# Gaussian fluctuations in the two-dimensional BCS-BEC crossover: finite temperature properties

Giacomo Bighin

in collaboration with:
Luca Salasnich and Flavio Toigo

Università degli Studi di Padova
and INFN

Shanghai, August 25th 2015

# Outline

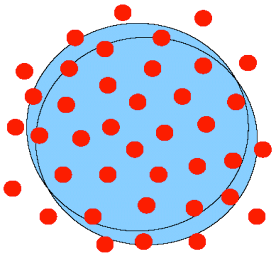
- Introduction and motivation: BCS-BEC crossover in 2D.
- Theoretical description of a 2D Fermi gas: mean-field and Gaussian fluctuations.
- Need for fluctuations: the composite boson limit.
- First and second sound.
- Berezinskii-Kosterlitz-Thouless critical temperature: comparison with experimental data.

**Main reference**: GB and L. Salasnich, arXiv:1507.07542.

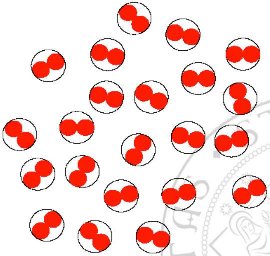
# The BCS-BEC crossover (1/2)

In 2004 the **BCS-BEC crossover** has been observed with ultracold gases made of fermionic $^{40}$K and $^{6}$Li alkali-metal atoms. The fermion-fermion attractive interaction can be tuned (using a Feshbach resonance), from weakly to strongly interacting.

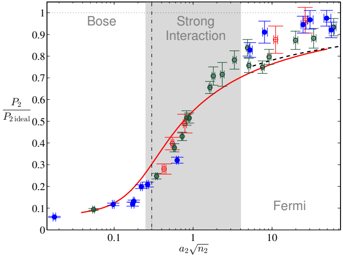
**BCS regime**: weakly interacting Cooper pairs.



**BEC regime**: tightly bound bosonic molecules.



# The BCS-BEC crossover (2/2)




In 2014 also the 2D BEC-BEC crossover has been achieved[1] with a **quasi-2D Fermi gas of $^6$Li atoms** with widely tunable s-wave interaction. The pressure $P$ *vs* the gas parameter $a_B n^{1/2}$. has been measured. In 2015 the BKT transition has been observed[2].

Why is the 2D case interesting?

- The fluctuations are more relevant for lower dimensionalities.
- Berezinskii-Kosterlitz-Thouless mechanism:
  - Mermin-Wagner-Hohenberg theorem: no condensation at finite temperature, no off-diagonal long-range order.
  - Algebraic decay of correlation functions $\langle \exp(i\theta(\mathbf{r})) \exp(i\theta(0)) \rangle \sim |\mathbf{r}|^{-\eta}$
  - Transition to the normal state at a finite temperature $T_{BKT}$.

[1] V. Makhalov, K. Martiyanov, and A. Turlapov, PRL **112**, 045301 (2014).

[2] P. A. Murthy et al., Phys. Rev. Lett. **115**, 010401 (2015).

# Formalism for a $D$-dimensional Fermi superfluid (1/4)

We adopt the path integral formalism. The partition function $\mathcal{Z}$ of the uniform system with fermionic fields $\psi_s(\mathbf{r},\tau)$ at temperature $T$, in a $D$-dimensional volume $L^D$, and with chemical potential $\mu$ reads

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \; \exp\left\{ -\frac{1}{\hbar} \; S \right\} ,$$

where ($\beta \equiv 1/(k_B T)$ with $k_B$ Boltzmann's constant)

$$S = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r} \; \mathcal{L}$$

is the Euclidean action functional with Lagrangian density:

$$\mathcal{L} = \bar{\psi}_s \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + g_0 \, \bar{\psi}_\uparrow \, \bar{\psi}_\downarrow \, \psi_\downarrow \, \psi_\uparrow$$

where $g_0$ is the attractive strength ($g_0 < 0$) of the s-wave coupling.

# Formalism for a $D$-dimensional Fermi superfluid (2/4)

In 2D the strength of the attractive s-wave potential is $g_0 < 0$, which can be implicitely related to the bound state energy:

$$-\frac{1}{g_0} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\epsilon_k + \frac{1}{2}\epsilon_b} \ .$$

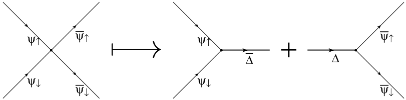
with $\epsilon_k = \hbar^2 k^2/(2m)$. In 2D, as opposed to the 3D case, a bound state exists even for arbitrarily weak interactions, making $\epsilon_B$ a good variable to describe the whole BCS-BEC crossover.

