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***Abstract*—The minimum variance distortionless response (MVDR) approach is very popular in array processing. It is also employed in spectral estimation where the Fourier matrix is used in the optimization process. First, we give a general form of the MVDR where any unitary matrix can be used to estimate the spectrum. Second and most importantly, we show how the MVDR method can be used to estimate the magnitude squared coherence function, which is very useful in so many applications but so few methods exist to estimate it. Simulations show that our algorithm gives much more reliable results than the one based on the popular Welch's method.**

***Index Terms*—Capon, coherence function, cross-spectrum, minimum variance distortionless response (MVDR), periodogram, spectral estimation, spectrum.**

I. Introduction

SPECTRAL estimation is a very important topic in signal processing, and applications demanding it are countless [1]–[3]. There are basically two broad categories of techniques for spectral estimation. One is the nonparametric approach, which is based on the concept of bandpass filtering. The other is the parametric method, which assumes a model for the data, and the spectral estimation then becomes a problem of estimating the parameters in the assumed model. If the model fits the data well, the latter may yield a more accurate spectral estimate than the former. However, in the case that the model does not satisfy the data, the parametric model will suffer significant performance degradation and lead to a biased estimate. Therefore, a great deal of research efforts are still devoted to the nonparametric approaches.

One of the most well-known nonparametric spectral estimation algorithms is the Capon's approach, which is also known as minimum variance distortionless response (MVDR) [4], [5]. This technique was extensively studied in the literature and is considered as a high-resolution method. The MVDR spectrum can be viewed as the output of a bank of filters, with each filter centered at one of the analysis frequencies. Its bandpass filters are both data and frequency dependent, which is the main difference with a periodogram-based approach where its bandpass filters are a discrete Fourier matrix, which is both data and frequency independent [3], [6].

The objective of this letter is twofold. First, we generalize the concept of the MVDR spectrum. Second and most importantly, we show how to use this approach to estimate the magnitude squared coherence (MSC) function as an alternative to the popular Welch's method [7].

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II. General Form of the Spectrum

Let $x(n)$ be a zero-mean stationary random process that is the input of K filters of length K

$$\mathbf{g}_k = [g_{k,0} \quad g_{k,1} \quad \cdots \quad g_{k,K-1}]^T, \qquad k = 0, 1, \ldots, K-1$$

where superscript T denotes transposition.

If we denote by $y_k(n)$ the output signal of the filter $\mathbf{g}_k$, its power is

$$\begin{aligned} E\{|y_k(n)|^2\} &= E\left\{\left|\mathbf{g}_k^H \mathbf{x}(n)\right|^2\right\} \\ &= \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k \end{aligned} \tag{1}$$

where $E\{\cdot\}$ is the mathematical expectation, superscript H denotes transpose conjugate of a vector or a matrix

$$\mathbf{R}_{xx} = E\{\mathbf{x}(n)\mathbf{x}^H(n)\} \tag{2}$$

is the covariance matrix of the input signal $x(n)$, and

$$\mathbf{x}(n) = [x(n) \quad x(n-1) \quad \cdots \quad x(n-K+1)]^T.$$

In the rest of this letter, we always assume that $\mathbf{R}_{xx}$ is positive definite.

Consider the unitary matrix

$$\mathbf{U} = [\mathbf{u}_0 \quad \mathbf{u}_1 \quad \cdots \quad \mathbf{u}_{K-1}]$$

with $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$. In the proposed generalized MVDR method, the filter coefficients are chosen so as to minimize the variance of the filter output, subject to the constraint

$$\mathbf{g}_k^H \mathbf{u}_k = \mathbf{u}_k^H \mathbf{g}_k = 1. \tag{3}$$

Under this constraint, the process $x(n)$ is passed through the filter $\mathbf{g}_k$ with no distortion along $\mathbf{u}_k$ and signals along other vectors than $\mathbf{u}_k$ tend to be attenuated. Mathematically, this is equivalent to minimizing the following cost function:

$$J_k = \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k + \mu\left[1 - \mathbf{g}_k^H \mathbf{u}_k\right] \tag{4}$$

where μ is a Lagrange multiplier. The minimization of (4) leads to the following solution:

$$\mathbf{g}_k = \frac{\mathbf{R}_{xx}^{-1}\mathbf{u}_k}{\mathbf{u}_k^H \mathbf{R}_{xx}^{-1}\mathbf{u}_k}. \tag{5}$$

