Problem 1

WTS. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Prove*

(a) *If $\{x_n\}$ is unbounded above, then $\limsup x_n = +\infty$, and there exists a subsequence y_k of x_n with $\lim y_k = +\infty$. Conversely, if $\{x_n\}$ is unbounded below, then $\liminf x_n = -\infty$, etc.*

(b) *$\{x_n\}$ converges to an $x \in \mathbb{R}$, or diverges to $\pm\infty$ if and only if*

$$\limsup x_n = \liminf x_n \tag{1}$$

Furthermore, we will denote the tail of $\{x_n\}$ by E_m, with

$$E_m = \{x_{n \geq m}\}$$

Let us utilize the following powerful Theorems

0.1 Eventual Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. We define the m-tail of the sequence,*

$$E_m = \{x_n,\, n \geq m\}$$

If $A \subseteq X$ is any set, the following are equivalent.

(a) *$x_n \in A$ eventually,*

(b) *$E_m \subseteq A$ eventually, (as $m \to \infty$),*

(c) *$A^c \cap E_m = \varnothing$ eventually, (as $m \to \infty$),*

(d) *$A^c \cap \{x_n\}$ is finite,*

(e) *it is false that $x_n \in A^c$ frequently,*

(f) *no subsequence x_{n_k} of x_n can lie in A^c eventually, (as $k \to \infty$),*

(g) *every subsequence of x_n can be found frequently in A*

Proof. Suppose (a) holds, then $\{x_{n\geq N}\} \subseteq A$. So $E_N \subseteq A$, and for every $m \geq N$, $E_m \subseteq E_N \subseteq A$, so (a) $\implies$ (b).

Suppose (b) holds, then

$$E_m \subseteq A \iff A^c \cap E_m = \varnothing, \qquad \text{eventually}$$

Hence (c) follows.

To show (c) $\implies$ (d), we assume (d) is false. So $A^c \cap \{x_n\}$ is infinite, and denote

$$\mathcal{N} = \left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\}$$

is an unbounded set. Now choose any $m \in \mathbb{N}^+$, so this m must not be an upper-bound of $\mathcal{N}$ (otherwise $\mathcal{N}$ would be bounded above, and therefore finite). For this m, there exists an $n > m'$, where $n \in \mathcal{N}$, with

$$x_n \in A^c \cap E_m \implies A^c \cap E_m \neq \varnothing$$

This holds for every m (we have proven a negation that is stronger than the negation of (c)), and (c) is invalid. Therefore (c) $\implies$ (d).

Suppose now (d) holds. Since $A^c \cap \{x_n\}$ is finite, there exists an $N \in \mathbb{N}^+$ where

$$N = \max\left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\} + 1$$

for every $n \geq N$, we have $x_n \notin A^c$. So $x_n \notin A^c$ eventually $\iff$ the claim that x_n is in A^c frequently is false, and (e) follows.

Now suppose (e), unboxing the quantifiers, reads

$$\neg\left(\forall N \in \mathbb{N}^+,\, \exists n \geq N,\, x_n \in A^c\right) \iff \exists N \in \mathbb{N}^+,\, \forall n \geq N,\, x_n \in A$$

The right member is equivalent to claim (a).

To show (f) is indeed equivalent with the rest. Suppose claim (d) does not hold. So $A^c \cap \{x_n\}$ is infinite. Let $\mathcal{K} = \{n \in \mathbb{N}^+,\, x_n \in A^c\}$ is an infinite set of natural numbers, and is therefore unbounded above. Following the argument within (c) $\implies$ (d), we can construct an increasing sequence of naturals $n_1 < n_2 < \dots$ such that $n_k \in \mathcal{K}$, and

$$\{x_{n_k}\} \subseteq A^c$$

This proves $\neg$(d) $\implies$ $\neg$(f). To show the converse, suppose that $x_{n_k} \in A^c$ eventually, then the set of naturals (also denoted by $\mathcal{K}$),

$$\mathcal{K} = \left\{k \in \mathbb{N}^+,\, x_{n_k} \in A^c\right\}$$

is an infinite set, so (d) is false.

