Problem 1

WTS. *Provide some divergent sequences with the following properties:*

Proof.

(a) For all $M \in \mathbb{N}$, $|x_n| \geq M$ eventually,

$$\{x_n\} = n$$

(b) x_n is unbounded, but there is some $M \in \mathbb{N}$ with $|x_n| \leq M$ frequently,

$$\{x_n\} = n(1 + (-1)^n)/2$$

(c) $\{x_n\}$ is periodic,

$$\{x_n\} = (-1)^n$$

(d) $\{x_n\}$ is bounded, but not eventually periodic.

$$\{x_n\} = 1, \text{ n is prime, 0 otherwise.}$$

□

Problem 2

WTS. *Prove two things*

(a) $x_n \to x \iff |x_n - x| \to 0$,

(b) *The convergence of* $\{x_n\}$ *implies the convergence of* $\{|x_n|\}$. *But not the converse.*

It is at this point we introduce a small Lemma.

Lemma 0.1 (Super Triangle Inequality)**.** *For any* $x, y, z \in X$*, where* X *denotes a metric space then,*

$$\left| d(x,z) - d(y,z) \right| \leq d(x,y)$$

Proof. From the regular Triangle Inequality,

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ d(y,z) &\leq d(x,y) + d(x,z) \end{aligned}$$

The proof is complete upon isolating $d(x,y)$ in both equations. □

Proof of Part A. A sequence $\{x_n\}$ converges to x, if and only if, for every $\varepsilon > 0$, eventually for large enough n

$$|x_n - x| < \varepsilon \iff ||x_n - x| - 0| < \varepsilon$$

And the right member above is equivalent to saying that $|x_n - x| \to 0$. This completes the proof. □

Proof of Part B. Suppose $x_n \to x$ and fix a $\varepsilon > 0$, then $d(x_n, x) < \varepsilon$ eventually. Using the Super Triangle Inequality in Lemma 0.1, and substituting $z = 0$, we obtain

$$|d(x_n, 0) - d(x, 0)| \leq d(x_n, x) < \varepsilon$$

Therefore $|x_n| \to |x|$. To show that the converse does not hold, simply take $x_n = (-1)^n$. □

Problem 3

WTS. *Prove three things*

(a) Give examples of divergent sequences $\{x_n\}$, and $\{y_n\}$ such that $\{x_n y_n\}$ converges.

(b) Prove that if $\{x_n\}$ and $\{x_n y_n\}$ are both convergent, and if $\lim x_n \neq 0$, then $\{y_n\}$ is convergent.

(c) Prove that if $\{nx_n\}$ converges, then $x_n \to 0$.

Proof of Part A. Let $x_n = y_n = (-1)^n$. □

We need two lemmas for the next proof.

Lemma 0.2. *If $\{x_n\} \in \mathbb{R}$, $x_n \to x \neq 0$, then $(x_n)^{-1} \to x^{-1}$.*

Proof. If $x_n \to x \neq 0$, then x_n lies in a ball about x of radius $|x|2^{-1}$ eventually, so $x_n x \neq 0$. Moreover, using the fact that x_n lies in said ball, $x_n x \geq 0$ eventually, and

$$\begin{aligned} |x_n - x| \leq |x|2^{-1} &\iff |x_n x - x^2| \leq |x|^2 2^{-1} \\ &\iff \Big| |x_n x| - |x|^2 \Big| \leq |x|^2 2^{-1} \\ &\iff |x|^2 2^{-1} \leq |x_n x| \leq 3|x|^2 2^{-1} \\ &\implies |x_n x|^{-1} \leq (|x^2| 2^{-1})^{-1} = M^{-1} \end{aligned}$$

Now for every $M\varepsilon > 0$, the following must hold eventually,

$$|x_n - x| M^{-1} < \varepsilon \implies \frac{|x_n - x|}{|x_n x|} \leq \frac{|x_n - x|}{M} < \varepsilon$$

Therefore,

$$\left| \frac{1}{x_n} - \frac{1}{x} \right| < \varepsilon \to 0$$

□

Lemma 0.3. *Let $\{x_n\}$, and $\{y_n\}$ be sequences of reals, and if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$.*