We define the spectrum of $x(n)$ along $\mathbf{u}_k$ as

$$S_{xx}(\mathbf{u}_k) = E\{|y_k(n)|^2\} = \mathbf{g}_k^H \mathbf{R}_{xx} \mathbf{g}_k. \tag{6}$$

Therefore, plugging (5) into (6), we find that

$$S_{xx}(\mathbf{u}_k) = \frac{1}{\mathbf{u}_k^H \mathbf{R}_{xx}^{-1} \mathbf{u}_k}. \tag{7}$$

Expression (7) is a general definition of the spectrum of the signal $x(n)$, which depends on the unitary matrix $\mathbf{U}$. Replacing the previous equation in (5), we get

$$\mathbf{R}_{xx}\mathbf{g}_k = S_{xx}(\mathbf{u}_k)\mathbf{u}_k. \tag{8}$$

Taking into account all vectors $\mathbf{u}_k$, $k = 0, 1, \ldots, K-1$, (8) has the general form

$$\mathbf{R}_{xx}\mathbf{G} = \mathbf{U}\mathbf{S}_{xx}(\mathbf{U}) \tag{9}$$

where

$$\mathbf{G} = [\mathbf{g}_0 \quad \mathbf{g}_1 \quad \cdots \quad \mathbf{g}_{K-1}]$$

and

$$\mathbf{S}_{xx}(\mathbf{U}) = \text{diag}\{S_{xx}(\mathbf{u}_0), S_{xx}(\mathbf{u}_1), \ldots, S_{xx}(\mathbf{u}_{K-1})\}$$

is a diagonal matrix.

Property 1: We have

$$\mathbf{S}_{xx}^2(\mathbf{U}) = \mathbf{G}^H \mathbf{R}_{xx}^2 \mathbf{G}. \tag{10}$$

Proof: This form follows immediately from (9).

Property 1 shows that there are an infinite number of ways to decompose matrix $\mathbf{R}_{xx}^2$, depending on how we choose the unitary matrix $\mathbf{U}$. Each one of these decompositions gives a representation of the square of the spectrum of the signal $x(n)$ in the subspace $\mathbf{U}$.

Property 2: We have

$$\text{tr}\left[\mathbf{R}_{xx}^{-1}\right] = \text{tr}\left[\mathbf{S}_{xx}^{-1}(\mathbf{U})\right]. \tag{11}$$

Proof: Indeed

$$\begin{aligned}\text{tr}\left[\mathbf{S}_{xx}^{-1}(\mathbf{U})\right] &= \sum_{k=0}^{K-1} S_{xx}^{-1}(\mathbf{u}_k) = \sum_{k=0}^{K-1} \text{tr}\left[\mathbf{u}_k^H \mathbf{R}_{xx}^{-1} \mathbf{u}_k\right] \\ &= \text{tr}\left[\mathbf{R}_{xx}^{-1} \sum_{k=0}^{K-1} \mathbf{u}_k \mathbf{u}_k^H\right] = \text{tr}\left[\mathbf{R}_{xx}^{-1}\right].\end{aligned}$$

Property 2 expresses the energy conservation. So no matter what we take for the unitary matrix $\mathbf{U}$, the sum of all values of the inverse spectrum is always the same.

A. Particular Cases

In this subsection, we propose to briefly discuss three important particular cases of the general form of the MVDR spectrum.

The first obvious choice for the unitary matrix $\mathbf{U}$ is the Fourier matrix

$$\mathbf{F} = [\mathbf{f}_0 \quad \mathbf{f}_1 \quad \cdots \quad \mathbf{f}_{K-1}]$$

where

$$\mathbf{f}_k = \frac{1}{\sqrt{K}}[1 \quad \exp(j\omega_k) \quad \cdots \quad \exp(j\omega_k(K-1))]^T$$

and $\omega_k = 2\pi k/K, k = 0, 1, \ldots, K-1$. Of course, $\mathbf{F}$ is a unitary matrix. With this choice, we obtain the classical Capon's method.

Now suppose $K \rightarrow \infty$. In this case, a Toeplitz matrix is asymptotically equivalent to a circulant matrix if its elements are absolutely summable [8], which is usually the case in most applications. Hence, we can decompose $\mathbf{R}_{xx}$ as

$$\mathbf{S}_{xx}(\mathbf{F}) = \mathbf{F}^H \mathbf{R}_{xx} \mathbf{F} \tag{12}$$

so that $\mathbf{G} = \mathbf{F}$. As a result, for a stationary signal and asymptotically, Capon's approach is equivalent to the periodogram. The difference between the MVDR and periodogram approaches can also be viewed as the difference between the eigenvalue decompositions of circulant and Toeplitz matrices. While for a circulant matrix, its corresponding unitary matrix is data independent, it is not for a Toeplitz matrix.

