MATH 254: Assignment 2

November 5, 2022

Problem 1

WTS. *Let $f : \mathbb{R} \to [0, +\infty)$ and $g : [0, +\infty) \to (-\infty, 0]$ be the functions defined by $f(x) \coloneqq x^2$ and $g(x) \coloneqq -\sqrt{x}$.*

(a) Explain why $f \circ g$ makes sense (can be defined) even though the codomain of g is not equal to the domain of f.

(b) Write down the domains and codomains of $f \circ g$ and $g \circ f$ and find explicit formulas for these functions.

(c) Is g the inverse of f? Explain your answer.

Answers:

(a) Fix any $x \in [0, +\infty)$, then $g(x) \in (-\infty, 0]$ means that $f(g(x)) = f \circ g(x)$ is well defined. Since x is mapped to exactly one element in $[0, +\infty)$. What matters here is that $\operatorname{range} g \subseteq \operatorname{dom} f$.

(b) Domains:
$\operatorname{dom}(f \circ g) = \operatorname{dom} g = [0, +\infty)$, and
$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$.

Codomains:
$\operatorname{codom}(f \circ g) = \operatorname{codom} f = [0, +\infty)$, and
$\operatorname{codom}(g \circ f) = \operatorname{codom} g = (-\infty, 0]$.

Formulas:

$$f \circ g(x) = (-\sqrt{x})^2 = |x| = x, \text{ For every } x \geq 0$$
$$g \circ f(x) = -\sqrt{x^2} = -|x| \neq x, \text{ For every } x \in \mathbb{R}$$

(c) g is not the inverse of f, fix $x = 1$, then $g \circ f(x) = -1 \neq 1$. Hence $g \circ f \neq \mathrm{id}_{\mathbb{R}}$.

Remark. *I do not know what the convention for the codomain is for this class. Here I assumed that for any function* $f : X \to Y$, $\operatorname{codom} f = Y$. *Its range however is the set of points in its codomain that it reaches, so* $\operatorname{range} f = \{f(x), x \in X\}$.

Problem 2

WTS. *Let $f : [0, 4) \to [0, 4)$ be defined by*

$$x \mapsto \begin{cases} x + 1 & \lfloor x \rfloor \in 2\mathbb{N} \\ x - 1 & \lfloor x \rfloor \notin 2\mathbb{N} \end{cases}$$

Where $k\mathbb{N} = \{kn : n \in \mathbb{N}\}$. Show that f is a bijection, and describe its inverse.

The proof for this question will resemble that of Homework 1 Q6(a). A few important lemmas must be stated.

Lemma 0.1. *For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|_A + f|_{A^c}$, and Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restriction of f onto A and A^c are bijections onto their direct images, then f is a bijection.*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where we shall omit the trivial case of them both belonging to the same A or A^c. Without loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by assumption $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f|_{A^c}(A^c)$. So $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in f_A(A)$ or $y \in f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets are bijections, there exists a corresponding $x \in X$ which will satisfy. This completes the proof. $\square$

Lemma 0.2. *Let f satisfy the hypothesis of the previous lemma, so that $(f|_A)^{-1}$ and $(f|_{A^c})^{-1}$ both exist, then $f^{-1} = (f|_A)^{-1} + (f|_{A^c})^{-1} = (f^{-1})|_{B_1} + (f^{-1})|_{B_2}$, where $f|_A(A) = B_1$, and $f|_{A^c}(A^c) = B_2$.*

Proof. Since B_1 and B_2 are disjoint, then fix any $y \in Y$. Without loss of generality, let us assume that $y \in B_1$. Then, $f^{-1}(y) = (f^{-1})|_{B_1}(y) = (f|_A)^{-1}(y)$. This inverse is indeed well defined, since $f|_A$ is a bijection onto its range, then there exists a unique $x \in A$ such that applying f on both sides yield

$$f\left((f|_A)^{-1}(y)\right) = f \circ (f|_A)^{-1}(y) = y$$

In the same manner, fix an $x \in A$ such that $f(x) = f|_A(x) \in B_1$, then applying $(f|_A)^{-1}$ on both sides

$$(f|_A)^{-1}(f(x)) = (f|_A)^{-1} \circ f(x) = x$$

Therefore the inverse of f can be written piecewise on two disjoint domains as follows.

$$f^{-1} = f^{-1}|_{B_1} + f^{-1}_{B_2}$$

□

Remark. *We adopt a slight abuse of notation with the 'restrictions' onto f, but they should be interpreted as piecewise functions. $f|_A + f|_{A^c}$ is equal to $f|_A\chi_A + f|_{A^c}\chi_{A^c}$ where χ is the indicator function.*

We begin the main part of the proof.

