MATH 254 Assignment 1

November 5, 2022

1a

WTS. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We can use a chain of equivalences. Suppose that both members are not empty.

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \land x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \land x \in B) \text{ or } (x \in A \land x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Now suppose one of the two members are empty. Then if the other member was not empty, it would imply that the original member was not empty, and this means that the two sets must be equal. $\square$

2

WTS. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Define $W = (A \setminus B) \cup (B \setminus A)$, then we will apply Q1a, and deMorgan's Theorem.

$$\begin{aligned} W^c &= (A^c \cup B) \cap (B^c \cup A) \\ &= ((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A) \\ &= (A^c \cap B^c) \cup (A \cap B) \\ &= (A \cup B)^c \cup (A \cap B) \\ &= [(A \cup B) \setminus (A \cap B)]^c \end{aligned}$$

Taking complements on both sides finishes the proof. $\square$

3a

WTS. $f \coloneqq \{(x, y) \in [-1, +1] \times [-1, +1] : x^2 + y^2 = 1\}$ *is not a function.*

Proof. f is not a function because $(0, 1) \in f$ and $(0, -1) \in f$, and $1 \neq -1$. $\square$

3b

WTS. *f is a function.*

Proof. Since $y \geq 0$, we can write $y = +\sqrt{1 - x^2}$. Fix an $x \in [-1, 1]$, then there is a unique $y \in [0, 1]$ that satisfies the above. Also, for every $x \in [-1, 1]$, $|y| \leq 1$. Therefore f is a function. $\square$

4a

WTS. $f(f^{-1}([-4,-1]\cup[1,4])) = [1,4]$, *where* $f = x^{-2}$ *for every* $x \neq 0$.

Proof. Write $W = f^{-1}([-4,-1]\cup[1,4])$, and because the inverse image preserves intersections and unions by Q5, and $[-4,-1]\cap\{f(x) : x \in \mathbb{R}\setminus\{0\}\} = \varnothing$, then $f^{-1}[-4,-1] = \varnothing$. Which means $W = f^{-1}[1,4]$ and hence $f(W) = [1,4]$, as $[1,4] \subseteq \{f(x) : x \in \mathbb{R}\setminus\{0\}\}$. $\square$

4b

WTS. $f^{-1}(f(1,2)) = [-2,-1] \cup [1,2]$. *Where* $f = x^{-}2$, *for every* $x \neq 0$.

Proof. The equality is obvious by inspection. $\square$

4cd

WTS. *$f(f^{-1}B) = B$ if f is a surjection, and $f^{-1}(f(B)) = B$ if f is an injection.*

We split this problem into two parts. We begin with the first assertion. Write $R = \{f(x) : x \in A\}$.

Lemma 0.1. *For every function $f : X \to Y$, $f(f^{-1}(B)) \subseteq B$.*

Proof. Use Q5a) onto the disjoint sets $f^{-1}(B \cap R)$ and $f^{-1}(B \cap R^c)$, then

$$f^{-1}(B) = f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)$$

Now $f^{-1}(B \cap R^c)$ must be empty, since no $x \in A$ satisfies $f(x) \in B \cap R^c$. Hence $f^{-1}(B) = f^{-1}(B \cap R)$.

$$\begin{aligned} f(f^{-1}(B)) &= f(f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \cup f(f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \\ &= \{f(x) : x \in f^{-1}(B \cap R)\} \\ &= \{y : y \in (B \cap R)\} \\ &= B \cap R \end{aligned}$$

Where for the second last equality we used the fact that f is always a surjection onto its range. Then $f(f^{-1}(B)) = B \cap R \subseteq B$. □

Remark. *If f is a surjection, then its range $R = Y$, then $f(f^{-1}(B)) = B \cap Y = B$.*

Lemma 0.2. *For every function $f : X \to Y$, $A \subseteq f^{-1}(f(A))$.*

Proof. Write $f^{-1}(f(A))$ as the disjoint union of $A \cap f^{-1}(f(A))$ and $A^c \cap f^{-1}(f(A))$. Then, we shall show that $f^{-1}(f(A)) = A$. For every $x \in A$,

$$\begin{aligned} f(x) \in f(A) \wedge x \in A &\iff x \in f^{-1}(f(A)) \wedge x \in A \\ &\iff x \in A \cap \left(f^{-1}(f(A))\right) \end{aligned}$$

Hence $A \cap f^{-1}(f(A)) = A$, and $A \subseteq f^{-1}(f(A))$

Remark. *If f is a injection, then for every $x \in A^c$, $f(x) \notin f(A)$, then $A^c \cap f^{-1}(f(A)) = \varnothing$, and*

$$f^{-1}(f(A)) = \left[A \cap f^{-1}(f(A))\right] \cup \left[A^c \cap f^{-1}(f(A))\right] = A$$

□

5a

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.*

Proof. Fix two subsets $B_1, B_2 \subseteq B$, then

$$\begin{aligned}
f^{-1}(B_1 \cup B_2) &= \{x \in A, f(x) \in B_1 \cup B_2\} \\
&= \{x \in A, f(x) \in B_1 \text{ or } f(x) \in B_2\} \\
&= \{x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)\} \\
&= f^{-1}(B_1) \cup f^{-1}(B_2)
\end{aligned}$$

□

5b

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.*

Proof. For any two sets $A_1, A_2 \subseteq A$,

$$\begin{aligned} f(A_1 \cup A_2) &= \{f(x) : x \in A_1 \cup A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } x \in A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } f(x) : x \in A_2\} \\ &= f(A_1) \cup f(A_2) \end{aligned}$$

$\square$

5c

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.*

Proof.

Lemma 0.3. *f^{-1} preserves complements.*

Proof. For every $E \subseteq B$,

$$\begin{aligned} f^{-1}(B \setminus E) &= \{x \in A : f(x) \in B \setminus E\} \\ &= \{x \in A, f(x) \in E^c\} \\ &= A \setminus f^{-1}(E) \end{aligned}$$

□

Lemma 0.4. *f^{-1} preserves intersections.*

Proof. Now we wish to prove that f^{-1} preserves intersections as well, for every pair of subsets, $B_1, B_2 \subseteq B$. Write their intersection as $(B_1^c \cup B_2^c)^c$, apply Q5a, and take complements.

$$\begin{aligned} f^{-1}\left((B_1^c \cup B_2^c)^c\right) &= (f^{-1}(B_1^c) \cup f^{-1}(B_2^c))^c \\ &= f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

□

To prove the assertion in Q5c, write $B_1 \setminus B_2 = B_1 \cap B_2^c$, and apply the two Lemmas. □

5d

WTS. *Provide an example such that $f(A_1 \setminus A_2) \neq f(A_1) \setminus f(A_2)$. Provide a condition on f that implies $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$.*

We begin our answer with the example. Suppose $f \in L^{p*}$, where f maps every element to 0. Take two subsets of $A \subseteq L^p$, $A_1 = \{g_1\} \neq \{g_2\} = A_2$. Then $f(A_1) \setminus f(A_2) = \varnothing$, but $A_1 \setminus A_2 = A_1$, and $f(A_1 \setminus A_2) = \{0\} \neq \varnothing$.

The condition we want to impose on f is that it must be an injection, we will prove that it satisfies the assertion.

Proof.

Lemma 0.5. *The direct image is monotonic. For every $E_1 \subseteq E_2 \subseteq A$, then $f(E_1) \subseteq f(E_2) \subseteq B$.*

Proof. Apply Q5b) to sets $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$, then $f(E_2) = f(E_2 \setminus E_1) \cup f(E_2 \cap E_1)$ implies that $f(E_1) \subseteq f(E_2)$. $\square$

Lemma 0.6. *For every pair of subsets, $E_1, E_2 \subseteq A$, then $f(E_1) \setminus f(E_2) \subseteq f(E_1 \setminus E_2)$.*

Proof. If the left member is empty, then it is trivial. If not, then fix an element $y \in f(E_1) \setminus f(E_2)$, then $y \in f(E_1)$ and $y \in f(E_2)^c$.

This is equivalent to saying that there exists a $x_1 \in E_1$ such that $f(x_1) = y$; and for every $x_2 \in E_2$, $f(x_2) \neq y$, and therefore x_1 is not a member of E_2, since f is a function. It follows that $x_1 \in E_1 \setminus E_2$, and $f(x_1) = y \in f(E_1 \setminus E_2)$. Since y is arbitrary, we are done. $\square$

Suppose f is an injection, then for every $x \neq p \in A$ implies that $f(x) \neq f(p)$. We wish to prove the reverse estimate in the second Lemma. Fix a member in $y \in f(E_1 \setminus E_2)$, then this y induces an $x \in E_1 \setminus E_2$. Since the two sets $(E_1 \setminus E_2)$ and E_2 are disjoint, for every $p \in E_2$, $x \neq p$ yields $f(x) \neq f(p)$; and $f(x) = y \in f(E_2)^c$. But this y is also a member of $f(E_1)$ by the first Lemma, if we simply take $E_1 \setminus E_2 \subseteq E_1 \subseteq A$. Therefore $y \in f(E_1) \setminus f(E_2)$. This completes the proof. $\square$

6a

WTS. *Show that $f(x) = x/(|x| + 1)$ is a bijection from $\mathbb{R}$ to $(-1, +1)$.*

Proof. We begin with an important Lemma.

Lemma 0.7. *For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|_A + f|_{A^c}$, and Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restriction of f onto A and A^c are bijections onto their direct images, then f is a bijection.*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where we shall omit the trivial case of them both belonging to the same A or A^c. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by assumption $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f|_{A^c}(A^c)$. So $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in f_A(A)$ or $y \in f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets are bijections, there exists a corresponding $x \in X$ which will satisfy. This completes the proof. □

Now we want to prove the original assertion. To use the lemma, take $A = [0, +\infty) \subseteq \mathbb{R}$. We will satisfy the assumptions of the Lemma. First, for any $x \in A$, $f(x) = 1 - 1/(x + 1)$. Injectivity is obvious at first glance, and we claim that $f|_A(A) = [0, 1)$. To show that $f|_A(A) \subseteq [0, 1)$, notice

$$f|_A = 1 - \frac{1}{x+1} \geq 0, \quad \forall x \in [0, +\infty)$$

$$f|_A \geq 1 \implies 1 - \frac{1}{x+1} \geq 1 \implies x \leq -1 \implies x \in A^c$$

Then $f|_A(A) \subseteq [0, 1)$ as required. Now to show the converse, fix any $y \in [0, 1)$, then there exists an $x = (1 - y)^{-1} - 1 \in A$. Thus we have proven that $f|_A$ is a bijection onto its direct image.

Next for $f|_{A^c}(x) = -1 + 1/(1 - x)$ for any $x \in A^c$. It is trivial to show that $f|_{A^c}$ is an injection. So, fix any $y \in (-1, 0)$ and there corresponds an $x = 1 - (y + 1)^{-1} \in A^c$. Hence $(-1, 0) \subseteq f|_{A^c}(A^c)$. To show the reverse, we will proceed by contradiction. So suppose there exists an $x \in A^c$ such that $f|_{A^c}(x) \geq 0$, which means that $f|_{A^c}(x) \in A$, then a cool way to arrive at a contradiction this would be to plug $y = f|_{A^c}(x)$ into $f|_A(y) \in [0,1)$, hence we have

$$\begin{aligned} f|_A(y) &= y/(y+1) \\ &= \frac{x/(1-x)}{x/(1-x)+1} \\ &= x \in [0,1) \end{aligned}$$

But $x \in A^c$ by assumption, so we have a contradiction. Suppose now, there exists an $x \in A^c$ such that $f|_{A^c}(x) \leq 1$, then

$$\begin{aligned} \frac{x}{1-x} &\leq 1 \\ -x/(1-x) &\geq 1 \\ 1 - 1/(1-x) &\geq 1 \\ 1/(1-x) &\leq 0 \\ 1 &\leq x \end{aligned}$$

And the contradiction establishes the bijection. Since $f|_A(A) = [0,1)$, and $f|_{A^c}(A^c) = (-1,0)$. $Y = (-1,1)$ is the disjoint union of these two sets, we can finally apply the Lemma, and the proof is complete. $\square$

6b

WTS. *Show that*

$$f(x) = (x+1)(m/2) + a$$

induces a bijection from $(-1, 1) \to (a, b)$ *for every* $m = b - a > 0$.

Proof. Since $m \neq 0$, f is obviously injective. And for every $y \in (a, b)$, one can easily find an

$$x = (y-a)(2/m) + (-1) \in (-1, 1)$$

To show that $f \in (a, b)$, we can attempt the contrapositive. $f \leq a$ or $f \geq b$ implies $|x| \geq 1$.

$$\begin{aligned} |f(x) - (a+b)/2| &\geq m/2 \\ |(x+1)(m/2) + 2a/2 - (a+b)/2| &\geq m/2 \\ |(x+1)m + 2a - a - b| &\geq m \\ |x| &\geq 1 \end{aligned}$$

This establishes the bijection. $\square$

MATH 254: Assignment 2

November 5, 2022

Problem 1

WTS. *Let* $f : \mathbb{R} \to [0, +\infty)$ *and* $g : [0, +\infty) \to (-\infty, 0]$ *be the functions defined by* $f(x) \coloneqq x^2$ *and* $g(x) \coloneqq -\sqrt{x}$*.*

(a) Explain why $f \circ g$ *makes sense (can be defined) even though the codomain of* g *is not equal to the domain of* f*.*

(b) Write down the domains and codomains of $f \circ g$ *and* $g \circ f$ *and find explicit formulas for these functions.*

(c) Is g *the inverse of* f*? Explain your answer.*

Answers:

(a) Fix any $x \in [0, +\infty)$, then $g(x) \in (-\infty, 0]$ means that $f(g(x)) = f \circ g(x)$ is well defined. Since x is mapped to exactly one element in $[0, +\infty)$. What matters here is that $\operatorname{range} g \subseteq \operatorname{dom} f$.