Proof. Since $\{x_n\}$ and $\{y_n\}$ are convergent sequences, they must be bounded. So there exists some $M_1, M_2 \in \mathbb{N}^+$ with $|y| \leq M_1$ and $|x_n| \leq M_2$. Next,

$$\begin{aligned} |x_n y_n - xy| &= |x_n(y_n - y) + y(x_n - x)| \\ &\leq M_2|y_n - y| + M_1|x_n - x| \end{aligned}$$

Now fix an $\varepsilon > 0$, and there exists N_1 and N_2 so large that

$$\begin{aligned} |y_n - y| &< \varepsilon M_2^{-1} \\ |x_n - x| &< \varepsilon M_1^{-1} \end{aligned}$$

for all $n \geq N_1 + N_2$. Therefore $|x_n y_n - xy| < \varepsilon$ eventually, and this completes the proof. $\square$

Proof of Part B. Apply the Lemmas 0.2 and 0.3, writing $\lim x_n y_n = xy$ for some y, then $y_n = x_n y_n (x_n)^{-1}$, and since $x_n \neq 0$ eventually, we can conclude that

$$x_n y_n (x_n)^{-1} = y_n \to y = xy(x)^{-1} \in \mathbb{R}$$

Therefore $\{y_n\}$ converges. $\square$

Proof of Part C. Let $a = \lim n x_n$, and $0 = \lim n^{-1}$. Then $\{n x_n n^{-1}\}$ converges by Lemma 0.3, and

$$n x_n n^{-1} = x_n \to 0 = 0a$$

$\square$

Problem 4

WTS. *Show that*

(a) $\dfrac{(-1)^n n}{n+1}$ *diverges,*

(b) $((n+1)^{1/2} - (n)^{1/2})n^{1/2} \to 2^{-1}$,

(c) $((n+1)^{1/2} - (n)^{1/2})n \to +\infty$,

(d) $(3\sqrt{n})^{1/2n} \to 1$,

(e) $(a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b$, *where* $0 < a < b$, *and*

(f) $(n!)^{1/n^2} \to 1$.

First we begin with an important lemma.

0.1 Lemma

Lemma 0.4. *If* $\{x_n\}$ *is a sequence in* $\mathbb{R}$, *and if* $x_{n+1}/x_n < 0$ *eventually, and if* $|x_n| \to a > 0$, *then* x_n *diverges.*

Proof. Using the fact that $|x_n| \to a$, and fix $\varepsilon = a/2 > 0$, then

$$\Big| |x_n| - a \Big| < a/2 \iff a/2 < |x_n| < 3a/2$$

Using the fact that $x_{n+1}/x_n < 0$ eventually,

- either $-x_{n+1} = |x_{n+1}|$,
- or $|-x_n + x_{n+1}| = |x_n| + |x_{n+1}|$,

We have,

$$\begin{aligned} d(x_n, x_{n+1}) &= \Big| x_n - x_{n+1} \Big| \\ &= |x_n| + |x_{n+1}| \\ &> a/2 + a/2 \\ &> a \end{aligned} \tag{1}$$

Now suppose that $x_n \to x$ for some $x \in \mathbb{R}$, then for any $\varepsilon = a/2 > 0$, we must have

$$d(x_n, x_{n+1}) \leq d(x_n, x) + d(x_{n+1}, x)$$

Using Equation (1), we get

$$a < a/2 + d(x_{n+1}, x) \implies a/2 < d(x_{n+1}, x) < a/2$$

Therefore $x_n \not\to x$, and this completes the proof. □

Proof of 4a. We want to show $\dfrac{(-1)^n n}{n+1}$ diverges.
Let $\{x_n\}$ be the sequence in question, and we want to use Lemma 0.4, since $|x_n| = n(n+1)^{-1}$, and

$$n(n+1)^{-1} = (1 + n^{-1})^{-1} \to 1 > 0$$

Also, $x_{n+1}/x_n < 0$ for all $n \geq 1$, therefore by Lemma 0.4, $\{x_n\}$ diverges. □

Proof of 4b. We want to show

$$((n+1)^{1/2} - (n)^{1/2})n^{1/2} \to 2^{-1}$$

Let $\{x_n\}$ be the sequence in question, and note that

$$\begin{aligned} x_n &= n^{1/2}\left(\frac{1}{(n+1)^{1/2} + (n)^{1/2}}\right) \\ &= \frac{1}{1 + (1 + n^{-1})^{1/2}} \end{aligned}$$