Proof. For every $x \in [0,1) \cup [2,3)$, $\lfloor x \rfloor \in 2\mathbb{N}$. Denote $A = [0,1) \cup [2,3)$, and $A^c = [0,4) \setminus A = [1,2) \cup [3,4)$. Then

$$f = f|_A + f|_{A^c} = (x+1)\chi_A + (x-1)\chi_{A^c}$$

To satisfy the assumptions of the two lemmas, we need to check if the direct images (ranges) of $f|_A$ and $f|_{A^c}$ are disjoint. This can be easily shown by

$$x \in A \iff x \in [0,1) \cup [2,3) \implies x+1 \in [1,2) \cup [3,4)$$

$$x \in A^c \iff x \in [1,2) \cup [3,4) \implies x+1 \in [0,1) \cup [2,3)$$

To avoid being overly verbose, we say that the two functions are bijections onto their ranges. Since $f|_A$ 'nudges' points $+1$ units to the right, while $f|_{A^c}$ does the exact opposite. And the range of each of the two functions is the complement of their domains. Even more is true:

$$(f|_A)^{-1} = x - 1 = f|_{A^c}$$

$$(f|_{A^c})^{-1} = x + 1 = f|_A$$

Therefore $f|_A$ and $f|_{A^c}$ are bijections onto their ranges (which are disjoint). Invoking the first lemma tells us that f is a bijection, and the second lemma gives us the following equality.

$$f^{-1} = f^{-1}|_{A^c} + f^{-1}|_A = (f|_A)^{-1} + (f|_{A^c})^{-1}$$

Plugging in our values for the inverses, we have

$$f^{-1} = f|_A + f|_{A^c} = f$$

$\square$

Problem 3

WTS. *Now it suffices to show that if* $f_1 : A \to A'$ *and* $f_2 : B \to B'$ *are bijections, then*

$$F : A \times B \to A' \times B'$$

is also a bijection if we define $\pi_{A'}(F(x)) = f_1(\pi_A(x))$ *and* $\pi_{B'}(F(x)) = f_2(\pi_B(x))$ *for every* $x \in A \times B$. *Where* π *denotes the coordinate map (or projection map).*

Proof. Fix two elements x_1, x_2 in $A \times B$, such that $x_1 \neq x_2$. Without loss of generality, let us assume that $\pi_A(x_1) \neq \pi_A(x_2)$, which implies

$$\pi_{A'}(F(x_1)) = f_1(\pi_A(x_1)) \neq f_1(\pi_A(x_2)) = \pi_{A'}(F(x_2))$$

Which means $F(x_1) \neq F(x_2)$. This proves injectivity.

To show that F is a surjection, fix any $y \in A' \times B'$, then this y induces a and b such that

$$a = f_1^{-1}(\pi_{A'}(y)) \in A$$
$$b = f_2^{-1}(\pi_{B'}(y)) \in B$$

Then denote an element of $A \times B$, and call it x, such that $\pi_A(x) = a$ and $\pi_B(x) = b$. Then it is an easy exercise to verify that $F(x) = y$, by taking the A' and B' projections of $F(x)$.