(b) Domains:
$\operatorname{dom}(f \circ g) = \operatorname{dom} g = [0, +\infty)$, and
$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$.

Codomains:
$\operatorname{codom}(f \circ g) = \operatorname{codom} f = [0, +\infty)$, and
$\operatorname{codom}(g \circ f) = \operatorname{codom} g = (-\infty, 0]$.

Formulas:

$$f \circ g(x) = (-\sqrt{x})^2 = |x| = x, \text{ For every } x \geq 0$$

$$g \circ f(x) = -\sqrt{x^2} = -|x| \neq x, \text{ For every } x \in \mathbb{R}$$

(c) g is not the inverse of f, fix $x = 1$, then $g \circ f(x) = -1 \neq 1$. Hence $g \circ f \neq \mathrm{id}_{\mathbb{R}}$.

Remark. *I do not know what the convention for the codomain is for this class. Here I assumed that for any function $f : X \to Y$, $\operatorname{codom} f = Y$. Its range however is the set of points in its codomain that it reaches, so $\operatorname{range} f = \{f(x), x \in X\}$.*

Problem 2

WTS. *Let $f : [0,4) \to [0,4)$ be defined by*

$$x \mapsto \begin{cases} x+1 & \lfloor x \rfloor \in 2\mathbb{N} \\ x-1 & \lfloor x \rfloor \notin 2\mathbb{N} \end{cases}$$

Where $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$. Show that f is a bijection, and describe its inverse.

The proof for this question will resemble that of Homework 1 Q6(a). A few important lemmas must be stated.

Lemma 0.1. *For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|_A + f|_{A^c}$, and Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restriction of f onto A and A^c are bijections onto their direct images, then f is a bijection.*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where we shall omit the trivial case of them both belonging to the same A or A^c. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by assumption $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f|_{A^c}(A^c)$. So $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in f_A(A)$ or $y \in f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets are bijections, there exists a corresponding $x \in X$ which will satisfy. This completes the proof. $\square$

Lemma 0.2. *Let f satisfy the hypothesis of the previous lemma, so that $(f|_A)^{-1}$ and $(f|_{A^c})^{-1}$ both exist, then $f^{-1} = (f|_A)^{-1} + (f|_{A^c})^{-1} = (f^{-1})|_{B_1} + (f^{-1})|_{B_2}$, where $f|_A(A) = B_1$, and $f|_{A^c}(A^c) = B_2$.*

Proof. Since B_1 and B_2 are disjoint, then fix any $y \in Y$. Without loss of generality, let us assume that $y \in B_1$. Then, $f^{-1}(y) = (f^{-1})|_{B_1}(y) = (f|_A)^{-1}(y)$. This inverse is indeed well defined, since $f|_A$ is a bijection onto its range, then there exists a unique $x \in A$ such that applying f on both sides yield

$$f\left((f|_A)^{-1}(y)\right) = f \circ (f|_A)^{-1}(y) = y$$

In the same manner, fix an $x \in A$ such that $f(x) = f|_A(x) \in B_1$, then applying $(f|_A)^{-1}$ on both sides

$$(f|_A)^{-1}(f(x)) = (f|_A)^{-1} \circ f(x) = x$$

Therefore the inverse of f can be written piecewise on two disjoint domains as follows.

$$f^{-1} = f^{-1}|_{B_1} + f_{B_2}^{-1}$$

$\square$

Remark. *We adopt a slight abuse of notation with the 'restrictions' onto f, but they should be interpreted as piecewise functions. $f|_A + f|_{A^c}$ is equal to $f|_A\chi_A + f|_{A^c}\chi_{A^c}$ where χ is the indicator function.*

We begin the main part of the proof.

Proof. For every $x \in [0,1) \cup [2,3)$, $\lfloor x \rfloor \in 2\mathbb{N}$. Denote $A = [0,1) \cup [2,3)$, and $A^c = [0,4) \setminus A = [1,2) \cup [3,4)$. Then

$$f = f|_A + f|_{A^c} = (x+1)\chi_A + (x-1)\chi_{A^c}$$

To satisfy the assumptions of the two lemmas, we need to check if the direct images (ranges) of $f|_A$ and $f|_{A^c}$ are disjoint. This can be easily shown by

$$x \in A \iff x \in [0,1) \cup [2,3) \implies x+1 \in [1,2) \cup [3,4)$$

$$x \in A^c \iff x \in [1,2) \cup [3,4) \implies x+1 \in [0,1) \cup [2,3)$$

To avoid being overly verbose, we say that the two functions are bijections onto their ranges. Since $f|_A$ 'nudges' points $+1$ units to the right, while $f|_{A^c}$ does the exact opposite. And the range of each of the two functions is the complement of their domains. Even more is true:

$$(f|_A)^{-1} = x - 1 = f|_{A^c}$$

$$(f|_{A^c})^{-1} = x + 1 = f|_A$$

Therefore $f|_A$ and $f|_{A^c}$ are bijections onto their ranges (which are disjoint). Invoking the first lemma tells us that f is a bijection, and the second lemma gives us the following equality.

$$f^{-1} = f^{-1}|_{A^c} + f^{-1}|_A = (f|_A)^{-1} + (f|_{A^c})^{-1}$$

Plugging in our values for the inverses, we have

$$f^{-1} = f|_A + f|_{A^c} = f$$

$\square$

Problem 3

WTS. *Now it suffices to show that if $f_1 : A \to A'$ and $f_2 : B \to B'$ are bijections, then*

$$F : A \times B \to A' \times B'$$

is also a bijection if we define $\pi_{A'}(F(x)) = f_1(\pi_A(x))$ and $\pi_{B'}(F(x)) = f_2(\pi_B(x))$ for every $x \in A \times B$. Where π denotes the coordinate map (or projection map).

Proof. Fix two elements x_1, x_2 in $A \times B$, such that $x_1 \neq x_2$. Without loss of generality, let us assume that $\pi_A(x_1) \neq \pi_A(x_2)$, which implies

$$\pi_{A'}(F(x_1)) = f_1(\pi_A(x_1)) \neq f_1(\pi_A(x_2)) = \pi_{A'}(F(x_2))$$

Which means $F(x_1) \neq F(x_2)$. This proves injectivity.

To show that F is a surjection, fix any $y \in A' \times B'$, then this y induces a and b such that

$$a = f_1^{-1}(\pi_{A'}(y)) \in A$$
$$b = f_2^{-1}(\pi_{B'}(y)) \in B$$

Then denote an element of $A \times B$, and call it x, such that $\pi_A(x) = a$ and $\pi_B(x) = b$. Then it is an easy exercise to verify that $F(x) = y$, by taking the A' and B' projections of $F(x)$.

Remark. *We implicitly defined 'equality' in the Cartesian (Direct) product by equality in each coordinate. So for every x_1, x_2 in $A \times B$, $x_1 = x_2$ if and only if $\pi_A(x_1) = \pi_A(x_2)$ and $\pi_B(x_1) = \pi_B(x_2)$.* $\square$

Problem 4

WTS. *Prove that every subset of $\mathbb{N}$ is countable. Conclude that if A has an injection into $\mathbb{N}$, then A is countable.*

Proof. Fix any $A \subseteq \mathbb{N}$, then if A is finite, then denote $|A| \coloneqq \sum_{a \in A} 1 < +\infty$, then $J_n = \{k \in \mathbb{N}, k < n\}$, then there exists a bijection between $J_{|A|}$ and A. Hence A is countable (countably finite). We also note that there exists an order preserving map between the two sets, namely $Y_A(0)$ denotes the least element in A, etc.

Now suppose that A is infinite. Since $\mathbb{N}$ is countable, there exists a map, $\mathbf{X}_A : \mathbb{N} \to A$ such that for every $n \geq 1$.

$$\mathbf{X}_A(n) \coloneqq \operatorname{least} \{k \in A, \mathbf{X}_A(n-1) < k\}$$

Where we also define $\mathbf{X}_A(0) = \operatorname{least} A$. Where we used the Well-Ordering Property of $\mathbb{N}$ twice, it is trivial to check that both of these sets are non-empty.

The map $\mathbf{X}$ is monotonic (and therefore an injection). A simple proof by induction will show this. Now, $\mathbf{X}_A(0) < \mathbf{X}_A(1)$ by inspection, and assume that $\mathbf{X}_A(n-1) < \mathbf{X}_A(n)$ for some $n \geq 1$, then

$$\mathbf{X}_A(n+1) = \operatorname{least} \{k \in A, \mathbf{X}_A(n-1) < \mathbf{X}_A(n) < k\}$$

Hence $\mathbf{X}_A(0) < \mathbf{X}_A(1) < \cdots < \mathbf{X}_A(n)$. Suppose that X_A is not a surjection, then there exists an non-empty set of elements of A that escape the range of $\mathbf{X}_A$. Take the least of this set, and call it m. Where this $m \neq \operatorname{least} A$, since $\mathbf{X}_A(0) = \operatorname{least} A$.

We then construct another subset of A and call it A^* which holds all the elements $k \in A$ such that $k < m$. Since A^* is obviously finite, we can use the same construct as shown in the earliest part of this proof. Define $N = |A^*|$, then we will show $\mathbf{X}_A(N) = m$.

$\mathbf{X}_A(n) \in A^* \iff 0 \leq n < N$. Since all the elements within A^* are strictly less than every element in A, and the number of elements within A^* is exactly N, and $\mathbf{X}_A(n)$ for $0 \leq n \leq N-1$ gives us the N smallest values in A. For any $n \geq N$, $\mathbf{X}_A(n) > q$ where q is any element in A^*. Hence $\mathbf{X}_A(n) \in A \setminus A^*$ for every $n \geq N$. Now, fix $n = N$, and this will retrieve the least element in $A \setminus A^*$, and this establishes the fact that every infinite subset of $\mathbb{N}$ is countably infinite.

$\square$

Moreover, the empty set is a subset of the natural numbers and is also countable. Now for the second part of the proof, if A is an injection into $\mathbb{N}$, then A must be countable.

Proof. Denote the injection of A into $\mathbb{N}$ by h, and its range by $\operatorname{range}(h)$, then since the range of any mapping is always a subset of its co-domain, $\operatorname{range}(h) \subseteq \mathbb{N}$, and h is actually a bijection onto its range (any mapping is a surjection onto its range), so

$$A \equiv \operatorname{range}(h)$$

But $\operatorname{range}(h)$ is a countable subset of $\mathbb{N}$, and this completes the proof. $\square$

Problem 5

WTS. *Prove that for each $n \in \mathbb{N}^+$,*

$$\sum_{j \leq n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

Where $j \leq n$ implicitly means that $1 \leq j \leq n$.

Proof. We will proceed by a proof by induction. Fix $n = 1$, and equality is trivial. Then suppose the assertion holds for a certain $n \geq 1$. Then

$$\begin{aligned}
\sum_{j \leq n+1} \frac{1}{j(j+1)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+1)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+1+1}
\end{aligned}$$

Where the second last equality is permissible because $n+1 \neq 0$ for every $n \geq 1$. This completes the proof. $\square$

Problem 6

WTS. *For every $n \geq 12$, $n \in \mathbb{N}$, there exists non-negative integers a, b such that*

$$n = (4, 5) \cdot (a, b)^T$$

Proof. I do not know how to show this rigourously for I lack training in Algebra, but regardless: For every $n \geq 12$, $(n - 12) \mod 4 \in [0, 3]$. Let us agree to define for each $j \in [0, 3]$.

$$W_j = \{4k + 12 + j, k \in \mathbb{N}^+ \cup \{0\}\}$$

Now these sets are not necessarily disjoint, but it will not matter for the purposes of this exercise, as every $n \geq 12$ must be contained in at least one of these sets. We also write

$$\lambda : (a, b) \mapsto (4, 5) \cdot (a, b)^T$$

Then we can write

- $12 = \lambda(3, 0)$
- $13 = \lambda(2, 1)$
- $14 = \lambda(1, 2)$
- $15 = \lambda(0, 3)$

For any fixed j, note that $(12+j) \in \lambda(\mathbb{N} \times \mathbb{N})$ as shown before. Now suppose that $(4k + (12 + j))$ is a member of $\lambda(\mathbb{N} \times \mathbb{N})$, then this induces some (a, b) such that $\lambda(a, b) = (4k+(12+j))$. But adding 4 to both sides of the equation, and by linearity in both arguments of the inner product over $\mathbb{R}$, we have

$$(4(k + 1) + (12 + j)) = \lambda(a + 1, b) \implies (4(k + 1) + (12_j)) \in \lambda(\mathbb{N} \times \mathbb{N})$$

Hence $W_j \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. But the union of all W_j contains every $n \geq 12$. Hence $\{n \in \mathbb{N}, n \geq 12\} \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. This completes the proof. $\square$

Problem 7

WTS. *A $n \times m$ grid always takes $nm - 1$ cuts to be decomposed into atomic cells.*

Proof. I am not sure why we need induction here. Take a finite set of pieces and call it $W^N = \{w_j, 1 \leq j \leq N\}$, then if $|W| = \sum_{w \in W} 1$, and if $|W^N| = N \neq nm$, then there exists a cut you can make on one of its pieces. Without loss of generality assume that this piece is $w_N \in W^N$, then $\operatorname{cut} w_N = \{a, b\}$, where a and b are pieces (atomic or not), then write the new state of the grid as

$W^{N+1} = W^N \cup \operatorname{cut}(w_N) \setminus \{w_N\}$, and relabelling indices, then we have $N+1$ elements in our new set.

