

# Chapter 3

## Notes on Chapter 3

### Proposition 1.1

Prove two things,

1.  $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2.  $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

*Proof.*



**Proposition 1.2**

If  $U \subseteq B(1, 0) = \{|x| < 1\}$ , and  $U \in \mathbb{B}$ , and if  $m(U) > 0$ , then the family of sets

$$E_r = \{x + r y, y \in U\}$$

shrinks nicely to  $x \in \mathbb{R}^n$ .

*Proof.* Let  $r > 0$  be fixed then  $\forall z \in E_r \leadsto z = x + r y$ . Hence,

$$\begin{aligned} d(x, z) &= d(x, x + r y) \\ &= |r| d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

**Definition 1.1: Signed measure**

Let  $\mathcal{M}$  be a  $\sigma$ -algebra and  $\nu: \mathcal{M} \rightarrow [-\infty, +\infty]$  be a set function on  $\mathcal{M}$ . It is a *signed measure* on  $\mathcal{M}$  if

- $\nu(\emptyset) = 0$ ,
- $\nu$  assumes at most one of the values  $\pm\infty$ ,
- If  $\{E_j\}_{j \geq 1}$  is a countable, disjoint sequence of sets, the expression

$$\sum_{j \geq 1} \nu(E_j) \quad \text{is unambiguous, and is equal to } \nu(\bigcup E_j)$$

More precisely,

- if  $|\nu(\bigcup E_j)| < +\infty$ , the series  $\sum \nu(E_j)$  converges absolutely,
- if  $\nu(\bigcup E_j) = \pm\infty$ , the series  $\sum \nu(E_j)$  diverges to  $\pm\infty$  on every permutation.

**Definition 1.2: Positive, negative, null sets**

Let  $\nu$  be a signed measure on  $\mathcal{M}$ . A measurable set  $E \in \mathcal{M}$  is called *positive* (resp. *negative*, *null*) if every measurable subset  $F \subseteq E$  satisfies  $\nu(F) \geq 0$  (resp.  $\nu(F) \leq 0$ ,  $\nu(F)=0$ ).

**Definition 1.3: Mutual singularity**

Two signed measures,  $\nu$  and  $\mu$  on a common  $\sigma$ -algebra  $\mathcal{M}$  are *mutually singular*, denoted by  $\nu \perp \mu$  if there exists disjoint, measurable sets  $E, F$  whose union is  $\mathbf{X}$ .

$$\mu \text{ is null on } E, \quad \text{and } \nu \text{ is null on } F$$

**Proposition 1.3**

Let  $\nu$  be a signed measure on  $(\mathbf{X}, \mathcal{M})$ . If  $\{E_j\}$  is an increasing sequence in  $\mathcal{M}$ ,  $\lim_{n \rightarrow +\infty} \nu(E_j) = \nu(\bigcup E_j)$ . If  $\{E_j\}$  is a decreasing sequence in  $\mathcal{M}$ ,  $\lim_{n \rightarrow +\infty} \nu(E_j) = \nu(\bigcap E_j)$  provided  $\nu(E_1)$  is of finite measure.

*Proof.* Let  $\nu$  be a signed measure, and fix any increasing sequence  $E_j \nearrow E = \bigcup E_{j \geq 1}$  of sets. This induces a disjoint sequence in  $\{F_n\}$ . Define  $F_1 = E_1$ , and if  $n \geq 2$ ,

$$F_n = E_n \setminus \bigcup_{j \leq n-1} E_j$$

Use  $\sigma$ -additivity of  $\nu$ , where the sum is 'defined' to be non-ambiguous.

