}$$
And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathfrak{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j+1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k-1 \neq 0$ and $k+1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathfrak{T}_X$, then there exists some open set $G \in \mathfrak{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathfrak{T}_X \times \mathfrak{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathfrak{T}_X \times \mathfrak{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathfrak{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since
$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathfrak{T}_X$
- $E_2^c \subseteq G_2 \in \mathfrak{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

Proposition 1.18: Folland Theorem 4.15: Urysohn's Lemma

Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.
For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) —

we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. ■

Notes on the construction of the countable 'onion' sequence within a normal space **X**.

If **X** is a normal space, and A and B are disjoint closed subsets, then we can easily find an open U with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{13}$$

We say that U hides in B^c if the closure of U is contained in B^c. Define $\Delta_n = \left\{k2^{-n},\ 1 < k < 2^n\right\}$, so that $\Delta_n \subseteq (0,1)$ for all $n \geq 1$. Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for Δ_{n+1} are contained in Δ_n. Suppose Δ_n is well defined, it suffices to choose the odd indices for Δ_{n+1}. If $r = j2^{-(n+1)}$, where j is odd, then r sits in between precisely two elements in $\Delta_n \cup \{0, 1\}$. If r sits between an endpoint, then define $\overline{U_0} = A$, and $B^c = U_1$. And denote the closest left and neighbours by s, t respectively. If $s < r < t$, it is clear that $\overline{U_s}$ and U_t^c are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of $\overline{U_s}$ that hides in U_t, denote this open set by U_r, and similar to Equation (13)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let A and B be disjoint closed sets. And define

$$f(x) = \inf\left\{r \in \Delta \cup \{1\},\ x \in U_r\right\}$$

where $U_1 = \mathbf{X}$. So that $0 \leq f(x) \leq 1$ is immediate. If $x \in A$, then x is in all U_r, and by density of $\Delta \subseteq (0,1)$, we have $f(x) = 0$. Conversely, if $x \in B$ then $x \notin U_r$ for all $r \in \Delta$, if E denotes the indices in Δ where $x \in U_s$ when $s \in E$,

$$(-\infty,\, r) \subseteq E^c \iff E \subseteq [r,\, +\infty) \iff \inf(E) \geq r \tag{14}$$

Send $r \to 1$ and $f(x) = 1$. Thus $f|_A = 0$ and $f|_B = 1$.

To show continuity, it suffices to show that the inverse images of the open half $\Big\{(x > \alpha),\, (x < \alpha)\Big\}_{\alpha \in \mathbb{R}}$ lines are indeed open in $\mathbf{X}$. Let α be fixed. And if $x \in \{f < \alpha\}$, we can 'wiggle' the infimum towards the right (towards α), and using density of Δ within $(0,1)$, there exists a $r \in E$ that satisfies $f(x) < r < \alpha$. This is equivalent to

$$x \in \bigcup_{r<\alpha} U_r$$

If there exists an $r < \alpha$ st x belongs to U_r as an element, then $f(x) \leq r < \alpha$.

If $f(x) > \alpha$, then $(-\infty,\, \alpha) \subseteq E^c$, by Equation (14). Suppose $\alpha < 1$, otherwise $\{f > \alpha\} = \varnothing$. Wiggle $f(x)$ to the left and obtain an $r \in \Delta$, $\alpha < r < f(x)$ with $x \notin U_r$. By density again, take any $s < r$ by a small amount (st $s > \alpha$, $s \in \Delta$), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that $x \in \overline{U}_s^c$ for some $s > \alpha$. This is equivalent to

$$x \in \bigcup_{s>\alpha} \overline{U}_s^c$$

Conversely, if $x \notin \overline{U}_s^c$ for some $s > \alpha$, since $\{U_r\}$ (thus $\{\overline{U}_r\}$) is increasing, and $x \notin U_r$ for every $r \leq s$. Hence,

$$(-\infty,\, s] \subseteq E^c \iff E \subseteq (s,\, +\infty) \iff f(x) \geq s > \alpha$$

Compactness

Compactness is one of the most important concepts in topology and analysis.

Definition 1.17: Compact topological space

A topological space $\mathbf{X}$ is compact if every open covering $\{U_\alpha\}$ contains a finite subcover. That is, if $\{U_\alpha\}$ is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup U_{\alpha \in A} \implies \bigcup_{j \leq n} U_{\alpha_j}$$

Definition 1.18: Compact set

$E \subseteq \mathbf{X}$ is compact if it is compact in the subspace topology.