Remark. *We implicitly defined 'equality' in the Cartesian (Direct) product by equality in each coordinate. So for every* x_1, x_2 *in* $A \times B$, $x_1 = x_2$ *if and only if* $\pi_A(x_1) = \pi_A(x_2)$ *and* $\pi_B(x_1) = \pi_B(x_2)$. $\square$

Problem 4

WTS. *Prove that every subset of* $\mathbb{N}$ *is countable. Conclude that if* A *has an injection into* $\mathbb{N}$*, then* A *is countable.*

Proof. Fix any $A \subseteq \mathbb{N}$, then if A is finite, then denote $|A| \coloneqq \sum_{a\in A} 1 < +\infty$, then $J_n = \{k \in \mathbb{N}, k < n\}$, then there exists a bijection between $J_{|A|}$ and A. Hence A is countable (countably finite). We also note that there exists an order preserving map between the two sets, namely $Y_A(0)$ denotes the least element in A, etc.

Now suppose that A is infinite. Since $\mathbb{N}$ is countable, there exists a map, $\mathbf{X}_A : \mathbb{N} \to A$ such that for every $n \geq 1$.

$$\mathbf{X}_A(n) \coloneqq \text{least}\,\{k \in A, \mathbf{X}_A(n-1) < k\}$$

Where we also define $\mathbf{X}_A(0) = \text{least}\,A$. Where we used the Well-Ordering Property of $\mathbb{N}$ twice, it is trivial to check that both of these sets are non-empty.

The map $\mathbf{X}$ is monotonic (and therefore an injection). A simple proof by induction will show this. Now, $\mathbf{X}_A(0) < \mathbf{X}_A(1)$ by inspection, and assume that $\mathbf{X}_A(n-1) < \mathbf{X}_A(n)$ for some $n \geq 1$, then

$$\mathbf{X}_A(n+1) = \text{least}\,\{k \in A, \mathbf{X}_A(n-1) < \mathbf{X}_A(n) < k\}$$

Hence $\mathbf{X}_A(0) < \mathbf{X}_A(1) < \cdots < \mathbf{X}_A(n)$. Suppose that X_A is not a surjection, then there exists an non-empty set of elements of A that escape the range of $\mathbf{X}_A$. Take the least of this set, and call it m. Where this $m \neq \text{least}\,A$, since $\mathbf{X}_A(0) = \text{least}\,A$.

We then construct another subset of A and call it A^* which holds all the elements $k \in A$ such that $k < m$. Since A^* is obviously finite, we can use the same construct as shown in the earliest part of this proof. Define $N = |A^*|$, then we will show $\mathbf{X}_A(N) = m$.

$\mathbf{X}_A(n) \in A^* \iff 0 \leq n < N$. Since all the elements within A^* are strictly less than every element in A, and the number of elements within A^* is exactly N, and $\mathbf{X}_A(n)$ for $0 \leq n \leq N-1$ gives us the N smallest values in A. For any $n \geq N$, $\mathbf{X}_A(n) > q$ where q is any element in A^*. Hence $\mathbf{X}_A(n) \in A \setminus A^*$ for every $n \geq N$. Now, fix $n = N$, and this will retrieve the least element in $A \setminus A^*$, and this establishes the fact that every infinite subset of $\mathbb{N}$ is countably infinite.

$\square$

Moreover, the empty set is a subset of the natural numbers and is also countable. Now for the second part of the proof, if A is an injection into $\mathbb{N}$, then A must be countable.

Proof. Denote the injection of A into $\mathbb{N}$ by h, and its range by $\operatorname{range}(h)$, then since the range of any mapping is always a subset of its co-domain, $\operatorname{range}(h) \subseteq \mathbb{N}$, and h is actually a bijection onto its range (any mapping is a surjection onto its range), so

$$A \equiv \operatorname{range}(h)$$

But $\operatorname{range}(h)$ is a countable subset of $\mathbb{N}$, and this completes the proof. $\square$

Problem 5

WTS. *Prove that for each $n \in \mathbb{N}^+$,*

$$\sum_{j \leq n} \frac{1}{j(j+1)} = \frac{n}{n+1}$$

Where $j \leq n$ implicitly means that $1 \leq j \leq n$.

