MATH 254: Assignment 2

November 5, 2022

Problem 1

WTS. *Let $f : \mathbb{R} \to [0, +\infty)$ and $g : [0, +\infty) \to (-\infty, 0]$ be defined by $f(x) \coloneqq x^2$ and $g(x) \coloneqq -\sqrt{x}$.*

(a) Explain why $f \circ g$ makes sense (can be defined) even though of g is not equal to the domain of f.

(b) Write down the domains and codomains of $f \circ g$ and g explicit formulas for these functions.

(c) Is g the inverse of f? Explain your answer.

Answers:

(a) Fix any $x \in [0, +\infty)$, then $g(x) \in (-\infty, 0]$ means tha $f \circ g(x)$ is well defined. Since x is mapped to exactly o $[0, +\infty)$. What matters here is that $\operatorname{range} g \subseteq \operatorname{dom} f$.

(b) Domains:
$\operatorname{dom}(f \circ g) = \operatorname{dom} g = [0, +\infty)$, and
$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$.

Codomains:
$\operatorname{codom}(f \circ g) = \operatorname{codom} f = [0, +\infty)$, and
$\operatorname{codom}(g \circ f) = \operatorname{codom} g = (-\infty, 0]$.

Formulas:

$$f \circ g(x) = (-\sqrt{x})^2 = |x| = x, \text{ For every } x \geq$$

$$g \circ f(x) = -\sqrt{x^2} = -|x| \neq x, \text{ For every } x \in$$

(c) g is not the inverse of f, fix $x = 1$, then $g \circ f(x) = -1$
$g \circ f \neq \mathrm{id}_{\mathbb{R}}$.

Remark. *I do not know what the convention for the co*
this class. Here I assumed that for any function $f : X \to Y$
Y*. Its range however is the set of points in its codomain th*
so $\operatorname{range} f = \{f(x), x \in X\}$.

Problem 2

WTS. *Let* $f : [0, 4) \to [0, 4)$ *be defined by*

$$x \mapsto \begin{cases} x + 1 & \lfloor x \rfloor \in 2\mathbb{N} \\ x - 1 & \lfloor x \rfloor \notin 2\mathbb{N} \end{cases}$$

Where $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$. *Show that* f *is a bijection, an*
inverse.

The proof for this question will resemble that of Homework 1
important lemmas must be stated.

Lemma 0.1. *For any* $f : X \to Y$, *if* $A \subseteq X$ *such that* $f = f|$
Y *is the disjoint union of* $f|_A(A)$ *and* $f|_{A^c}(A^c)$, *and the restrict*
A *and* A^c *are bijections onto their direct images, then* f *is a b*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where
the trivial case of them both belonging to the same A or
loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by
$f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in
$f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in$
$f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets a
there exists a corresponding $x \in X$ which will satisfy. This c
proof.

Lemma 0.2. *Let* f *satisfy the hypothesis of the previous le*
$(f|_A)^{-1}$ *and* $(f|_{A^c})^{-1}$ *both exist, then* $f^{-1} = (f|_A)^{-1} + (f|_{A^c})^{-1}$
$(f^{-1})|_{B_2}$, *where* $f|_A(A) = B_1$, *and* $f|_{A^c}(A^c) = B_2$.

Proof. Since B_1 and B_2 are disjoint, then fix any $y \in Y$.
of generality, let us assume that $y \in B_1$. Then, $f^{-1}(y) = ($
$(f|_A)^{-1}(y)$. This inverse is indeed well defined, since $f|_A$ is a b
its range, then there exists a unique $x \in A$ such that applyi
sides yield

$$f\left((f|_A)^{-1}(y)\right) = f \circ (f|_A)^{-1}(y) = y$$

In the same manner, fix an $x \in A$ such that $f(x) = f|_A(x$ applying $(f|_A)^{-1}$ on both sides

$$(f|_A)^{-1}(f(x)) = (f|_A)^{-1} \circ f(x) = x$$

Therefore the inverse of f can be written piecewise on two disj as follows.

$$f^{-1} = f^{-1}|_{B_1} + f^{-1}_{B_2}$$

Remark. *We adopt a slight abuse of notation with the 'restric but they should be interpreted as piecewise functions.* $f|_A + f|$ $f|_A\chi_A + f|_{A^c}\chi_{A^c}$ *where* χ *is the indicator function.*

We begin the main part of the proof.

