Problem 1

WTS. *An open ball $B_r(x)$, where $r \geq 0$ is equivalent to a bounded open interval.*

$$B_r(x) = \{y,\ d(x,y) < r\}$$

Proof. A bounded open interval (a,b) where $a, b \in \mathbb{R}$ is defined

$$\begin{aligned}
(a,b) &= \{y,\ a < y < b\} \\
&= \{y,\ m - b < y - m < b - m,\ m = (a+b)/2\} \\
&= \{y,\ |y - m| < b - m,\ m = (a+b)/2\} \\
&= \{y,\ d(y,m) < \max(b - m, 0)\} \\
&= B_{\max(b-m,0)}(m)
\end{aligned}$$

Fix any ball centered at $m \in \mathbb{R}$ with radius $r \geq 0$, then write $b = m + r$. The $\max(b-m, 0)$ is to ensure that the equivalent radius of the ball does not go negative, if $a > b$ then the open interval is equivalent to $(a,a) = B_0(m)$. $\square$

Problem 2

WTS. *Every metric space is T_2.*

Proof. Fix two elements $x \neq y$ then by definition of the metric $d(x, y) = 2r > 0$, then fix two open (balls) sets $V(x, r)$ and $V(y, r)$, for every element $z \in V(x, r)$ we have

$$d(x, y) \leq d(z, x) + d(z, y) \implies 2r < r + d(z, y)$$

Clearly this means that $V(x, r) \subseteq V^c(y, r)$, and $V(x, r) \cap V(y, r) = \varnothing$. $\square$

Problem 3

WTS. *Let x_n be a sequence of real numbers, and*

1. *Prove that (i) is always true.*
2. *Prove that (iv) $\iff$ (iii).*
3. *Prove that (iii) $\implies$ (ii) $\implies$ (i)*
4. *Give examples of sequences that satisfy: (iii), (ii) but not (ii), and do not satisfy (ii)*

Where (i) to (iv) are given by

(i) $\forall n \exists M |x_n| \leq M$

(ii) $\exists M \exists^\infty n |x_n| \leq M$

(iii) $\exists M \forall^\infty n |x_n| \leq M$

(iv) $\exists \forall n |x_n| \leq M$

We begin by first taking the problem apart in the abstract. This is encompassed in the following lemma.

Lemma 0.1. *If P is a proposition on the space of all sequences (onto, into? what is the right word) X, denoted by*

$$\Omega = \{x_n : \mathbb{N} \to X\}$$

And if

- $P(\forall n \geq 0) = \{x_n \in \Omega,\ \forall n \geq 0, P(x_n)\}$
- $P(\forall^\infty\, n) = \{x_n \in \Omega,\ \exists N,\ \forall n \geq N, P(x_n)\}$
- $P(\exists^\infty\, n) = \{x_n \in \Omega,\ \forall N,\ \exists n \geq N, P(x_n)\}$

Then

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \subseteq P(\exists^\infty n) \subseteq \Omega$$

Proof. Suppose that $x_n \in P(\forall n \geq 0)$, then $P(x_n)$ eventually is trivial, so

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n)$$

Now fix $x_n \in P(\forall^\infty n)$, this induces some $N \in \mathbb{N}$ such that for every $n \geq N$ means that $P(x_n)$. To show that $P(x_n)$ frequently, notice for every $M \in \mathbb{N}$ we can choose some $n = M + N$ such that $P(x_n)$ holds, so

$$P(\forall^\infty n) \subseteq P(\exists^\infty n)$$

The last inclusion is obvious as all three are subsets of Ω. $\square$

Problem 3a

Proof. For any sequence of reals x_n, and for every $n \in \mathbb{N}$, since the set of natural numbers is unbounded above, there exists some $M_n \in \mathbb{N}$ such that $|x_n| \leq M$. $\square$

Problem 3b

Proof. We define the proposition P on the reals

$$P(\alpha) \iff \exists M \in \mathbb{N}, \; |\alpha| \leq M$$

Then by Lemma 0.1,

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \text{ means that (iv)} \implies \text{(iii)}$$

To prove the converse, suppose that (iii) holds, then there exists some $M \in \mathbb{N}$ such that $|x_n| \leq M$ for all $n \geq N$, then let

$$\overline{M} \geq M + \sum_{n \leq N} |x_n|$$

Where we apply the Archimedean Property on the right member to obtain some $\overline{M} \in \mathbb{N}$. Then it is easy to verify that $|x_n| \leq \overline{M}$ for every $n \geq 0$, and (iii) $\implies$ (iv). And therefore (iii) $\iff$ (iv). $\square$

Problem 3c

Proof. Using Lemma 0.1, since

$$P(\forall n \geq 0) \subseteq P(\forall^\infty n) \subseteq P(\exists^\infty n) \subseteq \Omega$$

Therefore

$$\text{(iv)} \implies \text{(iii)} \implies \text{(ii)} \implies \text{(i)}$$

□

Problem 3d

Proof. Here are the sequences

1. $x_n = 0$ for every $n \geq 0$, then $x_n \in P(\forall n \geq 0) \subseteq P(\forall^\infty n)$. Since $|x_n| \leq 0$.

