Hyperuniformity of 1D Point Patterns

Week 1 Progress

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February 25, 2026

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Week 1 progress update on the 1D hyperuniformity project.

Background

A point pattern is **hyperuniform** if density fluctuations are suppressed relative to random (Torquato & Stillinger, 2003):

$$S(k) \sim |k|^{\alpha} \text{ as } k \to 0, \qquad \sigma^2(R) \sim \begin{cases} R^{d-\alpha} & 0 < \alpha < 1 \\ R^{d-1} \ln R & \alpha = 1 \\ R^{d-1} & \alpha > 1 \end{cases}$$

Three chains built from substitution rules on tiles $S{=}1$, $L{=}\mu$:

Chain	Rule	Metallic mean μ	ρ
Fibonacci	$S \to L,\ L \to LS$	$\tau \approx 1.618$	0.724
Silver	$S \to L,\ L \to LLS$	$\mu_2 \approx 2.414$	0.500
Bronze	$S \to L,\ L \to LLLS$	$\mu_3 \approx 3.303$	0.361

Goal: numerically verify the eigenvalue prediction $\boxed{\alpha = 3}$ (Oğuz et al., 2019) for all three chains, using diffusion spreadability (Torquato, 2021).

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Hyperuniformity: Torquato & Stillinger (2003). Eigenvalue formula: Oğuz et al. (2019), Acta Cryst. A 75. Spreadability method: Torquato (2021), Phys. Rev. E 104.

Code Validation: Poisson Benchmark

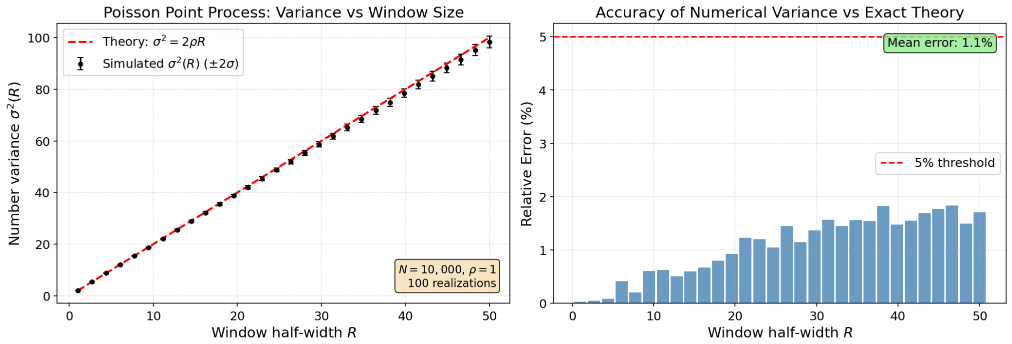
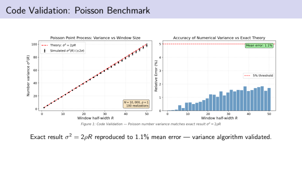


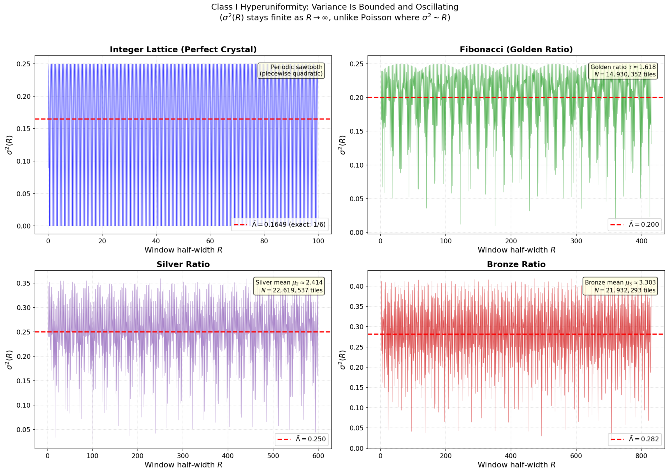
Figure 1: Code Validation — Poisson number variance matches exact result $\sigma^2 = 2\rho R$

Exact result $\sigma^2 = 2\rho R$ reproduced to 1.1% mean error — variance algorithm validated.



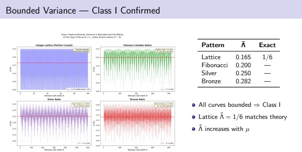
100 independent Poisson patterns, N=10,000. This validates the sliding-window algorithm used for all subsequent measurements.