Lastly, to show $(f) \iff (g)$, we unbox the quantifiers

$$\begin{aligned}
(g) &\iff \forall \{x_{n_k}\} \subseteq x_n, \neg\left(\exists K \in \mathbb{N}^+,\, \forall k \geq K,\, x_{n_k} \in A^c\right) \\
&\iff \forall \{x_{n_k}\} \subseteq x_n, \forall k \in \mathbb{N}^+,\, \exists k \geq K,\, x_{n_k} \in A^c \\
&\iff (f)
\end{aligned}$$

This completes the proof. $\square$

0.2 Frequent Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. Let E_m be the m-tail of the sequence as usual. If $A \subseteq X$ is any set, the following are equivalent.*

(a) $x_n \in A$ frequently,

(b) it is false that $x_n \in A^c$ eventually,

(c) $A \cap E_m$ is infinite, for every $m \geq 1$,

(d) there exists a subsequence x_{n_k} of x_n that lies in A eventually,

Proof. Notice that (a) is equivalent to the negation of Theorem 0.6a, but with A taking the place of A^c (within Theorem 0.6).

It immediately follows that

$$(a) \iff (b) \iff (c) \iff (d)$$

and the proof is complete. □

Corollary 0.0.1. *If x_n is in A eventually, then x_n lies in A frequently. Or the contrapositive: if x_n is in A^c frequently, then x_n does not lie in A eventually.*

0.3 sup, inf with unbounded sets

Lemma 0.1. *If A is a subset of $\mathbb{R}$, then $\sup(A) = +\infty$, if and only if $\inf(-1(A)) = -\infty$.*

Proof. Fix any $-M \in \mathbb{R}$, there exists some $x \in A$ with $x > -M \iff (-1)x < M$, for any arbitrary M, this proves $\implies$. The converse is trivial if we read the statement backwards. □

Lemma 0.2. *If $\{x_n\}$ is a real valued sequence, then*

$$\sup E_1 = +\infty \iff \sup E_m = +\infty, \quad \forall m \geq 1$$

Furthermore,

$$\inf E_1 = -\infty \iff \inf E_m = -\infty, \quad \forall m \geq 1$$

Proof. Suppose $\sup E_1 = +\infty$, and by contradiction that there exists some $m \geq 1$, with $\sup E_M < +\infty$. Then

$$|x_n| \leq \sum_{j<m} |x_j| + \sup E_m < +\infty$$

and it follows that $\sup E_1 < +\infty$, and the converse is trivial. Since $\sup E_m = +\infty$ directly implies the first claim.

The second statement is trivial upon applying the previous part and Lemma 0.1 □

Lemma 0.3. *If $\{x_n\}$ is any sequence in $\mathbb{R}$, then*

$$\begin{aligned} \limsup x_n = +\infty &\iff \sup E_1 = +\infty \\ \liminf x_n = -\infty &\iff \inf E_1 = -\infty \end{aligned}$$

that is to say, $\limsup x_n = +\infty$ *if and only if* $\{x_n\}$ *is unbounded above, and* $\liminf x_n$ *respectively.*

Proof. If $\sup E_1 = +\infty$, then $\limsup E_m = +\infty$ by Lemma 0.2. Conversely, if E_1 is bouned above, then $\sup E_m \leq \sup E_1 < +\infty$, and

$$\limsup E_m < +\infty \implies \limsup E_m \neq +\infty$$

The second statement follows after a simple modification of the proof above. □

Proof of Question 1 Part A. Suppose that $\{x_n\}$ is unbounded above, then $\sup E_1 = +\infty$. Using Lemma 0.2, $\sup E_m = +\infty$ for every $m \geq 1$, then

$$\lim_m \sup E_m \approxeq \lim_m +\infty \approxeq +\infty$$

We will construct a sequence that diverges to $+\infty$. Let us agree to define

$$\mathcal{N}(M) = \left\{ n \in \mathbb{N}^+,\, x_n > M \right\} \neq \varnothing,\, \forall M \in \mathbb{R}$$