And a moment's thought will show that $1 + (1 + n^{-1})^{1/2} \to 2$, and $x_n \to 2^{-1}$. □

Proof of 4c. We want to show

$$((n+1)^{1/2} - (n)^{1/2})n \to +\infty$$

Again, let $x_n = ((n+1)^{1/2} - (n)^{1/2})n$, notice that $x_n = n^{1/2} y_n$, where y_n is the same sequence in part 4b. Where $y_n \to 2^{-1}$. Now fix $\varepsilon = 4^{-1}$, then $|y_n - 2^{-1}| < 4^{-1}$ eventually.

Hence, $4^{-1} < |y_n|$ eventually. And multiplication by $n^{1/2}$ on both sides of the estimate gives

$$n^{1/2}4^{-1} < |x_n| = x_n$$

Now, fix some $(4M)^2 > 0$, by the Archimedean Property, there exists some n so large (frequently in $\mathbb{N}^+$) where

$$n > (4M)^2 \implies n^{1/2}4^{-1} > M \implies |x_n| = x_n > M$$

and sending $M \to +\infty$ gives $x_n \to +\infty$, and this finishes the proof. $\square$

Proof of 4d. We want to show

$$(3\sqrt{n})^{1/2n} \to 1$$

Let $x_n = (3\sqrt{n})^{1/2n}$, be the sequence in the question. Notice that

- $n^{1/n} \to 1$, and $(n^{1/n})^{1/4} = (n^{1/4})^{1/n} \to 1$,
- $3^{1/2} > 1$, therefore $(3^{1/2})^{1/n} \to 1$

The proof is complete upon multiplying the two convergent sequences above. $\square$

Remark (Remark about the two bullet points in 4d). *The first bullet point reuses a fact from Homework 4, and we took square roots on the positive sequence twice. The second bullet point is also from Homework 4.*

Proof of 4e. We want to show

$$(a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b, \quad \forall 0 < a < b$$

Let $x_n = (a^{n+1} + b^{n+1})(a^n + b^n)^{-1} \to b$, and $0 < a < b$, and define $r = b/a$ so that $r > 1, 0 < r^{-1} < 1$, with $r^{-n} \to 0$.

$$\begin{aligned} x_n &= \frac{a^{n+1}(1 + r^{n+1})}{a^n(1 + r^n)} \\ &= a(1 + r^{n+1})(1 + r^n)^{-1} \\ &= a(r^{-n} + r)(r^{-n} + 1)^{-1} \end{aligned}$$

To avoid being too verbose, $r^{-n} + 1 \to 1$, and $r^{-n} + r \to r$, therefore $x_n \to ar = b$. $\square$

Proof of 4f. We want to show

$$(n!)^{1/n^2} \to 1$$

Let $x_n = (n!)^{1/n^2}$, notice that for all $1 \leq n$,

$$1 \leq n! \implies 1 \leq x_n$$

More is true,

$$n! \leq n^n \implies (n!)^{1/n^2} \leq n^{1/n}$$

Since the constant sequence $a_n = 1$ converges to 1, and $n^{1/n} \to 1$ by Homework 4, x_n converges, and $x_n \to 1$. $\square$

Remark. *We used the Squeeze Theorem here.*

Problem 5

WTS. *Let $\{x_n\}$ be a sequence of reals, where $x_n > 0$ eventually, and if $x_{n+1}/x_n \to L > 1$, then $x_n \to +\infty$.*

Proof. This proof modifies that of the 'convergent ratio test'. Fix some $r = (1+L)/2$, then $1 < r < L$, and let $\varepsilon = L - r > 0$, so that eventually

$$\begin{aligned} \Big| x_{n+1}/x_n - L \Big| < \varepsilon &\iff -\varepsilon + L < x_{n+1}/x_n \\ &\iff r x_n < x_{n+1} \end{aligned} \tag{2}$$

Now, fix some $N \in \mathbb{N}$ such that x_n satisfies Equation (2) and $x_n > 0$ for every $n \geq N$. A simple induction will show that for all $k \geq 1$,

$$r^k x_N < x_{N+k} \tag{3}$$

Let $c = x_N > 0$, and $\Lambda = r > 1$, and the proof is complete upon showing that $r^k x_N \to +\infty$. Indeed, observe that $1 < r \iff 0 < r^{-1} < 1$, and $r^{-k} \to 0$, as $k \to +\infty$.