The second natural choice for $\mathbf{U}$ is the matrix containing the eigenvectors of the correlation matrix $\mathbf{R}_{xx}$. Indeed, it is well known that $\mathbf{R}_{xx}$ can be diagonalized as follows [9]:

$$\begin{aligned}\mathbf{R}_{xx} &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H \\ &= \sum_{k=0}^{K-1} \lambda_k \mathbf{v}_k \mathbf{v}_k^H\end{aligned} \tag{13}$$

where $\mathbf{V}$ is a unitary matrix containing the eigenvectors $\mathbf{v}_k$, and $\mathbf{\Lambda}$ is a diagonal matrix containing the corresponding eigenvalues λ_k. Thus, taking $\mathbf{U} = \mathbf{V}$, we find that $\mathbf{G} = \mathbf{V}$ and

$$\mathbf{S}_{xx}(\mathbf{V}) = \mathbf{\Lambda}. \tag{14}$$

In many applications, the process signal $x(n)$ is real, and it may be more convenient to select an orthogonal matrix instead of a unitary one. So our third and final particular case is the discrete cosine transform

$$\mathbf{C} = [\mathbf{c}_0 \quad \mathbf{c}_1 \quad \cdots \quad \mathbf{c}_{K-1}]$$

where the rest is shown in the equation at the bottom of the page, with $c(0) = \sqrt{1/K}$ and $c(k) = \sqrt{2/K}$ for $k \neq 0$. We can verify that $\mathbf{C}^T\mathbf{C} = \mathbf{C}\mathbf{C}^T = \mathbf{I}$. So with this orthogonal transform, the spectrum is

$$\mathbf{S}_{xx}(\mathbf{C}) = \text{diag}^{-1}\left\{\mathbf{C}^T \mathbf{R}_{xx}^{-1} \mathbf{C}\right\}. \tag{15}$$

$$\mathbf{c}_k = \left[c(0) \quad c(1)\cos\frac{\pi(2k+1)}{2K} \quad \cdots \quad c(K-1)\cos\frac{\pi(2k+1)(K-1)}{2K}\right]^T$$

III. Application to the Cross-Spectrum and MSC Function

In this section, we show how to use the generalized MVDR approach for the estimation of the cross-spectrum and the MSC function.

A. General Form of the Cross-Spectrum

We assume here that we have two zero-mean stationary random signals $x_1(n)$ and $x_2(n)$ with respective spectra $S_{x_1x_1}(\mathbf{u}_k)$ and $S_{x_2x_2}(\mathbf{u}_k)$. As explained in Section II, we can design two filters

$$\mathbf{g}_{p,k} = \frac{\mathbf{R}_{x_px_p}^{-1}\mathbf{u}_k}{\mathbf{u}_k^H\mathbf{R}_{x_px_p}^{-1}\mathbf{u}_k}, \qquad p = 1, 2 \tag{16}$$

to find the spectra of $x_1(n)$ and $x_2(n)$ along $\mathbf{u}_k$

$$S_{x_px_p}(\mathbf{u}_k) = \frac{1}{\mathbf{u}_k^H\mathbf{R}_{x_px_p}^{-1}\mathbf{u}_k}, \qquad p = 1, 2 \tag{17}$$

where

$$\mathbf{R}_{x_px_p} = E\left\{\mathbf{x}_p(n)\mathbf{x}_p^H(n)\right\} \tag{18}$$

is the covariance matrix of the signal $x_p(n)$ and

$$\mathbf{x}_p(n) = [x_p(n) \quad x_p(n-1) \quad \cdots \quad x_p(n-K+1)]^T, \qquad p = 1, 2.$$

Let $y_{1,k}(n)$ and $y_{2,k}(n)$ be the respective outputs of the filters $\mathbf{g}_{1,k}$ and $\mathbf{g}_{2,k}$. We define the cross-spectrum between $x_1(n)$ and $x_2(n)$ along $\mathbf{u}_k$ as

$$S_{x_1x_2}(\mathbf{u}_k) = E\{y_{1,k}(n)y_{2,k}^*(n)\} \tag{19}$$

where the superscript $*$ is the complex conjugate operator. Similarly

$$\begin{aligned} S_{x_2x_1}(\mathbf{u}_k) &= E\{y_{2,k}(n)y_{1,k}^*(n)\} \\ &= S_{x_1x_2}^*(\mathbf{u}_k). \end{aligned} \tag{20}$$

