Lost in Space

Binary search trees beyond dimension one

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Outline

$\mathbb{R}^1$

- BST

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$\mathbb{R}^d$

- What could go wrong?
- Quadtrees
- kd-trees
- Range trees
- More

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Binary Search Trees

- Is $p \in S$?
- Given x, what is closest point $p \in S$?
- Find $\{x \mid x \in [p_1, p_2]\}$.
- Given δ, find $\{x \mid \mathrm{d}(x, p) < \delta\}$.

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This is easy in $\mathbb{R}^1$. What about $\mathbb{R}^d$?

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What could go wrong?

The Curse of Dimensionality

What could go wrong?

The Curse of Dimensionality

Some things to think about:

- Volume of unit cube $\pm\epsilon$
- Distance from $(0, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$
- Physics: $1/r^{d-1}$

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Richard Ernest Bellman, Dynamic programming, Princeton University Press, 1957.

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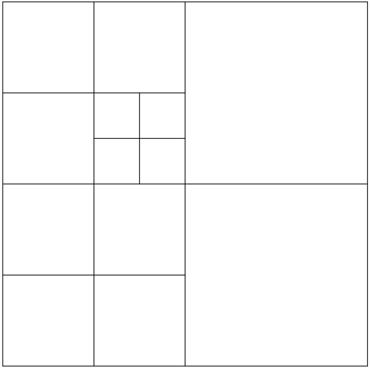
Quadtrees in Words

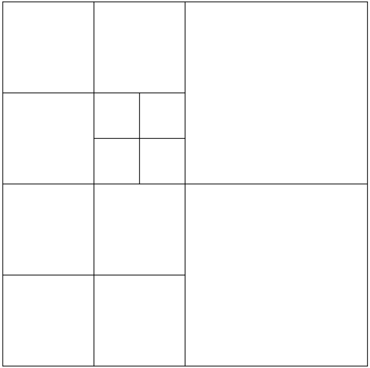
Properties (let's start in $\mathbb{R}^2$)

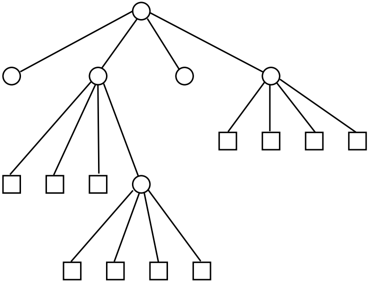
- Rooted tree
- Internal nodes have four children

Quadtrees in Pictures



Quadtrees in Pictures





Quadtrees in Mathematics

- Find a bounding box: $O(n)$
- Then,
  - Divide in four quadrants: $O(1)$
  - Partition the points: $O(n)$
  - Recursively build a quad tree: ??

Quadtrees in Equations

- Depth is at most $\delta = \left\lceil \log\left(\frac{s}{c}\right) + \frac{3}{2} \right\rceil$, where c is the smallest inter-point distance and s is the side length of the initial square.
- A quadtree of depth δ storing n points has $O((\delta + 1)n)$ nodes and requires $O((\delta + 1)n)$ time to construct.
- Finding neighbors in a given direction takes time $O(\delta + 1)$
- Balancing a quadtree takes time $O((\delta + 1)m)$, where m is the number of interior nodes.

Quadtrees in Popular Culture

- One of the first DS's for higher-dimensional data
- Finkel and Bentley, 1974 *(R. A. Finkel and J. L. Bentley, Quad trees: a data structure for retrieval on composite keys. Acta Inform, 4:1–9, 1974)*
- Still perform relatively well
- Easy to generalize to higher dimensions—called octrees

M. de Berg et al., *Computational Geometry: Algorithms and Applications*, second edition, chapter 14. Cf. also chapter end notes.