For the second part of the proof, notice if  $A \subseteq B$  are measurable sets, if  $\nu(A) = \pm\infty$ , then  $\nu(B) = \pm\infty$ , because of the second property of  $\nu$ . Indeed,

$$\nu(B) = \nu(A) + \nu(B \setminus A) = \pm\infty + c$$

where  $c \in \mathbb{R} \cup \{\pm\infty\}$ . Therefore  $\nu(B) = \nu(A)$ . By assumption  $\nu(E_1) \in \mathbb{R}$ , the contrapositive of the previous argument shows that the intersection  $\bigcap E_j$  is of finite measure as well. We can produce an increasing sequence  $G_n = E_1 \setminus E_n$  for  $n \in \mathbb{N}^+$ . Then

$$\bigcup G_n = \bigcup E_1 \setminus E_n = E_1 \cap \left[ \bigcup E_n^c \right] = \left[ \bigcap E_j \right]^c$$

We then write

$$E_1 = \left[ \bigcup G_n \right] \cup \left[ \bigcap E_n \right]$$

The finiteness of  $\nu(E_1)$  on the left hand side implies all the terms in the union converge absolutely. Therefore

$$\begin{aligned} \nu(E_1) - \nu\left(\bigcap E_n\right) &= \lim_{n \rightarrow +\infty} \nu(G_n) \\ &= \lim_{n \rightarrow +\infty} \nu(E_1) - \nu(E_n) \\ &= \nu(E_1) - \lim_{n \rightarrow +\infty} \nu(E_n) \end{aligned}$$

Cancelling terms finishes the proof. ■

**Proposition 1.4**

Any measurable subset of a positive set is again positive, and any countable union of positive sets is again positive. Similarly for negative, and null sets.

*Proof.* Trivial. ■

**Proposition 1.5: Hahn Decomposition Theorem**

Let  $\nu$  be a signed measure on the measurable space  $(\mathbf{X}, \mathcal{M})$ , then there exists positive and negative sets  $P, N \in \mathcal{M}$  where  $P \cup N = \mathbf{X}$ , and  $P \cap N = \emptyset$ . If  $P'$  and  $N'$  are another such decomposition,

$$P \Delta P' = N \Delta N' \quad \text{is } \nu\text{-null.}$$

*Proof.* There are multiple steps to this proof. Suppose  $\nu$  does not attain  $+\infty$ . Define

$$m = \sup \{ \nu(P), P \text{ is a positive set} \}$$

By assumption  $m < +\infty$ , let  $\{P_j\}$  be a sequence of positive sets with  $\nu(P_j) \nearrow m$ . We claim the supremum is attained. Indeed, if  $P \triangleq \cup P_j$ , then  $P$  is a positive set as well, by monotonicity  $\nu(P) \geq \nu(P_j)$ , taking the supremum on both sides reads  $\nu(P) = m$ .

Wanting to prove  $N \triangleq \mathbf{X} \setminus P$  is a  $\nu$ -negative set,

- Clearly  $N$  cannot contain any positive sets  $A \subseteq N$  with a non-null measure, since

$$\nu(A) > 0 \implies \nu(A) + \nu(P) = \nu(A + P) > m$$

contradicting the supremum.

- Let us examine the properties of subsets of  $N$  with *positive measure*. Call this set  $A \subseteq N$ , where  $\nu(A) > 0$ .

The previous bullet point tells us  $A$  cannot be a  $\nu$ -positive set. There exists a  $B \subseteq A$  of strictly negative measure,

$$\nu(A \setminus B) + \nu(B) = \nu(A) \implies \nu(A \setminus B) > \nu(A)$$

Notice the assumption  $\nu$  does not attain  $+\infty$  allows us to subtract  $B$  over. Summarizing,

existence of subset of positive measure  $\implies$  subset with even greater positive measure

We will use the above inductively to construct a measurable subset of  $N$ , that is 'small' but has 'large' positive measure at the same time.

- Suppose  $N$  is not  $\nu$ -negative, so it admits a set of positive measure in  $A_1 \subseteq N$ .