Now if we develop (19), we get

$$S_{x_1x_2}(\mathbf{u}_k) = \mathbf{g}_{1,k}^H\mathbf{R}_{x_1x_2}\mathbf{g}_{2,k} \tag{21}$$

where

$$\mathbf{R}_{x_1x_2} = E\left\{\mathbf{x}_1(n)\mathbf{x}_2^H(n)\right\} \tag{22}$$

is the cross-correlation matrix between $x_1(n)$ and $x_2(n)$. Replacing (16) in (21), we obtain the cross-spectrum

$$S_{x_1x_2}(\mathbf{u}_k) = \frac{\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k}{\left[\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{u}_k\right]\left[\mathbf{u}_k^H\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k\right]}. \tag{23}$$

For (23) to have the true sense of the cross-spectrum definition, the matrix $\mathbf{U}$ should be complex (and unitary).

Property 3: We have

$$\operatorname{tr}\left[\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\right] = \sum_{k=0}^{K-1} S_{x_1x_1}^{-1}(\mathbf{u}_k)S_{x_1x_2}(\mathbf{u}_k)S_{x_2x_2}^{-1}(\mathbf{u}_k). \tag{24}$$

Proof: This is easy to see from (23).

Property 3 is similar to property 2 and shows another form of energy conservation.

B. General Form of the MSC Function

We define the MSC function between two signals $x_1(n)$ and $x_2(n)$ as

$$\gamma_{x_1x_2}^2(\mathbf{u}_k) = \frac{|S_{x_1x_2}(\mathbf{u}_k)|^2}{S_{x_1x_1}(\mathbf{u}_k)S_{x_2x_2}(\mathbf{u}_k)}. \tag{25}$$

From (23), we deduce the magnitude squared cross-spectrum

$$|S_{x_1x_2}(\mathbf{u}_k)|^2 = \frac{\left|\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k\right|^2}{\left[\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{u}_k\right]^2\left[\mathbf{u}_k^H\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k\right]^2}. \tag{26}$$

Using expressions (17) and (26) in (25), the MSC becomes

$$\gamma_{x_1x_2}^2(\mathbf{u}_k) = \frac{\left|\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k\right|^2}{\left[\mathbf{u}_k^H\mathbf{R}_{x_1x_1}^{-1}\mathbf{u}_k\right]\left[\mathbf{u}_k^H\mathbf{R}_{x_2x_2}^{-1}\mathbf{u}_k\right]}. \tag{27}$$

Property 4: We have

$$0 \le \gamma_{x_1x_2}^2(\mathbf{u}_k) \le 1, \quad \forall \mathbf{u}_k. \tag{28}$$

Proof: Since matrices $\mathbf{R}_{x_1x_1}$ and $\mathbf{R}_{x_2x_2}$ are assumed to be positive definite, it is clear that $\gamma_{x_1x_2}^2(\mathbf{u}_k) \ge 0$. To prove that $\gamma_{x_1x_2}^2(\mathbf{u}_k) \le 1$, we need to rewrite the MSC function. Define the vectors

$$\mathbf{u}_{p,k} = \mathbf{R}_{x_px_p}^{-1/2}\mathbf{u}_k, \qquad p = 1, 2 \tag{29}$$

and the normalized cross-correlation matrix

$$\mathbf{R}_{\mathrm{n,x_1x_2}} = \mathbf{R}_{x_1x_1}^{-1/2}\mathbf{R}_{x_1x_2}\mathbf{R}_{x_2x_2}^{-1/2}. \tag{30}$$

Using the previous definitions in (27), the MSC is now

$$\gamma_{x_1x_2}^2(\mathbf{u}_k) = \frac{\left|\mathbf{u}_{1,k}^H\mathbf{R}_{\mathrm{n,x_1x_2}}\mathbf{u}_{2,k}\right|^2}{\left[\mathbf{u}_{1,k}^H\mathbf{u}_{1,k}\right]\left[\mathbf{u}_{2,k}^H\mathbf{u}_{2,k}\right]}. \tag{31}$$

Consider the Hermitian positive semidefinite matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{R}_{\mathrm{n,x_1x_2}} \\ \mathbf{R}_{\mathrm{n,x_1x_2}}^H & \mathbf{I} \end{bmatrix} \tag{32}$$

and the vectors

$$\mathbf{u}'_{1,k} = \begin{bmatrix} \mathbf{u}_{1,k} \\ \mathbf{0} \end{bmatrix} \tag{33}$$

$$\mathbf{u}'_{2,k} = \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_{2,k} \end{bmatrix}. \tag{34}$$

