MATH 254 Assignment 1

November 5, 2022

1a

WTS. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We can use a chain of equivalences. Suppose that both members are not empty.

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \land x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \land x \in B) \text{ or } (x \in A \land x \in C) \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Now suppose one of the two members are empty. Then if the other member was not empty, it would imply that the original member was not empty, and this means that the two sets must be equal. $\square$

2

WTS. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Define $W = (A \setminus B) \cup (B \setminus A)$, then we will apply Q1a, and deMorgan's Theorem.

$$
\begin{aligned}
W^c &= (A^c \cup B) \cap (B^c \cup A) \\
&= ((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A) \\
&= (A^c \cap B^c) \cup (A \cap B) \\
&= (A \cup B)^c \cup (A \cap B) \\
&= [(A \cup B) \setminus (A \cap B)]^c
\end{aligned}
$$

Taking complements on both sides finishes the proof. $\square$

3a

WTS. $f \coloneqq \{(x,y) \in [-1,+1] \times [-1,+1] : x^2 + y^2 = 1\}$ *is not a function.*

Proof. f is not a function because $(0,1) \in f$ and $(0,-1) \in f$, and $1 \neq -1$. $\square$

3b

WTS. *f is a function.*

Proof. Since $y \geq 0$, we can write $y = +\sqrt{1-x^2}$. Fix an $x \in [-1, 1]$, then there is a unique $y \in [0, 1]$ that satisfies the above. Also, for every $x \in [-1, 1]$, $|y| \leq 1$. Therefore f is a function. $\square$

4a

WTS. $f(f^{-1}([-4,-1] \cup [1,4])) = [1,4]$, *where* $f = x^{-2}$ *for every* $x \neq 0$.

Proof. Write $W = f^{-1}([-4,-1] \cup [1,4])$, and because the inverse image preserves intersections and unions by Q5, and $[-4,-1] \cap \{f(x) : x \in \mathbb{R} \setminus \{0\}\} = \varnothing$, then $f^{-1}[-4,-1] = \varnothing$. Which means $W = f^{-1}[1,4]$ and hence $f(W) = [1,4]$, as $[1,4] \subseteq \{f(x) : x \in \mathbb{R} \setminus \{0\}\}$. $\square$

4b

WTS. $f^{-1}(f(1,2)) = [-2,-1] \cup [1,2]$. *Where* $f = x^-2$*, for every* $x \neq 0$.

Proof. The equality is obvious by inspection. $\square$

4cd

WTS. *$f(f^{-1}B) = B$ if f is a surjection, and $f^{-1}(f(B)) = B$ if f is an injection.*

We split this problem into two parts. We begin with the first assertion. Write $R = \{f(x) : x \in A\}$.

Lemma 0.1. *For every function $f : X \to Y$, $f(f^{-1}(B)) \subseteq B$.*

Proof. Use Q5a) onto the disjoint sets $f^{-1}(B \cap R)$ and $f^{-1}(B \cap R^c)$, then

$$f^{-1}(B) = f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)$$

Now $f^{-1}(B \cap R^c)$ must be empty, since no $x \in A$ satisfies $f(x) \in B \cap R^c$. Hence $f^{-1}(B) = f^{-1}(B \cap R)$.

$$\begin{aligned} f(f^{-1}(B)) &= f(f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \cup f(f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \\ &= \{f(x) : x \in f^{-1}(B \cap R)\} \\ &= \{y : y \in (B \cap R)\} \\ &= B \cap R \end{aligned}$$

Where for the second last equality we used the fact that f is always a surjection onto its range. Then $f(f^{-1}(B)) = B \cap R \subseteq B$. $\square$

Remark. *If f is a surjection, then its range $R = Y$, then $f(f^{-1}(B)) = B \cap Y = B$.*

Lemma 0.2. *For every function $f : X \to Y$, $A \subseteq f^{-1}(f(A))$.*

Proof. Write $f^{-1}(f(A))$ as the disjoint union of $A \cap f^{-1}(f(A))$ and $A^c \cap f^{-1}(f(A))$. Then, we shall show that $f^{-1}(f(A)) = A$. For every $x \in A$,

$$\begin{aligned} f(x) \in f(A) \wedge x \in A &\iff x \in f^{-1}(f(A)) \wedge x \in A \\ &\iff x \in A \cap \left(f^{-1}(f(A))\right) \end{aligned}$$