2. $x_n = n \cdot (1 + (-1)^n)/2$, so that $x_n \in P(\exists^\infty n) \setminus P(\forall^\infty n)$. Since x_n visits 0 frequently at odd n but grows unbounded at even n.

3. $x_n = n$ is in $P(\exists^\infty n)^c$. Since for every natural M, we can choose some $N = M + 1$ such that for every $n \geq N$ implies that $x_n > M$.

□

Problem 4

WTS. *Show that*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = 0$$

Proof. Assume in good faith that we are indexing the sequence by $n \in \mathbb{N}^+$ so that $n \geq 1$. Then for every fixed $n \geq 1$,

$$0 \leq x_n = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

Now fix any $\varepsilon > 0$, and find a large M with $M \geq \varepsilon^{-1/2}$. Then it follows that for every $n \geq M$,

$$\frac{1}{\sqrt{\varepsilon}} \leq M \leq n \implies \frac{1}{n^2} < \varepsilon$$

It immediately follows that

$$|x_n - 0| = \left|\frac{1}{n} - \frac{1}{n+1} - 0\right| < \left|\frac{1}{n^2}\right| < \varepsilon$$

And $|x_n - 0| < \varepsilon$ eventually. $\square$

Problem 5

WTS. *Show that* $x_n = \dfrac{n}{\sqrt{n+5}}$ *diverges.*

We begin with an important Lemma.

Lemma 0.2. *Every convergent sequence in* $\mathbb{R}$ *is bounded.*

Proof. Fix $\{x_n\}_{n\geq 1} \to x$, also fix $\varepsilon = 1 > 0$, then there exists some $N \geq 0$ so large with

$$d(x_n, x) \leq 1 \quad \forall n \geq N$$

Then for every $n \geq N$ we have

$$d(x_n, 0) \leq d(x_n, x) + d(x, 0) \leq 1 + d(x, 0)$$

Now for every $n \geq 1$, obviously $d(x_n, 0) \leq 1 + d(x, 0) + \sum_{k \leq N} d(x_k, 0)$, and this establishes the Lemma. $\square$

Main Proof for Q5

Proof. We simply have to show that x_n is not bounded. Indeed, for any $M \geq 0$, take $n > 2M^2 + 5$ and $|x_n| > M$. The reasoning is as follows, if for every $n > 2M^2 + 5$, then

$$(n/2)^{1/2} > |M| = M$$

Also note that $n > 5 \implies 2n > n + 5$ therefore

$$(n+5)^{-1} > (2n)^{-1} \implies n(n+5)^{-1/2} > n(2n)^{-1/2}$$

And

$$x_n = |x_n| > n(2n)^{-1/2} = (n/2)^{1/2} > M$$

$\square$

Problem 6

WTS. *Let $\binom{n}{k}$ denote the number of k-element subsets of an n-element set. Prove that if $n < k$, then $\binom{n}{k} = 0$, and that*

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

Proof. We begin with some abstract notation.

- $J_n = \{1, 2 \cdots, n\} \subseteq \mathbb{N}^+$,
- $|E| \coloneqq \sum_{x \in E} 1$, the counting measure on E.
- X is any set where $|X| \geq 2$,
- $A \subseteq X$, $|A||A^c| \neq 0$, this implicitly means that neither set is empty,
- $\Omega_n = \{f : J_n \to X\}$,
- For every $f \in \Omega_n$, $f_{J_{n-1}}$ denotes the restriction of f onto J_{n-1}

With these definitions, it is clear that $\binom{n}{k}$, for every $n, k \in \mathbb{N}$,

$$\binom{n}{k} = \left|\left\{f \in \Omega_n,\ |f^{-1}(E)| = k\right\}\right|$$

Clearly, if $f \in \Omega_{n+1}$ and $|f^{-1}(E)| = k + 1$,

- If $f^{-1}(E) \cap J_{n+1} \setminus J_n = \varnothing$, then $|f_{J_n}^{-1}(E)| = |f^{-1}(E)| = k + 1$,
- If $f^{-1}(E) \cap J_{n+1} \setminus J_n \neq \varnothing$, then $|f_{J_n}^{-1}(E)| = k$,

Then we can write $E_1 = \left\{f \in \Omega_{n+1},\ |f^{-1}(E)| = k + 1\right\}$ as the disjoint union of $E_2 = \left\{f \in \Omega_{n+1},\ f^{-1}(E) \cap J_{n+1} \setminus J_n = \varnothing,\ f_{J_n}^{-1}(E)| = k + 1\right\}$ and $E_3 = \left\{f \in \Omega_{n+1},\ f^{-1}(E) \cap J_{n+1} \setminus J_n \neq \varnothing,\ |f_{J_n}^{-1}(E)| = k\right\}$.

Also note that $E_2 \equiv \{f \in \Omega_n, |f^{-1}(E)| = k{+}1\}$ and $E_3 \equiv \{f \in \Omega_n.|f^{-1}(E)| = k\}$. Since every $f \in E_2$ induces some $g \in \Omega_n$ with $|g^{-1}(E)| = k+1$, and respectively for $f \in E_3$. And for every $g \in \Omega_n$, $|g^{-1}(E)| = k+1$, there is a corresponding $f \in E_2$ with $f_{J_n} = g$.