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Kd-Trees in Words

- Range queries
- Partial match queries
- But not exact match queries (too easy)

Kd-Trees in Words

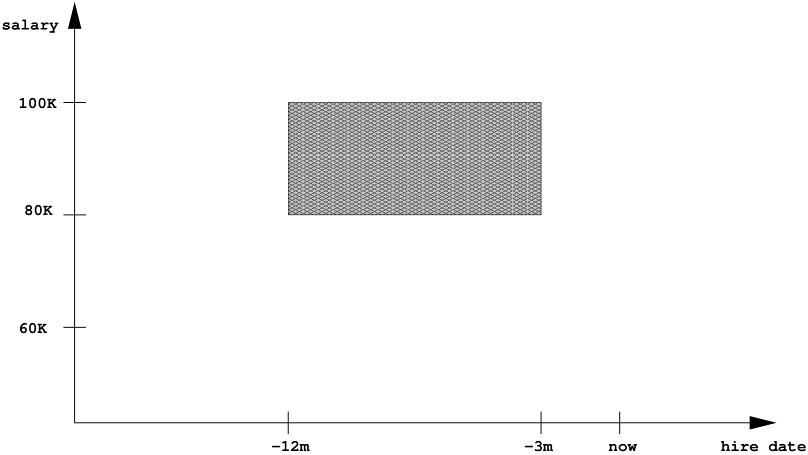
- Range queries
- Partial match queries
- But not exact match queries (too easy)
- Data bases! Records are points in space.

Kd-Trees in SQL

Find employees who are past their trial period but in their first year and who are close to hitting the £100K tax rule start.

```
SELECT salary, hire_date
FROM EMPLOYEES
WHERE 80000 < salary
  AND salary < 100000
AND now − 12m < hire_date
  AND hire_date < now − 3m
```

Kd-Trees in Pictures



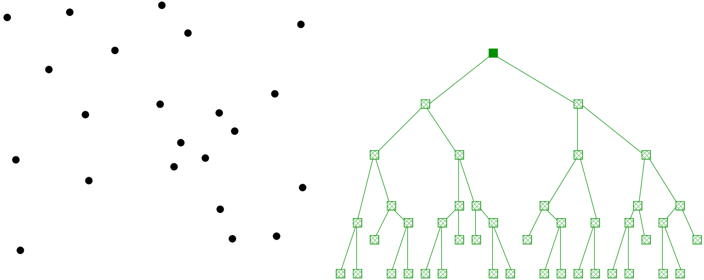
Kd-Trees in $\mathbb{R}^1$

- Binary search tree
- Find beginning of range
- Tree walk to end of range

Kd-Trees in $\mathbb{R}^2$

- At root, start with all points.
- Partition on x value to make level 1 nodes
- Partition those sets on y value to make level 2 nodes
- Repeat until done
- Result of a query is a forest of subtrees

Kd-Trees in Pictures



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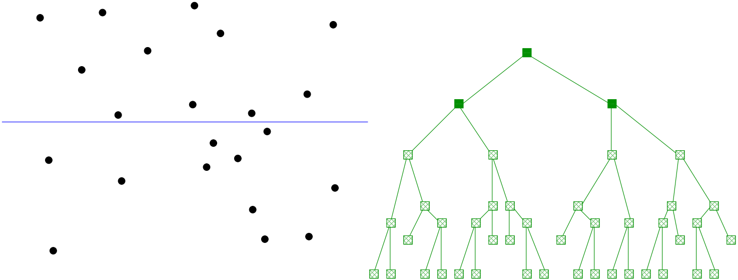

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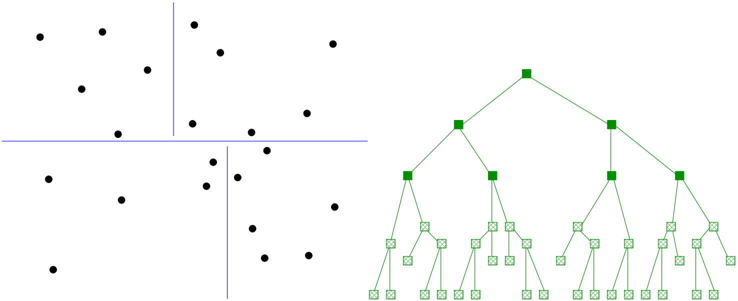

