Chapter 3

Notes on Chapter 3

Proposition 1.1

Prove two things,

1. $\limsup_{r\to R}\phi(r) = \lim_{\varepsilon\to 0}\sup_{0<|r-R|<\varepsilon}\phi(r) = \inf_{\varepsilon>0}\sup_{0<|r-R|<\varepsilon}\phi(r)$,
2. $\lim_{r\to R}\phi(r) = c \iff \limsup_{r\to R}|\phi(r)-c| = 0$

Proof. ∎

Proposition 1.2

If $U \subseteq B(1,0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets

$$E_r = \left\{ x + ry,\, y \in U \right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \looparrowright z = x + ry$. Hence,

$$\begin{aligned} d(x,z) &= d(x, x+ry) \\ &= |r| d(0,y) < |r| \end{aligned}$$

by translation invariance. ∎

Definition 1.1: Signed measure

Let $\mathcal{M}$ be a σ-algebra and $\nu : \mathcal{M} \to [-\infty, +\infty]$ be a set function on $\mathcal{M}$. It is a *signed measure* on $\mathcal{M}$ if

- $\nu(\varnothing) = 0$,
- ν assumes at most one of the values $\pm\infty$,
- If $\{E_j\}_{j\geq 1}$ is a countable, disjoint sequence of sets, the expression

$$\sum_{j\geq 1} \nu(E_j) \quad \text{is unambiguous, and is equal to } \nu\Big(\bigcup E_j\Big)$$

  More precisely,

  – if $|\nu(\bigcup E_j)| < +\infty$, the series $\sum \nu(E_j)$ converges absolutely,
  – if $\nu(\bigcup E_j) = \pm\infty$, the series $\sum \nu(E_j)$ diverges to $\pm\infty$ on every permutation.

Definition 1.2: Positive, negative, null sets

Let ν be a signed measure on $\mathcal{M}$. A measurable set $E \in \mathcal{M}$ is called *positive* (resp. *negative*, *null*) if every measurable subset $F \subseteq E$ satisfies $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$, $\nu(F)$=0).

Definition 1.3: Mutual singularity

Two signed measures, ν and μ on a common σ-algebra $\mathcal{M}$ are *mutually singular*, denoted by $\nu \perp \mu$ if there exists disjoint, measurable sets E, F whose union is $\mathbf{X}$.

$$\mu \text{ is null on } E, \quad \text{ and } \nu \text{ is null on } F$$

Proposition 1.3

Let ν be a signed measure on $(\mathbf{X}, \mathcal{M})$.

- If $\{E_j\}$ is an increasing sequence, $\lim_{n\to\infty} \nu(E_n) = \nu(\bigcup E_n)$, and
- if $\{E_j\}$ is a decreasing sequence — provided that $\nu(E_1) \in \mathbb{R}$ — then $\lim_{n\to\infty} \nu(E_n) = \nu(\bigcap E_n)$.

Proof. Let ν be a signed measure, and $E_j \nearrow E = \bigcup E_{j\geq 1}$. This induces a disjoint sequence in $\{F_n\}$, where $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup E_{j\leq n-1}$$

The first claim follows from the definition of ν, where $\sum_{n=1}^{\infty} \nu(E_n) = \lim_{n\to\infty} \sum_{j\leq n} \nu(E_j)$.

Next, for any measurable A, B, where $A \subseteq B$, if $\nu(A) = \pm\infty$, then $\nu(B) = \pm\infty$. It follows that $\mu(\cap E_n) \in \mathbb{R}$ as well. We can produce an increasing sequence through $G_n = E_1 \setminus E_n$ for $n \in \mathbb{N}^+$. Then,

$$\bigcup G_n = \bigcup E_1 \setminus E_n = E_1 \cap \left[\bigcup E_n^c\right] = \left[\bigcap E_j\right]^c.$$

We then write $E_1 = [\bigcup G_n] + [\bigcap E_n]$; and the finiteness of $\nu(E_1)$ on the left hand side implies all the terms in the union converge absolutely. Therefore

$$\begin{aligned}\nu(E_1) - \nu\left(\bigcap E_n\right) &= \lim_{n\to+\infty} \nu(G_n) = \lim_{n\to+\infty} \nu(E_1) - \nu(E_n) \\ &= \nu(E_1) - \lim_{n\to+\infty} \nu(E_n),\end{aligned}$$

and cancelling terms finishes the proof. ■

Proposition 1.4

Any measurable subset of a positive set is again positive, and any countable union of positive sets is again positive. Similarly for negative, and null sets.

