Hyperuniformity of 1D Point Patterns

Week 2 Progress: Expanding the Catalog

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Week 2 progress update. This week expanded the catalog from 4 patterns (lattice + 3 quasicrystals) to 25 patterns across all three hyperuniformity classes.

Week 1 Recap

Established:

- Validated variance algorithm (Poisson benchmark, 1.1% error)
- Generated Fibonacci, Silver, Bronze chains at $N \sim 10^7$
- Confirmed $\alpha = 3$ for all three via spreadability (Oğuz et al., 2019)
- Computed $\bar{\Lambda}$ — the surface-area coefficient

Pattern	α	$\bar{\Lambda}$	**Class**
Integer Lattice	∞	1/6	I
Fibonacci	3.05	0.200	I
Silver	2.99	0.250	I
Bronze	2.99	0.282	I

This week: expand from 4 to **25 patterns** across all three hyperuniformity classes.

Brief recap of week 1. Key gap: only 4 patterns, only Class I, only $\alpha = 3$ and $\alpha = \infty$.

Novel Result: Copper & Nickel Quasicrystals

Extended the metallic-mean family to $n = 4$ (Copper) and $n = 5$ (Nickel):

Chain	Rule ($L \to$)	μ_n	N	α	$\bar{\Lambda}$	Literature
Fibonacci	*LS*	1.618	14.9M	3.05	0.200	0.201[a]
Silver	*LLS*	2.414	22.6M	2.99	0.250	—
Bronze	*LLLS*	3.303	21.9M	2.99	0.282	—
Copper	*LLLLS*	4.236	39.1M	**3.03**	**0.293**	**novel**
Nickel	*LLLLLS*	5.193	16.4M	**2.88**	**0.310**	**novel**

[a] Zachary & Torquato (2009).

Key findings:

- $\alpha = 3$ confirmed for all five chains (eigenvalue formula: exact)
- $\bar{\Lambda}$ **increases monotonically** with n
- $\Delta\bar{\Lambda}$: 0.050, 0.032, 0.011, 0.017 $\Rightarrow$ converging toward a limit

Eigenvalue prediction (Oğuz et al., 2019):

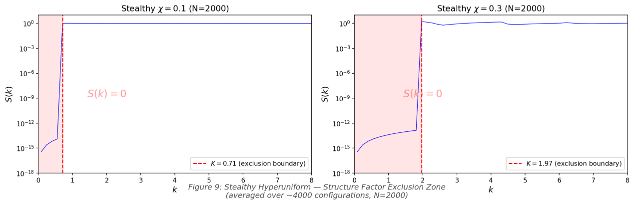
$$\alpha = 1 - \frac{2 \ln |\lambda_2|}{\ln \lambda_1}$$

For all metallic means, $\det M = -1$, so $|\lambda_2| = 1/\lambda_1$ and $\alpha = 3$ exactly.

Silver, Bronze, Copper, Nickel
$\bar{\Lambda}$ values appear **novel**.

Copper: $\mu_4 = 2 + \sqrt{5} \approx 4.236$; $\bar{\Lambda} = 0.293$. Nickel: $\mu_5 = (5 + \sqrt{29})/2 \approx 5.193$; $\bar{\Lambda} = 0.310$. Both novel. Alpha confirmed via eigenvalue formula (exact 3.000) and spreadability fit (3.03, 2.88). The trend suggests $\bar{\Lambda} \to$ limit $\lesssim 1/3$ as $n \to \infty$.

Stealthy Hyperuniform Patterns

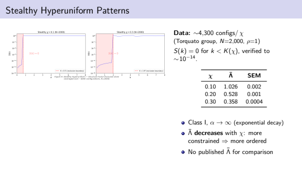


Data: $\sim$4,300 configs/ χ
(Torquato group, N=2,000, ρ=1)

$S(k) = 0$ for $k < K(\chi)$, verified to $\sim 10^{-14}$.

