Chapter 2

Theorem 2.1

Proposition 1.1

Proof. ■

Theorem 2.30

Proposition 1.2: Rudin 7.8

Let E be a metric space and $\{f_n\}$ be a uniformly Cauchy sequence of functions on E, then there exists a function f to which $f_n \to f$ uniformly.

Proof. Let $x \in E$ be fixed, then $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$. We define $f(x)$ to be the limit of $f_n(x)$, this is well defined because Cauchy limits (in $\mathbb{R}$ or $\mathbb{C}$) exist and are unique.

We isolate the mechanism that is used in the next part of the proof.

Note 1.1: Cauchy Sequences in Metric Spaces

Cauchy sequences in metric spaces behave like equivalence classes. We summarize some of the interesting and useful techniques regarding them.

Lemma 1.1: Dragging friends along

Let $\{x_n\} \subseteq E$ be Cauchy, suppose x^* is a point in E, and $i \in \mathbb{N}^+$ such that $d(x_i, x^*) < \varepsilon$. Then, we can find an entire subsequence of x_{n_k} where $d(x_{n_k}, x^*) < \varepsilon$.

Lemma 1.2: Replacing a sequence by its limit — Continuity Edition

Let $A : E \times E \to F$ be a continuous mapping between the product metric space $E \times E$ into another metric space F. Then $A(x, \cdot)$ is continuous from E into F, suppose further that $x_n \to x$ in E, then

$$A(x_n, x_m) \to A(x_n, x) \quad \text{as } m \to \infty.$$

Let $\varepsilon > 0$ be fixed, this induces a $N \in \mathbb{N}^+$ where $\|f_n - f_m\|_u \leq \varepsilon$ for $n, m \geq N$. For every $x \in E$, $|f_n(x) - f_m(x)|$ converges to $|f_n(x) - f(x)|$ as $m \to \infty$, but the topological argument:

$$\{|f_n(x) - f_m(x)|,\ \min(n, m) \geq N,\ x \in E\} \subseteq [0, \varepsilon]$$

implies that $\|f_n - f\|_u \leq \varepsilon$.

■

Note 1.2: Cauchy Sequences in Measure Spaces

Lemma 1.3: Cauchy Sequences derived from an l^1 bound

If $d(x_n, x_{n+1}) \leq c_n$ where $\{c_n\} \in l^1$, then x_n is Cauchy.

Proof. The tail $\sum_{j \geq n} |c_j|$ vanishes, and $d(x_{i_1}, x_{i_2}) \leq \sum_{\min(i_1, i_2)}^{\infty} c_j \to 0$. ■

Lemma 1.4: l^1 bounds derived from a Cauchy Sequence

If $\{x_n\} \subseteq E$ is a Cauchy sequence, for every $\{c_j\} \in l^1(\mathbb{R})$, there exists a subsequence x_k where $d(x_k, x_{k+1}) \leq c_k$.

Corollary 1.1: Limsup is null using l^1 bound

If E_j is a sequence of measurable sets with $\mu(E_j) \leq c_j$, where c_j is a l^1 sequence, then $\limsup E_j$ is null.

Proof. Let $C_n = \sum_{n \geq N} c_n$, and if $x \in \limsup E_j$, then $x \in E_j$ frequently. Set $F_n = \cup_{j \geq n} E_j$ to be the tail, and $x \in F_n$ for all n; and because each F_n has measure controlled by C_n, the proof is complete. ■

Proposition 1.3: Theorem 2.30 — Part 1

Let $\{f_n\}$ be Cauchy in measure, then there is a measurable function f, and a subsequence f_k where $f_k \to f$ pointwise a.e.

Proof. Let $\{c_k\}$ and $\{d_k\}$ be non-negative and summable. At each c_k, the sequence $E(k, n, m) = \{|f_n - f_m| \geq c_k\}$ has vanishing measure as $\min(n, m) \to \infty$. Choosing an increasing sequence of numbers $\{N_k\} \subseteq \mathbb{N}^+$ such that $\mu(|f_n - f_m| \geq c_k) \leq d_k$ for all $\min(n, m) \geq N_k$, we obtain a subsequence f_{N_k} where

$$\mu(E(k, N_k, N_k + m)) = \mu(|f_{N_k} - f_{N_k+m}| \geq c_k) \leq d_k \quad \forall m \geq 1.$$

We identify $N_k = k$ whenever it is a subscript, and we define $E_k = \{|f_k - f_{k+1}| \geq c_k\}$. If $x \notin \limsup E_k$, Lemma 1.3 shows that $\{f_k(x)\}$ is Cauchy, and because $\mu(\limsup E_k) = 0$ by Corollary 1.1, $f_k \to f$ pointwise a.e. ■

Remark 1.1: Commentary for Theorem 2.30 Part 1

The sequence $\{c_k\}$ assumes uniform pointwise control of our subsequence f_k over E_k, whereas $\{d_k\}$ allows us to control — in addition to the size of E_k — measure-theoretic properties of $\{f_k\}$.