Bounded Variance — Class I Confirmed



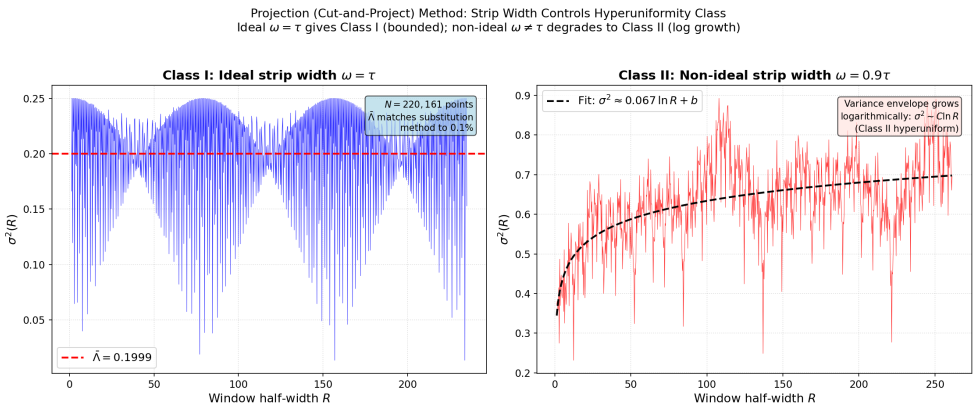
Pattern	$\bar{\Lambda}$	Exact
Lattice	0.165	1/6
Fibonacci	0.200	—
Silver	0.250	—
Bronze	0.282	—

- All curves bounded $\Rightarrow$ Class I
- Lattice $\bar{\Lambda} = 1/6$ matches theory
- $\bar{\Lambda}$ increases with μ

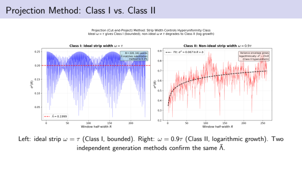


Lattice $\bar{\Lambda} = 1/6$ exact (Torquato & Stillinger, 2003). Fibonacci $\bar{\Lambda} = 0.201$ reported numerically in Zachary & Torquato (2009); our 0.200 matches to 0.5%.

Projection Method: Class I vs. Class II



Left: ideal strip $\omega = \tau$ (Class I, bounded). Right: $\omega = 0.9\tau$ (Class II, logarithmic growth). Two independent generation methods confirm the same $\bar{\Lambda}$.



The cut-and-project method provides an independent realization. Any deviation from ideal strip width degrades Class I to Class II.

Spreadability Method

Problem: $S(k)$ has dense Bragg peaks $\implies$ can't read off α directly (Torquato, 2021).

Solution: embed points in a two-phase medium (Torquato, 2002) and measure diffusion spreadability (Torquato, 2021):

1. Decorate each point with a solid rod (packing fraction $\phi_2 = 0.35$)
2. Compute spectral density $\tilde{\chi}_V(k) = \rho\,|\tilde{m}(k)|^2 S(k)$
3. Evaluate excess spreadability $E(t)$ via Gaussian-smoothed sum
4. Extract α from long-time power-law decay: $E(t) \sim t^{-(1+\alpha)/2}$

The Gaussian kernel $e^{-k^2 Dt}$ naturally smooths over the dense Bragg peaks, revealing the underlying α (Torquato, 2021; Hitin-Bialus et al., 2024).

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Two-phase media: Torquato (2002), Random Heterogeneous Materials, Ch. 2. Spreadability method: Torquato (2021), Phys. Rev. E 104, 054102. Applied to quasicrystals: Hitin-Bialus et al. (2024), Princeton thesis. Packing fraction $\phi_2 = 0.35$ follows from Hitin-Bialus (2024).

Main Result: $\alpha = 3$ for All Three Chains

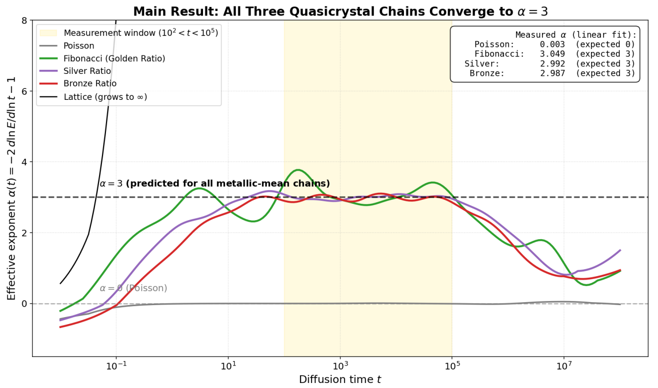
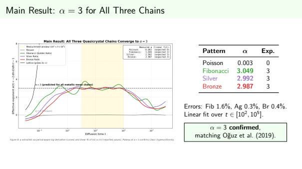


Figure 8: α extracted via period-aware log-derivative (curves) and linear fit of ln E vs ln t (reported values). Plateau at α = 3 confirms Class I hyperuniformity.

Pattern	α	Exp.
Poisson	0.003	0
Fibonacci	**3.049**	3
Silver	**2.992**	3
Bronze	**2.987**	3

Errors: Fib 1.6%, Ag 0.3%, Br 0.4%.
Linear fit over $t \in [10^2, 10^5]$.

$\alpha = 3$ **confirmed**,
matching Oğuz et al. (2019).



Main result: all three chains give $\alpha \approx 3$, matching the eigenvalue prediction $\alpha = 1 - 2\ln|\lambda_2|/\ln\lambda_1$ from Oğuz et al. (2019). Poisson baseline confirms $\alpha \approx 0$.