MATH 254 Assignment 1

November 5, 2022

1a

WTS. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. We can use a chain of equivalences. Suppose that both
not empty.

$$\begin{aligned} x \in A \cap (B \cup C) &\iff x \in A \wedge x \in (B \cup C) \\ &\iff x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\iff (x \in A \wedge x \in B) \text{ or } (x \in A \wedge x \\ &\iff x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Now suppose one of the two members are empty. Then if the c
was not empty, it would imply that the original member was nc
this means that the two sets must be equal.

2

WTS. $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Proof. Define $W = (A \setminus B) \cup (B \setminus A)$, then we will apply Q1a, an
Theorem.

$$\begin{aligned}
W^c &= (A^c \cup B) \cap (B^c \cup A) \\
&= ((A^c \cup B) \cap B^c) \cup ((A^c \cup B) \cap A) \\
&= (A^c \cap B^c) \cup (A \cap B) \\
&= (A \cup B)^c \cup (A \cap B) \\
&= [(A \cup B) \setminus (A \cap B)]^c
\end{aligned}$$

Taking complements on both sides finishes the proof.

3a

WTS. $f \coloneqq \{(x, y) \in [-1, +1] \times [-1, +1] : x^2 + y^2 = 1\}$ *is not*

Proof. f is not a function because $(0, 1) \in f$ and $(0, -1) \in f$, anc

3b

WTS. *f is a function.*

Proof. Since $y \geq 0$, we can write $y = +\sqrt{1 - x^2}$. Fix an $x \in$ there is a unique $y \in [0, 1]$ that satisfies the above. Also, for ever $|y| \leq 1$. Therefore f is a function.

4a

WTS. $f(f^{-1}([-4,-1] \cup [1,4])) = [1,4]$, *where* $f = x^{-2}$ *for eve*

Proof. Write $W = f^{-1}([-4,-1] \cup [1,4])$, and because the inver serves intersections and unions by Q5, and $[-4,-1] \cap \{f(x) : x$ $\varnothing$, then $f^{-1}[-4,-1] = \varnothing$. Which means $W = f^{-1}[1,4]$ and h $[1,4]$, as $[1,4] \subseteq \{f(x) : x \in \mathbb{R} \setminus \{0\}\}$.

4b

WTS. $f^{-1}(f(1,2)) = [-2,-1] \cup [1,2]$. *Where* $f = x^-2$, *for ev*

Proof. The equality is obvious by inspection.

4cd

WTS. *$f(f^{-1}B) = B$ if f is a surjection, and $f^{-1}(f(B)) =$ injection.*

We split this problem into two parts. We begin with the first ass $R = \{f(x) : x \in A\}$.

Lemma 0.1. *For every function $f : X \to Y$, $f(f^{-1}(B)) \subseteq B$.*

Proof. Use Q5a) onto the disjoint sets $f^{-1}(B \cap R)$ and $f^{-1}(B$

$$f^{-1}(B) = f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)$$

Now $f^{-1}(B \cap R^c)$ must be empty, since no $x \in A$ satisfies $f($ Hence $f^{-1}(B) = f^{-1}(B \cap R)$.

$$\begin{aligned} f(f^{-1}(B)) &= f(f^{-1}(B \cap R) \cup f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \cup f(f^{-1}(B \cap R^c)) \\ &= f(f^{-1}(B \cap R)) \\ &= \{f(x) : x \in f^{-1}(B \cap R)\} \\ &= \{y : y \in (B \cap R)\} \\ &= B \cap R \end{aligned}$$

Where for the second last equality we used the fact that f is jection onto its range. Then $f(f^{-1}(B)) = B \cap R \subseteq B$.

Remark. *If f is a surjection, then its range $R = Y$, then $B \cap Y = B$.*

Lemma 0.2. *For every function $f : X \to Y$, $A \subseteq f^{-1}(f(A))$.*

Proof. Write $f^{-1}(f(A))$ as the disjoint union of $A \cap f^{-1}(f($ $f^{-1}(f(A))$. Then, we shall show that $f^{-1}(f(A)) = A$. For eve

$$\begin{aligned} f(x) \in f(A) \wedge x \in A &\iff x \in f^{-1}(f(A)) \wedge x \in A \\ &\iff x \in A \cap \left(f^{-1}(f(A))\right) \end{aligned}$$