1. Choose $n_1 = \operatorname{least} \mathcal{N}(1)$,
2. Suppose $n_1 < n_2, \ldots < n_k$ have been chosen, and $n_1 > 1, \ldots n_k > k$,
3. We can select $n_{k+1} = \operatorname{least} \mathcal{N}(k + 1 + x_{n_k})$, so that $n_{k+1} > n_k$, and $x_{n_{k+1}} > k + 1$.

Clearly, $x_{n_k} \to +\infty$, and the proof is complete.
Likewise, suppose that $\{x_n\}$ is unbounded below, then $(-1)x_n$ is a sequence that is unbounded above (this is justified using Lemma 0.1). Using the same construct as above, obtain a sequence $\{(-1)x_{n_k}\}$ that diverges to $+\infty$, so that for every $-M \in \mathbb{R}$,

$$(-1)x_{n_k} > -M \implies x_{n_k} < M$$

And the subsequence $x_{n_k} \to -\infty$, and $\liminf x_n = -\infty$ is a matter of applying Lemma 0.2,

$$\inf E_1 = -\infty \iff \liminf E_m = -\infty$$

$\square$

For Part B of the proof, we equip ourselves with the following powerful lemmas.

0.4 sup, inf of A, B when A subset of B

WTS. *If* $A \subseteq B \subseteq \mathbb{R}$, *then* $\sup(A) \leq \sup(B)$, *and* $\inf(A) \geq \inf(B)$.

Proof. If we allow for the sup and inf of A and B to take on symbols in the extended reals. Then, $\sup(B)$ is an upper-bound for A and $\inf(B)$ is a lower-bound for A, therefore

$$\sup(A) \leq \sup(B), \quad \inf(A) \geq \inf(B)$$

□

0.5 Every element in A is less than every element in B

WTS. *If A, B are non-empty subsets of* $\mathbb{R}$,

$$\sup A \leq \inf B \iff \forall a \in A, \forall b \in B, \, a \leq b$$

Proof. Suppose that $\sup A \leq \inf B$, then for every $a \in A$, and we can safely assume that both $\sup A$ and $\inf B$ are finite (see remark),

$$a \leq \sup A \leq \inf B$$

so that a is a lower bound for B, but this is equivalent to saying that $a \leq b$ for every $b \in B$.

Now suppose that for every $a, b \in A, B$, $a \leq b$. Then every single $b \in B$ is an upper bound for the set A, therefore

$$\forall b \in B, \sup A \leq b \implies \sup A \leq \inf B$$

where the last estimate is due to $\sup A$ being a lower bound for B. □

Remark. *If* $\sup A = +\infty$, *then* $\inf B = +\infty$, *this can only happen if* $B = \varnothing$, *so* $\sup A = +\infty$ *is impossible, so is* $\inf B = -\infty$. *(We assume that A and B are subsets of* $\mathbb{R}$, *and not of* $\overline{\mathbb{R}}$*).*

Further, if $\sup A = -\infty$, *then either* $A = \{-\infty\}$ *which is not a subset of* $\mathbb{R}$, *or* $A = \varnothing$, *which is again impossible.*

Proof of Question 1 Part B. To begin, notice that for every $m \geq 1$, using the same notation as Part A, where $E_m = \{x_{n \geq m}\}$. Let us assume that E_1 is a bounded subset of $\mathbb{R}$. Then,

