# Chapter C: Algebraic Topology

# Homotopy

This section will follow Munkres Chapters 9 and 13 closely. Possibly other chapters as well.

**Definition 1.1: Path**

A *path* is a continuous function from the unit interval $f : [0,1] \to \mathbf{X}$. We say $f$ is a *path form $x_0$ to $x_1$* if $f(0) = x_0$ and $f(1) = x_1$.

We denote the set of paths from $x_0$ to $x_1$ by $\mathrm{Path}(x_0, x_1)$. If $f \in \mathrm{Path}(x_0, x_1)$, we sometimes denote the *reversal of $f$* by $\overline{f} \in \mathrm{Path}(x_1, x_0)$, where $\overline{f}(s) \triangleq f(1-s)$.

**Definition 1.2: Loop**

A *loop* at $x_0 \in \mathbf{X}$ is a path that begins and ends at $x_0$, and $\mathrm{Loop}(x_0) \triangleq \mathrm{Path}(x_0, x_0)$. The constant path (or loop) at $x_0$ is denoted by $e_{x_0} : [0,1] \to \mathbf{X}$.

$$e_{x_0}(s) = x_0, \quad \forall s \in [0,1]$$

**Definition 1.3: Homotopy of $C(\mathbf{X},\ \mathbf{Y})$**

Let $f$, and $g$ continuous functions from $\mathbf{X}$ to $\mathbf{Y}$. $f$ and $g$ are homotopic, denoted by $f \approx g$ if there exists a continuous function $F \in C(\mathbf{X} \times I,\ \mathbf{Y})$ where

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x) \tag{1}$$

where $I = [0,1]$.

The function $F$ is called the *homotopy between $f$ and $g$*.

If $f \simeq h$, where $h$ is the constant function, we say $f$ is *nulhomotopic*.

**Definition 1.4: Path Homotopy of $\mathrm{Path}(x_0, x_1)$**

Two paths $f_0$, $f_1 \in \mathrm{Path}(x_0, x_1)$ are said to be *path homotopic*, if there exists a continuous function $F \in C(I \times I,\ \mathbf{X})$, with

- $F$ is a *homotopy between $f_0$ and $f_1$* (in the sense of Definition 1.3). For every $s \in [0,1]$,

$$F(s,0) = f_0(s) \quad \text{and} \quad F(s,1) = f_1(s) \tag{2}$$

- $F$ leaves the endpoints fixed. For every $t \in [0,1]$, then

$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1 \tag{3}$$

If $f_0$ and $f_1$ are path-homotopic, we write $f_0 \simeq_p f_1$.

- The function $F \in C(I \times I,\ \mathbf{X})$ is called the *path homotopy between* $f_0$ *and* $f_1$.
- If $f \in \mathrm{Loop}(x_0)$ is path homotopic to the constant path $e_{x_0}$, then $f$ is *nulhomotopic.*
- The relation $\simeq_p$ is defined for paths that have the same initial and final points. So it is a relation *on* $\mathrm{Path}(x_0, x_1)$.

**Proposition 1.1: Munkres Lemma 51.1**

The relations $\simeq$ and $\simeq_p$ are equivalence relations on $C(\mathbf{X}, \mathbf{Y})$ and $\mathrm{Path}(x_0, x_1)$ respectively.

*Proof.* ($f \simeq f$): Let $f \in C(\mathbf{X}, \mathbf{Y})$. Define

$$F : \mathbf{X} \times I \to \mathbf{Y} \quad \text{For every } t \in [0,1], F(x,t) = f(x)$$

$F$ is continuous, since $F = \pi_{\mathbf{X}} \circ (f \times \mathrm{id}_{[0,1]})$, where $f \times \mathrm{id}_{[0,1]}$ is the product of two continuous functions, which is again continuous by Chapter A. Moreover, $F(x,0) = f(x) = F(x,1)$, so $F$ is a homotopy between $f$ and itself.

($f \simeq g \implies g \simeq f$): Let $F$ be the homotopy between $f$ and $g$. Let $G$ be the 'reversal' in the second coordinate of $F$, meaning

$$G(x,t) = F(x, 1-t) \text{ is continuous, since } G = F \circ (\mathrm{id}_{\mathbf{X}} \times c)$$

where $c : I \to I$ that maps $t \mapsto 1-t$ is continuous, so $\mathrm{id}_{\mathbf{X}} \times c$ is continuous; hence $G$ is continuous. Notice for every $x \in \mathbf{X}$,

$$G(x,0) = F(x,1) = g(x) \quad \text{and} \quad G(x,1) = F(x,0) = f(x)$$

therefore $G$ is a homotopy between $g$ and $f$.