χ	$\bar{\Lambda}$	SEM
0.10	1.026	0.002
0.20	0.528	0.001
0.30	0.358	0.0004

- Class I, $\alpha \to \infty$ (exponential decay)
- $\bar{\Lambda}$ **decreases** with χ: more constrained $\Rightarrow$ more ordered
- No published $\bar{\Lambda}$ for comparison



Stealthy patterns: Torquato, Zhang, Stillinger (2015), Phys. Rev. X 5. The exclusion zone $S(k) = 0$ for $k < K$ gives $\alpha \to \infty$ (faster than any power law). Lambda-bar from 500-config ensemble, SEM $< 0.2\%$.

Perturbed Lattice Models

Displace each site of $\mathbb{Z}$ by i.i.d. draw from distribution f (Klatt, Kim, Torquato, 2020):

Distribution	$\hat{f}(k)$	α	Class	Parameters
Uniform $[-a/2, a/2)$	$\mathrm{sinc}(ka/2\pi)$	2	I	$a = 0.1$–1.0
Gaussian $\mathcal{N}(0, \sigma^2)$	$e^{-\sigma^2 k^2/2}$	2	I	$\sigma = 0.1$–0.5
Cauchy$(0, \gamma)$	$e^{-\gamma\lvert k\rvert}$	1	II	$\gamma = 0.1$
Stable(s, c)	$e^{-c^s\lvert k\rvert^s}$	s	I/II/III	$s = 0.3$–1.7

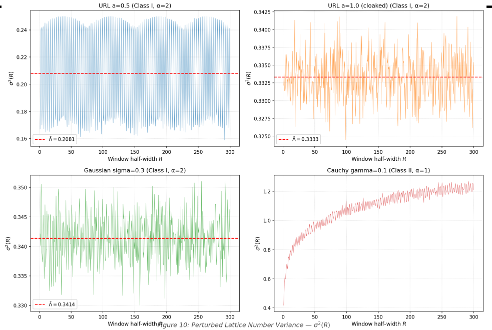
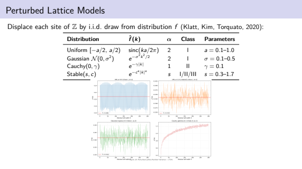


Figure 10: Perturbed Lattice Number Variance — $\sigma^2(R)$



Perturbed lattices: Klatt, Kim, Torquato (2020), Phys. Rev. E 101, 032118. The displacement distribution's characteristic function $\hat{f}(k)$ controls α: $S(k) \sim 1 - |\hat{f}(k)|^2$ near $k = 0$. URL and Gaussian give $\alpha = 2$ (Class I), Cauchy gives $\alpha = 1$ (Class II), stable with $s < 1$ gives Class III.

URL Model: Exact $\bar{\Lambda}(a)$ Validated

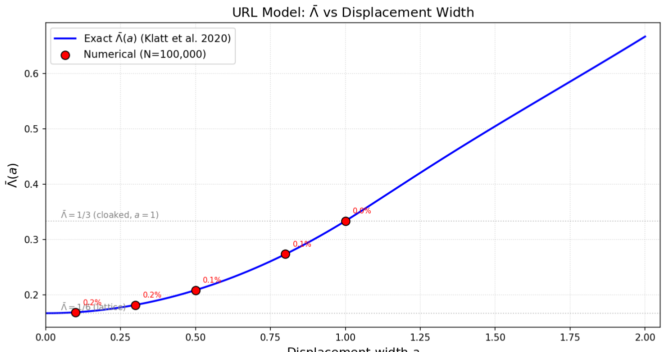


Figure 14: Exact $\bar{\Lambda}(a)$ for the Uniformly Randomized Lattice

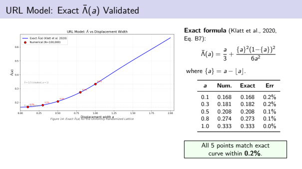
Exact formula (Klatt et al., 2020, Eq. B7):

$$\bar{\Lambda}(a) = \frac{a}{3} + \frac{\{a\}^2(1-\{a\})^2}{6a^2}$$

where $\{a\} = a - \lfloor a \rfloor$.

a	**Num.**	**Exact**	**Err**
0.1	0.168	0.168	0.2%
0.3	0.181	0.182	0.2%
0.5	0.208	0.208	0.1%
0.8	0.274	0.273	0.1%
1.0	0.333	0.333	0.0%

All 5 points match exact curve within **0.2%**.