The binding energy $\epsilon_b$ and the fermionic scattering length $a_F$ are related by the equation[3]:

$$\epsilon_B = \frac{4\hbar^2}{e^{2\gamma} m a_F^2}$$

[3] C. Mora and Y. Castin, Phys. Rev. A **67**, 053615 (2003).

# Formalism for a $D$-dimensional Fermi superfluid (3/4)




Through the usual Hubbard-Stratonovich transformation the Lagrangian density $\mathcal{L}$, quartic in the fermionic fields, can be rewritten as a quadratic form by introducing the auxiliary complex scalar field $\Delta(\mathbf{r},\tau)$ so that:

$$\mathcal{Z} = \int \mathcal{D}[\psi_s, \bar{\psi}_s]\, \mathcal{D}[\Delta, \bar{\Delta}] \;\exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})}{\hbar} \right\},$$

where

$$S_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta}) = \int_0^{\hbar\beta} d\tau \int_{L^D} d^D\mathbf{r}\; \mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$$

and the (exact) effective Euclidean Lagrangian density $\mathcal{L}_e(\psi_s, \bar{\psi}_s, \Delta, \bar{\Delta})$ reads

$$\mathcal{L}_e = \bar{\psi}_s \left[ \hbar\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu \right] \psi_s + \bar{\Delta}\, \psi_\downarrow\, \psi_\uparrow + \Delta \bar{\psi}_\uparrow\, \bar{\psi}_\downarrow - \frac{|\Delta|^2}{g_0}.$$

# Formalism for a $D$-dimensional Fermi superfluid (4/4)

We want to investigate the effect of fluctuations of the gap field $\Delta(\mathbf{r}, t)$ around its saddle-point value $\Delta_0$ which may be taken to be real. For this reason we set

$$\Delta(\mathbf{r}, \tau) = \Delta_0 + \eta(\mathbf{r}, \tau) \; ,$$

where $\eta(\mathbf{r}, \tau)$ is the complex field which describes pairing fluctuations. In particular, we are interested in the grand potential $\Omega$, given by

$$\Omega = -\frac{1}{\beta} \ln\left(\mathcal{Z}\right) \simeq -\frac{1}{\beta} \ln\left(\mathcal{Z}_{mf} \mathcal{Z}_g\right) = \Omega_{mf} + \Omega_g \; ,$$

where

$$\mathcal{Z}_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \exp\left\{ -\frac{S_e(\psi_s, \bar{\psi}_s, \Delta_0)}{\hbar} \right\}$$

is the mean-field partition function and

$$\mathcal{Z}_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \, \mathcal{D}[\eta, \bar{\eta}] \; \exp\left\{ -\frac{S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0)}{\hbar} \right\}$$

is the partition function of Gaussian pairing fluctuations.

# Single particle and collective excitations

One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{sp}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \, ,$$

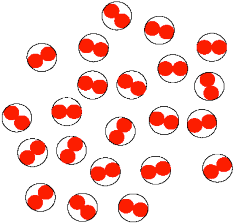
where $\Delta_0$ is the pairing gap, and bosonic collective excitations with energy

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \, \frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \, ,$$

where $\lambda$ is the first correction to the familiar low-momentum phonon dispersion $E_{col}(q) \simeq c_s \hbar q$ and $c_s$ is the sound velocity.

# The role of Gaussian fluctuations and collective excitations: composite bosons

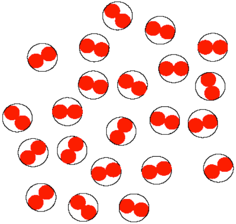
In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. Residual interaction.
Is this limit correctly recovered[4] at mean-field?
And at a Gaussian level?



[1]L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015)

# The role of Gaussian fluctuations and collective excitations: composite bosons

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. Residual interaction.
Is this limit correctly recovered[4] at mean-field?
And at a Gaussian level?



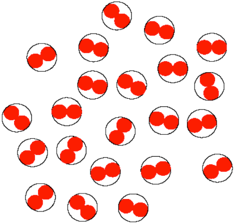
### Mean-field equation of state

$$\mu_B = \frac{8\pi\hbar^2}{m_B} n_B$$

This equation of state showing a bosonic chemical potential $\mu_B$ independent of the interaction between bosons is lacking important informations which must be encoded in the quantum fluctuations.