Hence $A \cap f^{-1}(f(A)) = A$, and $A \subseteq f^{-1}(f(A))$

Remark. *If f is a injection, then for every $x \in A^c$, $f(x) \notin f(A)$, then $A^c \cap f^{-1}(f(A)) = \varnothing$, and*

$$f^{-1}(f(A)) = \left[A \cap f^{-1}(f(A))\right] \cup \left[A^c \cap f^{-1}(f(A))\right] = A$$

$\square$

5a

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.*

Proof. Fix two subsets $B_1, B_2 \subseteq B$, then

$$\begin{aligned}
f^{-1}(B_1 \cup B_2) &= \{x \in A, f(x) \in B_1 \cup B_2\} \\
&= \{x \in A, f(x) \in B_1 \text{ or } f(x) \in B_2\} \\
&= \{x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)\} \\
&= f^{-1}(B_1) \cup f^{-1}(B_2)
\end{aligned}$$

$\square$

5b

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.*

Proof. For any two sets $A_1, A_2 \subseteq A$,

$$\begin{aligned} f(A_1 \cup A_2) &= \{f(x) : x \in A_1 \cup A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } x \in A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } f(x) : x \in A_2\} \\ &= f(A_1) \cup f(A_2) \end{aligned}$$

□

5c

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$. Show that $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.*

Proof.

Lemma 0.3. *f^{-1} preserves complements.*

Proof. For every $E \subseteq B$,

$$\begin{aligned} f^{-1}(B \setminus E) &= \{x \in A : f(x) \in B \setminus E\} \\ &= \{x \in A, f(x) \in E^c\} \\ &= A \setminus f^{-1}(E) \end{aligned}$$

□

Lemma 0.4. *f^{-1} preserves intersections.*

Proof. Now we wish to prove that f^{-1} preserves intersections as well, for every pair of subsets, $B_1, B_2 \subseteq B$. Write their intersection as $(B_1^c \cup B_2^c)^c$, apply Q5a, and take complements.

$$\begin{aligned} f^{-1}\left((B_1^c \cup B_2^c)^c\right) &= (f^{-1}(B_1^c) \cup f^{-1}(B_2^c))^c \\ &= f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

□

To prove the assertion in Q5c, write $B_1 \setminus B_2 = B_1 \cap B_2^c$, and apply the two Lemmas. □

5d

WTS. *Provide an example such that $f(A_1 \setminus A_2) \neq f(A_1) \setminus f(A_2)$. Provide a condition on f that implies $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$.*

We begin our answer with the example. Suppose $f \in L^{p*}$, where f maps every element to 0. Take two subsets of $A \subseteq L^p$, $A_1 = \{g_1\} \neq \{g_2\} = A_2$. Then $f(A_1) \setminus f(A_2) = \varnothing$, but $A_1 \setminus A_2 = A_1$, and $f(A_1 \setminus A_2) = \{0\} \neq \varnothing$.

The condition we want to impose on f is that it must be an injection, we will prove that it satisfies the assertion.

Proof.

Lemma 0.5. *The direct image is monotonic. For every $E_1 \subseteq E_2 \subseteq A$, then $f(E_1) \subseteq f(E_2) \subseteq B$.*

Proof. Apply Q5b) to sets $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$, then $f(E_2) = f(E_2 \setminus E_1) \cup f(E_2 \cap E_1)$ implies that $f(E_1) \subseteq f(E_2)$. □

Lemma 0.6. *For every pair of subsets, $E_1, E_2 \subseteq A$, then $f(E_1) \setminus f(E_2) \subseteq f(E_1 \setminus E_2)$.*

Proof. If the left member is empty, then it is trivial. If not, then fix an element $y \in f(E_1) \setminus f(E_2)$, then $y \in f(E_1)$ and $y \in f(E_2)^c$.

This is equivalent to saying that there exists a $x_1 \in E_1$ such that $f(x_1) = y$; and for every $x_2 \in E_2$, $f(x_2) \neq y$, and therefore x_1 is not a member of E_2, since f is a function. It follows that $x_1 \in E_1 \setminus E_2$, and $f(x_1) = y \in f(E_1 \setminus E_2)$. Since y is arbitrary, we are done. □

Suppose f is an injection, then for every $x \neq p \in A$ implies that $f(x) \neq f(p)$. We wish to prove the reverse estimate in the second Lemma. Fix a member in $y \in f(E_1 \setminus E_2)$, then this y induces an $x \in E_1 \setminus E_2$. Since the two sets $(E_1 \setminus E_2)$ and E_2 are disjoint, for every $p \in E_2$, $x \neq p$ yields $f(x) \neq f(p)$; and $f(x) = y \in f(E_2)^c$. But this y is also a member of $f(E_1)$ by the first Lemma, if we simply take $E_1 \setminus E_2 \subseteq E_1 \subseteq A$. Therefore $y \in f(E_1) \setminus f(E_2)$. This completes the proof. □

6a

WTS. *Show that $f(x) = x/(|x|+1)$ is a bijection from $\mathbb{R}$ to $(-1,+1)$.*

Proof. We begin with an important Lemma.