- If $m = k$, then
$$\inf E_m \leq \sup E_m$$
- If $m \leq k$, then
$$E_m \supseteq E_k \implies \inf E_m \leq \inf E_k \leq \sup E_k$$
by Lemma 0.4.
- If $m \geq k$, then
$$E_k \supseteq E_m \implies \inf E_m \leq \sup E_m \leq \sup E_k$$
also by by Lemma 0.4.
- Therefore for any $m, k \in \mathbb{N}^+$,
$$\inf E_m \leq \sup E_k$$
- Applying Lemma 0.5 gives
$$\sup \inf E_m \leq \inf \sup E_m \iff \liminf x_n \leq \limsup x_n \tag{2}$$
- Alternatively, we can prove Equation (2) by using the Monotone Convergence Theorem (because E_1 is bounded). Indeed, (2) reads
$$\lim_m \inf E_m \leq \lim_m \sup E_m \iff \liminf x_n \leq \limsup x_n$$

Suppose $x_n \to x \in \mathbb{R}$, then for any $\varepsilon > 0$, $x_n \in V_\varepsilon(x)$ eventually. By Lemma

0.6, $E_m \subseteq V_\varepsilon(x)$ eventually. Hence,

$$\begin{aligned}
E_m \subseteq V_\varepsilon(x) &\iff E_m \subseteq (x-\varepsilon, x+\varepsilon) \\
&\iff x-\varepsilon \leq \inf E_m \leq \sup E_m \leq x+\varepsilon \\
&\iff \begin{cases} x-\varepsilon \leq \inf E_m \leq \sup\inf E_m \leq \sup E_1 \\ \inf E_1 \leq \inf\sup E_m \leq \sup E_m \leq x+\varepsilon \end{cases} \\
&\iff \begin{cases} x - \left(\sup\inf E_m\right) \leq \varepsilon \\ \left(\inf\sup E_m\right) - x \leq \varepsilon \end{cases} \\
&\iff \inf\sup E_m \leq x \leq \sup\inf E_m \\
&\iff \inf\sup E_m \leq \sup\inf E_m
\end{aligned} \tag{3}$$

Combining (3) with (2) gives $\liminf x_n = \limsup x_n$.

On the other hand, if Equation (1) holds, and E_1 is bounded, let $\liminf x_n = x = \limsup x_n$. Then for every $\varepsilon > 0$, both $\sup E_m$ and $\inf E_m$ must belong within this ε-ball about x eventually. And by Theorem 0.6

$$\{\sup E_m\}_{m\geq N} \subseteq V_\varepsilon(x)$$
$$\{\inf E_m\}_{m\geq N} \subseteq V_\varepsilon(x)$$

Which reads, for every $m \geq N$,

$$x-\varepsilon \leq \sup E_m \leq x+\varepsilon$$
$$x-\varepsilon \leq \inf E_m \leq x+\varepsilon$$

Applying Equation (2) to the bounded sets $E_m \subseteq E_1$, yields

$$x-\varepsilon \leq \inf E_m \leq \sup E_m \leq x+\varepsilon$$

So that $E_m \subseteq [\inf E_m, \sup E_m] \subseteq V_\varepsilon(x)$ eventually. But by Theorem 0.6, this is to say that $x_n \in V_\varepsilon(x)$ eventually, so $x_n \to x$.

For the unbounded case, suppose $x_n \to +\infty$. Clearly $\sup E_1 = \infty$, and by Lemma 0.3,

$$x_n \geq L + \varepsilon_0 \text{ eventually} \implies x_{n_k} \geq L + \varepsilon_0 \text{ xeventually}$$

Now it suffices to show that $\liminf x_n = +\infty$. Notice that for every $M \in \mathbb{R}$, $E_m \subseteq [M, +\infty)$ eventually. So $\inf E_m \geq M$, but by Lemma 0.2, therefore

$$-\infty < \inf E_m \iff -\infty < \liminf E_m$$

Now, $\{\inf E_m\}_{m\geq 1}$ is a non-decreasing sequence that converges to its supremum. But no finite number can be an upper bound for $\inf E_m$, so

$$\sup_{m\geq 1} \inf E_m = \liminf E_m = \liminf x_n = +\infty$$

Conversely, let us assume that $\limsup x_n = \liminf x_n = +\infty$. It is obvious that $\sup E_1 = +\infty$, and $\inf E_1 > -\infty$ by Lemmas 0.3 and 0.2. Also,

(i) A monotonic sequence in $\{\inf E_m\}_{m\geq 1}$ increases towards its supremum, which in this case is $+\infty$.