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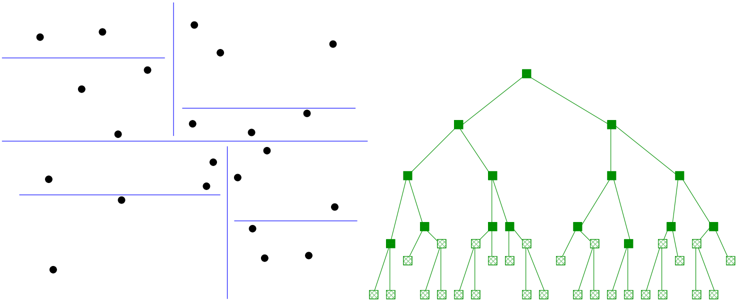

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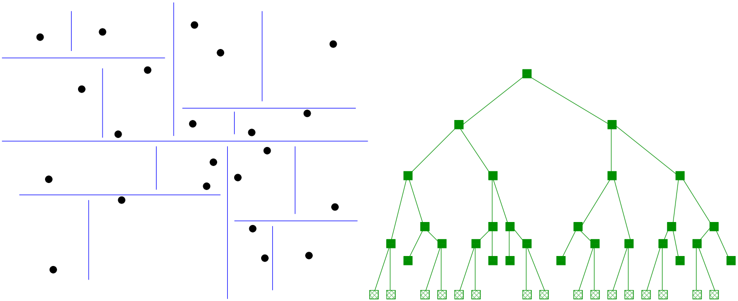

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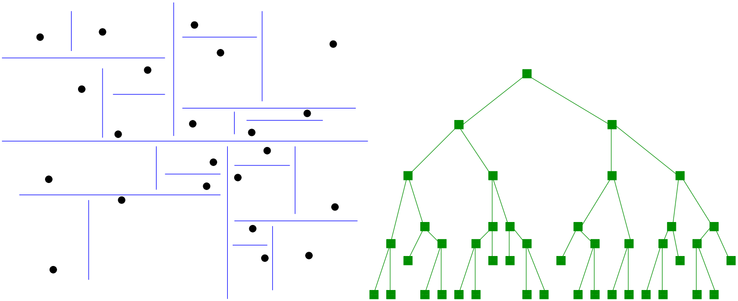

Kd-Trees in Mathematics

- Split the points into two equal sets with a horizontal separator
- If the set has only one point, return a leaf with that point
- Else recursively construct a kd-tree on each set, but splitting vertically

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More precisely, split horizontally at odd depth, vertically at even depth.

Kd-Trees in Equations

- $O(n)$ storage
- $O(n \log n)$ to build
- $O(\sqrt{n} + k)$ to query axis-parallel rectangles

Kd-Trees in Popular Culture

- Better worst-case behaviour than quadtrees
- Bentley, 1975 *(J. L. Bentley, Multidimensional binary search trees used for associative searching, Communications of the ACM, 18:509–517, 1975)*
- kNN *(for small dimension)*

M. de Berg et al., *Computational Geometry: Algorithms and Applications*, second edition, chapter 5. Cf. also chapter end notes.

Outline

Range trees in Words

- Range queries again
- In $\mathbb{R}^1$, a balanced binary search tree

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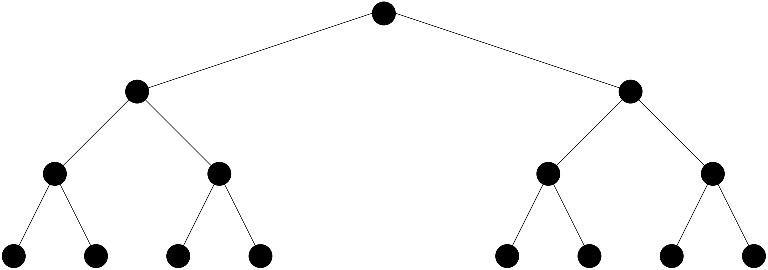
- In $\mathbb{R}^d$, split on dimension 1 with auxiliary range tree on remaining $d-1$ dimensions