Since the cutting process is complete if and only if $|W| = nm$, and the original state of the grid is at W^1, and each cut increases the number of elements in the set by 1, all it must take $nm - |W^1| = nm - 1$ cuts.

Now I guess you can shoehorn the induction in there by saying that for every $|W^n|$, $n - 1$ cuts must have been made for each and every step, but this is equivalent to the first argument I made above. $\square$

Problem 8

WTS. *Prove that for every $a < b$ in $\mathbb{R}$, the segment (a, b) is uncountable. Conclude that every segment in the form of (a, b) must contain an irrational number.*

Proof. From Lecture 4: $[0, 1)$ is uncountable. This is equivalent to saying that no $f : [0, 1) \to \mathbb{N}$ can be injective. Therefore no $f : (-1, +1) \to \mathbb{N}$ can be injective. Since if some f were to be injective from $(-1, +1)$ then it would have to be injective on $[0, 1)$.

Using the last part of Homework 1 Q6b, define a bijection f from $(-1, +1)$ to (a, b) where $a < b$.

$$f(x) = (x + 1)(m/2) + a, \quad m := b - a$$

It follows immediately that $(a, b) \equiv (-1, +1)$, which is an uncountable set. This proves the first claim.

To show the validity of the second claim, suppose not. So $(a, b) \subseteq \mathbb{Q}$, and $\mathbb{N} \equiv \mathbb{Q}$ (there exists a bijection between the two sets). So there exists an injection of (a, b) into $\mathbb{N}$, then (a, b) is countable and the proof is complete. $\square$

MATH 254: Assignment 3

November 5, 2022

Problem 1

WTS. *$\sqrt{3}$ is not rational.*

Proof. Suppose that $\sqrt{3} \in \mathbb{Q}$. Note that $\sqrt{3} \neq 0$ since $\sqrt{3} \cdot \sqrt{3} = 3 \neq 0$, and $\sqrt{3} > 0$. Then there exists a factorized form of $\sqrt{3} = a/b$ such that a and b are in $\mathbb{N}^+$ and have no common divisors other than 1. Now suppose that $a = 2m$ and $b = 2n + 1$.

$$\frac{2m}{2n+1} = \sqrt{3} \iff 2m = \sqrt{3}(2n+1) \tag{1}$$
$$\iff 4m^2 = 3(4n^2 + 4n + 1) \tag{2}$$
$$\iff (m^2 - 3n^2 - 3n) = 3/4 \in \mathbb{Z} \tag{3}$$

Where for the last assertion we used the fact that the integers are closed under addition and multiplication. And this contradiction establishes that a is odd.

To prove the fact that a cannot be odd as well, consider the following, (notice how we relabelled the coefficients), since $\sqrt{3} \neq 0$

$$\frac{2n+1}{2m} = \sqrt{3} \implies \frac{2m}{2n+1} = 1/\sqrt{3}$$

Mutiplying both sides by 3 yields

$$2(3m) = \sqrt{3}(2n+1)$$

Replace $m = 3m$ in (1), and the contradiction finishes the proof. (*) □

Remark. *In (*), consider the similarities between the above equation and (1), where m, n are arbitrary numbers in $\mathbb{N}^+$. Since no m, n in $\mathbb{N}^+$ can satisfy (1).*

Problem 2

WTS. $\mathbb{N}$ *is unbounded above.*

Proof. Consider $2 \in \mathbb{N}$. So $\mathbb{N} \neq \varnothing$, and suppose by contradiction that

$$2 \leq \sup \mathbb{N} = x < +\infty$$

Then there exists some $y \in \mathbb{N}$

$$x - 1 \leq y$$

But since $\mathbb{N}$ is closed under addition,

$$x < y + 2 \in \mathbb{N}$$

But $y + 2 > x$, and we are done. $\square$

Problem 3

WTS. *Show that* $\mathbb{Q}\sqrt{2}$ *and* $\mathbb{Q}+\sqrt{2}$ *are dense in* $\mathbb{R}$*.*

Proof. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, then fix any $\varnothing \neq (a,b) \subseteq \mathbb{R}$, if there exists some $q \in \mathbb{Q}$

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}} \implies a < q\sqrt{2} < b$$

Also,

$$a - \sqrt{2} < q < b - \sqrt{2} \implies a < q + \sqrt{2} < b$$

Since addition, subtraction, multiplication, division by $\sqrt{2} > 0$ preserves the order relation for $a < b$. This finishes the proof. $\square$

Problem 4

WTS. *We wish to show that* $\inf S = -1$ *and* $\sup S = +1$ *for*

$$S = \left\{\frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N}^+\right\}$$

We begin with a few important lemmas. For non-empty bounded subsets A, B of $\mathbb{R}$,

Lemma 0.1. *If A is a non-empty bounded above subset of $\mathbb{R}$, then* $(-1)(\sup A) = \inf(-1)A$.

Proof. Let $s = \sup(A)$, then

$$(-1)A = \{-x : x \in A\}$$

Evidently, for every $x \in A$

$$x \leq s \implies -s \leq -x \implies -s \leq (-1)A$$

Now fix any $\varepsilon > 0$, then there exists an $x \in A$ such that

$$s - \varepsilon \leq x \implies (-1)(s - \varepsilon) \geq -x \implies (-x) \leq (-s) + \varepsilon$$

Thus establishes $\inf(-1)A = (-1)\sup(A)$. □

Corollary 0.1.1. $(-1)\inf(A) = \sup(-1)A$. *The proof is trivial just replace A by $(-1)A$.*

Lemma 0.2. *If A and B are non-empty bounded above subsets of $\mathbb{R}$, then* $\sup A + \sup B = \sup(A + B)$

Proof. Define $s = \sup A$ and $t = \sup B$, then for every $(a, b) \in A \times B$

$$a \leq s,\ b \leq t \implies a + b \leq s + t \implies A + B \leq s + t$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq s - \varepsilon/2,\ b \leq t - \varepsilon/2 \implies s + t - \varepsilon \leq a + b$$

Therefore $\sup(A + B) = \sup(A) + \sup(B)$. □

Lemma 0.3. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. □

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Main Proof of Question 4

Now we are ready to tackle the question of Question 4

Proof. To prove Q4, define $A = \{n^{-1}, n \in \mathbb{N}^+\}$, then $\inf A = 0$ and $\sup A = 1$. Since 0 is a lower bound for A, it suffices to apply the Archimedean property for any $\varepsilon > 0$,

$$\exists n \in \mathbb{N}^+, 0 < \frac{1}{n} < 0 + \varepsilon = \varepsilon$$

Therefore $\inf A = 0$. To show $\sup A = 1$, notice that

$$n \geq 1 \implies 1/n \leq 1$$

But $1 \in A$ so applying Lemma 0.7 tells us that $\sup A = 1$.

We then construct two sets, S_1 and S_2 where $S_1 = S_2 = A$. By applying Lemma 0.1 we get

$$\sup(-S_2) = (-1)\inf(S_2) = 0$$

Now use Lemmas 0.2 (which allows us to add the supremums together) to obtain

$$\sup(S_1 - S_2) = 1 + 0 = 1$$

From here, let us turn our attention to the fact that

$$(-1)(S_1 - S_2) = (S_1 - S_2)$$

Apply Lemma 0.1 once again, then we can obtain

$$-1 = (-1)\sup\{(S_1 - S_2)\} = \inf\{(-1)(S_1 - S_2)\} = \inf\{(S_1 - S_2)\}$$

As a final step, recall that

$$S_1 - S_2 \coloneqq S = \left\{\frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N}^+\right\}$$

Therefore $\inf S = -1$ and $\sup S = +1$. $\square$

Problem 5

WTS. *Show that for every A, $B \subseteq \mathbb{Q}^+$,*

(a) $\inf A + \inf B = \inf A + B$

(b) $\inf A \inf B = \inf AB$

We will make use of two Lemmas proven in previous exercises.

Lemma 0.4. *If A and B are non-empty bounded below subsets of $\mathbb{R}$, then* $\inf A + \inf B = \inf(A + B)$

Proof. Define $w = \inf A$ and $q = \inf B$, then for every $(a, b) \in A \times B$

$$w \leq a,\ q \leq b \implies w + q \leq a + b \implies w + q \leq A + B$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq w + \varepsilon/2,\ b \leq q + \varepsilon/2 \implies a + b \leq w + q + \varepsilon$$

Therefore $\inf(A + B) = \inf(A) + \inf(B)$. $\square$

Lemma 0.5. *If A is a non-empty bounded below subset of $\mathbb{R}$, then for every* $c \geq 0, c(\inf A) = \inf cA$.

Proof. Let $w = \inf(A)$, then

$$cA = \{cx : x \in A\}$$

Then for every $x \in A$

$$w \leq x \implies cw \leq cx \implies cw \leq cA$$

If $c = 0$, then the equality is trivial since $cA = \{0\}$, if not, for every $\varepsilon/c > 0$, there exists an $x \in A$ such that

$$x \leq w + \frac{\varepsilon}{c} \implies cx \leq cw + \varepsilon$$

This establishes Lemma 0.5. $\square$

Main Proof for Question 5

Proof. Lemma 0.4 applies for all non-empty, bounded-below subsets of $\mathbb{R}$, since A and B are bounded below by $0 \in \mathbb{R}$, and we assume in good faith that neither of them are empty, then this establishes (a). If either one of them is empty, then there is nothing to prove as $A + B$ and AB will be empty as well.

Now let us prove (b). Since 0 is a lower bound for A and B,

- $0 \leq \inf_{a \in A} a$
- $0 \leq \inf_{b \in B} b$
- For every $a \in A$, $0 \leq a$
- For every $b \in B$, $0 \leq b$

Fix any member $a \in A$ such that $a \geq 0$ by Lemma 0.5

$$a \cdot \inf_{b \in B} b = \inf_{b \in B} ab$$

Taking the infimum with respect to A on both sides (equality of sets $\implies$ equality of inf)

$$\inf_{a \in A} \left(\left\{ a \cdot \inf_{b \in B} b \right\} \right) = \inf_{a \in A} \left(\left\{ \inf_{b \in B} ab \right\} \right)$$

Now since $\inf_{b \in B} b \geq 0$ we can apply Lemma 0.5 to the left member (without loss of generality, the inf commutes)

$$\inf_{b \in B} b \inf_{a \in A} a = \inf_{a \in A} \inf_{b \in B} ab$$

We claim that

$$\inf_{a \in A} \inf_{b \in B} (ab) = \inf_{(a,b) \in A \times B} (ab)$$

Let us take a step back and solve the problem in the abstract for a bit. We want to show that for any $f : X \times Y \to \mathbb{R}$ that is bounded below,

$$\inf_{a \in X} \inf_{b \in Y} f(a, b) = \inf_{(a,b) \in X \times Y} f(a, b)$$

Fix any $z = f(x, y)$, $\inf_{b \in Y} f_x(b) \leq f_x(y) = z$ where f_x and f^y denote the x and y sections of f, such that

$$f_x(y) = f(x, y) = f^y(x)$$

Then taking the inf over $a \in X$ gives

$$\inf_{a\in X}\inf_{b\in Y} f_a(b) \leq f(x,y), \quad \forall (x,y) \in X \times Y$$

So $\inf_{a\in X}\inf_{b\in Y} f_a(b)$ is a lower bound for f. To show that it is indeed the infimum, for every $\varepsilon/2 > 0$ we obtain some $x_0 \in X$ such that

$$\inf_{a\in X}\left(\left\{\inf_{b\in Y} f(a,b)\right\}\right) + \varepsilon/2 > \inf_{b\in Y} f_{x_0}(b)$$

Going through the same motion again, but this time for another $\varepsilon/2 > 0$ gives us some $y_o \in Y$

$$\inf_{b\in Y} f_{x_0}(b) + \varepsilon/2 > f(x_0, y_0)$$

Adding the two estimates together, there exists some $(x_0, y_0) \in X \times Y$ that satisfies, for every $\varepsilon > 0$

$$\inf_{a\in X}\left(\left\{\inf_{b\in Y} f(a,b)\right\}\right) + \varepsilon > f(x_0, y_0)$$

Now apply $f : A \times B \to \mathbb{R}$ with the mapping $(a,b) \mapsto a \cdot b$ (which is bounded below by 0, as the multiplication of two non-negative numbers is non-negative). This finishes the proof. $\square$

Problem 6

WTS. *Prove that if a bounded above set $S \subseteq \mathbb{R}$ contains one of its upper bounds u, then $\sup S = u$. And prove it for a bounded below set and its infimum.*

I already proved this in earlier questions as a Lemma.