Proof. Trivial. ∎

Proposition 1.5: Hahn Decomposition Theorem

Let ν be a signed measure on the measurable space $(\mathbf{X}, \mathcal{M})$, then there exists positive and negative sets $P, N \in \mathcal{M}$ where $P \cup N = \mathbf{X}$, and $P \cap N = \varnothing$. If P' and N' are another such decomposition,

$$P \Delta P' = N \Delta N' \quad \text{is } \nu\text{-null.}$$

Proof. There are multiple steps to this proof. Suppose ν does not attain $+\infty$. Define

$$m = \sup\left\{\nu(P),\ P \text{ is a positive set}\right\}$$

By assumption $m < +\infty$, let $\{P_j\}$ be a sequence of positive sets with $\nu(P_j) \nearrow m$. We claim the supremum is attained. Indeed, if $P \triangleq \cup P_j$, then P is a positive set as well, by monotonicity $\nu(P) \geq \nu(P_j)$, taking the supremum on both sides reads $\nu(P) = m$.

Wanting to prove $N \triangleq \mathbf{X} \setminus P$ is a ν-negative set,

- Clearly N cannot contain any positive sets $A \subseteq N$ with a non-null measure, since

  $$\nu(A) > 0 \implies \nu(A) + \nu(P) = \nu(A + P) > m$$

  contradicting the supremum.

- Let us examine the properties of subsets of N with *positive measure*. Call this set $A \subseteq N$, where $\nu(A) > 0$.

  The previous bullet point tells us A cannot be a ν-positive set. There exists a $B \subseteq A$ of strictly negative measure,

  $$\nu(A \setminus B) + \nu(B) = \nu(A) \implies \nu(A \setminus B) > \nu(A)$$

  Notice the assumption ν does not attain $+\infty$ allows us to subtract B over. Summarizing,

  $$\text{existence of subset of positive measure} \implies \text{subset with even greater positive measure}$$

  We will use the above inductively to construct a measurable subset of N, that is 'small' but has 'large' positive measure at the same time.

- Suppose N is not ν-negative, so it admits a set of positive measure in $A_1 \subseteq N$.

  Let $n_1 = \text{least}\left\{n \in \mathbb{N}^+,\ \exists B \subseteq A_1,\ \nu(B) > \nu(A) + n^{-1}\right\}$, since n_1 is attained, it corresponds to some $A_2 \subseteq A_1$ with $\nu(A_2) > \nu(A_1) + n_1^{-1}$.

  Repeating this process inductively, we see

  $$\nu(A_k) > \nu(A_{k-1}) + n_k^{-1}$$

Let $A = \bigcap A_k$, this should be a set of large positive measure. A simple induction will show

$$\nu(A_k) > \nu(A_1) + \sum_{j=1}^{k} n_j^{-1} > \sum_{j=1}^{k} n_j^{-1}$$

However, $\nu(A) < +\infty$ by assumption. Upon taking limits and using the estimate above,

$$\sum_{j\geq 1} n_j^{-1} = \lim_{n\to\infty} \nu(A_n) = \nu(A) < +\infty$$

The sum on the left is finite, so its terms must converge to 0. Notice $\nu(A)$ is a subset of N of positive measure, it admits a subset $B \subseteq A$ with $\nu(B) > \nu(A) + n^{-1}$ for $n \geq 1$.