Range trees in Words

- Range queries again
- In $\mathbb{R}^1$, a balanced binary search tree
- In $\mathbb{R}^d$, split on dimension 1 with auxiliary range tree on remaining $d-1$ dimensions
- Faster but bigger

Range trees in Pictures



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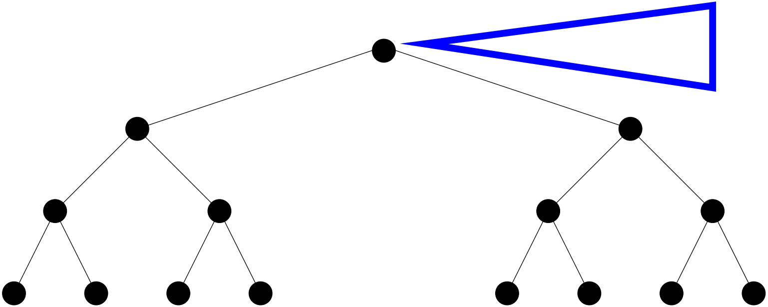

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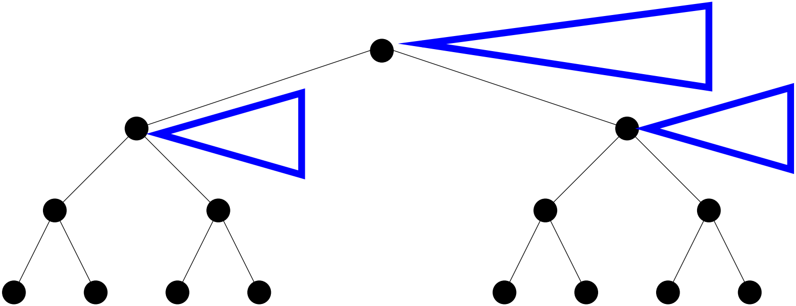

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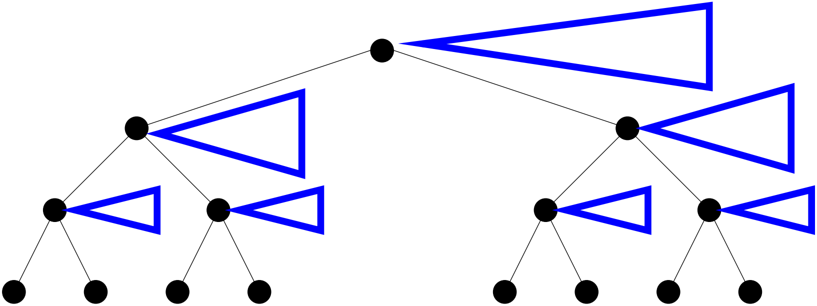

Range trees in Mathematics

- Build BST on first coordinate: $O(n \log n)$
- Recursively build range trees at each interior node on remaining $d - 1$ coordinates: ??

Range trees in Equations

Compared to kd-trees:

- Faster query times, $O(\log^d n + k)$
- Worse storage, $O(n \log^{d-1} n)$
- Worse construction, $O(n \log^{d-1} n)$

Range trees in Popular Culture

- Bentley, 1979 *J. L. Bentley, Decomposable searching problems, Information Processing Letters 8 (5): 244–251, 1979*
- Also: Lueker (1978); Lee, Wong (1980); Willard (1979)

M. de Berg et al., *Computational Geometry: Algorithms and Applications*, second edition, chapter 5. Cf. also chapter end notes.

Comparison

	Space	Time	Query
quadtree	$O\left(n \log\left(\frac{s}{c}\right)\right)$	$O\left(n \log\left(\frac{s}{c}\right)\right)$	$O\left(m \log\left(\frac{s}{c}\right)\right)$
kd-tree	$O(n)$	$O(n \log n)$	$O(\sqrt{n} + k)$
range-tree	$O(n \log^{d-1} n)$	$O(n \log^{d-1} n)$	$O(\log^{d} n + k)$