Lemma 0.6. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. □

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Problem 7

WTS. *Prove that every finite subset of $\mathbb{R}$ contains its infimum and supremum.*

We need an important Lemma.

Lemma 0.7. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. $\square$

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Main Proof for Question 7

Proof. If A is empty then this is a pathological case. If we are working in the extended reals $\overline{\mathbb{R}}$ then by convention

$$\inf \varnothing = +\infty, \quad \sup \varnothing = -\infty$$

Where both $\inf A \notin A$ and $\sup A \notin A$. Now, suppose that A is non-empty and finite. Define

$$|A| = \sum_{a \in A} 1$$

Then $1 \leq |A| < +\infty$. If $|A| = 1$, then $A = \{x\}$ for some $x \in \mathbb{R}$, then this x is an upper bound and a lower bound for A. Because

- For every $y \in A$, $y \leq x$.

- For every $y \in A$, $x \leq y$.

By Lemma (0.7) since $x \in A$, then $\sup A = \inf A = x$. Now let us show for the general case, where $|A| = N < +\infty$. Because A is a finite non-empty set, there exists a bijection between A and some initial segment of the countable ordinals. Let us agree to call this bijection

$$F : A \to [0, |A| - 1] \cap \mathbb{N}$$

Denote $a = F^{-1}(0)$. Then $E_1 := \{a\}$ is non-empty and contains its supremum and infimum. Now let us assume for the purposes of induction, that E_n where $1 \leq n < |A| - 1$ such that

$$\sup E_n \in E_n, \quad \inf E_n \in E_n$$

Then take $x = F^{-1}(n+1)$, so this $x \notin E_n$, since F^{-1} is also a function. Then we will show that $E_{n+1} = \{x\} \cup E_n$ contains its supremum and infimum.

Since $\sup E_n \in E_{n+1}$ and $\inf E_n \in E_{n+1}$, if

$$x \leq \sup E_n \implies \sup E_n = \sup E_{n+1}$$

Likewise for its infimum

$$\inf E_n \leq x \implies \inf E_n = \inf E_{n+1}$$

Because $\sup E_n$ is an upper bound of E_{n+1} that is contained in E_{n+1} (likewise for $\inf E_n$). However, if

$$\sup E_n < x \implies \forall y \in E_n,\ y \leq \sup E_n < x$$

Or

$$x < \inf E_{n+1} \implies \forall y \in E_n,\ x < \inf E_{n+1} \leq y$$

Then take $\sup E_{n+1} = x \in E_{n+1}$ or $\inf E_{n+1} = x \in E_{n+1}$. As a final 'step' we note that A can be written as

$$A = E_{|A|-1} \cup \{F^{-1}(|A| - 1)\}$$

This completes the proof. $\square$

Problem 8

WTS. *Prove the Triangle Inequality with $n \geq 2$.*

Lemma 0.8. *The Triangle Inequality, for every $a, b \in \mathbb{R}$*

$$|a+b| \leq |a| + |b|$$

Proof. Notice that for every $a, b \in \mathbb{R}$

- $-|a| \leq a \leq +|a|$
- $-|b| \leq b \leq +|b|$

Add two inequalities together, then we get

$$-(|a| + |b|) \leq (a+b) \leq +(|a| + |b|)$$

Since $|a| + |b| \geq 0$, we get $|a+b| \leq |a| + |b|$ as desired. Because for every $c \geq 0$,

$$\begin{aligned} c \geq |a| &\iff c \geq \max\{-a, +a\} \\ &\iff c \geq a \text{ and } c \geq -a \\ &\iff c \geq a \text{ and } -c \leq a \\ &\iff -c \leq a \leq +c \end{aligned}$$

$\square$

0.1 Main Proof for Q8

Proof. For every x_1, x_2 in $\mathbb{R}$,

$$\left|\sum_{j=1}^{2} x_j\right| \leq \sum_{j=1}^{2} |x_j|$$

This proves the non trivial base case, if $n = 1$ then obviously $|x_1| \leq |x_1|$. We now suppose that

$$\left|\sum_{j=1}^{k} x_j\right| \leq \sum_{j=1}^{k} |x_j|$$

Then write

$$\left|\sum_{j=1}^{k+1} x_j\right| = \left|\left(\sum_{j=1}^{k} x_j\right) + (x_{k+1})\right|$$

Applying the second estimate to the right member gives us

$$\left|\sum_{j=1}^{k+1} x_j\right| \leq \left|\sum_{j=1}^{k} x_j\right| + |x_{k+1}| \leq \sum_{j=1}^{k+1} |x_j|$$

This completes the proof. $\square$

Problem 1

WTS. *An open ball $B_r(x)$, where $r \geq 0$ is equivalent to a bounded open interval.*

$$B_r(x) = \{y,\ d(x,y) < r\}$$

Proof. A bounded open interval (a,b) where $a,b \in \mathbb{R}$ is defined

$$\begin{aligned}
(a,b) &= \{y,\ a < y < b\} \\
&= \{y,\ m-b < y-m < b-m,\ m = (a+b)/2\} \\
&= \{y,\ |y-m| < b-m,\ m = (a+b)/2\} \\
&= \{y,\ d(y,m) < \max(b-m,0)\} \\
&= B_{\max(b-m,0)}(m)
\end{aligned}$$

Fix any ball centered at $m \in \mathbb{R}$ with radius $r \geq 0$, then write $b = m+r$. The $\max(b-m,0)$ is to ensure that the equivalent radius of the ball does not go negative, if $a > b$ then the open interval is equivalent to $(a,a) = B_0(m)$. $\square$

Problem 2

WTS. *Every metric space is T_2.*

Proof. Fix two elements $x \neq y$ then by definition of the metric $d(x, y) = 2r > 0$, then fix two open (balls) sets $V(x, r)$ and $V(y, r)$, for every element $z \in V(x, r)$ we have

$$d(x, y) \leq d(z, x) + d(z, y) \implies 2r < r + d(z, y)$$

Clearly this means that $V(x, r) \subseteq V^c(y, r)$, and $V(x, r) \cap V(y, r) = \varnothing$. $\square$

Problem 3

WTS. *Let x_n be a sequence of real numbers, and*

1. *Prove that (i) is always true.*
2. *Prove that (iv) $\iff$ (iii).*
3. *Prove that (iii) $\implies$ (ii) $\implies$ (i)*
4. *Give examples of sequences that satisfy: (iii), (ii) but not (ii), and do not satisfy (ii)*

Where (i) to (iv) are given by

(i) $\forall n \exists M |x_n| \leq M$

(ii) $\exists M \exists^{\infty} n |x_n| \leq M$

(iii) $\exists M \forall^{\infty} n |x_n| \leq M$

(iv) $\exists \forall n |x_n| \leq M$

We begin by first taking the problem apart in the abstract. This is encompassed in the following lemma.

Lemma 0.1. *If P is a proposition on the space of all sequences (onto, into? what is the right word) X, denoted by*

$$\Omega = \{x_n : \mathbb{N} \to X\}$$

And if

- $P(\forall n \geq 0) = \{x_n \in \Omega, \, \forall n \geq 0, P(x_n)\}$
- $P(\forall^{\infty} \, n) = \{x_n \in \Omega, \, \exists N, \, \forall n \geq N, P(x_n)\}$
- $P(\exists^{\infty} \, n) = \{x_n \in \Omega, \, \forall N, \, \exists n \geq N, P(x_n)\}$

Then

$$P(\forall n \geq 0) \subseteq P(\forall^{\infty} n) \subseteq P(\exists^{\infty} n) \subseteq \Omega$$

Proof. Suppose that $x_n \in P(\forall n \geq 0)$, then $P(x_n)$ eventually is trivial, so

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n)$$

Now fix $x_n \in P(\forall^\infty n)$, this induces some $N \in \mathbb{N}$ such that for every $n \geq N$ means that $P(x_n)$. To show that $P(x_n)$ frequently, notice for every $M \in \mathbb{N}$ we can choose some $n = M + N$ such that $P(x_n)$ holds, so

$$P(\forall^\infty n) \subseteq P(\exists^\infty n)$$

The last inclusion is obvious as all three are subsets of Ω. $\square$

Problem 3a

Proof. For any sequence of reals x_n, and for every $n \in \mathbb{N}$, since the set of natural numbers is unbounded above, there exists some $M_n \in \mathbb{N}$ such that $|x_n| \leq M$. $\square$

Problem 3b

Proof. We define the proposition P on the reals

$$P(\alpha) \iff \exists M \in \mathbb{N}, \ |\alpha| \leq M$$

Then by Lemma 0.1,

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \text{ means that (iv)} \implies \text{(iii)}$$

To prove the converse, suppose that (iii) holds, then there exists some $M \in \mathbb{N}$ such that $|x_n| \leq M$ for all $n \geq N$, then let

$$\overline{M} \geq M + \sum_{n \leq N} |x_n|$$

Where we apply the Archimedean Property on the right member to obtain some $\overline{M} \in \mathbb{N}$. Then it is easy to verify that $|x_n| \leq \overline{M}$ for every $n \geq 0$, and (iii) $\implies$ (iv). And therefore (iii) $\iff$ (iv). $\square$

Problem 3c

Proof. Using Lemma 0.1, since

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \subseteq P(\exists^\infty n) \subseteq \Omega$$

Therefore

$$\text{(iv)} \implies \text{(iii)} \implies \text{(ii)} \implies \text{(i)}$$

$\square$

Problem 3d

Proof. Here are the sequences

1. $x_n = 0$ for every $n \geq 0$, then $x_n \in P(\forall n \geq 0) \subseteq P(\forall^\infty n)$. Since $|x_n| \leq 0$.
2. $x_n = n \cdot (1 + (-1)^n)/2$, so that $x_n \in P(\exists^\infty n) \setminus P(\forall^\infty n)$. Since x_n visits 0 frequently at odd n but grows unbounded at even n.
3. $x_n = n$ is in $P(\exists^\infty n)^c$. Since for every natural M, we can choose some $N = M + 1$ such that for every $n \geq N$ implies that $x_n > M$.

$\square$

Problem 4

WTS. *Show that*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$$

Proof. Assume in good faith that we are indexing the sequence by $n \in \mathbb{N}^+$ so that $n \geq 1$. Then for every fixed $n \geq 1$,

$$0 \leq x_n = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

Now fix any $\varepsilon > 0$, and find a large M with $M \geq \varepsilon^{-1/2}$. Then it follows that for every $n \geq M$,

$$\frac{1}{\sqrt{\varepsilon}} \leq M \leq n \implies \frac{1}{n^2} < \varepsilon$$

It immediately follows that

$$|x_n - 0| = \left|\frac{1}{n} - \frac{1}{n+1} - 0\right| < \left|\frac{1}{n^2}\right| < \varepsilon$$

And $|x_n - 0| < \varepsilon$ eventually. □

Problem 5

WTS. *Show that* $x_n = \dfrac{n}{\sqrt{n+5}}$ *diverges.*

We begin with an important Lemma.

Lemma 0.2. *Every convergent sequence in* $\mathbb{R}$ *is bounded.*

Proof. Fix $\{x_n\}_{n\geq 1} \to x$, also fix $\varepsilon = 1 > 0$, then there exists some $N \geq 0$ so large with

$$d(x_n, x) \leq 1 \quad \forall n \geq N$$

Then for every $n \geq N$ we have

$$d(x_n, 0) \leq d(x_n, x) + d(x, 0) \leq 1 + d(x, 0)$$

Now for every $n \geq 1$, obviously $d(x_n, 0) \leq 1 + d(x, 0) + \sum_{k\leq N} d(x_k, 0)$, and this establishes the Lemma. $\square$

Main Proof for Q5

Proof. We simply have to show that x_n is not bounded. Indeed, for any $M \geq 0$, take $n > 2M^2 + 5$ and $|x_n| > M$. The reasoning is as follows, if for every $n > 2M^2 + 5$, then

$$(n/2)^{1/2} > |M| = M$$

Also note that $n > 5 \implies 2n > n + 5$ therefore

$$(n+5)^{-1} > (2n)^{-1} \implies n(n+5)^{-1/2} > n(2n)^{-1/2}$$

And

$$x_n = |x_n| > n(2n)^{-1/2} = (n/2)^{1/2} > M$$

$\square$

Problem 6

WTS. *Let $\binom{n}{k}$ denote the number of k-element subsets of an n-element set. Prove that if $n < k$, then $\binom{n}{k} = 0$, and that*

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

Proof. We begin with some abstract notation.