Hence $A \cap f^{-1}(f(A)) = A$, and $A \subseteq f^{-1}(f(A))$

Remark. *If f is a injection, then for every $x \in A^c$, $f(x)$ $A^c \cap f^{-1}(f(A)) = \varnothing$, and*

$$f^{-1}(f(A)) = \left[A \cap f^{-1}(f(A))\right] \cup \left[A^c \cap f^{-1}(f(A))\right] =$$

5a

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and B_1, B_2 that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.*

Proof. Fix two subsets $B_1, B_2 \subseteq B$, then

$$\begin{aligned} f^{-1}(B_1 \cup B_2) &= \{x \in A, f(x) \in B_1 \cup B_2\} \\ &= \{x \in A, f(x) \in B_1 \text{ or } f(x) \in B_2\} \\ &= \{x \in f^{-1}(B_1) \text{ or } x \in f^{-1}(B_2)\} \\ &= f^{-1}(B_1) \cup f^{-1}(B_2) \end{aligned}$$

5b

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and B_1, B_2 that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.*

Proof. For any two sets $A_1, A_2 \subseteq A$,

$$\begin{aligned} f(A_1 \cup A_2) &= \{f(x) : x \in A_1 \cup A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } x \in A_2\} \\ &= \{f(x) : x \in A_1 \text{ or } f(x) : x \in A_2\} \\ &= f(A_1) \cup f(A_2) \end{aligned}$$

5c

WTS. *$f : A \to B$ is a function, and $A_1, A_2 \subseteq A$ and B_1, B_2 that $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.*

Proof.

Lemma 0.3. *f^{-1} preserves complements.*

Proof. For every $E \subseteq B$,

$$\begin{aligned} f^{-1}(B \setminus E) &= \{x \in A : f(x) \in B \setminus E\} \\ &= \{x \in A, f(x) \in E^c\} \\ &= A \setminus f^{-1}(E) \end{aligned}$$

Lemma 0.4. *f^{-1} preserves intersections.*

Proof. Now we wish to prove that f^{-1} preserves intersection every pair of subsets, $B_1, B_2 \subseteq B$. Write their intersection a apply Q5a, and take complements.

$$\begin{aligned} f^{-1}\left((B_1^c \cup B_2^c)^c\right) &= (f^{-1}(B_1^c) \cup f^{-1}(B_2^c))^c \\ &= f^{-1}(B_1) \cap f^{-1}(B_2) \end{aligned}$$

To prove the assertion in Q5c, write $B_1 \setminus B_2 = B_1 \cap B_2^c$, and a Lemmas.

5d

WTS. *Provide an example such that $f(A_1 \setminus A_2) \neq f(A_1) \setminus f(A$ condition on f that implies $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$.*

We begin our answer with the example. Suppose $f \in L^{p*}$, w every element to 0. Take two subsets of $A \subseteq L^p$, $A_1 = \{g_1\}$ Then $f(A_1) \setminus f(A_2) = \varnothing$, but $A_1 \setminus A_2 = A_1$, and $f(A_1 \setminus A_2) =$

The condition we want to impose on f is that it must be an will prove that it satisfies the assertion.

Proof.

Lemma 0.5. *The direct image is monotonic. For every $E_1 \subseteq$ $f(E_1) \subseteq f(E_2) \subseteq B$.*

Proof. Apply Q5b) to sets $E_2 = (E_2 \setminus E_1) \cup (E_2 \cap E_1)$, then $f($ $E_1) \cup f(E_2 \cap E_1)$ implies that $f(E_1) \subseteq f(E_2)$.

Lemma 0.6. *For every pair of subsets, $E_1, E_2 \subseteq A$, then $f(E$ $f(E_1 \setminus E_2)$.*

Proof. If the left member is empty, then it is trivial. If not, th ment $y \in f(E_1) \setminus f(E_2)$, then $y \in f(E_1)$ and $y \in f(E_2)^c$.

This is equivalent to saying that there exists a $x_1 \in E_1$ such th and for every $x_2 \in E_2$, $f(x_2) \neq y$, and therefore x_1 is not a m since f is a function. It follows that $x_1 \in E_1 \setminus E_2$, and $f(x_1) = y$ Since y is arbitrary, we are done.

Suppose f is an injection, then for every $x \neq p \in A$ implies that We wish to prove the reverse estimate in the second Lemma. F in $y \in f(E_1 \setminus E_2)$, then this y induces an $x \in E_1 \setminus E_2$. Since $(E_1 \setminus E_2)$ and E_2 are disjoint, for every $p \in E_2$, $x \neq p$ yields and $f(x) = y \in f(E_2)^c$. But this y is also a member of $f(E_1$ Lemma, if we simply take $E_1 \setminus E_2 \subseteq E_1 \subseteq A$. Therefore $y \in f$ This completes the proof.