# The role of Gaussian fluctuations and collective excitations: composite bosons

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. Residual interaction.
Is this limit correctly recovered[4] at mean-field?
And at a Gaussian level?



**Gaussian equation of state**

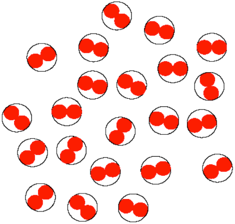
The zero-temperature total grand potential is

$$\Omega = \Omega_{mf} + \Omega_g = -\frac{mL^2}{64\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_b)^2 \ \ln\left(\frac{\epsilon_b}{2(\mu + \frac{1}{2}\epsilon_b)}\right) .$$

This is exactly Popov's equation of state of two-dimensional interacting composite bosons.

# The role of Gaussian fluctuations and collective excitations: composite bosons

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. Residual interaction.
Is this limit correctly recovered[4] at mean-field?
And at a Gaussian level?

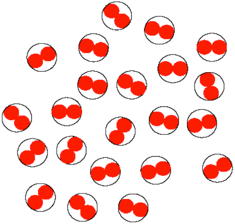


**Scattering length**

The Gaussian-level value for the scattering length $a_B$ of composite bosons is $a_B = a_F/(2^{\frac{1}{2}} e^{\frac{1}{4}}) \simeq 0.551 a_F$, in full agreement with Monte Carlo calculations (G. Bertaina and S. Giorgini, PRL **106**, 110403 (2011)).

# The role of Gaussian fluctuations and collective excitations: composite bosons

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_b/2)$ and mass $m_B = 2m$. Residual interaction.
Is this limit correctly recovered[4] at mean-field?
And at a Gaussian level?



Gaussian fluctuations are crucial in correctly describing the properties of a 2D Fermi gas in the BEC limit: it is also very important to use a Gaussian-level equation of state (examples will follow).

[1] L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015)

# Regularization

The contribution from fluctuations does not converge:

$$\Omega_g = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q)$$

Many regularization schemes:

- Dimensional regularization
  *Analytical results*[5] *in the BEC limit in 2D*
- Counterterms regularization
  *Analytical results*[6] *in the BEC limit in 3D*
- Convergence factor regularization
  *Numerics for the whole crossover*[7,8]

.

[4] L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015).
[5] L. Salasnich and GB, Phys. Rev. A **91**, 033610 (2015).
[6] R. B. Diener, R. Sensarma, and M. Randeria, Phys. Rev. A **77**, 023626 (2008)
[7] L. He, H. Lü, G. Cao, H. Hu and X.-J. Liu, arXiv:1506.07156

# Regularization

The contribution from fluctuations does not converge:

$$\Omega_g = \frac{1}{2} \sum_{\mathbf{q}} E_{col}(q)$$

Many regularization schemes:

- Dimensional regularization
  *Analytical results[5] in the BEC limit in 2D*
- Counterterms regularization
  *Analytical results[6] in the BEC limit in 3D*
- Convergence factor regularization
  *Numerics for the whole crossover[7,8]*

.

[4] L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015).
[5] L. Salasnich and GB, Phys. Rev. A **91**, 033610 (2015).
[6] R. B. Diener, R. Sensarma, and M. Randeria, Phys. Rev. A **77**, 023626 (2008)
[7] L. He, H. Lü, G. Cao, H. Hu and X.-J. Liu, arXiv:1506.07156

# First sound velocity (1/2)

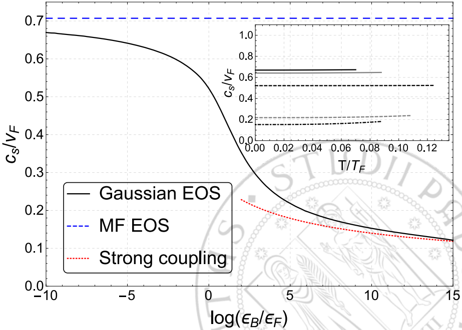
It can be read from the collective excitations spectrum:

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \frac{\hbar^2 q^2}{2m} + 2m{c_s}^2\right)} \simeq c_s \hbar q$$

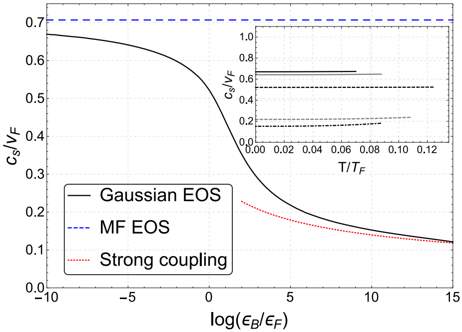
The sound velocity at $T = 0$ can be calculated through the thermodynamics formula:

$$c_s = \sqrt{\frac{n}{m}\frac{\partial \mu}{\partial n}}$$

We compare our result with the *"mean-field"* result, i.e. obtained with a mean-field equation of state, and with the composite boson limit.