($f \simeq g,\ g \simeq h \implies f \simeq h$): Let $F$ be the homotopy between $f$ and $g$, and $G$ be the homotopy between $g$ and $h$. Define a function $H : \mathbf{X} \times I \to \mathbf{Y}$ that morphs $f$ into $g$ on $[0, 2^{-1}]$, then $g$ into $h$ on $[2^{-1}, 1]$

$$H(x,t) = \begin{cases} F(x, 2t - \lfloor 2t \rfloor) & \text{for } 0 \le t \le 2^{-1} \\ G(x, 2t - \lfloor 2t \rfloor) & \text{for } 2^{-1} \le t \le 1 \end{cases} \tag{4}$$

where $\lfloor\cdot\rfloor$ denotes the *floor function*.

- $H$ is well defined on the overlap $\mathbf{X} \times 2^{-1}$, since $F(x,1) = G(x,0) = g(x)$ at every $x \in \mathbf{X}$.
- If $t = 0$, then $H(x,1) = F(x,0) = f(x)$, and $t = 1$ gives $H(x,1) = G(x,1) = h(x)$.
- Since $H|_{\mathbf{X}\times[0,2^{-1}]}$ and $H|_{\mathbf{X}\times[2^{-1},1]}$ are continuous functions, and they agree on the overlap, $H$ is continuous by the pasting Lemma, and defines a homotopy between $f$ and $h$.

Now consider paths $f$, $g$, $h$ in $\mathrm{Path}(x_0, x_1)$, $(f \simeq_p f)$ is trivial. So is symmetry of $\simeq_p$, as the reversal in the second coordinate (see above) of the path homotopy between $f$ and $g$ is path homotopy between $g$ and $f$.

Suppose $f \simeq_p g$, and $g \simeq_p g$. Let $F$, and $G$ be the path homotopies between $f, g$ and $g, h$. Write $H$ as in Equation (4), it is a continuous function on $I \times I \to \mathbf{X}$, that satisfies

$$H(s,0) = F(s,0) = f(s) \quad \text{and} \quad H(s,1) = G(s,1) = h(s) \text{ for every } s \in [0,1]$$

If $s = 0$, it is easy to see from Equation (4) that for every $t \in [0,1]$,

$$H(0,t) = \begin{cases} F(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 0 \le t \le 2^{-1} \\ G(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 2^{-1} \le t \le 1 \end{cases} = x_0 \quad \text{and}$$

$$H(1,t) = \begin{cases} F(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 0 \le t \le 2^{-1} \\ G(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 2^{-1} \le t \le 1 \end{cases} = x_1$$

So the endpoints remain fixed throughout the deformation in $t$, and $H$ is a path homotopy between $f$ and $h$. This proves transitivity. ■

## Path and PathClass Products

**Definition 1.5: Product of Paths $f * g$**

Let $f \in \mathrm{Path}(x_0, x_1)$ and $g \in \mathrm{Path}(x_1, x_2)$, the product of $f$ and $g$, denoted by $f * g$ is another path from $x_0$ to $x_2$. For $s \in [0,1]$,

$$(f * g)(s) \triangleq \begin{cases} f(2s - \lfloor 2s \rfloor) & \text{for } 0 \le s \le 2^{-1} \\ g(2s - \lfloor 2s \rfloor) & \text{for } 2^{-1} \le s \le 1 \end{cases} \tag{5}$$

Notice the similarities between Equations (4) and (5),

**Proposition 1.2: Properties of the Path Product**

Let $f \in \text{Path}(x_0, x_1)$ and $g \in \text{Path}(x_0, x_1)$, let $k \in C(\mathbf{X},\ \mathbf{Y})$, then

(i) Invariant under left-multiplication: $f \simeq_p g \implies k \circ f \simeq_p k \circ g$, where $k \circ f$ and $k \circ g$ are elements Paths from $k(x_0)$ to $k(x_1)$, and if $F$ be a path homotopy between $f$ and $g$, then $k \circ F$ is a path homotopy between $k \circ f$ and $k \circ g$.

(ii) If we redefine $f \in \text{Path}(x_0, x_1)$, $g \in \text{Path}(x_1, x_2)$, and $k$ be as above, then
$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

*Proof.*

Proof of Part (i): It is clear that $k \circ f$ and $k \circ g$ are elements of $\text{Path}(k(x_0), k(x_1))$, and see Part (ii) for the proof of $k \circ f \simeq_p k \circ g$.