The URL (Uniformly Randomized Lattice) has an exact $\bar{\Lambda}(a)$ formula from Klatt et al. (2020), Eq. B7. At $a = 0$: lattice (1/6). At $a = 1$: cloaked (1/3, all Bragg peaks vanish). Our numerics at $N = 100{,}000$ match to $< 0.2\%$ for all a values.

Filling the $1 < \alpha < 2$ Gap: Stable Perturbations

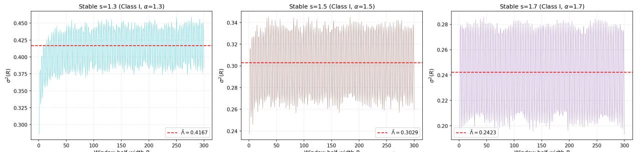


Figure 13: Stable Perturbations ($s > 1$) — Class I $\sigma^2(R)$

Bounded variance confirms Class I for all $s > 1$.

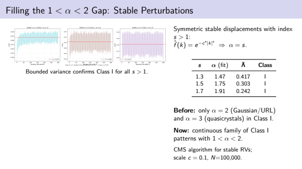
Symmetric stable displacements with index $s > 1$:
$\hat{f}(k) = e^{-c^s |k|^s} \Rightarrow \alpha = s$.

s	α (fit)	$\bar{\Lambda}$	Class
1.3	1.47	0.417	I
1.5	1.75	0.303	I
1.7	1.91	0.242	I

Before: only $\alpha = 2$ (Gaussian/URL) and $\alpha = 3$ (quasicrystals) in Class I.

Now: continuous family of Class I patterns with $1 < \alpha < 2$.

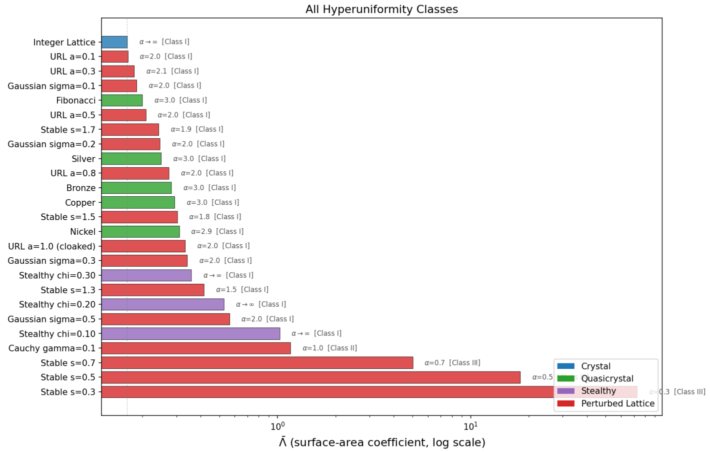
CMS algorithm for stable RVs;
scale $c = 0.1$, N=100,000.

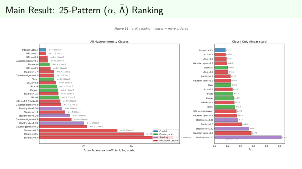


Stable distributions with $s > 1$ give Class I hyperuniform patterns with $\alpha = s$, filling the gap $1 < \alpha < 2$. Chambers-Mallows-Stuck algorithm generates symmetric stable RVs. The same code handles $s < 1$ (Class III) and $s > 1$ (Class I).

Main Result: 25-Pattern $(\alpha, \bar{\Lambda})$ Ranking

Figure 11: $(\alpha, \bar{\Lambda})$ ranking — lower = more ordered





The comprehensive ranking chart: 25 patterns sorted by $\bar{\Lambda}$. Left panel: log scale shows all classes. Right panel: linear scale zooms into Class I. Lower $\bar{\Lambda}$ = more ordered. Integer lattice is the most ordered ($\bar{\Lambda} = 1/6$). Stable $s = 0.3$ is the least ordered Class III pattern ($\bar{\Lambda} \sim 73$).