Proof. We will proceed by a proof by induction. Fix $n = 1$, and equality is trivial. Then suppose the assertion holds for a certain $n \geq 1$. Then

$$\begin{aligned}
\sum_{j \leq n+1} \frac{1}{j(j+1)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+1)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+1+1}
\end{aligned}$$

Where the second last equality is permissible because $n + 1 \neq 0$ for every $n \geq 1$. This completes the proof. $\square$

Problem 6

WTS. *For every $n \geq 12$, $n \in \mathbb{N}$, there exists non-negative integers a, b such that*

$$n = (4,5) \cdot (a,b)^T$$

Proof. I do not know how to show this rigourously for I lack training in Algebra, but regardless: For every $n \geq 12$, $(n-12) \mod 4 \in [0,3]$. Let us agree to define for each $j \in [0,3]$.

$$W_j = \{4k + 12 + j, k \in \mathbb{N}^+ \cup \{0\}\}$$

Now these sets are not necessarily disjoint, but it will not matter for the purposes of this exercise, as every $n \geq 12$ must be contained in at least one of these sets. We also write

$$\lambda : (a,b) \mapsto (4,5) \cdot (a,b)^T$$

Then we can write

- $12 = \lambda(3,0)$
- $13 = \lambda(2,1)$
- $14 = \lambda(1,2)$
- $15 = \lambda(0,3)$

For any fixed j, note that $(12+j) \in \lambda(\mathbb{N} \times \mathbb{N})$ as shown before. Now suppose that $(4k + (12+j))$ is a member of $\lambda(\mathbb{N} \times \mathbb{N})$, then this induces some (a,b) such that $\lambda(a,b) = (4k+(12+j))$. But adding 4 to both sides of the equation, and by linearity in both arguments of the inner product over $\mathbb{R}$, we have

$$(4(k+1) + (12+j)) = \lambda(a+1,b) \implies (4(k+1) + (12_j)) \in \lambda(\mathbb{N} \times \mathbb{N})$$

Hence $W_j \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. But the union of all W_j contains every $n \geq 12$. Hence $\{n \in \mathbb{N}, n \geq 12\} \subseteq \lambda(\mathbb{N} \times \mathbb{N})$. This completes the proof. $\square$

Problem 7

WTS. *A $n \times m$ grid always takes $nm - 1$ cuts to be decomposed into atomic cells.*

Proof. I am not sure why we need induction here. Take a finite set of pieces and call it $W^N = \{w_j, 1 \leq j \leq N\}$, then if $|W| = \sum_{w \in W} 1$, and if $|W^N| = N \neq nm$, then there exists a cut you can make on one of its pieces. Without loss of generality assume that this piece is $w_N \in W^N$, then $\operatorname{cut} w_N = \{a, b\}$, where a and b are pieces (atomic or not), then write the new state of the grid as

$W^{N+1} = W^N \cup \operatorname{cut}(w_N) \setminus \{w_N\}$, and relabelling indices, then we have $N + 1$ elements in our new set.

Since the cutting process is complete if and only if $|W| = nm$, and the original state of the grid is at W^1, and each cut increases the number of elements in the set by 1, all it must take $nm - |W^1| = nm - 1$ cuts.

Now I guess you can shoehorn the induction in there by saying that for every $|W^n|$, $n - 1$ cuts must have been made for each and every step, but this is equivalent to the first argument I made above. $\square$

Problem 8

WTS. *Prove that for every $a < b$ in $\mathbb{R}$, the segment (a, b) is uncountable. Conclude that every segment in the form of (a, b) must contain an irrational number.*

Proof. From Lecture 4: $[0, 1)$ is uncountable. This is equivalent to saying that no $f : [0, 1) \to \mathbb{N}$ can be injective. Therefore no $f : (-1, +1) \to \mathbb{N}$ can be injective. Since if some f were to be injective from $(-1, +1)$ then it would have to be injective on $[0, 1)$.

Using the last part of Homework 1 Q6b, define a bijection f from $(-1, +1)$ to (a, b) where $a < b$.

$$f(x) = (x + 1)(m/2) + a, \quad m := b - a$$

It follows immediately that $(a, b) \equiv (-1, +1)$, which is an uncountable set. This proves the first claim.

To show the validity of the second claim, suppose not. So $(a, b) \subseteq \mathbb{Q}$, and $\mathbb{N} \equiv \mathbb{Q}$ (there exists a bijection between the two sets). So there exists an injection of (a, b) into $\mathbb{N}$, then (a, b) is countable and the proof is complete. $\square$