Proof of Part (ii): Let $F$ be the path homotopy between $f$ and $g$. The composition $(k \circ F)$ is in $C(\mathbf{X} \times I,\ \mathbf{Y})$. Equation (2) reads

$$(k \circ F)(s, 0) = k(F(s, 0)) = (k \circ f)(s) \text{ and}$$
$$(k \circ F)(s, 1) = k(F(s, 1)) = (k \circ g)(s) \text{ for every } s \in [0, 1]$$

and Equation (3) gives

$$(k \circ F)(0, t) = k(F(0, t)) = k(x_0) \text{ and}$$
$$(k \circ F)(1, t) = k(F(1, t)) = k(x_1) \text{ for every } t \in [0, 1]$$

therefore $k \circ F$ is a path homotopy between the paths $k \circ f$ and $k \circ g$.

■

**Definition 1.6: Path Homotopy class $[f]$**

Let $f \in \text{Path}(x_0, x_1)$, we define the *path homotopy class of* $f$ as

$$[f] \triangleq \Big\{ g \in \text{Path}(x_0, x_1),\ g \simeq_p f \Big\}$$

**Definition 1.7: Product of PathClasses $[f] * [g]$**

Let $* : \text{PathClass}(x_0, x_1) \times \text{PathClass}(x_1, x_2) \to \text{PathClass}(x_0, x_2)$ be a binary operation, where
$$[f] * [g] \triangleq [f * g] \text{ is well defined.}$$

for arbitrary $[f] \in \text{PathClass}(x_0, x_1)$ and $[g] \in \text{PathClass}(x_1, x_2)$. This means it is independent of the representative chosen. More formally, if $f \simeq_p f' \in \text{Path}(x_0, x_1)$, and $g \simeq_p g' \in \text{Path}(x_1, x_2)$, then $f * g \simeq_p f' * g'$.

**Proposition 1.3: Properties of the PathClass product**

Let $[f]$, $[g]$ and $[h]$ be PathClasses from and to the points $x_0$, $x_1$, $x_2$. Then

1. Associativity: $([f] * [g]) * [h] = [f] * ([g] * [h])$,
2. Left and Right identities: if $[f] \in \text{PathClass}(x_0, x_1)$, $e_{x_0}$, $e_{x_1}$ denote the constant paths on $x_0$ and $x_1$ (the initial and final points of any $f \in [f]$), then
$$[e_{x_0}] * [f] = [f] \quad \text{and} \quad [f] * [e_{x_1}] = [f]$$
3. Left and Right inverses: let $[\overline{f}]$ be the PathClass containing the reversal of $f$ (see Definition 1.1) for the definition, then
$$[\overline{f}] * [f] = [e_{x_1}] \quad \text{and} \quad [f] * [\overline{f}] = [e_{x_0}]$$
4. Generalized Associativity: if $\{[f_j]\}_{j \leq n}$ is a sequence of PathClasses, such that $[f_j] \in \text{PathClass}(x_{j-1}, x_j)$, then
$$\prod [f_j] \triangleq [f_1] * [f_2] * \cdots * [f_n] \text{ is a well-defined object}$$
meaning we can place the brackets wherever we want.

*Proof.* We will give an outline for the proof of Generalized Associativity, the rest are trivial. Let $\{[f_j]\}$ be defined as above. If $\{a_j\}_{j=0}^n$, and $\{b_j\}_{j=0}^n$ are 'cell partitions' of the unit interval (in the sense of the Riemann integral), meaning

$$0 = a_0 < a_1 < \cdots < a_n = 1, \quad \text{and} \quad 0 = b_0 < b_1 < \cdots < b_n = 1$$

We agree to define the following

- the lengths of each cell $l_{a_j} \triangleq a_j - a_{j-1}$ and $l_{b_j} \triangleq b_j - b_{j-1}$, and
- the cells themselves are denoted by $\text{cell}(a_j) = [a_{j-1}, a_j]$, $\text{cell}(b_j) = [b_{j-1}, b_j]$,
- $p \in \text{Path}(0, 1)$, where $p$ is given explicitly by
$$p(s) = \sum_{j=1}^{n} \chi_{\text{cell}(a_j) \setminus \{a_{j-1}\}} \left( \frac{l_{b_j}}{l_{a_j}} (s - a_j) + b_j \right)$$

It is clear $p$ is continuous, and for $j = 1, \ldots, n$,

$$p|_{\text{cell}(a_j)} \text{ is the positive linear map from } \text{cell}(a_j) \text{ to } \text{cell}(b_j)$$

Using the same line of argumentation as in the proof for associativity, we see that any two 'ways' of bracketing the expression has no impact on the path-homotopy class. ■

## Fundamental Group

**Definition 1.8: Fundamental group $\pi_1(\mathbf{X}, x_0)$**

Let $x_0 \in \mathbf{X}$, the *fundamental group of* $\mathbf{X}$ *relative to (base point)* $x_0$ is denoted by $\pi_1(\mathbf{X}, x_0) = \text{PathClass}(x_0, x_0)$.

