MATH 254: Assignment 3

November 5, 2022

Problem 1

WTS. $\sqrt{3}$ *is not rational.*

Proof. Suppose that $\sqrt{3} \in \mathbb{Q}$. Note that $\sqrt{3} \neq 0$ since $\sqrt{3} \cdot \sqrt{3} = 3 \neq 0$, and $\sqrt{3} > 0$. Then there exists a factorized form of $\sqrt{3} = a/b$ such that a and b are in $\mathbb{N}^+$ and have no common divisors other than 1. Now suppose that $a = 2m$ and $b = 2n + 1$.

$$\begin{aligned}
\frac{2m}{2n+1} = \sqrt{3} &\iff 2m = \sqrt{3}(2n+1) && (1)\\
&\iff 4m^2 = 3(4n^2 + 4n + 1) && (2)\\
&\iff (m^2 - 3n^2 - 3n) = 3/4 \in \mathbb{Z} && (3)
\end{aligned}$$

Where for the last assertion we used the fact that the integers are closed under addition and multiplication. And this contradiction establishes that a is odd.

To prove the fact that a cannot be odd as well, consider the following, (notice how we relabelled the coefficients), since $\sqrt{3} \neq 0$

$$\frac{2n+1}{2m} = \sqrt{3} \implies \frac{2m}{2n+1} = 1/\sqrt{3}$$

Mutiplying both sides by 3 yields

$$2(3m) = \sqrt{3}(2n+1)$$

Replace $m = 3m$ in (1), and the contradiction finishes the proof. (*) $\square$

Remark. *In (*), consider the similarities between the above equation and (1), where m, n are arbitrary numbers in $\mathbb{N}^+$. Since no m, n in $\mathbb{N}^+$ can satisfy (1).*

Problem 2

WTS. $\mathbb{N}$ *is unbounded above.*

Proof. Consider $2 \in \mathbb{N}$. So $\mathbb{N} \neq \varnothing$, and suppose by contradiction that

$$2 \leq \sup \mathbb{N} = x < +\infty$$

Then there exists some $y \in \mathbb{N}$

$$x - 1 \leq y$$

But since $\mathbb{N}$ is closed under addition,

$$x < y + 2 \in \mathbb{N}$$

But $y + 2 > x$, and we are done. $\square$

Problem 3

WTS. *Show that* $\mathbb{Q}\sqrt{2}$ *and* $\mathbb{Q}+\sqrt{2}$ *are dense in* $\mathbb{R}$.

Proof. We know that $\mathbb{Q}$ is dense in $\mathbb{R}$, then fix any $\varnothing \neq (a, b) \subseteq \mathbb{R}$, if there exists some $q \in \mathbb{Q}$

$$\frac{a}{\sqrt{2}} < q < \frac{b}{\sqrt{2}} \implies a < q\sqrt{2} < b$$

Also,

$$a - \sqrt{2} < q < b - \sqrt{2} \implies a < q + \sqrt{2} < b$$

Since addition, subtraction, multiplication, division by $\sqrt{2} > 0$ preserves the order relation for $a < b$. This finishes the proof. $\square$

Problem 4

WTS. *We wish to show that* $\inf S = -1$ *and* $\sup S = +1$ *for*

$$S = \left\{ \frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N}^{+} \right\}$$

We begin with a few important lemmas. For non-empty bounded subsets A, B of $\mathbb{R}$,

Lemma 0.1. *If A is a non-empty bounded above subset of $\mathbb{R}$, then* $(-1)(\sup A) = \inf(-1)A$*.*

Proof. Let $s = \sup(A)$, then

$$(-1)A = \{-x : x \in A\}$$

Evidently, for every $x \in A$

$$x \leq s \implies -s \leq -x \implies -s \leq (-1)A$$

Now fix any $\varepsilon > 0$, then there exists an $x \in A$ such that

$$s - \varepsilon \leq x \implies (-1)(s - \varepsilon) \geq -x \implies (-x) \leq (-s) + \varepsilon$$

Thus establishes $\inf(-1)A = (-1)\sup(A)$. □

Corollary 0.1.1. $(-1)\inf(A) = \sup(-1)A$*. The proof is trivial just replace A by $(-1)A$.*