(ii) This is equivalent to saying $\inf E_m \geq M$ eventually, for every $M \in \mathbb{R}$.

(iii) Now, for all $x_n \in E_m \implies x_n \geq \inf E_m \geq M$ eventually, and sending $M \to +\infty$ proves $x_n \to +\infty$.

Let us prove $x_n \to -\infty \iff \limsup x_n = \liminf x_n = -\infty$. If $x_n \to -\infty$, it is clear that $(-1)x_n \to +\infty$. So that

$$(-1)x_n \to +\infty \iff \limsup(-1)E_m = \liminf_m(-1)E_m = +\infty$$

Apply Lemma 0.1 so that

$$\sup(-1)E_m = +\infty \iff \inf E_m = -\infty$$

Then, apply Lemmas 0.2 and 0.3 to the rightmost equality, which yields

$$\inf E_m = -\infty, \forall m \geq 1 \implies \liminf E_m = -\infty$$

Likewise, a simple application of the two Lemmas will give us $\limsup E_m = -\infty$. This proves $\implies$.

To show the converse, use Lemma 0.3, to obtain

$$\begin{aligned} \limsup E_m \neq +\infty &\iff \sup E_1 \neq +\infty \\ \liminf E_m = -\infty &\iff \inf E_1 = -\infty \end{aligned}$$

Now, modify the procedure (i), by forcing the m-tail of the sequence to live in $(-\infty, M]$ eventually, thus concluding that $x_n \to -\infty$. $\square$

Problem 2

WTS. *Show that* $x_n = (1 + n^{-2})^{2n^2} \to e^2$

Proof. Using the definition of e,

$$e = \lim(1 + k^{-1})^k, \quad e_k = (1 + k^{-1})^k \tag{4}$$

Now, let $\{k_n\}_{n \geq 1} = 1, 4, 9, 16, \ldots$. Clearly, $\{e_{k_n}\}$ is a subsequence of of e_k. Therefore $e_{k_n} \to e$ as $n \to \infty$. Now apply the multiplication rule two convergent sequences.

$$e_{k_n} \to e \implies e_{k_n} e_{k_n} = (1 + n^{-2})^{2n^2} \to e^2$$

$\square$

Problem 3

WTS. *If $\{x_n\}$ is a sequence in $\mathbb{R}$, show that if every subsequence of $\{x_n\}$ contains a further subsequence that converges to $L \in \mathbb{R}$, then $x_n \to L$.*

We will prove something that is much stronger. Let us consider the following two Theorems.

0.6 Eventual Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. We define the m-tail of the sequence,*

$$E_m = \{x_n,\, n \geq m\}$$

If $A \subseteq X$ is any set, the following are equivalent.

(a) *$x_n \in A$ eventually,*

(b) *$E_m \subseteq A$ eventually, (as $m \to \infty$),*

(c) *$A^c \cap E_m = \varnothing$ eventually, (as $m \to \infty$),*

(d) *$A^c \cap \{x_n\}$ is finite,*

(e) *it is false that $x_n \in A^c$ frequently,*

(f) *no subsequence x_{n_k} of x_n can lie in A^c eventually, (as $k \to \infty$),*

(g) *every subsequence of x_n can be found frequently in A*

Proof. Suppose (a) holds, then $\{x_{n \geq N}\} \subseteq A$. So $E_N \subseteq A$, and for every $m \geq N$, $E_m \subseteq E_N \subseteq A$, so (a) $\implies$ (b).