We can easily check that

$$\left|\mathbf{u}_{1,k}^{\prime H}\mathbf{M}\mathbf{u}_{2,k}^{\prime}\right|^2 = \left|\mathbf{u}_{1,k}^{H}\mathbf{R}_{\mathrm{n,x_1x_2}}\mathbf{u}_{2,k}\right|^2 \tag{35}$$

$$\mathbf{u}_{p,k}^{\prime H}\mathbf{M}\mathbf{u}_{p,k}^{\prime} = \mathbf{u}_{p,k}^{H}\mathbf{u}_{p,k}, \qquad p = 1, 2. \tag{36}$$

Inserting these expressions in the Cauchy–Schwartz inequality

$$\left|\mathbf{u}_{1,k}^{\prime H}\mathbf{M}\mathbf{u}_{2,k}^{\prime}\right|^2 \leq \left[\mathbf{u}_{1,k}^{\prime H}\mathbf{M}\mathbf{u}_{1,k}^{\prime}\right]\left[\mathbf{u}_{2,k}^{\prime H}\mathbf{M}\mathbf{u}_{2,k}^{\prime}\right] \tag{37}$$

we see that $\gamma_{x_1x_2}^2(\mathbf{u}_k) \leq 1, \forall \mathbf{u}_k$.

Property 4 was, of course, expected in order that the definition (27) of the MSC could have a sense.

C. Simulation Example

In this subsection, we compare the MSC function estimated with our approach and with the MATLAB function "cohere" that uses the Welch's averaged periodogram method [7]. We consider the illustrative example of two signals $x_1(n)$ and $x_2(n)$ that do not have that much in common, except for two sinusoids at frequencies ν_0 and ν_1

$$x_1(n) = w_1(n) + \cos(2\pi\nu_0 n) + \cos(2\pi\nu_1 n) \tag{38}$$

$$x_2(n) = w_2(n) + \cos[2\pi(\nu_0 n + \phi_0)] + \cos[2\pi(\nu_1 n + \phi_1)] \tag{39}$$

where $w_1(n)$ and $w_2(n)$ are two independent zero-mean (real) white Gaussian random processes with unit variance. The phases ϕ_0 and ϕ_1 in the signal $x_2(n)$ are random. In this example, the theoretical coherence should be equal to 1 at the two frequencies ν_0 and ν_1 and 0 at the others. Here we chose $\nu_0 = 0.15$ and $\nu_1 = 0.18$. For both algorithms, we took 1024 samples and a window of length $K = 100$. As for the choice of the unitary matrix in our approach, we took the Fourier matrix. Fig. 1(a) and (b) give the MSC estimated with MATLAB and our method, respectively. Clearly, the estimation of the coherence function with our algorithm is much closer to its theoretical values.

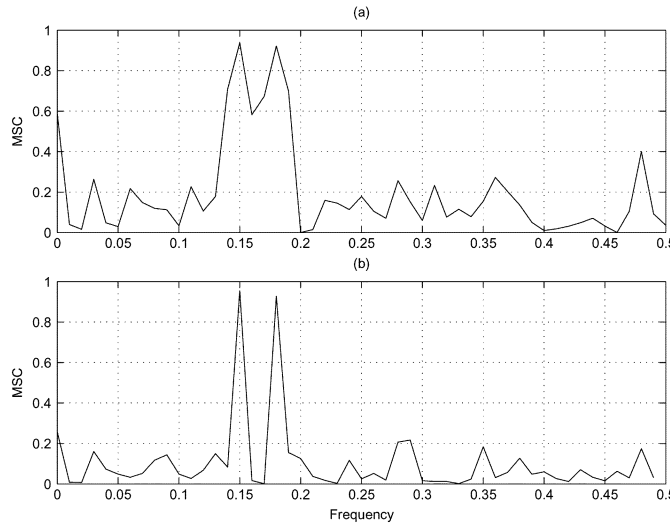

Fig. 1. Estimation of the MSC function. (a) MATLAB function "cohere." (b) Proposed algorithm with $\mathbf{U} = \mathbf{F}$. Conditions of simulations: $K = 100$ and a number of samples of 1024.

IV. Conclusion

The MVDR principle is very popular in array processing and spectral estimation. In this letter, we have shown that this concept can be generalized to unitary matrices other than the Fourier transform for spectrum evaluation. Most importantly, we have given an alternative to the popular Welch's method for the estimation of the MSC function. Simulations show that the new method works much better.

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