Lemma 0.7. *For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|_A + f|_{A^c}$, and Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restriction of f onto A and A^c are bijections onto their direct images, then f is a bijection.*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where we shall omit the trivial case of them both belonging to the same A or A^c. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by assumption $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f|_{A^c}(A^c)$. So $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in f_A(A)$ or $y \in f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets are bijections, there exists a corresponding $x \in X$ which will satisfy. This completes the proof. □

Now we want to prove the original assertion. To use the lemma, take $A = [0,+\infty) \subseteq \mathbb{R}$. We will satisfy the assumptions of the Lemma. First, for any $x \in A$, $f(x) = 1 - 1/(x+1)$. Injectivity is obvious at first glance, and we claim that $f|_A(A) = [0,1)$. To show that $f|_A(A) \subseteq [0,1)$, notice

$$f|_A = 1 - \frac{1}{x+1} \geq 0, \quad \forall x \in [0,+\infty)$$

$$f|_A \geq 1 \implies 1 - \frac{1}{x+1} \geq 1 \implies x \leq -1 \implies x \in A^c$$

Then $f|_A(A) \subseteq [0,1)$ as required. Now to show the converse, fix any $y \in [0,1)$, then there exists an $x = (1-y)^{-1} - 1 \in A$. Thus we have proven that $f|_A$ is a bijection onto its direct image.

Next for $f|_{A^c}(x) = -1 + 1/(1-x)$ for any $x \in A^c$. It is trivial to show that $f|_{A^c}$ is an injection. So, fix any $y \in (-1,0)$ and there corresponds an $x = 1 - (y+1)^{-1} \in A^c$. Hence $(-1,0) \subseteq f|_{A^c}(A^c)$. To show the reverse, we will proceed by contradiction. So suppose there exists an $x \in A^c$ such that $f|_{A^c}(x) \geq 0$, which means that $f|_{A^c}(x) \in A$, then a cool way to arrive at a contradiction this would be to plug $y = f|_{A^c}(x)$ into $f|_A(y) \in [0,1)$, hence we have

$$\begin{aligned} f|_A(y) &= y/(y+1) \\ &= \frac{x/(1-x)}{x/(1-x)+1} \\ &= x \in [0,1) \end{aligned}$$

But $x \in A^c$ by assumption, so we have a contradiction. Suppose now, there exists an $x \in A^c$ such that $f|_{A^c}(x) \leq 1$, then

$$\begin{aligned} \frac{x}{1-x} &\leq 1 \\ -x/(1-x) &\geq 1 \\ 1 - 1/(1-x) &\geq 1 \\ 1/(1-x) &\leq 0 \\ 1 &\leq x \end{aligned}$$

And the contradiction establishes the bijection. Since $f|_A(A) = [0,1)$, and $f|_{A^c}(A^c) = (-1,0)$. $Y = (-1,1)$ is the disjoint union of these two sets, we can finally apply the Lemma, and the proof is complete. $\square$

6b

WTS. *Show that*

$$f(x) = (x+1)(m/2) + a$$

induces a bijection from $(-1,1) \to (a,b)$ *for every* $m = b - a > 0$.

Proof. Since $m \neq 0$, f is obviously injective. And for every $y \in (a,b)$, one can easily find an

$$x = (y-a)(2/m) + (-1) \in (-1,1)$$

To show that $f \in (a,b)$, we can attempt the contrapositive. $f \leq a$ or $f \geq b$ implies $|x| \geq 1$.

$$\begin{aligned} |f(x) - (a+b)/2| &\geq m/2 \\ |(x+1)(m/2) + 2a/2 - (a+b)/2| &\geq m/2 \\ |(x+1)m + 2a - a - b| &\geq m \\ |x| &\geq 1 \end{aligned}$$

This establishes the bijection. □