Class I Ranking: $\bar{\Lambda}$ from Lattice to Disorder

#	**Pattern**	α	$\bar{\Lambda}$	**Category**
1	Integer Lattice	∞	0.167	crystal
2	URL a=0.1	2.0	0.168	perturbed
3	Gaussian σ=0.1	2.0	0.187	perturbed
4	Fibonacci	3.0	0.200	quasicrystal
5	URL a=0.5	2.0	0.208	perturbed
6	Stable s=1.7	1.9	0.242	perturbed
7	Silver	3.0	0.250	quasicrystal
8	**Copper**	**3.0**	**0.293**	quasicrystal
9	**Nickel**	**2.9**	**0.310**	quasicrystal
10	URL a=1.0 (cloaked)	2.0	0.333	perturbed
11	Stealthy χ=0.3	∞	0.358	stealthy
12	Stealthy χ=0.1	∞	1.026	stealthy

+ Cauchy (Class II) and 3 stable s<1 (Class III)

Lower $\bar{\Lambda}$ = more ordered. Lattice is ground state; quasicrystals intermediate; stealthy (despite $\alpha \to \infty$) have large $\bar{\Lambda}$ from residual short-range disorder.

Key insight: α and $\bar{\Lambda}$ are complementary metrics. Stealthy patterns have $\alpha \to \infty$ but $\bar{\Lambda} \gg 1/6$, while quasicrystals have finite $\alpha = 3$ but smaller $\bar{\Lambda}$. This shows why both metrics are needed for a complete ranking.

$\bar{\Lambda}$ comparisons:

Pattern	Ours	Lit.	Err
Lattice	0.167	1/6	exact
Fibonacci	0.200	0.201[a]	0.5%
URL a=1	0.333	1/3 [b]	0.0%
URL a=0.5	0.208	0.208[b]	0.1%
Silver	0.250	—	*novel*
Bronze	0.282	—	*novel*
Copper	0.293	—	*novel*
Nickel	0.310	—	*novel*

[a]Zachary & Torquato (2009).
[b]Klatt et al. (2020), Eq. B7.

α comparisons:

Pattern	Ours	Expected	Err
Fibonacci	3.05	3	1.6%
Silver	2.99	3	0.3%
Bronze	2.99	3	0.4%
Copper	3.03	3	0.9%
Nickel	2.88	3	4.0%
URL (all a)	≈2.0	2	<4%
Cauchy	0.95	1	4.9%

All published values reproduced within **5%**.

All α and $\bar{\Lambda}$ values validated against published results. Fibonacci $\bar{\Lambda} = 0.201$ from Zachary & Torquato (2009), Table 1. URL exact formula from Klatt et al. (2020), Eq. B7. Eigenvalue prediction from Oğuz et al. (2019). Silver, Bronze, Copper, Nickel $\bar{\Lambda}$ values appear to be novel.

Open Questions

1. The $2 < \alpha < 3$ gap:

- All metallic-mean substitutions give $\alpha = 3$ exactly (det $M = -1$)
- All finite-variance perturbations give $\alpha = 2$ exactly
- **No known 1D construction achieves** $2 < \alpha < 3$
- Is this a fundamental gap, or does a construction exist?

2. Metallic-mean $\bar{\Lambda}$ convergence:

- $\bar{\Lambda}(n)$: 0.200, 0.250, 0.282, 0.293, 0.310
- Successive differences decreasing — what is $\bar{\Lambda}_\infty$?
- Is the limit 1/3 (the cloaked-URL value)?

3. Next steps:

- Add period-doubling chains (Class II, $\alpha = 1$)
- Obtain more stealthy χ values from grad student
- Begin writing JP paper with the 25-pattern ranking table

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The $2 < \alpha < 3$ gap is a genuine open problem. Non-Pisot substitution matrices could potentially give $2 < \alpha < 3$, but these are rare and may not produce quasicrystalline patterns. The $\bar{\Lambda}$ convergence question could be addressed analytically or by computing $n = 6, 7, \ldots$ chains.