Suppose (b) holds, then

$$E_m \subseteq A \iff A^c \cap E_m = \varnothing, \quad \text{eventually}$$

Hence (c) follows.

To show (c) $\implies$ (d), we assume (d) is false. So $A^c \cap \{x_n\}$ is infinite, and denote

$$\mathcal{N} = \left\{ n \in \mathbb{N}^+,\, x_n \in A^c \right\}$$

is an unbounded set. Now choose any $m \in \mathbb{N}^+$, so this m must not be an upper-bound of $\mathcal{N}$ (otherwise $\mathcal{N}$ would be bounded above, and therefore finite). For this m, there exists an $n > m'$, where $n \in \mathcal{N}$, with

$$x_n \in A^c \cap E_m \implies A^c \cap E_m \neq \varnothing$$

This holds for every m (we have proven a negation that is stronger than the negation of (c)), and (c) is invalid. Therefore (c) $\implies$ (d).

Suppose now (d) holds. Since $A^c \cap \{x_n\}$ is finite, there exists an $N \in \mathbb{N}^+$ where

$$N = \max\left\{n \in \mathbb{N}^+,\, x_n \in A^c\right\} + 1$$

for every $n \geq N$, we have $x_n \notin A^c$. So $x_n \notin A^c$ eventually $\iff$ the claim that x_n is in A^c frequently is false, and (e) follows.

Now suppose (e), unboxing the quantifiers, reads

$$\neg\left(\forall N \in \mathbb{N}^+,\, \exists n \geq N,\, x_n \in A^c\right) \iff \exists N \in \mathbb{N}^+,\, \forall n \geq N,\, x_n \in A$$

The right member is equivalent to claim (a).

To show (f) is indeed equivalent with the rest. Suppose claim (d) does not hold. So $A^c \cap \{x_n\}$ is infinite. Let $\mathcal{K} = \{n \in \mathbb{N}^+,\, x_n \in A^c\}$ is an infinite set of natural numbers, and is therefore unbounded above. Following the argument within (c) $\implies$ (d), we can construct an increasing sequence of naturals $n_1 < n_2 < \ldots$ such that $n_k \in \mathcal{K}$, and

$$\{x_{n_k}\} \subseteq A^c$$

This proves $\neg$(d) $\implies$ $\neg$(f). To show the converse, suppose that $x_{n_k} \in A^c$ eventually, then the set of naturals (also denoted by $\mathcal{K}$),

$$\mathcal{K} = \left\{k \in \mathbb{N}^+,\, x_{n_k} \in A^c\right\}$$

is an infinite set, so (d) is false.

Lastly, to show $(f) \iff (g)$, we unbox the quantifiers

$$
\begin{aligned}
(g) &\iff \forall\{x_{n_k}\} \subseteq x_n, \neg\Big(\exists K \in \mathbb{N}^+, \forall k \geq K, x_{n_k} \in A^c\Big) \\
&\iff \forall\{x_{n_k}\} \subseteq x_n, \forall k \in \mathbb{N}^+, \exists k \geq K, x_{n_k} \in A^c \\
&\iff (f)
\end{aligned}
$$

This completes the proof. $\square$

0.7 Frequent Behaviour of Sequences

WTS. *Let $\{x_n\}$ be a sequence in an arbitrary space X. Let E_m be the m-tail of the sequence as usual. If $A \subseteq X$ is any set, the following are equivalent.*

(a) $x_n \in A$ frequently,

(b) it is false that $x_n \in A^c$ eventually,

(c) $A \cap E_m$ is infinite, for every $m \geq 1$,

(d) there exists a subsequence x_{n_k} of x_n that lies in A eventually,

Proof. Notice that (a) is equivalent to the negation of Theorem 0.6a, but with A taking the place of A^c (within Theorem 0.6).