Definition 1.19: Precompact set

$E \subseteq \mathbf{X}$ is precompact if its closure is compact (as a subset).

Definition 1.20: Paracompact space

A topological space $\mathbf{X}$ is paracompact if every open covering of $\mathbf{X}$ has a locally finite open refinement that covers $\mathbf{X}$.

Definition 1.21: Locally finite collection of sets

Let $\mathcal{A}$ be a collection of subsets of $\mathbf{X}$. It is called locally finite, if at every point $p \in \mathbf{X}$, we can find a neighbourhood U of p (not necessarily open), that intersects only finitely many members of $\mathcal{A}$. In symbols,

$$U \cap E = \varnothing \quad \text{for all but finitely many} \quad E \in \mathcal{A}$$

We do not require $\mathcal{A}$ to be a cover of $\mathbf{X}$, nor do we require $\mathcal{A}$ to be a collection of open sets.

Definition 1.22: Countably locally finite

A collection $\mathbb{B}$ is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \overbrace{\bigcup_{\mathbb{N}}}^{\substack{\text{countable}\\ \text{union}}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

Definition 1.23: Refinement

If $\mathcal{A}$ is a collection of sets, $\mathbb{B}$ is a refinement of $\mathcal{A}$ if every element $B \in \mathbb{B}$, induces an element $A \in \mathcal{A}$, such that $B \subseteq A$.

Remark 1.2: Intuition for refinements

If $\mathbb{B}$ is a refinement of $\mathcal{A}$, we can use the 'absolute continuity' muscle. For each element in $\mathbb{B}$ is dominated by some element (through subset inclusion) in $\mathcal{A}$. Recall, if ν and μ are non-negative measures, then $\nu \ll \mu$ if for every measurable set $E \in \mathcal{M}$, $\mu(E) = 0 \implies \nu(E) = 0$.

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*

Proposition 1.19

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

Properties of Compact Spaces

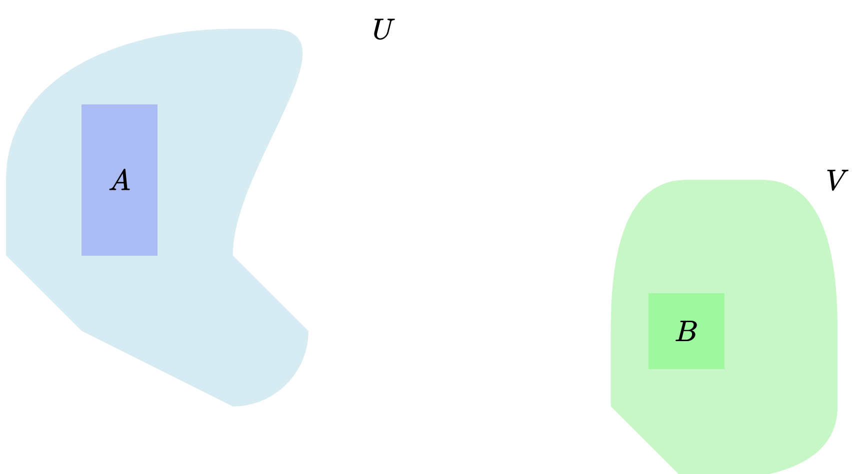

Figure 1: Closed sets A and B within open sets U and V, respectively.

Proposition 1.20

Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces.

(a) If $F \in C(\mathbf{X}, \mathbf{Y})$, and $\mathbf{X}$ is compact, then $F(\mathbf{X})$ is compact.

(b) If $\mathbf{X}$ is compact and $F \in C(\mathbf{X}, \mathbb{R})$, then $F(\mathbf{X})$ is bounded, and F attains its supremum and infimum on $\mathbf{X}$.

(c) A finite union of compact subspaces of $\mathbf{X}$ is again compact.

(d) If $\mathbf{X}$ is Hausdorff, and A, B are disjoint, compact subspaces of $\mathbf{X}$, there exists open U and V, (see fig. 1).

(e) Every closed subset of a compact space is compact.

(f) Every compact subset of a Hausdorff space is closed.

(g) Every compact subset of a metric space is bounded.

(h) Every finite product of compact spaces is compact.