- $J_n = \{1, 2 \cdots, n\} \subseteq \mathbb{N}^+$,
- $|E| := \sum_{x \in E} 1$, the counting measure on E.
- X is any set where $|X| \geq 2$,
- $A \subseteq X$, $|A||A^c| \neq 0$, this implicitly means that neither set is empty,
- $\Omega_n = \{f : J_n \to X\}$,
- For every $f \in \Omega_n$, $f_{J_{n-1}}$ denotes the restriction of f onto J_{n-1}

With these definitions, it is clear that $\binom{n}{k}$, for every $n, k \in \mathbb{N}$,

$$\binom{n}{k} = \left|\left\{f \in \Omega_n,\ |f^{-1}(E)| = k\right\}\right|$$

Clearly, if $f \in \Omega_{n+1}$ and $|f^{-1}(E)| = k + 1$,

- If $f^{-1}(E) \cap J_{n+1} \setminus J_n = \varnothing$, then $|f^{-1}_{J_n}(E)| = |f^{-1}(E)| = k + 1$,
- If $f^{-1}(E) \cap J_{n+1} \setminus J_n \neq \varnothing$, then $|f^{-1}_{J_n}(E)| = k$,

Then we can write $E_1 = \left\{f \in \Omega_{n+1},\ |f^{-1}(E)| = k + 1\right\}$ as the disjoint union of $E_2 = \left\{f \in \Omega_{n+1},\ f^{-1}(E) \cap J_{n+1} \setminus J_n = \varnothing,\ f^{-1}_{J_n}(E)| = k + 1\right\}$ and $E_3 = \left\{f \in \Omega_{n+1},\ f^{-1}(E) \cap J_{n+1} \setminus J_n \neq \varnothing,\ |f^{-1}_{J_n}(E)| = k\right\}$.

Also note that $E_2 \equiv \{f \in \Omega_n, |f^{-1}(E)| = k{+}1\}$ and $E_3 \equiv \{f \in \Omega_n.|f^{-1}(E)| = k\}$. Since every $f \in E_2$ induces some $g \in \Omega_n$ with $|g^{-1}(E)| = k+1$, and respectively for $f \in E_3$. And for every $g \in \Omega_n$, $|g^{-1}(E)| = k+1$, there is a corresponding $f \in E_2$ with $f_{J_n} = g$.

Therefore $|E_2| = \binom{n}{k+1}$ and $|E_3| = \binom{n}{k}$. Since $|\cdot|$ is just the counting measure on finite sets, and E_1 is the disjoint union, it follows that

$$|E_1| = \binom{n+1}{k+1} = |E_2| + |E_3| = \binom{n}{k+1} + \binom{n}{k}$$

$\square$

Problem 7

WTS. *Prove three things*

(a) The Binomial formula, for every $n \in \mathbb{N}$*,* $a, b \in \mathbb{R}$

$$(a+b)^n = \sum_{k\geq 0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

(b) The Generalized Bernoulli inequality, for every $n \in \mathbb{N}^+$*,* $k \leq n$

$$(1+b)^n \geq 1 + \binom{n}{k} b^k$$

(c) A special case of the Generalized Bernoulli inequality, for any $b \geq 0$*,* $n \in \mathbb{N}^+$

$$(1+b)^n \geq 1 + \frac{n(n-1)}{2} b^2$$

Proof. We begin by showing that (a) $\implies$ (b). For every $n \geq 1$, we have

$$(1+b)^n = \sum_{j\geq 0}^{n} \binom{n}{j} b^j = 1 + \binom{n}{k} b^k + \sum_{j\geq 1,\, j\neq k}^{n} \binom{n}{j} b^j$$

Since $\binom{n}{k} b^j \geq 0$, (b) holds.

Now to show that (b) $\implies$ (c), simply substitute $k = 2$ if $2 \geq n$, if $n = 1$ then the inequality is trivial.

The proof for (a) also quite straight forward, if $n = 0$ then

$$(a+b)^0 = 1 = \sum_{k=0}^{0} \binom{n}{k} a^k n^{n-k} = \binom{0}{0} a^0 b^{0-0} = 1$$

Assume that (a) holds for some $n \in \mathbb{N}$, then

$$
\begin{aligned}
(a+b)^{n+1} &= \sum_{k\geq 0}^{n} \binom{n}{k}\left(a^{k+1}b^{n-k} + a^k b^{(n+1)-k}\right) \\
&= \binom{n}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n} \binom{n}{k-1}\left(a^k b^{(n+1)-k}\right) + \sum_{k\geq 1}^{n} \binom{n}{k}\left(a^k b^{(n+1)-k}\right) \\
&= \binom{n+1}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n}\left(\binom{n}{k} + \binom{n}{k-1}\right)\left(a^k b^{(n+1)-k}\right) \\
&= \binom{n+1}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n} \binom{n+1}{k}\left(a^k b^{(n+1)-k}\right) \\
&= \sum_{k\geq 0}^{n} \binom{n+1}{k} a^k b^{(n+1)-k}
\end{aligned}
$$

For the third equality we used the fact that $\binom{\alpha}{0} = \binom{\beta}{0}$ for every $\alpha, \beta \in \mathbb{N}^+$. $\square$

Problem 8

WTS. *Prove two things.*

(a) *Prove that for every* $a \in \mathbb{R}$, $a > 1$ *we have* $\lim na^{-n} = 0$

(b) *Prove that* $\lim n^{1/n} = 1$

Proof. Let us start with (a). Assume that there exists some $a > 1$ with $\lim na^{-n} \neq 0$. So that there exists some $\varepsilon > 0$ and for every $N \in \mathbb{N}$, some $n \geq N$ with

$$\varepsilon \leq |na^{-n}| \implies (1 + n(n-1)(a-1)^2/2)\varepsilon \leq a^n \varepsilon \leq n$$

Dividing by n across both sides and noting that $1/n \geq 0$,

$$\varepsilon(n-1)(a-1)^2/2 \leq 1 \implies n \leq \left(\varepsilon(a-1)^2/2\right)^{-1} + 1$$

Which is obviously false, because $\mathbb{R}$ is Archemedian. This establishes (a).

For (b), we write $x_n = n^{1/n}$, where $x_n \geq 1$ for every $n \geq 1$. Indeed, if $x_n < 1$ for some $n \in \mathbb{N}^+$ then $x_n^n < 1$ by induction on n.

By applying Bernoulli's Inequality again,

$$x_n^n = n \geq 1 + n(n-1)(1-x_n)^2/2 \implies 2/n \geq (1-x_n)^2$$

So that $(1-x_n)^2 = |1-x_n|^2 \to 0$. We claim that if any sequence $|a_n|^2 \to 0$ then $a_n \to 0$. Fix an arbitrary $\varepsilon > 0$ then

$$|a_n - 0|^2 < \varepsilon^2 \implies |a_n - 0| < \varepsilon, \exists N \forall n \geq N$$

Therefore $|1 - x_n| \to 0$, and $x_n = n^{1/n} \to 1$. $\square$

Problem 1

WTS. *Provide some divergent sequences with the following properties:*

Proof.

(a) For all $M \in \mathbb{N}$, $|x_n| \geq M$ eventually,

$$\{x_n\} = n$$

(b) x_n is unbounded, but there is some $M \in \mathbb{N}$ with $|x_n| \leq M$ frequently,

$$\{x_n\} = n(1 + (-1)^n)/2$$

(c) $\{x_n\}$ is periodic,

$$\{x_n\} = (-1)^n$$

(d) $\{x_n\}$ is bounded, but not eventually periodic.

$$\{x_n\} = 1, \text{ n is prime, 0 otherwise.}$$

$\square$

Problem 2

WTS. *Prove two things*

(a) $x_n \to x \iff |x_n - x| \to 0$,

(b) *The convergence of* $\{x_n\}$ *implies the convergence of* $\{|x_n|\}$. *But not the converse.*

It is at this point we introduce a small Lemma.

Lemma 0.1 (Super Triangle Inequality). *For any* $x, y, z \in X$, *where* X *denotes a metric space then,*

$$\Big|d(x,z) - d(y,z)\Big| \leq d(x,y)$$

Proof. From the regular Triangle Inequality,

$$d(x,z) \leq d(x,y) + d(y,z)$$
$$d(y,z) \leq d(x,y) + d(x,z)$$

The proof is complete upon isolating $d(x,y)$ in both equations. □

Proof of Part A. A sequence $\{x_n\}$ converges to x, if and only if, for every $\varepsilon > 0$, eventually for large enough n

$$|x_n - x| < \varepsilon \iff ||x_n - x| - 0| < \varepsilon$$

And the right member above is equivalent to saying that $|x_n - x| \to 0$. This completes the proof. □

Proof of Part B. Suppose $x_n \to x$ and fix a $\varepsilon > 0$, then $d(x_n, x) < \varepsilon$ eventually. Using the Super Triangle Inequality in Lemma 0.1, and substituting $z = 0$, we obtain

$$|d(x_n, 0) - d(x, 0)| \leq d(x_n, x) < \varepsilon$$

Therefore $|x_n| \to |x|$. To show that the converse does not hold, simply take $x_n = (-1)^n$. □

Problem 3

WTS. *Prove three things*

(a) Give examples of divergent sequences $\{x_n\}$, and $\{y_n\}$ such that $\{x_n y_n\}$ converges.

(b) Prove that if $\{x_n\}$ and $\{x_n y_n\}$ are both convergent, and if $\lim x_n \neq 0$, then $\{y_n\}$ is convergent.

(c) Prove that if $\{nx_n\}$ converges, then $x_n \to 0$.

Proof of Part A. Let $x_n = y_n = (-1)^n$. □

We need two lemmas for the next proof.

Lemma 0.2. *If $\{x_n\} \in \mathbb{R}$, $x_n \to x \neq 0$, then $(x_n)^{-1} \to x^{-1}$.*

Proof. If $x_n \to x \neq 0$, then x_n lies in a ball about x of radius $|x|2^{-1}$ eventually, so $x_n x \neq 0$. Moreover, using the fact that x_n lies in said ball, $x_n x \geq 0$ eventually, and

$$\begin{aligned} |x_n - x| \leq |x|2^{-1} &\iff |x_n x - x^2| \leq |x|^2 2^{-1} \\ &\iff \left| |x_n x| - |x|^2 \right| \leq |x|^2 2^{-1} \\ &\iff |x|^2 2^{-1} \leq |x_n x| \leq 3|x|^2 2^{-1} \\ &\implies |x_n x|^{-1} \leq (|x^2|2^{-1})^{-1} = M^{-1} \end{aligned}$$

Now for every $M\varepsilon > 0$, the following must hold eventually,

$$|x_n - x|M^{-1} < \varepsilon \implies \frac{|x_n - x|}{|x_n x|} \leq \frac{|x_n - x|}{M} < \varepsilon$$

Therefore,

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| < \varepsilon \to 0$$

□

Lemma 0.3. *Let $\{x_n\}$, and $\{y_n\}$ be sequences of reals, and if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.*

Proof. Since $\{x_n\}$ and $\{y_n\}$ are convergent sequences, they must be bounded. So there exists some $M_1, M_2 \in \mathbb{N}^+$ with $|y| \leq M_1$ and $|x_n| \leq M_2$. Next,

$$\begin{aligned} |x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq M_2|y_n - y| + M_1|x_n - x| \end{aligned}$$

Now fix an $\varepsilon > 0$, and there exists N_1 and N_2 so large that

$$\begin{aligned} |y_n - y| &< \varepsilon M_2^{-1} \\ |x_n - x| &< \varepsilon M_1^{-1} \end{aligned}$$

for all $n \geq N_1 + N_2$. Therefore $|x_n y_n - xy| < \varepsilon$ eventually, and this completes the proof. □

Proof of Part B. Apply the Lemmas 0.2 and 0.3, writing $\lim x_n y_n = xy$ for some y, then $y_n = x_n y_n (x_n)^{-1}$, and since $x_n \neq 0$ eventually, we can conclude that

$$x_n y_n (x_n)^{-1} = y_n \to y = xy(x)^{-1} \in \mathbb{R}$$

Therefore $\{y_n\}$ converges. □

Proof of Part C. Let $a = \lim n x_n$, and $0 = \lim n^{-1}$. Then $\{n x_n n^{-1}\}$ converges by Lemma 0.3, and

$$n x_n n^{-1} = x_n \to 0 = 0a$$

□

Problem 4

WTS. *Show that*

(a) $\dfrac{(-1)^n n}{n+1}$ *diverges,*

(b) $((n+1)^{1/2} - (n)^{1/2})n^{1/2} \to 2^{-1}$,

(c) $((n+1)^{1/2} - (n)^{1/2})n \to +\infty$,

(d) $(3\sqrt{n})^{1/2n} \to 1$,

(e) $(a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b$, *where* $0 < a < b$, *and*

(f) $(n!)^{1/n^2} \to 1$.

First we begin with an important lemma.