Lemma 0.2. *If A and B are non-empty bounded above subsets of $\mathbb{R}$, then* $\sup A + \sup B = \sup(A + B)$

Proof. Define $s = \sup A$ and $t = \sup B$, then for every $(a, b) \in A \times B$

$$a \leq s,\ b \leq t \implies a + b \leq s + t \implies A + B \leq s + t$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq s - \varepsilon/2,\ b \leq t - \varepsilon/2 \implies s + t - \varepsilon \leq a + b$$

Therefore $\sup(A + B) = \sup(A) + \sup(B)$. □

Lemma 0.3. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. □

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Main Proof of Question 4

Now we are ready to tackle the question of Question 4

Proof. To prove Q4, define $A = \{n^{-1}, n \in \mathbb{N}^+\}$, then $\inf A = 0$ and $\sup A = 1$. Since 0 is a lower bound for A, it suffices to apply the Archimedean property for any $\varepsilon > 0$,

$$\exists n \in \mathbb{N}^+, 0 < \frac{1}{n} < 0 + \varepsilon = \varepsilon$$

Therefore $\inf A = 0$. To show $\sup A = 1$, notice that

$$n \geq 1 \implies 1/n \leq 1$$

But $1 \in A$ so applying Lemma 0.7 tells us that $\sup A = 1$.

We then construct two sets, S_1 and S_2 where $S_1 = S_2 = A$. By applying Lemma 0.1 we get

$$\sup(-S_2) = (-1)\inf(S_2) = 0$$

Now use Lemmas 0.2 (which allows us to add the supremums together) to obtain

$$\sup(S_1 - S_2) = 1 + 0 = 1$$

From here, let us turn our attention to the fact that

$$(-1)(S_1 - S_2) = (S_1 - S_2)$$

Apply Lemma 0.1 once again, then we can obtain

$$-1 = (-1)\sup\{(S_1 - S_2)\} = \inf\{(-1)(S_1 - S_2)\} = \inf\{(S_1 - S_2)\}$$

As a final step, recall that

$$S_1 - S_2 \coloneqq S = \left\{\frac{1}{m} - \frac{1}{n} : m, n \in \mathbb{N}^+\right\}$$

Therefore $\inf S = -1$ and $\sup S = +1$. $\square$

Problem 5

WTS. *Show that for every A, $B \subseteq \mathbb{Q}^+$,*

(a) $\inf A + \inf B = \inf A + B$

(b) $\inf A \inf B = \inf AB$

We will make use of two Lemmas proven in previous exercises.

Lemma 0.4. *If A and B are non-empty bounded below subsets of $\mathbb{R}$, then* $\inf A + \inf B = \inf(A + B)$

Proof. Define $w = \inf A$ and $q = \inf B$, then for every $(a, b) \in A \times B$

$$w \leq a,\ q \leq b \implies w + q \leq a + b \implies w + q \leq A + B$$

Now for every $\varepsilon/2 > 0$, there exists $(a, b) \in A \times B$ such that

$$a \leq w + \varepsilon/2,\ b \leq q + \varepsilon/2 \implies a + b \leq w + q + \varepsilon$$

Therefore $\inf(A + B) = \inf(A) + \inf(B)$. □

Lemma 0.5. *If A is a non-empty bounded below subset of $\mathbb{R}$, then for every $c \geq 0, c(\inf A) = \inf cA$.*

Proof. Let $w = \inf(A)$, then

$$cA = \{cx : x \in A\}$$

Then for every $x \in A$

$$w \leq x \implies cw \leq cx \implies cw \leq cA$$

If $c = 0$, then the equality is trivial since $cA = \{0\}$, if not, for every $\varepsilon/c > 0$, there exists an $x \in A$ such that

$$x \leq w + \frac{\varepsilon}{c} \implies cx \leq cw + \varepsilon$$

This establishes Lemma 0.5. □

Main Proof for Question 5

Proof. Lemma 0.4 applies for all non-empty, bounded-below subsets of $\mathbb{R}$, since A and B are bounded below by $0 \in \mathbb{R}$, and we assume in good faith that neither of them are empty, then this establishes (a). If either one of them is empty, then there is nothing to prove as $A + B$ and AB will be empty as well.

