CSC236 winter 2020, week 8: Proving termination

Recommended supplementary reading: Chapter 2 Vassos course notes

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Proving termination

```
1  def imax(A):
2    """Pre: A is non-empty and contains comparable items.
3    Post: return the maximum element in A
4    """
5    curr = A[0]
6    i = 1
7    while i < len(A):
8      if A[i] > curr:
9        curr = A[i]
10     i += 1
11   return curr
```

Eventually *i* must reach `len(A)`...

A corollary of principle of well-ordering

All decreasing sequences of natural numbers are finite.

Spot the decreasing sequence

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6    i = 1
7    while i < len(A):
8      if A[i] > curr:
9        curr = A[i]
10     i += 1
11   return curr
```

Another corollary of PWO

Any increasing sequence of natural numbers with an upper bound is finite.

Recipe: proving termination

Define some quantity q_j associated with each iteration j of the loop.

- will be defined in terms of one or more variables that change inside the loop
- e.g. $q_j = \text{len}(A) - i_j$

Show that

- Every $q_j \in \mathbb{N}$
- the sequence $\langle q_0, q_1, q_2, \ldots \rangle$ is decreasing.

Example: A2Q3 appendix

```
1   def R(A):
2       B = []
3       i = 0
4       while i < len(A):
5           a = A[i]
6           b = A[(i+1) % len(A)]
7           if a == b:
8               B.append(a)
9           i += 1
10      return B
```

Lemma (R termination)

R terminates on any $A \in \mathbb{N}^*$

Proof.

Let $q_j = \text{len}(A) - i_j$ be a quantity associated with each loop iteration j. By Lemma 1.3 (a), $q_j \in \mathbb{N}$. By line 9, $q_{j+1} = q_j - 1$. Thus $q_0, q_1, q_2, \ldots$ is a decreasing sequence of natural numbers, and therefore finite. Therefore, R terminates. □

merge

```
1  def merge(A, B):
2    """Pre: A and B are sorted lists of numbers.
3    Post: return a sorted permutation of A+B
4    """
5    i = j = 0
6    C = []
7    while i < len(A) and j < len(B):
8      if A[i] <= B[j]:
9        C.append(A[i])
10       i += 1
11     else:
12       C.append(B[j])
13       j += 1
14   return C + A[i:] + B[j:]
```

bitcount (week 7 tutorial exercise)

```
1  def bitcount(n):
2      """Pre: n is a positive int.
3      Post: return the number of digits in the binary representation of n
4      """
5      i = 1
6      while n > 1:
7          n = n//2
8          i += 1
9      return i
```

Tricky example

```
1  def mystery(n):
2    """Pre: n is a positive int
3    """
4    i = 0
5    while n > 1:
6      if n % 10 == 0:
7        n = n // 10
8      else:
9        n += 1
10     i += 1
11   return i
```