Proof. For every $x \in [0,1) \cup [2,3)$, $\lfloor x \rfloor \in 2\mathbb{N}$. Denote $A = [0,1$ $A^c = [0,4) \setminus A = [1,2) \cup [3,4)$. Then

$$f = f|_A + f|_{A^c} = (x+1)\chi_A + (x-1)\chi_{A^c}$$

To satisfy the assumptions of the two lemmas, we need to checl images (ranges) of $f|_A$ and $f|_{A^c}$ are disjoint. This can be easily

$$x \in A \iff x \in [0,1) \cup [2,3) \implies x+1 \in [1,2) \cup [3$$

$$x \in A^c \iff x \in [1,2) \cup [3,4) \implies x+1 \in [0,1) \cup [2$$

To avoid being overly verbose, we say that the two functions a onto their ranges. Since $f|_A$ 'nudges' points $+1$ units to the rig does the exact opposite. And the range of each of the two fu complement of their domains. Even more is true:

$$(f|_A)^{-1} = x - 1 = f|_{A^c}$$

$$(f|_{A^c})^{-1} = x + 1 = f|_A$$

Therefore $f|_A$ and $f|_{A^c}$ are bijections onto their ranges (which Invoking the first lemma tells us that f is a bijection, and the s gives us the following equality.

$$f^{-1} = f^{-1}|_{A^c} + f^{-1}|_A = (f|_A)^{-1} + (f|_{A^c})^{-1}$$

Plugging in our values for the inverses, we have

$$f^{-1} = f|_A + f|_{A^c} = f$$

Problem 3

WTS. *Now it suffices to show that if* $f_1 : A \to A'$ *and* f_2 :
bijections, then

$$F : A \times B \to A' \times B'$$

is also a bijection if we define $\pi_{A'}(F(x)) = f_1(\pi_A(x))$ *and*
$f_2(\pi_B(x))$ *for every* $x \in A \times B$. *Where* π *denotes the coordi*
projection map).

Proof. Fix two elements x_1, x_2 in $A \times B$, such that $x_1 \neq x_2$. W
generality, let us assume that $\pi_A(x_1) \neq \pi_A(x_2)$, which implies

$$\pi_{A'}(F(x_1)) = f_1(\pi_A(x_1)) \neq f_1(\pi_A(x_2)) = \pi_{A'}(F(x_2)$$

Which means $F(x_1) \neq F(x_2)$. This proves injectivity.

To show that F is a surjection, fix any $y \in A' \times B'$, then thi
and b such that

$$a = f_1^{-1}(\pi_{A'}(y)) \in A$$
$$b = f_2^{-1}(\pi_{B'}(y)) \in B$$

Then denote an element of $A \times B$, and call it x, such that $\pi_{,}$
$\pi_B(x) = b$. Then it is an easy exercise to verify that $F(x) = y$,
A' and B' projections of $F(x)$.

Remark. *We implicitly defined 'equality' in the Cartesian (D*
by equality in each coordinate. So for every x_1, x_2 *in* $A \times B$, x
only if $\pi_A(x_1) = \pi_A(x_2)$ *and* $\pi_B(x_1) = \pi_B(x_2)$.

Problem 4

WTS. *Prove that every subset of* $\mathbb{N}$ *is countable. Conclude th* *injection into* $\mathbb{N}$*, then* A *is countable.*

Proof. Fix any $A \subseteq \mathbb{N}$, then if A is finite, then denote $|A| \coloneqq \sum$ then $J_n = \{k \in \mathbb{N}, k < n\}$, then there exists a bijection betwee Hence A is countable (countably finite). We also note that th order preserving map between the two sets, namely $Y_A(0)$ den element in A, etc.

Now suppose that A is infinite. Since $\mathbb{N}$ is countable, there $\mathbf{X}_A : \mathbb{N} \to A$ such that for every $n \geq 1$.

$$\mathbf{X}_A(n) \coloneqq \text{least}\,\{k \in A, \mathbf{X}_A(n-1) < k\}$$

Where we also define $\mathbf{X}_A(0) = \text{least}\, A$. Where we used the V Property of $\mathbb{N}$ twice, it is trivial to check that both of these empty.
The map $\mathbf{X}$ is monotonic (and therefore an injection). A sin induction will show this. Now, $\mathbf{X}_A(0) < \mathbf{X}_A(1)$ by inspection that $\mathbf{X}_A(n-1) < \mathbf{X}_A(n)$ for some $n \geq 1$, then

$$\mathbf{X}_A(n+1) = \text{least}\,\{k \in A, \mathbf{X}_A(n-1) < \mathbf{X}_A(n) < k$$

Hence $\mathbf{X}_A(0) < \mathbf{X}_A(1) < \cdots < \mathbf{X}_A(n)$. Suppose that X_A is not then there exists an non-empty set of elements of A that escape $\mathbf{X}_A$. Take the least of this set, and call it m. Where this $m \neq$ $\mathbf{X}_A(0) = \text{least}\, A$.

We then construct another subset of A and call it A^* which elements $k \in A$ such that $k < m$. Since A^* is obviously finite, w same construct as shown in the earliest part of this proof. Defi then we will show $\mathbf{X}_A(N) = m$.