Therefore $|E_2| = \binom{n}{k+1}$ and $|E_3| = \binom{n}{k}$. Since $|\cdot|$ is just the counting measure on finite sets, and E_1 is the disjoint union, it follows that

$$|E_1| = \binom{n+1}{k+1} = |E_2| + |E_3| = \binom{n}{k+1} + \binom{n}{k}$$

$\square$

Problem 7

WTS. *Prove three things*

(a) The Binomial formula, for every $n \in \mathbb{N}$, $a, b \in \mathbb{R}$

$$(a+b)^n = \sum_{k\geq 0}^{n} \binom{n}{k} a^k\, b^{n-k}$$

(b) The Generalized Bernoulli inequality, for every $n \in \mathbb{N}^+$, $k \leq n$

$$(1+b)^n \geq 1 + \binom{n}{k} b^k$$

(c) A special case of the Generalized Bernoulli inequality, for any $b \geq 0$, $n \in \mathbb{N}^+$

$$(1+b)^n \geq 1 + \frac{n(n-1)}{2} b^2$$

Proof. We begin by showing that (a) $\implies$ (b). For every $n \geq 1$, we have

$$(1+b)^n = \sum_{j\geq 0}^{n} \binom{n}{j} b^j = 1 + \binom{n}{k} b^k + \sum_{j\geq 1,\, j\neq k}^{n} \binom{n}{j} b^j$$

Since $\binom{n}{k} b^j \geq 0$, (b) holds.

Now to show that (b) $\implies$ (c), simply substitute $k = 2$ if $2 \geq n$, if $n = 1$ then the inequality is trivial.

The proof for (a) also quite straight forward, if $n = 0$ then

$$(a+b)^0 = 1 = \sum_{k=0}^{0} \binom{n}{k} a^k n^{n-k} = \binom{0}{0} a^0 b^{0-0} = 1$$

Assume that (a) holds for some $n \in \mathbb{N}$, then

$$
\begin{aligned}
(a+b)^{n+1} &= \sum_{k\geq 0}^{n} \binom{n}{k}\left(a^{k+1}b^{n-k} + a^{k}b^{(n+1)-k}\right) \\
&= \binom{n}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n} \binom{n}{k-1}\left(a^{k}b^{(n+1)-k}\right) + \sum_{k\geq 1}^{n} \binom{n}{k}\left(a^{k}b^{(n+1)-k}\right) \\
&= \binom{n+1}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n} \left(\binom{n}{k} + \binom{n}{k-1}\right)\left(a^{k}b^{(n+1)-k}\right) \\
&= \binom{n+1}{0} a^0 b^{(n+1)-0} + \sum_{k\geq 1}^{n} \binom{n+1}{k}\left(a^{k}b^{(n+1)-k}\right) \\
&= \sum_{k\geq 0}^{n} \binom{n+1}{k} a^{k}b^{(n+1)-k}
\end{aligned}
$$

For the third equality we used the fact that $\binom{\alpha}{0} = \binom{\beta}{0}$ for every $\alpha, \beta \in \mathbb{N}^{+}$. $\square$

Problem 8

WTS. *Prove two things.*

(a) Prove that for every $a \in \mathbb{R}$, $a > 1$ we have $\lim na^{-n} = 0$

(b) Prove that $\lim n^{1/n} = 1$

Proof. Let us start with (a). Assume that there exists some $a > 1$ with $\lim na^{-n} \neq 0$. So that there exists some $\varepsilon > 0$ and for every $N \in \mathbb{N}$, some $n \geq N$ with

$$\varepsilon \leq |na^{-n}| \implies (1 + n(n-1)(a-1)^2/2)\varepsilon \leq a^n \varepsilon \leq n$$

Dividing by n across both sides and noting that $1/n \geq 0$,

$$\varepsilon(n-1)(a-1)^2/2 \leq 1 \implies n \leq \left(\varepsilon(a-1)^2/2\right)^{-1} + 1$$

Which is obviously false, because $\mathbb{R}$ is Archemedian. This establishes (a).

For (b), we write $x_n = n^{1/n}$, where $x_n \geq 1$ for every $n \geq 1$. Indeed, if $x_n < 1$ for some $n \in \mathbb{N}^+$ then $x_n^n < 1$ by induction on n.

By applying Bernoulli's Inequality again,

$$x_n^n = n \geq 1 + n(n-1)(1-x_n)^2/2 \implies 2/n \geq (1-x_n)^2$$

So that $(1-x_n)^2 = |1-x_n|^2 \to 0$. We claim that if any sequence $|a_n|^2 \to 0$ then $a_n \to 0$. Fix an arbitrary $\varepsilon > 0$ then

$$|a_n - 0|^2 < \varepsilon^2 \implies |a_n - 0| < \varepsilon, \, \exists N \forall n \geq N$$

Therefore $|1 - x_n| \to 0$, and $x_n = n^{1/n} \to 1$. □