Proposition 1.4: Theorem 2.30 — Part 2

Let f_n, f_k and f be as in Part 1. The subsequential pointwise a.e. limit is also a subsequential limit in measure. That is: $f_k \to f$ in measure.

Proof. We can afford a bit of wiggle room since $C_k = \sum c_{j \geq k}$ and $D_k = \sum d_{j \geq k}$ both vanish as $k \to \infty$. If x is not in $\bigcup E_{j \geq k}$, one obtains from subadditivity $|f_k(x) - f_l(x)| \leq C_k$ for every $l = N_l \geq N_k$. Using Lemma 1.2, we can replace $f_l(x)$ by $f(x)$ and $\mu(|f_k - f| \leq C_k) \leq D_k$ as needed. ■

Proposition 1.5: Theorem 2.30 — Part 3

Let f_n, f_k, and f be as in Part 2, then $f_n \to f$ in measure.

Proof. The proof imitates that of Lemma 1.1. We demonstrate a useful mechanism involving finite, non-negative sums.

Note 1.3: Summation over non-negative, finite lists

Let $x_{\underline{n}}$ be non-negative.

1. Because '$\exists \leq \max \leq \forall$', if $a < \sum x_{\underline{n}} \leq b$, then $an^{-1} < \max(x_{\underline{n}}) \leq b$,
2. and '$\forall \leq \min \leq \exists$' means if $a \leq \sum x_{\underline{n}} < b$, then $a \leq \min(x_{\underline{n}}) < bn^{-1}$.

By the Triangle Inequality (there is potentially more to be said here), we see that

$$\varepsilon \leq |f_n - f| \leq |f_n - f_k| + |f_k - f| \quad \text{implies} \quad \varepsilon 2^{-1} \leq \max(|f_k - f|, |f_n - f_k|),$$

where a nudge in ε has been suppressed. We can represent this with sets,

$$\{|f_n - f| \geq \varepsilon\} \subseteq \{|f_n - f_k| \geq \varepsilon 2^{-1}\} \cup \{|f_k - f| \geq \varepsilon 2^{-1}\}. \tag{1}$$

We conclude that $f_n \to f$ in measure, as the measures of the right members vanish as $n, k \to \infty$. ■

Proposition 1.6: Theorem 2.30 — Part 4

Let f_n, f_k, and f be as in Part 3. The limit in measure is also unique, that is: if $f_n \to g$ in measure, then $g = f$ pointwise a.e.

Proof. Given that $f_n \to g$ in measure, if $\varepsilon_m \to 0$ is a positive sequence, then

$$\{|f - g| \geq \varepsilon_m\} \subseteq \{|f_n - f| \geq \varepsilon_m 2^{-1}\} \cup \{|f - g| \geq \varepsilon_m 2^{-1}\},$$

So that $\mu(|f - g| \geq \varepsilon_m) = 0$, and σ-subadditivity tell us $\mu(|f - g| \neq 0) = 0$. ■

Remark 1.2: Cauchy + Pointwise = Pointwise: Part 1

The construction in Equation (1) occurs quite often. Elements of a Cauchy sequence lump together at infinity, and given that some subsequence converges to a limit, we are able to drag the entire sequence in by means of the Triangle Inequality. Convergence requires

$$\varepsilon \in \bigcup_{N \in \mathbb{N}^+} \bigcap_{n \geq N} (|x_n - x|, \infty)$$

whereas the Cauchy Criterion gives us

$$\varepsilon \in \bigcup_{N\in\mathbb{N}^+} \bigcap_{\min(m,n)\geq N} (|x_n - x_m|, \infty).$$

Remark 1.3: a.e sets behave like open sets

One can reuse the same mental muscle, since the family of a.e. sets are closed under finite unions (follows from σ-subadditivity).

Remark 1.4: Two failed attempts at Theorem 2.30 Part 2

Note 1.4: Attempt 1: Replacing sequence with limit — DCT Edition

A direct appeal to Lemma 1.2 will not work. Given $\varepsilon > 0$, the mapping

$$A_\varepsilon : \mathcal{M} \times \mathcal{M} \to [0, +\infty) \quad \text{where} \quad A_\varepsilon(f, g) = \mu(|f - g| \geq \varepsilon)$$

does not admit an obvious 'continuity' criterion where we can exchange

$$A(f_n, f_k) \to A(f_n, f) \quad \text{as } f_k \to f \quad \text{pwae.}$$

An easy fix to this would be to write the measure as an integral, and pass to a DCT argument, that is:

$$\mu(|f_n - f_k| \geq \varepsilon) = \int \chi_{|f_n - f_k|\geq\varepsilon} \quad \text{and} \quad \chi_{|f_n - f_k|\geq\varepsilon} \to \chi_{|f_n - f|\geq\varepsilon}.$$

However, this would require the pwae convergence of the indicator functions, as $|f_n(x) - f_k(x)| \to |f_n(x) - f_k(x)|$ does not imply $|f_n(x) - f_k(x)|$ approaches its limit from above.

Note 1.5: Attempt 2: Replacing sequence with limit — L^1 Edition

The idea behind the proof is to utilize the 'in-measure closeness' (brought about by $\{f_n\}$ being Cauchy in measure). First, we provide a bit of motivation to the technique used.