0.1 Lemma

Lemma 0.4. *If* $\{x_n\}$ *is a sequence in* $\mathbb{R}$, *and if* $x_{n+1}/x_n < 0$ *eventually, and if* $|x_n| \to a > 0$, *then* x_n *diverges.*

Proof. Using the fact that $|x_n| \to a$, and fix $\varepsilon = a/2 > 0$, then

$$\Big||x_n| - a\Big| < a/2 \iff a/2 < |x_n| < 3a/2$$

Using the fact that $x_{n+1}/x_n < 0$ eventually,

- either $-x_{n+1} = |x_{n+1}|$,
- or $|-x_n + x_{n+1}| = |x_n| + |x_{n+1}|$,

We have,

$$\begin{aligned} d(x_n, x_{n+1}) &= \Big|x_n - x_{n+1}\Big| \\ &= |x_n| + |x_{n+1}| \\ &> a/2 + a/2 \\ &> a \end{aligned} \tag{1}$$

Now suppose that $x_n \to x$ for some $x \in \mathbb{R}$, then for any $\varepsilon = a/2 > 0$, we must have

$$d(x_n, x_{n+1}) \leq d(x_n, x) + d(x_{n+1}, x)$$

Using Equation (1), we get

$$a < a/2 + d(x_{n+1}, x) \implies a/2 < d(x_{n+1}, x) < a/2$$

Therefore $x_n \not\to x$, and this completes the proof. $\square$

Proof of 4a. We want to show $\dfrac{(-1)^n n}{n+1}$ diverges.
Let $\{x_n\}$ be the sequence in question, and we want to use Lemma 0.4, since $|x_n| = n(n+1)^{-1}$, and

$$n(n+1)^{-1} = (1 + n^{-1})^{-1} \to 1 > 0$$

Also, $x_{n+1}/x_n < 0$ for all $n \geq 1$, therefore by Lemma 0.4, $\{x_n\}$ diverges. $\square$

Proof of 4b. We want to show

$$((n+1)^{1/2} - (n)^{1/2})n^{1/2} \to 2^{-1}$$

Let $\{x_n\}$ be the sequence in question, and note that

$$\begin{aligned} x_n &= n^{1/2}\left(\frac{1}{(n+1)^{1/2} + (n)^{1/2}}\right) \\ &= \frac{1}{1 + (1 + n^{-1})^{1/2}} \end{aligned}$$

And a moment's thought will show that $1 + (1 + n^{-1})^{1/2} \to 2$, and $x_n \to 2^{-1}$. $\square$

Proof of 4c. We want to show

$$((n+1)^{1/2} - (n)^{1/2})n \to +\infty$$

Again, let $x_n = ((n+1)^{1/2} - (n)^{1/2})n$, notice that $x_n = n^{1/2} y_n$, where y_n is the same sequence in part 4b. Where $y_n \to 2^{-1}$. Now fix $\varepsilon = 4^{-1}$, then $|y_n - 2^{-1}| < 4^{-1}$ eventually.

Hence, $4^{-1} < |y_n|$ eventually. And multiplication by $n^{1/2}$ on both sides of the estimate gives

$$n^{1/2}4^{-1} < |x_n| = x_n$$

Now, fix some $(4M)^2 > 0$, by the Archimedean Property, there exists some n so large (frequently in $\mathbb{N}^+$) where

$$n > (4M)^2 \implies n^{1/2}4^{-1} > M \implies |x_n| = x_n > M$$

and sending $M \to +\infty$ gives $x_n \to +\infty$, and this finishes the proof. □

Proof of 4d. We want to show

$$(3\sqrt{n})^{1/2n} \to 1$$

Let $x_n = (3\sqrt{n})^{1/2n}$, be the sequence in the question. Notice that

- $n^{1/n} \to 1$, and $(n^{1/n})^{1/4} = (n^{1/4})^{1/n} \to 1$,
- $3^{1/2} > 1$, therefore $(3^{1/2})^{1/n} \to 1$

The proof is complete upon multiplying the two convergent sequences above. □

Remark (Remark about the two bullet points in 4d)**.** *The first bullet point reuses a fact from Homework 4, and we took square roots on the positive sequence twice. The second bullet point is also from Homework 4.*

Proof of 4e. We want to show

$$(a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b, \quad \forall 0 < a < b$$

Let $x_n = (a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b$, and $0 < a < b$, and define $r = b/a$ so that $r > 1, 0 < r^{-1} < 1$, with $r^{-n} \to 0$.

$$\begin{aligned} x_n &= \frac{a^{n+1}(1 + r^{n+1})}{a^n(1 + r^n)} \\ &= a(1 + r^{n+1})(1 + r^n)^{-1} \\ &= a(r^{-n} + r)(r^{-n} + 1)^{-1} \end{aligned}$$

To avoid being too verbose, $r^{-n} + 1 \to 1$, and $r^{-n} + r \to r$, therefore $x_n \to ar = b$. □

Proof of 4f. We want to show

$$(n!)^{1/n^2} \to 1$$

Let $x_n = (n!)^{1/n^2}$, notice that for all $1 \leq n$,

$$1 \leq n! \implies 1 \leq x_n$$

More is true,

$$n! \leq n^n \implies (n!)^{1/n^2} \leq n^{1/n}$$

Since the constant sequence $a_n = 1$ converges to 1, and $n^{1/n} \to 1$ by Homework 4, x_n converges, and $x_n \to 1$. $\square$

Remark. *We used the Squeeze Theorem here.*

Problem 5

WTS. *Let $\{x_n\}$ be a sequence of reals, where $x_n > 0$ eventually, and if $x_{n+1}/x_n \to L > 1$, then $x_n \to +\infty$.*

Proof. This proof modifies that of the 'convergent ratio test'. Fix some $r = (1 + L)/2$, then $1 < r < L$, and let $\varepsilon = L - r > 0$, so that eventually

$$\begin{aligned} \Big| x_{n+1}/x_n - L \Big| < \varepsilon &\iff -\varepsilon + L < x_{n+1}/x_n \\ &\iff r x_n < x_{n+1} \end{aligned} \tag{2}$$

Now, fix some $N \in \mathbb{N}$ such that x_n satisfies Equation (2) and $x_n > 0$ for every $n \geq N$. A simple induction will show that for all $k \geq 1$,

$$r^k x_N < x_{N+k} \tag{3}$$

Let $c = x_N > 0$, and $\Lambda = r > 1$, and the proof is complete upon showing that $r^k x_N \to +\infty$. Indeed, observe that $1 < r \iff 0 < r^{-1} < 1$, and $r^{-k} \to 0$, as $k \to +\infty$.

Let $M/x_N > 0$ be arbitrary, and multiply M/x_N with the sequence r^{-k}. So that $M/x_N r^{-k} \to 0$ as well, and choosing $\varepsilon = 1$,

$$|M/x_N r^{-k}| < 1$$

and it immediately follows that

$$\begin{aligned} M/x_N r^{-k} < 1 &\iff x_N/M > r^{-k} \\ &\iff M/x_N < r^k \\ &\iff M < r^k x_N < x_{N+k} \end{aligned}$$

$\square$

Problem 6

WTS. *Apply the ratio test to determine the convergence of*

(a) $b^n/n!$,

(b) b^n/n^2

Proof of 6a. Let $x_n = b^n/n!$, and using the ratio test yields

$$|x_{n+1}/x_n| = b(n!/(n+1)!)^{-1} \to 0 < 1$$

Therefore $x_n \to 0$. $\square$

Proof of 6b. Let $x_n = b^n/n^2$, and

$$|x_{n+1}/x_n| = b(n/(n+1))^2 = b(1+n^{-1})^{-2} \to b > 1$$

Using Question 5, since $x_n > 0$ eventually, $x_n \to +\infty$. $\square$

Problem 7

WTS. *Show that if* $\{x_n\}$ *is a monotone sequence, then*

- $x_n \to \sup\{x_n\}$ *if* $\{x_n\}$ *is non-decreasing, and*
- $x_n \to \inf\{x_n\}$ *if* $\{x_n\}$ *is non-increasing.*

where we allow $\sup\{x_n\}$ *and* $\inf\{x_n\}$ *to also be symbols in the extended reals.*

We need a small lemma before proceeding.

0.2 Lemma

Lemma 0.5. *If* A *is a non-empty bounded above subset of* $\mathbb{R}$*, then* $(-1)(\sup A) = \inf(-1)A$.

Proof. Let $s = \sup(A)$, then

$$(-1)A = \{-x : x \in A\}$$

Evidently, for every $x \in A$

$$x \leq s \implies -s \leq -x \implies -s \leq (-1)A$$

Now fix any $\varepsilon > 0$, then there exists an $x \in A$ such that

$$s - \varepsilon \leq x \implies (-1)(s - \varepsilon) \geq -x \implies (-x) \leq (-s) + \varepsilon$$

Thus establishes $\inf(-1)A = (-1)\sup(A)$. $\square$

Corollary 0.5.1. $(-1)\inf(A) = \sup(-1)A$. *The proof is trivial just replace* A *by* $(-1)A$.

Proof of Problem 7. Suppose that $\{x_n\}$ is non-decreasing, if $\sup\{x_n\} = +\infty$ it means that for any M $x_n \geq M$ eventually, so that $x_n \to +\infty = \sup\{x_n\}$. If $\sup\{x_n\}$ is finite, then for any $\varepsilon > 0$, there exists $x_N \in \{x_n\}$ with $\sup\{x_n\} - \varepsilon < x_N$, but $\{x_n\}$ is non-decreasing, this implies that for every $n \geq N$,

$$\sup\{x_n\} - \varepsilon < x_N \leq x_{n \geq N} \leq \sup\{x_n\} < \sup\{x_n\} + \varepsilon$$

and $x_n \in V_\varepsilon(\sup\{x_n\})$ eventually; therefore $x_n \to \sup\{x_n\}$.

If $\{x_n\}$ is non-increasing, and if $\inf\{x_n\} = -\infty$ then it is trivial to modify the above proof to show that $x_n \to -\infty = \inf\{x_n\}$. For the case where $\inf\{x_n\}$ is finite, notice that $\sup\{(-1)x_n\} = (-1)\inf\{x_n\} \neq +\infty$ by Lemma 0.5, and $\{(-1)x_n\}$ is a monotone, non-decreasing sequence, hence

$$(-1)x_n \to \sup\{(-1)x_n\} \implies x_n \to (-1)\sup\{(-1)x_n\} = \inf\{x_n\}$$

$\square$

Problem 1

WTS. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove*

(a) *If $\{x_n\}$ is unbounded above, then $\limsup x_n = +\infty$, and there exists a subsequence y_k of x_n with $\lim y_k = +\infty$. Conversely, if $\{x_n\}$ is unbounded below, then $\liminf x_n = -\infty$, etc.*

(b) *$\{x_n\}$ converges to an $x \in \mathbb{R}$, or diverges to $\pm\infty$ if and only if*

$$\limsup x_n = \liminf x_n \tag{1}$$

Furthermore, we will denote the tail of $\{x_n\}$ by E_m, with

$$E_m = \{x_{n \geq m}\}$$

Let us utilize the following powerful Theorems

0.1 Eventual Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. We define the m-tail of the sequence,*

$$E_m = \{x_n,\, n \geq m\}$$

If $A \subseteq X$ is any set, the following are equivalent.

(a) *$x_n \in A$ eventually,*

(b) *$E_m \subseteq A$ eventually, (as $m \to \infty$),*

(c) *$A^c \cap E_m = \varnothing$ eventually, (as $m \to \infty$),*

(d) *$A^c \cap \{x_n\}$ is finite,*

(e) *it is false that $x_n \in A^c$ frequently,*

(f) *no subsequence x_{n_k} of x_n can lie in A^c eventually, (as $k \to \infty$),*

(g) *every subsequence of x_n can be found frequently in A*

Proof. Suppose (a) holds, then $\{x_{n\geq N}\} \subseteq A$. So $E_N \subseteq A$, and for every $m \geq N$, $E_m \subseteq E_N \subseteq A$, so (a) $\implies$ (b).

Suppose (b) holds, then

$$E_m \subseteq A \iff A^c \cap E_m = \varnothing, \qquad \text{eventually}$$

Hence (c) follows.

To show (c) $\implies$ (d), we assume (d) is false. So $A^c \cap \{x_n\}$ is infinite, and denote

$$\mathcal{N} = \left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\}$$

is an unbounded set. Now choose any $m \in \mathbb{N}^+$, so this m must not be an upper-bound of $\mathcal{N}$ (otherwise $\mathcal{N}$ would be bounded above, and therefore finite). For this m, there exists an $n > m'$, where $n \in \mathcal{N}$, with

$$x_n \in A^c \cap E_m \implies A^c \cap E_m \neq \varnothing$$

This holds for every m (we have proven a negation that is stronger than the negation of (c)), and (c) is invalid. Therefore (c) $\implies$ (d).

Suppose now (d) holds. Since $A^c \cap \{x_n\}$ is finite, there exists an $N \in \mathbb{N}^+$ where

$$N = \max\left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\} + 1$$

for every $n \geq N$, we have $x_n \notin A^c$. So $x_n \notin A^c$ eventually $\iff$ the claim that x_n is in A^c frequently is false, and (e) follows.

Now suppose (e), unboxing the quantifiers, reads

$$\neg\left(\forall N \in \mathbb{N}^+,\, \exists n \geq N,\, x_n \in A^c\right) \iff \exists N \in \mathbb{N}^+,\, \forall n \geq N,\, x_n \in A$$

The right member is equivalent to claim (a).

To show (f) is indeed equivalent with the rest. Suppose claim (d) does not hold. So $A^c \cap \{x_n\}$ is infinite. Let $\mathcal{K} = \{n \in \mathbb{N}^+,\, x_n \in A^c\}$ is an infinite set of natural numbers, and is therefore unbounded above. Following the argument within (c) $\implies$ (d), we can construct an increasing sequence of naturals $n_1 < n_2 < \dots$ such that $n_k \in \mathcal{K}$, and

$$\{x_{n_k}\} \subseteq A^c$$

This proves $\neg$(d) $\implies$ $\neg$(f). To show the converse, suppose that $x_{n_k} \in A^c$ eventually, then the set of naturals (also denoted by $\mathcal{K}$),

$$\mathcal{K} = \left\{ k \in \mathbb{N}^+,\, x_{n_k} \in A^c \right\}$$

is an infinite set, so (d) is false.