$\mathbf{X}_A(n) \in A^* \iff 0 \leq n < N$. Since all the elements within A less than every element in A, and the number of elements within N, and $\mathbf{X}_A(n)$ for $0 \leq n \leq N-1$ gives us the N smallest valu any $n \geq N$, $\mathbf{X}_A(n) > q$ where q is any element in A^*. Hence $\mathbf{X}$ for every $n \geq N$. Now, fix $n = N$, and this will retrieve the

in $A \setminus A^*$, and this establishes the fact that every infinite su
countably infinite.

Moreover, the empty set is a subset of the natural numbers and
able. Now for the second part of the proof, if A is an injection
A must be countable.

Proof. Denote the injection of A into $\mathbb{N}$ by h, and its range
then since the range of any mapping is always a subset of it
range$(h) \subseteq \mathbb{N}$, and h is actually a bijection onto its range (any
surjection onto its range), so

$$A \equiv \text{range}\,(h)$$

But range(h) is a countable subset of $\mathbb{N}$, and this completes th

Problem 5

WTS. *Prove that for each* $n \in \mathbb{N}^+$,

$$\sum_{j \leq n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

Where $j \leq n$ *implicitly means that* $1 \leq j \leq n$.

Proof. We will proceed by a proof by induction. Fix $n = 1$, a
trivial. Then suppose the assertion holds for a certain $n \geq 1$.

$$\begin{aligned}
\sum_{j \leq n+1} \frac{1}{j(j+1)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+1)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+1+1}
\end{aligned}$$

Where the second last equality is permissible because $n + 1$
$n \geq 1$. This completes the proof.

Problem 6

WTS. *For every $n \geq 12$, $n \in \mathbb{N}$, there exists non-negative inte*
that

$$n = (4, 5) \cdot (a, b)^T$$

Proof. I do not know how to show this rigourously for I lac
Algebra, but regardless: For every $n \geq 12$, $(n - 12) \mod 4 \in$
agree to define for each $j \in [0, 3]$.

$$W_j = \{4k + 12 + j, k \in \mathbb{N}^+ \cup \{0\}\}$$

Now these sets are not necessarily disjoint, but it will not m
purposes of this exercise, as every $n \geq 12$ must be contained i
of these sets. We also write

$$\lambda : (a, b) \mapsto (4, 5) \cdot (a, b)^T$$

Then we can write

- $12 = \lambda(3, 0)$
- $13 = \lambda(2, 1)$
- $14 = \lambda(1, 2)$
- $15 = \lambda(0, 3)$

For any fixed j, note that $(12 + j) \in \lambda(\mathbb{N} \times \mathbb{N})$ as shown before.
that $(4k + (12 + j))$ is a member of $\lambda(\mathbb{N} \times \mathbb{N})$, then this induc
such that $\lambda(a, b) = (4k + (12 + j))$. But adding 4 to both sides of
and by linearity in both arguments of the inner product over $\mathbb{F}$

$$(4(k + 1) + (12 + j)) = \lambda(a + 1, b) \implies (4(k + 1) + (12_j)) \in$$

Hence $W_j \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. But the union of all W_j contains every n
$\{n \in \mathbb{N}, n \geq 12\} \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. This completes the proof.

Problem 7

WTS. *A $n \times m$ grid always takes $nm - 1$ cuts to be decompose cells.*

Proof. I am not sure why we need induction here. Take a pieces and call it $W^N = \{w_j, 1 \leq j \leq N\}$, then if $|W| =$ if $|W^N| = N \neq nm$, then there exists a cut you can make pieces. Without loss of generality assume that this piece is w_N cut $w_N = \{a, b\}$, where a and b are pieces (atomic or not), tl new state of the grid as

$W^{N+1} = W^N \cup \text{cut}(w_N) \setminus \{w_N\}$, and relabelling indices, then w elements in our new set.

Since the cutting process is complete if and only if $|W| = nm$, nal state of the grid is at W^1, and each cut increases the numbe in the set by 1, all it must take $nm - |W^1| = nm - 1$ cuts.

Now I guess you can shoehorn the induction in there by saying $|W^n|$, $n - 1$ cuts must have been made for each and every ste equivalent to the first argument I made above.

Problem 8

WTS. *Prove that for every $a < b$ in $\mathbb{R}$, the segment (a, b) is*
Conclude that every segment in the form of (a, b) must contain
number.

Proof. From Lecture 4: $[0, 1)$ is uncountable. This is equival
that no $f : [0, 1) \to \mathbb{N}$ can be injective. Therefore no $f : (-1,$
be injective. Since if some f were to be injective from $(-1, +1)$
have to be injective on $[0, 1)$.

Using the last part of Homework 1 Q6b, define a bijection f f
to (a, b) where $a < b$.

$$f(x) = (x + 1)(m/2) + a, \quad m := b - a$$

It follows immediately that $(a, b) \equiv (-1, +1)$, which is an unc
This proves the first claim.

To show the validity of the second claim, suppose not. So (a, b)
$\mathbb{Q}$ (there exists a bijection between the two sets). So there exist
of (a, b) into $\mathbb{N}$, then (a, b) is countable and the proof is compl