# First sound velocity (2/2)




- In the BEC limit $c_s$ is strongly affected by the Gaussian equation of state.
- The temperature dependence (inset) is very weak.
- Strong coupling: composite boson limit.
- Very recent developments in 2D ultracold Fermi gas should make this theoretical prediction open to verification (hopefully) quite soon.

# BKT critical temperature (1/2)

The BKT critical temperature is found using the Kosterlitz-Nelson condition:

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT})$$

The superfluid density is obtained using Landau's quasiparticle excitations formula for fermionic and bosonic excitations:

$$n_{n,f} = \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_k}}{(e^{\beta E_k}+1)^2} \quad \text{and} \quad n_{n,b} = \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta \omega_q}}{(e^{\beta \omega_q}-1)^2} \, ,$$

then $n_s = n - n_{n,f} - n_{n,b}$.

- **Approximation:** no hybridization due to Landau damping.
- **What we expect:** Combining $a_B = \frac{1}{2^{1/2} e^{1/4}} a_F$, $\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m a_F{}^2}$ we get:

$$\frac{\epsilon_B}{\epsilon_F} = \frac{\kappa}{n_B a_B^2} \qquad \kappa \simeq 0.061$$

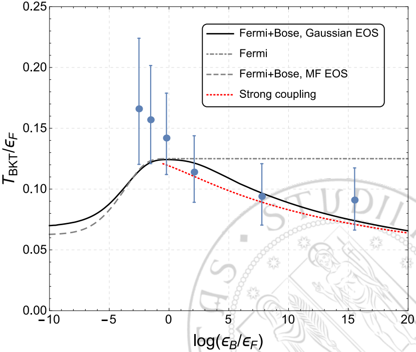
# BKT critical temperature (2/2)

We can compare the theory with very recently obtained experimental data[9]:

- Within error bars for $\epsilon_B/\epsilon_F \gtrsim 1$
- Worse agreement for $\epsilon_B/\epsilon_F \lesssim 1$
- In the strong coupling limit:

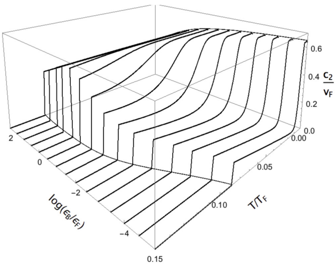
$$k_B T_{BKT} \approx \frac{\mu_B^{\frac{2}{3}} \epsilon_F^{\frac{1}{3}}}{\sqrt[3]{12\zeta(3)}} - \frac{8}{3} \frac{\mu_B^{\frac{4}{3}} \epsilon_F^{-\frac{1}{3}}}{(12\zeta(3))^{\frac{2}{3}}}$$

- Caveat: we model a uniform system, experiments are done in a trap.




[1] P. A. Murthy et al., Phys. Rev. Lett. **115**, 010401 (2015).

# Second sound velocity




A superfluid can also sustain the second sound (entropy wave as opposed to density wave):

$$F_{sp} = -\frac{2}{\beta}\sum_{\mathbf{k}} \ln\left[1 + e^{-\beta E_{sp}(k)}\right]$$

$$F_{col} = \frac{1}{\beta}\sum_{\mathbf{q}} \ln\left[1 - e^{-\beta E_{col}(q)}\right]$$

$$S = -(\partial F/\partial T)_{N,L^2}$$

$$c_2 = \sqrt{\frac{1}{m}\frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,L^2}}\frac{n_s}{n_n}}$$

# Conclusions

- The theoretical treatment of a 2D Fermi gas needs the inclusion of Gaussian fluctuations, which in turn require a proper regularization.
- This approach shows good agreement with experimental data (BKT critical temperature), other predictions are open to verification (first sound, second sound): two-dimensional BCS-BEC is a young field.
- This treatment can be extended to 2D systems with BCS-like pairing (bilayers of polar molecules, exciton condensates, etc.)

Thanks for your attention.