$n_j^{-1} \to 0$ implies $n_j \to \infty$. So $n < n_j$ for large j. Notice $B \subseteq A \subseteq A_j$, and $\nu(B) > \nu(A_j) + n^{-1}$. This contradicts our definition of n_j, stated below for convenience

$$n_j = \text{least}\left\{ n \in \mathbb{N}^+,\ \exists B \subseteq A_j,\ \nu(B) > \nu(A_j) + n^{-1} \right\}$$

This proves N is ν-negative.

To show this composition is ν-unique, let P' and N' be disjoint, measurable positive and negative sets of $\mathbf{X}$. Then

$$P \setminus P' \subseteq P \quad \text{and} \quad P \setminus P' \setminus \mathbf{X} \setminus P' \subseteq N'$$

So $P \setminus P'$ is at the same time a ν-positive and a ν-negative set, hence it is ν-null by Lemma 3.2.

Finally, the case for when ν attains $+\infty$ can be handled if we consider $-\nu$. P is positive for $-\nu$ iff it is negative for ν, and similarly for N. Relabelling P and N finishes the proof. ■

Theorem 3.4

Proposition 1.6

Proof. ■

Theorem 3.5

Proposition 1.7

Proof. ∎

Theorem 3.6

Proposition 1.8

Proof. ■

Theorem 3.7

Proposition 1.9

Proof. ∎

Theorem 3.8

Proposition 1.10

Proof. ■

Theorem 3.9

Proposition 1.11

Proof. ■

Theorem 3.10

Proposition 1.12

Proof. ∎

Theorem 3.11

Proposition 1.13

Proof. ■

Theorem 3.12

Proposition 1.14

Proof. ■

Theorem 3.13

Proposition 1.15

Proof. ■

Theorem 3.14

Proposition 1.16

Proof. ■

Theorem 3.15

Proposition 1.17

Proof. ■

Theorem 3.16

Proposition 1.18

Proof. ■

Theorem 3.17

Proposition 1.19

Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1$$

for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{Hf > \alpha\right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{A_{r_x}|f| > \alpha\right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r,x)} |f|dz\right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \looparrowright r_x > 0 \looparrowright A_{r_x}|f|$,
- If $U = \bigcup_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \ldots, x_N \in E_\alpha$$

  so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$
- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

  by finite additivity,

$$c3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

  and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on $\mathbb{B}$, since

$$m(E_\alpha) = \sup \left\{ m(K),\ K \subseteq \mathbb{R}^n, \text{ compact. } \ K \subseteq E_\alpha \right\}$$

  for any compact K, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

> **Remark 1.1**
>
> We used the properties of a Radon Measure here, without relying on the phrase ‘sending $c \to E_\alpha$‘, which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.

■

Theorem 3.18

Proposition 1.20

Proof. ■

Theorem 3.19

Proposition 1.21

Proof. ∎

Theorem 3.20

Proposition 1.22

Proof. ■

Theorem 3.21

Proposition 1.23

The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have

1. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$,
2. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$,

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n\prime}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r,x)) \implies m(E_r) > \alpha \cdot m(B(r,x))$$

for some $\alpha > 0$, independent on r. Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r,x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r,x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0<r<\varepsilon} m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \to 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
\lim_{r\to 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
&\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
&\leq \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
&= \lim_{r\to 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
&= 0
\end{aligned}$$

∎

Theorem 3.22

Proposition 1.24

Proof. ■

Theorem 3.23

Proposition 1.25

Proof. ■

Theorem 3.24

Proposition 1.26

Proof. ∎

Theorem 3.25

Proposition 1.27

Proof. ■

Theorem 3.26

Proposition 1.28

Proof. ■

Theorem 3.27

Proposition 1.29

Proof. ■

Theorem 3.28

Proposition 1.30

Proof. ■

Theorem 3.29

Proposition 1.31

Proof. ■

Theorem 3.30

Proposition 1.32

Proof. ■

Theorem 3.31

Proposition 1.33

Proof. ■

Theorem 3.32

Proposition 1.34

Proof. ■

Theorem 3.33

Proposition 1.35

Proof. ■

Theorem 3.34

Proposition 1.36

Proof. ∎

Theorem 3.35

Proposition 1.37

Proof. ∎

Theorem 3.36

Proposition 1.38

Proof. ■