Lastly, to show $(f) \iff (g)$, we unbox the quantifiers

$$\begin{aligned}
(g) &\iff \forall \{x_{n_k}\} \subseteq x_n, \neg\left(\exists K \in \mathbb{N}^+,\, \forall k \geq K,\, x_{n_k} \in A^c \right) \\
&\iff \forall \{x_{n_k}\} \subseteq x_n, \forall k \in \mathbb{N}^+,\, \exists k \geq K,\, x_{n_k} \in A^c \\
&\iff (f)
\end{aligned}$$

This completes the proof. $\square$

0.2 Frequent Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. Let E_m be the m-tail of the sequence as usual. If $A \subseteq X$ is any set, the following are equivalent.*

(a) $x_n \in A$ frequently,

(b) it is false that $x_n \in A^c$ eventually,

(c) $A \cap E_m$ is infinite, for every $m \geq 1$,

(d) there exists a subsequence x_{n_k} of x_n that lies in A eventually,

Proof. Notice that (a) is equivalent to the negation of Theorem 0.6a, but with A taking the place of A^c (within Theorem 0.6).

It immediately follows that

$$(a) \iff (b) \iff (c) \iff (d)$$

and the proof is complete. □

Corollary 0.0.1. *If x_n is in A eventually, then x_n lies in A frequently. Or the contrapositive: if x_n is in A^c frequently, then x_n does not lie in A eventually.*

0.3 sup, inf with unbounded sets

Lemma 0.1. *If A is a subset of $\mathbb{R}$, then $\sup(A) = +\infty$, if and only if $\inf(-1(A)) = -\infty$.*

Proof. Fix any $-M \in \mathbb{R}$, there exists some $x \in A$ with $x > -M \iff (-1)x < M$, for any arbitrary M, this proves $\implies$. The converse is trivial if we read the statement backwards. □

Lemma 0.2. *If $\{x_n\}$ is a real valued sequence, then*

$$\sup E_1 = +\infty \iff \sup E_m = +\infty, \quad \forall m \geq 1$$

Furthermore,

$$\inf E_1 = -\infty \iff \inf E_m = -\infty, \quad \forall m \geq 1$$

Proof. Suppose $\sup E_1 = +\infty$, and by contradiction that there exists some $m \geq 1$, with $\sup E_M < +\infty$. Then

$$|x_n| \leq \sum_{j<m} |x_j| + \sup E_m < +\infty$$

and it follows that $\sup E_1 < +\infty$, and the converse is trivial. Since $\sup E_m = +\infty$ directly implies the first claim.

The second statement is trivial upon applying the previous part and Lemma 0.1 □

Lemma 0.3. *If $\{x_n\}$ is any sequence in $\mathbb{R}$, then*

$$\begin{aligned} \limsup x_n = +\infty &\iff \sup E_1 = +\infty \\ \liminf x_n = -\infty &\iff \inf E_1 = -\infty \end{aligned}$$

that is to say, $\limsup x_n = +\infty$ *if and only if* $\{x_n\}$ *is unbounded above, and* $\liminf x_n$ *respectively.*

Proof. If $\sup E_1 = +\infty$, then $\limsup E_m = +\infty$ by Lemma 0.2. Conversely, if E_1 is bouned above, then $\sup E_m \leq \sup E_1 < +\infty$, and

$$\limsup E_m < +\infty \implies \limsup E_m \neq +\infty$$

The second statement follows after a simple modification of the proof above. □

Proof of Question 1 Part A. Suppose that $\{x_n\}$ is unbounded above, then $\sup E_1 = +\infty$. Using Lemma 0.2, $\sup E_m = +\infty$ for every $m \geq 1$, then

$$\lim_m \sup E_m \approxeq \lim_m +\infty \approxeq +\infty$$

We will construct a sequence that diverges to $+\infty$. Let us agree to define

$$\mathcal{N}(M) = \left\{ n \in \mathbb{N}^+,\, x_n > M \right\} \neq \varnothing,\, \forall M \in \mathbb{R}$$

1. Choose $n_1 = \operatorname{least} \mathcal{N}(1)$,
2. Suppose $n_1 < n_2, \ldots < n_k$ have been chosen, and $n_1 > 1, \ldots n_k > k$,
3. We can select $n_{k+1} = \operatorname{least} \mathcal{N}(k + 1 + x_{n_k})$, so that $n_{k+1} > n_k$, and $x_{n_{k+1}} > k + 1$.

Clearly, $x_{n_k} \to +\infty$, and the proof is complete.
Likewise, suppose that $\{x_n\}$ is unbounded below, then $(-1)x_n$ is a sequence that is unbounded above (this is justified using Lemma 0.1). Using the same construct as above, obtain a sequence $\{(-1)x_{n_k}\}$ that diverges to $+\infty$, so that for every $-M \in \mathbb{R}$,

$$(-1)x_{n_k} > -M \implies x_{n_k} < M$$

And the subsequence $x_{n_k} \to -\infty$, and $\liminf x_n = -\infty$ is a matter of applying Lemma 0.2,

$$\inf E_1 = -\infty \iff \liminf E_m = -\infty$$

□

For Part B of the proof, we equip ourselves with the following powerful lemmas.

0.4 sup, inf of A, B when A subset of B

WTS. *If* $A \subseteq B \subseteq \mathbb{R}$*, then* $\sup(A) \leq \sup(B)$*, and* $\inf(A) \geq \inf(B)$*.*

Proof. If we allow for the sup and inf of A and B to take on symbols in the extended reals. Then, $\sup(B)$ is an upper-bound for A and $\inf(B)$ is a lower-bound for A, therefore

$$\sup(A) \leq \sup(B), \quad \inf(A) \geq \inf(B)$$

□

0.5 Every element in A is less than every element in B

WTS. *If* A*,* B *are non-empty subsets of* $\mathbb{R}$*,*

$$\sup A \leq \inf B \iff \forall a \in A, \forall b \in B, \, a \leq b$$

Proof. Suppose that $\sup A \leq \inf B$, then for every $a \in A$, and we can safely assume that both $\sup A$ and $\inf B$ are finite (see remark),

$$a \leq \sup A \leq \inf B$$

so that a is a lower bound for B, but this is equivalent to saying that $a \leq b$ for every $b \in B$.

Now suppose that for every $a, b \in A, B$, $a \leq b$. Then every single $b \in B$ is an upper bound for the set A, therefore

$$\forall b \in B, \sup A \leq b \implies \sup A \leq \inf B$$

where the last estimate is due to $\sup A$ being a lower bound for B. □

Remark. *If* $\sup A = +\infty$*, then* $\inf B = +\infty$*, this can only happen if* $B = \varnothing$*, so* $\sup A = +\infty$ *is impossible, so is* $\inf B = -\infty$*. (We assume that* A *and* B *are subsets of* $\mathbb{R}$*, and not of* $\overline{\mathbb{R}}$*).*

Further, if $\sup A = -\infty$*, then either* $A = \{-\infty\}$ *which is not a subset of* $\mathbb{R}$*, or* $A = \varnothing$*, which is again impossible.*

Proof of Question 1 Part B. To begin, notice that for every $m \geq 1$, using the same notation as Part A, where $E_m = \{x_{n \geq m}\}$. Let us assume that E_1 is a bounded subset of $\mathbb{R}$. Then,

- If $m = k$, then
$$\inf E_m \leq \sup E_m$$
- If $m \leq k$, then
$$E_m \supseteq E_k \implies \inf E_m \leq \inf E_k \leq \sup E_k$$
by Lemma 0.4.
- If $m \geq k$, then
$$E_k \supseteq E_m \implies \inf E_m \leq \sup E_m \leq \sup E_k$$
also by by Lemma 0.4.
- Therefore for any $m, k \in \mathbb{N}^+$,
$$\inf E_m \leq \sup E_k$$
- Applying Lemma 0.5 gives
$$\sup \inf E_m \leq \inf \sup E_m \iff \liminf x_n \leq \limsup x_n \tag{2}$$
- Alternatively, we can prove Equation (2) by using the Monotone Convergence Theorem (because E_1 is bounded). Indeed, (2) reads
$$\lim_m \inf E_m \leq \lim_m \sup E_m \iff \liminf x_n \leq \limsup x_n$$

Suppose $x_n \to x \in \mathbb{R}$, then for any $\varepsilon > 0$, $x_n \in V_\varepsilon(x)$ eventually. By Lemma

0.6, $E_m \subseteq V_\varepsilon(x)$ eventually. Hence,

$$\begin{aligned}
E_m \subseteq V_\varepsilon(x) &\iff E_m \subseteq (x-\varepsilon, x+\varepsilon) \\
&\iff x-\varepsilon \leq \inf E_m \leq \sup E_m \leq x+\varepsilon \\
&\iff \begin{cases} x-\varepsilon \leq \inf E_m \leq \sup\inf E_m \leq \sup E_1 \\ \inf E_1 \leq \inf\sup E_m \leq \sup E_m \leq x+\varepsilon \end{cases} \\
&\iff \begin{cases} x - \left(\sup\inf E_m\right) \leq \varepsilon \\ \left(\inf\sup E_m\right) - x \leq \varepsilon \end{cases} \\
&\iff \inf\sup E_m \leq x \leq \sup\inf E_m \\
&\iff \inf\sup E_m \leq \sup\inf E_m
\end{aligned} \tag{3}$$

Combining (3) with (2) gives $\liminf x_n = \limsup x_n$.

On the other hand, if Equation (1) holds, and E_1 is bounded, let $\liminf x_n = x = \limsup x_n$. Then for every $\varepsilon > 0$, both $\sup E_m$ and $\inf E_m$ must belong within this ε-ball about x eventually. And by Theorem 0.6

$$\begin{aligned}
\{\sup E_m\}_{m\geq N} \subseteq V_\varepsilon(x) \\
\{\inf E_m\}_{m\geq N} \subseteq V_\varepsilon(x)
\end{aligned}$$

Which reads, for every $m \geq N$,

$$\begin{aligned}
x-\varepsilon \leq \sup E_m \leq x+\varepsilon \\
x-\varepsilon \leq \inf E_m \leq x+\varepsilon
\end{aligned}$$

Applying Equation (2) to the bounded sets $E_m \subseteq E_1$, yields

$$x-\varepsilon \leq \inf E_m \leq \sup E_m \leq x+\varepsilon$$

So that $E_m \subseteq [\inf E_m, \sup E_m] \subseteq V_\varepsilon(x)$ eventually. But by Theorem 0.6, this is to say that $x_n \in V_\varepsilon(x)$ eventually, so $x_n \to x$.

For the unbounded case, suppose $x_n \to +\infty$. Clearly $\sup E_1 = \infty$, and by Lemma 0.3,

$$x_n \geq L + \varepsilon_0 \text{ eventually} \implies x_{n_k} \geq L + \varepsilon_0 \text{ xeventually}$$

Now it suffices to show that $\liminf x_n = +\infty$. Notice that for every $M \in \mathbb{R}$, $E_m \subseteq [M, +\infty)$ eventually. So $\inf E_m \geq M$, but by Lemma 0.2, therefore

$$-\infty < \inf E_m \iff -\infty < \liminf E_m$$

Now, $\{\inf E_m\}_{m\geq 1}$ is a non-decreasing sequence that converges to its supremum. But no finite number can be an upper bound for $\inf E_m$, so

$$\sup_{m\geq 1} \inf E_m = \liminf E_m = \liminf x_n = +\infty$$

Conversely, let us assume that $\limsup x_n = \liminf x_n = +\infty$. It is obvious that $\sup E_1 = +\infty$, and $\inf E_1 > -\infty$ by Lemmas 0.3 and 0.2. Also,

(i) A monotonic sequence in $\{\inf E_m\}_{m\geq 1}$ increases towards its supremum, which in this case is $+\infty$.

(ii) This is equivalent to saying $\inf E_m \geq M$ eventually, for every $M \in \mathbb{R}$.

(iii) Now, for all $x_n \in E_m \implies x_n \geq \inf E_m \geq M$ eventually, and sending $M \to +\infty$ proves $x_n \to +\infty$.

Let us prove $x_n \to -\infty \iff \limsup x_n = \liminf x_n = -\infty$. If $x_n \to -\infty$, it is clear that $(-1)x_n \to +\infty$. So that

$$(-1)x_n \to +\infty \iff \limsup(-1)E_m = \liminf_m(-1)E_m = +\infty$$

Apply Lemma 0.1 so that

$$\sup(-1)E_m = +\infty \iff \inf E_m = -\infty$$

Then, apply Lemmas 0.2 and 0.3 to the rightmost equality, which yields

$$\inf E_m = -\infty, \forall m \geq 1 \implies \liminf E_m = -\infty$$

Likewise, a simple application of the two Lemmas will give us $\limsup E_m = -\infty$. This proves $\implies$.