6a

WTS. *Show that $f(x) = x/(|x|+1)$ is a bijection from $\mathbb{R}$ to (*

Proof. We begin with an important Lemma.

Lemma 0.7. *For any $f : X \to Y$, if $A \subseteq X$ such that $f = f|$ Y is the disjoint union of $f|_A(A)$ and $f|_{A^c}(A^c)$, and the restrict A and A^c are bijections onto their direct images, then f is a b*

Proof. To prove injectivity, suppose we have $x_1 \neq x_2$, where the trivial case of them both belonging to the same A or loss of generality, suppose $x_1 \in A$ and $x_2 \in A^c$. Then by $f(x_1) = f|_A(x) \in f|_A(A)$ which implies that $f(x_1)$ is not in $f(x_1) \neq f(x_2)$.

Now to show surjectivity, simply take any $y \in Y$, and either $y \in$ $f_{A^c}(A^c)$, and since the two restrictions of f onto the two sets a there exists a corresponding $x \in X$ which will satisfy. This c proof.

Now we want to prove the original assertion. To use the lemn $[0, +\infty) \subseteq \mathbb{R}$. We will satisfy the assumptions of the Lemma. $x \in A$, $f(x) = 1 - 1/(x+1)$. Injectivity is obvious at first gl claim that $f|_A(A) = [0, 1)$. To show that $f|_A(A) \subseteq [0, 1)$, notic

$$f|_A = 1 - \frac{1}{x+1} \geq 0, \quad \forall x \in [0, +\infty)$$

$$f|_A \geq 1 \implies 1 - \frac{1}{x+1} \geq 1 \implies x \leq -1 \implies x \in$$

Then $f|_A(A) \subseteq [0, 1)$ as required. Now to show the converse $[0, 1)$, then there exists an $x = (1-y)^{-1} - 1 \in A$. Thus we hav $f|_A$ is a bijection onto its direct image.

Next for $f|_{A^c}(x) = -1 + 1/(1-x)$ for any $x \in A^c$. It is tr that $f|_{A^c}$ is an injection. So, fix any $y \in (-1, 0)$ and there cc $x = 1 - (y+1)^{-1} \in A^c$. Hence $(-1, 0) \subseteq f|_{A^c}(A^c)$. To show th will proceed by contradiction. So suppose there exists an $x \in$ $f|_{A^c}(x) \geq 0$, which means that $f|_{A^c}(x) \in A$, then a cool way

contradiction this would be to plug $y = f|_{A^c}(x)$ into $f|_A(y) \in$
we have

$$\begin{aligned} f|_A(y) &= y/(y+1) \\ &= \frac{x/(1-x)}{x/(1-x)+1} \\ &= x \in [0,1) \end{aligned}$$

But $x \in A^c$ by assumption, so we have a contradiction. Suppo
exists an $x \in A^c$ such that $f|_{A^c}(x) \leq 1$, then

$$\begin{aligned} \frac{x}{1-x} &\leq 1 \\ -x/(1-x) &\geq 1 \\ 1 - 1/(1-x) &\geq 1 \\ 1/(1-x) &\leq 0 \\ 1 &\leq x \end{aligned}$$

And the contradiction establishes the bijection. Since $f|_A(A)$
$f|_{A^c}(A^c) = (-1, 0)$. $Y = (-1, 1)$ is the disjoint union of these
can finally apply the Lemma, and the proof is complete.

6b

WTS. *Show that*

$$f(x) = (x+1)(m/2) + a$$

induces a bijection from $(-1, 1) \to (a, b)$ *for every* $m = b - a >$

Proof. Since $m \neq 0$, f is obviously injective. And for every y can easily find an

$$x = (y-a)(2/m) + (-1) \in (-1, 1)$$

To show that $f \in (a, b)$, we can attempt the contrapositive. f implies $|x| \geq 1$.

$$\begin{aligned}
|f(x) - (a+b)/2| &\geq m/2 \\
|(x+1)(m/2) + 2a/2 - (a+b)/2| &\geq m/2 \\
|(x+1)m + 2a - a - b| &\geq m \\
|x| &\geq 1
\end{aligned}$$

This establishes the bijection.