It immediately follows that

$$(a) \iff (b) \iff (c) \iff (d)$$

and the proof is complete. □

Corollary 0.3.1. *If x_n is in A eventually, then x_n lies in A frequently. Or the contrapositive: if x_n is in A^c frequently, then x_n does not lie in A eventually.*

0.8 Main Proof of Q3

Proof. Let us simplify the subsequence notation for a bit, and write x_{nk} as a subsequence for x_n, and x_{nkj} as a subsequence of x_{nk} (which makes x_{nkj} a subsubsequence of x_n).

If for every x_{nk}, there exists a $x_{nkj} \to L$. This is equivalent to: for every h^{-1}, where $h \in \mathbb{N}^+$,

$$d(x_{nkj}, L) < h^{-1} \iff x_{nkj} \in V_{h^{-1}}(L), \text{ eventually}$$

And for every $V_{h^{-1}}$, Theorem 0.7(d) holds for some subsequence $x_{n_{k_j}}$ of every subsequence x_{n_k}. This is equivalent to saying that x_{n_k} lies in $V_{h^{-1}}(L)$. But Theorem 0.6g holds x_n, therefore $x_n \in V_{h^{-1}}(L)$ eventually.

But this is true if and only if $d(x_n, L) < h^{-1}$ eventually. Since this holds for every $h \geq 1$, we must conclude that $x_n \to L$. □

Problem 4

WTS. *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $\mathbb{R}$. Show that*

(a) $\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$,

(b) Give an example for when (a) is a strict inequality.

Proof. Since x_n and y_n are bounded, this makes $x_n + y_n$ bounded too. Indeed, if $|x_n| \leq M2^{-1}$ for a large M, and similarly for $|y_n|$. An application of the Triangle Inequality will show that $|x_n + y_n| \leq M$.

Notice also for any fixed $m \geq 1$,

$$\Big\{x_n + y_n,\, n \geq m\Big\} \subseteq \Big\{x_j + y_k,\, j, k \geq m\Big\}$$

Taking the supremum across both sets yields

$$\sup_{n \geq m}(x_n + y_n) \leq \sup_{n \geq m} x_n + \sup_{n \geq m} y_n$$

Finally, let $m \to +\infty$. Since this inequality holds for every $m \geq 1$, we have the following estimates for their limits

$$\limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

This proves (a). Now let $x_n = (-1)^n$, and $y_n = -x_n$. Both are bounded sequences and $\limsup x_n = \limsup y_n = 1$, but $x_n + y_n = 0$ at every n; the strict inequality follows. □

Problem 5

WTS. *Prove two things,*

(a) *Let $x_n = n^{1/2}$. Show that $|x_{n+1} - x_n| \to 0$, but x_n is not Cauchy.*

(b) *Answer the following*

(b) Even more strikingly, let (x_n) be the sequence $(\underbrace{\sqrt{m}, \sqrt{m}, \dots, \sqrt{m}}_{m})_m$, i.e.

$$(\sqrt{1}, \sqrt{2}, \sqrt{2}, \sqrt{3}, \sqrt{3}, \sqrt{3}, \sqrt{4}, \sqrt{4}, \sqrt{4}, \sqrt{4}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \sqrt{5}, \dots).$$

Prove that for each fixed $k \in \mathbb{N}$, we have $\lim_n |x_n - x_{n+k}| = 0$ but (x_n) is not Cauchy.

Remark: The moral here is that demanding for each **fixed** k separately that the distance $d(x_n, x_{n+k})$ goes to 0 as $n \to \infty$ **does not** guarantee Cauchyness, which demands that $\lim_n \sup_k d(x_n, x_{n+k}) = 0$. In particular, we **cannot switch the order** of $\lim_n$ and $\sup_k$; indeed, $\sup_k \lim_n d(x_n, x_{n+k}) = 0$ while $\lim_n \sup_k d(x_n, x_{n+k}) = \infty$.

