Chapter 6

Theorem 6.1

Proposition 1.1

For every a, $b \geq 0$, and $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

Theorem 6.2

Proposition 1.2

Proof. ■

Theorem 6.3

Proposition 1.3

Proof. ∎

Theorem 6.4

Proposition 1.4

Proof. ■

Theorem 6.5

Proposition 1.5

Proof. ■

Theorem 6.6

Proposition 1.6

Theorem 6.7

Proposition 1.7

Theorem 6.8

Proposition 1.8

Theorem 6.9

Proposition 1.9

Proof. ■

Theorem 6.10

Proposition 1.10

Proof. ∎

Theorem 6.11

Proposition 1.11

Proof. ■

Theorem 6.12

Proposition 1.12

Proof. ■

Theorem 6.13

Proposition 1.13

Theorem 6.14

Proposition 1.14

Theorem 6.15

Proposition 1.15

Proof. First suppose that $(X, \mathcal{M}, \mu)$ is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in L^p.

It follows that every simple function is also in L^p, since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to μ.

To show σ-additivity, fix any disjoint sequence $\{E_j\}_{j\geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the p-norm

$$\chi_E = \sum_{j\geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup E_{j\leq n}\right) = \left(\bigcup E_{j\geq 1}\right) \setminus \left(\bigcup E_{j\leq n}\right) = \bigcup E_{j\geq n+1}$$

and define F_{n+1} as the rightmost member above. Then $\{F_{n\geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j\leq n}) = \mu(F_{n+1}) \to 0$.

Now, for any fixed $n \geq 1$,

$$\left|\chi_E - \sum \chi_{E_{j\leq n}}\right| = \left|\sum \chi_{E_{j\geq n+1}}\right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the pth power does not change pointwise, and

$$\left|\sum \chi_{E_{j\geq n+1}}\right|^p = \left|\sum \chi_{E_{j\geq n+1}}\right| = \sum \chi_{E_{j\geq n+1}}$$

Convergence in p-norm is given by

$$\left\|\chi_E - \sum \chi_{E_{j\leq n}}\right\| = \left\|\sum \chi_{E_{j\geq n+1}}\right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned}
\nu(E) &= \phi(\chi_E) \\
&= \phi\left(\lim_{n\to\infty} \sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \phi\left(\sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \phi\left(\chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \nu(E_{j\le n})
\end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \overline{\text{sgn}(\left\|\phi(\chi_{E_j})\right\|}$) then multiplication yields

$$\left\|\phi(\chi_{E_j})\right\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the p-norm.

$$\left\| \sum_{j\ge 1} \beta_j \chi_{E_j} - \sum_{j\le n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j\ge n+1} \beta_j \chi_{E_j} \right\|_p$$

And because $\left|\sum_{j\ge n+1} \beta_j \chi_{E_j}\right|$ is pointwise equal to $\left|\sum_{j\ge n+1} \chi_{E_j}\right|$, since $|\beta_j| = 1$ for every $j \ge 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j\ge 1} \beta_j \chi_{E_j} \in L^p$ since its p-norm is equal to $\mu(E)^{1/p}$.

$$
\begin{aligned}
\sum_{j\geq 1} \left|\nu(E_j)\right| &= \sup_{n\geq 1} \sum_{j\leq n} \|\nu(E_{j\leq n})\| \\
&= \lim_{n\to\infty} \sum_{j\leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n\to\infty} \sum_{j\leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n\to\infty} \phi\left(\sum_{j\leq n} \beta_j \chi_{E_j}\right) \\
&= \phi\left(\lim_{n\to\infty} \sum_{j\leq n} \beta_j \chi_{E_j}\right) \\
&\leq \|\phi\| \left\| \sum_{j\geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}
$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ (ν is now a measure and $\nu \ll \mu$), As the indicator of a null set is equal to the zero element in L^p. Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function g. For every χ_E measurable, $\left\|\chi_E g\right\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in L^+. So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that αg is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left|\int \alpha g\right| = \left|\phi(\alpha)\right| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup\left\{\left|\int \alpha \cdot g\right|, \|\alpha\|_p = 1, \quad \text{and } \alpha \text{ is simple, and vanishes outside a set of finite measure.}\right\} < \infty$$

Since $S_g = \{x \in X, g(x) \neq 0\}$ is σ-finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know g is in L^q we can use the density of α in L^p to show, for every single $f \in L^p$

$$\phi(f) = \int fg d\mu$$

Conjure a sequence of 'α's, and call them $\{f_n\} \to f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int fg d\mu = \phi(f)$$

This completes the proof for when μ is finite.

Let us upgrade our μ into a σ-finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each E_n is of finite measure. Define

$$P_n = \left\{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\right\}$$

So every function in P_n vanishes outside a set of finite measure and is also in L^p. And Q_n is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of g_n towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on E_n pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto E_n and E_m spawns two functions g_n and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on E_n. Now we define g measurable such that $g|_{E_n} = g_n$ for every n. And

$$\begin{aligned}
|g_n| = \chi_{E_n} \cdot |g_m| &\implies \\
|g_n| \le |g_{n+1}| &\implies \\
&\quad \|g_n\|_q \le \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \le \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with $(P_n \subseteq P_{n+1})$. If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \le |g|$ pointwise for every n, and for each $x \in X$, there exists a N, where $n \ge N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n \ge 1}\{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$\begin{aligned}\int \lim_{n\to\infty} |g_n|^q \, d\mu &= \int \sup_{n\geq 1} |g_n|^q \, d\mu \\ &= \int |g|^q \, d\mu \\ &= \lim \int |g_n|^q \, d\mu\end{aligned}$$

Which yields $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \to f$ in the p-norm.

$$\begin{aligned}|f_n - f| &\leq |f_n| + |f| \\ &\leq |f| + |f| \\ &\leq 2|f|\end{aligned}$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\ \lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\ &= 0\end{aligned}$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}\phi(f) &= \lim \phi|_{P_n}(f_n) \\ &= \lim \int f_n \cdot g_n \\ &= \lim \int f \cdot g \cdot \chi_{E_n} \\ &= \int \lim \left(fg \cdot \chi_{E_n}\right) \\ &= \int fg\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \to fg$ pointwise and Holder's Inequality. This completes the proof for when μ is of σ-finite measure, and $p \in [1, +\infty)$.

Suppose now μ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every σ-finite $E \subseteq X$

$$P_E = \left\{ L^p, |f| = |f| \cdot \chi_E \right\}$$

Where Q_E does not hold any surprises. Then for each E we have a $\phi|_E$ which induces a g_E that vanishes outside E. We are ready for the final part of the proof.

First, if $E \subseteq F$ and both E and F are σ-finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in L^+ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$W = \left\{ \|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E \right\}$$

Let M be the supremum of W, then there exists a sequence of σ-finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \to M \leq \|\phi\|_{p*}$. Take a set $F = \cup E_{n \geq 1}$, which is also σ-finite, so that $\|g_F\|_q = M$. Now assume there exists another σ-finite superset of F, let us call it A. Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A \setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where f does not vanish is σ-finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this σ-finite set. But we already know, even if this set A contains F as a subset, $\int f g_F = \int f g_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating f with g. ■