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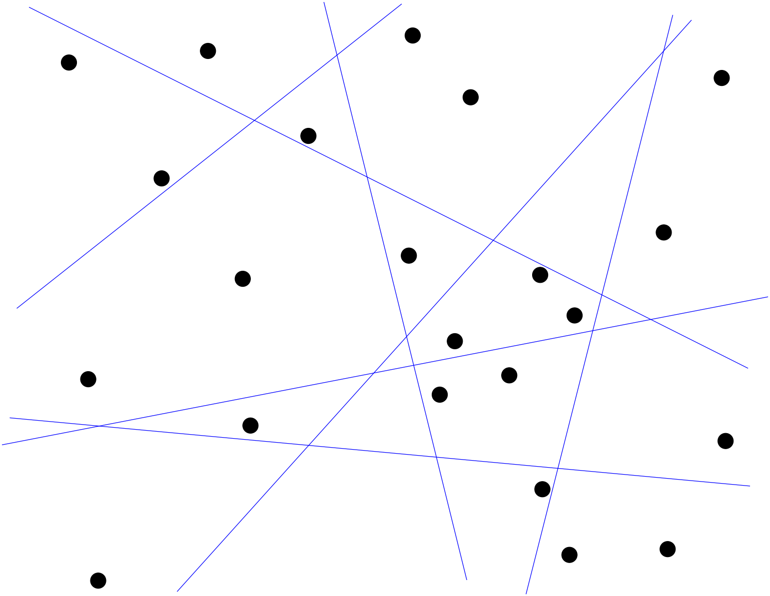
More. . .

- Voronoi diagrams
- database range searches
- kNN *(Use kd-trees for small d)*
- aNN (locality sensitive hashing)
- What if we want to insert or delete points?

aNN and LSH

- N points
- h hyperplanes, $h \ll N$
- planes are oriented
- Hamming distance, e.g.

aNN and LSH



Voronoi Diagrams in Words

- 1-nearest-neighbour
- Grass fire function

Voronoi Diagrams in Pictures

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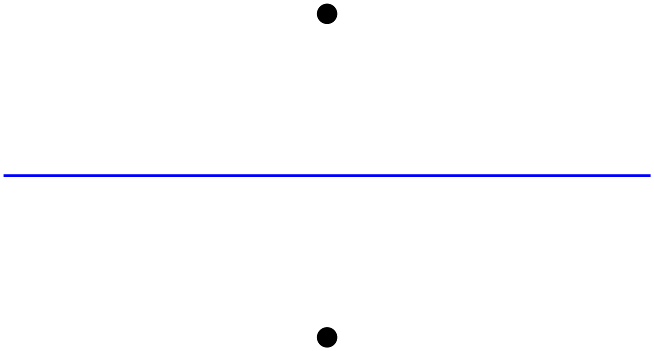

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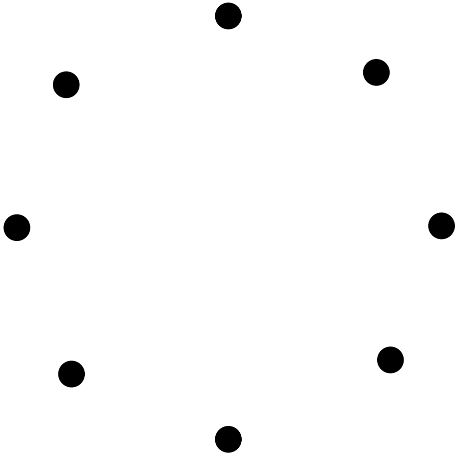

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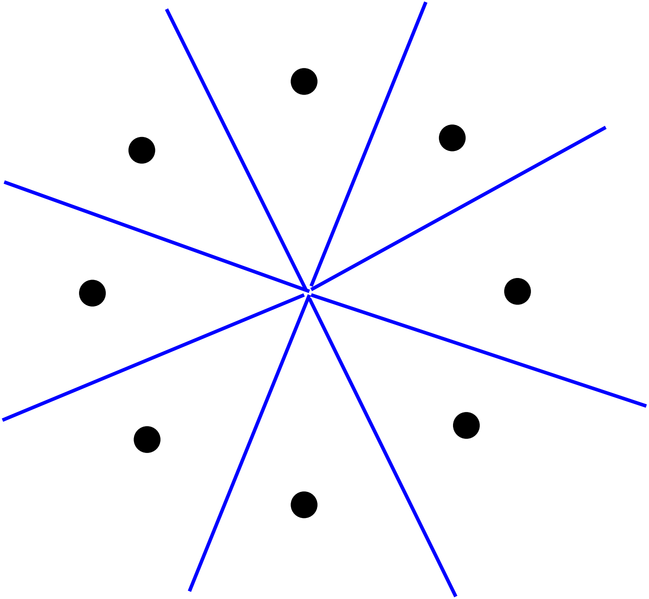