Proof of Part A. Notice for every $k \geq 1$

$$\begin{aligned} a_n &= \left| k\left((n+k)^{1/2} + (n)^{1/2} \right)^{-1} \right| \\ &= \left| \frac{(n+k) - (n)}{(n+1)^{1/2} + (n)^{1/2}} \right| \\ &= |(n+k)^{1/2} - (n)^{1/2}| \end{aligned}$$

A simple consequence of $k^{-1}\sqrt{n} \leq a_n^{-1}$ is that $a_n \to 0$, and

$$|(n+k)^{1/2} - (n)^{1/2}| \to 0, \quad \forall k \geq 1 \tag{5}$$

Hence $|x_{n+1} - x_n| \to 0$ (by taking $k = 1$ within (5)).

x_n is obviously not Cauchy, because it is unbounded. Indeed for every $\varepsilon^2 > 0$ you can find a large $N \in \mathbb{N}^+$ where $N > \varepsilon^2$ eventually. And

$$n \geq N > \varepsilon^2 \implies x_n > \varepsilon$$

□

Proof of Part B. It is clear that $|x_{n+k} - x_n| \leq |(n+k)^{1/2} - (n)^{1/2}|$. Sending $n \to +\infty$ reads

$$|x_{n+k} - x_n| \to 0, \forall k \geq 1$$

x_n is not Cauchy because it contains an unbounded subsequence

$$\{x_{n_k}\}, \quad k \mapsto \sqrt{k}$$

We will outine the construction, for any $k \geq 1$, apply the Well Ordering Property to obtain $n_k = \text{least}\{q \in \mathbb{N}, x_n = \sqrt{q}\}$.

□

Problem 6

WTS. *Let $y_0 < y_1 \in \mathbb{R}$. Define $\{y_n\}$ for every $n \geq 2$*

$$y_n = (1/3)y_{n-1} + (2/3)y_{n-2}$$

Prove two things,

(i) Prove that $\{y_n\}$ is contractive,

(ii) Prove that $y_n \to \frac{2}{5}y_0 + \frac{3}{5}y_1$

Proof of Part A. The sequence is clearly contractive, fix any $n \geq 0$, and

$$y_{n+2} - y_{n+1} = \frac{-2}{3}\left(y_{n+1} - y_n\right) \tag{6}$$

Taking absolute values on both sides of (6), and replacing 2/3 with 1/3 finishes the proof. □

Proof of Part B. We know from Part A, that y_n is contractive, and hence Cauchy. To show that $y_n \to (2/5)y_0 + (3/5)y_1$, replace the left and right members above by

$$x_{n+2} = (-2/3)x_{n+1}, \quad x_n = y_n - y_{n-1}, \quad \forall n \geq 1 \tag{7}$$

A simple induction on $n \geq 1$ will yield

- $x_2 = \frac{-2}{3}x_1$,
- and suppose $x_j = (-2/3)^j x_1$ for every $j \geq 1$, then
- $x_{j+1} = (-2/3)^{j+1}x_1$, and this completes the induction

We require a second induction to extract y_{n+2}, and we will omit the details here. From (7), we have

$$y_{n+2} - y_0 = \sum_{j=1}^{n+2} x_j$$

$$y_{n+2} = y_0 + x_1 \sum_{j=1}^{n+2} \left(\frac{-2}{3}\right)^{j-1}$$

$$y_{n+2} = y_0 + x_1 \sum_{j=0}^{n+1} \left(\frac{-2}{3}\right)^{j}$$

Sending $n \to \infty$, noting that every subsequence of y_n must converge to the same limit, and

$$y_n \to y_0 + (y_1 - y_0)\frac{1}{1-(-2/3)} = y_0 + (y_1 - y_0)(3/5)$$

Simplifying yields

$$y_n \to \frac{2}{5}y_0 + \frac{3}{5}y_1$$

$\square$