Let us recall the fact that we have the measure theoretic bound d_k, and

$$\mu(|f_n - f_k| \geq \varepsilon) \leq d_k \quad \forall n \geq k \quad \text{and} \quad \mu(|f_k - f| \geq k).$$

We can control the convergence using two pieces: $|\chi_{|f_n - f_k|\geq\varepsilon} - \chi_{|f_n - f|\geq\varepsilon}| = \chi_{\{|f_n - f_k|\geq\varepsilon\}\Delta\{|f_n - f|\geq\varepsilon\}}$; and one of the two pieces are difficult to deal with.

Definition 1.1: Almost Uniform Convergence

A sequence of measurable functions f_n is said to converge *almost uniformly* if for every $\varepsilon > 0$, there exists $E \in \mathcal{M}$ where $f_n \to f$ uniformly on E^c (as an equivalence class), and $\mu(E) < \varepsilon$.

Proposition 1.7: Egorov's Theorem

Let $\{f_n\}$ be a sequence of complex-valued measurable functions, and $f_n \to f$ pointwise a.e, and $|f_n| \leq g \in L^1 \cap L^+$, then $f_n \to f$ almost uniformly.

Remark 1.5: Egorov's Theorem on Finite Measure Spaces

If $\mu(X)$ is finite, then the requirement that f_n is dominated can be dropped.

Proof. First, notice that if $f_n \to f$ pointwise a.e, and f_n is dominated, then $|f_n| \to |f|$. An application of Lebesgue's Theorem with $|f| \leq |g|$ tells us $\|f_n - f\|_1 \to 0$ and $f \in L^1$ by completeness.

Let $\{c_j\}$, $\{d_j\}$, $\{e_j\}$ be positive, $c_j \to 0$, $\{d_j\}, \{e_j\} \in l^1$ and $\sum d_j \leq 1$. We define $A(j, n) = \{|f_n - f| \geq c_j\}$, and since $f_n \to f$ pointwise a.e, we see that

$$\mu(\limsup_{n \to \infty} A(j, n)) = 0 \quad \forall j \geq 1.$$

Note 1.6: Finite Measure Space or f_n is dominated

If the measure space is finite, continuity from above reads

$$\mu(\cup_{\geq n} A(j, \cdot)) \to 0 \quad \text{as } n \to \infty.$$

If $\mu(X) = +\infty$, since $\|f_n - f\|_1 \to 0$, we can choose a subsequence f_k where $\|f_k - f\|_1$ is controlled by e_k. This means

$$\mu(\cup_{\geq n} A(j, \cdot)) \leq c_j^{-1} \sum e_i < +\infty,$$

and continuity from above gives us $\mu(\cup_{\geq n} A(j, \cdot)) \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$, at every $j \geq 1$, we can choose $K_j \in \mathbb{N}^+$ where $\mu(\cup_{k \geq K_j} A(j, k)) \leq \varepsilon d_j$, and we set

$$\widetilde{A} = \bigcup_{j \geq 1} \bigcup_{k \geq K_j} A(j, k) \quad \text{where} \quad \mu(\widetilde{A}) \leq \varepsilon.$$

If $x \notin \widetilde{A}$, for every $j \geq 1$, we see that $x \in \cap_{k \geq K_j} A(j, k)^c$, and $\sup_{\substack{x \notin \widetilde{A}, \\ k \geq K_j}} |f_k(x) - f(x)| < c_j$. Since $c_j \to 0$, this establishes the claim for the subsequence f_k.

Our goal is to extend something that is valid on a subsequence of f_n to the entire sequence. To wit, we write

$$B(j) = \bigcap_{N \geq 1} \bigcup_{\min(n,m) \geq N} \{|f_n - f_m| \geq c_j\},$$

which describes the set of points where Cauchyness fails. We note that $\mu(B(j)) = 0$, since $x \in B(j)$ means that $\{f_n(x)\}$ is not Cauchy. Since $\widetilde{B} = \bigcup B(j)$ has measure zero, and $\mu(\widetilde{A} \cup \widetilde{B}) \leq \varepsilon$. The set $\widetilde{A} \cup \widetilde{B}$ is the set where the 1) the subsequence f_k does not converge uniformly to f or 2) the sequence $f_n(x)$ fails to be Cauchy. If $x \notin \widetilde{A} \cup \widetilde{B}$, one sees that

- for every $j \geq 1$, there exists $K_j \in \mathbb{N}^+$ where $k \geq K_j$, $|f_k(x) - f(x)| < c_j$, and
- there exists N_j where $\min(n, m) \geq N_j$ implies $|f_n(x) - f_m(x)| < c_j$;

and the proof is complete upon combining the two estimates. ■

Remark 1.6: Pointwise + Cauchy = Pointwise: Part 2

We motivate the construction of $\widetilde{A} \cup \widetilde{B}$ reminding the reader that $\widetilde{A}^c$ (resp. $\widetilde{B}^c$) is responsible for subsequential pointwise convergence (resp. Cauchy criterion of the tail).