Now let us prove (b). Since 0 is a lower bound for A and B,

- $0 \leq \inf_{a \in A} a$
- $0 \leq \inf_{b \in B} b$
- For every $a \in A$, $0 \leq a$
- For every $b \in B$, $0 \leq b$

Fix any member $a \in A$ such that $a \geq 0$ by Lemma 0.5

$$a \cdot \inf_{b \in B} b = \inf_{b \in B} ab$$

Taking the infimum with respect to A on both sides (equality of sets $\implies$ equality of inf)

$$\inf_{a \in A} \left(\left\{ a \cdot \inf_{b \in B} b \right\} \right) = \inf_{a \in A} \left(\left\{ \inf_{b \in B} ab \right\} \right)$$

Now since $\inf_{b \in B} b \geq 0$ we can apply Lemma 0.5 to the left member (without loss of generality, the inf commutes)

$$\inf_{b \in B} b \inf_{a \in A} a = \inf_{a \in A} \inf_{b \in B} ab$$

We claim that

$$\inf_{a \in A} \inf_{b \in B} (ab) = \inf_{(a,b) \in A \times B} (ab)$$

Let us take a step back and solve the problem in the abstract for a bit. We want to show that for any $f : X \times Y \to \mathbb{R}$ that is bounded below,

$$\inf_{a \in X} \inf_{b \in Y} f(a, b) = \inf_{(a,b) \in X \times Y} f(a, b)$$

Fix any $z = f(x, y)$, $\inf_{b \in Y} f_x(b) \leq f_x(y) = z$ where f_x and f^y denote the x and y sections of f, such that

$$f_x(y) = f(x, y) = f^y(x)$$

Then taking the inf over $a \in X$ gives

$$\inf_{a\in X}\inf_{b\in Y} f_a(b) \le f(x,y), \quad \forall (x,y) \in X \times Y$$

So $\inf_{a\in X}\inf_{b\in Y} f_a(b)$ is a lower bound for f. To show that it is indeed the infimum, for every $\varepsilon/2 > 0$ we obtain some $x_0 \in X$ such that

$$\inf_{a\in X}\left(\left\{\inf_{b\in Y} f(a,b)\right\}\right) + \varepsilon/2 > \inf_{b\in Y} f_{x_0}(b)$$

Going through the same motion again, but this time for another $\varepsilon/2 > 0$ gives us some $y_o \in Y$

$$\inf_{b\in Y} f_{x_0}(b) + \varepsilon/2 > f(x_0, y_0)$$

Adding the two estimates together, there exists some $(x_0, y_0) \in X \times Y$ that satisfies, for every $\varepsilon > 0$

$$\inf_{a\in X}\left(\left\{\inf_{b\in Y} f(a,b)\right\}\right) + \varepsilon > f(x_0, y_0)$$

Now apply $f : A \times B \to \mathbb{R}$ with the mapping $(a,b) \mapsto a \cdot b$ (which is bounded below by 0, as the multiplication of two non-negative numbers is non-negative). This finishes the proof. $\square$

Problem 6

WTS. *Prove that if a bounded above set $S \subseteq \mathbb{R}$ contains one of its upper bounds u, then $\sup S = u$. And prove it for a bounded below set and its infimum.*

I already proved this in earlier questions as a Lemma.

Lemma 0.6. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. □

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Problem 7

WTS. *Prove that every finite subset of $\mathbb{R}$ contains its infimum and supremum.*

We need an important Lemma.

Lemma 0.7. *If A is a non-empty bounded subset of $\mathbb{R}$, if s and t are upper and lower bounds of A, and if $s \in A$ then $s = \sup A$. Also if $t \in A$, then $t = \inf A$*

Proof. Suppose that s and s' are upper bounds of A, then

$$s \in A \implies s \leq s'$$

So $s = \sup A$, now if t and t' are lower bounds of A, then

$$t \in A \implies t' \leq t$$

This completes the proof. □

Remark. *We only require A to be bounded above for the supremum part of the proof, and A to be bounded below for the infimum of the proof. We omit the caveat because it is a needless distraction.*

Main Proof for Question 7

Proof. If A is empty then this is a pathological case. If we are working in the extended reals $\overline{\mathbb{R}}$ then by convention

$$\inf \varnothing = +\infty, \quad \sup \varnothing = -\infty$$

Where both $\inf A \notin A$ and $\sup A \notin A$. Now, suppose that A is non-empty and finite. Define

$$|A| = \sum_{a \in A} 1$$

Then $1 \leq |A| < +\infty$. If $|A| = 1$, then $A = \{x\}$ for some $x \in \mathbb{R}$, then this x is an upper bound and a lower bound for A. Because

- For every $y \in A$, $y \leq x$.