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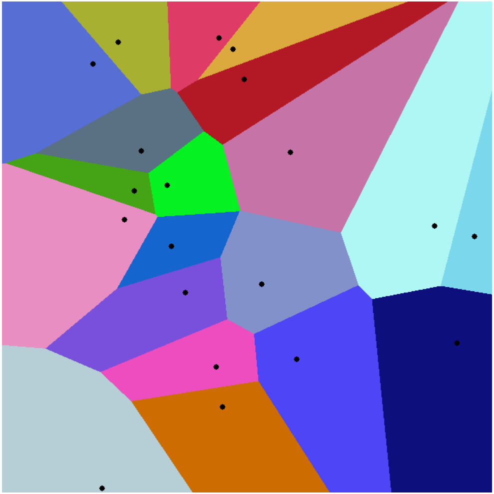
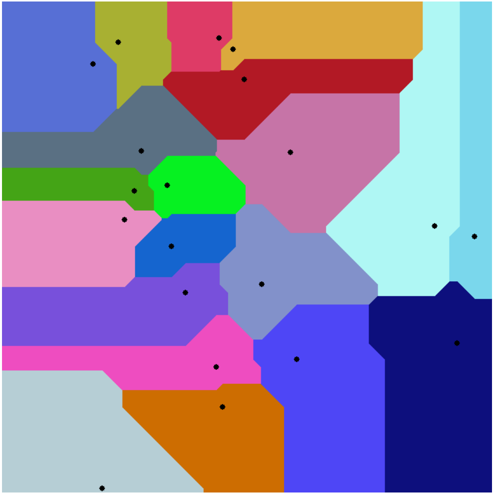


https://en.wikipedia.org/wiki/File:Euclidean_Voronoi_Diagram.png

Voronoi Diagrams in Pictures



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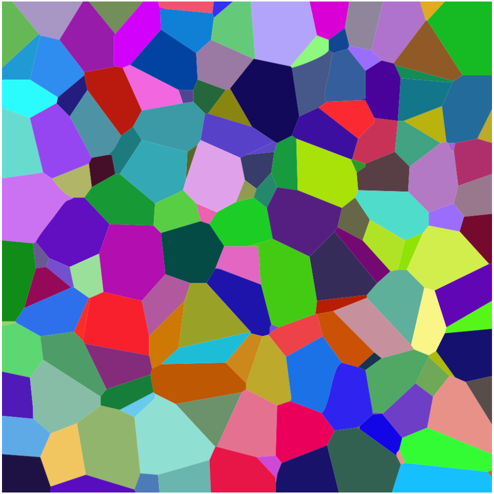

A slice of a 3D voronoi diagram on random points.

`https://en.wikipedia.org/wiki/File:Coloured_Voronoi_3D_slice.svg.png`

Voronoi Diagrams in Mathematics

- Sweep line algorithm
- We don't have enough time today...

Voronoi Diagrams in Equations

- $O(n)$ to store
- $O(n \log n)$ to construct

Voronoi Diagrams in Popular Culture

- Shamos and Hoey, 1975 *(M.I. Shamos and D. Hoey, Closest-point problems, in Proc. 16th Annual IEEE Sympos. Found. Comput. Sci., pp 151–162, 1975)*
- Delaunay triangulations *(2D)*

M. de Berg et al., *Computational Geometry: Algorithms and Applications*, second edition, chapter 7. Cf. also chapter end notes.

Summary

- Trade-offs
- Every problem is different
- Dimension matters
- This is all quite simplified

Questions