**Definition 1.9: Isomorphism induced by $\text{Path}(x_0, x_1)$**

Suppose $\alpha \in \text{Path}(x_0, x_1)$, we define a map $\hat{\alpha} : \pi_1(\mathbf{X}, x_0) \to \pi_1(\mathbf{X}, x_1)$, with

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$$

where $\overline{\alpha}$ is the reversal of $\alpha$. We call $\hat{\alpha}$ the *isomorphism induced by* $\alpha$ (Munkres Theorem 52.1).

*Isomorphism proof.* Let $[f]$ and $[g]$ be elements of $\pi_1(\mathbf{X}, x_0)$, then

$$\begin{aligned}
\hat{\alpha}([f] * [g]) &= ([\overline{\alpha}] * [f] * [\alpha]) * ([\overline{\alpha}] * [g] * [\alpha]) \\
&= [\overline{\alpha}] * ([f] * [g]) * [\alpha] \\
&= \hat{\alpha}([f]) * \hat{\alpha}([g])
\end{aligned}$$

and $\hat{\alpha}$ is a homomorphism. We claim inverse of $\hat{\alpha}$ is $\hat{\overline{\alpha}}$. Fix $[f] \in \pi_1(\mathbf{X}, x_0)$, $[g] \in \pi_1(\mathbf{X}, x_1)$, then

$$(\hat{\overline{\alpha}} \circ \hat{\alpha})([f]) = [\alpha] * ([\overline{\alpha}] * [f] * [\alpha]) * [\overline{\alpha}] = [f]$$

so $\hat{\overline{\alpha}}$ is the left-inverse for $\hat{\alpha}$. A similar argument shows it is the right inverse as well with $(\hat{\alpha} \circ \hat{\overline{\alpha}})([g]) = [g]$. Therefore $\pi_1(\mathbf{X}, x_0)$ is group isomorphic to $\pi_1(\mathbf{X}, x_1)$. ■

## Homomorphisms

**Definition 1.10: Homomorphism induced by a continuous map**

Let $h \in C(\mathbf{X},\ \mathbf{Y})$, and $y_0 = h(x_0)$, it induces a map between loops at $x_0$ and

$y_0$.

$$h_* : \text{Loop}(x_0) \to \text{Loop}(y_0), f \mapsto h \circ f$$

It is a also a group homomorphism between fundamental groups. We use the same symbol for the two maps, relying on context to distinguish between the two.

$$h_* : \pi_1(\mathbf{X}, x_0) \to \pi_1(\mathbf{Y}, y_0), [f] \mapsto [h \circ f]$$

is well defined because of Proposition 1.3, it is a homomorphism (again by Proposition 1.3) because $h$ 'distributes' over $*$

$$h \circ (f * g) = (h \circ f) * (h \circ g)$$

**Remark 1.1: Functorial properties of the $h_*$**

If $x_0 \in \mathbf{X}$, the tuple $(x_0, \mathbf{X})$ is an object in the category of *pointed topological spaces*, and the map $h_*$ is a *covariant functor* from the category of pointed topological spaces to the category of groups.

Follows from Munkres Theorem 52.4, if the expressions below make sense,

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad h_* \circ (g \circ f)_* = (h \circ g)_* \circ f_*$$

And the identity map $i : \mathbf{X} \to \mathbf{X}$ gets 'sent' to the identity homomorphism in $\text{Hom}(\pi_1(\mathbf{X}, x_0), \pi_1(\mathbf{X}, x_0))$. And if $h$ is a homeomorphism between $\mathbf{X}$ and $\mathbf{Y}$, then $h_*$ is an isomorphism at every point.

## Simply connected space

**Definition 1.11: Simply connected space**

A topological space $\mathbf{X}$ is *simply connected* if it is path-connected, and $\pi_1(\mathbf{X}, x_0) = \{[e_{x_0}]\}$ for some $x_0 \in \mathbf{X}$. Notice this implies every fundamental group of $\mathbf{X}$ is trivial.

**Proposition 1.4: Properties of simply connected spaces**

If $\mathbf{X}$ is a simply connected space, then $\text{PathClass}(x_0, x_1)$ consists of one element. That is to say, if $f$ and $g$ are Paths from $x_0$ to $x_1$, then $f \simeq_p g$.

## Covering maps

**Definition 1.12: Covering maps and spaces**