Let $M/x_N > 0$ be arbitrary, and multiply M/x_N with the sequence r^{-k}. So that $M/x_N r^{-k} \to 0$ as well, and choosing $\varepsilon = 1$,

$$|M/x_N r^{-k}| < 1$$

and it immediately follows that

$$\begin{aligned} M/x_N r^{-k} < 1 &\iff x_N/M > r^{-k} \\ &\iff M/x_N < r^k \\ &\iff M < r^k x_N < x_{N+k} \end{aligned}$$

□

Problem 6

WTS. *Apply the ratio test to determine the convergence of*

(a) $b^n/n!$,

(b) b^n/n^2

Proof of 6a. Let $x_n = b^n/n!$, and using the ratio test yields

$$|x_{n+1}/x_n| = b(n!/(n+1)!)^{-1} \to 0 < 1$$

Therefore $x_n \to 0$. $\square$

Proof of 6b. Let $x_n = b^n/n^2$, and

$$|x_{n+1}/x_n| = b(n/(n+1))^2 = b(1+n^{-1})^{-2} \to b > 1$$

Using Question 5, since $x_n > 0$ eventually, $x_n \to +\infty$. $\square$

Problem 7

WTS. *Show that if* $\{x_n\}$ *is a monotone sequence, then*

- $x_n \to \sup\{x_n\}$ *if* $\{x_n\}$ *is non-decreasing, and*
- $x_n \to \inf\{x_n\}$ *if* $\{x_n\}$ *is non-increasing.*

where we allow $\sup\{x_n\}$ *and* $\inf\{x_n\}$ *to also be symbols in the extended reals.*

We need a small lemma before proceeding.

0.2 Lemma

Lemma 0.5. *If* A *is a non-empty bounded above subset of* $\mathbb{R}$, *then* $(-1)(\sup A) = \inf(-1)A$.

Proof. Let $s = \sup(A)$, then

$$(-1)A = \{-x : x \in A\}$$

Evidently, for every $x \in A$

$$x \leq s \implies -s \leq -x \implies -s \leq (-1)A$$

Now fix any $\varepsilon > 0$, then there exists an $x \in A$ such that

$$s - \varepsilon \leq x \implies (-1)(s - \varepsilon) \geq -x \implies (-x) \leq (-s) + \varepsilon$$

Thus establishes $\inf(-1)A = (-1)\sup(A)$. □

Corollary 0.5.1. $(-1)\inf(A) = \sup(-1)A$. *The proof is trivial just replace* A *by* $(-1)A$.

Proof of Problem 7. Suppose that $\{x_n\}$ is non-decreasing, if $\sup\{x_n\} = +\infty$ it means that for any M $x_n \geq M$ eventually, so that $x_n \to +\infty = \sup\{x_n\}$. If $\sup\{x_n\}$ is finite, then for any $\varepsilon > 0$, there exists $x_N \in \{x_n\}$ with $\sup\{x_n\} - \varepsilon < x_N$, but $\{x_n\}$ is non-decreasing, this implies that for every $n \geq N$,

$$\sup\{x_n\} - \varepsilon < x_N \leq x_{n \geq N} \leq \sup\{x_n\} < \sup\{x_n\} + \varepsilon$$

and $x_n \in V_\varepsilon(\sup\{x_n\})$ eventually; therefore $x_n \to \sup\{x_n\}$.

If $\{x_n\}$ is non-increasing, and if $\inf\{x_n\} = -\infty$ then it is trivial to modify the above proof to show that $x_n \to -\infty = \inf\{x_n\}$. For the case where $\inf\{x_n\}$ is finite, notice that $\sup\{(-1)x_n\} = (-1)\inf\{x_n\} \neq +\infty$ by Lemma 0.5, and $\{(-1)x_n\}$ is a monotone, non-decreasing sequence, hence

$$(-1)x_n \to \sup\{(-1)x_n\} \implies x_n \to (-1)\sup\{(-1)x_n\} = \inf\{x_n\}$$

$\square$