(i) Every quotient of a compact space is compact.

Proof of Proposition 1.20 Part A. Let $f \in C(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X}$ compact. Fix an open cover of $f(\mathbf{X})$ in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X},\ U_\alpha \text{ open in } \mathbf{Y}$$

So that $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$. Since $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for $\mathbf{X}$, this induces a finite subcollection of indices $\{\alpha_1, \ldots, \alpha_n\}$ with

$$\bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^{n} f^{-1}(U_\alpha \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^{n} f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^{n} U_{\alpha_j}$$

■

Proof of Proposition 1.20 Part B. Let $\mathbf{X}$ be compact, and $f \in C(\mathbf{X}, \mathbb{R})$, so that $f(\mathbf{X}) \subseteq \mathbb{R}$ is compact. Compact subsets are closed and bounded in $\mathbb{R}$, let $A = \sup f(\mathbf{X})$ and $B = \inf f(\mathbf{X})$. Both A and B are accumulation points of $f(\mathbf{X})$, so $A = f(x)$ and $B = f(y)$ for some x, y in $\mathbf{X}$. ■

Proof of Proposition 1.20 Part C. Let $\mathbf{X}$ be a topological space, and $K_1, \ldots K_n$ be compact subspaces. Denote $K = \bigcup_{j=1}^{n} K_j$. Let $\{U_\alpha \cap K\}_{\alpha \in A}$ be an open cover for K, where U_α is open in $\mathbf{X}$. We can pass the argument to each individual K_j as follows. Let $1 \leq j \leq n$, then $\{U_\alpha \cap K_j\}_{\alpha \in A}$ is an oepn cover for K_j, so there exists as finite subcollection of indices $I_j \subseteq A$, (a finite subset of A) whose open sets cover K_j. Repeat this process for each j and

$$I = \bigcup_{j=1}^{n} I_j \text{ is a finite subset of } A$$

with $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$. Taking the union over all K_j reads

$$K = \bigcup_{j=1}^{n} K_j \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap I$$

■

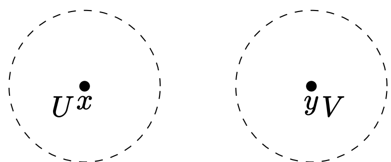

Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V.

Proof of Proposition 1.20 Part D. Let $\mathbf{X}$ be Hausdorff. We first prove that compact subspaces of $\mathbf{X}$ are closed. Indeed, if K is compact in $\mathbf{X}$, fix any $x \in K^c$. Let y range through the elements of K, then $x \neq y$ induces a pair of disjoint open sets U_y and V_y, such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \varnothing$
- See fig. 2

Let V_y range through all possible $y \in K$, So that $\{V_y\}_{y \in K}$ is an open cover. There exists a finite subcollection of 'anchor points' of K, $y_1, \ldots, y_n$ that corresponds with $\{V_{y_j}\}_{j=1}^n$.
A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^{n} U_{y_j} \text{ is open}$$

Define $V = \bigcup_{j=1}^{n} V_{y_j}$, so $V \subseteq K$ and $U \cap V = \varnothing$ and $x \in U \subseteq K^c$ (see fig. 3). Therefore K is closed.