- For every $y \in A$, $x \leq y$.

By Lemma (0.7) since $x \in A$, then $\sup A = \inf A = x$. Now let us show for the general case, where $|A| = N < +\infty$. Because A is a finite non-empty set, there exists a bijection between A and some initial segment of the countable ordinals. Let us agree to call this bijection

$$F : A \to [0, |A| - 1] \cap \mathbb{N}$$

Denote $a = F^{-1}(0)$. Then $E_1 := \{a\}$ is non-empty and contains its supremum and infimum. Now let us assume for the purposes of induction, that E_n where $1 \leq n < |A| - 1$ such that

$$\sup E_n \in E_n, \quad \inf E_n \in E_n$$

Then take $x = F^{-1}(n+1)$, so this $x \notin E_n$, since F^{-1} is also a function. Then we will show that $E_{n+1} = \{x\} \cup E_n$ contains its supremum and infimum.

Since $\sup E_n \in E_{n+1}$ and $\inf E_n \in E_{n+1}$, if

$$x \leq \sup E_n \implies \sup E_n = \sup E_{n+1}$$

Likewise for its infimum

$$\inf E_n \leq x \implies \inf E_n = \inf E_{n+1}$$

Because $\sup E_n$ is an upper bound of E_{n+1} that is contained in E_{n+1} (likewise for $\inf E_n$). However, if

$$\sup E_n < x \implies \forall y \in E_n,\ y \leq \sup E_n < x$$

Or

$$x < \inf E_{n+1} \implies \forall y \in E_n,\ x < \inf E_{n+1} \leq y$$

Then take $\sup E_{n+1} = x \in E_{n+1}$ or $\inf E_{n+1} = x \in E_{n+1}$. As a final 'step' we note that A can be written as

$$A = E_{|A|-1} \cup \{F^{-1}(|A| - 1)\}$$

This completes the proof. $\square$

Problem 8

WTS. *Prove the Triangle Inequality with $n \geq 2$.*

Lemma 0.8. *The Triangle Inequality, for every $a, b \in \mathbb{R}$*

$$|a+b| \leq |a| + |b|$$

Proof. Notice that for every $a, b \in \mathbb{R}$

- $-|a| \leq a \leq +|a|$
- $-|b| \leq b \leq +|b|$

Add two inequalities together, then we get

$$-\left(|a| + |b|\right) \leq (a+b) \leq +\left(|a| + |b|\right)$$

Since $|a| + |b| \geq 0$, we get $|a+b| \leq |a| + |b|$ as desired. Because for every $c \geq 0$,

$$\begin{aligned} c \geq |a| &\iff c \geq \max\{-a, +a\} \\ &\iff c \geq a \text{ and } c \geq -a \\ &\iff c \geq a \text{ and } -c \leq a \\ &\iff -c \leq a \leq +c \end{aligned}$$

□

0.1 Main Proof for Q8

Proof. For every x_1, x_2 in $\mathbb{R}$,

$$\left|\sum_{j=1}^{2} x_j\right| \leq \sum_{j=1}^{2} |x_j|$$

This proves the non trivial base case, if $n = 1$ then obviously $|x_1| \leq |x_1|$. We now suppose that

$$\left|\sum_{j=1}^{k} x_j\right| \leq \sum_{j=1}^{k} |x_j|$$

Then write

$$\left|\sum_{j=1}^{k+1} x_j\right| = \left|\left(\sum_{j=1}^{k} x_j\right) + (x_{k+1})\right|$$

Applying the second estimate to the right member gives us

$$\left|\sum_{j=1}^{k+1} x_j\right| \leq \left|\sum_{j=1}^{k} x_j\right| + |x_{k+1}| \leq \sum_{j=1}^{k+1} |x_j|$$

This completes the proof. $\square$