To show the converse, use Lemma 0.3, to obtain

$$\limsup E_m \neq +\infty \iff \sup E_1 \neq +\infty$$
$$\liminf E_m = -\infty \iff \inf E_1 = -\infty$$

Now, modify the procedure (i), by forcing the m-tail of the sequence to live in $(-\infty, M]$ eventually, thus concluding that $x_n \to -\infty$. $\square$

Problem 2

WTS. *Show that* $x_n = (1 + n^{-2})^{2n^2} \to e^2$

Proof. Using the definition of e,

$$e = \lim(1 + k^{-1})^k, \quad e_k = (1 + k^{-1})^k \tag{4}$$

Now, let $\{k_n\}_{n\geq 1} = 1, 4, 9, 16, \ldots$. Clearly, $\{e_{k_n}\}$ is a subsequence of of e_k. Therefore $e_{k_n} \to e$ as $n \to \infty$. Now apply the multiplication rule two convergent sequences.

$$e_{k_n} \to e \implies e_{k_n} e_{k_n} = (1 + n^{-2})^{2n^2} \to e^2$$

$\square$

Problem 3

WTS. *If $\{x_n\}$ is a sequence in $\mathbb{R}$, show that if every subsequence of $\{x_n\}$ contains a further subsequence that converges to $L \in \mathbb{R}$, then $x_n \to L$.*

We will prove something that is much stronger. Let us consider the following two Theorems.

0.6 Eventual Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. We define the m-tail of the sequence,*

$$E_m = \{x_n,\, n \geq m\}$$

If $A \subseteq X$ is any set, the following are equivalent.

(a) $x_n \in A$ *eventually,*

(b) $E_m \subseteq A$ *eventually, (as* $m \to \infty$*),*

(c) $A^c \cap E_m = \varnothing$ *eventually, (as* $m \to \infty$*),*

(d) $A^c \cap \{x_n\}$ *is finite,*

(e) it is false that $x_n \in A^c$ *frequently,*

(f) no subsequence x_{n_k} *of* x_n *can lie in* A^c *eventually, (as* $k \to \infty$*),*

(g) every subsequence of x_n *can be found frequently in* A

Proof. Suppose (a) holds, then $\{x_{n \geq N}\} \subseteq A$. So $E_N \subseteq A$, and for every $m \geq N$, $E_m \subseteq E_N \subseteq A$, so (a) $\implies$ (b).

Suppose (b) holds, then

$$E_m \subseteq A \iff A^c \cap E_m = \varnothing, \quad \text{eventually}$$

Hence (c) follows.

To show (c) $\implies$ (d), we assume (d) is false. So $A^c \cap \{x_n\}$ is infinite, and denote

$$\mathcal{N} = \left\{ n \in \mathbb{N}^+,\, x_n \in A^c \right\}$$

is an unbounded set. Now choose any $m \in \mathbb{N}^+$, so this m must not be an upper-bound of $\mathcal{N}$ (otherwise $\mathcal{N}$ would be bounded above, and therefore finite). For this m, there exists an $n > m'$, where $n \in \mathcal{N}$, with

$$x_n \in A^c \cap E_m \implies A^c \cap E_m \neq \varnothing$$

This holds for every m (we have proven a negation that is stronger than the negation of (c)), and (c) is invalid. Therefore (c) $\implies$ (d).

Suppose now (d) holds. Since $A^c \cap \{x_n\}$ is finite, there exists an $N \in \mathbb{N}^+$ where

$$N = \max\left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\} + 1$$

for every $n \geq N$, we have $x_n \notin A^c$. So $x_n \notin A^c$ eventually $\iff$ the claim that x_n is in A^c frequently is false, and (e) follows.

Now suppose (e), unboxing the quantifiers, reads

$$\neg\left(\forall N \in \mathbb{N}^+,\, \exists n \geq N,\, x_n \in A^c\right) \iff \exists N \in \mathbb{N}^+,\, \forall n \geq N,\, x_n \in A$$

The right member is equivalent to claim (a).

To show (f) is indeed equivalent with the rest. Suppose claim (d) does not hold. So $A^c \cap \{x_n\}$ is infinite. Let $\mathcal{K} = \{n \in \mathbb{N}^+,\, x_n \in A^c\}$ is an infinite set of natural numbers, and is therefore unbounded above. Following the argument within (c) $\implies$ (d), we can construct an increasing sequence of naturals $n_1 < n_2 < \ldots$ such that $n_k \in \mathcal{K}$, and

$$\{x_{n_k}\} \subseteq A^c$$

This proves $\neg$(d) $\implies$ $\neg$(f). To show the converse, suppose that $x_{n_k} \in A^c$ eventually, then the set of naturals (also denoted by $\mathcal{K}$),

$$\mathcal{K} = \left\{k \in \mathbb{N}^+,\, x_{n_k} \in A^c\right\}$$

is an infinite set, so (d) is false.

Lastly, to show $(f) \iff (g)$, we unbox the quantifiers

$$\begin{aligned}(g) &\iff \forall \{x_{n_k}\} \subseteq x_n, \neg\left(\exists K \in \mathbb{N}^+, \forall k \geq K, x_{n_k} \in A^c\right) \\ &\iff \forall \{x_{n_k}\} \subseteq x_n, \forall k \in \mathbb{N}^+, \exists k \geq K, x_{n_k} \in A^c \\ &\iff (f)\end{aligned}$$

This completes the proof. $\square$

0.7 Frequent Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. Let E_m be the m-tail of the sequence as usual. If $A \subseteq X$ is any set, the following are equivalent.*

(a) $x_n \in A$ frequently,

(b) it is false that $x_n \in A^c$ eventually,

(c) $A \cap E_m$ is infinite, for every $m \geq 1$,

(d) there exists a subsequence x_{n_k} of x_n that lies in A eventually,

Proof. Notice that (a) is equivalent to the negation of Theorem 0.6a, but with A taking the place of A^c (within Theorem 0.6).

It immediately follows that

$$(a) \iff (b) \iff (c) \iff (d)$$

and the proof is complete. □

Corollary 0.3.1. *If x_n is in A eventually, then x_n lies in A frequently. Or the contrapositive: if x_n is in A^c frequently, then x_n does not lie in A eventually.*

0.8 Main Proof of Q3

Proof. Let us simplify the subsequence notation for a bit, and write x_{nk} as a subsequence for x_n, and x_{nkj} as a subsequence of x_{nk} (which makes x_{nkj} a subsubsequence of x_n).

If for every x_{nk}, there exists a $x_{nkj} \to L$. This is equivalent to: for every h^{-1}, where $h \in \mathbb{N}^+$,

$$d(x_{nkj}, L) < h^{-1} \iff x_{nkj} \in V_{h^{-1}}(L), \text{ eventually}$$

And for every $V_{h^{-1}}$, Theorem 0.7(d) holds for some subsequence $x_{n_{k_j}}$ of every subsequence x_{n_k}. This is equivalent to saying that x_{n_k} lies in $V_{h^{-1}}(L)$. But Theorem 0.6g holds x_n, therefore $x_n \in V_{h^{-1}}(L)$ eventually.

But this is true if and only if $d(x_n, L) < h^{-1}$ eventually. Since this holds for every $h \geq 1$, we must conclude that $x_n \to L$. □

Problem 4

WTS. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that*

(a) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$,

(b) Give an example for when (a) is a strict inequality.

Proof. Since x_n and y_n are bounded, this makes x_n+y_n bounded too. Indeed, if $|x_n| \leq M2^{-1}$ for a large M, and similarly for $|y_n|$. An application of the Triangle Inequality will show that $|x_n + y_n| \leq M$.

Notice also for any fixed $m \geq 1$,

$$\left\{x_n + y_n,\, n \geq m\right\} \subseteq \left\{x_j + y_k,\, j, k \geq m\right\}$$

Taking the supremum across both sets yields

$$\sup_{n \geq m}(x_n + y_n) \leq \sup_{n \geq m} x_n + \sup_{n \geq m} y_n$$

Finally, let $m \to +\infty$. Since this inequality holds for every $m \geq 1$, we have the following estimates for their limits

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

This proves (a). Now let $x_n = (-1)^n$, and $y_n = -x_n$. Both are bounded sequences and $\limsup x_n = \limsup y_n = 1$, but $x_n + y_n = 0$ at every n; the strict inequality follows. $\square$

Problem 5

WTS. *Prove two things,*

(a) *Let $x_n = n^{1/2}$. Show that $|x_{n+1} - x_n| \to 0$, but x_n is not Cauchy.*

(b) *Answer the following*

(b) Even more strikingly, let (x_n) be the sequence $(\underbrace{\sqrt{m}, \sqrt{m}, \dots, \sqrt{m}}_{m})_m$, i.e.

$$(\sqrt{1}, \sqrt{2}, \sqrt{2}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{4}, \sqrt{4}, \sqrt{4}, \sqrt{4}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \dots).$$

Prove that for each fixed $k \in \mathbb{N}$, we have $\lim_n |x_n - x_{n+k}| = 0$ but (x_n) is not Cauchy.

REMARK: The moral here is that demanding for each **fixed** k separately that the distance $d(x_n, x_{n+k})$ goes to 0 as $n \to \infty$ **does not** guarantee Cauchyness, which demands that $\lim_n \sup_k d(x_n, x_{n+k}) = 0$. In particular, we **cannot switch the order** of $\lim_n$ and $\sup_k$; indeed, $\sup_k \lim_n d(x_n, x_{n+k}) = 0$ while $\lim_n \sup_k d(x_n, x_{n+k}) = \infty$.

Proof of Part A. Notice for every $k \geq 1$

$$\begin{aligned} a_n &= \left| k\left((n+k)^{1/2} + (n)^{1/2} \right)^{-1} \right| \\ &= \left| \frac{(n+k) - (n)}{(n+1)^{1/2} + (n)^{1/2}} \right| \\ &= |(n+k)^{1/2} - (n)^{1/2}| \end{aligned}$$

A simple consequence of $k^{-1}\sqrt{n} \leq a_n^{-1}$ is that $a_n \to 0$, and

$$|(n+k)^{1/2} - (n)^{1/2}| \to 0, \quad \forall k \geq 1 \tag{5}$$

Hence $|x_{n+1} - x_n| \to 0$ (by taking $k = 1$ within (5)).

x_n is obviously not Cauchy, because it is unbounded. Indeed for every $\varepsilon^2 > 0$ you can find a large $N \in \mathbb{N}^+$ where $N > \varepsilon^2$ eventually. And

$$n \geq N > \varepsilon^2 \implies x_n > \varepsilon$$

$\square$

Proof of Part B. It is clear that $|x_{n+k} - x_n| \leq |(n+k)^{1/2} - (n)^{1/2}|$. Sending $n \to +\infty$ reads

$$|x_{n+k} - x_n| \to 0, \forall k \geq 1$$

x_n is not Cauchy because it contains an unbounded subsequence

$$\{x_{n_k}\}, \quad k \mapsto \sqrt{k}$$

We will outine the construction, for any $k \geq 1$, apply the Well Ordering Property to obtain $n_k = \text{least}\{q \in \mathbb{N}, x_n = \sqrt{q}\}$.

□

Problem 6

WTS. *Let $y_0 < y_1 \in \mathbb{R}$. Define $\{y_n\}$ for every $n \geq 2$*

$$y_n = (1/3)y_{n-1} + (2/3)y_{n-2}$$

Prove two things,

(i) Prove that $\{y_n\}$ is contractive,

(ii) Prove that $y_n \to \frac{2}{5}y_0 + \frac{3}{5}y_1$

Proof of Part A. The sequence is clearly contractive, fix any $n \geq 0$, and

$$y_{n+2} - y_{n+1} = \frac{-2}{3}\left(y_{n+1} - y_n\right) \tag{6}$$

Taking absolute values on both sides of (6), and replacing 2/3 with 1/3 finishes the proof. □

Proof of Part B. We know from Part A, that y_n is contractive, and hence Cauchy. To show that $y_n \to (2/5)y_0 + (3/5)y_1$, replace the left and right members above by

$$x_{n+2} = (-2/3)x_{n+1}, \quad x_n = y_n - y_{n-1}, \quad \forall n \geq 1 \tag{7}$$

A simple induction on $n \geq 1$ will yield

- $x_2 = \frac{-2}{3}x_1$,
- and suppose $x_j = (-2/3)^j x_1$ for every $j \geq 1$, then
- $x_{j+1} = (-2/3)^{j+1}x_1$, and this completes the induction

We require a second induction to extract y_{n+2}, and we will omit the details here. From (7), we have

$$y_{n+2} - y_0 = \sum_{j=1}^{n+2} x_j$$

$$y_{n+2} = y_0 + x_1 \sum_{j=1}^{n+2} \left(\frac{-2}{3}\right)^{j-1}$$

$$y_{n+2} = y_0 + x_1 \sum_{j=0}^{n+1} \left(\frac{-2}{3}\right)^{j}$$

Sending $n \to \infty$, noting that every subsequence of y_n must converge to the same limit, and

$$y_n \to y_0 + (y_1 - y_0)\frac{1}{1 - (-2/3)} = y_0 + (y_1 - y_0)(3/5)$$

Simplifying yields

$$y_n \to \frac{2}{5}y_0 + \frac{3}{5}y_1$$

$\square$