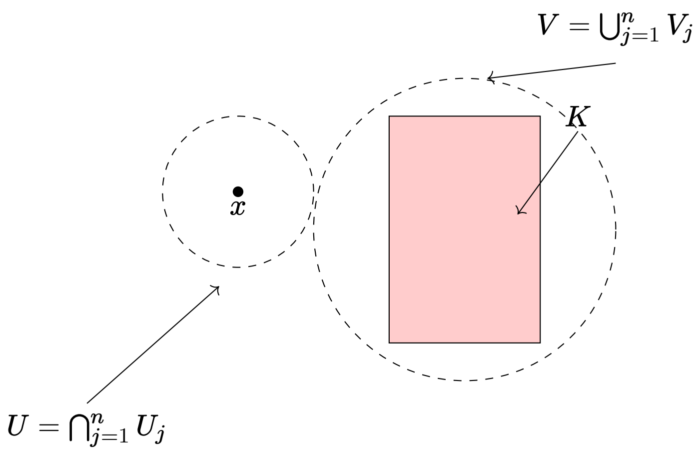

Figure 3: Compact sets are closed in Hausdorff spaces

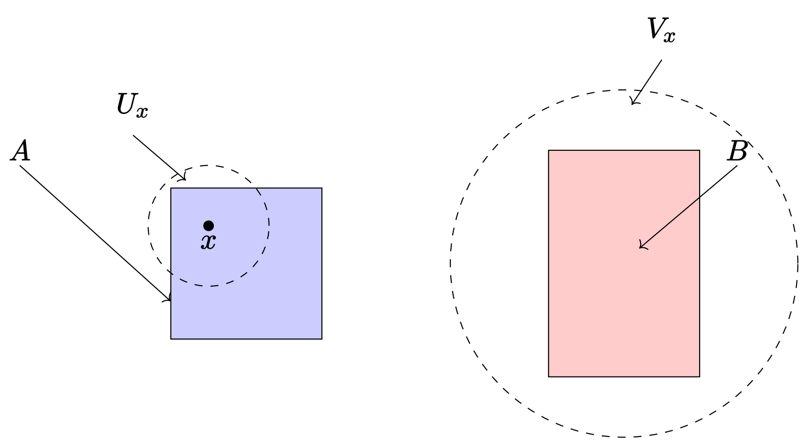

Figure 4: Closed sets A and B, point x in A, and disjoint neighbourhoods U around x and V around B.

Finally, if A and B are disjoint compact sets, then each $x \in A \subseteq B^c$ induces neighbourhoods $x \in U_x$, and $B \subseteq V_x$ (see fig. 4), let x range through all the elements of A. By compactness of A, this produces a finite subcover, and

$$U = \bigcup_{j=1}^{n} U_{x_j} \quad V = \bigcap_{j=1}^{n} V_{x_j}$$

are disjoint open sets that contain A and B respectively.

■

Proof of Proposition 1.20 Part E. Let $K \subseteq \mathbf{X}$ be a closed set of a compact space. Let $\{U_\alpha \cap K\}$ be an open cover for K, where each U_α is open in $\mathbf{X}$. We can append an extra set K^c which is open in $\mathbf{X}$. The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of $W_1, \ldots, W_n$ that cover $\mathbf{X}$ (since $\mathbf{X}$ is compact by itself). Remove K^c from this finite subcollection if it exists, and take the intersection with K for each element W_j, and

$$\{W_1 \cap K, \ldots, W_n \cap K\} = \{U_1 \cap K, \ldots, U_n \cap K\} \text{ covers } K$$

so K is compact. ■

Proof of Proposition 1.20 Part F. Proven in Part D. ■

Proof of Proposition 1.20 Part G. let $K \subseteq \mathbf{X}$ be a compact subset of the metric space $(\mathbf{X}, d)$. Compact subsets of $\mathbf{X}$ are totally bounded, and hence bounded. ■

Proof of Proposition 1.20 Part H. See Tynchonoff's Theorem in Folland Chapter 4. ■

Proof of Proposition 1.20 Part I. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $\pi : \mathbf{X} \to \mathbf{Y}$ be a quotient map. So that $\mathbf{Y}$ is endowed with the quotient topology. So that π is a surjective continuous map. and $\pi(\mathbf{X}) = \mathbf{Y}$. Apply Part A, and we see that $\mathbf{Y}$ is compact. ■

Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and

2) whether we can embed a given topological space in a larger one to force it to be compact.

Definition 1.24: LCH space

Let $\mathbf{X}$ be a Hausdorff space. We call $\mathbf{X}$ a LCH space if every point $p \in \mathbf{X}$ admits a compact neighbourhood. That is, a compact set K whose interior contains p.

We note in passing that the above definition differs slightly from the usual 'local' definitions.

Definition 1.25: Locally connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 1.26: Locally path-connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a path-connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 1.27: Local homeomorphism

$\mathbf{X}$ locally homeomorphic to $\mathbb{R}^n$ if every point $x \in \mathbf{X}$ belongs to a coordinate chart (U, ϕ), where U is an open neighbourhood of x and ϕ is a homeomorphism from $U \to \phi(U) \subseteq \mathbb{R}^n$.

Definition 1.28: Local diffeomorphism

Let M be a smooth manifold and $F \in C^\infty(M, N)$. F is a local diffeomorphism if every $p \in M$ in its domain induces a neighbourhood $U \subseteq M$ with $F|U : U \to F(U)$ is a diffeomorphism (in the sense of two open sub-manifolds).