Let  $n_1 = \text{least} \{ n \in \mathbb{N}^+, \exists B \subseteq A_1, \nu(B) > \nu(A) + n^{-1} \}$ , since  $n_1$  is attained, it corresponds to some  $A_2 \subseteq A_1$  with  $\nu(A_2) > \nu(A_1) + n_1^{-1}$ .

Repeating this process inductively, we see

$$\nu(A_k) > \nu(A_{k-1}) + n_k^{-1}$$

Let  $A = \bigcap A_k$ , this should be a set of large positive measure. A simple induction will show

$$\nu(A_k) > \nu(A_1) + \sum_{j=1}^k n_j^{-1} > \sum_{j=1}^k n_j^{-1}$$

However,  $\nu(A) < +\infty$  by assumption. Upon taking limits and using the estimate above,

$$\sum_{j \geq 1} n_j^{-1} = \lim_{n \rightarrow \infty} \nu(A_n) = \nu(A) < +\infty$$

The sum on the left is finite, so its terms must converge to 0. Notice  $\nu(A)$  is a subset of  $N$  of positive measure, it admits a subset  $B \subseteq A$  with  $\nu(B) > \nu(A) + n^{-1}$  for  $n \geq 1$ .

$n_j^{-1} \rightarrow 0$  implies  $n_j \rightarrow \infty$ . So  $n < n_j$  for large  $j$ . Notice  $B \subseteq A \subseteq A_j$ , and  $\nu(B) > \nu(A_j) + n^{-1}$ . This contradicts our definition of  $n_j$ , stated below for convenience

$$n_j = \text{least} \left\{ n \in \mathbb{N}^+, \exists B \subseteq A_j, \nu(B) > \nu(A_j) + n^{-1} \right\}$$

This proves  $N$  is  $\nu$ -negative.

To show this composition is  $\nu$ -unique, let  $P'$  and  $N'$  be disjoint, measurable positive and negative sets of  $\mathbf{X}$ . Then

$$P \setminus P' \subseteq P \quad \text{and} \quad P \setminus P' \setminus \mathbf{X} \setminus P' \subseteq N'$$

So  $P \setminus P'$  is at the same time a  $\nu$ -positive and a  $\nu$ -negative set, hence it is  $\nu$ -null by Lemma 3.2.

Finally, the case for when  $\nu$  attains  $+\infty$  can be handled if we consider  $-\nu$ .  $P$  is positive for  $-\nu$  iff it is negative for  $\nu$ , and similarly for  $N$ . Relabelling  $P$  and  $N$  finishes the proof. ■



**Theorem 3.4**

**Proposition 1.6**

*Proof.*



**Theorem 3.5**

Proposition 1.7

*Proof.*



**Theorem 3.6**

Proposition 1.8

*Proof.*



**Theorem 3.7**

Proposition 1.9

*Proof.*



**Theorem 3.8**

Proposition 1.10

*Proof.*



**Theorem 3.9**

Proposition 1.11

*Proof.*



**Theorem 3.10****Proposition 1.12***Proof.*

**Theorem 3.11****Proposition 1.13***Proof.*



**Theorem 3.12**

Proposition 1.14

*Proof.*



**Theorem 3.13****Proposition 1.15***Proof.*

**Theorem 3.14**

Proposition 1.16

*Proof.*



**Theorem 3.15**

Proposition 1.17

*Proof.*



**Theorem 3.16**

Proposition 1.18

*Proof.*



**Theorem 3.17****Proposition 1.19**

Let the maximal function of any measurable  $f \in \mathbb{B}_{\mathbb{R}^n}$  be denoted by  $Hf(x)$ , more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy$$

where  $A_r|f|$  is the average of  $|f|$  on a ball with radius  $r > 0$  centered at  $x \in \mathbb{R}^n$ . In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy$$

The maximal theorem makes two claims:

1.  $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , and  $Hf$  is measurable for every  $f \in L^1_{loc}$ .
2. There exists a  $C > 0$ , for every  $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every  $\alpha > 0$ .

*Proof.* Let  $\alpha > 0$  and fix  $z \in (Hf)^{-1}((\alpha, +\infty))$ , so  $Hf(z) > \alpha$  and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with  $Hf(z) - \alpha > 0$ , we get some  $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let  $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some  $r_0 > 0$ . Since  $A_r|f|$  (a function indexed by  $r > 0$ ) is continuous in  $x \in \mathbb{R}^n$ ,  $(A_r|f|)^{-1}((\alpha, +\infty))$  is open, and  $Hf$  is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \{Hf > \alpha\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let  $x \in E_\alpha$ , this induces a  $r_x > 0$  where  $x \in \{A_{r_x}|f| > \alpha\}$ . Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r,x)} |f| dz\right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each  $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x}|f|$ ,
- If  $U = \bigcup_{x \in E_\alpha} B(r_x, x)$ , then  $E_\alpha \subseteq U$ ,
- Choose  $c < m(E_\alpha) \leq m(U)$  (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that  $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$ , and  $c < 3^n \sum_{j \leq k} m(B_j)$

- Define  $B_j = B(r_{x_j}, x_j)$  for all  $j \leq k$ , and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of  $m$  on  $\mathbb{B}$ , since

$$m(E_\alpha) = \sup \left\{ m(K), K \subseteq \mathbb{R}^n, \text{ compact. } K \subseteq E_\alpha \right\}$$

for any compact  $K$ ,  $K \subseteq E_\alpha$ , we have  $m(K) < +\infty$ ,  $m(K) \leq m(E_\alpha)$  and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

### Remark 1.1

We used the properties of a Radon Measure here, without relying on the phrase ‘sending  $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases  $m(E_\alpha) < +\infty$  and  $m(E_\alpha) = +\infty$ .

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■



**Theorem 3.18**

Proposition 1.20

*Proof.*



**Theorem 3.19**

Proposition 1.21

*Proof.*



**Theorem 3.20****Proposition 1.22***Proof.*

## Theorem 3.21

### Proposition 1.23

The Lebesgue Differentiation Theorem. Suppose  $f \in L^1_{loc}$ , and for every  $x \in \mathcal{L}_f$ , (so that  $x \in \mathbb{R}^n$  a.e.). We have

1.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family  $\{E_r\}_{r>0}$  that shrinks nicely to  $x \in \mathbb{R}^{n'}$ .

*Proof.* Since the family  $\{E_r\}_{r>0}$  shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some  $\alpha > 0$ , independent on  $r$ . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every  $\varepsilon > 0$ , if  $0 < r < \varepsilon$ , then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending  $\varepsilon \rightarrow 0$ , proves the first claim. The second claim is immediate upon applying the  $L^1$  inequality.

Fix any  $\varepsilon > 0$ , and

$$\begin{aligned}
 \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
 &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
 &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= 0
 \end{aligned}$$

■

**Theorem 3.22**

Proposition 1.24

*Proof.*



**Theorem 3.23**

Proposition 1.25

*Proof.*



**Theorem 3.24**

Proposition 1.26

*Proof.*





**Theorem 3.25**

Proposition 1.27

*Proof.*



**Theorem 3.26**

Proposition 1.28

*Proof.*



**Theorem 3.27**

Proposition 1.29

*Proof.*



**Theorem 3.28**

Proposition 1.30

*Proof.*



**Theorem 3.29**

Proposition 1.31

*Proof.*



**Theorem 3.30**

Proposition 1.32

*Proof.*



**Theorem 3.31****Proposition 1.33***Proof.*

**Theorem 3.32**

Proposition 1.34

*Proof.*





**Theorem 3.33**

Proposition 1.35

*Proof.*



**Theorem 3.34**

Proposition 1.36

*Proof.*



**Theorem 3.35**

Proposition 1.37

*Proof.*



**Theorem 3.36**

Proposition 1.38

*Proof.*

